MISCELLANEOUS SEQUENCES & SERIES QUESTIONS

Question 1 (***+)

Show that

$$\sum_{r=1}^{12}\left[2r + 7 + 2^r\right] = 8430.$$

Detailed workings must be shown in this question.

$\boxed{}$, $\boxed{\text{proof}}$

$$\boxed{\sum_{r=1}^{12}(2r+7+2^r) = 8430}$$

"USUALLY" WE GENERATE A FEW TERMS TO SEE A PATTERN

$$2\times1 + 7 + 2^1 = 11$$
$$2\times2 + 7 + 2^2 = 15$$
$$2\times3 + 7 + 2^3 = 21$$
$$2\times4 + 7 + 2^4 = 31$$
$$2\times5 + 7 + 2^5 = 49$$

NO OBVIOUS PATTERN !?

SPLIT THE SUM

$$\sum_{r=1}^{12}(2r+7) + \sum_{r=1}^{12}2^r$$

$$= \underbrace{(9+11+13+\dots+31)}_{\text{THIS IS AN A.P}} + \underbrace{(2+4+8+\dots+4096)}_{\text{G.P}}$$

A.P: $a=9$, $d=2$, $L=31$, $n=12$

G.P: $a=2$, $r=2$, $n=12$

USING $S_n = \frac{n}{2}[a+L]$ & $S_n = \frac{a(1-r^n)}{1-r}$

$$= \frac{12}{2}[9+31] + \frac{2(1-2^{12})}{1-2}$$

$$= 240 + 8190$$

$$= 8430$$

Question 2 (***+)

Evaluate the following sum

$$\sum_{r=13}^{30}\left[(-2)^r - 4r - 78\right].$$

Detailed workings must support the answer.

$\boxed{}$, $\boxed{715,822,200}$

MANIPULATE AS FOLLOWS

$$\sum_{r=13}^{30}\left[(-2)^r - 4r - 78\right] = \sum_{r=1}^{30}\left[(-2)^r - 4r - 78\right] - \sum_{r=1}^{12}\left[(-2)^r - 4r - 78\right]$$

SPLIT EACH SUMMATION INTO ARITHMETIC & GEOMETRIC PARTS

$$= \left[\sum_{r=1}^{30}(-2)^r - \sum_{r=1}^{30}(4r+78)\right] - \left[\sum_{r=1}^{12}(-2)^r - \sum_{r=1}^{12}(4r+78)\right]$$

G.P. $-2+4-8+16-\dots$ $\quad S_n = \dfrac{a(r^n-1)}{r-1}$

A.P. $82+86+90+\dots$ $\quad S_n = \dfrac{n}{2}\left[2a+(n-1)d\right]$

$$= \left[\frac{-2\left[(-2)^{30}-1\right]}{-2-1} - \frac{30}{2}\left[164+29\times4\right]\right] - \left[\frac{-2\left[(-2)^{12}-1\right]}{-2-1} - \frac{12}{2}\left[164+11\times4\right]\right]$$

$$= (715\,827\,882 - 4200) - (2730 - 1248)$$

$$= 715,822,200$$

Question 3 (***+)

Three numbers, A, B, C in that order, are in geometric progression with common ratio r.

Given further that A, $2B$, C in that order are in arithmetic progression, determine the possible values of r.

$\boxed{}$, $\boxed{r = 2 \pm \sqrt{3}}$

WITHOUT LOSS OF GENERALITY LET THE A, B & C BE

$A = A \neq 0$
$B = Ar$
$C = Ar^2$ (AS THEY ARE IN GEOMETRIC PROGRESSION)

NOW $A, 2B$ & C ARE IN ARITHMETIC PROGRESSION

$\Rightarrow 2B - A = C - 2B$
$\Rightarrow 4B = A + C$
$\Rightarrow 4Ar = A + Ar^2$
$\Rightarrow 4r = 1 + r^2$ $(A \neq 0)$
$\Rightarrow r^2 - 4r + 1 = 0$
$\Rightarrow (r-2)^2 - 4 + 1 = 0$
$\Rightarrow (r-2)^2 = 3$
$\Rightarrow r - 2 = \pm\sqrt{3}$
$\Rightarrow r = 2 + \sqrt{3}$ or $2 - \sqrt{3}$

Question 4 (***+)

An arithmetic series has common difference 2.

The 3^{rd}, 6^{th} and 10^{th} terms of the arithmetic series are the respective first three terms of a geometric series.

Determine in any order the first term of the arithmetic series and the common ratio of the geometric series.

$\boxed{}$, $\boxed{a = 14}$, $\boxed{r = \frac{4}{3}}$

THE n^{th} TERM OF AN ARITHMETIC SERIES OF COMMON DIFFERENCE 2 IS GIVEN BY

$u_n = a + (n-1) \times 2$
$u_n = a + 2(n-1)$

HENCE WE NOW HAVE

u_3	u_6	u_{10}
$a+4$	$a+10$	$a+18$

AS THESE ARE IN GEOMETRIC PROGRESSION

$\frac{a+10}{a+4} = \frac{a+18}{a+10} \longrightarrow (a+10)^2 = (a+4)(a+18)$
$\Rightarrow \cancel{a^2} + 20a + 100 = \cancel{a^2} + 22a + 72$
$\Rightarrow 28 = 2a$
$\Rightarrow a = 14$

SO THE COMMON RATIO IS

$r = \frac{a+10}{a+4} = \frac{24}{18} = \frac{4}{3}$

Question 5 (***+)

Each of the terms of an arithmetic series is added to the corresponding terms of a geometric series, forming a new series with first term $\frac{3}{8}$ and second term $\frac{13}{16}$.

The common difference of the arithmetic series is four times as large as the first term of the geometric series. The common ratio of the geometric series is twice as large as the first term of the arithmetic series.

Determine the possible values of the first term of the geometric series.

$\boxed{}$, $\boxed{\frac{1}{8} \cup \frac{7}{4}}$

- START FORMING SOME EQUATIONS – LET a = 1ST TERM OF A.P.
  b = 1ST TERM OF G.P
  d = COMMON DIFFERENCE
  r = COMMON RATIO

- AP : $a + (a+d)$
  G.P : $b + br$
  NEW SERIES : $[a+b] + [a+d+br]$

- (I) $a+b = \frac{3}{8}$
  (II) $a+d+br = \frac{13}{16}$
  (III) $r = 2a$
  (IV) $d = 4b$

  SUBSTITUTE (III) & (IV) INTO (II) GIVES
  $a + 4b + 2ab = \frac{13}{16}$

- $\left\{\begin{array}{l} a+b = \frac{3}{8} \\ a+4b+2ab = \frac{13}{16} \end{array}\right\} \Rightarrow a = \frac{3}{8} - b$

$$\Rightarrow \frac{3}{8} - b + 4b + 2b\left(\frac{3}{8} - b\right) = \frac{13}{16}$$
$$\Rightarrow \frac{3}{8} + 3b + \frac{3}{4}b - 2b^2 = \frac{13}{16} \quad \times 16$$
$$\Rightarrow 6 + 48b + 12b - 32b^2 = 13$$
$$\Rightarrow 0 = 32b^2 - 60b + 7$$
$$\Rightarrow (8b-1)(4b-7) = 0$$
$$\Rightarrow b = \begin{cases} 1/8 \\ 7/4 \end{cases}$$

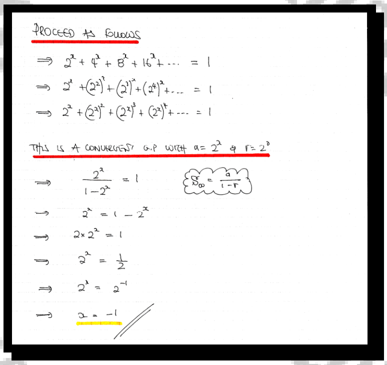
Question 6 **(****)**

Solve the following equation

$$2^x + 4^x + 8^x + 16^x + 32^x + \ldots = 1.$$

You may assume that the left hand side of the equation converges.

$\boxed{}$, $\boxed{x = -1}$



PROCEED AS FOLLOWS

$$\Rightarrow 2^x + 4^x + 8^x + 16^x + \ldots = 1$$

$$\Rightarrow 2^x + (2^2)^x + (2^3)^x + (2^4)^x + \ldots = 1$$

$$\Rightarrow 2^x + (2^x)^2 + (2^x)^3 + (2^x)^4 + \ldots = 1$$

THIS IS A CONVERGENT G.P WITH $a = 2^x$ & $r = 2^x$

$$\Rightarrow \frac{2^x}{1 - 2^x} = 1 \qquad S_\infty = \frac{a}{1-r}$$

$$\Rightarrow 2^x = 1 - 2^x$$

$$\Rightarrow 2 \times 2^x = 1$$

$$\Rightarrow 2^x = \frac{1}{2}$$

$$\Rightarrow 2^x = 2^{-1}$$

$$\Rightarrow x = -1$$

Question 7 **(****)**

Solve the following equation

$$x - 2x^2 + x^3 - 2x^4 + x^5 - 2x^6 + \ldots = -\frac{2}{5}.$$

You may assume that the left hand side of the equation converges.

$\boxed{}$, $\boxed{x = \frac{2}{3} \cup x = -\frac{1}{4}}$

START BY REGROUPING THE TERMS

$$\Rightarrow x - 2x^2 + x^3 - 2x^4 + x^5 - 2x^6 + \ldots = -\frac{2}{5}$$

$$\Rightarrow (x + x^3 + x^5 + \ldots) - 2(x^2 + x^4 + x^6 + \ldots) = -\frac{2}{5}$$

G.P WITH $a = x$, $r = x^2$: $S_\infty = \frac{x}{1-x^2}$

G.P. WITH $a = x^2$, $r = x^2$: $S_\infty = \frac{x^2}{1-x^2}$

HENCE WE NOW HAVE

$$\Rightarrow \frac{x}{1-x^2} - 2\left(\frac{x^2}{1-x^2}\right) = -\frac{2}{5}$$

$$\Rightarrow \frac{x}{1-x^2} - \frac{2x^2}{1-x^2} = -\frac{2}{5}$$

$$\Rightarrow \frac{x - 2x^2}{1-x^2} = \frac{-2}{5}$$

$$\Rightarrow 5x - 10x^2 = -2 + 2x^2$$

$$\Rightarrow 0 = 12x^2 - 5x - 2$$

$$\Rightarrow 0 = (4x+1)(3x-2)$$

$$\Rightarrow x = \begin{cases} \frac{2}{3} \\ -\frac{1}{4} \end{cases}$$

Question 8 (****)

Find in simplified form, in terms of n, the value of

$$\sum_{r=1}^{2n}\left[(3r-2)(-1)^r\right].$$

$\boxed{}$, $\boxed{3n}$

- $\sum_{r=1}^{2n}\left[(-1)^r(3r-2)\right] = -1+4-7+10-13+16-\dots-(3(2n-1)-2)+(6n-2)$

  $= -1+4-7+10-13+16-\dots-(6n-5)+(6n-2)$

- NOW PROCEED BY GROUPING

  $= \underbrace{(-1+4)+(-7+10)+(-13+16)+\dots+(-6n+5+6n-2)}_{n \text{ terms}}$

  $= 3n$

ALTERNATIVE/REGROUP AS FOLLOWS

$= \underbrace{\left[4+10+16+\dots+(6n-2)\right]}_{\substack{n \text{ terms} \\ a=4 \\ d=3 \\ L=6n-2}} - \underbrace{\left[1+7+13+\dots+6n-5\right]}_{\substack{n \text{ terms} \\ a=1 \\ d=3 \\ L=6n-5}}$

$= \frac{n}{2}\left[4+6n-2\right] - \frac{n}{2}\left[1+6n-5\right]$

$= \frac{n}{2}(6n+2) - \frac{n}{2}(6n-4)$

$= \cancel{3n^2}+n-\cancel{3n^2}+2n$

$= 3n$

AS BEFORE

Question 9 (****)

The 1^{st}, 3^{rd} and 11^{th} term of an arithmetic progression are the first three terms of a geometric progression.

It is further given that the sum of the first 13 terms of the arithmetic progression is 260.

Find, in any order, the common ratio of the geometric progression and the first term and common difference of the arithmetic progression.

$\boxed{}$, $\boxed{r=4}$

STARTING THE MODELLING WITH A DIAGRAM

A.P. $u_1 \quad u_3 \quad u_{11}$

$$a \xrightarrow{\times r} a+2d \xrightarrow{\times r} a+10d$$

FROM THE ABOVE WE HAVE

$$\left.\begin{aligned} ar &= a+2d \\ (a+2d)r &= a+10d \end{aligned}\right\}$$

DIVIDING THE EQUATIONS "UPWARDS"

$$\Rightarrow \frac{a+2d}{a} = \frac{a+10d}{a+2d}$$
$$\Rightarrow (a+2d)^2 = a(a+10d)$$
$$\Rightarrow \cancel{a^2} + 4ad + 4d^2 = \cancel{a^2} + 10ad$$
$$\Rightarrow 4d^2 = 6ad$$
$$\Rightarrow 2d^2 = 3ad$$
$$\Rightarrow 2d = 3a \qquad d \neq 0$$

NOW WE MAKE USE OF $S_{13} = 260$ FOR THE ARITHMETIC SERIES

$$\Rightarrow \frac{13}{2}[2a+12d] = 260$$
$$\Rightarrow 13[a+6d] = 260$$
$$\Rightarrow a+6d = 20$$

SOLVING SIMULTANEOUSLY

$$\left.\begin{aligned} 2d &= 3a \\ a+6d &= 20 \end{aligned}\right\} \Rightarrow \left.\begin{aligned} 6d &= 9a \\ a+6d &= 20 \end{aligned}\right\}$$

$$\Rightarrow a = 20 - 9a$$
$$\Rightarrow 10a = 20$$
$$\Rightarrow a = 2$$
$$\Rightarrow d = 3$$

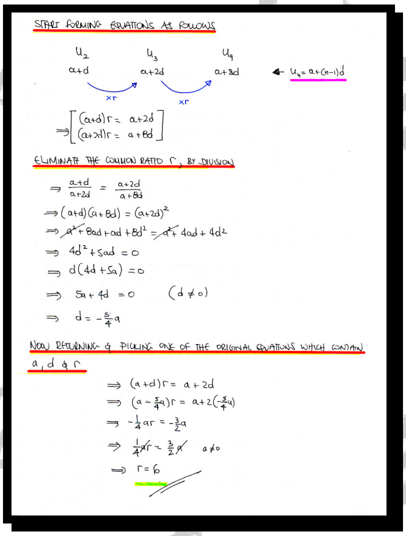
$$\Rightarrow ar = a + 2d$$
$$\Rightarrow 2r = 2 + 6$$
$$\Rightarrow r = 4$$

Question 10 (****)

The 2^{nd}, 3^{rd} and 9^{th} term of an arithmetic progression are three consecutive terms of a geometric progression.

Find the common ratio of the geometric progression.

$\boxed{r=6}$



START FORMING EQUATIONS AS FOLLOWS

$$\begin{array}{ccc} u_2 & u_3 & u_9 \\ a+d & a+2d & a+8d \end{array} \quad \leftarrow u_n = a+(n-1)d$$

$\times r$ $\quad$ $\times r$

$$\Rightarrow \left[\begin{array}{l} (a+d)r = a+2d \\ (a+2d)r = a+8d \end{array}\right]$$

ELIMINATE THE COMMON RATIO r, BY DIVISION

$$\Rightarrow \frac{a+d}{a+2d} = \frac{a+2d}{a+8d}$$

$$\Rightarrow (a+d)(a+8d) = (a+2d)^2$$

$$\Rightarrow \cancel{a^2} + 8ad + ad + 8d^2 = \cancel{a^2} + 4ad + 4d^2$$

$$\Rightarrow 4d^2 + 5ad = 0$$

$$\Rightarrow d(4d+5a) = 0$$

$$\Rightarrow 5a + 4d = 0 \qquad (d \neq 0)$$

$$\Rightarrow d = -\frac{5}{4}a$$

NOW RETURNING & PICKING ONE OF THE ORIGINAL EQUATIONS WHICH CONTAIN a, d & r

$$\Rightarrow (a+d)r = a+2d$$

$$\Rightarrow \left(a - \frac{5}{4}a\right)r = a + 2\left(-\frac{5}{4}a\right)$$

$$\Rightarrow -\frac{1}{4}ar = -\frac{3}{2}a$$

$$\Rightarrow \frac{1}{4}\cancel{a}r = \frac{3}{2}\cancel{a} \qquad a \neq 0$$

$$\Rightarrow r = 6$$

Question 11 (****+)

It is given that

$$\frac{1}{n}\sum_{r=1}^{n} x_r = 2 \qquad \text{and} \qquad \sqrt{\frac{1}{n}\sum_{r=1}^{n}(x_r)^2 - \frac{1}{n^2}\left(\sum_{r=1}^{n} x_r\right)^2} = 3 .$$

Determine, in terms of n, the value of

$$\sum_{r=1}^{n}(x_r+1)^2 .$$

$$\boxed{} , \quad \boxed{\sum_{r=1}^{n}(x_r+1)^2 = 18n}$$

REWRITING THE FIRST EQUATION

$$\frac{1}{n}\sum_{r=1}^{n} x_r = 2 \implies \boxed{\sum_{r=1}^{n} x_r = 2n}$$

NOW PROCEED AS FOLLOWS

$$\implies \sqrt{\frac{1}{n}\sum_{r=1}^{n}(x_r)^2 - \frac{1}{n^2}\left[\sum_{r=1}^{n} x_r\right]^2} = 3$$

$$\implies \frac{1}{n}\sum_{r=1}^{n}(x_r)^2 - \frac{1}{n^2}\left[\sum_{r=1}^{n} x_r\right]^2 = 9$$

$$\implies \frac{1}{n}\sum_{r=1}^{n}(x_r)^2 - \frac{1}{n^2}(2n)^2 = 9$$

$$\implies \frac{1}{n}\sum_{r=1}^{n}(x_r)^2 - 4 = 9$$

$$\implies \frac{1}{n}\sum_{r=1}^{n}(x_r)^2 = 13$$

$$\implies \boxed{\sum_{r=1}^{n}(x_r)^2 = 13n}$$

FINALLY WE HAVE

$$\implies \sum_{r=1}^{n}(x_r+1)^2 = \sum_{r=1}^{n}\left[(x_r)^2 + 2x_r + 1\right]$$

$$= \sum_{r=1}^{n}(x_r)^2 + 2\sum_{r=1}^{n}(x_r) + \sum_{r=1}^{n} 1$$

$$= 13n + 2(2n) + n$$

$$= 18n$$

Question 12 (****+)

It is given that

$$\sum_{r=1}^{20}\left[f(r)-10\right]=200 \quad \text{and} \quad \sum_{r=1}^{20}\left[f(r)-10\right]^2=2800.$$

Find the value of

$$\sum_{r=1}^{20}\left[f(r)\right]^2.$$

$$\boxed{}, \quad \boxed{\sum_{r=1}^{20}\left[f(r)\right]^2=8800}$$

MANIPULATE THE SUMS AS FOLLOWS

$$\sum_{r=1}^{20}(f(r)-10)=\left[\sum_{r=1}^{20}f(r)\right]-\sum_{r=1}^{20}10$$

$$200=\left[\sum_{r=1}^{20}f(r)\right]-10\times 20$$

$$\sum_{r=1}^{20}f(r)=400$$

NEXT WE HAVE

$$\sum_{r=1}^{20}(f(r)-10)^2=\sum_{r=1}^{20}\left[[f(r)]^2-20f(r)+100\right]$$

$$2800=\sum_{r=1}^{20}[f(r)]^2-20\sum_{r=1}^{20}[f(r)]+100\sum_{r=1}^{20}1$$

$$2800=\sum_{r=1}^{20}(f(r))^2-20\times 400+100\times 20$$

$$2800=\sum_{r=1}^{20}(f(r))^2-6000$$

$$\sum_{r=1}^{20}[f(r)]^2=8800$$

Question 13 (****+)

Solve the following equation

$$\sum_{r=2}^{\infty}\left(2^{x-r}\right) = \sqrt{1+3\times 2^{x-2}}\ .$$

You may assume that the left hand side of the equation converges.

$\boxed{}$, $\boxed{x=2}$

WRITE IN EXPLICIT FORM

$$\Rightarrow \sum_{r=2}^{\infty}\left(2^{x-r}\right) = \sqrt{1+3\times 2^{x-2}}$$

$$\Rightarrow 2^{x-2}+2^{x-3}+2^{x-3}+2^{x-4}+\dots = \sqrt{1+3\times 2^{x-2}}$$

USING $S_{\infty}=\frac{a}{1-r}$ WITH $a=2^{x-2}$ & $r=\frac{1}{2}$

$$\Rightarrow \frac{2^{x-2}}{1-\frac{1}{2}} = \sqrt{1+3\times 2^{x-2}}$$

$$\Rightarrow 2\times 2^{x-2} = \sqrt{1+3\times 2^{x-2}}$$

SQUARING BOTH SIDES

$$\Rightarrow 4\times\left(2^{x-2}\right)^2 = 1+3\times 2^{x-2}$$

$$\Rightarrow 2^2\times 2^{2x-4} = 1+3\times 2^{x-2}$$

$$\Rightarrow 2^{2x-2} = 1+3\times 2^{x-2}$$

MULTIPLY BOTH SIDES BY $4=2^2$

$$\Rightarrow 2^{2x} = 4+3\times 2^{x}$$

$$\Rightarrow \left(2^{x}\right)^2-3\left(2^{x}\right)-4=0$$

$$\Rightarrow \left(2^{x}+1\right)\left(2^{x}-4\right)=0$$

$$\Rightarrow 2^{x} = \cancel{-1},\ 4$$

$\therefore\ x=2$

Question 14 (****+)

The sum of the first 2 terms of an arithmetic progression is 40.

The sum of the first 4 terms of the same arithmetic progression is 130.

a) Determine the sum of the first 5 terms of the arithmetic progression.

The sum of the first 2 terms of a geometric progression is 40.

The sum of the first 4 terms of the same geometric progression is 130.

b) Find the two possible values of the sum of the first 5 terms of the geometric progression.

$\boxed{}$, $\boxed{S_5 = \frac{775}{4} = 193.75}$, $\boxed{S_5 = 211 \quad \text{or} \quad S_5 = -275}$

a) FORMING TWO EQUATIONS FROM THE INFORMATION GIVEN

- $S_2 = 40$

$\Rightarrow a + (a+d) = 40$

$\Rightarrow 2a + d = 40$

- $S_4 = 130$

$\Rightarrow a + (a+d) + (a+2d) + (a+3d) = 130$

$\Rightarrow 4a + 6d = 130$

$\Rightarrow 2a + 3d = 65$

SUBTRACTING

$$\begin{aligned} 2a + 3d &= 65 \\ 2a + d &= 40 \\ \hline 2d &= 25 \\ d &= 12.5 \end{aligned}$$

& $2a + d = 40$

$2a + 12.5 = 40$

$2a = 27.5$

$a = 13.75$

$\therefore$ USING $S_n = \frac{n}{2}\left[2a + (n-1)d\right]$

$S_5 = \frac{5}{2}\left[2 \times 13.75 + 4 \times 12.5\right]$

$S_5 = 193.75$

b) REPEATING BUT FOR A GEOMETRIC PROGRESSION NOTING $S_n = \frac{a(r^n - 1)}{r-1}$

- $S_2 = 40$

$\Rightarrow a + ar = 40$

$\Rightarrow a(1+r) = 40$

- $S_4 = 130$

$\Rightarrow \frac{a(r^4-1)}{r-1} = 130$

$\Rightarrow \frac{a(r^2-1)(r^2+1)}{r-1} = 130$

$\Rightarrow \frac{a(r+1)(r-1)(r^2+1)}{(r-1)} = 130$

AS $r - 1 \neq 0$ WE MAY CANCEL

$\Rightarrow a(r+1)(r^2+1) = 130$

$\Rightarrow 40(r^2+1) = 130$

$\Rightarrow r^2 + 1 = \frac{13}{4}$

$\Rightarrow r^2 = \frac{9}{4}$

$\Rightarrow r = \frac{3}{2}$ or $-\frac{3}{2}$

NOW IF $r = \frac{3}{2}$

$a = \frac{40}{1+r}$

$a = \frac{40}{1+\frac{3}{2}}$

$a = \frac{40}{2.5}$

$a = 16$

$S_5 = \frac{16(1.5^5 - 1)}{1.5 - 1}$

$S_5 = 211$

AND IF $r = -\frac{3}{2}$

$a = \frac{40}{1+r}$

$a = \frac{40}{1-\frac{3}{2}}$

$a = \frac{40}{-0.5}$

$a = -80$

$S_5 = \frac{-80((-1.5)^5 - 1)}{-1.5 - 1}$

$S_5 = -275$

Question 15 (****+)

Consider the following 2 sequences.

$$10, 13, 16, 19, 22, \ldots \qquad \text{and} \qquad 6, 12, 24, 48, 96, \ldots$$

The sum of the n^{th} term of the first sequence and the n^{th} term of the second sequence is denoted by U_n.

Show algebraically that

$$U_{n+1} = U_n + 3\left(1 + 2^n\right).$$

, $\boxed{\text{proof}}$

$$\boxed{\begin{array}{ll} f(n) = 10, 13, 16, 19, 22, \ldots & 3n+7 \\ g(n) = 6, 12, 24, 48, 96, \ldots & 3 \times 2^n \end{array}}$$

ADDING THE n^{th} TERMS

$$u_n = f(n) + g(n) = 3n + 7 + 3 \times 2^n$$

$$u_{n+1} = 3(n+1) + 7 + 3 \times 2^{n+1} = 3n + 10 + 3 \times 2^{n+1}$$

SUBTRACTING GIVES

$$\begin{aligned} u_{n+1} - u_n &= \left(3n + 10 + 3 \times 2^{n+1}\right) - \left(3n + 7 + 3 \times 2^n\right) \\ &= \cancel{3n} + 10 + 3 \times 2^{n+1} - \cancel{3n} - 7 - 3 \times 2^n \\ &= 3 + 3 \times 2 \times 2^n - 3 \times 2^n \\ &= 3 + 6 \times 2^n - 3 \times 2^n \\ &= 3 + 3 \times 2^n \\ &= 3(1 + 2^n) \end{aligned}$$

FINALLY WE HAVE

$$u_{n+1} - u_n = 3(1 + 2^n)$$

$$u_{n+1} = u_n + 3(2^n + 1)$$

AS REQUIRED

Question 16 (****+)

Solve the following equation

$$\sum_{r=0}^{\infty} (\sin x)^{2r} = 2\tan x .$$

You may assume that the left hand side of the equation converges.

$$\boxed{x = \frac{1}{4}\pi \pm n\pi, \quad n = 0,1,2,3,\ldots}$$

WRITE THE L.H.S EXPLICITLY

$\Rightarrow \sum_{r=0}^{\infty} (\sin x)^{2r} = 2\tan x$

$\Rightarrow 1 + \sin^2 x + \sin^4 x + \sin^6 x + \ldots = 2\tan x$

THIS IS A G.P WITH $a=1$ & $r = \sin^2 x$ & $S_\infty = \frac{a}{1-r}$

$\Rightarrow \frac{1}{1-\sin^2 x} = 2\tan x$

$\Rightarrow \frac{1}{\cos^2 x} = 2\tan x$

$\Rightarrow \sec^2 x = 2\tan x$

USING $1 + \tan^2 x \equiv \sec^2 x$

$\Rightarrow 1 + \tan^2 x = 2\tan x$

$\Rightarrow \tan^2 x - 2\tan x + 1 = 0$

$\Rightarrow (\tan x - 1)^2 = 0$

$\Rightarrow \tan x = 1$

$\therefore x = \frac{\pi}{4} \pm n\pi \quad n = 0,1,2,3,4,\ldots$

Question 17 (*****)

The product operator $\prod$, is defined as

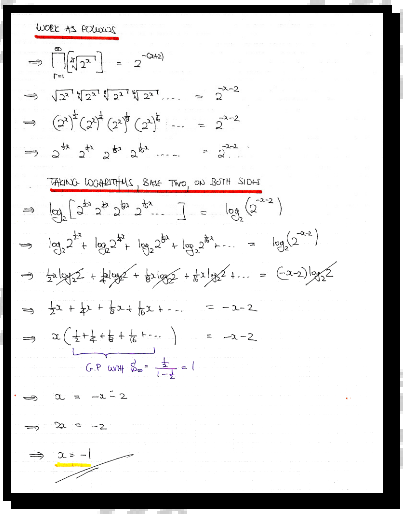
$$\prod_{i=1}^{k}\left[u_i\right] = u_1 \times u_2 \times u_3 \times u_4 \times \ldots \times u_{k-1} \times u_k .$$

Solve the equation

$$\prod_{r=1}^{\infty}\left[\sqrt[2r]{2^x}\right] = 2^{-(x+2)} .$$

You may assume that the left hand side of the equation converges.

$\boxed{\quad}$, $\boxed{x=-1}$



WORK AS FOLLOWS

$\Rightarrow \prod_{r=1}^{\infty}\left[\sqrt[2r]{2^x}\right] = 2^{-(x+2)}$

$\Rightarrow \sqrt{2^x}\,\sqrt[4]{2^x}\,\sqrt[8]{2^x}\,\sqrt[16]{2^x}\ldots = 2^{-x-2}$

$\Rightarrow (2^x)^{\frac{1}{2}}(2^x)^{\frac{1}{4}}(2^x)^{\frac{1}{8}}(2^x)^{\frac{1}{16}}\ldots = 2^{-x-2}$

$\Rightarrow 2^{\frac{1}{2}x}\,2^{\frac{1}{4}x}\,2^{\frac{1}{8}x}\,2^{\frac{1}{16}x}\ldots = 2^{-x-2}$

TAKING LOGARITHMS, BASE TWO, ON BOTH SIDES

$\Rightarrow \log_2\left[2^{\frac{1}{2}x}\,2^{\frac{1}{4}x}\,2^{\frac{1}{8}x}\,2^{\frac{1}{16}x}\ldots\right] = \log_2\left(2^{-x-2}\right)$

$\Rightarrow \log_2 2^{\frac{1}{2}x} + \log_2 2^{\frac{1}{4}x} + \log_2 2^{\frac{1}{8}x} + \log_2 2^{\frac{1}{16}x} + \ldots = \log_2\left(2^{-x-2}\right)$

$\Rightarrow \frac{1}{2}x\cancel{\log_2 2} + \frac{1}{4}x\cancel{\log_2 2} + \frac{1}{8}x\cancel{\log_2 2} + \frac{1}{16}x\cancel{\log_2 2} + \ldots = (-x-2)\cancel{\log_2 2}$

$\Rightarrow \frac{1}{2}x + \frac{1}{4}x + \frac{1}{8}x + \frac{1}{16}x + \ldots = -x-2$

$\Rightarrow x\Big(\underbrace{\tfrac{1}{2}+\tfrac{1}{4}+\tfrac{1}{8}+\tfrac{1}{16}+\ldots}_{\text{G.P WITH } S_\infty = \frac{\frac{1}{2}}{1-\frac{1}{2}} = 1}\Big) = -x-2$

$\Rightarrow x = -x-2$

$\Rightarrow 2x = -2$

$\Rightarrow x = -1$

Question 18 (*****)

It is given that

$$\sum_{r=1}^{n} u_r = \frac{1+3^{2n+2}-2\times 5^{n+1}}{8},$$

where u_n is the n^{th} term of a sequence.

Find a simplified expression for u_n.

$\boxed{u_n = 9^n - 5^n}$

$$\boxed{\sum_{r=1}^{n} u_r = \frac{1+3^{2n+2}-2\times 5^{n+1}}{8}}$$

PROCEED AS FOLLOWS

$$S_n = \sum_{r=1}^{n} u_r = \frac{1+3^{2n+2}-2\times 5^{n+1}}{8}$$

$$S_{n-1} = \sum_{r=1}^{n-1} u_r = \frac{1+3^{2(n-1)+2}-2\times 5^{(n-1)+1}}{8} = \frac{1+3^{2n}-2\times 5^{n}}{8}$$

SUBTRACTING WOULD YIELD u_n

$$\Rightarrow u_n = S_n - S_{n-1} = \frac{1+3^{2n+2}-2\times 5^{n+1}}{8} - \frac{1+3^{2n}-2\times 5^{n}}{8}$$

$$\Rightarrow 8u_n = [\cancel{1}+3^{2n+2}-2\times 5^{n+1}] - [\cancel{1}+3^{2n}-2\times 5^{n}]$$

$$\Rightarrow 8u_n = 3^{2n+2}-2\times 5^{n+1}-3^{2n}+2\times 5^{n}$$

$$\Rightarrow 8u_n = 3^2\times 3^{2n}-3^{2n}+2\times 5^{n}-2\times 5^1\times 5^{n}$$

$$\Rightarrow 8u_n = (9\times 3^{2n}-3^{2n})+(2\times 5^{n}-10\times 5^{n})$$

$$\Rightarrow \cancel{8}u_n = \cancel{8}\times 3^{2n}-\cancel{8}\times 5^{n}$$

$$\Rightarrow u_n = 3^{2n}-5^{n}$$

$$\Rightarrow u_n = (3^2)^n-5^n$$

$$\Rightarrow u_n = 9^n-5^n$$

Question 19 (*****)

It is given that

$$\sum_{r=1}^{n} u_r = 3n^2 - 2n + 4 + (3n-2) \times 2^{n+1},$$

where u_n is the n^{th} term of a sequence.

Find a simplified expression for u_n.

$\boxed{}$, $\boxed{u_n = (3n+1)\times 2^n + 6n - 5}$

$$\boxed{\sum_{r=1}^{\infty} u_r = 2^{n+1} \times (3n-2) + 3n^2 - 2n + 4}$$

- FIND SIMPLIFIED EXPRESSION. FOR S_{n-1}

$\Rightarrow S_{n-1} = 2^{(n-1)+1} \times [3(n-1)-2] + 3(n-1)^2 - 2(n-1) + 4$

$\Rightarrow S_{n-1} = (3n-5) \times 2^n + 3n^2 - 6n + 3 - 2n + 2 + 4$

$\Rightarrow S_{n-1} = (3n-5) \times 2^n + 3n^2 - 8n + 9$

- HENCE WE OBTAIN

$\Rightarrow u_n = S_n - S_{n-1}$

$\Rightarrow u_n = \left[(3n-2)\times 2^{n+1} + \cancel{3n^2} - 2n + 4\right] - \left[(3n-5)\times 2^n + \cancel{3n^2} - 8n + 9\right]$

$\Rightarrow u_n = (3n-2)\times 2^{n+1} - (3n-5)\times 2^n + 6n - 5$

$\Rightarrow u_n = 2(3n-2)\times 2^n - (3n-5)\times 2^n + 6n - 5$

$\Rightarrow u_n = \left[2(3n-2) - (3n-5)\right]\times 2^n + 6n - 5$

$\Rightarrow u_n = (3n+1)\times 2^n + 6n - 5$

Question 20 (*****)

It is given that

$$\sum_{r=1}^{n} u_r = 6^{n+1} - 10\times 2^n + 4,$$

where u_n is the n^{th} term of a sequence.

Show clearly that

$$u_{n+2} = Au_{n+1} + Bu_n,$$

where A and B are integers to be found.

$\boxed{}$, $\boxed{u_{n+2} = 8u_{n+1} - 12u_n}$

$$\boxed{\sum_{r=1}^{n} u_r = 6^{n+1} - 10\times 2^n + 4}$$

- TRYING TO FIND AN EXPRESSION FOR THE nth TERM FIRST

$$\begin{cases} S_n = 6^{n+1} - 10\times 2^n + 4 \\ S_{n-1} = 6^n - 10\times 2^{n-1} + 4 \end{cases}$$

$$\begin{aligned}
\Rightarrow u_n = S_n - S_{n-1} &= \left[6^{n+1} - 10\times 2^n + 4\right] - \left[6^n - 10\times 2^{n-1} + 4\right] \\
&= 6^{n+1} - 6^n - 10\times 2^n + 10\times 2^{n-1} \\
&= (6\times 6^n - 6^n) + 10\times 2^{n-1} - 10\times 2^n \\
&= 5\times 6^n + (5\times 2\times 2^{n-1} - 10\times 2^n) \\
&= 5\times 6^n + (5\times 2^n - 10\times 2^n) \\
&= 5\times 6^n - 5\times 2^n
\end{aligned}$$

$$\therefore \boxed{u_n = 5[6^n - 2^n]}$$

- NOW WE MAY ELIMINATE THE POWERS OF 6 & 2, AS FOLLOWS

$u_n = 5[6^n - 2^n]$

$u_{n+1} = 5[6^{n+1} - 2^{n+1}] = 5[6\times 6^n - 2\times 2^n] = 30\times 6^n - 10\times 2^n$

$u_{n+2} = 30\times 6^{n+1} - 10\times 2^{n+1} = 30\times 6\times 6^n - 10\times 2\times 2^n = 180\times 6^n - 20\times 2^n$

- FOR SIMPLICITY LET $P = 6^n$ & $Q = 2^n$

$$\left.\begin{aligned} u_n &= 5P - 5Q \\ u_{n+1} &= 30P - 10Q \\ u_{n+2} &= 180P - 20Q \end{aligned}\right\} \Rightarrow \begin{aligned} -2u_{n+1} &= -60P + 20Q \\ u_{n+2} &= 180P - 20Q \\ \hline u_{n+2} - 2u_{n+1} &= 120P \end{aligned}$$

ALSO ELIMINATING P FROM THE LAST 2 EQUATIONS

$$\begin{aligned} -6u_{n+1} &= -180P + 60Q \\ u_{n+2} &= 180P - 20Q \\ \hline u_{n+2} - 6u_{n+1} &= 40Q \end{aligned}$$

FINALLY TAKE THE FIRST EQUATION & MULTIPLY IT BY 24

$\Rightarrow 24u_n = 120P - 120Q$

$\Rightarrow 24u_n = 120P - 3(40Q)$

$\Rightarrow 24u_n = u_{n+2} - 2u_{n+1} - 3(u_{n+2} - 6u_{n+1})$

$\Rightarrow 24u_n = u_{n+2} - 2u_{n+1} - 3u_{n+2} + 18u_{n+1}$

$\Rightarrow 2u_{n+2} = 16u_{n+1} - 24u_n$

$\Rightarrow u_{n+2} = 8u_{n+1} - 12u_n$

$\therefore A = 8$ & $B = -12$

Question 21 (*****)

Find in exact simplified form an exact expression for the sum of the first n terms of the following series

$$1 + 11 + 111 + 1111 + 11111 + \ldots$$

$\boxed{}$, $\boxed{S_n = \frac{1}{81}\left[10^{n+1} - 10 - 9n\right]}$

- LET $S_n = 1 + 11 + 111 + 1111 + \ldots + \underbrace{111\ldots111}_{\text{N DIGITS}}$

$$\Rightarrow S_n = \left(\tfrac{1}{9}\times 9\right) + \left(\tfrac{1}{9}\times 99\right) + \left(\tfrac{1}{9}\times 999\right) + \ldots + \left(\tfrac{1}{9}\times \underbrace{999\ldots999}_{\text{N DIGITS}}\right)$$

$$\Rightarrow S_n = \tfrac{1}{9}\left[9 + 99 + 999 + \ldots + \underbrace{999\ldots999}_{n\text{ DIGITS}}\right]$$

$$\Rightarrow S_n = \tfrac{1}{9}\left[(10^1-1) + (10^2-1) + (10^3-1) + \ldots + (10^n-1)\right]$$

$$\Rightarrow S_n = \tfrac{1}{9}\left[\underbrace{(10^1+10^2+10^3+\ldots+10^n)} - (\underbrace{1+1+1+\ldots+1}_{n\text{ terms}})\right]$$

GEOMETRIC PROGRESSION
$a=10$
$r=10$
"$S_n = \frac{a(r^n-1)}{r-1}$"

$$\Rightarrow S_n = \tfrac{1}{9}\left[\frac{10(10^n-1)}{10-1} - (n\times 1)\right]$$

$$\Rightarrow S_n = \tfrac{1}{9}\left[\tfrac{10}{9}(10^n-1) - n\right]$$

$$\Rightarrow S_n = \tfrac{1}{81}\left[10(10^n-1) - 9n\right]$$

$$\Rightarrow S_n = \tfrac{1}{81}\left[10^{n+1} - 10 - 9n\right]$$

$$\Rightarrow S_n = \tfrac{1}{81}\left[10^{n+1} - 9n - 10\right]$$

ALTERNATIVE BY LOOKING AT DIFFERENT PATTERNS

- LET $S_n = 1 + 11 + 111 + 1111 + \ldots + \underbrace{111\ldots111}_{n\text{ DIGITS}}$

$$\Rightarrow S_n = \begin{array}{l} 10^0 \leftarrow 1 \\ 10^1 + 10^0 \leftarrow 11 \\ 10^2 + 10^1 + 10^0 \leftarrow 111 \\ 10^3 + 10^2 + 10^1 + 10^0 \leftarrow 1111 \\ \vdots \\ 10^{n-1} + 10^{n-2} + 10^{n-3} + 10^{n-4} + \ldots + 10^2 + 10^1 + 10^0 \leftarrow \underbrace{111\ldots111}_{n\text{ digits}} \end{array}$$

G.P.	G.P.	G.P.	G.P.	G.P.	G.P.	G.P.
$a=1$	$a=1$	$a=1$	$a=1$	$a=1$	$a=1$	$a=1$
$r=10$	$r=10$	$r=10$	$r=10$	$r=10$	$r=10$	$r=10$
n TERMS	$n-1$ TERMS	$n-2$ TERMS	$n-3$ TERMS	3 TERMS	2 TERMS	1 TERM

- NOW LOOKING AT ANY OF THESE G.Ps VERTICALLY DOWN

$$S_N = \frac{a(r^N-1)}{r-1} = \frac{1(10^N-1)}{10-1} = \tfrac{1}{9}(10^N-1)$$

$$\Rightarrow S_n = \sum \text{ALL THESE G.Ps}$$

$$\Rightarrow S_n = \tfrac{1}{9}\left[10^n-1\right] + \tfrac{1}{9}\left[10^{n-1}-1\right] + \tfrac{1}{9}\left[10^{n-2}-1\right] + \ldots + \tfrac{1}{9}\left[10^3-1\right] + \tfrac{1}{9}\left[10^2-1\right] + \tfrac{1}{9}\left[10^1-1\right]$$

$$\Rightarrow S_n = \tfrac{1}{9}\left[(10^n + 10^{n-1} + 10^{n-2} + \ldots + 10^3 + 10^2 + 10^1) - (\underbrace{1+1+1+\ldots+1}_{n\text{ terms}})\right]$$

$$\Rightarrow S_n = \tfrac{1}{9}\left[(10 + 10^2 + 10^3 + \ldots + 10^n) - n\times 1\right]$$

$$\Rightarrow S_n = \tfrac{1}{9}\times\left[\frac{10(10^n-1)}{10-1} - n\right] \quad \text{(SUMMING THE SAME G.P AS BEFORE)}$$

$$\Rightarrow S_n = \tfrac{1}{9}\left[\frac{10^{n+1}-10}{9} - n\right]$$

$$\Rightarrow S_n = \tfrac{1}{81}\left[10^{n+1} - 9n - 10\right]$$

AS BEFORE

Question 22 (*****)

The first three terms of a geometric progression are the respective 7^{th} term, 4^{th} term, and 2^{nd} term of an arithmetic progression.

Determine the common ratio of the geometric progression.

$\boxed{\text{MP2-S}}$, $\boxed{r = \frac{2}{3}}$

- DEFINE SOME VARIABLES IN ORDER TO FORM SOME EQUATIONS

a = 1ST TERM OF A.P.
d = COMMON DIFFERENCE OF A.P.
A = 1ST TERM OF G.P.
r = COMMON RATIO OF G.P.

Nth TERM OF A.P
$u_n = a + (n-1)d$

Nth TERM OF A G.P.
$u_n = ar^{n-1}$

- HENCE WE NOW HAVE

$$A = a + 6d \quad \text{— I}$$
$$Ar = a + 3d \quad \text{— II}$$
$$Ar^2 = a + d \quad \text{— III}$$

- SOLVE III FOR d: $d = Ar^2 - a$

- SUBSTITUTE INTO I & II

$$\left.\begin{array}{l} A = a + 6Ar^2 - 6a \\ Ar = a + 3Ar^2 - 3a \end{array}\right\} \Rightarrow$$

$$\left.\begin{array}{l} 5a = 6Ar^2 - A \\ 2a = 3Ar^2 - Ar \end{array}\right\} \Rightarrow$$

- DIVIDE EQUATIONS

$$\Rightarrow \frac{5a}{2a} = \frac{6Ar^2 - A}{3Ar^2 - Ar}$$
$$\Rightarrow \frac{5}{2} = \frac{6r^2 - 1}{3r^2 - r}$$
$$\Rightarrow 15r^2 - 5r = 12r^2 - 2$$
$$\Rightarrow 3r^2 - 5r + 2 = 0$$
$$\Rightarrow (3r - 2)(r - 1) = 0$$
$$\Rightarrow r = \begin{cases} \cancel{1} & (r \neq \pm 1, 0) \\ \frac{2}{3} \end{cases}$$

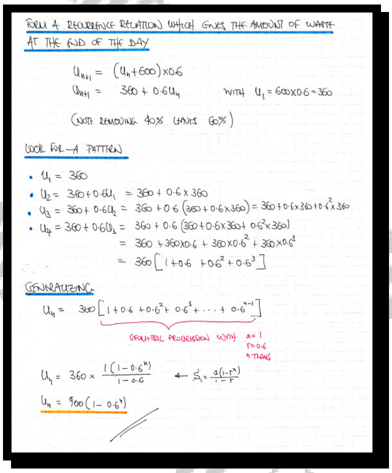
Question 23 (*****)

A factory gets permission to dispose, at the start of every day, 600 kg of waste into a stream of water.

The running stream removes 40% of the any waste present, by the end of the day.

Determine a simplified expression for the amount of waste present in the stream at the end of the n^{th} day.

$$u_n = 900\left[1-\left(\tfrac{3}{5}\right)^n\right]$$



FORM A RECURRENCE RELATION WHICH GIVES THE AMOUNT OF WASTE AT THE END OF THE DAY

$u_{n+1} = (u_n + 600) \times 0.6$

$u_{n+1} = 360 + 0.6u_n$ WITH $u_1 = 600 \times 0.6 = 360$

(NOTE REMOVING 40% LEAVES 60%)

LOOK FOR A PATTERN

- $u_1 = 360$
- $u_2 = 360 + 0.6u_1 = 360 + 0.6 \times 360$
- $u_3 = 360 + 0.6u_2 = 360 + 0.6(360 + 0.6 \times 360) = 360 + 0.6 \times 360 + 0.6^2 \times 360$
- $u_4 = 360 + 0.6u_3 = 360 + 0.6(360 + 0.6 \times 360 + 0.6^2 \times 360)$
  $= 360 + 360 \times 0.6 + 360 \times 0.6^2 + 360 \times 0.6^3$
  $= 360\left[1 + 0.6 + 0.6^2 + 0.6^3\right]$

GENERALIZING

$$u_n = 360\underbrace{\left[1 + 0.6 + 0.6^2 + 0.6^3 + \dots + 0.6^{n-1}\right]}$$

GEOMETRIC PROGRESSION WITH $a = 1$, $r = 0.6$, n TERMS

$u_n = 360 \times \dfrac{1(1 - 0.6^n)}{1 - 0.6}$ $\leftarrow S_n = \dfrac{a(1-r^n)}{1-r}$

$u_n = 900(1 - 0.6^n)$

Question 24 (*****)

$$3 + 33 + 333 + 3333 + 33333 + \ldots$$

Express the sum of the first n terms of the above series in sigma notation.

You are not required to sum the series.

$$\boxed{S_n = \sum_{r=1}^{n} \left[\tfrac{1}{3}\left(10^r - 1\right) \right]}$$

$$S = 3 + 33 + 333 + 3333 + 33333 + \ldots$$

$$S = \left(\tfrac{3}{9} \times 9\right) + \left(\tfrac{3}{9} \times 99\right) + \left(\tfrac{3}{9} \times 999\right) + \left(\tfrac{3}{9} \times 9999\right) + \ldots$$

$$S = \tfrac{3}{9}\left[9 + 99 + 999 + 9999 + \ldots\right]$$

$$S = \tfrac{1}{3}\left[(10^1 - 1) + (10^2 - 1) + (10^3 - 1) + (10^4 - 1) + \ldots\right]$$

$$S = \tfrac{1}{3}\left[(10^1 + 10^2 + 10^3 + 10^4 + \ldots) + (-1 \underbrace{-1 -1 \ldots -1}_{n \text{ of them}})\right.$$

$$S = \tfrac{1}{3}\sum_{r=1}^{n}\left(10^r - 1\right)$$

Question 25 (*****)

A rectangle has perimeter P and area A.

Show that

$$A \le f(P),$$

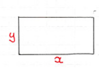
where $f(P)$ is a simplified expression to be found.

$\boxed{}$, $\boxed{A \le \frac{1}{16}P^2}$

START FORMING EXPRESSIONS FOR THE PERIMETER AND AREA

$P = 2(x+y)$

$A = xy$



REARRANGE THE EXPRESSIONS, AS THEY RESEMBLE THE AM – GM INEQUALITY WHICH STATES THAT

$\Rightarrow AM \geqslant GM$

$\Rightarrow \frac{x+y}{2} \geqslant \sqrt[2]{xy}$

THUS WE HAVE

$P = 2(x+y)$

$x+y = \frac{P}{2}$

$\frac{x+y}{2} = \frac{P}{4}$

&

$A = xy$

$\sqrt{xy} = \sqrt{A}$

COMBINING THESE RESULTS INTO THE INEQUALITY WE OBTAIN

$\Rightarrow \frac{P}{4} \geqslant \sqrt{A}$

$\Rightarrow \frac{P^2}{16} \geqslant A$

$\Rightarrow A \leqslant \frac{1}{16}P^2$

Question 26 (*****)

A sequence of positive integers is generated by the formula

$$u_n = 2n^3 - 57n^2 - 120n + 9200, \quad n \in \mathbb{N}.$$

Determine the largest value of n, such that $u_n > u_{n+1}$.

$\boxed{}$, $\boxed{n = 19}$

$\boxed{u_n = 2n^3 - 57n^2 - 120n + 9200, \quad n \in \mathbb{N}}$

- CONSIDER THE RECURRENCE RELATION AS A CONTINUOUS CURVE BUT ONLY CONSIDER INTEGER VALUES OF n

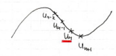
I.E $f(n) = 2n^3 - 57n^2 - 120n + 9200$

- DIFFERENTIATE W.R.T n, AND SOLVE FOR ZERO

$f'(n) = 6n^2 - 114n - 120$

- SOLVE FOR ZERO — IF IT DOES NOT SOLVE NICELY WE SHALL CONSIDER THE SIGN OF THE GRADIENT AS THE REQUIRED POINT WILL OCCUR BEFORE THE LOCAL MINIMUM OF THE CUBIC

$\Rightarrow 6n^2 - 114n - 120 = 0$
$\Rightarrow 3n^2 - 57n - 60 = 0$
$\Rightarrow n^2 - 19n - 20 = 0$
$\Rightarrow (n+1)(n-20) = 0$
$\Rightarrow n = \begin{cases} -1 \\ 20 \end{cases}$



- DUE TO THE SHAPE OF THE CUBIC THE LOCAL MINIMUM IS EXACTLY ON u_{20}, SO THAT THE LARGEST VALUE OF n IS 19

IE $u_{19} > u_{20}$

Question 27 (*****)

The geometric mean of two positive numbers a and b is denoted by G.

The arithmetic mean of $\frac{1}{a}$ and $\frac{1}{b}$ is denoted by A.

Given further that the ratio $\frac{1}{A}:G = 4:5$, determine the ratio between a and b.

$\boxed{}$, $\boxed{4:1}$

$$\boxed{G = \sqrt[2]{ab}}$$

$$\boxed{A = \frac{\frac{1}{a}+\frac{1}{b}}{2} \qquad A = \frac{b+a}{2ab} \qquad \frac{1}{A} = \frac{2ab}{a+b}}$$

WE EXPECT SYMMETRICAL SOLUTIONS AS SOLUTIONS AS SWAPPING a & b MAKES NO DIFFERENCE

$$\frac{\frac{1}{A}}{G} = \frac{4}{5} \Rightarrow \frac{1}{AG} = \frac{4}{5}$$
$$\Rightarrow 4AG = 5$$
$$\Rightarrow 4\left(\frac{b+a}{2ab}\right)(ab)^{\frac{1}{2}} = 5$$
$$\Rightarrow \frac{2(b+a)}{(ab)^{\frac{1}{2}}} = 5$$
$$\Rightarrow 2(a+b) = 5(ab)^{\frac{1}{2}}$$
$$\Rightarrow 4(a^2+2ab+b^2) = 25(ab)$$
$$\Rightarrow 4a^2+8ab+4b^2 = 25ab$$
$$\Rightarrow 4a^2-17ab+4b^2 = 0$$
$$\Rightarrow (4a-b)(a-4b) = 0$$
$$\Rightarrow 4a = b \quad \underline{\text{OR}} \quad a = 4b$$
$$\Rightarrow \frac{a}{b} = \frac{1}{4} \quad \underline{\text{OR}} \quad \frac{a}{b} = \frac{4}{1}$$
$$\Rightarrow a:b = 1:4 \quad \text{OR} \quad a:b = 4:1$$

(SYMMETRICAL SOLUTIONS)

Question 28 (*****)

A function is defined as

$$[x] \equiv \{\text{the greatest integer less or equal to } x\}.$$

The function f is defined as

$$f(n) = n\left[\frac{3}{5} + \frac{3n}{100}\right], \quad n \in \mathbb{N}.$$

Determine the value of

$$\sum_{n=1}^{82} f(n).$$

$\boxed{}$, $\boxed{5877}$

$$\boxed{[x] \equiv \{\text{GREATEST INTEGER LESS OR EQUAL TO } x\}}$$

- $f(n) = n\left[\frac{3}{5} + \frac{3n}{100}\right], \ n \in \mathbb{N}$.
- WE NEED TO INVESTIGATE THE SIZE OF THE FIRST 82 TERMS
WE NEED TO WORK IN GROUPS

$\frac{3}{5} + \frac{3n}{100} \leqslant 1$	$\frac{3}{5} + \frac{3n}{100} \leqslant 2$	$\frac{3}{5} + \frac{3n}{100} \leqslant 3$
$\frac{3n}{100} \leqslant \frac{2}{5}$	$\frac{3n}{100} \leqslant \frac{7}{5}$	$\frac{3n}{100} \leqslant \frac{12}{5}$
$3n \leqslant 40$	$3n \leqslant 140$	$3n \leqslant 240$
$n \leqslant \frac{40}{3} = 13\frac{1}{3}$	$n \leqslant \frac{140}{3} = 46\frac{2}{3}$	$n \leqslant 80$
∴ THE FIRST 13 "TERMS" OF [...] ARE ALL ZERO	THE "TERMS" OF [...] FROM 14th TO 46th ARE 1	THE TERMS OF [...] FROM 47th TO 79TH ARE 2

[EVIDENTLY THE $80^{\text{TH}}, 81^{\text{ST}}, 82^{\text{ND}}$ TERMS OF [...] ARE 3]

- SUMMING UP THE SERIES

$$\sum_{n=1}^{82} f(n) = \sum_{n=1}^{82} n\left[\frac{3}{5} + \frac{3n}{100}\right]$$

$$= \cancel{\left[\sum_{n=1}^{13} (n \times 0)\right]} + \left[\sum_{n=14}^{46} (n \times 1)\right] + \left[\sum_{n=47}^{79} (n \times 2)\right] + \left[\sum_{n=80}^{82} (n \times 3)\right]$$

$46 - 13 = 33$ $\quad 79 - 46 = 33$ $\quad 3$

$a = 14$, $d = 1$, $L = 46$, $n = 33$

$a = 47 \times 2 = 96$, $d = 2$, $L = 79 \times 2 = 158$, $n = 33$

NO NEED FOR FORMULAS FOR JUST 3 TERMS

- USING $S_n = \frac{n}{2}[a + L]$

- $\sum_{n=14}^{46} n = \frac{33}{2}[14 + 46] = \frac{33}{2} \times 60 = 33 \times 30 = 990$
- $\sum_{n=47}^{79} 2n = \frac{33}{2}[94 + 158] = \frac{33}{2} \times 252 = 33 \times 126 = \begin{array}{r} 3780 \\ 378 \\ \hline 4158 \end{array}$
- $\sum_{n=80}^{82} 3n = (80 + 81 + 82) \times 3 = 243 \times 3 = 729$

$$\therefore \sum_{n=1}^{82} nf(n) = 990 + 4158 + 729 = \begin{array}{r} 4158 \\ 990 \\ 729 \\ \hline \end{array} = 5877$$

Question 29 (*****)

Evaluate the following expression

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\left[\frac{1}{3^{m+n}}\right].$$

Detailed workings must be shown.

$\boxed{}$, $\boxed{\frac{9}{4}}$

WORK AS FOLLOWS

$$\sum_{n=0}^{\infty}\left[\sum_{m=0}^{\infty}\left[\frac{1}{3^{m+n}}\right]\right] = \sum_{n=0}^{\infty}\left[\sum_{m=0}^{\infty}\left[\frac{1}{3^m}\times\frac{1}{3^n}\right]\right]$$

$$= \sum_{n=0}^{\infty}\left[\frac{1}{3^n}\sum_{m=0}^{\infty}\left[\frac{1}{3^m}\right]\right]$$

$$= \sum_{n=0}^{\infty}\left[\frac{1}{3^n}\underbrace{\left(1+\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\dots\right)}_{\text{G.P.}}\right]$$

THIS IS A GEOMETRIC PROGRESSION WITH $a=1$, $r=\frac{1}{3}$ & $S_\infty = \frac{a}{1-r}$

$$= \sum_{n=0}^{\infty}\left[\frac{1}{3^n}\times\frac{1}{1-\frac{1}{3}}\right]$$

$$= \sum_{n=0}^{\infty}\left[\frac{3}{2}\times\frac{1}{3^n}\right]$$

$$= \frac{3}{2}\sum_{n=0}^{\infty}\left[\frac{1}{3^n}\right]$$

$$= \frac{3}{2}\underbrace{\left(1+\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\dots\right)}_{\text{Same G.P. as above with } S_\infty = \frac{1}{1-}}$$

$$= \frac{3}{2}\times\frac{3}{2}$$

$$= \frac{9}{4}$$

Question 30 (*****)

The sum to infinity S of the convergent geometric series is given by

$$S = 1 + x + x^2 + x^3 + x^4 + \dots, \quad |x| < 1,$$

By integrating the above equation between suitable limits, or otherwise, find

$$\sum_{r=1}^{\infty}\left[\frac{1}{r \times 2^r}\right].$$

You may assume that integration and summation commute.

$\boxed{}$, $\boxed{\ln 2}$

WRITE THE GEOMETRIC SERIES COMPACTLY

$$\Rightarrow S = 1 + x + x^2 + x^3 + x^4 + \dots \qquad |x| < 1$$

$$\Rightarrow \sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \qquad |x| < 1$$

IN ORDER TO PRODUCE THE REQUIRED SERIES WE REWRITE THE LHS AS

$$\Rightarrow \sum_{n=1}^{\infty} x^{n-1} = \frac{1}{1-x}$$

$$\Rightarrow \int_0^{\frac{1}{2}} \left[\sum_{n=1}^{\infty} x^{n-1}\right] dx = \int_0^{\frac{1}{2}} \frac{1}{1-x}\, dx$$

$$\Rightarrow \sum_{n=1}^{\infty}\left[\int_0^{\frac{1}{2}} x^{n-1}\, dx\right] = \Big[-\ln|1-x|\Big]_0^{\frac{1}{2}}$$

$$\Rightarrow \sum_{n=1}^{\infty}\left[\frac{x^n}{n}\right]_0^{\frac{1}{2}} = \Big[\ln|1-x|\Big]_{\frac{1}{2}}^{0}$$

$$\Rightarrow \sum_{n=1}^{\infty}\left[\frac{1}{n}\left(\left(\frac{1}{2}\right)^n - 0\right)\right] = \cancel{\ln 1} - \ln\frac{1}{2}$$

$$\Rightarrow \sum_{n=1}^{\infty}\left[\frac{1}{n \times 2^n}\right] = -\ln\frac{1}{2}$$

$$\Rightarrow \sum_{r=1}^{\infty}\left[\frac{1}{2^r r}\right] = \ln 2$$

ALTERNATIVE SOLUTION USING STANDARD EXPANSIONS

CONSIDERING THE EXPANSION OF $\ln(1-x)$

$$\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \dots$$

LET $x = \frac{1}{2}$

$$\ln\frac{1}{2} = -\left[\frac{1}{2} + \frac{1}{2}\left(\frac{1}{2}\right)^2 + \frac{1}{3}\left(\frac{1}{2}\right)^3 + \frac{1}{4}\left(\frac{1}{2}\right)^4 + \dots\right]$$

$$-\ln 2 = -\left[\frac{1}{2} + \frac{1}{2 \cdot 2^2} + \frac{1}{3 \cdot 2^3} + \frac{1}{4 \cdot 2^4} + \dots\right]$$

$$\sum_{r=1}^{\infty}\left[\frac{1}{2^r r}\right] = \ln 2$$

Question 31 (*****)

Evaluate the following expression

$$\sum_{k=1}^{\infty}\left[\sum_{r=1}^{k} r\right]^{-1}.$$

SP, $\boxed{2}$

- REWRITE FOR SIMPLICITY AS FOLLOWS

$$\sum_{k=1}^{\infty}\left[\sum_{r=1}^{k} r\right]^{-1} = \sum_{k=1}^{\infty}\left[\frac{1}{\sum_{r=1}^{k} r}\right]$$

$$= \frac{1}{1} + \frac{1}{1+2} + \frac{1}{1+2+3} + \frac{1}{1+2+3+4} + \dots$$

- INTRODUCE A FINITE LIMIT FOR THE SUMMATION, SAY n

$$= \lim_{n\to\infty}\left[1 + \frac{1}{1+2} + \frac{1}{1+2+3} + \frac{1}{1+2+3+4} + \dots + \frac{1}{1+2+3+\dots+n}\right]$$

$$= \lim_{n\to\infty}\left[\sum_{r=1}^{n}\frac{1}{\frac{1}{2}r(r+1)}\right]$$

$$= 2\lim_{n\to\infty}\left[\sum_{r=1}^{n}\frac{1}{r(r+1)}\right]$$

- SPLIT INTO TWO FRACTIONS BY INSPECTION

$$= 2\lim_{n\to\infty}\left[\sum_{r=1}^{n}\frac{1}{r} - \frac{1}{r+1}\right]$$

$$= 2\lim_{n\to\infty}\left[\left(1-\cancel{\tfrac{1}{2}}\right)+\left(\cancel{\tfrac{1}{2}}-\cancel{\tfrac{1}{3}}\right)+\left(\cancel{\tfrac{1}{3}}-\cancel{\tfrac{1}{4}}\right)+\dots+\left(\cancel{\tfrac{1}{n}}-\tfrac{1}{n+1}\right)\right]$$

$$= 2\lim_{n\to\infty}\left[1-\frac{1}{n+1}\right]$$

$$= 2$$

Question 32 (*****)

Evaluate the following expression

$$\sum_{n=0}^{\infty}\sum_{m=0}^{n}\left[\frac{1}{2^{m+n}}\right].$$

Detailed workings must be shown.

$\boxed{}$, $\boxed{\frac{8}{3}}$

WORK AS FOLLOWS

$$\sum_{n=0}^{\infty}\sum_{m=0}^{n}\left[\frac{1}{2^{m+n}}\right] = \sum_{n=0}^{\infty}\left[\sum_{m=0}^{n}\left[\frac{1}{2^m}\times\frac{1}{2^n}\right]\right]$$

$$= \sum_{n=0}^{\infty}\left[\frac{1}{2^n}\sum_{m=0}^{n}\left(\frac{1}{2^m}\right)\right]$$

$$= \sum_{n=0}^{\infty}\left[\frac{1}{2^n}\left(\underbrace{1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\dots+\frac{1}{2^n}}_{\text{G.P WITH } a=1,\ r=\frac{1}{2},\ n+1 \text{ TERMS}}\right)\right]$$

$$S_n = \frac{a(1-r^n)}{1-r}$$

$$S_{n+1} = \frac{a(1-r^{n+1})}{1-r}$$

THUS WE SIMPLIFY TO

$$= \sum_{n=0}^{\infty}\left[\frac{1}{2^n}\times\frac{1\left(1-\left(\frac{1}{2}\right)^{n+1}\right)}{1-\frac{1}{2}}\right]$$

$$= \sum_{n=0}^{\infty}\left[\frac{1}{2^n}\times 2\times\left(1-\left(\frac{1}{2}\right)^{n+1}\right)\right]$$

$$= 2\sum_{n=0}^{\infty}\left[\frac{1}{2^n}\left(1-\frac{1}{2^{n+1}}\right)\right]$$

$$= 2\sum_{n=0}^{\infty}\left[\frac{1}{2^n}-\frac{1}{2^{2n+1}}\right]$$

WRITE THE GEOMETRIC PROGRESSIONS EXPLICITLY

$$\dots = 2\sum_{n=0}^{\infty}\frac{1}{2^n} - 2\sum_{n=0}^{\infty}\frac{1}{2^{2n+1}}$$

$$= 2\left[1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\dots\right] - 2\left[\frac{1}{2}+\frac{1}{8}+\frac{1}{32}+\frac{1}{128}+\dots\right]$$

$a=1$, $r=\frac{1}{2}$ $\qquad$ $a=\frac{1}{2}$, $r=\frac{1}{4}$

USING $S_{\infty}=\frac{a}{1-r}$ IN EACH CASE

$$= 2\times\frac{1}{1-\frac{1}{2}} - 2\times\frac{\frac{1}{2}}{1-\frac{1}{4}}$$

$$= 2\times\frac{1}{\frac{1}{2}} - 2\times\frac{2}{4-1}$$

$$= 2\times 2 - \frac{4}{3}$$

$$= 4 - \frac{4}{3}$$

$$= \frac{8}{3}$$

Question 33 (*****)

It is given that L is the sum to infinity of the following convergent series

$$\sum_{r=0}^{\infty}\left[\frac{1}{r!}\right].$$

Use this fact to find, in terms of L, the sum to infinity of this convergent series

$$\sum_{r=1}^{\infty}\left[\frac{r^3}{r!}\right].$$

$\boxed{}$, $\boxed{5L}$

$$\boxed{\sum_{r=0}^{\infty}\left(\frac{1}{r!}\right) = L \text{ , GIVEN}}$$

- LET THE SUM TO INFINITY OF THE REQUIRED SERIES BE S

$$\Rightarrow S = \sum_{r=1}^{\infty}\left[\frac{r^3}{r!}\right] = \sum_{r=1}^{\infty}\left[\frac{r^3}{r\times(r-1)!}\right] = \sum_{r=1}^{\infty}\left[\frac{r^2}{(r-1)!}\right] = \sum_{r=0}^{\infty}\left[\frac{(r+1)^2}{r!}\right]$$

- EXPAND THE NUMERATOR, SPLIT THE FRACTION, CANCEL DOWN AND RE-ADJUST THE DUMMY VARIABLE

$$\Rightarrow S = \sum_{r=0}^{\infty}\frac{r^2+2r+1}{r!} = \sum_{r=0}^{\infty}\left[\frac{r^2}{r!}\right] + \sum_{r=0}^{\infty}\left[\frac{2r}{r!}\right] + \left[\sum_{r=0}^{\infty}\left[\frac{1}{r!}\right]\right]$$

FIRST TERM IS ZERO SO WE MAY START FROM $r=1$ L

$$\Rightarrow S = \sum_{r=1}^{\infty}\left[\frac{r^2}{r\times(r-1)!}\right] + 2\sum_{r=1}^{\infty}\left[\frac{r}{r\times(r-1)!}\right] + L$$

$$\Rightarrow S = \sum_{r=1}^{\infty}\left[\frac{r}{(r-1)!}\right] + 2\sum_{r=1}^{\infty}\left(\frac{1}{(r-1)!}\right] + L$$

$$\Rightarrow S = \sum_{r=0}^{\infty}\left[\frac{r+1}{r!}\right] + 2\underbrace{\sum_{r=0}^{\infty}\left[\frac{1}{r!}\right]}_{L} + L$$

$$\Rightarrow S = \sum_{r=0}^{\infty}\left[\frac{r+1}{r!}\right] + 3L$$

- REPEAT THE PROCESS ONCE MORE WITH THE SUM

$$\Rightarrow S = \sum_{r=0}^{\infty}\left[\frac{r}{r!}\right] + \underbrace{\sum_{r=0}^{\infty}\left[\frac{1}{r!}\right]}_{L} + 3L$$

FIRST TERM IS ZERO

$$\Rightarrow S = \sum_{r=1}^{\infty}\left[\frac{r}{r!}\right] + 4L$$

$$\Rightarrow S = \sum_{r=1}^{\infty}\left[\frac{r}{r\times(r-1)!}\right] + 4L$$

$$\Rightarrow S = \sum_{r=1}^{\infty}\left[\frac{1}{(r-1)!}\right] + 4L$$

$$\Rightarrow S = \underbrace{\sum_{r=0}^{\infty}\left[\frac{1}{r!}\right]}_{L} + 4L$$

$$\Rightarrow S = 5L$$

$$\therefore \sum_{r=1}^{\infty}\left(\frac{r^3}{r!}\right) = 5L$$