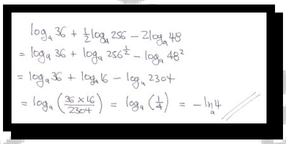
LOGARITHMS EXAM QUESTIONS

Question 1 (**)

Show clearly that

$$\log_a 36 + \frac{1}{2}\log_a 256 - 2\log_a 48 = -\log_a 4\,.$$

proof



$\log_a 36 + \frac{1}{2}\log_a 256 - 2\log_a 48$

$= \log_a 36 + \log_a 256^{\frac{1}{2}} - \log_a 48^2$

$= \log_a 36 + \log_a 16 - \log_a 2304$

$= \log_a \left(\frac{36 \times 16}{2304}\right) = \log_a \left(\frac{1}{4}\right) = -\ln_a 4$ //

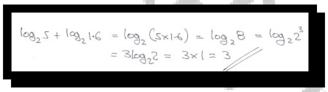
Question 2 (**)

Simplify

$$\log_2 5 + \log_2 1.6\,,$$

giving the final answer as an integer.

$\boxed{3}$



$\log_2 5 + \log_2 1{\cdot}6 = \log_2 (5 \times 1{\cdot}6) = \log_2 8 = \log_2 2^3$

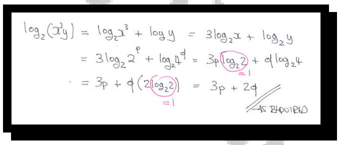
$= 3\log_2 2 = 3 \times 1 = 3$ //

Question 3 (+)**

Given that $x = 2^p$ and $y = 4^q$, show clearly that

$$\log_2 (x^3 y) = 3p + 2q\,.$$

proof



$\log_2 (x^3 y) = \log_2 x^3 + \log y = 3\log_2 x + \log_2 y$

$= 3\log_2 2^p + \log_2 4^q = 3p\underset{=1}{\boxed{\log_2 2}} + q\log_2 4$

$= 3p + q\left(2\underset{=1}{\boxed{\log_2 2}}\right) = 3p + 2q$

// AS REQUIRED

Question 4 **(**+)**

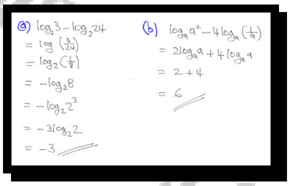
Simplify each of the following expressions, giving the final answer as an integer.

a) $\log_2 3 - \log_2 24$.

b) $\log_a a^2 - 4\log_a\left(\frac{1}{a}\right)$, $a > 0$, $a \neq 1$.

Full workings, justifying every step, must support each answer.

$\boxed{-3}$, $\boxed{6}$



(a) $\log_2 3 - \log_2 24$
$= \log\left(\frac{3}{24}\right)$
$= \log_2\left(\frac{1}{8}\right)$
$= -\log_2 8$
$= -\log_2 2^3$
$= -3\log_2 2$
$= -3$

(b) $\log_a a^2 - 4\log_a\left(\frac{1}{a}\right)$
$= 2\log_a a + 4\log_a a$
$= 2 + 4$
$= 6$

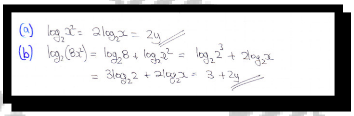
Question 5 **(**+)**

Given that $y = \log_2 x$, write each of the following expressions in terms of y.

a) $\log_2 x^2$

b) $\log_2\left(8x^2\right)$

$\boxed{2y}$, $\boxed{3+2y}$

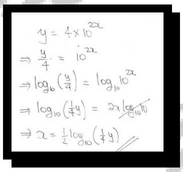


(a) $\log_2 x^2 = 2\log_2 x = 2y$
(b) $\log_2(8x^2) = \log_2 8 + \log_2 x^2 = \log_2 2^3 + 2\log_2 x$
$= 3\log_2 2 + 2\log_2 x = 3 + 2y$

Question 6 (+)**

Given that $y = 4\times10^{2x}$ express x in terms of y, giving an exact simplified answer in terms of logarithms base 10.

$$\boxed{x = \frac{1}{2}\log_{10}\left(\frac{1}{4}y\right)}$$



$$y = 4\times10^{2x}$$
$$\Rightarrow \frac{y}{4} = 10^{2x}$$
$$\Rightarrow \log_{10}\left(\frac{y}{4}\right) = \log_{10}10^{2x}$$
$$\Rightarrow \log_{10}\left(\frac{1}{4}y\right) = 2x\cancel{\log_{10}10}$$
$$\Rightarrow x = \frac{1}{2}\log_{10}\left(\frac{1}{4}y\right)$$

Question 7 (+)**

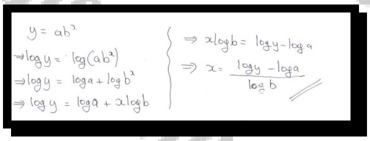
An exponential curve has equation

$$y = ab^x,\ x\in\mathbb{R},$$

where a and b are non zero constants.

Make x the subject of the above equation, giving the final answer in terms of logarithms base 10.

$$\boxed{x = \frac{\log y - \log a}{\log b}}$$



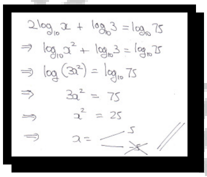
$$y = ab^x$$
$$\Rightarrow \log y = \log(ab^x)$$
$$\Rightarrow \log y = \log a + \log b^x$$
$$\Rightarrow \log y = \log a + x\log b$$
$$\Rightarrow x\log b = \log y - \log a$$
$$\Rightarrow x = \frac{\log y - \log a}{\log b}$$

Question 8 **(**+)**

Solve the following logarithmic equation

$$2\log_{10} x + \log_{10} 3 = \log_{10} 75 .$$

$\boxed{x = 5,\ x \neq -5}$



$2\log_{10} x + \log_{10} 3 = \log_{10} 75$

$\Rightarrow \log_{10} x^2 + \log_{10} 3 = \log_{10} 75$

$\Rightarrow \log\left(3x^2\right) = \log_{10} 75$

$\Rightarrow 3x^2 = 75$

$\Rightarrow x^2 = 25$

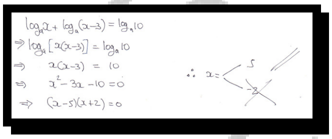
$\Rightarrow x = \left\langle \begin{matrix} 5 \\ \cancel{-5} \end{matrix} \right. \quad //$

Question 9 **(**+)**

Solve the following logarithmic equation

$$\log_a x + \log_a (x-3) = \log_a 10 .$$

$\boxed{}$, $\boxed{x = 5,\ x \neq -2}$



$\log_a x + \log_a (x-3) = \log_a 10$

$\Rightarrow \log_a [x(x-3)] = \log_a 10$

$\Rightarrow x(x-3) = 10$

$\Rightarrow x^2 - 3x - 10 = 0$

$\Rightarrow (x-5)(x+2) = 0$

$\therefore x = \left\langle \begin{matrix} 5 \\ \cancel{-2} \end{matrix} \right. \quad //$

Question 10 (***)

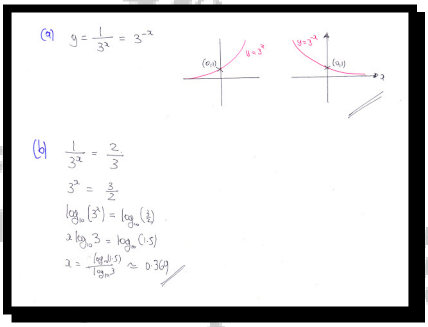
An exponential curve C has equation

$$y = \frac{1}{3^x}, \quad x \in \mathbb{R}.$$

a) Sketch the graph of C.

b) Solve the equation $y = \frac{2}{3}$, giving the answer correct to 3 significant figures.

$\boxed{}$, $\boxed{0.369}$



(a) $y = \frac{1}{3^x} = 3^{-x}$

(b) $\frac{1}{3^x} = \frac{2}{3}$

$3^x = \frac{3}{2}$

$\log_{10}(3^x) = \log_{10}\left(\frac{3}{2}\right)$

$x\log_{10}3 = \log_{10}(1.5)$

$x = \frac{\log_{10}(1.5)}{\log_{10}3} \simeq 0.369$

Question 11 (***)

Given that

$$p = \log_a 4 \quad \text{and} \quad q = \log_a 5,$$

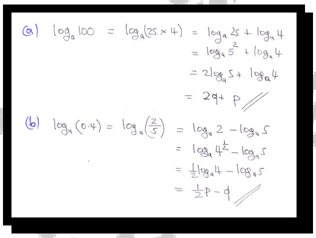
express each of the following logarithms in terms of p and q.

a) $\log_a 100$

b) $\log_a 0.4$

The final answers may not contain any logarithms.

$\boxed{\quad}$, $\boxed{p + 2q}$, $\boxed{\frac{1}{2}p - q}$



(a) $\log_a 100 = \log_a(25 \times 4) = \log_a 25 + \log_a 4$
$= \log_a 5^2 + \log_a 4$
$= 2\log_a 5 + \log_a 4$
$= 2q + p$ //

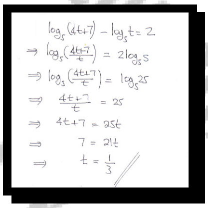
(b) $\log_a(0.4) = \log_a\left(\frac{2}{5}\right) = \log_a 2 - \log_a 5$
$= \log_a 4^{\frac{1}{2}} - \log_a 5$
$= \frac{1}{2}\log_a 4 - \log_a 5$
$= \frac{1}{2}p - q$ //

Question 12 (***)

Solve the following logarithmic equation

$$\log_5(4t + 7) - \log_5 t = 2.$$

$\boxed{\text{C2}}$, $\boxed{t = \frac{1}{3}}$



$\log_5(4t+7) - \log_5 t = 2$
$\Rightarrow \log_5\left(\frac{4t+7}{t}\right) = 2\log_5 5$
$\Rightarrow \log_5\left(\frac{4t+7}{t}\right) = \log_5 25$
$\Rightarrow \frac{4t+7}{t} = 25$
$\Rightarrow 4t + 7 = 25t$
$\Rightarrow 7 = 21t$
$\Rightarrow t = \frac{1}{3}$ //

Question 13 **(***)**

Given that

$$p = \log_2 3 \quad \text{and} \quad q = \log_2 5,$$

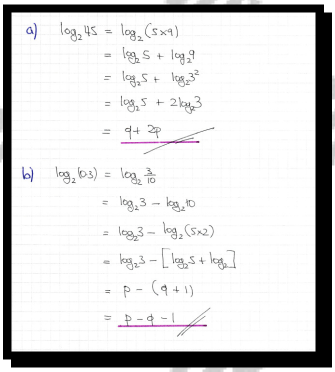
express each of the following logarithms in terms of p and q.

a) $\log_2 45$

b) $\log_2 0.3$

The final answers may not contain any logarithms.

$\boxed{\quad\quad}$, $\boxed{2p+q}$, $\boxed{p-q-1}$



a) $\log_2 45 = \log_2(5\times9)$

$= \log_2 5 + \log_2 9$

$= \log_2 5 + \log_2 3^2$

$= \log_2 5 + 2\log_2 3$

$= q + 2p$

b) $\log_2(0.3) = \log_2 \frac{3}{10}$

$= \log_2 3 - \log_2 10$

$= \log_2 3 - \log_2(5\times2)$

$= \log_2 3 - [\log_2 5 + \log_2]$

$= p - (q+1)$

$= p - q - 1$

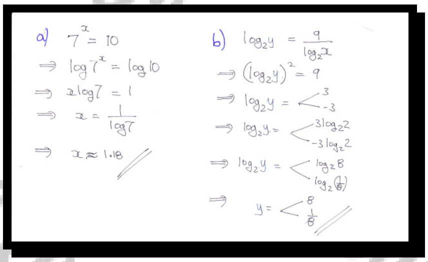
Question 14 **(***)**

Solve each of the following equations, giving the final answers correct to three significant figures, where appropriate.

a) $7^x = 10$.

b) $\log_2 y = \dfrac{9}{\log_2 y}$.

$\boxed{}$, $\boxed{x \approx 1.18}$, $\boxed{y = \frac{1}{8}, 8}$



a) $7^x = 10$

$\Rightarrow \log 7^x = \log 10$

$\Rightarrow x \log 7 = 1$

$\Rightarrow x = \dfrac{1}{\log 7}$

$\Rightarrow x \approx 1.18$

b) $\log_2 y = \dfrac{9}{\log_2 x}$

$\Rightarrow (\log_2 y)^2 = 9$

$\Rightarrow \log_2 y = < \begin{matrix} 3 \\ -3 \end{matrix}$

$\Rightarrow \log_2 y = < \begin{matrix} 3\log_2 2 \\ -3\log_2 2 \end{matrix}$

$\Rightarrow \log_2 y = < \begin{matrix} \log_2 8 \\ \log_2 \left(\frac{1}{8}\right) \end{matrix}$

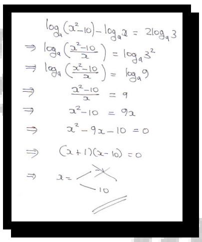
$\Rightarrow y = < \begin{matrix} 8 \\ \frac{1}{8} \end{matrix}$

Question 15 **(***)**

Solve the following logarithmic equation for x.

$$\log_a(x^2 - 10) - \log_a x = 2\log_a 3.$$

$\boxed{x = 10,\ x \neq -1}$



$\log_a(x^2-10) - \log_a x = 2\log_a 3$

$\Rightarrow \log_a\left(\dfrac{x^2-10}{x}\right) = \log_a 3^2$

$\Rightarrow \log_a\left(\dfrac{x^2-10}{x}\right) = \log_a 9$

$\Rightarrow \dfrac{x^2-10}{x} = 9$

$\Rightarrow x^2 - 10 = 9x$

$\Rightarrow x^2 - 9x - 10 = 0$

$\Rightarrow (x+1)(x-10) = 0$

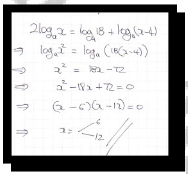
$\Rightarrow x = < \begin{matrix} \cancel{-1} \\ 10 \end{matrix}$

Question 16 (***)

Solve the following logarithmic equation for x.

$$2\log_a x = \log_a 18 + \log_a (x-4) .$$

$\boxed{x = 6,\ 12}$



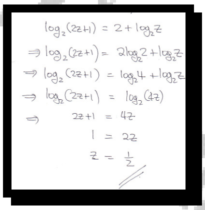
$$2\log_a x = \log_a 18 + \log_a (x-4)$$
$$\Rightarrow \log_a x^2 = \log_a (18(x-4))$$
$$\Rightarrow x^2 = 18x - 72$$
$$\Rightarrow x^2 - 18x + 72 = 0$$
$$\Rightarrow (x-6)(x-12) = 0$$
$$\Rightarrow x = \begin{cases} 6 \\ 12 \end{cases} //$$

Question 17 (***)

Solve the following logarithmic equation

$$\log_2 (2z+1) = 2 + \log_2 z .$$

$\boxed{z = \frac{1}{2}}$



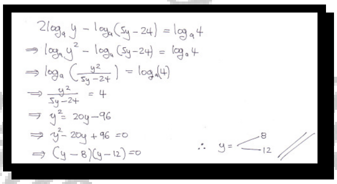
$$\log_2 (2z+1) = 2 + \log_2 z$$
$$\Rightarrow \log_2 (2z+1) = 2\log_2 2 + \log_2 z$$
$$\Rightarrow \log_2 (2z+1) = \log_2 4 + \log_2 z$$
$$\Rightarrow \log_2 (2z+1) = \log_2 (4z)$$
$$\Rightarrow 2z+1 = 4z$$
$$1 = 2z$$
$$z = \frac{1}{2} //$$

Question 18 (***)

Solve the following logarithmic equation for y.

$$2\log_a y - \log_a(5y-24) = \log_a 4 .$$

$$\boxed{x = 8,\ 12}$$



$2\log_a y - \log_a(5y-24) = \log_a 4$

$\Rightarrow \log_a y^2 - \log_a(5y-24) = \log_a 4$

$\Rightarrow \log_a\left(\frac{y^2}{5y-24}\right) = \log_a(4)$

$\Rightarrow \frac{y^2}{5y-24} = 4$

$\Rightarrow y^2 = 20y - 96$

$\Rightarrow y^2 - 20y + 96 = 0$

$\Rightarrow (y-8)(y-12) = 0$

$\therefore\ y = 8,\ 12$ //

Question 19 (***)

It is given that x satisfies the logarithmic equation

$$\log_a x = 2(\log_a k - \log_a 2),$$

where $k > 0$, $a > 0$, $a \neq 1$.

a) Find x in terms of k, giving the answer in a form not involving logarithms.

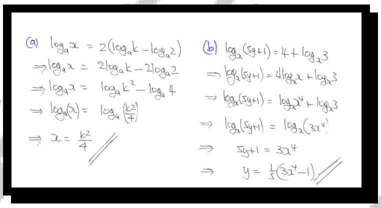
Suppose instead that x satisfies

$$\log_x(5y+1) = 4 + \log_x 3$$

where $x > 0$, $x \neq 1$ and $y > 0$, $y \neq 1$.

b) Solve the above equation expressing y in terms of x, giving the answer in a form not involving logarithms.

$$\boxed{x = \frac{k^2}{4}},\ \boxed{y = \frac{3x^4 - 1}{5}}$$



(a) $\log_a x = 2(\log_a k - \log_a 2)$

$\Rightarrow \log_a x = 2\log_a k - 2\log_a 2$

$\Rightarrow \log_a x = \log_a k^2 - \log_a 4$

$\Rightarrow \log_a(x) = \log_a\left(\frac{k^2}{4}\right)$

$\Rightarrow x = \frac{k^2}{4}$ //

(b) $\log_x(5y+1) = 4 + \log_x 3$

$\Rightarrow \log_x(5y+1) = 4\log_x x + \log_x 3$

$\Rightarrow \log_x(5y+1) = \log_x x^4 + \log_x 3$

$\Rightarrow \log_x(5y+1) = \log_x(3x^4)$

$\Rightarrow 5y+1 = 3x^4$

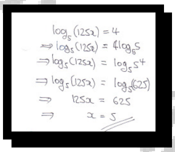
$\Rightarrow y = \frac{1}{5}(3x^4 - 1)$ //

Question 20 **(***)**

Solve the following logarithmic equation

$$\log_5(125x) = 4 .$$

$$\boxed{x = 5}$$



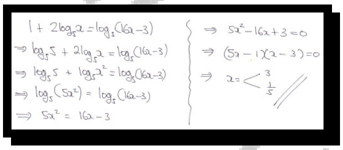
$$\log_5(125x) = 4$$
$$\Rightarrow \log_5(125x) = 4\log_5 5$$
$$\Rightarrow \log_5(125x) = \log_5 5^4$$
$$\Rightarrow \log_5(125x) = \log_5(625)$$
$$\Rightarrow 125x = 625$$
$$\Rightarrow x = 5$$

Question 21 **(***)**

Solve the following logarithmic equation

$$1 + 2\log_5 x = \log_5(16x - 3) .$$

$$\boxed{x = 3,\ x = \frac{1}{5}}$$



$$1 + 2\log_5 x = \log_5(16x - 3)$$
$$\Rightarrow \log_5 5 + 2\log_5 x = \log_5(16x - 3)$$
$$\Rightarrow \log_5 5 + \log_5 x^2 = \log_5(16x - 3)$$
$$\Rightarrow \log_5(5x^2) = \log_5(16x - 3)$$
$$\Rightarrow 5x^2 = 16x - 3$$
$$\Rightarrow 5x^2 - 16x + 3 = 0$$
$$\Rightarrow (5x - 1)(x - 3) = 0$$
$$\Rightarrow x = \begin{cases} 3 \\ \frac{1}{5} \end{cases}$$

Question 22 (***)

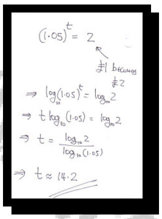
Every £1 invested in a saving scheme gains interest at the rate of 5% per annum so that the total value of this £1 investment after t years is £ y.

This is modelled by the equation

$$y = 1.05^t \, , \; t \geq 0 \, .$$

Find after how many years the investment will double.

$\boxed{t \approx 14.2}$



$(1.05)^t = 2$

£1 becomes £2

$\Rightarrow \log_{10}(1.05)^t = \log_{10} 2$

$\Rightarrow t \log_{10}(1.05) = \log_{10} 2$

$\Rightarrow t = \dfrac{\log_{10} 2}{\log_{10}(1.05)}$

$\Rightarrow t \approx 14.2$

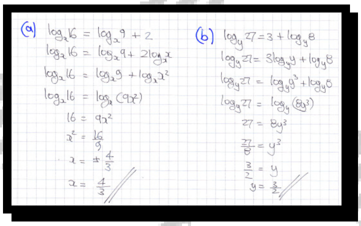
Question 23 (***)

Solve each of the following logarithmic equations.

a) $\log_x 16 = \log_x 9 + 2$.

b) $\log_y 27 = 3 + \log_y 8$.

$\boxed{x = \frac{4}{3}, \; x \neq -\frac{4}{3}}$, $\boxed{y = \frac{3}{2}}$



(a) $\log_x 16 = \log_x 9 + 2$

$\log_x 16 = \log_x 9 + 2\log_x x$

$\log_x 16 = \log_x 9 + \log_x x^2$

$\log_x 16 = \log_x (9x^2)$

$16 = 9x^2$

$x^2 = \frac{16}{9}$

$x = \pm \frac{4}{3}$

$x = \frac{4}{3}$

(b) $\log_y 27 = 3 + \log_y 8$

$\log_y 27 = 3\log_y y + \log_y 8$

$\log_y 27 = \log_y y^3 + \log_y 8$

$\log_y 27 = \log_y (8y^3)$

$27 = 8y^3$

$\frac{27}{8} = y^3$

$\frac{3}{2} = y$

$y = \frac{3}{2}$

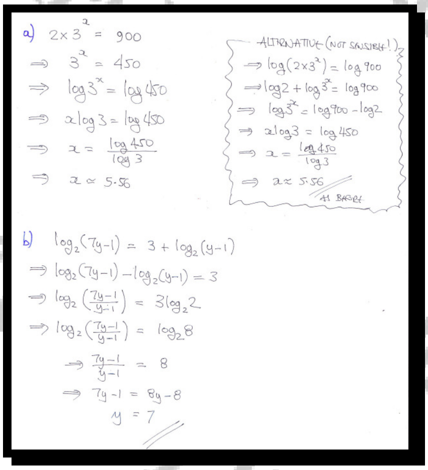
Question 24 **(***)**

Solve each of the following equations, giving the final answers correct to three significant figures, where appropriate.

a) $2\times 3^{x}=900$.

b) $\log_{2}(7y-1)=3+\log_{2}(y-1)$.

$\boxed{}$, $\boxed{x\approx 5.56}$, $\boxed{y=7}$



a) $2\times 3^{x} = 900$

$\Rightarrow 3^{x} = 450$

$\Rightarrow \log 3^{x} = \log 450$

$\Rightarrow x\log 3 = \log 450$

$\Rightarrow x = \dfrac{\log 450}{\log 3}$

$\Rightarrow x \simeq 5.56$

ALTERNATIVE (NOT SENSIBLE!)

$\Rightarrow \log(2\times 3^{x}) = \log 900$

$\Rightarrow \log 2 + \log 3^{x} = \log 900$

$\Rightarrow \log 3^{x} = \log 900 - \log 2$

$\Rightarrow x\log 3 = \log 450$

$\Rightarrow x = \dfrac{\log 450}{\log 3}$

$\Rightarrow x \approx 5.56$

AS BEFORE

b) $\log_{2}(7y-1) = 3 + \log_{2}(y-1)$

$\Rightarrow \log_{2}(7y-1) - \log_{2}(y-1) = 3$

$\Rightarrow \log_{2}\left(\dfrac{7y-1}{y-1}\right) = 3\log_{2}2$

$\Rightarrow \log_{2}\left(\dfrac{7y-1}{y-1}\right) = \log_{2}8$

$\Rightarrow \dfrac{7y-1}{y-1} = 8$

$\Rightarrow 7y-1 = 8y-8$

$y = 7$

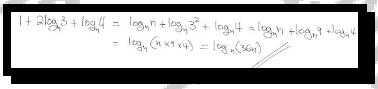
Question 25 **(***+)**

Simplify fully

$$1+2\log_{n}3+\log_{n}4\,,$$

giving the final answer as a single logarithm.

$\boxed{\log_{n}(36n)}$



$1 + 2\log_{n}3 + \log_{n}4 = \log_{n}n + \log_{n}3^{2} + \log_{n}4 = \log_{n}n + \log_{n}9 + \log_{n}4$

$= \log_{n}(n\times 9\times 4) = \log_{n}(36n)$

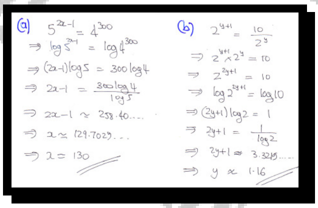
Question 26 **(***+)**

Solve each of the following exponential equations, giving the final answers correct to 3 significant figures.

a) $5^{2x-1} = 4^{300}$.

b) $2^{y+1} = \dfrac{10}{2^{y}}$.

$\boxed{}$, $\boxed{x \approx 130}$, $\boxed{y \approx 1.16}$



(a) $5^{2x-1} = 4^{300}$

$\Rightarrow \log 5^{2x-1} = \log 4^{300}$

$\Rightarrow (2x-1)\log 5 = 300\log 4$

$\Rightarrow 2x-1 = \dfrac{300\log 4}{\log 5}$

$\Rightarrow 2x-1 \simeq 258.40\ldots$

$\Rightarrow x \simeq 129.7029\ldots$

$\Rightarrow x = 130$

(b) $2^{y+1} = \dfrac{10}{2^{y}}$

$\Rightarrow 2^{y+1} \times 2^{y} = 10$

$\Rightarrow 2^{2y+1} = 10$

$\Rightarrow \log 2^{2y+1} = \log 10$

$\Rightarrow (2y+1)\log 2 = 1$

$\Rightarrow 2y+1 = \dfrac{1}{\log 2}$

$\Rightarrow 2y+1 \approx 3.3219\ldots$

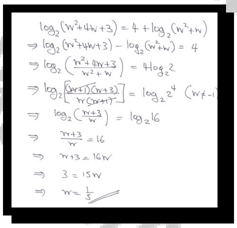
$\Rightarrow y \approx 1.16$

Question 27 **(***+)**

Solve the following logarithmic equation

$$\log_2(w^2+4w+3) = 4+\log_2(w^2+w), \; w \neq -1.$$

$\boxed{w = \frac{1}{5}}$



$\log_2(w^2+4w+3) = 4+\log_2(w^2+w)$

$\Rightarrow \log_2(w^2+4w+3) - \log_2(w^2+w) = 4$

$\Rightarrow \log_2\left(\dfrac{w^2+4w+3}{w^2+w}\right) = 4\log_2 2$

$\Rightarrow \log_2\left[\dfrac{\cancel{(w+1)}(w+3)}{w\cancel{(w+1)}}\right] = \log_2 2^4 \quad (w \neq -1)$

$\Rightarrow \log_2\left(\dfrac{w+3}{w}\right) = \log_2 16$

$\Rightarrow \dfrac{w+3}{w} = 16$

$\Rightarrow w+3 = 16w$

$\Rightarrow 3 = 15w$

$\Rightarrow w = \frac{1}{5}$

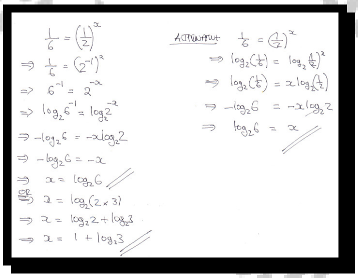
Question 28 **(***+)**

Solve the following exponential equation

$$\frac{1}{6}=\left(\frac{1}{2}\right)^{x},$$

giving the answer as single logarithm of base 2.

$$\boxed{x=\log_2 6=1+\log_2 3}$$



$\frac{1}{6}=\left(\frac{1}{2}\right)^{x}$

$\Rightarrow \frac{1}{6}=\left(2^{-1}\right)^{x}$

$\Rightarrow 6^{-1}=2^{-x}$

$\Rightarrow \log_2 6^{-1}=\log_2 2^{-x}$

$\Rightarrow -\log_2 6=-x\log_2 2$

$\Rightarrow -\log_2 6=-x$

$\Rightarrow x=\log_2 6$ //

OR $\Rightarrow x=\log_2(2\times 3)$

$\Rightarrow x=\log_2 2+\log_2 3$

$\Rightarrow x=1+\log_2 3$ //

ALTERNATIVE $\frac{1}{6}=\left(\frac{1}{2}\right)^{x}$

$\Rightarrow \log_2\left(\frac{1}{6}\right)=\log_2\left(\frac{1}{2}\right)^{x}$

$\Rightarrow \log_2\left(\frac{1}{6}\right)=x\log_2\left(\frac{1}{2}\right)$

$\Rightarrow -\log_2 6=-x\log_2 2$

$\Rightarrow \log_2 6=x$ //

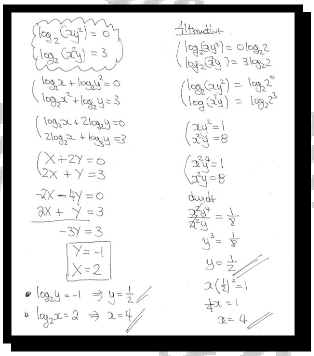
Question 29 **(***+)**

Solve the following simultaneous logarithmic equations

$$\log_2\left(xy^2\right)=0$$

$$\log_2\left(x^2y\right)=3.$$

$$\boxed{\text{C4U}},\ \boxed{x=4,\ y=\frac{1}{2}}$$



$\log_2\left(xy^2\right)=0$

$\log_2\left(x^2y\right)=3$

$\log_2 x+\log_2 y^2=0$

$\log_2 x^2+\log_2 y=3$

$\log_2 x+2\log_2 y=0$

$2\log_2 x+\log_3 y=3$

$X+2Y=0$

$2X+Y=3$

$-2X-4Y=0$

$2X+Y=3$

$-3Y=3$

$\boxed{\begin{array}{l}Y=-1\\X=2\end{array}}$

- $\log_2 y=-1 \Rightarrow y=\frac{1}{2}$ //
- $\log_2 x=2 \Rightarrow x=4$ //

Alternative

$\log_2\left(xy^2\right)=0\log_2 2$

$\log_2\left(x^2y\right)=3\log_2 2$

$\log_2\left(xy^2\right)=\log_2 2^0$

$\log\left(x^2y\right)=\log_2 2^3$

$xy^2=1$

$x^2y=8$

$x^2y^4=1$

$x^2y=8$

divide

$\frac{\cancel{x^2}y^4}{\cancel{x^2}y}=\frac{1}{8}$

$y^3=\frac{1}{8}$

$y=\frac{1}{2}$ //

$x\left(\frac{1}{2}\right)^2=1$

$\frac{1}{4}x=1$

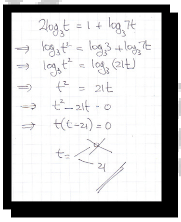
$x=4$ //

Question 30 (***+)

Solve the following logarithmic equation

$$2\log_3 t = 1 + \log_3 7t\,.$$

$\boxed{t = 21,\ t \neq 0}$



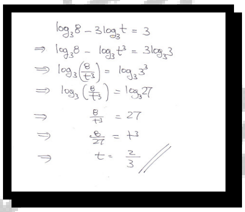
$$2\log_3 t = 1 + \log_3 7t$$
$$\Rightarrow \log_3 t^2 = \log_3 3 + \log_3 7t$$
$$\Rightarrow \log_3 t^2 = \log_3 (21t)$$
$$\Rightarrow t^2 = 21t$$
$$\Rightarrow t^2 - 21t = 0$$
$$\Rightarrow t(t-21) = 0$$
$$t = \begin{cases} \cancel{0} \\ 21 \end{cases} //$$

Question 31 (***+)

Solve the following logarithmic equation

$$\log_3 8 - 3\log_3 t = 3\,.$$

$\boxed{\quad}$, $\boxed{t = \frac{2}{3}}$



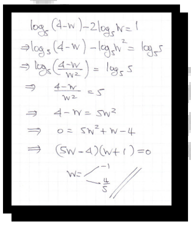
$$\log_3 8 - 3\log_3 t = 3$$
$$\Rightarrow \log_3 8 - \log_3 t^3 = 3\log_3 3$$
$$\Rightarrow \log_3\left(\frac{8}{t^3}\right) = \log_3 3^3$$
$$\Rightarrow \log_3\left(\frac{8}{t^3}\right) = \log_3 27$$
$$\Rightarrow \frac{8}{t^3} = 27$$
$$\Rightarrow \frac{8}{27} = t^3$$
$$\Rightarrow t = \frac{2}{3} //$$

Question 32 **(***+)**

Solve the following logarithmic equation

$$\log_5(4-w)-2\log_5 w=1.$$

$\boxed{}$, $\boxed{w=\frac{4}{5},\ w\neq -1}$



$\log_5(4-w)-2\log_5 w=1$

$\Rightarrow \log_5(4-w)-\log_5 w^2=\log_5 5$

$\Rightarrow \log_5\left(\frac{4-w}{w^2}\right)=\log_5 5$

$\Rightarrow \frac{4-w}{w^2}=5$

$\Rightarrow 4-w=5w^2$

$\Rightarrow 0=5w^2+w-4$

$\Rightarrow (5w-4)(w+1)=0$

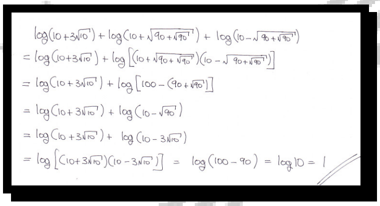
$w = \begin{cases} -1 \\ \frac{4}{5} \end{cases}$ //

Question 33 **(***+)**

Simplify fully the following logarithmic expression, showing clearly all the workings.

$$\log\left(10+3\sqrt{10}\right)+\log\left(10+\sqrt{90+\sqrt{90}}\right)+\log\left(10-\sqrt{90+\sqrt{90}}\right).$$

$\boxed{1}$



$\log(10+3\sqrt{10})+\log(10+\sqrt{90+\sqrt{90}})+\log(10-\sqrt{90+\sqrt{90}})$

$=\log(10+3\sqrt{10})+\log\left[(10+\sqrt{90+\sqrt{90}})(10-\sqrt{90+\sqrt{90}})\right]$

$=\log(10+3\sqrt{10})+\log\left[100-(90+\sqrt{90})\right]$

$=\log(10+3\sqrt{10})+\log(10-\sqrt{90})$

$=\log(10+3\sqrt{10})+\log(10-3\sqrt{10})$

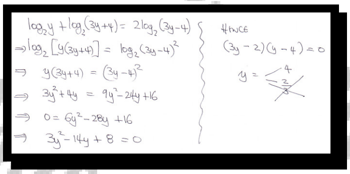
$=\log\left[(10+3\sqrt{10})(10-3\sqrt{10})\right] = \log(100-90)=\log 10=1$ //

Question 34 **(***+)**

Solve the following logarithmic equation

$$\log_2 y + \log_2(3y+4) = 2\log_2(3y-4).$$

$$\boxed{y=4,\ y \neq \frac{2}{3}}$$



$\log_2 y + \log_2(3y+4) = 2\log_2(3y-4)$

$\Rightarrow \log_2\left[y(3y+4)\right] = \log_2(3y-4)^2$

$\Rightarrow y(3y+4) = (3y-4)^2$

$\Rightarrow 3y^2+4y = 9y^2-24y+16$

$\Rightarrow 0 = 6y^2-28y+16$

$\Rightarrow 3y^2-14y+8=0$

Hence

$(3y-2)(y-4)=0$

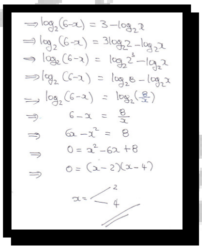
$y = \begin{cases} 4 \\ \cancel{\frac{2}{3}} \end{cases}$

Question 35 **(***+)**

Solve the following logarithmic equation

$$\log_2(6-x) = 3-\log_2 x.$$

$$\boxed{x=2,\ 4}$$



$\Rightarrow \log_2(6-x) = 3-\log_2 x$

$\Rightarrow \log_2(6-x) = 3\log_2 2-\log_2 x$

$\Rightarrow \log_2(6-x) = \log_2 2^3-\log_2 x$

$\Rightarrow \log_2(6-x) = \log_2 8-\log_2 x$

$\Rightarrow \log_2(6-x) = \log_2\left(\frac{8}{x}\right)$

$\Rightarrow 6-x = \frac{8}{x}$

$\Rightarrow 6x-x^2 = 8$

$\Rightarrow 0 = x^2-6x+8$

$\Rightarrow 0 = (x-2)(x-4)$

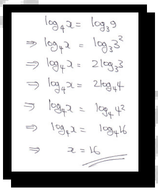
$x = \begin{cases} 2 \\ 4 \end{cases}$

Question 36 **(***+)**

Solve the following logarithmic equation

$$\log_4 x = \log_3 9 .$$

$\boxed{x = 16}$



$\log_4 x = \log_3 9$

$\Rightarrow \log_4 x = \log_3 3^2$

$\Rightarrow \log_4 x = 2\log_3 3$

$\Rightarrow \log_4 x = 2\log_4 4$

$\Rightarrow \log_4 x = \log_4 4^2$

$\Rightarrow \log_4 x = \log_4 16$

$\Rightarrow x = 16$

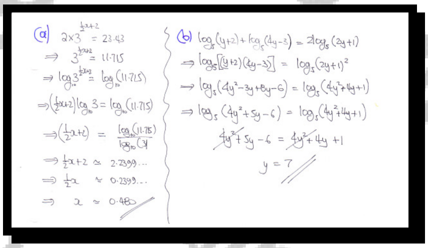
Question 37 **(***+)**

Solve each of the following equations.

a) $2 \times 3^{\frac{1}{2}x+2} = 23.43$.

b) $\log_5 (y+2) + \log_5 (4y-3) = 2\log_5 (2y+1)$.

$\boxed{x \approx 0.480}$, $\boxed{y = 7}$



(a) $2 \times 3^{\frac{1}{2}x+2} = 23.43$

$\Rightarrow 3^{\frac{1}{2}x+2} = 11.715$

$\Rightarrow \log_{10} 3^{\frac{1}{2}x+2} = \log_{10}(11.715)$

$\Rightarrow \left(\frac{1}{2}x+2\right)\log_{10} 3 = \log_{10}(11.715)$

$\Rightarrow \left(\frac{1}{2}x+2\right) = \frac{\log_{10}(11.715)}{\log_{10}(3)}$

$\Rightarrow \frac{1}{2}x+2 \simeq 2.2399\ldots$

$\Rightarrow \frac{1}{2}x \approx 0.2399\ldots$

$\Rightarrow x \approx 0.480$

(b) $\log_5 (y+2) + \log_5 (4y-3) = 2\log_5 (2y+1)$

$\Rightarrow \log_5 [(y+2)(4y-3)] = \log_5 (2y+1)^2$

$\Rightarrow \log_5 (4y^2-3y+8y-6) = \log_5 (4y^2+4y+1)$

$\Rightarrow \log_5 (4y^2+5y-6) = \log_5 (4y^2+4y+1)$

$\cancel{4y^2} + 5y - 6 = \cancel{4y^2} + 4y + 1$

$y = 7$

Question 38 **(***+)**

The population P of a certain town in time t years is modelled by the equation

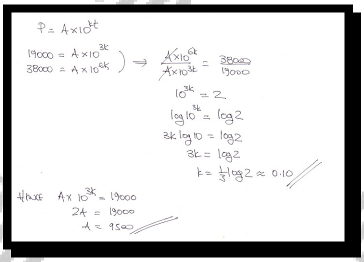
$$P = A \times 10^{kt}, \; t \geq 0,$$

where A and k are non zero constants.

When $t = 3$, $P = 19000$ and when $t = 6$, $P = 38000$.

Find the value of A and the value of k, correct to 2 significant figures.

$\boxed{\text{C2Y}}$, $\boxed{A = 9500, \; k = 0.10}$



$P = A \times 10^{kt}$

$\left.\begin{array}{l} 19000 = A \times 10^{3k} \\ 38000 = A \times 10^{6k} \end{array}\right) \Rightarrow \frac{\cancel{A} \times 10^{6k}}{\cancel{A} \times 10^{3k}} = \frac{38000}{19000}$

$10^{3k} = 2$

$\log 10^{3k} = \log 2$

$3k \log 10 = \log 2$

$3k = \log 2$

$k = \frac{1}{3} \log 2 \approx 0.10$ //

Hence $A \times 10^{3k} = 19000$

$2A = 19000$

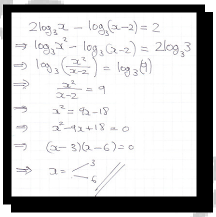
$A = 9500$ //

Question 39 **(***+)**

Solve the following logarithmic equation

$$2\log_3 x - \log_3 (x - 2) = 2.$$

$\boxed{}$, $\boxed{x = 3, \; 6}$



$2\log_3 x - \log_3 (x-2) = 2$

$\Rightarrow \log_3 x^2 - \log_3 (x-2) = 2\log_3 3$

$\Rightarrow \log_3 \left(\frac{x^2}{x-2}\right) = \log_3 (9)$

$\Rightarrow \frac{x^2}{x-2} = 9$

$\Rightarrow x^2 = 9x - 18$

$\Rightarrow x^2 - 9x + 18 = 0$

$\Rightarrow (x-3)(x-6) = 0$

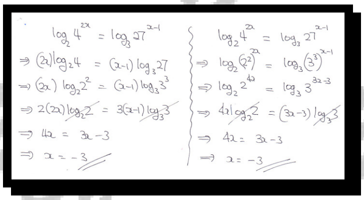
$\Rightarrow x = \begin{cases} 3 \\ 6 \end{cases}$ //

Question 40 **(***+)**

Solve the following logarithmic equation

$$\log_2 4^{2x} = \log_3 27^{x-1}.$$

$$\boxed{x = -3}$$



$$\log_2 4^{2x} = \log_3 27^{x-1}$$

$$\Rightarrow (2x)\log_2 4 = (x-1)\log_3 27$$

$$\Rightarrow (2x)\log_2 2^2 = (x-1)\log_3 3^3$$

$$\Rightarrow 2(2x)\cancel{\log_2 2} = 3(x-1)\cancel{\log_3 3}$$

$$\Rightarrow 4x = 3x-3$$

$$\Rightarrow x = -3$$

$$\log_2 4^{2x} = \log_3 27^{x-1}$$

$$\Rightarrow \log_2 (2^2)^{2x} = \log_3 (3^3)^{x-1}$$

$$\Rightarrow \log_2 2^{4x} = \log_3 3^{3x-3}$$

$$\Rightarrow 4x\cancel{\log_2 2} = (3x-3)\cancel{\log_3 3}$$

$$\Rightarrow 4x = 3x-3$$

$$\Rightarrow x = -3$$

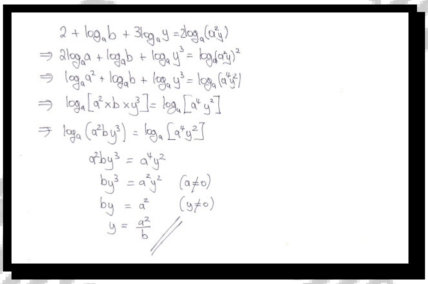
Question 41 **(***+)**

Given that $a \neq 0$, $b \neq 0$, $y \neq 0$ and

$$2 + \log_a b + 3\log_a y = 2\log_a\left(a^2 y\right),$$

express y in terms of a and b, in a form **not** involving logarithms.

$$\boxed{y = \frac{a^2}{b}}$$



$$2 + \log_a b + 3\log_a y = 2\log_a(a^2y)$$

$$\Rightarrow 2\log_a a + \log_a b + \log_a y^3 = \log_a(a^2y)^2$$

$$\Rightarrow \log_a a^2 + \log_a b + \log_a y^3 = \log_a(a^4y^2)$$

$$\Rightarrow \log_a[a^2 \times b \times y^3] = \log_a[a^4y^2]$$

$$\Rightarrow \log_a(a^2by^3) = \log_a[a^4y^2]$$

$$a^2by^3 = a^4y^2$$

$$by^3 = a^2y^2 \quad (a \neq 0)$$

$$by = a^2 \quad (y \neq 0)$$

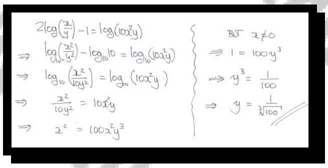
$$y = \frac{a^2}{b}$$

Question 42 **(***+)**

$$2\log\left(\frac{x}{y}\right) - 1 = \log(10x^2 y), \; x \neq 0, \; y \neq 0 .$$

Find the exact value of y.

$$\boxed{y = \frac{1}{\sqrt[3]{100}}}$$



$2\log\left(\frac{x}{y}\right) - 1 = \log(10x^2y)$

$\Rightarrow \log_{10}\left(\frac{x^2}{y^2}\right) - \log_{10}10 = \log_{10}(10x^2y)$

$\Rightarrow \log_{10}\left(\frac{x^2}{10y^2}\right) = \log_{10}(10x^2y)$

$\Rightarrow \frac{x^2}{10y^2} = 10x^2y$

$\Rightarrow x^2 = 100x^2y^3$

BUT $x \neq 0$

$\Rightarrow 1 = 100y^3$

$\Rightarrow y^3 = \frac{1}{100}$

$\Rightarrow y = \frac{1}{\sqrt[3]{100}}$

Question 43 **(***+)** **non calculator**

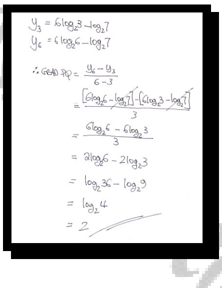
The points P and Q lie on the curve with equation

$$y = 6\log_2 x - \log_2 7, \; x > 0 .$$

The x coordinates of P and Q are 3 and 6, respectively.

Find the gradient of the straight line segment PQ.

$$\boxed{2}$$



$y_3 = 6\log_2 3 - \log_2 7$

$y_6 = 6\log_2 6 - \log_2 7$

$\therefore$ GRAD PQ $= \frac{y_6 - y_3}{6-3}$

$= \frac{[6\log_2 6 - \cancel{\log_2 7}] - [6\log_2 3 - \cancel{\log_2 7}]}{3}$

$= \frac{6\log_2 6 - 6\log_2 3}{3}$

$= 2\log_2 6 - 2\log_2 3$

$= \log_2 36 - \log_2 9$

$= \log_2 4$

$= 2$

Question 44 (***+)

$$y = 3 \times 2^{x}.$$

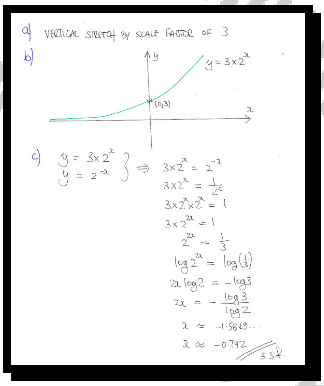
a) Describe the geometric transformation which maps the graph of the curve with equation $y = 2^{x}$, onto the graph of the curve with equation $y = 3 \times 2^{x}$.

b) Sketch the graph of $y = 3 \times 2^{x}$.

The curve with equation $y = 2^{-x}$ intersects the curve with equation $y = 3 \times 2^{x}$ at the point P.

c) Determine, correct to 3 decimal places, the x coordinate of P.

$\boxed{}$, $\boxed{\text{vertical stretch by scale factor 3}}$, $\boxed{x \approx -0.792}$



a) VERTICAL STRETCH BY SCALE FACTOR OF 3

b)

c) $\left.\begin{array}{l} y = 3 \times 2^{x} \\ y = 2^{-x} \end{array}\right\} \Rightarrow 3 \times 2^{x} = 2^{-x}$

$3 \times 2^{x} = \frac{1}{2^{x}}$

$3 \times 2^{x} \times 2^{x} = 1$

$3 \times 2^{2x} = 1$

$2^{2x} = \frac{1}{3}$

$\log 2^{2x} = \log\left(\frac{1}{3}\right)$

$2x \log 2 = -\log 3$

$2x = -\frac{\log 3}{\log 2}$

$x \approx -1.5849\ldots$

$x \approx -0.792$

3 sf.

Question 45 **(***+)**

It is given that $p = \log_6 25$ and $q = \log_6 2$.

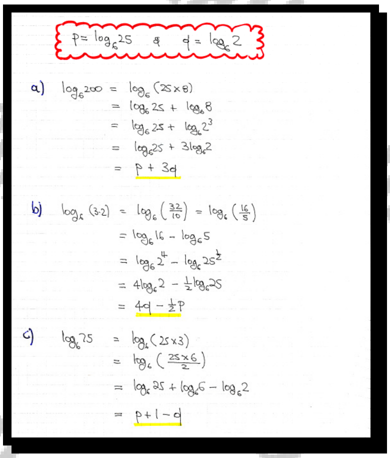
Express in terms of p and q each of the following expressions

a) $\log_6 200$

b) $\log_6 3.2$

c) $\log_6 75$

$\boxed{\quad\quad}$, $\boxed{\log_6 200 = p + 3q}$, $\boxed{\log_6 3.2 = -\frac{1}{2}p + 4q}$, $\boxed{\log_6 75 = p - q + 1}$



$p = \log_6 25$ & $q = \log_6 2$

a) $\log_6 200 = \log_6 (25 \times 8)$

$= \log_6 25 + \log_6 8$

$= \log_6 25 + \log_6 2^3$

$= \log_6 25 + 3\log_6 2$

$= p + 3q$

b) $\log_6 (3.2) = \log_6 \left(\frac{32}{10}\right) = \log_6 \left(\frac{16}{5}\right)$

$= \log_6 16 - \log_6 5$

$= \log_6 2^4 - \log_6 25^{\frac{1}{2}}$

$= 4\log_6 2 - \frac{1}{2}\log_6 25$

$= 4q - \frac{1}{2}p$

c) $\log_6 75 = \log_6 (25 \times 3)$

$= \log_6 \left(\frac{25 \times 6}{2}\right)$

$= \log_6 25 + \log_6 6 - \log_6 2$

$= p + 1 - q$

Question 46 (****)

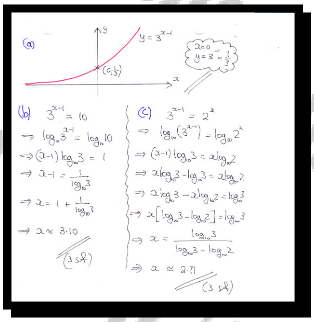
$$y = 3^{x-1}, \ x \in \mathbb{R}.$$

a) Sketch the graph of $y = 3^{x-1}$ showing the coordinates of all intercepts with the coordinate axes.

b) Find to 3 significant figures the x coordinate of the point where the curve $y = 3^{x-1}$ intersects with the straight line with equation $y = 10$.

c) Determine to 3 significant figures the x coordinate of the point where the curve $y = 3^{x-1}$ intersects with the curve $y = 2^x$.

$\boxed{3.10}$, $\boxed{2.71}$



(a)

$y = 3^{x-1}$

$(0, \frac{1}{3})$

$x=0$
$y = 3^{-1} = \frac{1}{3}$

(b) $3^{x-1} = 10$
$\Rightarrow \log_{10} 3^{x-1} = \log_{10} 10$
$\Rightarrow (x-1)\log_{10} 3 = 1$
$\Rightarrow x - 1 = \frac{1}{\log_{10} 3}$
$\Rightarrow x = 1 + \frac{1}{\log_{10} 3}$
$\Rightarrow x \approx 3.10$
(3 sf)

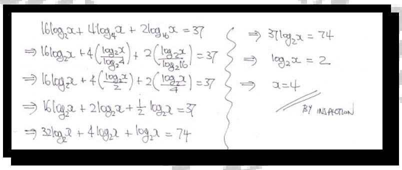
(c) $3^{x-1} = 2^x$
$\Rightarrow \log_{10}(3^{x-1}) = \log_{10} 2^x$
$\Rightarrow (x-1)\log_{10} 3 = x\log_{10} 2$
$\Rightarrow x\log_{10} 3 - \log_{10} 3 = x\log_{10} 2$
$\Rightarrow x\log_{10} 3 - x\log_{10} 2 = \log_{10} 3$
$\Rightarrow x[\log_{10} 3 - \log_{10} 2] = \log_{10} 3$
$\Rightarrow x = \frac{\log_{10} 3}{\log_{10} 3 - \log_{10} 2}$
$\Rightarrow x \approx 2.71$
(3 sf)

Question 47 (****)

Solve the following logarithmic equation

$$16\log_2 x + 4\log_4 x + 2\log_{16} x = 37, \quad x > 0.$$

$\boxed{x = 4}$



$16\log_2 x + 4\log_4 x + 2\log_{16} x = 37$
$\Rightarrow 16\log_2 x + 4\left(\frac{\log_2 x}{\log_2 4}\right) + 2\left(\frac{\log_2 x}{\log_2 16}\right) = 37$
$\Rightarrow 16\log_2 x + 4\left(\frac{\log_2 x}{2}\right) + 2\left(\frac{\log_2 x}{4}\right) = 37$
$\Rightarrow 16\log_2 x + 2\log_2 x + \frac{1}{2}\log_2 x = 37$
$\Rightarrow 32\log_2 x + 4\log_2 x + \log_2 x = 74$
$\Rightarrow 37\log_2 x = 74$
$\Rightarrow \log_2 x = 2$
$\Rightarrow x = 4$
BY INSPECTION

Question 48 (****)

In 1970 the average weekly pay of footballers in a certain club was £100.

The average weekly pay, £P, is modelled by the equation

$$P = A \times b^t,$$

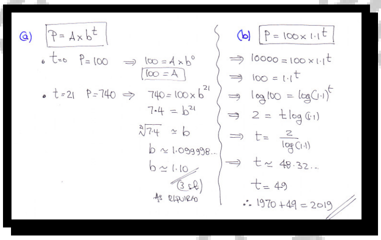
where t is the number of years since 1970, and A and b are positive constants.

In 1991 the average weekly pay of footballers in the same club had risen to £740.

a) Find the value of A and show that $b = 1.10$, correct to three significant figures.

b) Determine the year when the average weekly pay of footballers in this club will first exceed £10000.

$\boxed{}$, $\boxed{A = 100}$, $\boxed{2019}$



(a) $\boxed{P = A \times b^t}$

- $t=0$ $P=100$ $\Rightarrow$ $100 = A \times b^0$
  $\boxed{100 = A}$
- $t=21$ $P=740$ $\Rightarrow$ $740 = 100 \times b^{21}$
  $7.4 = b^{21}$
  $\sqrt[21]{7.4} \simeq b$
  $b \simeq 1.099998\ldots$
  $b \simeq 1.10$ (3 sf)
  AS REQUIRED

(b) $\boxed{P = 100 \times 1.1^t}$

$\Rightarrow 10000 = 100 \times 1.1^t$

$\Rightarrow 100 = 1.1^t$

$\Rightarrow \log 100 = \log(1.1)^t$

$\Rightarrow 2 = t \log(1.1)$

$\Rightarrow t = \frac{2}{\log(1.1)}$

$\Rightarrow t \simeq 48.32\ldots$

$t = 49$

$\therefore 1970 + 49 = 2019$

Question 49 (****)

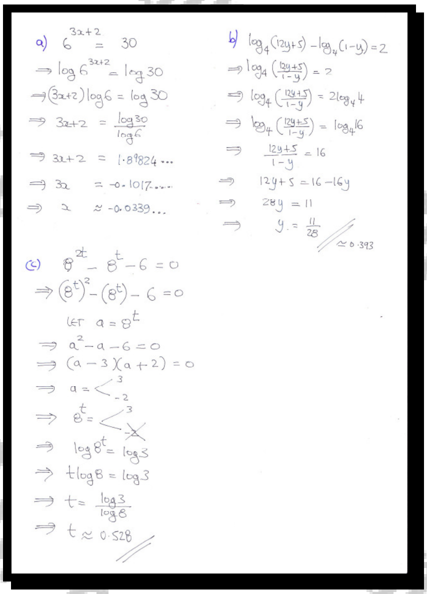
Solve each of the following equations, giving the final answers correct to three significant figures, where appropriate.

a) $6^{3x+2} = 30$.

b) $\log_4(12y+5) - \log_4(1-y) = 2$.

c) $8^{2t} - 8^t - 6 = 0$.

$\boxed{x \approx -0.0339}$, $\boxed{y = \frac{11}{28} \approx 0.393}$, $\boxed{t \approx 0.528}$



a) $6^{3x+2} = 30$

$\Rightarrow \log 6^{3x+2} = \log 30$

$\Rightarrow (3x+2)\log 6 = \log 30$

$\Rightarrow 3x+2 = \frac{\log 30}{\log 6}$

$\Rightarrow 3x+2 = 1.89824\ldots$

$\Rightarrow 3x = -0.1017\ldots$

$\Rightarrow x \approx -0.0339\ldots$

b) $\log_4(12y+5) - \log_4(1-y) = 2$

$\Rightarrow \log_4\left(\frac{12y+5}{1-y}\right) = 2$

$\Rightarrow \log_4\left(\frac{12y+5}{1-y}\right) = 2\log_4 4$

$\Rightarrow \log_4\left(\frac{12y+5}{1-y}\right) = \log_4 16$

$\Rightarrow \frac{12y+5}{1-y} = 16$

$\Rightarrow 12y+5 = 16-16y$

$\Rightarrow 28y = 11$

$\Rightarrow y = \frac{11}{28} \approx 0.393$

(c) $8^{2t} - 8^t - 6 = 0$

$\Rightarrow (8^t)^2 - (8^t) - 6 = 0$

LET $a = 8^t$

$\Rightarrow a^2 - a - 6 = 0$

$\Rightarrow (a-3)(a+2) = 0$

$\Rightarrow a = \begin{cases} 3 \\ -2 \end{cases}$

$\Rightarrow 8^t = \begin{cases} 3 \\ \cancel{-2} \end{cases}$

$\Rightarrow \log 8^t = \log 3$

$\Rightarrow t\log 8 = \log 3$

$\Rightarrow t = \frac{\log 3}{\log 8}$

$\Rightarrow t \approx 0.528$

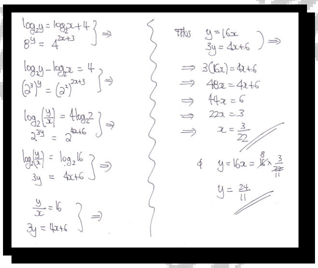
Question 50 **(****)**

Solve the following simultaneous equations, giving your answers as exact fractions

$$\log_2 y = \log_2 x + 4$$

$$8^y = 4^{2x+3}.$$

$$\boxed{x = \frac{3}{22},\ \ y = \frac{24}{11}}$$



$$\left.\begin{array}{l}\log_2 y = \log_2 x + 4\\ 8^y = 4^{2x+3}\end{array}\right\}\Rightarrow$$

$$\left.\begin{array}{l}\log_2 y - \log_2 x = 4\\ (2^3)^y = (2^2)^{2x+3}\end{array}\right\}\Rightarrow$$

$$\left.\begin{array}{l}\log_2\left(\frac{y}{x}\right) = 4\log_2 2\\ 2^{3y} = 2^{4x+6}\end{array}\right\}\Rightarrow$$

$$\left.\begin{array}{l}\log_2\left(\frac{y}{x}\right) = \log_2 16\\ 3y = 4x+6\end{array}\right\}\Rightarrow$$

$$\left.\begin{array}{l}\frac{y}{x} = 16\\ 3y = 4x+6\end{array}\right\}\Rightarrow$$

Thus $\left.\begin{array}{l}y = 16x\\ 3y = 4x+6\end{array}\right)\Rightarrow$

$\Rightarrow 3(16x) = 4x+6$

$\Rightarrow 48x = 4x+6$

$\Rightarrow 44x = 6$

$\Rightarrow 22x = 3$

$\Rightarrow x = \frac{3}{22}$

& $y = 16x = \overset{8}{\cancel{16}} \times \frac{3}{\underset{11}{\cancel{22}}}$

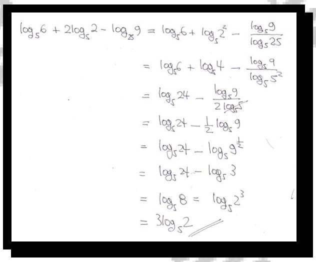
$y = \frac{24}{11}$

Question 51 **(****)**

Show clearly that

$$\log_5 6 + 2\log_5 2 - \log_{25} 9 = 3\log_5 2.$$

$$\boxed{\text{proof}}$$



$\log_5 6 + 2\log_5 2 - \log_{25} 9 = \log_5 6 + \log_5 2^2 - \frac{\log_5 9}{\log_5 25}$

$= \log_5 6 + \log_5 4 - \frac{\log_5 9}{\log_5 5^2}$

$= \log_5 24 - \frac{\log_5 9}{2\cancel{\log_5 5}}$

$= \log_5 24 - \frac{1}{2}\log_5 9$

$= \log_5 24 - \log_5 9^{\frac{1}{2}}$

$= \log_5 24 - \log_5 3$

$= \log_5 8 = \log_5 2^3$

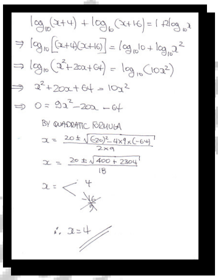
$= 3\log_5 2$

Question 52 **(****)**

Solve the following logarithmic equation

$$\log_{10}(x+4)+\log_{10}(x+16)=1+2\log_{10}x\,.$$

$$\boxed{x=4,\ x\neq-\frac{16}{9}}$$



$\log_{10}(x+4)+\log_{10}(x+16)=1+2\log_{10}x$

$\Rightarrow \log_{10}\left[(x+4)(x+16)\right]=\log_{10}10+\log_{10}x^2$

$\Rightarrow \log_{10}(x^2+20x+64)=\log_{10}(10x^2)$

$\Rightarrow x^2+20x+64=10x^2$

$\Rightarrow 0=9x^2-20x-64$

BY QUADRATIC FORMULA

$x=\dfrac{20\pm\sqrt{(-20)^2-4\times9\times(-64)}}{2\times9}$

$x=\dfrac{20\pm\sqrt{400+2304}}{18}$

$x=\begin{cases}4\\ \cancel{-\frac{16}{9}}\end{cases}$

$\therefore\ x=4$ //

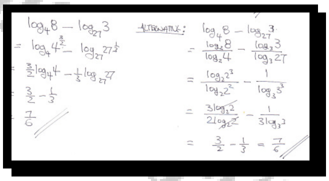
Question 53 **(****)**

Simplify

$$\log_4 8-\log_{27}3,$$

giving the final answer as a simplified fraction.

$$\boxed{\frac{7}{6}}$$



$\log_4 8-\log_{27}3$

$=\log_4 4^{\frac{3}{2}}-\log_{27}27^{\frac{1}{3}}$

$=\frac{3}{2}\log_4 4-\frac{1}{3}\log_{27}27$

$=\frac{3}{2}-\frac{1}{3}$

$=\frac{7}{6}$ //

ALTERNATIVE: $\log_4 8-\log_{27}3$

$=\dfrac{\log_2 8}{\log_2 4}-\dfrac{\log_3 3}{\log_3 27}$

$=\dfrac{\log_2 2^3}{\log_2 2^2}-\dfrac{1}{\log_3 3^3}$

$=\dfrac{3\cancel{\log_2 2}}{2\cancel{\log_2 2}}-\dfrac{1}{3\log_3 3}$

$=\frac{3}{2}-\frac{1}{3}=\frac{7}{6}$ //

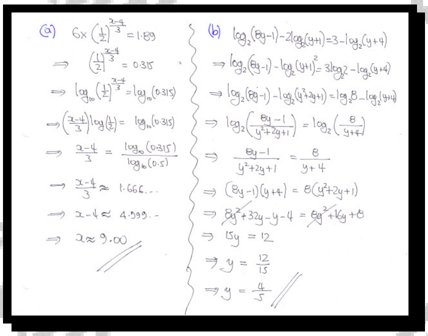
Question 54 **(****)**

Solve each of the following equations.

a) $6\times\left(\frac{1}{2}\right)^{\frac{x-4}{3}}=1.89$.

b) $\log_2(8y-1)-2\log_2(y+1)=3-\log_2(y+4)$.

$\boxed{}$, $\boxed{x\approx 9.00}$, $\boxed{y=\frac{4}{5}}$



(a) $6\times\left(\frac{1}{2}\right)^{\frac{x-4}{3}}=1.89$

$\Rightarrow \left(\frac{1}{2}\right)^{\frac{x-4}{3}}=0.315$

$\Rightarrow \log_{10}\left(\frac{1}{2}\right)^{\frac{x-4}{3}}=\log_{10}(0.315)$

$\Rightarrow \left(\frac{x-4}{3}\right)\log\left(\frac{1}{2}\right)=\log_{10}(0.315)$

$\Rightarrow \frac{x-4}{3}=\frac{\log_{10}(0.315)}{\log_{10}(0.5)}$

$\rightarrow \frac{x-4}{3}\approx 1.666\ldots$

$\rightarrow x-4\approx 4.999\ldots$

$\Rightarrow x\approx 9.00$

(b) $\log_2(8y-1)-2\log_2(y+1)=3-\log_2(y+4)$

$\Rightarrow \log_2(8y-1)-\log_2(y+1)^2=3\log_2 2-\log_2(y+4)$

$\Rightarrow \log_2(8y-1)-\log_2(y^2+2y+1)=\log_2 8-\log_2(y+4)$

$\Rightarrow \log_2\left(\frac{8y-1}{y^2+2y+1}\right)=\log_2\left(\frac{8}{y+4}\right)$

$\Rightarrow \frac{8y-1}{y^2+2y+1}=\frac{8}{y+4}$

$\Rightarrow (8y-1)(y+4)=8(y^2+2y+1)$

$\Rightarrow \cancel{8y^2}+32y-y-4=\cancel{8y^2}+16y+8$

$\Rightarrow 15y=12$

$\Rightarrow y=\frac{12}{15}$

$\Rightarrow y=\frac{4}{5}$

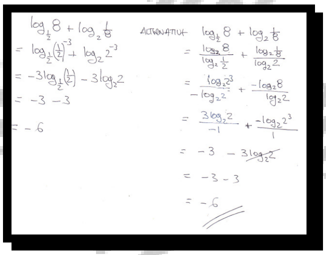
Question 55 **(****)**

Simplify

$$\log_{\frac{1}{2}}8+\log_2\frac{1}{8},$$

giving the final answer as an integer.

$\boxed{-6}$



$\log_{\frac{1}{2}}8+\log_2\frac{1}{8}$

$=\log_{\frac{1}{2}}\left(\frac{1}{2}\right)^{-3}+\log_2 2^{-3}$

$=-3\log_{\frac{1}{2}}\left(\frac{1}{2}\right)-3\log_2 2$

$=-3-3$

$=-6$

ALTERNATIVE $\log_{\frac{1}{2}}8+\log_2\frac{1}{8}$

$=\frac{\log_2 8}{\log_2\frac{1}{2}}+\frac{\log_2\frac{1}{8}}{\log_2 2}$

$=\frac{\log_2 2^3}{-\log_2 2}+\frac{-\log_2 8}{\log_2 2}$

$=\frac{3\log_2 2}{-1}+\frac{-\log_2 2^3}{1}$

$=-3-3\cancel{\log_2 2}$

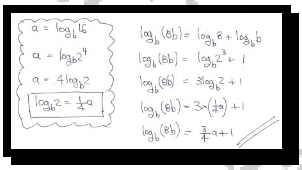
$=-3-3$

$=-6$

Question 56 **(****)**

Given that $a = \log_b 16$, express $\log_b (8b)$ in terms of a.

$$\boxed{1 + \frac{3}{4}a}$$



$a = \log_b 16$

$a = \log_b 2^4$

$a = 4\log_b 2$

$\boxed{\log_b 2 = \frac{1}{4}a}$

$\log_b (8b) = \log_b 8 + \log_b b$

$\log_b (8b) = \log_b 2^3 + 1$

$\log_b (8b) = 3\log_b 2 + 1$

$\log_b (8b) = 3 \times \left(\frac{1}{4}a\right) + 1$

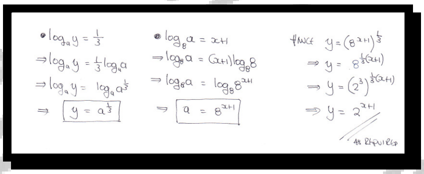
$\log_b (8b) = \frac{3}{4}a + 1$ //

Question 57 **(****)**

$$\log_a y = \frac{1}{3} \quad \text{and} \quad \log_8 a = x + 1 .$$

Show clearly that $y = 2^{x+1}$

$$\boxed{\text{proof}}$$



• $\log_a y = \frac{1}{3}$

$\Rightarrow \log_a y = \frac{1}{3}\log_a a$

$\Rightarrow \log_a y = \log_a a^{\frac{1}{3}}$

$\Rightarrow \boxed{y = a^{\frac{1}{3}}}$

• $\log_8 a = x+1$

$\Rightarrow \log_8 a = (x+1)\log_8 8$

$\Rightarrow \log_8 a = \log_8 8^{x+1}$

$\Rightarrow \boxed{a = 8^{x+1}}$

HENCE $y = \left(8^{x+1}\right)^{\frac{1}{3}}$

$\Rightarrow y = 8^{\frac{1}{3}(x+1)}$

$\Rightarrow y = \left(2^3\right)^{\frac{1}{3}(x+1)}$

$\Rightarrow y = 2^{x+1}$ //

AS REQUIRED

Question 58 (****)

It is given that

$$p = \log_6 25 \qquad \text{and} \qquad q = \log_6 2 .$$

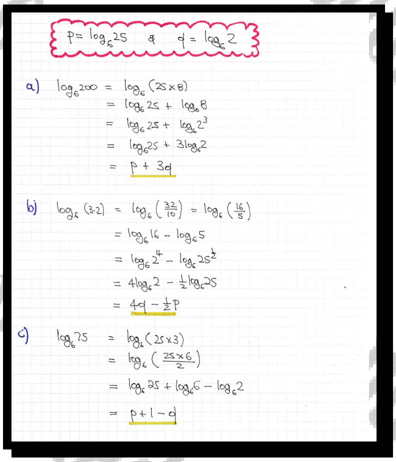
Simplify each of the following logarithms, giving the final answers in terms of p, q and positive integers, where appropriate.

i. $\log_6(200)$.

ii. $\log_6(3.2)$.

iii. $\log_6(75)$.

$\boxed{\text{[illegible]}}$, $\boxed{p+3q}$, $\boxed{4q-\frac{1}{2}p}$, $\boxed{p+1-q}$



$p = \log_6 25$ & $q = \log_6 2$

a) $\log_6 200 = \log_6(25\times 8)$
$= \log_6 25 + \log_6 8$
$= \log_6 25 + \log_6 2^3$
$= \log_6 25 + 3\log_6 2$
$= \underline{p + 3q}$

b) $\log_6(3.2) = \log_6\left(\frac{32}{10}\right) = \log_6\left(\frac{16}{5}\right)$
$= \log_6 16 - \log_6 5$
$= \log_6 2^4 - \log_6 25^{\frac{1}{2}}$
$= 4\log_6 2 - \frac{1}{2}\log_6 25$
$= \underline{4q - \frac{1}{2}p}$

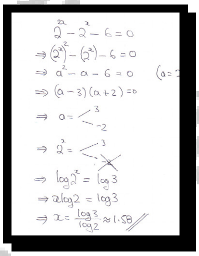
c) $\log_6 75 = \log_6(25\times 3)$
$= \log_6\left(\frac{25\times 6}{2}\right)$
$= \log_6 25 + \log_6 6 - \log_6 2$
$= \underline{p+1-q}$

Question 59 **(****)**

Solve the following exponential equation, giving the answer correct to 3 s.f.

$$2^{2x} - 2^{x} - 6 = 0 .$$

$\boxed{x \approx 1.58}$



$$2^{2x} - 2^{x} - 6 = 0$$
$$\Rightarrow (2^{x})^{2} - (2^{x}) - 6 = 0$$
$$\Rightarrow a^{2} - a - 6 = 0 \qquad (a = 2$$
$$\Rightarrow (a-3)(a+2) = 0$$
$$\Rightarrow a = \begin{cases} 3 \\ -2 \end{cases}$$
$$\Rightarrow 2^{x} = \begin{cases} 3 \\ \cancel{-2} \end{cases}$$
$$\Rightarrow \log 2^{x} = \log 3$$
$$\Rightarrow x \log 2 = \log 3$$
$$\Rightarrow x = \frac{\log 3}{\log 2} \approx 1.58 //$$

Question 60 **(****)**

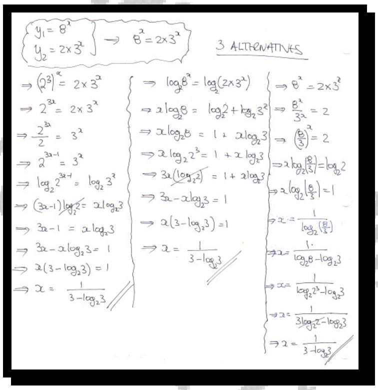
Two curves C_1 and C_2 are defined for all values of x and have respective equations

$$y_1 = 8^{x} \quad \text{and} \quad y_2 = 2 \times 3^{x} .$$

Show that the x coordinate of the point of intersection of the two curves is given by

$$\frac{1}{3 - \log_2 3} .$$

$\boxed{\text{proof}}$



$$\left.\begin{aligned} y_1 &= 8^{x} \\ y_2 &= 2 \times 3^{x} \end{aligned}\right\} \Rightarrow 8^{x} = 2 \times 3^{x}$$

3 ALTERNATIVES

$$\Rightarrow (2^{3})^{x} = 2 \times 3^{x}$$
$$\Rightarrow 2^{3x} = 2 \times 3^{x}$$
$$\Rightarrow \frac{2^{3x}}{2} = 3^{x}$$
$$\Rightarrow 2^{3x-1} = 3^{x}$$
$$\Rightarrow \log_2 2^{3x-1} = \log_2 3^{x}$$
$$\Rightarrow (3x-1)\cancel{\log_2 2} = x \log_2 3$$
$$\Rightarrow 3x - 1 = x \log_2 3$$
$$\Rightarrow 3x - x \log_2 3 = 1$$
$$\Rightarrow x(3 - \log_2 3) = 1$$
$$\Rightarrow x = \frac{1}{3 - \log_2 3} //$$

$$\Rightarrow \log_2 8^{x} = \log_2 (2 \times 3^{x})$$
$$\Rightarrow x \log_2 8 = \log_2 2 + \log_2 3^{x}$$
$$\Rightarrow x \log_2 8 = 1 + x \log_2 3$$
$$\Rightarrow x \log_2 2^{3} = 1 + x \log_2 3$$
$$\Rightarrow 3x \cancel{(\log_2 2)} = 1 + x \log_2 3$$
$$\Rightarrow 3x - x \log_2 3 = 1$$
$$\Rightarrow x(3 - \log_2 3) = 1$$
$$\Rightarrow x = \frac{1}{3 - \log_2 3} //$$

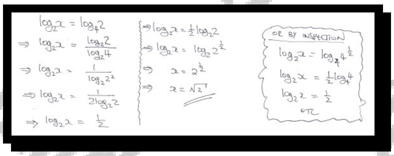
$$\Rightarrow 8^{x} = 2 \times 3^{x}$$
$$\Rightarrow \frac{8^{x}}{3^{x}} = 2$$
$$\Rightarrow \left(\frac{8}{3}\right)^{x} = 2$$
$$\Rightarrow x \log_2 \left(\frac{8}{3}\right) = \log_2 2$$
$$\Rightarrow x \log_2 \left(\frac{8}{3}\right) = 1$$
$$\Rightarrow x = \frac{1}{\log_2 \left(\frac{8}{3}\right)}$$
$$\Rightarrow x = \frac{1}{\log_2 8 - \log_2 3}$$
$$\Rightarrow x = \frac{1}{\log_2 2^{3} - \log_2 3}$$
$$\Rightarrow x = \frac{1}{3\cancel{\log_2 2} - \log_2 3}$$
$$\Rightarrow x = \frac{1}{3 - \log_2 3} //$$

Question 61 (****)

Solve the following logarithmic equation

$$\log_2 x = \log_4 2 .$$

$$\boxed{x = \sqrt{2}}$$



$\log_2 x = \log_4 2$

$\Rightarrow \log_2 x = \frac{\log_2 2}{\log_2 4}$

$\Rightarrow \log_2 x = \frac{1}{\log_2 2^2}$

$\Rightarrow \log_2 x = \frac{1}{2\log_2 2}$

$\Rightarrow \log_2 x = \frac{1}{2}$

$\Rightarrow \log_2 x = \frac{1}{2}\log_2 2$

$\Rightarrow \log_2 x = \log_2 2^{\frac{1}{2}}$

$\Rightarrow x = 2^{\frac{1}{2}}$

$\Rightarrow x = \sqrt{2}$

OR BY INSPECTION

$\log_2 x = \log_4 4^{\frac{1}{2}}$

$\log_2 x = \frac{1}{2}\log_4 4$

$\log_2 x = \frac{1}{2}$

etc

Question 62 (****)

The functions f and g are defined as

$$f(x) = 3\left(2^{-x}\right) - 1, \quad x \in \mathbb{R}, \quad x \geq 0$$

$$g(x) = \log_2 x, \quad x \in \mathbb{R}, \quad x \geq 1.$$

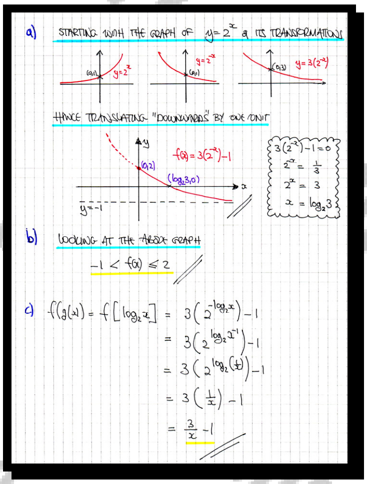
a) Sketch the graph of f.

- Mark clearly the exact coordinates of any points where the curve meets the coordinate axes. Give the answers, where appropriate, in exact form in terms of logarithms base 2.
- Mark and label the equation of the asymptote to the curve.

b) State the range of f.

c) Find $f(g(x))$ in its simplest form.

$\boxed{\quad}$, $\boxed{(0,2)}$, $\boxed{(\log_2 3, 0)}$, $\boxed{y = -1}$, $\boxed{-1 < f(x) \leq 2}$, $\boxed{f(g(x)) = \dfrac{6}{x} - 1}$



a) STARTING WITH THE GRAPH OF $y = 2^x$ & ITS TRANSFORMATIONS

HENCE TRANSLATING "DOWNWARDS" BY ONE UNIT

$3(2^{-x}) - 1 = 0$

$2^{-x} = \frac{1}{3}$

$2^{x} = 3$

$x = \log_2 3$

b) LOOKING AT THE ABOVE GRAPH

$-1 < f(x) \leqslant 2$

c) $f(g(x)) = f[\log_2 x] = 3\left(2^{-\log_2 x}\right) - 1$

$= 3\left(2^{\log_2 x^{-1}}\right) - 1$

$= 3\left(2^{\log_2 \left(\frac{1}{x}\right)}\right) - 1$

$= 3\left(\frac{1}{x}\right) - 1$

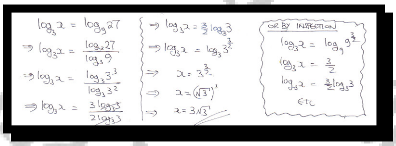
$= \frac{3}{x} - 1$

Question 63 (****)

Solve the following logarithmic equation

$$\log_3 x = \log_9 27 .$$

$$\boxed{x = 3\sqrt{3}}$$



$\log_3 x = \log_9 27$

$\Rightarrow \log_3 x = \frac{\log_3 27}{\log_3 9}$

$\Rightarrow \log_3 x = \frac{\log_3 3^3}{\log_3 3^2}$

$\Rightarrow \log_3 x = \frac{3 \cancel{\log_3 3}}{2 \cancel{\log_3 3}}$

$\Rightarrow \log_3 x = \frac{3}{2} \log_3 3$

$\Rightarrow \log_3 x = \log_3 3^{\frac{3}{2}}$

$\Rightarrow x = 3^{\frac{3}{2}}$

$\Rightarrow x = (\sqrt{3})^3$

$\Rightarrow x = 3\sqrt{3}$

OR BY INSPECTION

$\log_3 x = \log_9 9^{\frac{3}{2}}$

$\log_3 x = \frac{3}{2}$

$\log_3 x = \frac{3}{2} \log_3 3$

ETC

Question 64 (****)

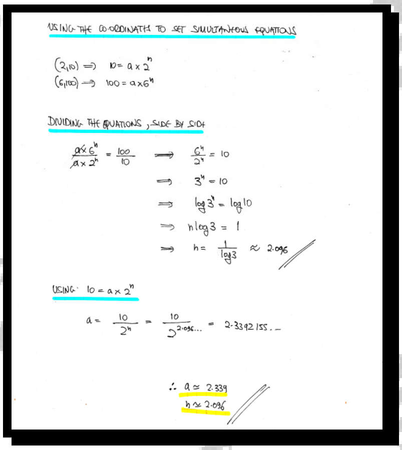
The points $(2,10)$ and $(6,100)$ lie on the curve with equation

$$y = ax^n ,$$

where a and n are non zero constants.

Find, to three decimal places, the value of a and the value of n.

$\boxed{\quad}$, $\boxed{a = 2.339}$, $\boxed{n \approx 2.096}$



USING THE CO-ORDINATES TO SET SIMULTANEOUS EQUATIONS

$(2,10) \Rightarrow 10 = a \times 2^n$

$(6,100) \Rightarrow 100 = a \times 6^n$

DIVIDING THE EQUATIONS, SIDE BY SIDE

$\frac{\cancel{a} \times 6^n}{\cancel{a} \times 2^n} = \frac{100}{10} \Longrightarrow \frac{6^n}{2^n} = 10$

$\Longrightarrow 3^n = 10$

$\Longrightarrow \log 3^n = \log 10$

$\Longrightarrow n \log 3 = 1$

$\Longrightarrow n = \frac{1}{\log 3} \approx 2.096$

USING $10 = a \times 2^n$

$a = \frac{10}{2^n} = \frac{10}{2^{2.096\ldots}} = 2.3392155\ldots$

$\therefore a \simeq 2.339$

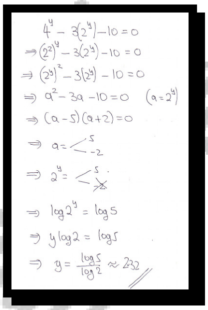
$n \simeq 2.096$

Question 65 **(****)**

Solve the following exponential equation, giving the answer correct to 3 s.f.

$$4^y - 3\left(2^y\right) - 10 = 0 .$$

$\boxed{\quad}$, $\boxed{y \approx 2.32}$



$4^y - 3(2^y) - 10 = 0$

$\Rightarrow (2^2)^y - 3(2^y) - 10 = 0$

$\Rightarrow (2^y)^2 - 3(2^y) - 10 = 0$

$\Rightarrow a^2 - 3a - 10 = 0 \quad (a = 2^y)$

$\Rightarrow (a-5)(a+2) = 0$

$\Rightarrow a = \begin{cases} 5 \\ -2 \end{cases}$

$\Rightarrow 2^y = \begin{cases} 5 \\ \cancel{-2} \end{cases}$

$\Rightarrow \log 2^y = \log 5$

$\Rightarrow y \log 2 = \log 5$

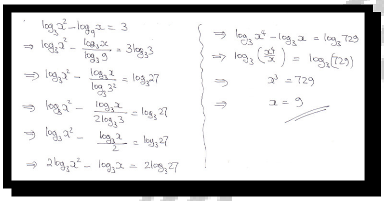
$\Rightarrow y = \frac{\log 5}{\log 2} \approx 2.32$ //

Question 66 **(****)**

Solve the following logarithmic equation

$$\log_3 x^2 - \log_9 x = 3 .$$

$\boxed{x = 9, x \neq 0}$



$\log_3 x^2 - \log_9 x = 3$

$\Rightarrow \log_3 x^2 - \frac{\log_3 x}{\log_3 9} = 3\log_3 3$

$\Rightarrow \log_3 x^2 - \frac{\log_3 x}{\log_3 3^2} = \log_3 27$

$\Rightarrow \log_3 x^2 - \frac{\log_3 x}{2\log_3 3} = \log_3 27$

$\Rightarrow \log_3 x^2 - \frac{\log_3 x}{2} = \log_3 27$

$\Rightarrow 2\log_3 x^2 - \log_3 x = 2\log_3 27$

$\Rightarrow \log_3 x^4 - \log_3 x = \log_3 729$

$\Rightarrow \log_3\left(\frac{x^4}{x}\right) = \log_3(729)$

$\Rightarrow x^3 = 729$

$\Rightarrow x = 9$ //

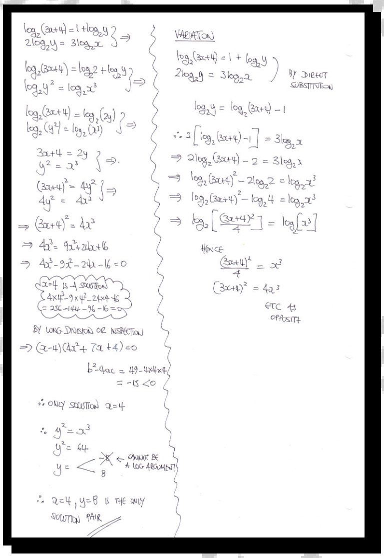
Question 67 (****)

Show that $x=4$ and $y=8$ is the only solution pair of the following logarithmic simultaneous equations

$$\log_2(3x+4)=1+\log_2 y\ .$$

$$2\log_2 y=3\log_2 x\ .$$

$\boxed{\text{proof}}$



$$\left.\begin{aligned}\log_2(3x+4)&=1+\log_2 y\\ 2\log_2 y&=3\log_2 x\end{aligned}\right\}\Rightarrow$$

$$\left.\begin{aligned}\log_2(3x+4)&=\log_2 2+\log_2 y\\ \log_2 y^2&=\log_2 x^3\end{aligned}\right\}\Rightarrow$$

$$\left.\begin{aligned}\log_2(3x+4)&=\log_2(2y)\\ \log_2(y^2)&=\log_2(x^3)\end{aligned}\right\}\Rightarrow$$

$$\left.\begin{aligned}3x+4&=2y\\ y^2&=x^3\end{aligned}\right\}\Rightarrow.$$

$$\left.\begin{aligned}(3x+4)^2&=4y^2\\ 4y^2&=4x^3\end{aligned}\right\}\Rightarrow$$

$$\Rightarrow (3x+4)^2=4x^3$$

$$\Rightarrow 4x^3=9x^2+24x+16$$

$$\Rightarrow 4x^3-9x^2-24x-16=0$$

$x=4$ IS A SOLUTION

$4\times4^3-9\times4^2-24\times4-16$

$=256-144-96-16=0$

BY LONG DIVISION OR INSPECTION

$$\Rightarrow (x-4)(4x^2+7x+4)=0$$

$$b^2-4ac=49-4\times4\times4$$
$$=-15<0$$

$\therefore$ ONLY SOLUTION $x=4$

$$\therefore\ y^2=x^3$$
$$y^2=64$$
$$y=<\begin{matrix}\cancel{-8}\\ 8\end{matrix}\leftarrow \text{CANNOT BE A LOG ARGUMENT}$$

$\therefore\ x=4,\ y=8$ IS THE ONLY SOLUTION PAIR

VARIATION

$$\left.\begin{aligned}\log_2(3x+4)&=1+\log_2 y\\ 2\log_2 y&=3\log_2 x\end{aligned}\right)\ \text{BY DIRECT SUBSTITUTION}$$

$$\log_2 y=\log_2(3x+4)-1$$

$$\therefore 2\left[\log_2(3x+4)-1\right]=3\log_2 x$$

$$\Rightarrow 2\log_2(3x+4)-2=3\log_2 x$$

$$\Rightarrow \log_2(3x+4)^2-2\log_2 2=\log_2 x^3$$

$$\Rightarrow \log_2(3x+4)^2-\log_2 4=\log_2 x^3$$

$$\Rightarrow \log_2\left[\frac{(3x+4)^2}{4}\right]=\log_2\left[x^3\right]$$

HENCE

$$\frac{(3x+4)^2}{4}=x^3$$

$$(3x+4)^2=4x^3$$

ETC AS OPPOSITE

Question 68 (****)

A population P of an endangered species of animals was introduced to a park.

The population obeys the equation

$$P = \frac{125ka^t}{k + 2a^t}, \ t \geq 0,$$

where k and a are positive constants, and t is the time in years since the species was introduced to the park .

Initially 100 individual animals were introduced to the park, and this population doubled after 5 years.

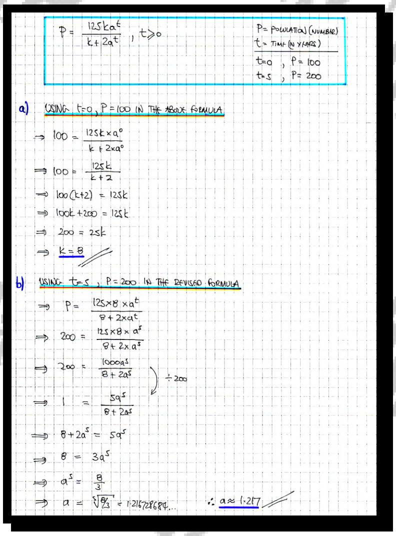
a) Show that $k = 8$.

b) Find the value of a, correct to 4 significant figures.

c) Determine the value of t when the $P = 400$.

d) Explain why this population cannot exceed 500.

$\boxed{}$, $\boxed{a \approx 1.217}$, $\boxed{t \approx 14.13}$



$P = \frac{125ka^t}{k+2a^t}$, $t \geqslant 0$

P = POPULATION (NUMBER)
t = TIME (IN YEARS)

$t=0$, $P=100$
$t=5$, $P=200$

a) USING $t=0$, $P=100$ IN THE ABOVE FORMULA

$\Rightarrow 100 = \frac{125k \times a^0}{k + 2 \times a^0}$

$\Rightarrow 100 = \frac{125k}{k+2}$

$\Rightarrow 100(k+2) = 125k$

$\Rightarrow 100k + 200 = 125k$

$\Rightarrow 200 = 25k$

$\Rightarrow k = 8$ //

b) USING $t=5$, $P=200$ IN THE REVISED FORMULA

$\Rightarrow P = \frac{125 \times 8 \times a^t}{8 + 2 \times a^t}$

$\Rightarrow 200 = \frac{125 \times 8 \times a^5}{8 + 2 \times a^5}$

$\Rightarrow 200 = \frac{1000a^5}{8+2a^5}$) $\div 200$

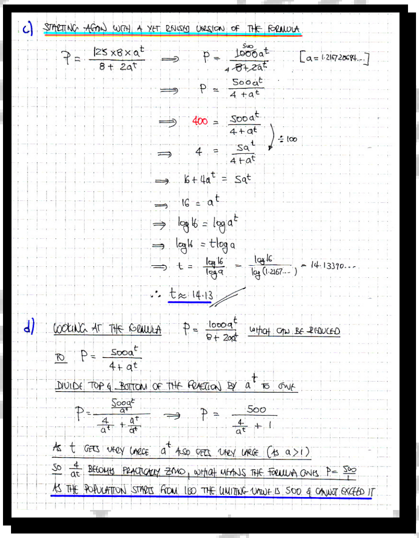
$\Rightarrow 1 = \frac{5a^5}{8+2a^5}$

$\Rightarrow 8 + 2a^5 = 5a^5$

$\Rightarrow 8 = 3a^5$

$\Rightarrow a^5 = \frac{8}{3}$

$\Rightarrow a = \sqrt[5]{8/3} = 1.216728684...$ $\therefore a \approx 1.217$ //



c) STARTING AGAIN WITH A YET REVISED VERSION OF THE FORMULA

$P = \frac{125 \times 8 \times a^t}{8 + 2a^t} \Longrightarrow P = \frac{\overset{500}{\cancel{1000}}a^t}{\underset{4}{\cancel{8+2a^t}}}$ $[a = 1.216720684...]$

$\Longrightarrow P = \frac{500a^t}{4+a^t}$

$\Longrightarrow 400 = \frac{500a^t}{4+a^t}$) $\div 100$

$\Longrightarrow 4 = \frac{5a^t}{4+a^t}$

$\Longrightarrow 16 + 4a^t = 5a^t$

$\Longrightarrow 16 = a^t$

$\Rightarrow \log 16 = \log a^t$

$\Rightarrow \log 16 = t \log a$

$\Rightarrow t = \frac{\log 16}{\log a} = \frac{\log 16}{\log(1.2167...)} = 14.13390...$

$\therefore t \approx 14.13$ //

d) LOOKING AT THE FORMULA $P = \frac{1000a^t}{8+2a^t}$ WHICH CAN BE REDUCED TO $P = \frac{500a^t}{4+a^t}$

DIVIDE TOP & BOTTOM OF THE FRACTION BY a^t TO GIVE

$P = \frac{\frac{500a^t}{a^t}}{\frac{4}{a^t} + \frac{a^t}{a^t}} \Longrightarrow P = \frac{500}{\frac{4}{a^t} + 1}$

AS t GETS VERY LARGE a^t ALSO GETS VERY LARGE (AS $a > 1$)

SO $\frac{4}{at}$ BECOMES PRACTICALLY ZERO, WHICH MEANS THE FORMULA GIVES $P = \frac{500}{1}$

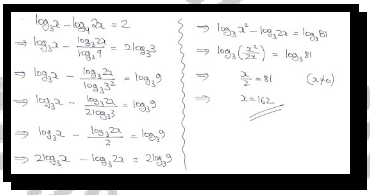
AS THE POPULATION STARTS FROM 100 THE LIMITING VALUE IS 500 & CANNOT EXCEED IT

Question 69 **(****)**

Solve the following logarithmic equation

$$\log_3 x - \log_9 2x = 2 .$$

$$\boxed{x = 162,\ \ x \neq 0}$$



$\log_3 x - \log_9 2x = 2$

$\Rightarrow \log_3 x - \dfrac{\log_3 2x}{\log_3 9} = 2\log_3 3$

$\Rightarrow \log_3 x - \dfrac{\log_3 2x}{\log_3 3^2} = \log_3 9$

$\Rightarrow \log_3 x - \dfrac{\log_3 2x}{2\log_3 3} = \log_3 9$

$\Rightarrow \log_3 x - \dfrac{\log_3 2x}{2} = \log_3 9$

$\Rightarrow 2\log_3 x - \log_3 2x = 2\log_3 9$

$\Rightarrow \log_3 x^2 - \log_3 2x = \log_3 81$

$\Rightarrow \log_3 \left(\dfrac{x^2}{2x}\right) = \log_3 81$

$\Rightarrow \dfrac{x}{2} = 81 \quad (x \neq 0)$

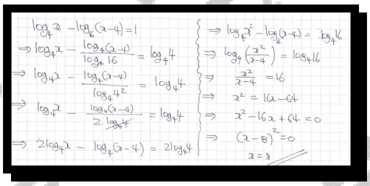
$\Rightarrow x = 162$

Question 70 **(****)**

Solve the following logarithmic equation

$$\log_4 x - \log_{16}(x-4) = 1 .$$

$\boxed{}$, $\boxed{x = 8}$



$\log_4 x - \log_{16}(x-4) = 1$

$\Rightarrow \log_4 x - \dfrac{\log_4 (x-4)}{\log_4 16} = \log_4 4$

$\Rightarrow \log_4 x - \dfrac{\log_4 (x-4)}{\log_4 4^2} = \log_4 4$

$\Rightarrow \log_4 x - \dfrac{\log_4 (x-4)}{2\cancel{\log_4 4}} = \log_4 4$

$\Rightarrow 2\log_4 x - \log_4 (x-4) = 2\log_4 4$

$\Rightarrow \log_4 x^2 - \log_4 (x-4) = \log_4 16$

$\Rightarrow \log_4 \left(\dfrac{x^2}{x-4}\right) = \log_4 16$

$\Rightarrow \dfrac{x^2}{x-4} = 16$

$\Rightarrow x^2 = 16x - 64$

$\Rightarrow x^2 - 16x + 64 = 0$

$\Rightarrow (x-8)^2 = 0$

$x = 8$

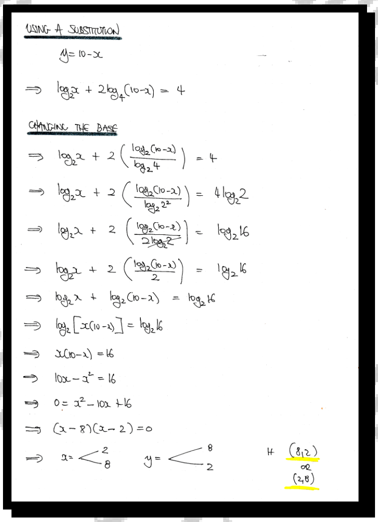
Question 71 (****)

Solve the following simultaneous equations

$$\log_2 x + 2\log_4 y = 4$$

$$x + y = 10$$

$\boxed{\text{MP-G}}$, $\boxed{x = 2,\ y = 8 \quad \text{or} \quad x = 8,\ y = 2}$



USING A SUBSTITUTION

$y = 10 - x$

$\Rightarrow \log_2 x + 2\log_4(10-x) = 4$

CHANGING THE BASE

$\Rightarrow \log_2 x + 2\left(\frac{\log_2(10-x)}{\log_2 4}\right) = 4$

$\Rightarrow \log_2 x + 2\left(\frac{\log_2(10-x)}{\log_2 2^2}\right) = 4\log_2 2$

$\Rightarrow \log_2 x + 2\left(\frac{\log_2(10-x)}{2\cancel{\log_2 2}}\right) = \log_2 16$

$\Rightarrow \log_2 x + 2\left(\frac{\log_2(10-x)}{2}\right) = \log_2 16$

$\Rightarrow \log_2 x + \log_2(10-x) = \log_2 16$

$\Rightarrow \log_2[x(10-x)] = \log_2 16$

$\Rightarrow x(10-x) = 16$

$\Rightarrow 10x - x^2 = 16$

$\Rightarrow 0 = x^2 - 10x + 16$

$\Rightarrow (x-8)(x-2) = 0$

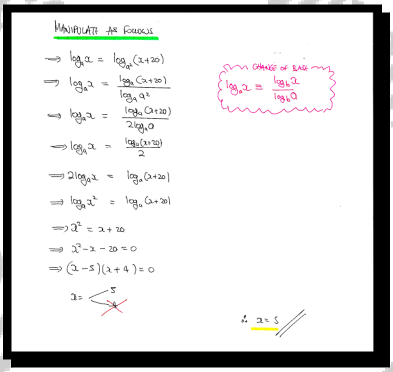
$\Rightarrow x = \genfrac{}{}{0pt}{}{2}{8} \qquad y = \genfrac{}{}{0pt}{}{8}{2} \qquad \therefore \ (8,2) \text{ or } (2,8)$

Question 72 (****)

Given that a is positive constant greater than 1, solve the following logarithmic equation

$$\log_a x = \log_{a^2}(x+20).$$

$\boxed{\text{SYN-P}}$, $\boxed{x = 5}$



MANIPULATE AS FOLLOWS

$\Rightarrow \log_a x = \log_{a^2}(x+20)$

$\Rightarrow \log_a x = \frac{\log_a(x+20)}{\log_a a^2}$

$\Rightarrow \log_a x = \frac{\log_a(x+20)}{2\log_a a}$

$\Rightarrow \log_a x = \frac{\log_a(x+20)}{2}$

$\Rightarrow 2\log_a x = \log_a(x+20)$

$\Rightarrow \log_a x^2 = \log_a(x+20)$

$\Rightarrow x^2 = x + 20$

$\Rightarrow x^2 - x - 20 = 0$

$\Rightarrow (x-5)(x+4) = 0$

$x = \genfrac{}{}{0pt}{}{5}{\cancel{-4}}$

CHANGE OF BASE

$\log_a x \equiv \frac{\log_b x}{\log_b a}$

$\therefore x = 5$

Question 73 **(****)**

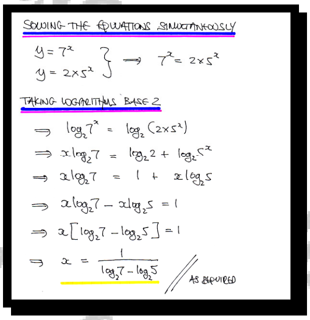
Two curves C_1 and C_2 are defined for all values of x and have respective equations

$$y_1 = 7^x \quad \text{and} \quad y_2 = 2 \times 5^x .$$

Show that the x coordinate of the point of intersection of the two curves is given by

$$\frac{1}{\log_2 7 - \log_2 5} .$$

$\boxed{}$, $\boxed{\text{proof}}$



SOLVING THE EQUATIONS SIMULTANEOUSLY

$$\left.\begin{array}{l} y = 7^x \\ y = 2\times5^x \end{array}\right\} \Rightarrow 7^x = 2\times5^x$$

TAKING LOGARITHMS BASE 2

$\Rightarrow \log_2 7^x = \log_2 (2\times5^x)$

$\Rightarrow x\log_2 7 = \log_2 2 + \log_2 5^x$

$\Rightarrow x\log_2 7 = 1 + x\log_2 5$

$\Rightarrow x\log_2 7 - x\log_2 5 = 1$

$\Rightarrow x\left[\log_2 7 - \log_2 5\right] = 1$

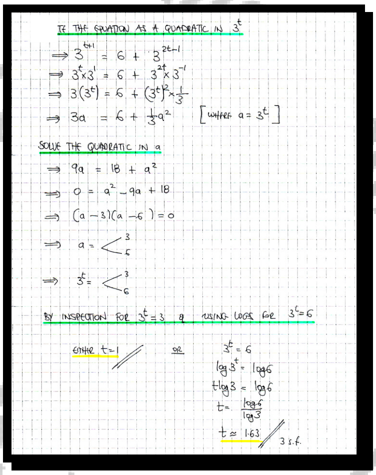
$\Rightarrow x = \dfrac{1}{\log_2 7 - \log_2 5}$ // AS REQUIRED

Question 74 **(****)**

Solve the following exponential equation, giving the answer correct to 3 s.f.

$$3^{t+1} = 6 + 3^{2t-1}$$

$\boxed{}$, $\boxed{t = 1 \text{ or } t \approx 1.63}$



TE THE EQUATION AS A QUADRATIC IN 3^t

$\Rightarrow 3^{t+1} = 6 + 3^{2t-1}$

$\Rightarrow 3^t\times3^1 = 6 + 3^{2t}\times3^{-1}$

$\Rightarrow 3(3^t) = 6 + (3^t)^2\times\frac{1}{3}$

$\Rightarrow 3a = 6 + \frac{1}{3}a^2$ $\left[\text{WHERE } a = 3^t\right]$

SOLVE THE QUADRATIC IN a

$\Rightarrow 9a = 18 + a^2$

$\Rightarrow 0 = a^2 - 9a + 18$

$\Rightarrow (a-3)(a-6) = 0$

$\Rightarrow a = \begin{cases} 3 \\ 6 \end{cases}$

$\Rightarrow 3^t = \begin{cases} 3 \\ 6 \end{cases}$

BY INSPECTION FOR $3^t = 3$ & USING LOGS FOR $3^t = 6$

EITHER $t = 1$ // OR

$3^t = 6$

$\log 3^t = \log 6$

$t\log 3 = \log 6$

$t = \dfrac{\log 6}{\log 3}$

$t \approx 1.63$ // 3 s.f.

Question 75 (****)

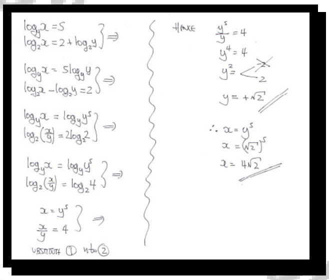
Solve the following simultaneous logarithmic equations

$$\log_y x = 5$$

$$\log_2 x = 2 + \log_2 y \,.$$

Give the answer as exact simplified surds.

$\boxed{}$, $\boxed{x = 4\sqrt{2}, \quad y = \sqrt{2}}$



$$\left.\begin{aligned}\log_y x &= 5\\ \log_2 x &= 2+\log_2 y\end{aligned}\right\} \Rightarrow$$

$$\left.\begin{aligned}\log_y x &= 5\log_y y\\ \log_2 x - \log_2 y &= 2\end{aligned}\right\} \Rightarrow$$

$$\left.\begin{aligned}\log_y x &= \log_y y^5\\ \log_2\left(\frac{x}{y}\right) &= 2\log_2 2\end{aligned}\right\} \Rightarrow$$

$$\left.\begin{aligned}\log_y x &= \log_y y^5\\ \log_2\left(\frac{x}{y}\right) &= \log_2 4\end{aligned}\right\} \Rightarrow$$

$$\left.\begin{aligned}x &= y^5\\ \frac{x}{y} &= 4\end{aligned}\right\} \Rightarrow$$

ubstitute ① into ②

HENCE $\frac{y^5}{y} = 4$

$$y^4 = 4$$

$$y^2 = \cancel{-2},\ 2$$

$$y = +\sqrt{2}$$

$$\therefore x = y^5$$

$$x = \left(\sqrt{2}\right)^5$$

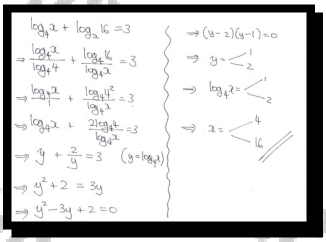
$$x = 4\sqrt{2}$$

Question 76 (****)

Solve the following logarithmic equation

$$\log_4 x + \log_x 16 = 3\,, \;\; x > 0\,, \;\; x \neq 1\,.$$

$\boxed{x = 4,\ 16}$



$$\log_4 x + \log_x 16 = 3$$

$$\Rightarrow \frac{\log_4 x}{\log_4 4} + \frac{\log_4 16}{\log_4 x} = 3$$

$$\Rightarrow \frac{\log_4 x}{1} + \frac{\log_4 4^2}{\log_4 x} = 3$$

$$\Rightarrow \log_4 x + \frac{2\log_4 4}{\log_4 x} = 3$$

$$\Rightarrow y + \frac{2}{y} = 3 \quad (y = \log_4 x)$$

$$\Rightarrow y^2 + 2 = 3y$$

$$\Rightarrow y^2 - 3y + 2 = 0$$

$$\Rightarrow (y-2)(y-1) = 0$$

$$\Rightarrow y = 1,\ 2$$

$$\Rightarrow \log_4 x = 1,\ 2$$

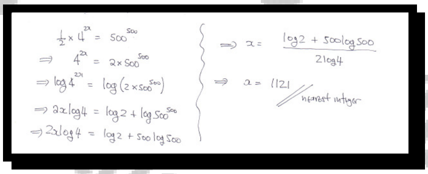
$$\Rightarrow x = 4,\ 16$$

Question 77 (****)

Find, to the nearest integer, the solution of the following exponential equation

$$\frac{1}{2}\times 4^{2x} = 500^{500}.$$

$\boxed{x \approx 1121}$



$\frac{1}{2}\times 4^{2x} = 500^{500}$

$\Rightarrow 4^{2x} = 2\times 500^{500}$

$\Rightarrow \log 4^{2x} = \log\left(2\times 500^{500}\right)$

$\Rightarrow 2x\log 4 = \log 2 + \log 500^{500}$

$\Rightarrow 2x\log 4 = \log 2 + 500\log 500$

$\Rightarrow x = \dfrac{\log 2 + 500\log 500}{2\log 4}$

$\Rightarrow x = 1121$ // nearest integer

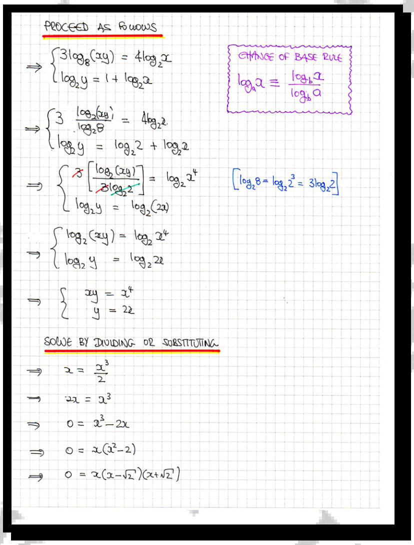
Question 78 (****)

Solve the following simultaneous logarithmic equations.

$$3\log_8(xy) = 4\log_2 x$$

$$\log_2 y = 1 + \log_2 x$$

$\boxed{\phantom{x=\sqrt{2}}}$, $\boxed{x=\sqrt{2},\quad y=\sqrt{8}}$



PROCEED AS FOLLOWS

$\Rightarrow \begin{cases} 3\log_8(xy) = 4\log_2 x \\ \log_2 y = 1 + \log_2 x \end{cases}$

CHANGE OF BASE RULE

$\log_a x \equiv \dfrac{\log_b x}{\log_b a}$

$\Rightarrow \begin{cases} 3\,\dfrac{\log_2(xy)}{\log_2 8} = 4\log_2 x \\ \log_2 y = \log_2 2 + \log_2 x \end{cases}$

$\Rightarrow \begin{cases} \cancel{3}\left[\dfrac{\log_2(xy)}{\cancel{3}\cancel{\log_2 2}}\right] = \log_2 x^4 \\ \log_2 y = \log_2(2x) \end{cases}$

$\left[\log_2 8 = \log_2 2^3 = 3\log_2 2\right]$

$\Rightarrow \begin{cases} \log_2(xy) = \log_2 x^4 \\ \log_2 y = \log_2 2x \end{cases}$

$\Rightarrow \begin{cases} xy = x^4 \\ y = 2x \end{cases}$

SOLVE BY DIVIDING OR SUBSTITUTING

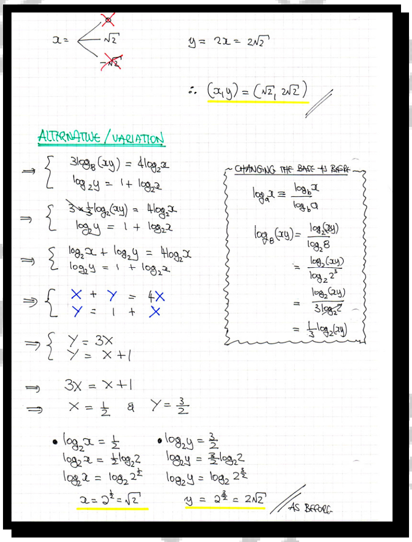
$\Rightarrow x = \dfrac{x^3}{2}$

$\Rightarrow 2x = x^3$

$\Rightarrow 0 = x^3 - 2x$

$\Rightarrow 0 = x(x^2-2)$

$\Rightarrow 0 = x(x-\sqrt{2})(x+\sqrt{2})$



$x = \begin{cases} \cancel{0} \\ \sqrt{2} \\ \cancel{-\sqrt{2}} \end{cases}$ $\quad y = 2x = 2\sqrt{2}$

$\therefore (x,y) = (\sqrt{2}, 2\sqrt{2})$ //

ALTERNATIVE / VARIATION

$\Rightarrow \begin{cases} 3\log_8(xy) = 4\log_2 x \\ \log_2 y = 1 + \log_2 x \end{cases}$

$\Rightarrow \begin{cases} \cancel{3}\times\frac{1}{\cancel{3}}\log_2(xy) = 4\log_2 x \\ \log_2 y = 1 + \log_2 x \end{cases}$

$\Rightarrow \begin{cases} \log_2 x + \log_2 y = 4\log_2 x \\ \log_2 y = 1 + \log_2 x \end{cases}$

$\Rightarrow \begin{cases} X + Y = 4X \\ Y = 1 + X \end{cases}$

$\Rightarrow \begin{cases} Y = 3X \\ Y = X + 1 \end{cases}$

$\Rightarrow 3X = X + 1$

$\Rightarrow X = \frac{1}{2}$ & $Y = \frac{3}{2}$

CHANGING THE BASE AS BEFORE

$\log_a x \equiv \dfrac{\log_b x}{\log_b a}$

$\log_8(xy) = \dfrac{\log_2(xy)}{\log_2 8} = \dfrac{\log_2(xy)}{\log_2 2^3} = \dfrac{\log_2(xy)}{3\cancel{\log_2 2}} = \frac{1}{3}\log_2(xy)$

- $\log_2 x = \frac{1}{2}$

  $\log_2 x = \frac{1}{2}\log_2 2$

  $\log_2 x = \log_2 2^{\frac{1}{2}}$

  $x = 2^{\frac{1}{2}} = \sqrt{2}$

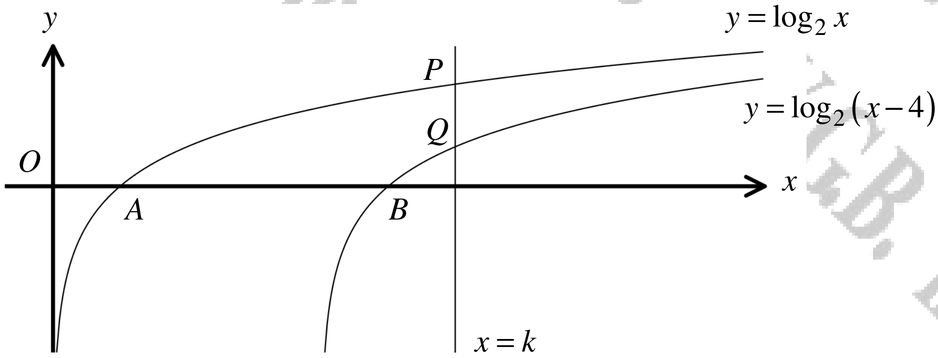
- $\log_2 y = \frac{3}{2}$

  $\log_2 y = \frac{3}{2}\log_2 2$

  $\log_2 y = \log_2 2^{\frac{3}{2}}$

  $y = 2^{\frac{3}{2}} = 2\sqrt{2}$ // AS BEFORE

Question 79 (****)



The figure above shows the graphs of the curves with equations

$$y=\log_2 x, \quad \text{and} \quad y=\log_2(x-4).$$

The points A and B are the respective x intercepts of the two graphs.

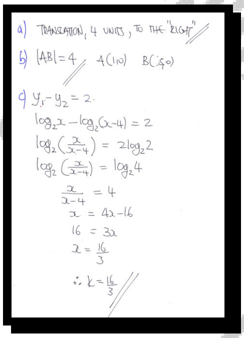
a) Describe the geometric transformation which maps the graph of $y=\log_2 x$ onto the graph of $y=\log_2(x-4)$.

b) State the distance AB.

The straight line with equation $x=k$, where k is a positive constant, meets the graph of $y=\log_2 x$ at the point P and the graph of $y=\log_2(x-4)$ at the point Q.

c) Given that the distance PQ is 2 units determine the value of k.

$\boxed{\text{C2V}}$, $\boxed{\text{translation, 4 units to the "right"}}$, $\boxed{|AB|=4}$, $\boxed{k=\frac{16}{3}}$



a) TRANSLATION, 4 UNITS, TO THE "RIGHT" //

b) $|AB|=4$ // $A(1,0)$ $B(5,0)$

c) $y_1-y_2=2$

$\log_2 x-\log_2(x-4)=2$

$\log_2\left(\frac{x}{x-4}\right)=2\log_2 2$

$\log_2\left(\frac{x}{x-4}\right)=\log_2 4$

$\frac{x}{x-4}=4$

$x=4x-16$

$16=3x$

$x=\frac{16}{3}$

$\therefore k=\frac{16}{3}$ //

Question 80 (****)

The radioactive decay of a phosphorus isotope is modelled by the equation

$$m = m_0 \times 2^{-0.2t}, \ t \geq 0$$

where m is the mass of phosphorus left, in grams, and t is the time in days since the decay started. The initial mass of phosphorus is m_0.

a) Find the mass of the phosphorus left, when an initial mass of 20 grams is left to decay for 10 days, according to this model.

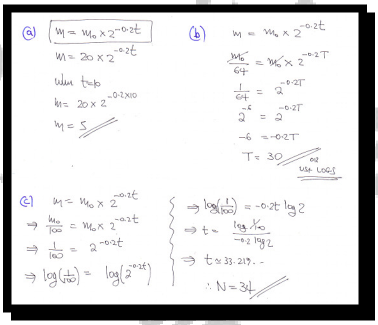
An initial mass, m_0 grams, of this type of phosphorus decays to $\frac{m_0}{64}$ grams in T days.

b) Find the value of T.

After N days have elapsed, less than 1% of this type of phosphorus remains from its initial mass m_0.

c) Find the smallest integer value of N.

$\boxed{}$, $\boxed{m = 5}$, $\boxed{T = 30}$, $\boxed{N = 34}$



(a) $\boxed{m = m_0 \times 2^{-0.2t}}$

$m = 20 \times 2^{-0.2t}$

when $t=10$

$m = 20 \times 2^{-0.2\times10}$

$m = 5$

(b) $m = m_0 \times 2^{-0.2t}$

$\frac{\cancel{m_0}}{64} = \cancel{m_0} \times 2^{-0.2T}$

$\frac{1}{64} = 2^{-0.2T}$

$2^{-6} = 2^{-0.2T}$

$-6 = -0.2T$

$T = 30$

OR USE LOGS

(c) $m = m_0 \times 2^{-0.2t}$

$\Rightarrow \frac{m_0}{100} = m_0 \times 2^{-0.2t}$

$\Rightarrow \frac{1}{100} = 2^{-0.2t}$

$\Rightarrow \log\left(\frac{1}{100}\right) = \log\left(2^{-0.2t}\right)$

$\Rightarrow \log\left(\frac{1}{100}\right) = -0.2t \log 2$

$\Rightarrow t = \frac{\log 1/100}{-0.2 \log 2}$

$\Rightarrow t \simeq 33.219...$

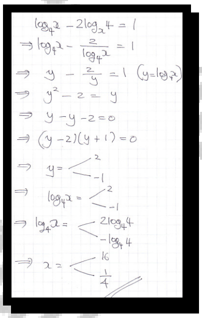
$\therefore N = 34$

Question 81 (****)

Solve the following logarithmic equation

$$\log_4 x - 2\log_x 4 = 1 .$$

, $\boxed{x = \frac{1}{4}, 16}$



$\log_4 x - 2\log_x 4 = 1$

$\Rightarrow \log_4 x - \frac{2}{\log_4 x} = 1$

$\Rightarrow y - \frac{2}{y} = 1 \quad (y = \log_4 x)$

$\Rightarrow y^2 - 2 = y$

$\Rightarrow y - y - 2 = 0$

$\Rightarrow (y-2)(y+1) = 0$

$\Rightarrow y = \begin{cases} 2 \\ -1 \end{cases}$

$\Rightarrow \log_4 x = \begin{cases} 2 \\ -1 \end{cases}$

$\Rightarrow \log_4 x = \begin{cases} 2\log_4 4 \\ -\log_4 4 \end{cases}$

$\Rightarrow x = \begin{cases} 16 \\ \frac{1}{4} \end{cases}$

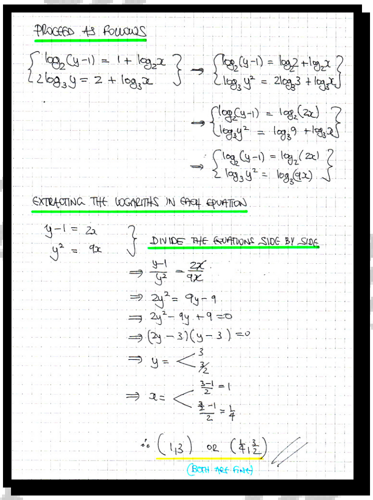
Question 82 (****)

Solve the following simultaneous logarithmic equations.

$$\log_2 (y-1) = 1 + \log_2 x$$

$$2\log_3 y = 2 + \log_3 x .$$

, $\boxed{x = 1,\ y = 3 \text{ or } x = \frac{1}{4}, y = \frac{3}{2}}$



PROCEED AS FOLLOWS

$\left\{\begin{array}{l} \log_2(y-1) = 1 + \log_2 x \\ 2\log_3 y = 2 + \log_3 x \end{array}\right\} \Rightarrow \left\{\begin{array}{l} \log_2(y-1) = \log_2 2 + \log_2 x \\ \log_3 y^2 = 2\log_3 3 + \log_3 x \end{array}\right\}$

$\Rightarrow \left\{\begin{array}{l} \log_2(y-1) = \log_2(2x) \\ \log_3 y^2 = \log_3 9 + \log_3 x \end{array}\right\}$

$\Rightarrow \left\{\begin{array}{l} \log_2(y-1) = \log_2(2x) \\ \log_3 y^2 = \log_3(9x) \end{array}\right\}$

EXTRACTING THE LOGARITHS IN EACH EQUATION

$\left.\begin{array}{l} y-1 = 2x \\ y^2 = 9x \end{array}\right\}$ DIVIDE THE EQUATIONS SIDE BY SIDE

$\Rightarrow \frac{y-1}{y^2} = \frac{2\cancel{x}}{9\cancel{x}}$

$\Rightarrow 2y^2 = 9y - 9$

$\Rightarrow 2y^2 - 9y + 9 = 0$

$\Rightarrow (2y-3)(y-3) = 0$

$\Rightarrow y = \begin{cases} 3 \\ 3/2 \end{cases}$

$\Rightarrow x = \begin{cases} \frac{3-1}{2} = 1 \\ \frac{\frac{3}{2}-1}{2} = \frac{1}{4} \end{cases}$

$\therefore (1,3)$ OR $(\frac{1}{4}, \frac{3}{2})$

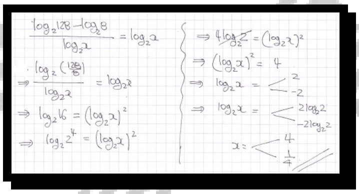
(BOTH ARE FINE)

Question 83 **(****)**

Solve the following logarithmic equation

$$\frac{\log_2 128 - \log_2 8}{\log_2 x} = \log_2 x\ .$$

$\boxed{\quad}$, $\boxed{x = 4, \frac{1}{4}}$



$$\frac{\log_2 128 - \log_2 8}{\log_2 x} = \log_2 x$$

$$\Rightarrow \frac{\log_2\left(\frac{128}{8}\right)}{\log_2 x} = \log_2 x$$

$$\Rightarrow \log_2 16 = (\log_2 x)^2$$

$$\Rightarrow \log_2 2^4 = (\log_2 x)^2$$

$$\Rightarrow 4\cancel{\log_2 2} = (\log_2 x)^2$$

$$\Rightarrow (\log_2 x)^2 = 4$$

$$\Rightarrow \log_2 x = \left\langle \begin{array}{l} 2 \\ -2 \end{array} \right.$$

$$\Rightarrow \log_2 x = \left\langle \begin{array}{l} 2\log_2 2 \\ -2\log_2 2 \end{array} \right.$$

$$x = \left\langle \begin{array}{l} 4 \\ \frac{1}{4} \end{array} \right.$$

Question 84 **(****)**

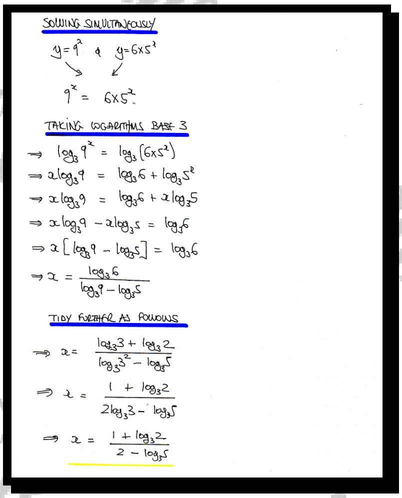
Two curves C_1 and C_2 are defined for all values of x and have respective equations

$$y_1 = 9^x \quad \text{and} \quad y_2 = 6 \times 5^x\ .$$

Show that the x coordinate of the point of intersection of the two curves is given by :

$$\frac{1 + \log_3 2}{2 - \log_3 5}$$

$\boxed{\quad}$, $\boxed{\text{proof}}$



SOLVING SIMULTANEOUSLY

$$y = 9^x \quad \& \quad y = 6 \times 5^x$$

$$9^x = 6 \times 5^x$$

TAKING LOGARITHMS BASE 3

$$\Rightarrow \log_3 9^x = \log_3 (6 \times 5^x)$$

$$\Rightarrow x\log_3 9 = \log_3 6 + \log_3 5^x$$

$$\Rightarrow x\log_3 9 = \log_3 6 + x\log_3 5$$

$$\Rightarrow x\log_3 9 - x\log_3 5 = \log_3 6$$

$$\Rightarrow x\left[\log_3 9 - \log_3 5\right] = \log_3 6$$

$$\Rightarrow x = \frac{\log_3 6}{\log_3 9 - \log_3 5}$$

TIDY FURTHER AS FOLLOWS

$$\Rightarrow x = \frac{\log_3 3 + \log_3 2}{\log_3 3^2 - \log_3 5}$$

$$\Rightarrow x = \frac{1 + \log_3 2}{2\log_3 3 - \log_3 5}$$

$$\Rightarrow x = \frac{1 + \log_3 2}{2 - \log_3 5}$$

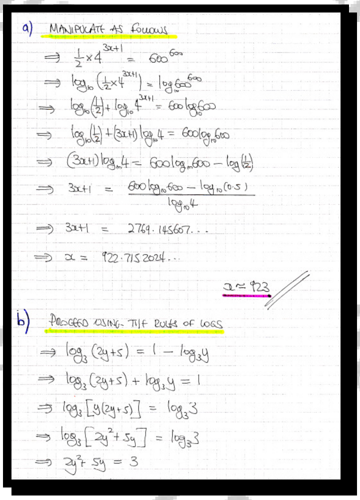
Question 85 **(****+)**

Solve each of the following equations.

a) $\frac{1}{2}\times 4^{3x+1} = 600^{600}$.

b) $\log_3(2y+5) = 1 - \log_3 y$.

$\boxed{\text{M.P}}$, $\boxed{x = 922.7152024...}$, $\boxed{y = \frac{1}{2}}$



a) MANIPULATE AS FOLLOWS

$$\Rightarrow \frac{1}{2}\times 4^{3x+1} = 600^{600}$$

$$\Rightarrow \log_{10}\left(\frac{1}{2}\times 4^{3x+1}\right) = \log_{10}600^{600}$$

$$\Rightarrow \log_{10}\left(\frac{1}{2}\right) + \log_{10}4^{3x+1} = 600\log_{10}600$$

$$\Rightarrow \log_{10}\left(\frac{1}{2}\right) + (3x+1)\log_{10}4 = 600\log_{10}600$$

$$\Rightarrow (3x+1)\log_{10}4 = 600\log_{10}600 - \log\left(\frac{1}{2}\right)$$

$$\Rightarrow 3x+1 = \frac{600\log_{10}600 - \log_{10}(0.5)}{\log_{10}4}$$

$$\Rightarrow 3x+1 = 2769.145607...$$

$$\Rightarrow x = 922.7152024...$$

$$x \simeq 923$$

b) PROCEED USING THE RULES OF LOGS

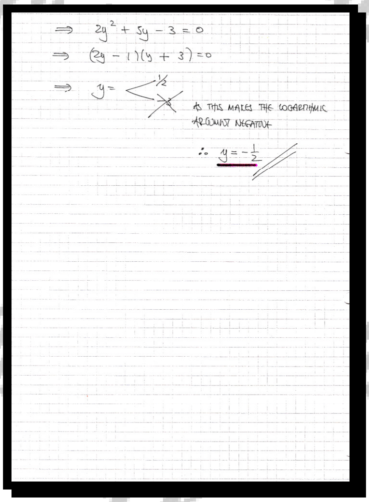
$$\Rightarrow \log_3(2y+5) = 1 - \log_3 y$$

$$\Rightarrow \log_3(2y+5) + \log_3 y = 1$$

$$\Rightarrow \log_3[y(2y+5)] = \log_3 3$$

$$\Rightarrow \log_3[2y^2+5y] = \log_3 3$$

$$\Rightarrow 2y^2 + 5y = 3$$



$$\Rightarrow 2y^2 + 5y - 3 = 0$$

$$\Rightarrow (2y-1)(y+3) = 0$$

$$\Rightarrow y = \begin{cases} 1/2 \\ \cancel{-3} \end{cases}$$

AS THIS MAKES THE LOGARITHMIC ARGUMENT NEGATIVE

$$\therefore y = -\frac{1}{2}$$

Question 86 **(****+)**

$$y = \left(\tfrac{1}{2}\right)^x .$$

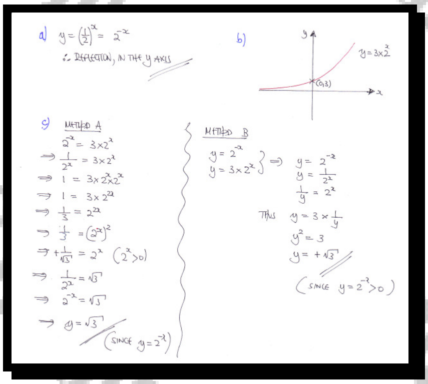
a) Describe the geometric transformation which maps the graph of the curve with equation $y = 2^x$, onto the graph of the curve with equation $y = \left(\tfrac{1}{2}\right)^x$.

b) Sketch the graph of $y = 3\times 2^x$.

The curve with equation $y = \left(\tfrac{1}{2}\right)^x$ intersects the curve with equation $y = 3\times 2^x$ at the point P.

c) Determine, as an exact simplified surd, the y coordinate of P.

$\boxed{}$, $\boxed{\text{reflection in the } y \text{ axis}}$, $\boxed{y=\sqrt{3}}$



a) $y=\left(\frac{1}{2}\right)^x = 2^{-x}$

∴ REFLECTION, IN THE y AXIS

b)

c) METHOD A

$2^{-x} = 3\times2^x$

$\Rightarrow \frac{1}{2^x} = 3\times2^x$

$\Rightarrow 1 = 3\times2^x\times2^x$

$\Rightarrow 1 = 3\times2^{2x}$

$\Rightarrow \frac{1}{3} = 2^{2x}$

$\Rightarrow \frac{1}{3} = \left(2^x\right)^2$

$\Rightarrow +\frac{1}{\sqrt{3}} = 2^x \quad (2^x>0)$

$\Rightarrow \frac{1}{2^x} = \sqrt{3}$

$\Rightarrow 2^{-x} = \sqrt{3}$

$\Rightarrow y = \sqrt{3}$

(SINCE $y=2^{-x}$)

METHOD B

$\left.\begin{array}{l} y=2^{-x} \\ y=3\times2^x \end{array}\right\} \Rightarrow \begin{array}{l} y=2^{-x} \\ y=\frac{1}{2^x} \\ \frac{1}{y}=2^x \end{array}$

THUS $y = 3\times\frac{1}{y}$

$y^2=3$

$y=+\sqrt{3}$

(SINCE $y=2^{-x}>0$)

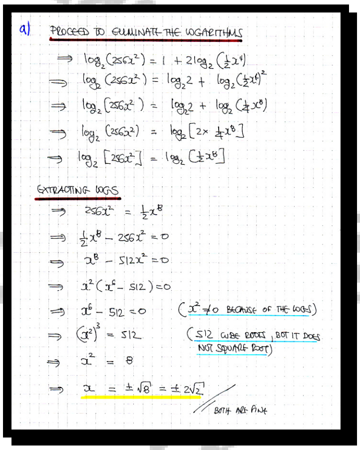
Question 87 **(****+)**

Solve each of the following logarithmic equations, giving the answers in exact simplified form where appropriate.

a) $\log_2(256x^2) = 1 + 2\log_2\left(\frac{1}{2}x^4\right)$.

b) $2\log_2\left(\frac{y}{2}\right) + \log_2\sqrt{y} = 8$.

$\boxed{\text{S.N}}$, $\boxed{x = \pm\sqrt{8}}$, $\boxed{y = 16}$



a) PROCEED TO ELIMINATE THE LOGARITHMS

$\Rightarrow \log_2(256x^2) = 1 + 2\log_2\left(\tfrac{1}{2}x^4\right)$

$\Rightarrow \log_2(256x^2) = \log_2 2 + \log_2\left(\tfrac{1}{2}x^4\right)^2$

$\Rightarrow \log_2(256x^2) = \log_2 2 + \log_2\left(\tfrac{1}{4}x^8\right)$

$\Rightarrow \log_2(256x^2) = \log_2\left[2 \times \tfrac{1}{4}x^8\right]$

$\Rightarrow \log_2\left[256x^2\right] = \log_2\left[\tfrac{1}{2}x^8\right]$

EXTRACTING LOGS

$\Rightarrow 256x^2 = \tfrac{1}{2}x^8$

$\Rightarrow \tfrac{1}{2}x^8 - 256x^2 = 0$

$\Rightarrow x^8 - 512x^2 = 0$

$\Rightarrow x^2(x^6 - 512) = 0$

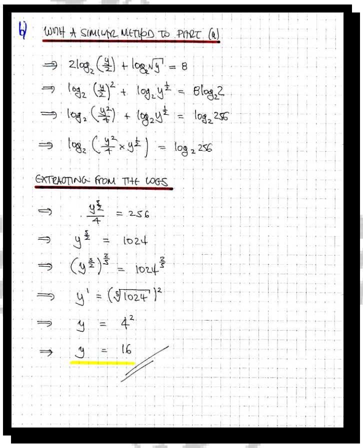
$\Rightarrow x^6 - 512 = 0$ ($x^2 \neq 0$ BECAUSE OF THE LOGS)

$\Rightarrow (x^2)^3 = 512$ (512 CUBE ROOTS, BUT IT DOES NOT SQUARE ROOT)

$\Rightarrow x^2 = 8$

$\Rightarrow x = \pm\sqrt{8} = \pm 2\sqrt{2}$ //

BOTH ARE FINE



b) WITH A SIMILAR METHOD TO PART (a)

$\Rightarrow 2\log_2\left(\tfrac{y}{2}\right) + \log_2\sqrt{y} = 8$

$\Rightarrow \log_2\left(\tfrac{y}{2}\right)^2 + \log_2 y^{\frac{1}{2}} = 8\log_2 2$

$\Rightarrow \log_2\left(\tfrac{y^2}{4}\right) + \log_2 y^{\frac{1}{2}} = \log_2 256$

$\Rightarrow \log_2\left(\tfrac{y^2}{4} \times y^{\frac{1}{2}}\right) = \log_2 256$

EXTRACTING FROM THE LOGS

$\Rightarrow \frac{y^{\frac{5}{2}}}{4} = 256$

$\Rightarrow y^{\frac{5}{2}} = 1024$

$\Rightarrow \left(y^{\frac{5}{2}}\right)^{\frac{2}{5}} = 1024^{\frac{2}{5}}$

$\Rightarrow y^1 = \left(\sqrt[5]{1024}\right)^2$

$\Rightarrow y = 4^2$

$\Rightarrow y = 16$ //

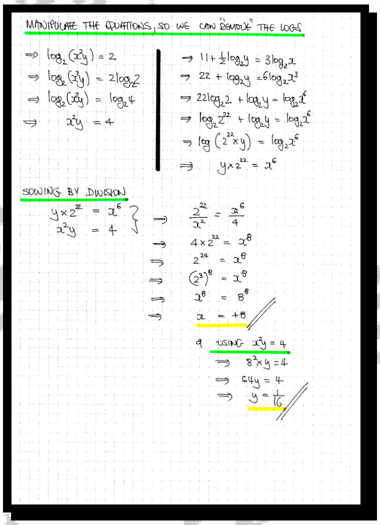
Question 88 (****+)

Solve the following simultaneous logarithmic equations.

$$\log_2\left(x^2y\right)=2$$

$$11+\frac{1}{2}\log_2 y=3\log_2 x\,.$$

$\boxed{}$, $\boxed{x=8,\quad y=\frac{1}{16}}$



MANIPULATE THE EQUATIONS, SO WE CAN "REMOVE" THE LOGS

$\Rightarrow \log_2(x^2y)=2$

$\Rightarrow \log_2(x^2y)=2\log_2 2$

$\Rightarrow \log_2(x^2y)=\log_2 4$

$\Rightarrow x^2y=4$

$\Rightarrow 11+\frac{1}{2}\log_2 y=3\log_2 x$

$\Rightarrow 22+\log_2 y=6\log_2 x^3$

$\Rightarrow 22\log_2 2+\log_2 y=\log_2 x^6$

$\Rightarrow \log_2 2^{22}+\log_2 y=\log_2 x^6$

$\Rightarrow \log(2^{22}\times y)=\log_2 x^6$

$\Rightarrow y\times 2^{22}=x^6$

SOWING BY DIVISION

$$\left.\begin{aligned} y\times 2^{22}&=x^6\\ x^2y&=4\end{aligned}\right\}\Rightarrow \frac{2^{22}}{x^2}=\frac{x^6}{4}$$

$\Rightarrow 4\times 2^{22}=x^8$

$\Rightarrow 2^{24}=x^8$

$\Rightarrow (2^3)^8=x^8$

$\Rightarrow x^8=8^8$

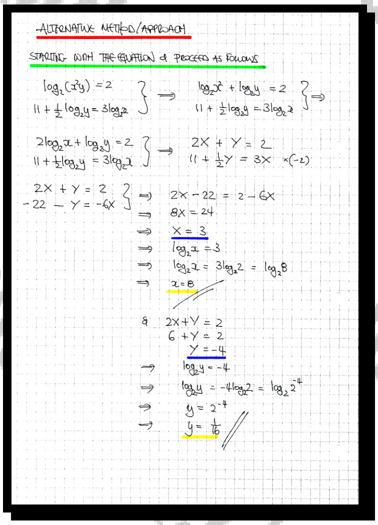
$\Rightarrow x=+8$

& USING $x^2y=4$

$\Rightarrow 8^2\times y=4$

$\Rightarrow 64y=4$

$\Rightarrow y=\frac{1}{16}$



ALTERNATIVE METHOD/APPROACH

STARTING WITH THE EQUATION & PROCEED AS FOLLOWS

$$\left.\begin{aligned} \log_2(x^2y)&=2\\ 11+\tfrac{1}{2}\log_2 y&=3\log_2 x\end{aligned}\right\}\Rightarrow \left.\begin{aligned} \log_2 x^2+\log_2 y&=2\\ 11+\tfrac{1}{2}\log_2 y&=3\log_2 x\end{aligned}\right\}\Rightarrow$$

$$\left.\begin{aligned} 2\log_2 x+\log_2 y&=2\\ 11+\tfrac{1}{2}\log_2 y&=3\log_2 x\end{aligned}\right\}\Rightarrow \begin{aligned} 2X+Y&=2\\ 11+\tfrac{1}{2}Y&=3X \quad \times(-2)\end{aligned}$$

$$\left.\begin{aligned} 2X+Y&=2\\ -22-Y&=-6X\end{aligned}\right\}\Rightarrow 2X-22=2-6X$$

$\Rightarrow 8X=24$

$\Rightarrow X=3$

$\Rightarrow \log_2 x=3$

$\Rightarrow \log_2 x=3\log_2 2=\log_2 8$

$\Rightarrow x=8$

& $2X+Y=2$

$6+Y=2$

$Y=-4$

$\Rightarrow \log_2 y=-4$

$\Rightarrow \log_2 y=-4\log_2 2=\log_2 2^{-4}$

$\Rightarrow y=2^{-4}$

$\Rightarrow y=\frac{1}{16}$

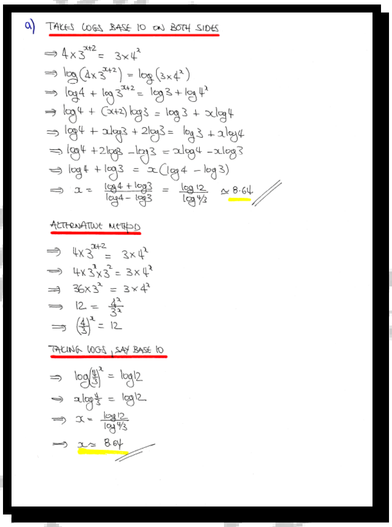
Question 89 (****+)

Solve each of the following equations, giving the final answers correct to three significant figures, where appropriate.

a) $4\times 3^{x+2} = 3\times 4^{x}$.

b) $\log_a\left(1+\sqrt{x}\right) = \frac{1}{2}\log_a\left(9+\sqrt{16x}\right)$.

$\boxed{}$, $\boxed{x \approx 8.64}$, $\boxed{y = 16}$



a) TAKES LOGS BASE 10 ON BOTH SIDES

$\Rightarrow 4\times 3^{x+2} = 3\times 4^{x}$

$\Rightarrow \log(4\times 3^{x+2}) = \log(3\times 4^{x})$

$\Rightarrow \log 4 + \log 3^{x+2} = \log 3 + \log 4^{x}$

$\Rightarrow \log 4 + (x+2)\log 3 = \log 3 + x\log 4$

$\Rightarrow \log 4 + x\log 3 + 2\log 3 = \log 3 + x\log 4$

$\Rightarrow \log 4 + 2\log 3 - \log 3 = x\log 4 - x\log 3$

$\Rightarrow \log 4 + \log 3 = x(\log 4 - \log 3)$

$\Rightarrow x = \frac{\log 4 + \log 3}{\log 4 - \log 3} = \frac{\log 12}{\log 4/3} \simeq 8.64$ //

ALTERNATIVE METHOD

$\Rightarrow 4\times 3^{x+2} = 3\times 4^{x}$

$\Rightarrow 4\times 3^{x}\times 3^{2} = 3\times 4^{x}$

$\Rightarrow 36\times 3^{x} = 3\times 4^{x}$

$\Rightarrow 12 = \frac{4^{x}}{3^{x}}$

$\Rightarrow \left(\frac{4}{3}\right)^{x} = 12$

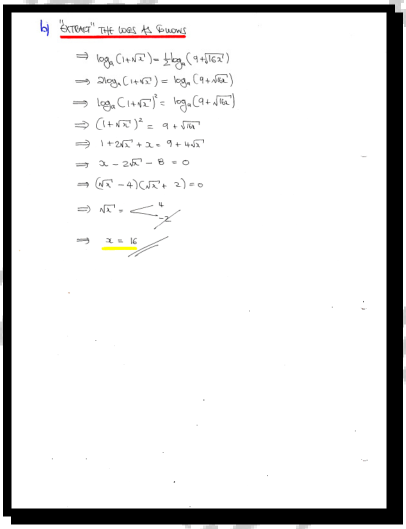
TAKING LOGS, SAY BASE 10

$\Rightarrow \log\left(\frac{4}{3}\right)^{x} = \log 12$

$\Rightarrow x\log\frac{4}{3} = \log 12$

$\Rightarrow x = \frac{\log 12}{\log 4/3}$

$\Rightarrow x \simeq 8.64$ //



b) "EXTRACT" THE LOGS AS FOLLOWS

$\Rightarrow \log_a(1+\sqrt{x}) = \frac{1}{2}\log_a(9+\sqrt{16x})$

$\Rightarrow 2\log_a(1+\sqrt{x}) = \log_a(9+\sqrt{16x})$

$\Rightarrow \log_a(1+\sqrt{x})^{2} = \log_a(9+\sqrt{16x})$

$\Rightarrow (1+\sqrt{x})^{2} = 9+\sqrt{16x}$

$\Rightarrow 1+2\sqrt{x}+x = 9+4\sqrt{x}$

$\Rightarrow x - 2\sqrt{x} - 8 = 0$

$\Rightarrow (\sqrt{x}-4)(\sqrt{x}+2) = 0$

$\Rightarrow \sqrt{x} = \begin{cases} 4 \\ \cancel{-2} \end{cases}$

$\Rightarrow x = 16$ //

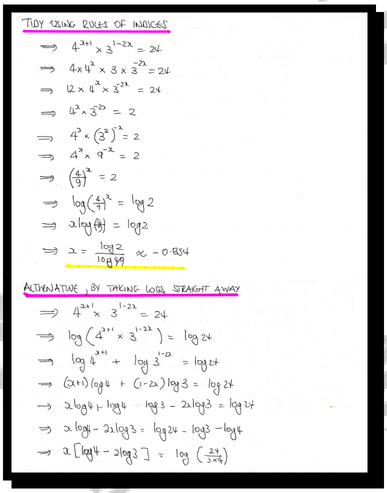
Question 90 (****+)

Solve the following exponential equation.

$$4^{x+1} \times 3^{1-2x} = 24 .$$

Give the answer correct to 3 decimal places.

$\boxed{}$, $\boxed{x \approx -0.855}$



TIDY USING RULES OF INDICES

$\Rightarrow 4^{x+1} \times 3^{1-2x} = 24$

$\Rightarrow 4 \times 4^{x} \times 3 \times 3^{-2x} = 24$

$\Rightarrow 12 \times 4^{x} \times 3^{-2x} = 24$

$\Rightarrow 4^{x} \times 3^{-2x} = 2$

$\Rightarrow 4^{x} \times (3^{2})^{-x} = 2$

$\Rightarrow 4^{x} \times 9^{-x} = 2$

$\Rightarrow \left(\frac{4}{9}\right)^{x} = 2$

$\Rightarrow \log\left(\frac{4}{9}\right)^{x} = \log 2$

$\Rightarrow x \log\left(\frac{4}{9}\right) = \log 2$

$\Rightarrow x = \frac{\log 2}{\log 4/9} \approx -0.854$

ALTERNATIVE, BY TAKING LOGS STRAIGHT AWAY

$\Rightarrow 4^{x+1} \times 3^{1-2x} = 24$

$\Rightarrow \log\left(4^{x+1} \times 3^{1-2x}\right) = \log 24$

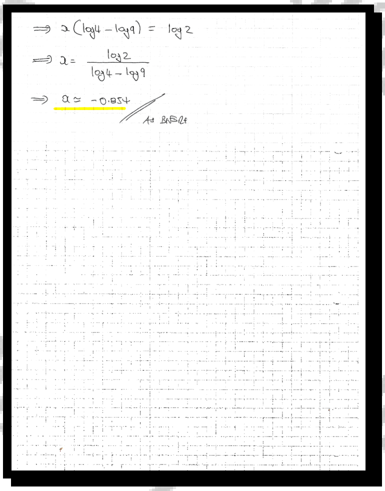
$\Rightarrow \log 4^{x+1} + \log 3^{1-2x} = \log 24$

$\Rightarrow (x+1)\log 4 + (1-2x)\log 3 = \log 24$

$\rightarrow x\log 4 + \log 4 \quad \log 3 - 2x\log 3 = \log 24$

$\Rightarrow x\log 4 - 2x\log 3 = \log 24 - \log 3 - \log 4$

$\rightarrow x\left[\log 4 - 2\log 3\right] = \log\left(\frac{24}{3\times 4}\right)$



$\Rightarrow x(\log 4 - \log 9) = \log 2$

$\Rightarrow x = \frac{\log 2}{\log 4 - \log 9}$

$\Rightarrow x \simeq -0.854$

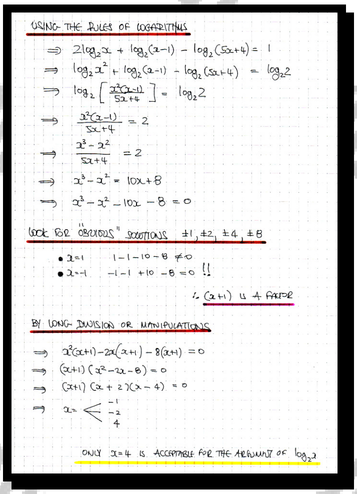
AS BEFORE

Question 91 (****+)

Solve the following logarithmic equation.

$$2\log_2 x+\log_2(x-1)-\log_2(5x+4)=1.$$

[illegible], $\boxed{x=4}$



USING THE RULES OF LOGARITHMS

$\Rightarrow 2\log_2 x+\log_2(x-1)-\log_2(5x+4)=1$

$\Rightarrow \log_2 x^2+\log_2(x-1)-\log_2(5x+4)=\log_2 2$

$\Rightarrow \log_2\left[\frac{x^2(x-1)}{5x+4}\right]=\log_2 2$

$\Rightarrow \frac{x^2(x-1)}{5x+4}=2$

$\Rightarrow \frac{x^3-x^2}{5x+4}=2$

$\Rightarrow x^3-x^2=10x+8$

$\Rightarrow x^3-x^2-10x-8=0$

LOOK FOR "OBVIOUS" SOLUTIONS $\pm1, \pm2, \pm4, \pm8$

- $x=1$ $\quad 1-1-10-8\neq0$
- $x=-1$ $\quad -1-1+10-8=0$!!

$\therefore (x+1)$ IS A FACTOR

BY LONG DIVISION OR MANIPULATIONS

$\Rightarrow x^2(x+1)-2x(x+1)-8(x+1)=0$

$\Rightarrow (x+1)(x^2-2x-8)=0$

$\Rightarrow (x+1)(x+2)(x-4)=0$

$\Rightarrow x = \left\langle \begin{array}{l} -1 \\ -2 \\ 4 \end{array} \right.$

ONLY $x=4$ IS ACCEPTABLE FOR THE ARGUMENT OF $\log_2 x$

Question 92 (****+)

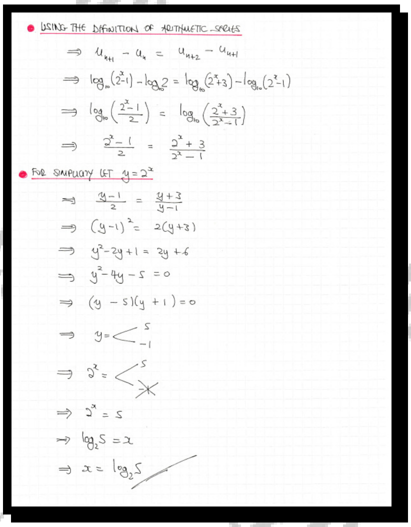
The following three numbers

$$\log_{10} 2, \quad \log_{10}\left(2^x - 1\right), \quad \log_{10}\left(2^x + 3\right),$$

are consecutive terms in an arithmetic progression.

Determine the value of x as an exact logarithm, of base 2.

$\boxed{}$, $\boxed{x = \log_2 5}$



- USING THE DEFINITION OF ARITHMETIC SERIES

$\Rightarrow u_{n+1} - u_n = u_{n+2} - u_{n+1}$

$\Rightarrow \log_{10}(2^x-1) - \log_{10}2 = \log_{10}(2^x+3) - \log_{10}(2^x-1)$

$\Rightarrow \log_{10}\left(\frac{2^x-1}{2}\right) = \log_{10}\left(\frac{2^x+3}{2^x-1}\right)$

$\Rightarrow \frac{2^x-1}{2} = \frac{2^x+3}{2^x-1}$

- FOR SIMPLICITY LET $y = 2^x$

$\Rightarrow \frac{y-1}{2} = \frac{y+3}{y-1}$

$\Rightarrow (y-1)^2 = 2(y+3)$

$\Rightarrow y^2 - 2y + 1 = 2y + 6$

$\Rightarrow y^2 - 4y - 5 = 0$

$\Rightarrow (y-5)(y+1) = 0$

$\Rightarrow y = \begin{cases} 5 \\ -1 \end{cases}$

$\Rightarrow 2^x = \begin{cases} 5 \\ \cancel{-1} \end{cases}$

$\Rightarrow 2^x = 5$

$\Rightarrow \log_2 5 = x$

$\Rightarrow x = \log_2 5$

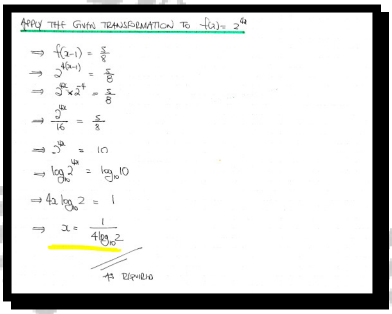
Question 93 (****+)

$$f(x) = 2^{4x}$$

Show that the solution of the equation $f(x-1) = \frac{5}{8}$ is given by

$$x = \frac{1}{4\log_{10} 2}.$$

$\boxed{}$, $\boxed{\text{proof}}$



APPLY THE GIVEN TRANSFORMATION TO $f(x) = 2^{4x}$

$\Rightarrow f(x-1) = \frac{5}{8}$

$\Rightarrow 2^{4(x-1)} = \frac{5}{8}$

$\Rightarrow 2^{4x} \times 2^{-4} = \frac{5}{8}$

$\Rightarrow \frac{2^{4x}}{16} = \frac{5}{8}$

$\Rightarrow 2^{4x} = 10$

$\Rightarrow \log_{10} 2^{4x} = \log_{10} 10$

$\Rightarrow 4x \log_{10} 2 = 1$

$\Rightarrow x = \frac{1}{4\log_{10} 2}$

AS REQUIRED

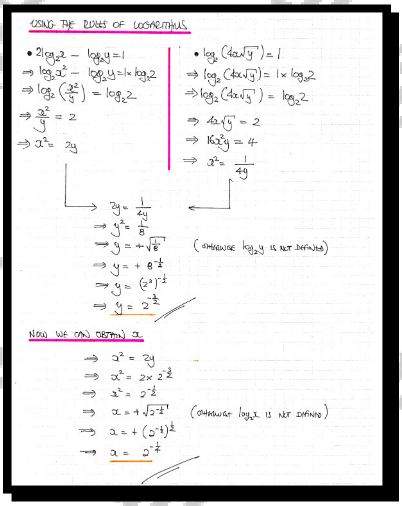
Question 94 **(****+)**

$$2\log_2 x - \log_2 y = 1$$

$$\log_2\left(4x\sqrt{y}\right) = 1.$$

Solve the above simultaneous logarithmic equations, giving the final answers as exact powers of 2.

$\boxed{}$, $\boxed{x = 2^{-\frac{1}{4}},\ y = 2^{-\frac{3}{2}}}$



USING THE RULES OF LOGARITHMS

- $2\log_2 x - \log_2 y = 1$
- $\Rightarrow \log_2 x^2 - \log_2 y = 1 \times \log_2 2$
- $\Rightarrow \log_2\left(\frac{x^2}{y}\right) = \log_2 2$
- $\Rightarrow \frac{x^2}{y} = 2$
- $\Rightarrow x^2 = 2y$

- $\log_2\left(4x\sqrt{y}\right) = 1$
- $\Rightarrow \log_2\left(4x\sqrt{y}\right) = 1 \times \log_2 2$
- $\Rightarrow \log_2\left(4x\sqrt{y}\right) = \log_2 2$
- $\Rightarrow 4x\sqrt{y} = 2$
- $\Rightarrow 16x^2y = 4$
- $\Rightarrow x^2 = \frac{1}{4y}$

$2y = \frac{1}{4y}$

$\Rightarrow y^2 = \frac{1}{8}$

$\Rightarrow y = +\sqrt{\frac{1}{8}}$ (OTHERWISE $\log_2 y$ IS NOT DEFINED)

$\Rightarrow y = +8^{-\frac{1}{2}}$

$\Rightarrow y = (2^3)^{-\frac{1}{2}}$

$\Rightarrow y = 2^{-\frac{3}{2}}$

NOW WE CAN OBTAIN x

$\Rightarrow x^2 = 2y$

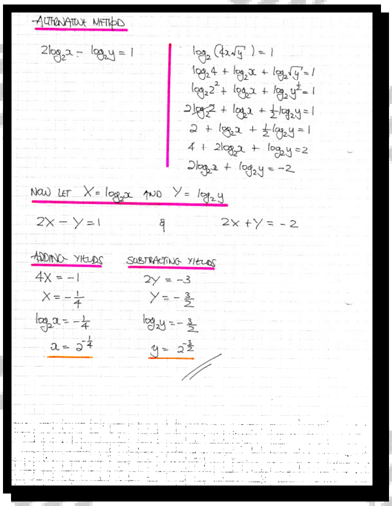
$\Rightarrow x^2 = 2 \times 2^{-\frac{3}{2}}$

$\Rightarrow x^2 = 2^{-\frac{1}{2}}$

$\Rightarrow x = +\sqrt{2^{-\frac{1}{2}}}$ (OTHERWISE $\log_2 x$ IS NOT DEFINED)

$\Rightarrow x = +(2^{-\frac{1}{2}})^{\frac{1}{2}}$

$\Rightarrow x = 2^{-\frac{1}{4}}$



ALTERNATIVE METHOD

$2\log_2 x - \log_2 y = 1$

$\log_2\left(4x\sqrt{y}\right) = 1$

$\log_2 4 + \log_2 x + \log_2\sqrt{y} = 1$

$\log_2 2^2 + \log_2 x + \log_2 y^{\frac{1}{2}} = 1$

$2\cancel{\log_2 2} + \log_2 x + \frac{1}{2}\log_2 y = 1$

$2 + \log_2 x + \frac{1}{2}\log_2 y = 1$

$4 + 2\log_2 x + \log_2 y = 2$

$2\log_2 x + \log_2 y = -2$

NOW LET $X = \log_2 x$ AND $Y = \log_2 y$

$2X - Y = 1$ & $2X + Y = -2$

ADDING YIELDS

$4X = -1$

$X = -\frac{1}{4}$

$\log_2 x = -\frac{1}{4}$

$x = 2^{-\frac{1}{4}}$

SUBTRACTING YIELDS

$2Y = -3$

$Y = -\frac{3}{2}$

$\log_2 y = -\frac{3}{2}$

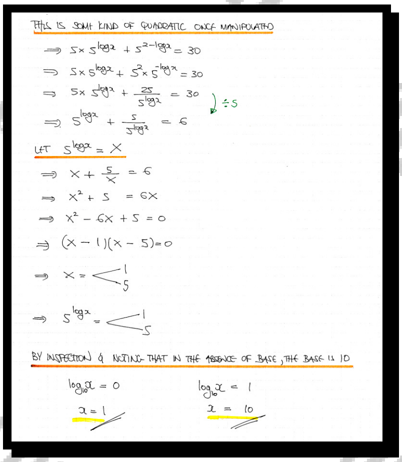
$y = 2^{-\frac{3}{2}}$

Question 95 (****+)

Solve the following logarithmic equation

$$5\times 5^{\log x}+5^{2-\log x}=30, \quad x>0.$$

$\boxed{}$, $\boxed{x=1,\ x=10}$



THIS IS SOME KIND OF QUADRATIC ONCE MANIPULATED

$\Rightarrow 5\times 5^{\log x}+5^{2-\log x}=30$

$\Rightarrow 5\times 5^{\log x}+5^{2}\times 5^{-\log x}=30$

$\Rightarrow 5\times 5^{\log x}+\frac{25}{5^{\log x}}=30$ $\downarrow \div 5$

$\Rightarrow 5^{\log x}+\frac{5}{5^{\log x}}=6$

LET $5^{\log x}=X$

$\Rightarrow X+\frac{5}{X}=6$

$\Rightarrow X^2+5=6X$

$\Rightarrow X^2-6X+5=0$

$\Rightarrow (X-1)(X-5)=0$

$\Rightarrow X=\begin{cases}1\\5\end{cases}$

$\Rightarrow 5^{\log x}=\begin{cases}1\\5\end{cases}$

BY INSPECTION & NOTING THAT IN THE ABSENCE OF BASE, THE BASE IS 10

$\log_{10}x=0$ $\quad$ $\log_{10}x=1$

$x=1$ $\quad$ $x=10$

Question 96 (****+)

It is given that

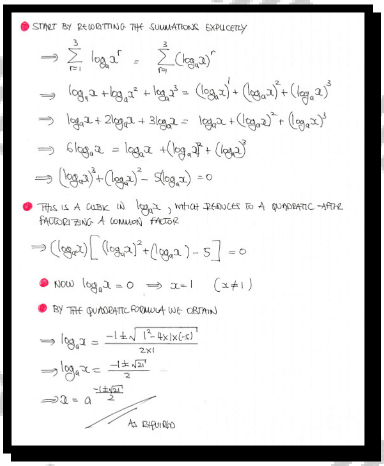
$$\sum_{r=1}^{3} \log_a x^r = \sum_{r=1}^{3} \left(\log_a x\right)^r ,$$

where a and x are positive numbers such that $x \neq a$, $x \neq 1$ and $a > 1$.

Show clearly that

$$x = a^{\frac{-1 \pm \sqrt{21}}{2}} .$$

$\boxed{}$, $\boxed{\text{proof}}$



● START BY REWRITTING THE SUMMATIONS EXPLICETLY

$$\Rightarrow \sum_{r=1}^{3} \log_a x^r = \sum_{r=1}^{3} (\log_a x)^r$$

$$\Rightarrow \log_a x + \log_a x^2 + \log_a x^3 = (\log_a x)^1 + (\log_a x)^2 + (\log_a x)^3$$

$$\Rightarrow \log_a x + 2\log_a x + 3\log_a x = \log_a x + (\log_a x)^2 + (\log_a x)^3$$

$$\Rightarrow 6\log_a x = \log_a x + (\log_a x)^2 + (\log_a x)^3$$

$$\Rightarrow (\log_a x)^3 + (\log_a x)^2 - 5(\log_a x) = 0$$

● THIS IS A CUBIC IN $\log_a x$, WHICH REDUCES TO A QUADRATIC AFTER FACTORIZING A COMMON FACTOR

$$\Rightarrow (\log_a x)\left[(\log_a x)^2 + (\log_a x) - 5\right] = 0$$

● NOW $\log_a x = 0 \Rightarrow x = 1$ $(x \neq 1)$

● BY THE QUADRATIC FORMULA WE OBTAIN

$$\Rightarrow \log_a x = \frac{-1 \pm \sqrt{1^2 - 4 \times 1 \times (-5)}}{2 \times 1}$$

$$\Rightarrow \log_a x = \frac{-1 \pm \sqrt{21}}{2}$$

$$\Rightarrow x = a^{\frac{-1 \pm \sqrt{21}}{2}}$$

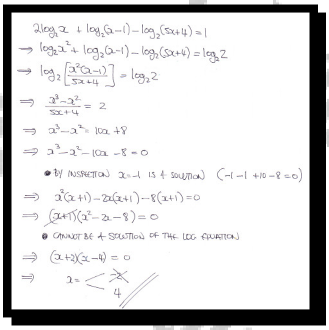
AS REQUIRED

Question 97 (**+)**

$$2\log_2 x + \log_2(x-1) - \log_2(5x+4) = 1 .$$

Find the only real root of the above logarithmic equation.

$\boxed{\text{SYN-V}}$, $\boxed{x = 4}$



$2\log_2 x + \log_2(x-1) - \log_2(5x+4) = 1$

$\Rightarrow \log_2 x^2 + \log_2(x-1) - \log_2(5x+4) = \log_2 2$

$\Rightarrow \log_2\left[\frac{x^2(x-1)}{5x+4}\right] = \log_2 2$

$\Rightarrow \frac{x^3-x^2}{5x+4} = 2$

$\Rightarrow x^3 - x^2 = 10x + 8$

$\Rightarrow x^3 - x^2 - 10x - 8 = 0$

- BY INSPECTION $x=-1$ IS A SOLUTION $(-1-1+10-8=0)$

$\Rightarrow x^2(x+1) - 2x(x+1) - 8(x+1) = 0$

$\Rightarrow \cancel{(x+1)}(x^2-2x-8) = 0$

- CANNOT BE A SOLUTION OF THE LOG EQUATION

$\Rightarrow (x+2)(x-4) = 0$

$\Rightarrow x = \cancel{-2}, 4$ //

Question 98 **(****+)**

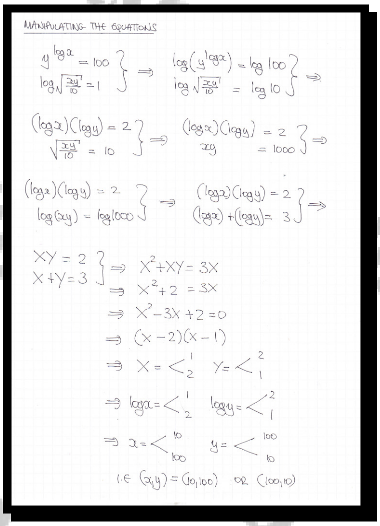
Solve the following simultaneous logarithmic equations

$$y^{\log x} = 100$$

$$\log\sqrt{\frac{xy}{10}} = 1,$$

given further that x, $y \in \mathbb{R}$, with $x > 0$, $y > 0$.

$\boxed{}$, $\boxed{x = 10,\ \ y = 100,\ \text{ or the other way round}}$



MANIPULATING THE EQUATIONS

$$\left.\begin{array}{l} y^{\log x} = 100 \\ \log\sqrt{\frac{xy}{10}} = 1 \end{array}\right\} \Rightarrow \left.\begin{array}{l} \log\left(y^{\log x}\right) = \log 100 \\ \log\sqrt{\frac{xy}{10}} = \log 10 \end{array}\right\} \Rightarrow$$

$$\left.\begin{array}{l} (\log x)(\log y) = 2 \\ \sqrt{\frac{xy}{10}} = 10 \end{array}\right\} \Rightarrow \left.\begin{array}{l} (\log x)(\log y) = 2 \\ xy = 1000 \end{array}\right\} \Rightarrow$$

$$\left.\begin{array}{l} (\log x)(\log y) = 2 \\ \log(xy) = \log 1000 \end{array}\right\} \Rightarrow \left.\begin{array}{l} (\log x)(\log y) = 2 \\ (\log x) + (\log y) = 3 \end{array}\right\} \Rightarrow$$

$$\left.\begin{array}{l} XY = 2 \\ X + Y = 3 \end{array}\right\} \Rightarrow X^2 + XY = 3X$$

$$\Rightarrow X^2 + 2 = 3X$$

$$\Rightarrow X^2 - 3X + 2 = 0$$

$$\Rightarrow (X-2)(X-1)$$

$$\Rightarrow X = <^{1}_{2} \quad Y = <^{2}_{1}$$

$$\Rightarrow \log x = <^{1}_{2} \quad \log y = <^{2}_{1}$$

$$\Rightarrow x = <^{10}_{100} \quad y = <^{100}_{10}$$

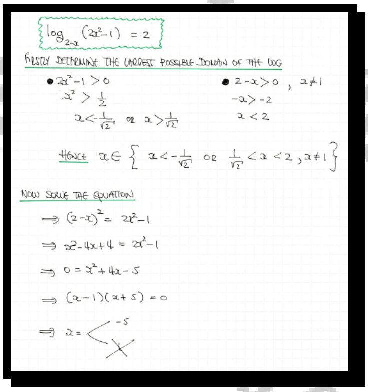
I.E $(x,y) = (10,100)$ OR $(100,10)$

Question 99 **(****+)**

Solve the following logarithmic equation, over the largest real domain

$$\log_{2-x}\left[2x^2-1\right]=2,\ x\in\mathbb{R}.$$

$\boxed{\quad}$, $\boxed{x=-5}$



$\log_{2-x}(2x^2-1)=2$

FIRSTLY DETERMINE THE LARGEST POSSIBLE DOMAIN OF THE LOG

- $2x^2-1>0$
  $x^2>\frac{1}{2}$
  $x<-\frac{1}{\sqrt{2}}$ or $x>\frac{1}{\sqrt{2}}$
- $2-x>0$, $x\neq1$
  $-x>-2$
  $x<2$

HENCE $x\in\left\{x<-\frac{1}{\sqrt{2}}\ \text{or}\ \frac{1}{\sqrt{2}}<x<2, x\neq1\right\}$

NOW SOLVE THE EQUATION

$\Rightarrow (2-x)^2=2x^2-1$

$\Rightarrow x^2-4x+4=2x^2-1$

$\Rightarrow 0=x^2+4x-5$

$\Rightarrow (x-1)(x+5)=0$

$\Rightarrow x=\begin{cases}-5\\ \cancel{1}\end{cases}$

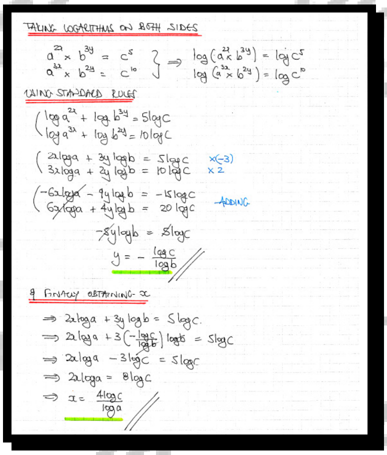
Question 100 **(****+)**

Solve the following simultaneous equations

$$a^{2x}\times b^{3y}=c^5$$
$$a^{3x}\times b^{2y}=c^{10}$$

Give the answers in exact form in terms of $\log a$, $\log b$ and $\log c$.

$\boxed{\quad}$, $\boxed{x=\frac{4\log c}{\log a}}$, $\boxed{y=-\frac{\log c}{\log b}}$



TAKING LOGARITHMS ON BOTH SIDES

$\left.\begin{array}{l}a^{2x}\times b^{3y}=c^5\\ a^{3x}\times b^{2y}=c^{10}\end{array}\right\}\Rightarrow \begin{array}{l}\log(a^{2x}\times b^{3y})=\log c^5\\ \log(a^{3x}\times b^{2y})=\log c^{10}\end{array}$

USING STANDARD RULES

$\left(\begin{array}{l}\log a^{2x}+\log b^{3y}=5\log c\\ \log a^{3x}+\log b^{2y}=10\log c\end{array}\right.$

$\left(\begin{array}{ll}2x\log a+3y\log b=5\log c & \times(-3)\\ 3x\log a+2y\log b=10\log c & \times 2\end{array}\right.$

$\left(\begin{array}{ll}\cancel{-6x\log a}-9y\log b=-15\log c & \text{ADDING}\\ \cancel{6x\log a}+4y\log b=20\log c\end{array}\right.$

$-\cancel{5}y\log b=\cancel{5}\log c$

$y=-\frac{\log c}{\log b}$

& FINALLY OBTAINING x

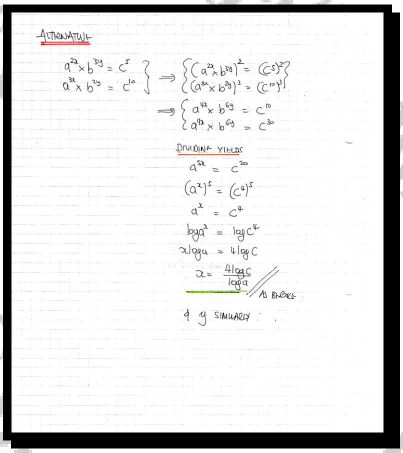
$\Rightarrow 2x\log a+3y\log b=5\log c.$

$\Rightarrow 2x\log a+3\left(-\frac{\log c}{\cancel{\log b}}\right)\cancel{\log b}=5\log c$

$\Rightarrow 2x\log a-3\log c=5\log c$

$\Rightarrow 2x\log a=8\log c$

$\Rightarrow x=\frac{4\log c}{\log a}$



ALTERNATIVE

$\left.\begin{array}{l}a^{2x}\times b^{3y}=c^5\\ a^{3x}\times b^{2y}=c^{10}\end{array}\right\}\Rightarrow\left\{\begin{array}{l}(a^{2x}\times b^{3y})^2=(c^5)^2\\ (a^{3x}\times b^{2y})^3=(c^{10})^3\end{array}\right\}$

$\Rightarrow\left\{\begin{array}{l}a^{4x}\times b^{6y}=c^{10}\\ a^{9x}\times b^{6y}=c^{30}\end{array}\right.$

DIVIDING YIELDS

$a^{5x}=c^{20}$

$(a^x)^5=(c^4)^5$

$a^x=c^4$

$\log a^x=\log c^4$

$x\log a=4\log c$

$x=\frac{4\log c}{\log a}$ AS BEFORE

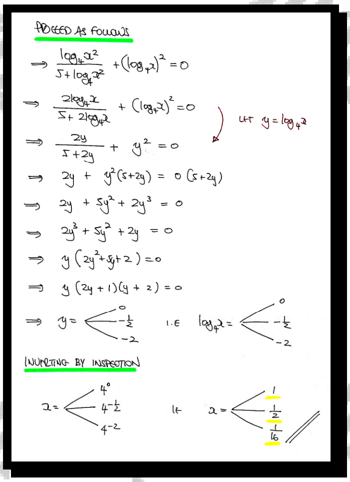
& y SIMILARLY

Question 101 **(****+)**

Solve the following logarithmic equation

$$\frac{\log_4 x^2}{5+\log_4 x^2}+\left(\log_4 x\right)^2=0.$$

$\boxed{}$, $\boxed{x=1,\ \frac{1}{2},\ \frac{1}{16}}$



PROCEED AS FOLLOWS

$$\Longrightarrow \frac{\log_4 x^2}{5+\log_4 x^2}+(\log_4 x)^2=0$$

$$\Longrightarrow \frac{2\log_4 x}{5+2\log_4 x}+(\log_4 x)^2=0$$

LET $y=\log_4 x$

$$\Longrightarrow \frac{2y}{5+2y}+y^2=0$$

$$\Longrightarrow 2y+y^2(5+2y)=0(5+2y)$$

$$\Longrightarrow 2y+5y^2+2y^3=0$$

$$\Longrightarrow 2y^3+5y^2+2y=0$$

$$\Longrightarrow y(2y^2+5y+2)=0$$

$$\Longrightarrow y(2y+1)(y+2)=0$$

$$\Longrightarrow y=\left\langle\begin{array}{l}0\\-\frac{1}{2}\\-2\end{array}\right. \quad \text{i.e} \quad \log_4 x=\left\langle\begin{array}{l}0\\-\frac{1}{2}\\-2\end{array}\right.$$

INVERTING BY INSPECTION

$$x=\left\langle\begin{array}{l}4^0\\4^{-\frac{1}{2}}\\4^{-2}\end{array}\right. \quad \text{ie} \quad x=\left\langle\begin{array}{l}1\\\frac{1}{2}\\\frac{1}{16}\end{array}\right. //$$

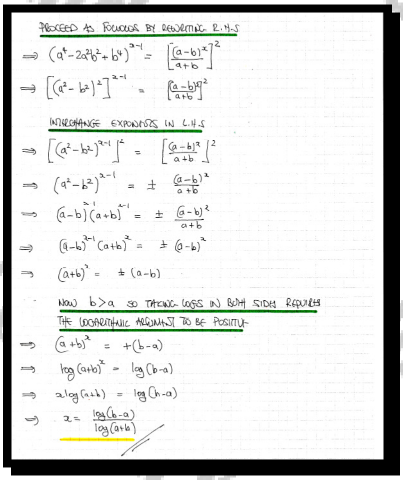
Question 102 **(****+)**

Solve the following equation

$$\left(a^4 - 2a^2b^2 + b^4\right)^{x-1} = (a-b)^{2x}(a+b)^{-2}$$

Give the answers in exact form in terms of a and b.

$\boxed{}$, $\boxed{x = \dfrac{\log(b-a)}{\log(b+a)}}$



PROCEED AS FOLLOWS BY REWRITING R.H.S

$\Rightarrow \left(a^4-2a^2b^2+b^4\right)^{x-1} = \left[\frac{(a-b)^x}{a+b}\right]^2$

$\Rightarrow \left[\left(a^2-b^2\right)^2\right]^{x-1} = \left[\frac{(a-b)^x}{a+b}\right]^2$

INTERCHANGE EXPONENTS IN L.H.S

$\Rightarrow \left[\left(a^2-b^2\right)^{x-1}\right]^2 = \left[\frac{(a-b)^x}{a+b}\right]^2$

$\Rightarrow \left(a^2-b^2\right)^{x-1} = \pm\frac{(a-b)^x}{a+b}$

$\Rightarrow (a-b)^{x-1}(a+b)^{x-1} = \pm\frac{(a-b)^x}{a+b}$

$\Rightarrow (a-b)^{x-1}(a+b)^x = \pm(a-b)^x$

$\Rightarrow (a+b)^x = \pm(a-b)$

NOW $b>a$ SO TAKING LOGS IN BOTH SIDES REQUIRES THE LOGARITHMIC ARGUMENT TO BE POSITIVE

$\Rightarrow (a+b)^x = +(b-a)$

$\Rightarrow \log(a+b)^x = \log(b-a)$

$\Rightarrow x\log(a+b) = \log(b-a)$

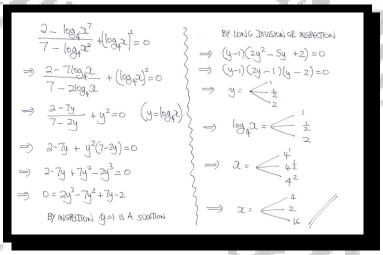
$\Rightarrow x = \frac{\log(b-a)}{\log(a+b)}$

Question 103 **(*****)**

Solve the following logarithmic equation

$$\frac{2-\log_4 x^7}{7-\log_4 x^2} + \left(\log_4 x\right)^2 = 0$$

$\boxed{}$, $\boxed{x = 2,\ x = 4,\ x = 16}$



$\frac{2-\log_4 x^7}{7-\log_4 x^2} + \left(\log_4 x\right)^2 = 0$

$\Rightarrow \frac{2-7\log_4 x}{7-2\log_4 x} + \left(\log_4 x\right)^2 = 0$

$\Rightarrow \frac{2-7y}{7-2y} + y^2 = 0 \quad (y = \log_4 x)$

$\Rightarrow 2-7y + y^2(7-2y) = 0$

$\Rightarrow 2-7y+7y^2-2y^3 = 0$

$\Rightarrow 0 = 2y^3-7y^2+7y-2$

BY INSPECTION $y=1$ IS A SOLUTION

BY LONG DIVISION OR INSPECTION

$\Rightarrow (y-1)(2y^2-5y+2) = 0$

$\Rightarrow (y-1)(2y-1)(y-2) = 0$

$\Rightarrow y = 1, \frac{1}{2}, 2$

$\Rightarrow \log_4 x = 1, \frac{1}{2}, 2$

$\Rightarrow x = 4^1, 4^{\frac{1}{2}}, 4^2$

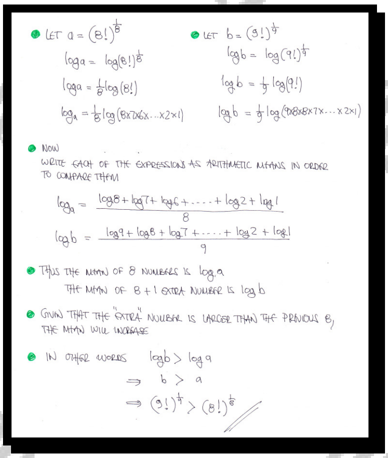
$\Rightarrow x = 4, 2, 16$

Question 104 (*****)

Show by valid mathematical arguments that

$$\sqrt[8]{8!} < \sqrt[9]{9!}$$

$\boxed{\quad}$, $\boxed{\text{proof}}$



- LET $a = (8!)^{\frac{1}{8}}$

  $\log a = \log(8!)^{\frac{1}{8}}$

  $\log a = \frac{1}{8}\log(8!)$

  $\log_a = \frac{1}{8}\log(8\times7\times6\times\ldots\times2\times1)$

- LET $b = (9!)^{\frac{1}{9}}$

  $\log b = \log(9!)^{\frac{1}{9}}$

  $\log b = \frac{1}{9}\log(9!)$

  $\log b = \frac{1}{9}\log(9\times8\times8\times7\times\ldots\times2\times1)$

- NOW

  WRITE EACH OF THE EXPRESSIONS AS ARITHMETIC MEANS IN ORDER TO COMPARE THEM

$$\log_a = \frac{\log8+\log7+\log6+\ldots\ldots+\log2+\log1}{8}$$

$$\log b = \frac{\log9+\log8+\log7+\ldots\ldots+\log2+\log1}{9}$$

- THUS THE MEAN OF 8 NUMBERS IS $\log a$

  THE MEAN OF 8 + 1 EXTRA NUMBER IS $\log b$

- GIVEN THAT THE "EXTRA" NUMBER IS LARGER THAN THE PREVIOUS 8, THE MEAN WILL INCREASE

- IN OTHER WORDS $\log b > \log a$

$$\Rightarrow \quad b > a$$

$$\Rightarrow \quad (9!)^{\frac{1}{9}} > (8!)^{\frac{1}{8}} \;//$$

Question 105 (*****)

i. Simplify the following expression.

$$9\log_{24}2+\log_{24}27\,.$$

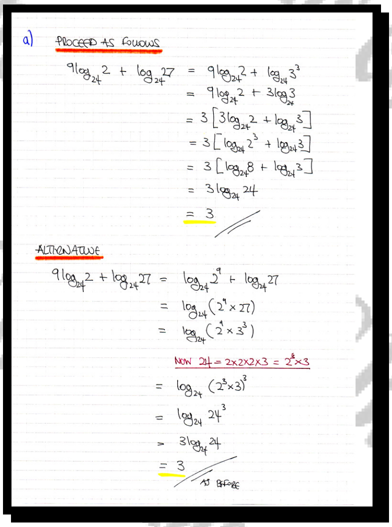
Show detailed workings in this simplification.

ii. It is given that

$$5\times 2^{t-1} = 2\times t^{2t} \quad \Rightarrow \quad (10k)^t = k$$

Determine the value of k.

$\boxed{}$, $\boxed{3}$, $\boxed{k=1.25}$



a) PROCEED AS FOLLOWS

$$9\log_{24}2+\log_{24}27 = 9\log_{24}2+\log_{24}3^3$$
$$= 9\log_{24}2+3\log_{24}3$$
$$= 3\left[3\log_{24}2+\log_{24}3\right]$$
$$= 3\left[\log_{24}2^3+\log_{24}3\right]$$
$$= 3\left[\log_{24}8+\log_{24}3\right]$$
$$= 3\log_{24}24$$
$$= 3$$

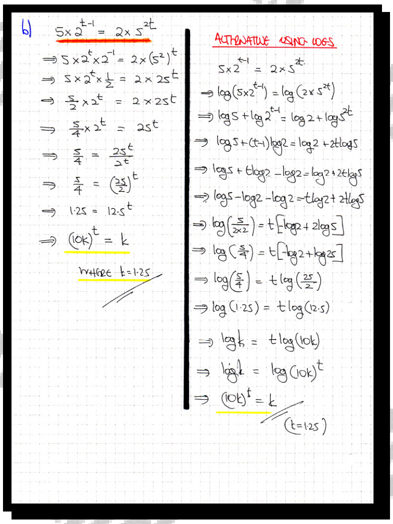
ALTERNATIVE

$$9\log_{24}2+\log_{24}27 = \log_{24}2^9+\log_{24}27$$
$$= \log_{24}(2^9\times 27)$$
$$= \log_{24}(2^9\times 3^3)$$

NOW $24 = 2\times2\times2\times3 = 2^3\times3$

$$= \log_{24}(2^3\times3)^3$$
$$= \log_{24}24^3$$
$$> 3\log_{24}24$$
$$= 3$$

AS BEFORE



b) $5\times 2^{t-1} = 2\times 5^{2t}$

$$\Rightarrow 5\times2^t\times2^{-1} = 2\times(5^2)^t$$
$$\Rightarrow 5\times2^t\times\tfrac{1}{2} = 2\times25^t$$
$$\Rightarrow \tfrac{5}{2}\times2^t = 2\times25^t$$
$$\Rightarrow \tfrac{5}{4}\times2^t = 25^t$$
$$\Rightarrow \tfrac{5}{4} = \frac{25^t}{2^t}$$
$$\Rightarrow \tfrac{5}{4} = \left(\tfrac{25}{2}\right)^t$$
$$\Rightarrow 1.25 = 12.5^t$$
$$\Rightarrow (10k)^t = k$$

WHERE $k=1.25$

ALTERNATIVE USING LOGS

$$5\times2^{t-1} = 2\times5^{2t}$$
$$\Rightarrow \log(5\times2^{t-1}) = \log(2\times5^{2t})$$
$$\Rightarrow \log5+\log2^{t-1} = \log2+\log5^{2t}$$
$$\Rightarrow \log5+(t-1)\log2 = \log2+2t\log5$$
$$\Rightarrow \log5+t\log2-\log2 = \log2+2t\log5$$
$$\Rightarrow \log5-\log2-\log2 = -t\log2+2t\log5$$
$$\Rightarrow \log\left(\tfrac{5}{2\times2}\right) = t\left[-\log2+2\log5\right]$$
$$\Rightarrow \log\left(\tfrac{5}{4}\right) = t\left[-\log2+\log25\right]$$
$$\Rightarrow \log\left(\tfrac{5}{4}\right) = t\log\left(\tfrac{25}{2}\right)$$
$$\Rightarrow \log(1.25) = t\log(12.5)$$
$$\Rightarrow \log k = t\log(10k)$$
$$\Rightarrow \log k = \log(10k)^t$$
$$\Rightarrow (10k)^t = k$$

$(k=1.25)$

Question 106 (*****)

On the 1st January 2000 a rare stamp was purchased at an auction for £16384 and by the 1st January 2010 its value was four times as large as its purchase price.

The future value of this stamp, £V, t years after the 1st January 2000 is modelled by the equation

$$V = Ap^t, \ t \geq 0,$$

where A and p are positive constants.

On the 1st January 1990 a different stamp was purchased for £2.

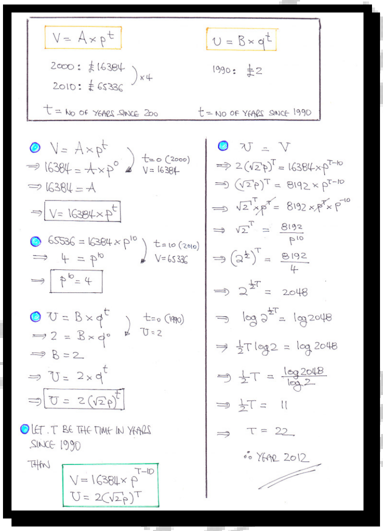
The future value of this stamp, £U, t years after the 1st January 1990 is modelled by the equation

$$U = Bq^t, \ t \geq 0,$$

where B and q are positive constants.

Given further that $q = p\sqrt{2}$, determine the year when the two stamps will achieve the same value according to their modelling equations.

$\boxed{\text{S-V}}$, $\boxed{2012}$



$\boxed{V = A \times p^t}$

2000: £16384
2010: £65336 $\Big) \times 4$

t = NO OF YEARS SINCE 200

$\boxed{U = B \times q^t}$

1990: £2

t = NO OF YEARS SINCE 1990

● $V = A \times p^t$
$\Rightarrow 16384 = A \times p^0$ $t=0$ (2000), $V = 16384$
$\Rightarrow 16384 = A$
$\Rightarrow \boxed{V = 16384 \times p^t}$

● $65536 = 16384 \times p^{10}$ $t=10$ (2010), $V=65336$
$\Rightarrow 4 = p^{10}$
$\Rightarrow \boxed{p^{10} = 4}$

● $U = B \times q^t$ $t=0$ (1990), $U=2$
$\Rightarrow 2 = B \times q^0$
$\Rightarrow B = 2$
$\Rightarrow U = 2 \times q^t$
$\Rightarrow \boxed{U = 2(\sqrt{2}p)^t}$

● LET. T BE THE TIME IN YEARS SINCE 1990

THEN

$$\boxed{\begin{array}{l} V = 16384 \times p^{T-10} \\ U = 2(\sqrt{2}p)^T \end{array}}$$

● $U = V$
$\Rightarrow 2(\sqrt{2}p)^T = 16384 \times p^{T-10}$
$\Rightarrow (\sqrt{2}p)^T = 8192 \times p^{T-10}$
$\Rightarrow \sqrt{2}^T \times \cancel{p^T} = 8192 \times \cancel{p^T} \times p^{-10}$
$\Rightarrow \sqrt{2}^T = \frac{8192}{p^{10}}$
$\Rightarrow (2^{\frac{1}{2}})^T = \frac{8192}{4}$
$\Rightarrow 2^{\frac{1}{2}T} = 2048$
$\Rightarrow \log 2^{\frac{1}{2}T} = \log 2048$
$\Rightarrow \frac{1}{2}T \log 2 = \log 2048$
$\Rightarrow \frac{1}{2}T = \frac{\log 2048}{\log 2}$
$\Rightarrow \frac{1}{2}T = 11$
$\Rightarrow T = 22$

$\therefore$ YEAR 2012

Question 107 (*****)

It is given that

- a and x are positive real numbers such that $x \neq a$, $x \neq 1$ and $a > 1$.
- n is a positive integer such that $n > 1$.

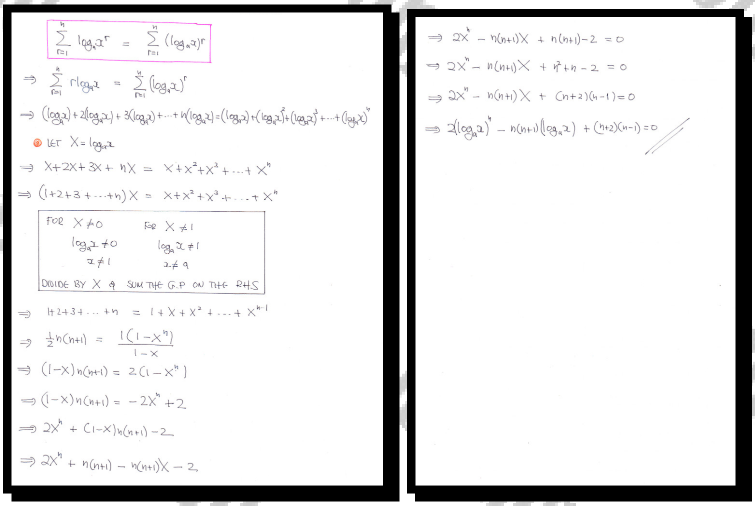
Show that the equation

$$\sum_{r=1}^{n} \log_a x^r = \sum_{r=1}^{n} (\log_a x)^r ,$$

can be written as

$$2(\log_a x)^n - n(n+1)\log_a x + (n-1)(n+2) = 0 .$$

$\boxed{}$, $\boxed{\text{proof}}$



$$\boxed{\sum_{r=1}^{n} \log_a x^r = \sum_{r=1}^{n} (\log_a x)^r}$$

$\Rightarrow \sum_{r=1}^{n} r\log_a x = \sum_{r=1}^{n} (\log_a x)^r$

$\Rightarrow (\log_a x) + 2(\log_a x) + 3(\log_a x) + \dots + n(\log_a x) = (\log_a x) + (\log_a x)^2 + (\log_a x)^3 + \dots + (\log_a x)^n$

● LET $X = \log_a x$

$\Rightarrow X + 2X + 3X + nX = X + X^2 + X^3 + \dots + X^n$

$\Rightarrow (1+2+3+\dots+n)X = X + X^2 + X^3 + \dots + X^n$

FOR $X \neq 0$ FOR $X \neq 1$

$\log_a x \neq 0$ $\log_a x \neq 1$

$x \neq 1$ $x \neq a$

DIVIDE BY X & SUM THE G.P ON THE RHS

$\Rightarrow 1+2+3+\dots+n = 1 + X + X^2 + \dots + X^{n-1}$

$\Rightarrow \frac{1}{2}n(n+1) = \frac{1(1-X^n)}{1-X}$

$\Rightarrow (1-X)n(n+1) = 2(1-X^n)$

$\Rightarrow (1-X)n(n+1) = -2X^n + 2$

$\Rightarrow 2X^n + (1-X)n(n+1) - 2$

$\Rightarrow 2X^n + n(n+1) - n(n+1)X - 2$

$\Rightarrow 2X^n - n(n+1)X + n(n+1) - 2 = 0$

$\Rightarrow 2X^n - n(n+1)X + n^2 + n - 2 = 0$

$\Rightarrow 2X^n - n(n+1)X + (n+2)(n-1) = 0$

$\Rightarrow 2(\log_a x)^n - n(n+1)(\log_a x) + (n+2)(n-1) = 0$ //

Question 108 (*****)

It is given that

$$a^{\log b} \equiv b^{\log a}, \; a > 0, \; b > 0.$$

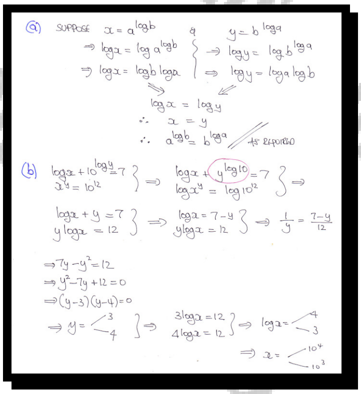
i. Prove the validity of the above result.

ii. Hence, or otherwise, solve the following simultaneous equations

$$\log x + 10^{\log y} = 7,$$

$$x^y = 10^{12}.$$

$\boxed{}$, $\boxed{(10000,3),(1000,4)}$



(a) SUPPOSE $x = a^{\log b}$ & $y = b^{\log a}$

$\Rightarrow \log x = \log a^{\log b}$ $\Rightarrow \log y = \log b^{\log a}$

$\Rightarrow \log x = \log b \log a$ $\Rightarrow \log y = \log a \log b$

$\log x = \log y$

$\therefore \; x = y$

$\therefore \; a^{\log b} = b^{\log a}$ // AS REQUIRED

(b) $\left.\begin{array}{l}\log x + 10^{\log y} = 7\\ x^y = 10^{12}\end{array}\right\} \Rightarrow \left.\begin{array}{l}\log x + \boxed{y^{\log 10}} = 7\\ \log x^y = \log 10^{12}\end{array}\right\} \Rightarrow$

$\left.\begin{array}{l}\log x + y = 7\\ y \log x = 12\end{array}\right\} \Rightarrow \left.\begin{array}{l}\log x = 7 - y\\ y \log x = 12\end{array}\right\} \Rightarrow \frac{1}{y} = \frac{7-y}{12}$

$\Rightarrow 7y - y^2 = 12$

$\Rightarrow y^2 - 7y + 12 = 0$

$\Rightarrow (y-3)(y-4) = 0$

$\Rightarrow y = \begin{cases}3\\4\end{cases} \Rightarrow \left.\begin{array}{l}3\log x = 12\\ 4\log x = 12\end{array}\right\} \Rightarrow \log x = \begin{cases}4\\3\end{cases}$

$\Rightarrow x = \begin{cases}10^4\\10^3\end{cases}$

Question 109 **(*****)**

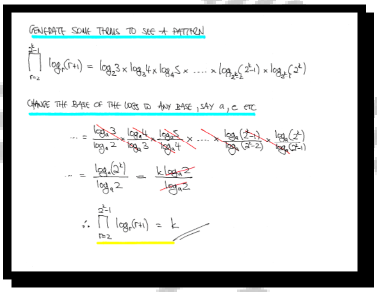
The product operator $\prod$, is defined as

$$\prod_{i=1}^{k}\left[u_i\right] = u_1 \times u_2 \times u_3 \times u_4 \times \ldots \times u_{k-1} \times u_k .$$

Given that $k \in \mathbb{N}$, use a detailed method to find the value of

$$\prod_{r=2}^{2^k-1}\left[\log_r (r+1)\right].$$

$\boxed{}$, $\boxed{k}$



GENERATE SOME TERMS TO SEE A PATTERN

$$\prod_{r=2}^{2^k-1} \log_r(r+1) = \log_2 3 \times \log_3 4 \times \log_4 5 \times \ldots \times \log_{2^k-2}(2^k-1) \times \log_{2^k-1}(2^k)$$

CHANGE THE BASE OF THE LOGS TO ANY BASE, SAY a, e ETC

$$\ldots = \frac{\cancel{\log_a 3}}{\log_a 2} \times \frac{\cancel{\log_a 4}}{\cancel{\log_a 3}} \times \frac{\cancel{\log_a 5}}{\cancel{\log_a 4}} \times \ldots \times \frac{\cancel{\log_a (2^k-1)}}{\cancel{\log_a (2^k-2)}} \times \frac{\log_a (2^k)}{\cancel{\log_a (2^k-1)}}$$

$$\ldots = \frac{\log_a (2^k)}{\log_a 2} = \frac{k\cancel{\log_a 2}}{\cancel{\log_a 2}}$$

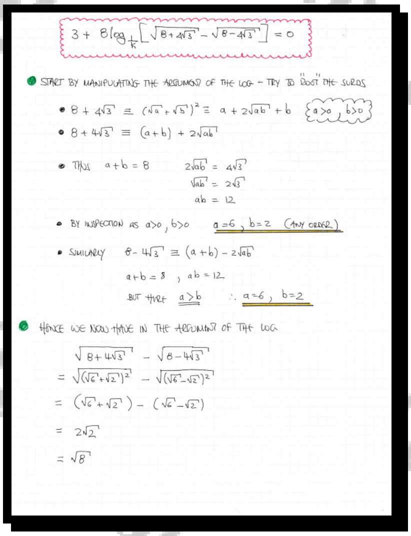
$$\therefore \prod_{r=2}^{2^k-1} \log_r(r+1) = k$$

Question 110 (*****)

Solve the following logarithmic equation.

$$3+8\log_{\frac{1}{k}}\left[\sqrt{8+4\sqrt{3}}-\sqrt{8-4\sqrt{3}}\right]=0\,,\qquad k>0\,,\ \ k\neq 1\,.$$

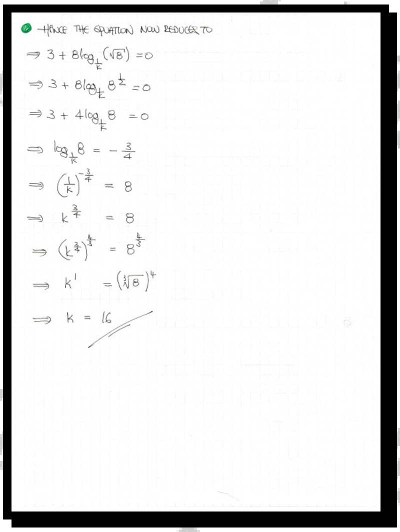
$\boxed{}$, $\boxed{k=16}$



$$\boxed{3+8\log_{\frac{1}{k}}\left[\sqrt{8+4\sqrt{3}}-\sqrt{8-4\sqrt{3}}\right]=0}$$

- START BY MANIPULATING THE ARGUMENT OF THE LOG – TRY TO "ROOT" THE SURDS
  - $8+4\sqrt{3}\equiv(\sqrt{a}+\sqrt{b})^2\equiv a+2\sqrt{ab}+b$ $\{a>0,\ b>0\}$
  - $8+4\sqrt{3}\equiv(a+b)+2\sqrt{ab}$
  - THUS $a+b=8$ $\quad 2\sqrt{ab}=4\sqrt{3}$
    $\sqrt{ab}=2\sqrt{3}$
    $ab=12$
  - BY INSPECTION AS $a>0,\ b>0$ $\quad \underline{a=6,\ b=2}$ (ANY ORDER)
  - SIMILARLY $8-4\sqrt{3}\equiv(a+b)-2\sqrt{ab}$
    $a+b=8\ ,\ ab=12$
    BUT THAT $\underline{\underline{a>b}}$ $\quad\therefore\ \underline{\underline{a=6,\ b=2}}$
- HENCE WE NOW HAVE IN THE ARGUMENT OF THE LOG

$$\begin{aligned}&\sqrt{8+4\sqrt{3}}-\sqrt{8-4\sqrt{3}}\\=&\sqrt{(\sqrt{6}+\sqrt{2})^2}-\sqrt{(\sqrt{6}-\sqrt{2})^2}\\=&(\sqrt{6}+\sqrt{2})-(\sqrt{6}-\sqrt{2})\\=&2\sqrt{2}\\=&\sqrt{8}\end{aligned}$$



- HENCE THE EQUATION NOW REDUCES TO

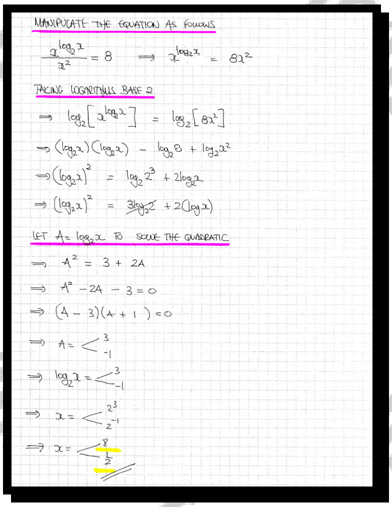
$$\begin{aligned}&\Rightarrow 3+8\log_{\frac{1}{k}}(\sqrt{8})=0\\&\Rightarrow 3+8\log_{\frac{1}{k}}8^{\frac{1}{2}}=0\\&\Rightarrow 3+4\log_{\frac{1}{k}}8=0\\&\Rightarrow \log_{\frac{1}{k}}8=-\frac{3}{4}\\&\Rightarrow \left(\frac{1}{k}\right)^{-\frac{3}{4}}=8\\&\Rightarrow k^{\frac{3}{4}}=8\\&\Rightarrow \left(k^{\frac{3}{4}}\right)^{\frac{4}{3}}=8^{\frac{4}{3}}\\&\Rightarrow k^{1}=\left(\sqrt[3]{8}\right)^{4}\\&\Rightarrow k=16\end{aligned}$$

Question 111 (*****)

Solve the following logarithmic equation.

$$\frac{x^{\log_2 x}}{x^2} = 8, \ x \in (0, \infty)$$

$\boxed{}$, $\boxed{x = \frac{1}{2} \cup x = 8}$



MANIPULATE THE EQUATION AS FOLLOWS

$$\frac{x^{\log_2 x}}{x^2} = 8 \implies x^{\log_2 x} = 8x^2$$

TAKING LOGARITHMS BASE 2

$$\implies \log_2\left[x^{\log_2 x}\right] = \log_2\left[8x^2\right]$$

$$\implies (\log_2 x)(\log_2 x) - \log_2 8 + \log_2 x^2$$

$$\implies (\log_2 x)^2 = \log_2 2^3 + 2\log_2 x$$

$$\implies (\log_2 x)^2 = 3\cancel{\log_2 2} + 2(\log x)$$

LET $A = \log_2 x$ TO SOLVE THE QUADRATIC

$$\implies A^2 = 3 + 2A$$

$$\implies A^2 - 2A - 3 = 0$$

$$\implies (A - 3)(A + 1) = 0$$

$$\implies A = \begin{cases} 3 \\ -1 \end{cases}$$

$$\implies \log_2 x = \begin{cases} 3 \\ -1 \end{cases}$$

$$\implies x = \begin{cases} 2^3 \\ 2^{-1} \end{cases}$$

$$\implies x = \begin{cases} 8 \\ \frac{1}{2} \end{cases}$$

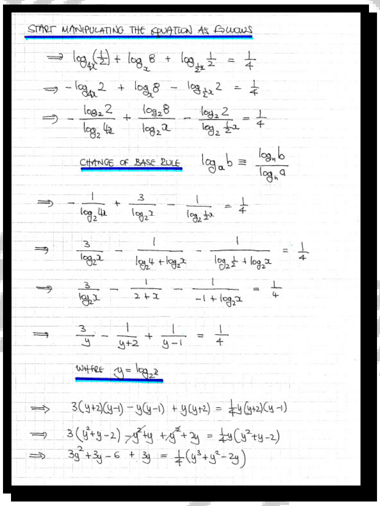
Question 112 **(*****)**

Solve the following logarithmic equation.

$$\log_{4x}\left(\frac{1}{2}\right)+\log_{x}8+\log_{\frac{1}{2}x}\left(\frac{1}{2}\right)=\frac{1}{4}\,,\ x>0\,,\ x\neq 1\,.$$

Give the answers in exact simplified form where appropriate.

$\boxed{}$, $\boxed{x=16,\quad x=2^{\frac{1}{2}\left(7\pm\sqrt{73}\right)}}$



START MANIPULATING THE EQUATION AS FOLLOWS

$\Rightarrow \log_{4x}\left(\frac{1}{2}\right)+\log_{x}8+\log_{\frac{1}{2}x}\frac{1}{2}=\frac{1}{4}$

$\Rightarrow -\log_{4x}2+\log_{x}8-\log_{\frac{1}{2}x}2=\frac{1}{4}$

$\Rightarrow -\frac{\log_2 2}{\log_2 4x}+\frac{\log_2 8}{\log_2 x}-\frac{\log_2 2}{\log_2 \frac{1}{2}x}=\frac{1}{4}$

CHANGE OF BASE RULE $\log_a b \equiv \frac{\log_n b}{\log_n a}$

$\Rightarrow -\frac{1}{\log_2 4x}+\frac{3}{\log_2 x}-\frac{1}{\log_2 \frac{1}{2}x}=\frac{1}{4}$

$\Rightarrow \frac{3}{\log_2 x}-\frac{1}{\log_2 4+\log_2 x}-\frac{1}{\log_2\frac{1}{2}+\log_2 x}=\frac{1}{4}$

$\Rightarrow \frac{3}{\log_2 x}-\frac{1}{2+x}-\frac{1}{-1+\log_2 x}=\frac{1}{4}$

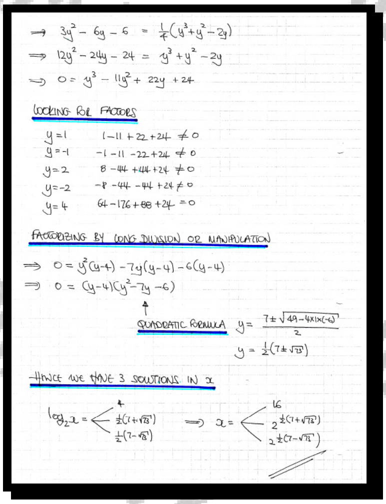
$\Rightarrow \frac{3}{y}-\frac{1}{y+2}+\frac{1}{y-1}=\frac{1}{4}$

WHERE $y=\log_2 x$

$\Rightarrow 3(y+2)(y-1)-y(y-1)+y(y+2)=\frac{1}{4}y(y+2)(y-1)$

$\Rightarrow 3(y^2+y-2)\cancel{-y^2}+y+\cancel{y^2}+2y=\frac{1}{4}y(y^2+y-2)$

$\Rightarrow 3y^2+3y-6+3y=\frac{1}{4}(y^3+y^2-2y)$



$\Rightarrow 3y^2-6y-6=\frac{1}{4}(y^3+y^2-2y)$

$\Rightarrow 12y^2-24y-24=y^3+y^2-2y$

$\Rightarrow 0=y^3-11y^2+22y+24$

LOOKING FOR FACTORS

$y=1 \quad 1-11+22+24\neq 0$

$y=-1 \quad -1-11-22+24\neq 0$

$y=2 \quad 8-44+44+24\neq 0$

$y=-2 \quad -8-44-44+24\neq 0$

$y=4 \quad 64-176+88+24=0$

FACTORIZING BY LONG DIVISION OR MANIPULATION

$\Rightarrow 0=y^2(y-4)-7y(y-4)-6(y-4)$

$\Rightarrow 0=(y-4)(y^2-7y-6)$

$\uparrow$

QUADRATIC FORMULA $y=\frac{7\pm\sqrt{49-4\times1\times(-6)}}{2}$

$y=\frac{1}{2}(7\pm\sqrt{73})$

HENCE WE HAVE 3 SOLUTIONS IN x

$\log_2 x = \begin{cases} 4 \\ \frac{1}{2}(7+\sqrt{73}) \\ \frac{1}{2}(7-\sqrt{73}) \end{cases} \Rightarrow x=\begin{cases} 16 \\ 2^{\frac{1}{2}(7+\sqrt{73})} \\ 2^{\frac{1}{2}(7-\sqrt{73})} \end{cases}$

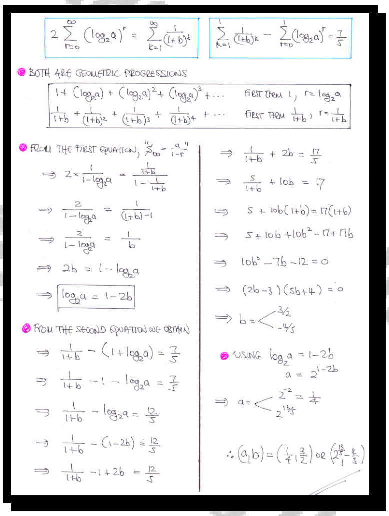
Question 113 (*****)

Solve the following simultaneous equations.

$$2\sum_{r=0}^{\infty}\left[\log_2 a\right]^r = \sum_{k=1}^{\infty}(1+b)^{-k} \quad \text{and} \quad \sum_{k=1}^{1}(1+b)^{-k} - \sum_{r=0}^{1}\left[\log_2 a\right]^r = \frac{7}{5}.$$

You may leave the answers as indices in their simplest form, where appropriate.

$$\boxed{}, \ \boxed{[a,b]=\left[\frac{3}{2},\frac{1}{4}\right]=\left[-\frac{4}{5},2^{\frac{13}{5}}\right]}$$



$$\boxed{2\sum_{r=0}^{\infty}(\log_2 a)^r = \sum_{k=1}^{\infty}\frac{1}{(1+b)^k}} \quad \boxed{\sum_{k=1}^{1}\frac{1}{(1+b)^k} - \sum_{r=0}^{1}(\log_2 a)^r = \frac{7}{5}}$$

- BOTH ARE GEOMETRIC PROGRESSIONS

$$\boxed{\begin{array}{ll} 1+(\log_2 a)+(\log_2 a)^2+(\log_2 a)^3+\dots & \text{FIRST TERM } 1,\ r=\log_2 a \\ \frac{1}{1+b}+\frac{1}{(1+b)^2}+\frac{1}{(1+b)^3}+\frac{1}{(1+b)^4}+\dots & \text{FIRST TERM } \frac{1}{1+b},\ r=\frac{1}{1+b} \end{array}}$$

- FROM THE FIRST EQUATION, "$S_\infty = \frac{a}{1-r}$"

$\Rightarrow 2\times\frac{1}{1-\log_2 a} = \frac{\frac{1}{1+b}}{1-\frac{1}{1+b}}$

$\Rightarrow \frac{2}{1-\log_2 a} = \frac{1}{(1+b)-1}$

$\Rightarrow \frac{2}{1-\log_2 a} = \frac{1}{b}$

$\Rightarrow 2b = 1-\log_2 a$

$\Rightarrow \boxed{\log_2 a = 1-2b}$

- FROM THE SECOND EQUATION WE OBTAIN

$\Rightarrow \frac{1}{1+b} - (1+\log_2 a) = \frac{7}{5}$

$\Rightarrow \frac{1}{1+b} - 1 - \log_2 a = \frac{7}{5}$

$\Rightarrow \frac{1}{1+b} - \log_2 a = \frac{12}{5}$

$\Rightarrow \frac{1}{1+b} - (1-2b) = \frac{12}{5}$

$\Rightarrow \frac{1}{1+b} - 1 + 2b = \frac{12}{5}$

$\Rightarrow \frac{1}{1+b} + 2b = \frac{17}{5}$

$\Rightarrow \frac{5}{1+b} + 10b = 17$

$\Rightarrow 5 + 10b(1+b) = 17(1+b)$

$\Rightarrow 5 + 10b + 10b^2 = 17 + 17b$

$\Rightarrow 10b^2 - 7b - 12 = 0$

$\Rightarrow (2b-3)(5b+4) = 0$

$\Rightarrow b = \begin{cases} 3/2 \\ -4/5 \end{cases}$

- USING $\log_2 a = 1-2b$

$a = 2^{1-2b}$

$\Rightarrow a = \begin{cases} 2^{-2} = \frac{1}{4} \\ 2^{13/5} \end{cases}$

$\therefore (a,b) = \left(\frac{1}{4},\frac{3}{2}\right) \text{ OR } \left(2^{\frac{13}{5}}, -\frac{4}{5}\right)$

Question 114 (*****)

It is given that

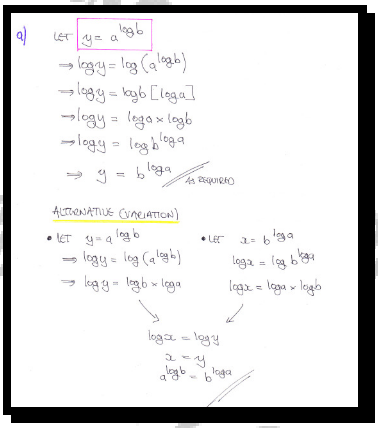
$$a^{\log b} \equiv b^{\log a}, \ a > 0, \ b > 0.$$

a) Prove the validity of the above result.

b) Hence, or otherwise, solve the equation

$$3^{\log x} + 3 \times x^{\log 3} = 36.$$

$\boxed{}$, $\boxed{x = 100}$



a) LET $\boxed{y = a^{\log b}}$

$\Rightarrow \log y = \log \left(a^{\log b}\right)$

$\Rightarrow \log y = \log b \left[\log a\right]$

$\Rightarrow \log y = \log a \times \log b$

$\Rightarrow \log y = \log b^{\log a}$

$\Rightarrow y = b^{\log a}$ // AS REQUIRED

ALTERNATIVE (VARIATION)

- LET $y = a^{\log b}$

  $\Rightarrow \log y = \log \left(a^{\log b}\right)$

  $\Rightarrow \log y = \log b \times \log a$

- LET $x = b^{\log a}$

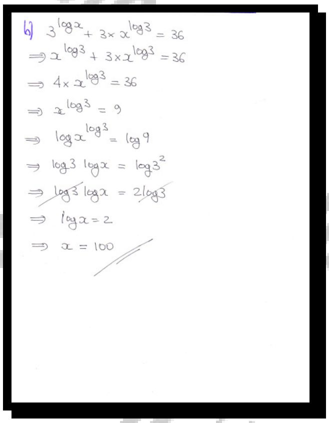
  $\log x = \log b^{\log a}$

  $\log x = \log a \times \log b$

$\log x = \log y$

$x = y$

$a^{\log b} = b^{\log a}$ //



b) $3^{\log x} + 3 \times x^{\log 3} = 36$

$\Rightarrow x^{\log 3} + 3 \times x^{\log 3} = 36$

$\Rightarrow 4 \times x^{\log 3} = 36$

$\Rightarrow x^{\log 3} = 9$

$\Rightarrow \log x^{\log 3} = \log 9$

$\Rightarrow \log 3 \log x = \log 3^2$

$\Rightarrow \cancel{\log 3} \log x = 2 \cancel{\log 3}$

$\Rightarrow \log x = 2$

$\Rightarrow x = 100$ //

Question 115 **(*****)**

It is given that

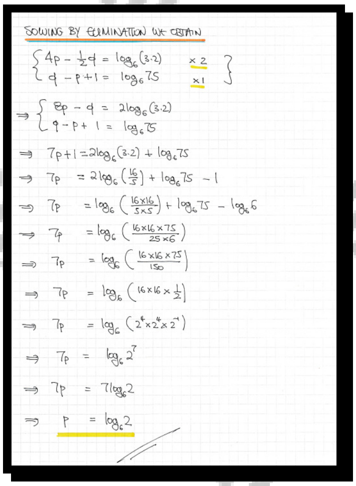
$$4p-\frac{1}{2}q=\log_6(3.6) \qquad \text{and} \qquad q-p+1=\log_6(75).$$

Solve these simultaneous equations, to show that

$$p=\log_6 k,$$

where k is a positive integer to be found.

$\boxed{\quad\quad}$, $\boxed{k=2}$



SOLVING BY ELIMINATION WE OBTAIN

$$\begin{cases} 4p-\frac{1}{2}q=\log_6(3.2) & \times 2 \\ q-p+1=\log_6 75 & \times 1 \end{cases}$$

$$\Rightarrow \begin{cases} 8p-q=2\log_6(3.2) \\ q-p+1=\log_6 75 \end{cases}$$

$$\Rightarrow 7p+1=2\log_6(3.2)+\log_6 75$$

$$\Rightarrow 7p=2\log_6\left(\frac{16}{5}\right)+\log_6 75-1$$

$$\Rightarrow 7p=\log_6\left(\frac{16\times16}{5\times5}\right)+\log_6 75-\log_6 6$$

$$\Rightarrow 7p=\log_6\left(\frac{16\times16\times75}{25\times6}\right)$$

$$\Rightarrow 7p=\log_6\left(\frac{16\times16\times75}{150}\right)$$

$$\Rightarrow 7p=\log_6\left(16\times16\times\frac{1}{2}\right)$$

$$\Rightarrow 7p=\log_6\left(2^4\times2^4\times2^{-1}\right)$$

$$\Rightarrow 7p=\log_6 2^7$$

$$\Rightarrow 7p=7\log_6 2$$

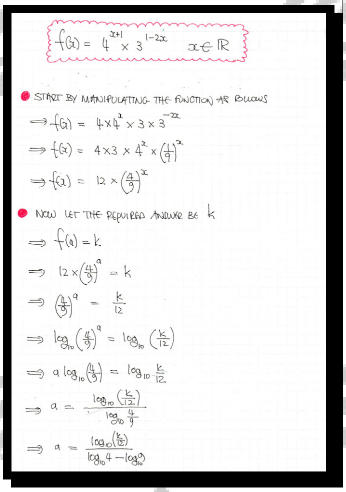
$$\Rightarrow p=\log_6 2$$

Question 116 (*****)

$$f(x) \equiv 4^{x+1} \times 3^{1-2x}, \; x \in \mathbb{R}.$$

Determine the value of $f(a)$, where $a = \dfrac{\log_{10} 2}{\log_{10} 4 - \log_{10} 9}$.

$\boxed{}$, $\boxed{f(a) = 24}$



$$f(x) = 4^{x+1} \times 3^{1-2x} \qquad x \in \mathbb{R}$$

- START BY MANIPULATING THE FUNCTION AS FOLLOWS

$\Rightarrow f(x) = 4 \times 4^{x} \times 3 \times 3^{-2x}$

$\Rightarrow f(x) = 4 \times 3 \times 4^{x} \times \left(\frac{1}{9}\right)^{x}$

$\Rightarrow f(x) = 12 \times \left(\frac{4}{9}\right)^{x}$

- NOW LET THE REQUIRED ANSWER BE k

$\Rightarrow f(a) = k$

$\Rightarrow 12 \times \left(\frac{4}{9}\right)^{a} = k$

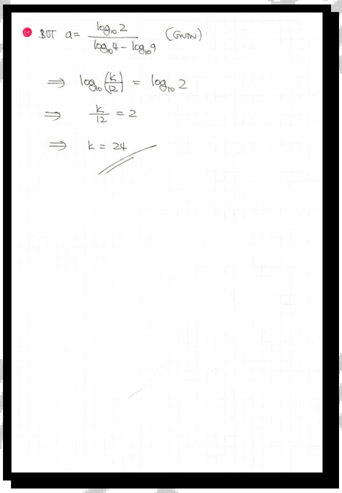
$\Rightarrow \left(\frac{4}{9}\right)^{a} = \frac{k}{12}$

$\Rightarrow \log_{10}\left(\frac{4}{9}\right)^{a} = \log_{10}\left(\frac{k}{12}\right)$

$\Rightarrow a \log_{10}\left(\frac{4}{9}\right) = \log_{10} \frac{k}{12}$

$\Rightarrow a = \dfrac{\log_{10}\left(\frac{k}{12}\right)}{\log_{10} \frac{4}{9}}$

$\Rightarrow a = \dfrac{\log_{10}\left(\frac{k}{12}\right)}{\log_{10} 4 - \log_{10} 9}$



- BUT $a = \dfrac{\log_{10} 2}{\log_{10} 4 - \log_{10} 9}$ (GIVEN)

$\Rightarrow \log_{10}\left(\frac{k}{12}\right) = \log_{10} 2$

$\Rightarrow \frac{k}{12} = 2$

$\Rightarrow k = 24$

Question 117 (*****)

The positive solution of the quadratic equation $x^2 - x - 1 = 0$ is denoted by ϕ, and is commonly known as the golden section or golden number.

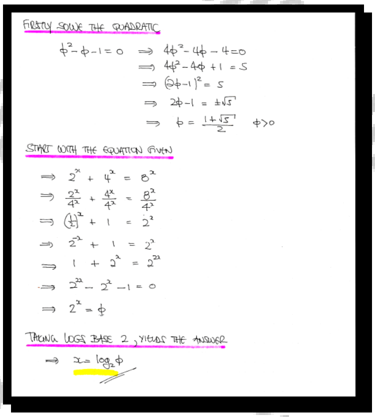
Show, with a detailed method, that the real solution of the following exponential equation

$$2^x + 4^x = 8^x,$$

can be written in exact form as

$$\log_2 \phi .$$

$\boxed{}$, $\boxed{\text{proof}}$



FIRSTLY SOLVE THE QUADRATIC

$$\phi^2 - \phi - 1 = 0 \implies 4\phi^2 - 4\phi - 4 = 0$$
$$\implies 4\phi^2 - 4\phi + 1 = 5$$
$$\implies (2\phi - 1)^2 = 5$$
$$\implies 2\phi - 1 = \pm\sqrt{5}$$
$$\implies \phi = \frac{1+\sqrt{5}}{2} \qquad \phi > 0$$

START WITH THE EQUATION GIVEN

$$\implies 2^x + 4^x = 8^x$$
$$\implies \frac{2^x}{4^x} + \frac{4^x}{4^x} = \frac{8^x}{4^x}$$
$$\implies \left(\tfrac{1}{2}\right)^x + 1 = 2^x$$
$$\implies 2^{-x} + 1 = 2^x$$
$$\implies 1 + 2^x = 2^{2x}$$
$$\implies 2^{2x} - 2^x - 1 = 0$$
$$\implies 2^x = \phi$$

TAKING LOGS BASE 2, YIELDS THE ANSWER

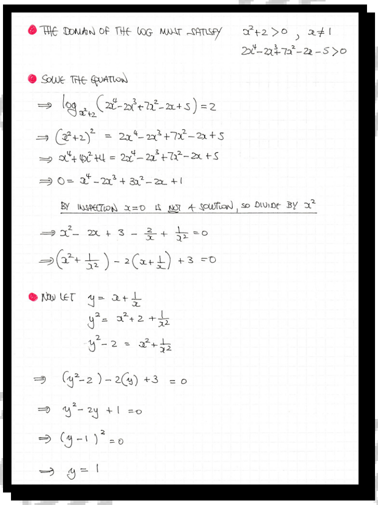
$$\implies x = \log_2 \phi$$

Question 118 **(*****)**

Show that the following logarithmic equation has no real solutions.

$$\log_{x^2+2}\left[2x^4-2x^3+7x^2-2x+5\right]=2\,,\ x\in\mathbb{R}\,.$$

$\boxed{}$, $\boxed{\text{proof}}$



THE DOMAIN OF THE LOG MUST SATISFY $x^2+2>0$, $x \neq 1$

$2x^4-2x^3+7x^2-2x-5>0$

SOLVE THE EQUATION

$\Rightarrow \log_{x^2+2}\left(2x^4-2x^3+7x^2-2x+5\right)=2$

$\Rightarrow (x^2+2)^2 = 2x^4-2x^3+7x^2-2x+5$

$\Rightarrow x^4+4x^2+4 = 2x^4-2x^3+7x^2-2x+5$

$\Rightarrow 0 = x^4-2x^3+3x^2-2x+1$

BY INSPECTION $x=0$ IS NOT A SOLUTION, SO DIVIDE BY x^2

$\Rightarrow x^2-2x+3-\frac{2}{x}+\frac{1}{x^2}=0$

$\Rightarrow \left(x^2+\frac{1}{x^2}\right)-2\left(x+\frac{1}{x}\right)+3=0$

NOW LET $y=x+\frac{1}{x}$

$y^2=x^2+2+\frac{1}{x^2}$

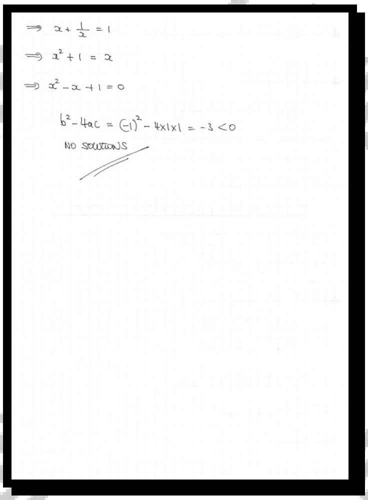
$y^2-2=x^2+\frac{1}{x^2}$

$\Rightarrow (y^2-2)-2(y)+3=0$

$\Rightarrow y^2-2y+1=0$

$\Rightarrow (y-1)^2=0$

$\Rightarrow y=1$



$\Rightarrow x+\frac{1}{x}=1$

$\Rightarrow x^2+1=x$

$\Rightarrow x^2-x+1=0$

$b^2-4ac=(-1)^2-4\times1\times1=-3<0$

NO SOLUTIONS

Question 119 **(*****)**

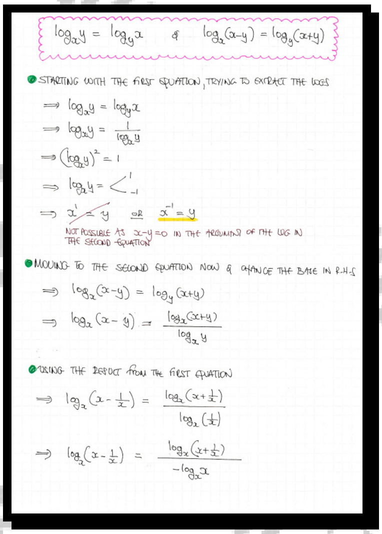
It is given that for $x > 0$, $x \neq 1$ and $y > 0$, $y \neq 1$

$$\log_x y = \log_y x \qquad \text{and} \qquad \log_x (x - y) = \log_y (x + y).$$

Show that

$$x^4 - x^2 - 1 = 0.$$

$\boxed{}$, $\boxed{\text{proof}}$



$$\boxed{\log_x y = \log_y x \quad \& \quad \log_x(x-y) = \log_y(x+y)}$$

- STARTING WITH THE FIRST EQUATION, TRYING TO EXTRACT THE LOGS

$\Rightarrow \log_x y = \log_y x$

$\Rightarrow \log_x y = \frac{1}{\log_x y}$

$\Rightarrow (\log_x y)^2 = 1$

$\Rightarrow \log_x y = \begin{cases} 1 \\ -1 \end{cases}$

$\Rightarrow \cancel{x^1 = y}$ OR $x^{-1} = y$

NOT POSSIBLE AS $x-y=0$ IN THE ARGUMENT OF THE LOG IN THE SECOND EQUATION

- MOVING TO THE SECOND EQUATION NOW & CHANGE THE BASE IN R.H.S

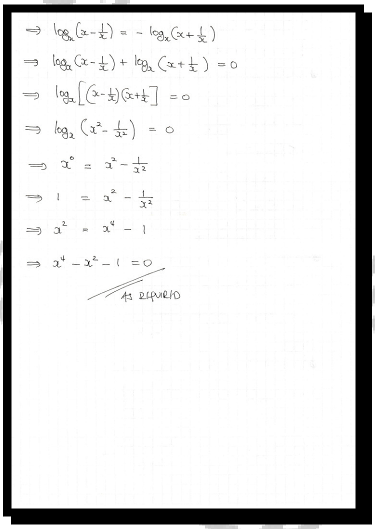
$\Rightarrow \log_x(x-y) = \log_y(x+y)$

$\Rightarrow \log_x(x-y) = \frac{\log_x(x+y)}{\log_x y}$

- USING THE RESULT FROM THE FIRST EQUATION

$\Rightarrow \log_x\left(x-\frac{1}{x}\right) = \frac{\log_x\left(x+\frac{1}{x}\right)}{\log_x\left(\frac{1}{x}\right)}$

$\Rightarrow \log_x\left(x-\frac{1}{x}\right) = \frac{\log_x\left(x+\frac{1}{x}\right)}{-\log_x x}$



$\Rightarrow \log_x\left(x-\frac{1}{x}\right) = -\log_x\left(x+\frac{1}{x}\right)$

$\Rightarrow \log_x\left(x-\frac{1}{x}\right) + \log_x\left(x+\frac{1}{x}\right) = 0$

$\Rightarrow \log_x\left[\left(x-\frac{1}{x}\right)\left(x+\frac{1}{x}\right)\right] = 0$

$\Rightarrow \log_x\left(x^2-\frac{1}{x^2}\right) = 0$

$\Rightarrow x^0 = x^2 - \frac{1}{x^2}$

$\Rightarrow 1 = x^2 - \frac{1}{x^2}$

$\Rightarrow x^2 = x^4 - 1$

$\Rightarrow x^4 - x^2 - 1 = 0$

AS REQUIRED

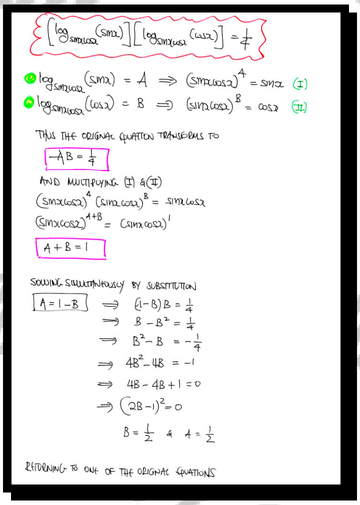
Question 120 (*****)

$$\log_{\sin x\cos x}\left(\sin x\right)\times\log_{\sin x\cos x}\left(\cos x\right)=\frac{1}{4}.$$

Show that the solution of the above equation is given by

$$x=\frac{1}{4}\pi\left(8n-7\right),\ n\in\mathbb{N}.$$

☐ , proof



$$\boxed{\left[\log_{\sin x\cos x}(\sin x)\right]\left[\log_{\sin x\cos x}(\cos x)\right]=\frac{1}{4}}$$

● $\log_{\sin x\cos x}(\sin x) = A \Rightarrow (\sin x\cos x)^{A} = \sin x$ (I)

● $\log_{\sin x\cos x}(\cos x) = B \Rightarrow (\sin x\cos x)^{B} = \cos x$ (II)

THUS THE ORIGINAL EQUATION TRANSFORMS TO

$$\boxed{AB=\frac{1}{4}}$$

AND MULTIPLYING (I) & (II)

$$(\sin x\cos x)^{A}(\sin x\cos x)^{B} = \sin x\cos x$$

$$(\sin x\cos x)^{A+B} = (\sin x\cos x)^{1}$$

$$\boxed{A+B=1}$$

SOLVING SIMULTANEOUSLY BY SUBSTITUTION

$$\boxed{A=1-B} \Rightarrow (1-B)B=\frac{1}{4}$$

$$\Rightarrow B-B^{2}=\frac{1}{4}$$

$$\Rightarrow B^{2}-B=-\frac{1}{4}$$

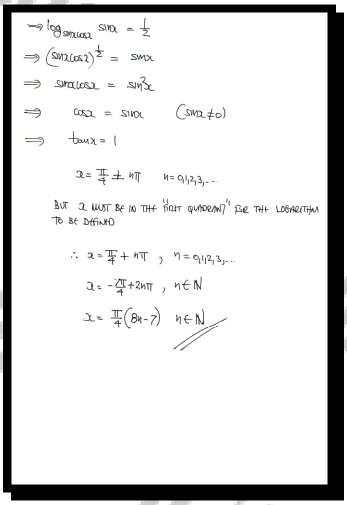
$$\Rightarrow 4B^{2}-4B=-1$$

$$\Rightarrow 4B-4B+1=0$$

$$\Rightarrow (2B-1)^{2}=0$$

$$B=\frac{1}{2}\ \&\ A=\frac{1}{2}$$

RETURNING TO ONE OF THE ORIGINAL EQUATIONS



$$\Rightarrow \log_{\sin x\cos x}\sin x=\frac{1}{2}$$

$$\Rightarrow (\sin x\cos x)^{\frac{1}{2}}=\sin x$$

$$\Rightarrow \sin x\cos x=\sin^{2}x$$

$$\Rightarrow \cos x=\sin x \quad (\sin x\neq 0)$$

$$\Rightarrow \tan x=1$$

$$x=\frac{\pi}{4}\pm n\pi \quad n=0,1,2,3,\ldots$$

BUT x MUST BE IN THE "FIRST QUADRANT" FOR THE LOGARITHM TO BE DEFINED

$$\therefore x=\frac{\pi}{4}+n\pi,\ n=0,1,2,3,\ldots$$

$$x=-\frac{7\pi}{4}+2n\pi,\ n\in\mathbb{N}$$

$$x=\frac{\pi}{4}(8n-7)\ \ n\in\mathbb{N}$$