2D VECTORS

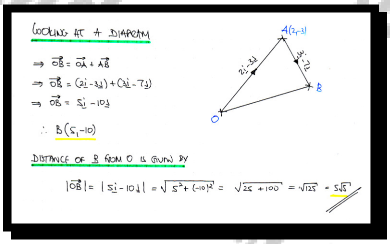
Question 1 (**)

Relative to a fixed origin O, the point A has coordinates $(2,-3)$.

The point B is such so that $\overrightarrow{AB} = 3\mathbf{i} - 7\mathbf{j}$, where $\mathbf{i}$ and $\mathbf{j}$ are mutually perpendicular unit vectors lying on the same plane.

Determine the distance of B from O.

$\boxed{}$, $\boxed{|OB| = 5\sqrt{5}}$



Question 2 (**+)

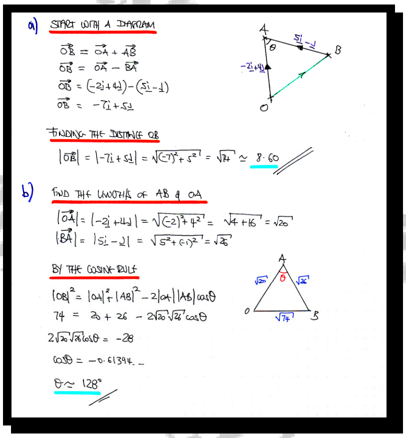
Relative to a fixed origin O, the point A has coordinates $(-2,4)$.

The point B is such so that $\overrightarrow{BA} = 5\mathbf{i} - \mathbf{j}$, where $\mathbf{i}$ and $\mathbf{j}$ are mutually perpendicular unit vectors lying on the same plane.

a) Determine the distance of B from O.

b) Calculate the angle OAB.

$\boxed{}$, $\boxed{|OB| = \sqrt{74}}$, $\boxed{\measuredangle OAB \approx 128°}$



a) START WITH A DIAGRAM

$\vec{OB} = \vec{OA} + \vec{AB}$

$\vec{OB} = \vec{OA} - \vec{BA}$

$\vec{OB} = (-2\mathbf{i}+4\mathbf{j}) - (5\mathbf{i}-\mathbf{j})$

$\vec{OB} = -7\mathbf{i}+5\mathbf{j}$

FINDING THE DISTANCE OB

$|\vec{OB}| = |-7\mathbf{i}+5\mathbf{j}| = \sqrt{(-7)^2+5^2} = \sqrt{74} \simeq 8.60$

b) FIND THE LENGTHS OF AB & OA

$|\vec{OA}| = |-2\mathbf{i}+4\mathbf{j}| = \sqrt{(-2)^2+4^2} = \sqrt{4+16} = \sqrt{20}$

$|\vec{BA}| = |5\mathbf{i}-\mathbf{j}| = \sqrt{5^2+(-1)^2} = \sqrt{26}$

BY THE COSINE RULE

$|OB|^2 = |OA|^2 + |AB|^2 - 2|OA||AB|\cos\theta$

$74 = 20 + 26 - 2\sqrt{20}\sqrt{26}\cos\theta$

$2\sqrt{20}\sqrt{26}\cos\theta = -28$

$\cos\theta = -0.61394\ldots$

$\theta \simeq 128°$

Question 3 (***)

The points A, B and C lie on a plane so that

$$\overrightarrow{AB} = 2\mathbf{i} + 7\mathbf{j} \quad \text{and} \quad \overrightarrow{AC} = 4\mathbf{i} - 5\mathbf{j},$$

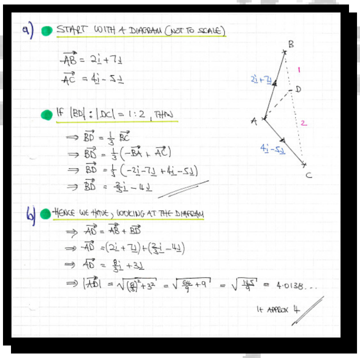
where $\mathbf{i}$ and $\mathbf{j}$ are mutually perpendicular unit vectors lying on the same plane.

The point D lies on the straight line segment BC, so that $|BD|:|DC| = 1:2$.

a) Determine a simplified expression, in terms of $\mathbf{i}$ and $\mathbf{j}$, for $\overrightarrow{BD}$.

b) Show that the $|\overrightarrow{AD}|$ is approximately 4 units.

$\boxed{}$, $\boxed{\overrightarrow{BD} = \frac{2}{3}\mathbf{i} - 4\mathbf{j}}$


a) START WITH A DIAGRAM (NOT TO SCALE)

$\overrightarrow{AB} = 2\underline{i} + 7\underline{j}$

$\overrightarrow{AC} = 4\underline{i} - 5\underline{j}$

IF $|BD| : |DC| = 1:2$, THEN

$\Rightarrow \overrightarrow{BD} = \frac{1}{3}\overrightarrow{BC}$

$\Rightarrow \overrightarrow{BD} = \frac{1}{3}(-\overrightarrow{BA} + \overrightarrow{AC})$

$\Rightarrow \overrightarrow{BD} = \frac{1}{3}(-2\underline{i} - 7\underline{j} + 4\underline{i} - 5\underline{j})$

$\Rightarrow \overrightarrow{BD} = \frac{2}{3}\underline{i} - 4\underline{j}$

b) HENCE WE HAVE, LOOKING AT THE DIAGRAM

$\Rightarrow \overrightarrow{AD} = \overrightarrow{AB} + \overrightarrow{BD}$

$\Rightarrow \overrightarrow{AD} = (2\underline{i} + 7\underline{j}) + (\frac{2}{3}\underline{i} - 4\underline{j})$

$\Rightarrow \overrightarrow{AD} = \frac{8}{3}\underline{i} + 3\underline{j}$

$\Rightarrow |\overrightarrow{AD}| = \sqrt{(\frac{8}{3})^2 + 3^2} = \sqrt{\frac{64}{9} + 9} = \sqrt{\frac{145}{9}} = 4.0138\ldots$

IE APPROX 4

Question 4 (***)

The following information is given for four points which lie on the same plane.

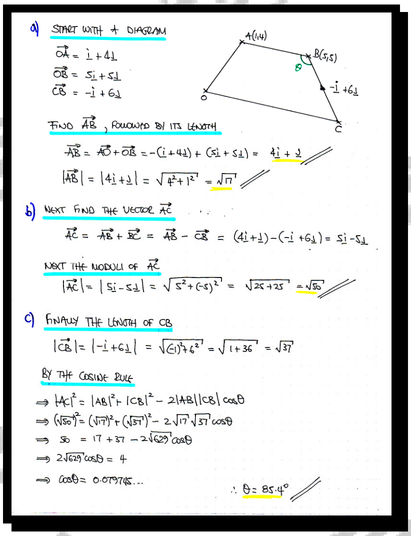
$$\overrightarrow{OA} = \mathbf{i} + 4\mathbf{j}, \qquad \overrightarrow{OB} = 5\mathbf{i} + 5\mathbf{j} \qquad \text{and} \qquad \overrightarrow{CB} = -\mathbf{i} + 6\mathbf{j},$$

a) Find the vector $\overrightarrow{AB}$ and hence state its length

b) Determine the length of $\overrightarrow{AC}$.

c) Calculate the size of the angle ABC.

$\boxed{}$, $\boxed{\overrightarrow{AB} = 4\mathbf{i} + \mathbf{j}}$, $\boxed{\left|\overrightarrow{AB}\right| = \sqrt{17}}$, $\boxed{\left|\overrightarrow{AC}\right| = \sqrt{50}}$, $\boxed{\measuredangle ABC \approx 85.4°}$


a) START WITH A DIAGRAM

$\overrightarrow{OA} = \underline{i} + 4\underline{j}$

$\overrightarrow{OB} = 5\underline{i} + 5\underline{j}$

$\overrightarrow{CB} = -\underline{i} + 6\underline{j}$

FIND $\overrightarrow{AB}$, FOLLOWED BY ITS LENGTH

$\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB} = -(\underline{i} + 4\underline{j}) + (5\underline{i} + 5\underline{j}) = 4\underline{i} + \underline{j}$ //

$|\overrightarrow{AB}| = |4\underline{i} + \underline{j}| = \sqrt{4^2 + 1^2} = \sqrt{17}$ //

b) NEXT FIND THE VECTOR $\overrightarrow{AC}$

$\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AB} - \overrightarrow{CB} = (4\underline{i} + \underline{j}) - (-\underline{i} + 6\underline{j}) = 5\underline{i} - 5\underline{j}$

NEXT THE MODULI OF $\overrightarrow{AC}$

$|\overrightarrow{AC}| = |5\underline{i} - 5\underline{j}| = \sqrt{5^2 + (-5)^2} = \sqrt{25 + 25} = \sqrt{50}$ //

c) FINALLY THE LENGTH OF CB

$|\overrightarrow{CB}| = |-\underline{i} + 6\underline{j}| = \sqrt{(-1)^2 + 6^2} = \sqrt{1 + 36} = \sqrt{37}$

BY THE COSINE RULE

$\Rightarrow |AC|^2 = |AB|^2 + |CB|^2 - 2|AB||CB|\cos\theta$

$\Rightarrow (\sqrt{50})^2 = (\sqrt{17})^2 + (\sqrt{37})^2 - 2\sqrt{17}\sqrt{37}\cos\theta$

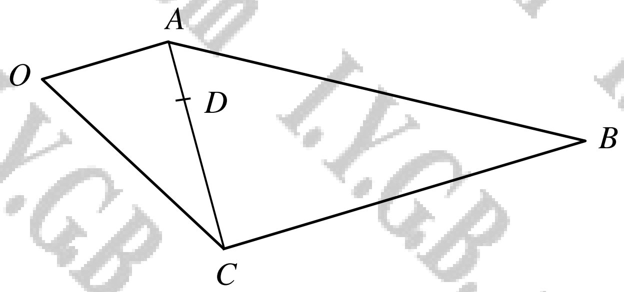
$\Rightarrow 50 = 17 + 37 - 2\sqrt{629}\cos\theta$

$\Rightarrow 2\sqrt{629}\cos\theta = 4$

$\Rightarrow \cos\theta = 0.079745\ldots$

$\therefore \theta = 85.4°$ //

Question 5 (***)



The figure above shows a trapezium $OABC$, where O is a fixed origin.

The position vectors of A and C are $12\mathbf{i}+4\mathbf{j}$ and $18\mathbf{i}-21\mathbf{j}$, respectively.

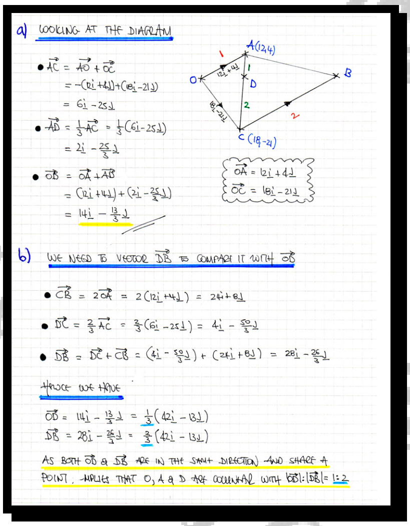
CB is parallel to OA, so that $|\overrightarrow{CB}| = 2|\overrightarrow{OA}|$.

The point D lies on AC so that $AD : DC = 1:2$.

a) Find a simplified expression, in terms of $\mathbf{i}$ and $\mathbf{j}$, for the position vector of D.

b) Show that that O, D and B are collinear and state the ratio of $OD : DB$.

, $\boxed{\overrightarrow{OD} = 14\mathbf{i} - \frac{13}{3}\mathbf{j}}$, $\boxed{OD : DB = 1:2}$



a) LOOKING AT THE DIAGRAM

- $\overrightarrow{AC} = \overrightarrow{AO} + \overrightarrow{OC}$
  $= -(12\underline{i}+4\underline{j}) + (18\underline{i}-21\underline{j})$
  $= 6\underline{i} - 25\underline{j}$
- $\overrightarrow{AD} = \frac{1}{3}\overrightarrow{AC} = \frac{1}{3}(6\underline{i}-25\underline{j})$
  $= 2\underline{i} - \frac{25}{3}\underline{j}$
- $\overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AB}$
  $= (12\underline{i}+4\underline{j}) + (2\underline{i}-\frac{25}{3}\underline{j})$
  $= 14\underline{i} - \frac{13}{3}\underline{j}$

$\overrightarrow{OA} = 12\underline{i}+4\underline{j}$
$\overrightarrow{OC} = 18\underline{i}-21\underline{j}$

b) WE NEED TO VECTOR $\overrightarrow{DB}$ TO COMPARE IT WITH $\overrightarrow{OD}$

- $\overrightarrow{CB} = 2\overrightarrow{OA} = 2(12\underline{i}+4\underline{j}) = 24\underline{i}+8\underline{j}$
- $\overrightarrow{DC} = \frac{2}{3}\overrightarrow{AC} = \frac{2}{3}(6\underline{i}-25\underline{j}) = 4\underline{i} - \frac{50}{3}\underline{j}$
- $\overrightarrow{DB} = \overrightarrow{DC} + \overrightarrow{CB} = (4\underline{i} - \frac{50}{3}\underline{j}) + (24\underline{i}+8\underline{j}) = 28\underline{i} - \frac{26}{3}\underline{j}$

HENCE WE HAVE

$\overrightarrow{OD} = 14\underline{i} - \frac{13}{3}\underline{j} = \frac{1}{3}(42\underline{i} - 13\underline{j})$
$\overrightarrow{DB} = 28\underline{i} - \frac{26}{3}\underline{j} = \frac{2}{3}(42\underline{i} - 13\underline{j})$

AS BOTH $\overrightarrow{OD}$ & $\overrightarrow{DB}$ ARE IN THE SAME DIRECTION AND SHARE A POINT, IMPLIES THAT O, A & D ARE COLLINEAR WITH $|\overrightarrow{OD}| : |\overrightarrow{DB}| = 1:2$

Question 6 (***)

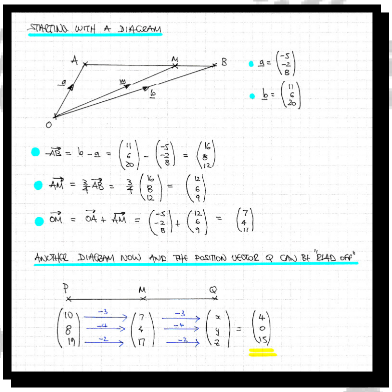
The points A and B have position vectors $\begin{pmatrix} -5 \\ -2 \\ 8 \end{pmatrix}$ and $\begin{pmatrix} 11 \\ 6 \\ 20 \end{pmatrix}$, respectively.

The point M lies on AB so that $|AM|:|MB| = 3:1$

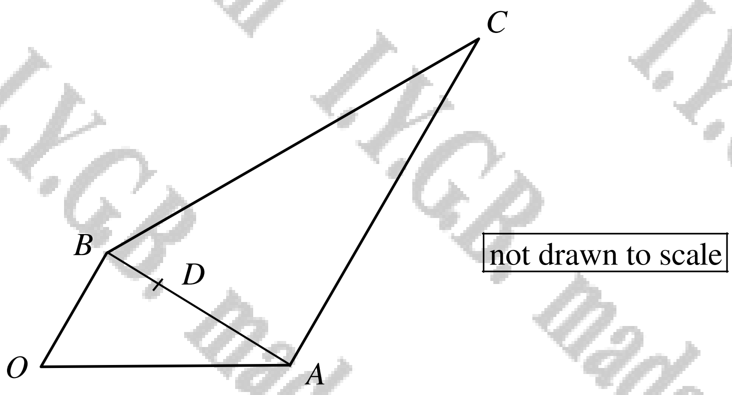
The point P has position vector $\begin{pmatrix} 10 \\ 8 \\ 19 \end{pmatrix}$.

Determine the position vector of the point Q, if M is the midpoint of PQ.

SYN-A, $\boxed{\mathbf{q} = \begin{pmatrix} 4 \\ 0 \\ 15 \end{pmatrix}}$



Question 7 (***+)



The figure above shows a trapezium $OBCA$ where OB is parallel to AC.

The point D lies on BA so that $BD:DA = 1:2$.

It is further given that $\overrightarrow{OA} = 7\mathbf{i} - 4\mathbf{j}$, $\overrightarrow{OB} = 3\mathbf{i} + 2\mathbf{j}$ and $\overrightarrow{AC} = 2\,\overrightarrow{OB}$, where $\mathbf{i}$ and $\mathbf{j}$ are mutually perpendicular unit vectors lying on the same plane.

a) Determine simplified expressions, in terms of $\mathbf{i}$ and $\mathbf{j}$, for each of the vectors $\overrightarrow{OC}$, $\overrightarrow{AB}$, $\overrightarrow{AD}$ and $\overrightarrow{OD}$.

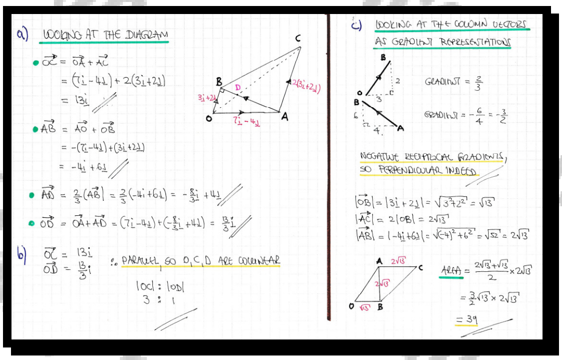
b) Deduce, showing your reasoning, that O, D and C are collinear and state the ratio of $OC:OD$.

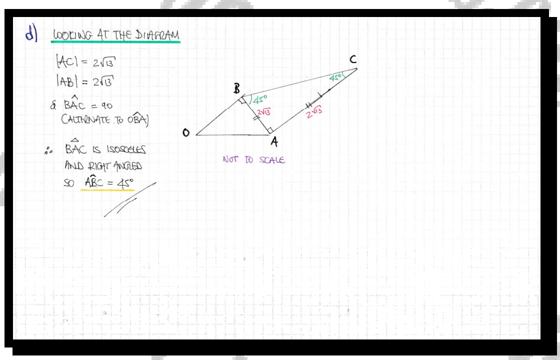
c) Show that $\measuredangle OBA = 90°$ and hence find the area of the trapezium $OBCA$.

d) State the size of the angle $\measuredangle ABC$.

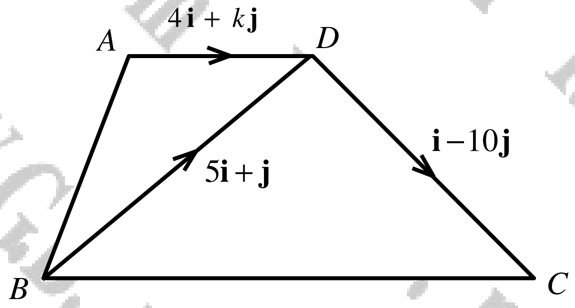
$\boxed{}$, $\boxed{\overrightarrow{OC} = 13\mathbf{i}}$, $\boxed{\overrightarrow{AB} = -4\mathbf{i} + 6\mathbf{j}}$, $\boxed{\overrightarrow{AD} = -\frac{8}{3}\mathbf{i} + 4\mathbf{j}}$, $\boxed{\overrightarrow{OD} = \frac{13}{3}\mathbf{i}}$, $\boxed{OC:OD = 3:1}$,

$\boxed{\text{area} = 39}$, $\boxed{\measuredangle ABC = 45°}$





Question 8 (***+)



The figure above shows a trapezium $ABCD$, where AD is parallel to BC.

The following information is given for this trapezium.

$$\overrightarrow{BD} = 5\mathbf{i} + \mathbf{j}, \quad \overrightarrow{DC} = \mathbf{i} - 10\mathbf{j} \quad \text{and} \quad \overrightarrow{AD} = 4\mathbf{i} + k\mathbf{j},$$

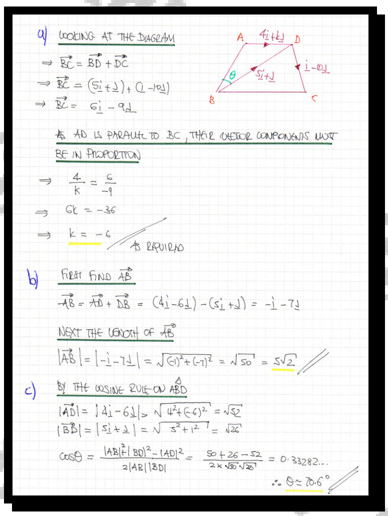
where k is an integer.

a) Use vector algebra to show that $k = -6$.

b) Find the length of $\overrightarrow{AB}$.

c) Calculate the size of the angle ABD.

$\boxed{}$, $\boxed{\left|\overrightarrow{AB}\right| = \sqrt{50} = 5\sqrt{2}}$, $\boxed{\measuredangle ABD \approx 70.6°}$



a) LOOKING AT THE DIAGRAM

$\Rightarrow \overrightarrow{BC} = \overrightarrow{BD} + \overrightarrow{DC}$

$\Rightarrow \overrightarrow{BC} = (5\mathbf{i} + \mathbf{j}) + (\mathbf{i} - 10\mathbf{j})$

$\Rightarrow \overrightarrow{BC} = 6\mathbf{i} - 9\mathbf{j}$

AS AD IS PARALLEL TO BC, THEIR VECTOR COMPONENTS MUST BE IN PROPORTION

$\Rightarrow \frac{4}{k} = \frac{6}{-9}$

$\Rightarrow 6k = -36$

$\Rightarrow k = -6$ // AS REQUIRED

b) FIRST FIND $\overrightarrow{AB}$

$\overrightarrow{AB} = \overrightarrow{AD} + \overrightarrow{DB} = (4\mathbf{i} - 6\mathbf{j}) - (5\mathbf{i} + \mathbf{j}) = -\mathbf{i} - 7\mathbf{j}$

NEXT THE LENGTH OF $\overrightarrow{AB}$

$|\overrightarrow{AB}| = |-\mathbf{i} - 7\mathbf{j}| = \sqrt{(-1)^2 + (-7)^2} = \sqrt{50} = 5\sqrt{2}$ //

c) BY THE COSINE RULE ON $\hat{ABD}$

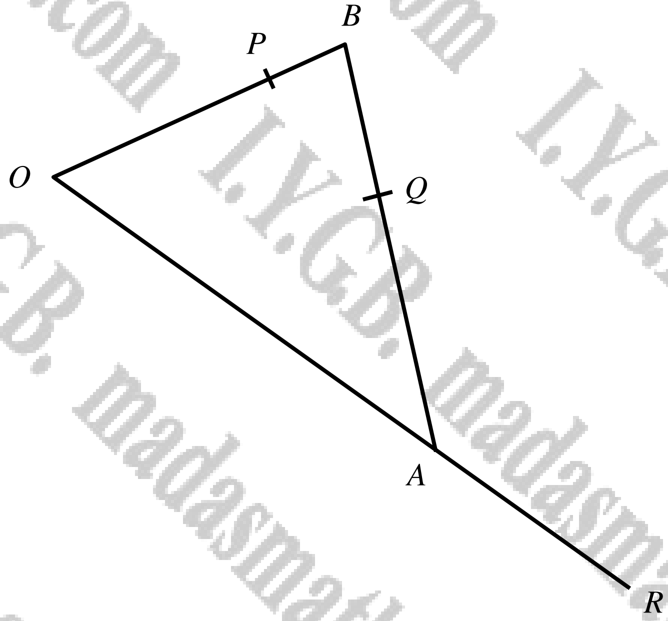
$|\overrightarrow{AD}| = |4\mathbf{i} - 6\mathbf{j}| = \sqrt{4^2 + (-6)^2} = \sqrt{52}$

$|\overrightarrow{BD}| = |5\mathbf{i} + \mathbf{j}| = \sqrt{5^2 + 1^2} = \sqrt{26}$

$\cos\theta = \frac{|AB|^2 + |BD|^2 - |AD|^2}{2|AB||BD|} = \frac{50 + 26 - 52}{2 \times \sqrt{50}\sqrt{26}} = 0.33282\ldots$

$\therefore \theta \simeq 70.6°$ //

Question 9 **(***+)**



The figure above shows a triangle OAB, where O is a fixed origin.

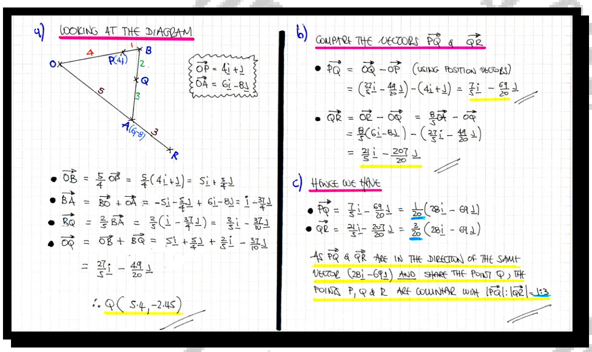
- The point A has coordinates $(6,-8)$.
- The point P, whose coordinates are $(4,1)$, lies on OB so that $OP:PB = 4:1$.
- The point Q lies on AB so that $AQ : QB = 3:2$
- The side OA is extended to the point R so that $OA:AR = 5:3$.

a) Use vector methods to determine the coordinates of Q.

b) Determine expressions, in terms of $\mathbf{i}$ and $\mathbf{j}$, for the vectors $\overrightarrow{PQ}$ and $\overrightarrow{QR}$.

c) Deduce, showing your reasoning, that P, Q and R are collinear and state the ratio of $PQ:QR$.

$\boxed{}$, $\boxed{Q\left(\frac{27}{5},-\frac{49}{20}\right)}$, $\boxed{\overrightarrow{PQ}=\frac{7}{5}\mathbf{i}-\frac{69}{20}\mathbf{j}}$, $\boxed{\overrightarrow{QR}=\frac{21}{5}\mathbf{i}-\frac{207}{20}\mathbf{j}}$, $\boxed{PQ:QR=1:3}$



Question 10 (***+)

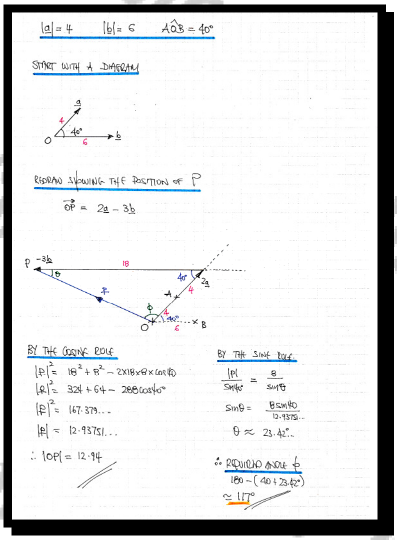
The points A, B and P lie on the x-y plane, where the point O is the origin.

It is further given that

$$|OA| = 4, \quad |OB| = 6 \quad \text{and} \quad \measuredangle AOB = 40°.$$

If $\overrightarrow{OP} = 2\left(\overrightarrow{OA}\right) - 3\left(\overrightarrow{OB}\right)$ determine the distance of P from the origin and the angle between $\overrightarrow{OP}$ and $\overrightarrow{OA}$.

$\boxed{}$, $\boxed{|OP| \approx 12.94}$, $\boxed{\approx 117°}$



Question 11 (****)

The points $A(-1,4)$, $B(2,3)$ and $C(8,1)$ lie on the x-y plane, where O is the origin.

a) Show that A, B and C are collinear.

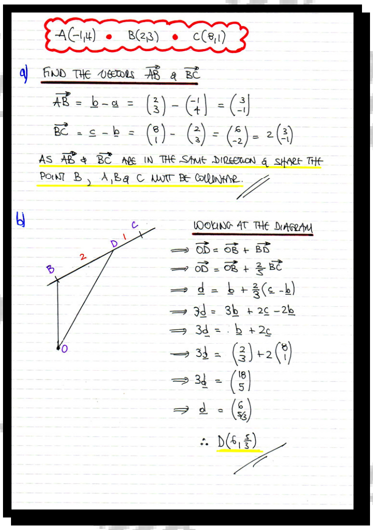
The point D lies on BC so that $\overrightarrow{BD} : \overrightarrow{BC} = 2:3$.

b) Find the coordinates of D.

The straight line OB is extended to the point P, so that $\overrightarrow{AP}$ is parallel to $\overrightarrow{OC}$.

c) Determine the coordinates of P.

$\boxed{}$, $\boxed{D\left(6,\frac{5}{3}\right)}$, $\boxed{P\left(3,\frac{9}{2}\right)}$



A(−1,4) • B(2,3) • C(8,1)

a) FIND THE VECTORS $\overrightarrow{AB}$ & $\overrightarrow{BC}$

$\overrightarrow{AB} = \underline{b} - \underline{a} = \begin{pmatrix}2\\3\end{pmatrix} - \begin{pmatrix}-1\\4\end{pmatrix} = \begin{pmatrix}3\\-1\end{pmatrix}$

$\overrightarrow{BC} = \underline{c} - \underline{b} = \begin{pmatrix}8\\1\end{pmatrix} - \begin{pmatrix}2\\3\end{pmatrix} = \begin{pmatrix}6\\-2\end{pmatrix} = 2\begin{pmatrix}3\\-1\end{pmatrix}$

AS $\overrightarrow{AB}$ & $\overrightarrow{BC}$ ARE IN THE SAME DIRECTION & SHARE THE POINT B, A, B & C MUST BE COLLINEAR.

b)

LOOKING AT THE DIAGRAM

$\Rightarrow \overrightarrow{OD} = \overrightarrow{OB} + \overrightarrow{BD}$

$\Rightarrow \overrightarrow{OD} = \overrightarrow{OB} + \frac{2}{3}\overrightarrow{BC}$

$\Rightarrow \underline{d} = \underline{b} + \frac{2}{3}(\underline{c} - \underline{b})$

$\Rightarrow 3\underline{d} = 3\underline{b} + 2\underline{c} - 2\underline{b}$

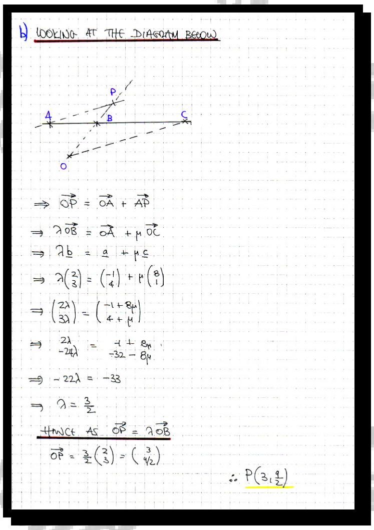
$\Rightarrow 3\underline{d} = \underline{b} + 2\underline{c}$

$\Rightarrow 3\underline{d} = \begin{pmatrix}2\\3\end{pmatrix} + 2\begin{pmatrix}8\\1\end{pmatrix}$

$\Rightarrow 3\underline{d} = \begin{pmatrix}18\\5\end{pmatrix}$

$\Rightarrow \underline{d} = \begin{pmatrix}6\\5/3\end{pmatrix}$

$\therefore D\left(6, \frac{5}{3}\right)$



b) LOOKING AT THE DIAGRAM BELOW

$\Rightarrow \overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{AP}$

$\Rightarrow \lambda\overrightarrow{OB} = \overrightarrow{OA} + \mu\overrightarrow{OC}$

$\Rightarrow \lambda\underline{b} = \underline{a} + \mu\underline{c}$

$\Rightarrow \lambda\begin{pmatrix}2\\3\end{pmatrix} = \begin{pmatrix}-1\\4\end{pmatrix} + \mu\begin{pmatrix}8\\1\end{pmatrix}$

$\Rightarrow \begin{pmatrix}2\lambda\\3\lambda\end{pmatrix} = \begin{pmatrix}-1+8\mu\\4+\mu\end{pmatrix}$

$\Rightarrow \begin{matrix}2\lambda \\ -24\lambda\end{matrix} = \begin{matrix}-1 + 8\mu \\ -32 - 8\mu\end{matrix}$

$\Rightarrow -22\lambda = -33$

$\Rightarrow \lambda = \frac{3}{2}$

HENCE AS $\overrightarrow{OP} = \lambda\overrightarrow{OB}$

$\overrightarrow{OP} = \frac{3}{2}\begin{pmatrix}2\\3\end{pmatrix} = \begin{pmatrix}3\\9/2\end{pmatrix}$

$\therefore P\left(3, \frac{9}{2}\right)$

Question 12 (****+)

Relative to a fixed origin O on a horizontal plane, the points A and B have respective position vectors $3\mathbf{i}-2\mathbf{j}$ and $5\mathbf{i}+4\mathbf{j}$.

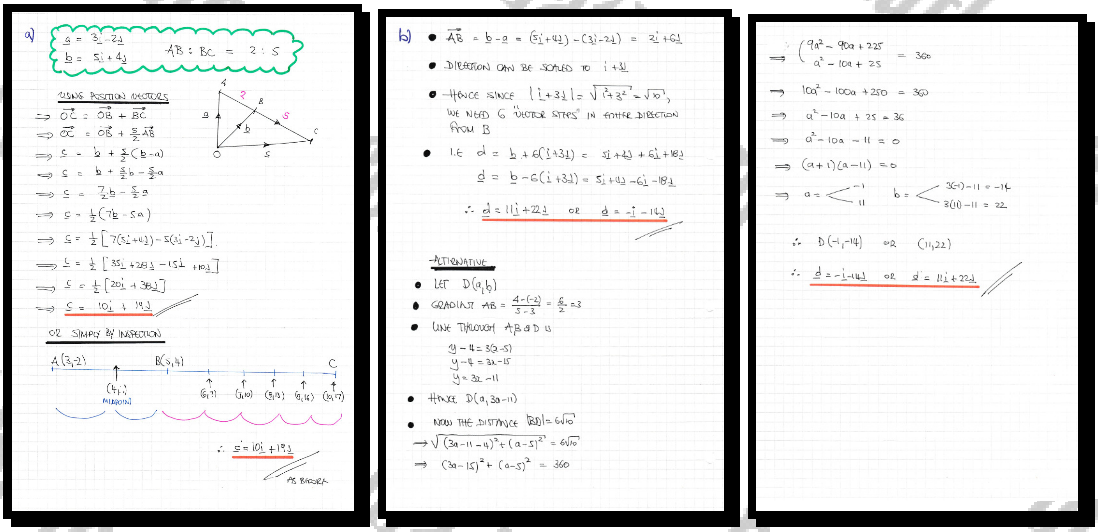
The point C lies on the same plane as A and B so that $\overrightarrow{AB} : \overrightarrow{BC} = 2:5$.

a) Find the position vector of C.

The point D lies on the same plane as A and B so that A, B and D are collinear.

b) Given that $|BD| = 6\sqrt{10}$, determine the possible position vectors of D.

$\boxed{}$, $\boxed{\mathbf{c} = 10\mathbf{i}+19\mathbf{j}}$, $\boxed{\mathbf{d} = -\mathbf{i}-14\mathbf{j} \ \cup \ \mathbf{d} = 11\mathbf{i}+22\mathbf{j}}$



Question 13 (****+)

The four vertices of a quadrilateral $ABCD$ lie on the same plane.

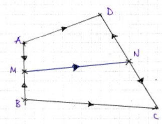
The points M and N are the midpoints of AB and CD, respectively.

Determine the possible values of the scalar constant λ, given further that

$$\left(\lambda^2 - 6\lambda + 10\right)\overrightarrow{MN} = \overrightarrow{AD} + \overrightarrow{BC}.$$

$\boxed{\lambda = 2 \ \cup \ \lambda = 4}$

STARTING WITH A DIAGRAM AND APPROACHING THE PROBLEM AS FOLLOWS...



$\overrightarrow{AD} = \overrightarrow{AM} + \overrightarrow{MN} + \overrightarrow{ND}$

$\overrightarrow{BC} = \overrightarrow{BM} + \overrightarrow{MN} + \overrightarrow{NC}$

ADDING THE EQUATIONS

$\overrightarrow{AD} + \overrightarrow{BC} = \overrightarrow{AM} + \overrightarrow{BM} + 2\overrightarrow{MN} + \overrightarrow{ND} + \overrightarrow{NC}$

BUT AS "M" & "N" ARE MIDPOINTS

$\overrightarrow{AM} + \overrightarrow{BM} = \overrightarrow{AM} - \overrightarrow{MB} = \overrightarrow{AM} - \overrightarrow{AM} =$ "ZERO VECTOR"

AND SIMILARLY

$\overrightarrow{ND} + \overrightarrow{NC} =$ "ZERO VECTOR"

HENCE WE NOW HAVE

$\Rightarrow \overrightarrow{AD} + \overrightarrow{BC} = 2\overrightarrow{MN}$

$\Rightarrow (\lambda^2 - 6\lambda + 10)\overrightarrow{MN} = 2\overrightarrow{MN}$

$\Rightarrow \lambda^2 - 6\lambda + 10 = 2$

$\Rightarrow \lambda^2 - 6\lambda + 8 = 0$

$\Rightarrow (\lambda - 4)(\lambda - 2) = 0$

$\Rightarrow \lambda = \begin{cases} 2 \\ 4 \end{cases}$ //

Question 14 (****+)

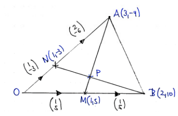
Relative to a fixed origin O, the points A and B have position vectors $3\mathbf{i}-9\mathbf{j}$ and $2\mathbf{i}+10\mathbf{j}$, respectively.

The point M is the midpoint of OB and the point N lies on OA so that $\overrightarrow{OA} = 3\overrightarrow{ON}$.

The point P is the point of intersection of AM and BN.

Determine the ratio $\overrightarrow{NP} : \overrightarrow{PB}$.

$\boxed{\text{SYN-Q}}$, $\boxed{\overrightarrow{NP} : \overrightarrow{PB} = 2 : 3}$



START WITH A DIAGRAM (NOT TO SCALE) AND LABEL IT

- $\overrightarrow{NB} = \overrightarrow{NO} + \overrightarrow{OB} = \binom{-1}{3} + \binom{2}{10} = \binom{1}{13}$

NOW PROCEED AS FOLLOWS

$\overrightarrow{AP} = \overrightarrow{AN} + \overrightarrow{NP} = \overrightarrow{AN} + k(\overrightarrow{NB}) = \binom{-2}{6} + k\binom{1}{13} = \binom{k-2}{13k+6}$

$\overrightarrow{PM} = \overrightarrow{PA} + \overrightarrow{AO} + \overrightarrow{OM} = \binom{2-k}{-13k-6} + \binom{-3}{9} + \binom{1}{5} = \binom{-k}{8-13k}$

BUT A−P−M IS A STRAIGHT LINE, SO $\overrightarrow{AP}$ & $\overrightarrow{PM}$ MUST BE IN PROPORTION

$\Longrightarrow \dfrac{k-2}{13k+6} = \dfrac{-k}{8-13k}$

$\Longrightarrow \dfrac{k-2}{13k+6} = \dfrac{k}{13k-8}$

$\Longrightarrow (k-2)(13k-8) = k(13k+6)$

$\Longrightarrow \cancel{13k^2} - 8k - 26k + 16 = \cancel{13k^2} + 6k$

$\Longrightarrow 16 = 40k$

$\Longrightarrow k = \frac{2}{5}$

∴ REQUIRED RATIO $\overrightarrow{NP} : \overrightarrow{PM} = 2:3$

Question 15 (*****)

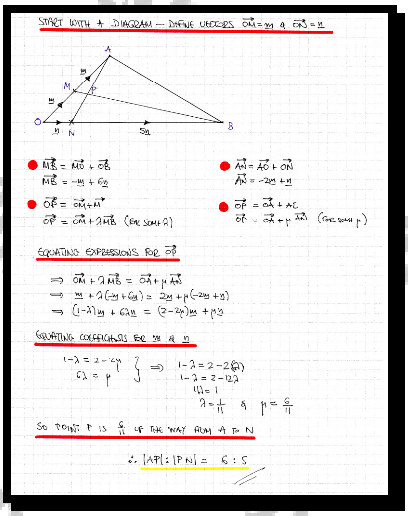
A triangle OAB is given.

The point M is the midpoint of OA.

The point N lies on OB so that $|ON|:|NB| = 1:5$

If the point P is the intersection of the straight lines AN and BM, use vector algebra to find the ratio of $|AP|:|EP|$.

, $\boxed{|AP|:|PN| = 6:5}$



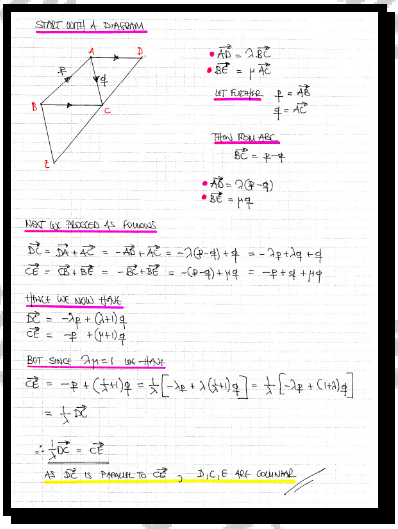
Question 16 (*****)

The triangle ABC is given.

The points D and E are such so that $\overrightarrow{AD} = \lambda \overrightarrow{BC}$ and $\overrightarrow{BE} = \mu \overrightarrow{AC}$, where λ and μ are positive scalar constants.

Given further that $\lambda\mu = 1$, show that D, C and E are collinear.

$\boxed{}$, $\boxed{\text{proof}}$



START WITH A DIAGRAM

- $\overrightarrow{AD} = \lambda \overrightarrow{BC}$
- $\overrightarrow{BE} = \mu \overrightarrow{AC}$

LET FURTHER $\underline{p} = \overrightarrow{AB}$, $\underline{q} = \overrightarrow{AC}$

THEN FROM ABC

$\overrightarrow{BC} = \underline{p} - \underline{q}$

- $\overrightarrow{AD} = \lambda(\underline{p} - \underline{q})$
- $\overrightarrow{BE} = \mu \underline{q}$

NEXT WE PROCEED AS FOLLOWS

$\overrightarrow{DC} = \overrightarrow{DA} + \overrightarrow{AC} = -\overrightarrow{AD} + \overrightarrow{AC} = -\lambda(\underline{p} - \underline{q}) + \underline{q} = -\lambda\underline{p} + \lambda\underline{q} + \underline{q}$

$\overrightarrow{CE} = \overrightarrow{CB} + \overrightarrow{BE} = -\overrightarrow{BC} + \overrightarrow{BE} = -(\underline{p} - \underline{q}) + \mu\underline{q} = -\underline{p} + \underline{q} + \mu\underline{q}$

HENCE WE NOW HAVE

$\overrightarrow{DC} = -\lambda\underline{p} + (\lambda+1)\underline{q}$

$\overrightarrow{CE} = -\underline{p} + (\mu+1)\underline{q}$

BUT SINCE $\lambda\mu = 1$ WE HAVE

$\overrightarrow{CE} = -\underline{p} + \left(\frac{1}{\lambda}+1\right)\underline{q} = \frac{1}{\lambda}\left[-\lambda\underline{p} + \lambda\left(\frac{1}{\lambda}+1\right)\underline{q}\right] = \frac{1}{\lambda}\left[-\lambda\underline{p} + (1+\lambda)\underline{q}\right]$

$= \frac{1}{\lambda}\overrightarrow{DC}$

$\therefore \frac{1}{\lambda}\overrightarrow{DC} = \overrightarrow{CE}$

AS $\overrightarrow{DC}$ IS PARALLEL TO $\overrightarrow{CE}$, D, C, E ARE COLLINEAR

Question 17 (*****)

It is given that

$$\overrightarrow{AP} + 4\overrightarrow{BP} + 3\overrightarrow{PC} = \vec{0}.$$

Show that

$$\overrightarrow{AP} = \frac{1}{2}\left[\overrightarrow{AB} - 3\overrightarrow{BC}\right].$$

$\boxed{}$, $\boxed{\text{proof}}$

TAKE "A" TO BE THE ORIGIN

$$\overrightarrow{AP} + 4\overrightarrow{BP} + 3\overrightarrow{PC} = \vec{0}$$
$$\overrightarrow{AP} + 4(\overrightarrow{BA}+\overrightarrow{AP}) + 3(\overrightarrow{PA}+\overrightarrow{AC}) = \vec{0}$$
$$\overrightarrow{AP} + 4\overrightarrow{BA} + 4\overrightarrow{AP} + 3\overrightarrow{PA} + 3\overrightarrow{AC} = \vec{0}$$
$$\overrightarrow{AP} + 4\overrightarrow{AP} + 3\overrightarrow{PA} = -4\overrightarrow{BA} - 3\overrightarrow{AC}$$
$$\overrightarrow{AP} + 4\overrightarrow{AP} - 3\overrightarrow{AP} = -\overrightarrow{BA} - 3\overrightarrow{BA} - 3\overrightarrow{AC}$$
$$2\overrightarrow{AP} = -\overrightarrow{BA} - 3(\overrightarrow{BA}+\overrightarrow{AC})$$
$$2\overrightarrow{AP} = \overrightarrow{AB} - 3\overrightarrow{BC}$$
$$\overrightarrow{AP} = \frac{1}{2}(\overrightarrow{AB} - 3\overrightarrow{BC})$$

AS REQUIRED

Question 18 (*****)

$$\mathbf{a} = \left(\frac{1}{2}x^2 + y^2 + 3\right)\mathbf{i} + 4\mathbf{j} \quad \text{and} \quad \mathbf{b} = (x+y)\mathbf{i} + 2\mathbf{j}.$$

Determine the value of x and the value of y given that $\mathbf{a}$ and $\mathbf{b}$ are parallel.

$\boxed{\text{SYN}}$, $\boxed{x = 2, \quad y = 1}$

$\underline{a} = \left(\frac{1}{2}x^2+y^2+3\right)\underline{i} + 4\underline{j}$ $\quad$ $\underline{b} = (x+y)\underline{i} + 2\underline{j}$

AS THE VECTORS ARE PARALLEL

$$\Rightarrow \frac{\frac{1}{2}x^2+y^2+3}{x+y} = \frac{4}{2}$$
$$\Rightarrow x^2 + 2y^2 + 6 = 4x + 4y$$
$$\Rightarrow x^2 - 4x + 6 + 2y^2 - 4y = 0$$
$$\Rightarrow (x-2)^2 - 4 + 6 + 2(y^2-2y) = 0$$
$$\Rightarrow (x-2)^2 + 2 + 2\left[(y-1)^2 - 1\right] = 0$$
$$\Rightarrow (x-2)^2 + \cancel{2} + 2(y-1)^2 - \cancel{2} = 0$$
$$\Rightarrow (x-2)^2 + 2(y-1)^2 = 0$$
$$\therefore \ x = 2 \ \& \ y = 1$$

Introducing Elementary 3D Vectors

Question 1 (**)

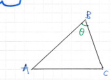
Relative to a fixed origin O, the points A, B and C have respective position vectors

$$-3\mathbf{i}+\mathbf{k}, \quad -\mathbf{i}+4\mathbf{j}+\mathbf{k} \quad \text{and} \quad 5\mathbf{i}+4\mathbf{j}.$$

Calculate the size of the angle ABC and hence find the area of the triangle ABC.

[illegible], $\boxed{\measuredangle ABC \approx 116°}$, $\boxed{\text{area} \approx 12.2}$

$A(-3,0,1) \quad B(-1,4,1) \quad C(5,4,0)$



- $|\overrightarrow{AB}| = |\underline{b}-\underline{a}| = |(-1,4,1)-(-3,0,1)|$
  $= |2,4,0| = \sqrt{4+16+0}$
  $= \sqrt{20}$
- $|\overrightarrow{BC}| = |\underline{c}-\underline{b}| = |(5,4,0)-(-1,4,1)| = |6,0,-1| = \sqrt{36+0+1} = \sqrt{37}$
- $|\overrightarrow{CA}| = |\underline{a}-\underline{c}| = |(-3,0,1)-(5,4,0)| = |-8,-4,1| = \sqrt{64+16+1} = 9$

BY THE COSINE RULE

$\Longrightarrow |AC|^2 = |AB|^2 + |BC|^2 - 2|AB||BC|\cos\theta$

$\Longrightarrow 9^2 = 20 + 37 - 2\sqrt{20}\sqrt{37}\cos\theta$

$\Longrightarrow 2\sqrt{20}\sqrt{37}\cos\theta = 20+37-81$

$\Longrightarrow \cos\theta = 0.441128\ldots$

$\Longrightarrow \theta \simeq 116°$

FINALLY THE AREA IS GIVEN BY

$\frac{1}{2}|AB||BC|\sin\theta = \frac{1}{2}\sqrt{20} \times \sqrt{37}\sin(116.^\circ..) \simeq 12.2$

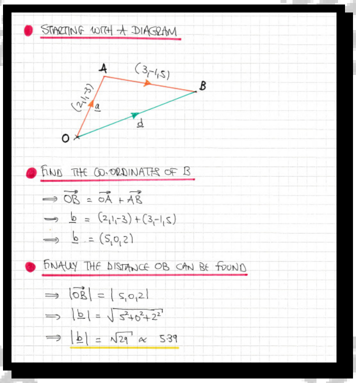
Question 2 **(**)**

Relative to a fixed origin O, the point A has coordinates $(2,1,-3)$.

The point B is such so that $\overrightarrow{AB} = 3\mathbf{i} - \mathbf{j} + 5\mathbf{k}$.

Determine the distance of B from O.

$\boxed{}$, $\boxed{|OB| = \sqrt{29}}$



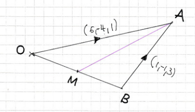
Question 3 (**)

Relative to a fixed origin O, the point A has coordinates $(6,-4,1)$.

The point B is such so that $\overrightarrow{BA} = \mathbf{i} - \mathbf{j} + 3\mathbf{k}$.

If the point M is the midpoint of OB, show that $\left|\overrightarrow{AM}\right| = k\sqrt{10}$, where k is a rational constant to be found.

$\boxed{}$, $\boxed{k = \frac{3}{2}}$


PUT THE INFORMATION INTO A DIAGRAM

FIND THE POSITION VECTOR (CO-ORDINATES) OF B

$\Rightarrow \overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AB}$

$\Rightarrow \overrightarrow{OB} = (6,-4,1) - (1,-1,3)$

$\Rightarrow \overrightarrow{OB} = (5,-3,-2)$ $\quad \therefore B(5,-3,-2)$

NEXT THE COORDINATES OF M

$\Rightarrow \overrightarrow{OM} = \frac{1}{2}\overrightarrow{OB} = \frac{1}{2}(5,-3,-2) = \left(\frac{5}{2},-\frac{3}{2},-1\right)$ $\quad \therefore M\left(\frac{5}{2},-\frac{3}{2},-1\right)$

THEN FIND THE VECTOR $\overrightarrow{AM}$

$\Rightarrow \overrightarrow{AM} = \overrightarrow{AO} + \overrightarrow{OM} = -(6,-4,1) + \left(\frac{5}{2},-\frac{3}{2},-1\right)$

$\Rightarrow \overrightarrow{AM} = \left(-\frac{7}{2},\frac{5}{2},-2\right)$

FINALLY THE DISTANCE AM

$\Rightarrow \left|\overrightarrow{AM}\right| = \left|-\frac{7}{2},\frac{5}{2},-2\right| = \sqrt{\frac{49}{4}+\frac{25}{4}+4} = \sqrt{\frac{90}{4}} = \frac{3}{2}\sqrt{10}$

$\therefore k = \frac{3}{2}$

Question 4 **(**+)**

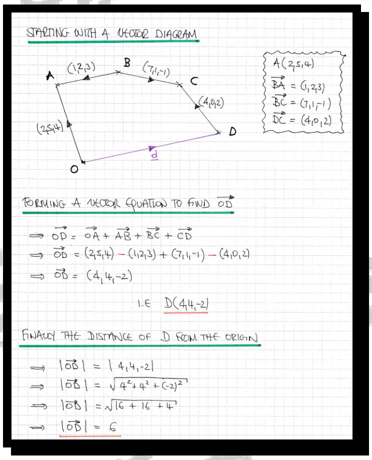
Relative to a fixed origin O, the point A has coordinates $(2,5,4)$.

The points B, C and D are such so that

$$\overrightarrow{BA} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}, \quad \overrightarrow{BC} = 7\mathbf{i} + \mathbf{j} - \mathbf{k} \quad \text{and} \quad \overrightarrow{DC} = 4\mathbf{i} + 2\mathbf{k}.$$

Determine the distance of D from the origin.

$\boxed{}$, $\boxed{|OD| = 6}$


STARTING WITH A VECTOR DIAGRAM

$A(2,5,4)$
$\overrightarrow{BA} = (1,2,3)$
$\overrightarrow{BC} = (7,1,-1)$
$\overrightarrow{DC} = (4,0,2)$

FORMING A VECTOR EQUATION TO FIND $\overrightarrow{OD}$

$\Rightarrow \overrightarrow{OD} = \overrightarrow{OA} + \overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD}$
$\Rightarrow \overrightarrow{OD} = (2,5,4) - (1,2,3) + (7,1,-1) - (4,0,2)$
$\Rightarrow \overrightarrow{OD} = (4,4,-2)$

I.E $D(4,4,-2)$

FINALLY THE DISTANCE OF D FROM THE ORIGIN

$\Rightarrow |\overrightarrow{OD}| = |4,4,-2|$
$\Rightarrow |\overrightarrow{OD}| = \sqrt{4^2+4^2+(-2)^2}$
$\Rightarrow |\overrightarrow{OD}| = \sqrt{16+16+4}$
$\Rightarrow |\overrightarrow{OD}| = 6$

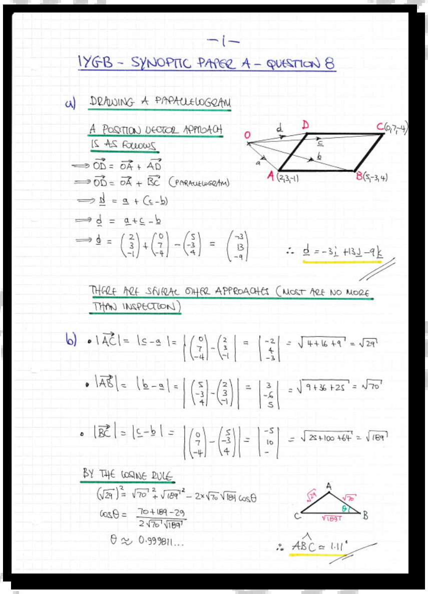
Question 5 (**+)

Relative to a fixed origin O, the points A, B and C have respective position vectors $2\mathbf{i}+3\mathbf{j}-\mathbf{k}$, $5\mathbf{i}-3\mathbf{j}+4\mathbf{k}$ and $7\mathbf{j}-4\mathbf{k}$.

a) Given that $ABCD$ is a parallelogram, determine the position vector of D.

b) Determine the distance AC and hence calculate the angle ABC.

$\boxed{}$, $\boxed{\mathbf{d}=-3\mathbf{i}+13\mathbf{j}-9\mathbf{k}}$, $\boxed{|AC|=\sqrt{29}}$, $\boxed{\measuredangle ABC \approx 1.11°}$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IYGB – SYNOPTIC PAPER A – QUESTION 8

a) DRAWING A PARALLELOGRAM

A POSITION VECTOR APPROACH IS AS FOLLOWS

$\Longrightarrow \vec{OD} = \vec{OA} + \vec{AD}$

$\Longrightarrow \vec{OD} = \vec{OA} + \vec{BC}$ (PARALLELOGRAM)

$\Longrightarrow \underline{d} = \underline{a} + (\underline{c} - \underline{b})$

$\Longrightarrow \underline{d} = \underline{a} + \underline{c} - \underline{b}$

$\Longrightarrow \underline{d} = \begin{pmatrix}2\\3\\-1\end{pmatrix} + \begin{pmatrix}0\\7\\-4\end{pmatrix} - \begin{pmatrix}5\\-3\\4\end{pmatrix} = \begin{pmatrix}-3\\13\\-9\end{pmatrix}$

$\therefore \underline{d} = -3\underline{i} + 13\underline{j} - 9\underline{k}$

THERE ARE SEVERAL OTHER APPROACHES (MOST ARE NO MORE THAN INSPECTION)

b) • $|\vec{AC}| = |\underline{c} - \underline{a}| = \left|\begin{pmatrix}0\\7\\-4\end{pmatrix} - \begin{pmatrix}2\\3\\-1\end{pmatrix}\right| = \left|\begin{matrix}-2\\4\\-3\end{matrix}\right| = \sqrt{4+16+9} = \sqrt{29}$

• $|\vec{AB}| = |\underline{b} - \underline{a}| = \left|\begin{pmatrix}5\\-3\\4\end{pmatrix} - \begin{pmatrix}2\\3\\-1\end{pmatrix}\right| = \left|\begin{matrix}3\\-6\\5\end{matrix}\right| = \sqrt{9+36+25} = \sqrt{70}$

• $|\vec{BC}| = |\underline{c} - \underline{b}| = \left|\begin{pmatrix}0\\7\\-4\end{pmatrix} - \begin{pmatrix}5\\-3\\4\end{pmatrix}\right| = \left|\begin{matrix}-5\\10\\-\end{matrix}\right| = \sqrt{25+100+64} = \sqrt{189}$

BY THE COSINE RULE

$(\sqrt{29})^2 = \sqrt{70}^2 + \sqrt{189}^2 - 2\times\sqrt{70}\sqrt{189}\cos\theta$

$\cos\theta = \frac{70+189-29}{2\sqrt{70}\sqrt{189}}$

$\theta \approx 0.999811\ldots$

$\therefore \hat{ABC} \approx 1.11°$

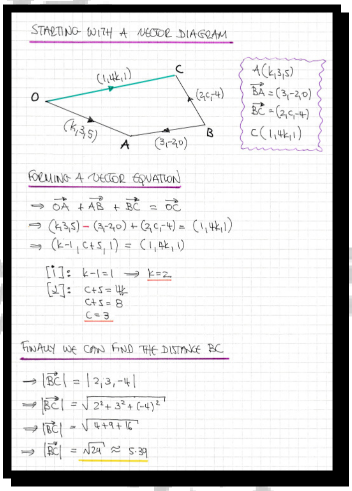
Question 6 (***)

Relative to a fixed origin O, the point A has coordinates $(k,3,5)$, where k is a scalar constant.

The points B and C are such so that $\overrightarrow{BA} = 3\mathbf{i} - 2\mathbf{j}$ and $\overrightarrow{BC} = 2\mathbf{i} + c\mathbf{j} - 4\mathbf{k}$, where c is a scalar constant.

If the coordinates of C are $(1,4k,1)$, determine the distance BC.

$\boxed{}$, $\boxed{|BC| = \sqrt{29}}$


STARTING WITH A VECTOR DIAGRAM

$A(k,3,5)$
$\overrightarrow{BA} = (3,-2,0)$
$\overrightarrow{BC} = (2,c,-4)$
$C(1,4k,1)$

FORMING A VECTOR EQUATION

$\Rightarrow \overrightarrow{OA} + \overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{OC}$

$\Rightarrow (k,3,5) - (3,-2,0) + (2,c,-4) = (1,4k,1)$

$\Rightarrow (k-1, c+5, 1) = (1,4k,1)$

$[\mathbf{i}]$: $k-1=1 \Rightarrow k=2$

$[\mathbf{j}]$: $c+5=4k$
$c+5=8$
$c=3$

FINALLY WE CAN FIND THE DISTANCE BC

$\Rightarrow |\overrightarrow{BC}| = |2,3,-4|$

$\Rightarrow |\overrightarrow{BC}| = \sqrt{2^2+3^2+(-4)^2}$

$\Rightarrow |\overrightarrow{BC}| = \sqrt{4+9+16}$

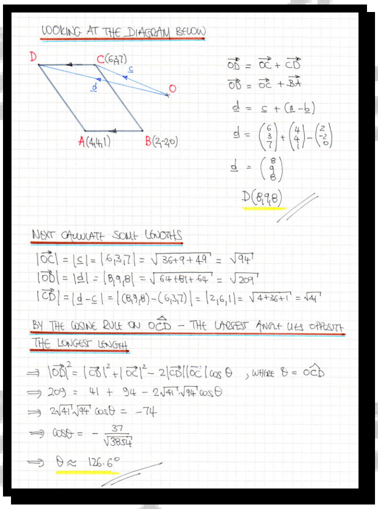
$\Rightarrow |\overrightarrow{BC}| = \sqrt{29} \approx 5.39$

Question 7 (***)

The points $A(4,4,1)$, $B(2,-2,0)$ and $C(6,3,7)$ are referred relative to a fixed origin O.

If A, B, C and the point D form the parallelogram $ABCD$, use vector algebra to find the coordinates of D and hence calculate the angle OCD.

$\boxed{}$, $\boxed{D(8,9,8)}$, $\boxed{\measuredangle OCD \approx 126.6°}$



Question 8 (***)

$OABC$ is a square, where O is the origin, and the vertices A and C have respective position vectors $2\mathbf{i}+4\mathbf{j}+4\mathbf{k}$ and $4\mathbf{i}+2\mathbf{j}-4\mathbf{k}$.

The point M is the midpoint of AB and the point N is the midpoint of MC.

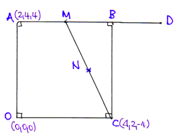
The point D is such so that $\overrightarrow{AD}=\frac{3}{2}\overrightarrow{AB}$.

a) Find the position vectors of the points B, D and N.

b) Deduce, showing your reasoning, that O, N and D are collinear.

$\boxed{}$, $\boxed{\overrightarrow{OB}=6\mathbf{i}+6\mathbf{j}}$, $\boxed{\overrightarrow{OD}=8\mathbf{i}+7\mathbf{j}-2\mathbf{k}}$, $\boxed{\overrightarrow{ON}=4\mathbf{i}+\frac{7}{2}\mathbf{j}-\mathbf{k}}$

a) STARTING WITH A DIAGRAM FOR THE SQUARE



- $\overrightarrow{OB}=\overrightarrow{OA}+\overrightarrow{AB}$

  $\overrightarrow{OB}=\overrightarrow{OA}+\overrightarrow{OC}$

  $\overrightarrow{OB}=(2,4,4)+(4,2,-4)$

  $\overrightarrow{OB}=(6,6,0)$

  $\therefore\ \underline{b}=6\underline{i}+6\underline{j}$

- $\overrightarrow{OD}=\overrightarrow{OA}+\overrightarrow{AD}$

  $\overrightarrow{OD}=\overrightarrow{OA}+\frac{3}{2}\overrightarrow{AB}$

  $\overrightarrow{OD}=\overrightarrow{OA}+\frac{3}{2}\overrightarrow{OC}$

  $\overrightarrow{OD}=(2,4,4)+\frac{3}{2}(4,2,-4)$

  $\overrightarrow{OD}=(8,7,-2)$

  $\therefore\ \underline{d}=8\underline{i}+7\underline{j}-2\underline{k}$

- $\overrightarrow{ON}=\overrightarrow{OC}+\frac{1}{2}\overrightarrow{CM}$

  $\overrightarrow{ON}=\overrightarrow{OC}+\frac{1}{2}[\overrightarrow{CO}+\overrightarrow{OA}+\frac{1}{2}\overrightarrow{AB}]$

  $\overrightarrow{ON}=\overrightarrow{OC}+\frac{1}{2}\overrightarrow{CO}+\frac{1}{2}\overrightarrow{OA}+\frac{1}{4}\overrightarrow{AB}$

  $\overrightarrow{ON}=(4,2,-4)+\frac{1}{2}(-4,-2,+4)+\frac{1}{2}(2,4,4)+\frac{1}{4}\overrightarrow{OC}$

  $\overrightarrow{ON}=(4,2,-4)+(-2,-1,2)+(1,2,2)+\frac{1}{4}(4,2,-4)$

  $\overrightarrow{ON}=(3,3,0)+(1,\frac{1}{2},-1)$

  $\overrightarrow{ON}=(4,\frac{7}{2},-1)$

  $\therefore\ \underline{n}=4\underline{i}+\frac{7}{2}\underline{j}-\underline{k}$

b) COMPARING VECTORS FOUND EARLIER

$\overrightarrow{OD}=8\underline{i}+7\underline{j}-2\underline{k}$

$\overrightarrow{OD}=2(4\underline{i}+\frac{7}{2}\underline{j}-\underline{k})$

$\overrightarrow{OD}=2\overrightarrow{ON}$

$\therefore$ O, N, D ARE COLLINEAR

Question 9 (***)

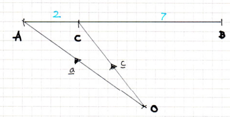
The points $A(-2,-10,-17)$ and $B(25,-1,19)$ are referred relative to a fixed origin O.

The point C is such so that ACB forms a straight line.

Given further that $\dfrac{|\overrightarrow{AC}|}{|\overrightarrow{CB}|}=\dfrac{2}{7}$, determine the coordinates of C.

$\boxed{}$, $\boxed{C(4,-8,-9)}$

$A(-2,-10,-17)$ $B(25,-1,19)$ $|\overrightarrow{AC}|:|\overrightarrow{CB}|$
$2 : 7$



LOOKING AT THE DIAGRAM

$\Rightarrow \overrightarrow{OC} = \overrightarrow{OA} + \overrightarrow{AC}$

$\Rightarrow \overrightarrow{OC} = \overrightarrow{OA} + \frac{2}{9}\overrightarrow{AB}$

$\Rightarrow \underline{c} = \underline{a} + \frac{2}{9}(\underline{b}-\underline{a})$

$\Rightarrow \underline{c} = \underline{a} + \frac{2}{9}\underline{b} - \frac{2}{9}\underline{a}$

$\Rightarrow \underline{c} = \frac{7}{9}\underline{a} + \frac{2}{9}\underline{b}$

$\Rightarrow \underline{c} = \frac{1}{9}\left[7\underline{a} + 2\underline{b}\right]$

$\Rightarrow \underline{c} = \frac{1}{9}\left[7(-2,-10,-17) + 2(25,-1,19)\right]$

$\Rightarrow \underline{c} = \frac{1}{9}(36,-72,-81)$

$\Rightarrow \underline{c} = (4,-8,-9)$ $\quad\therefore C(4,-8,-9)$

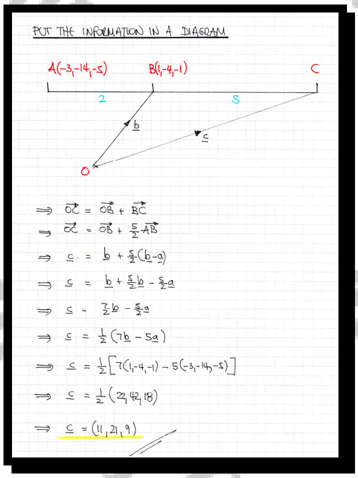
Question 10 (***)

The points $A(-3,-14,-5)$ and $B(1,-4,-1)$ are referred relative to a fixed origin O.

The point C is such so that ABC forms a straight line.

Given further that $\dfrac{|\overrightarrow{AB}|}{|\overrightarrow{BC}|} = \dfrac{2}{5}$, determine the coordinates of C.

$\boxed{}$, $\boxed{C(11,21,9)}$



Question 11 (***)

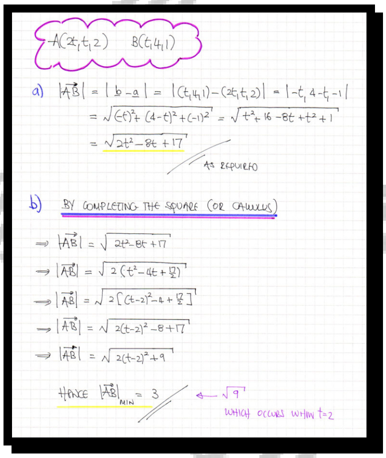
The variable points $A(2t,t,2)$ and $B(t,4,1)$, where t is a scalar variable, are referred relative to a fixed origin O.

a) Show that

$$\left|\overrightarrow{AB}\right| = \sqrt{2t^2 - 8t + 17}.$$

b) Hence find the shortest distance between A and B, as t varies.

$\boxed{}$, $\boxed{\left|\overrightarrow{AB}\right|_{\min} = 3}$



$A(2t,t,2)$ $B(t,4,1)$

a) $|\overrightarrow{AB}| = |\underline{b} - \underline{a}| = |(t,4,1)-(2t,t,2)| = |-t, 4-t, -1|$

$= \sqrt{(-t)^2 + (4-t)^2 + (-1)^2} = \sqrt{t^2 + 16 - 8t + t^2 + 1}$

$= \sqrt{2t^2 - 8t + 17}$

AS REQUIRED

b) BY COMPLETING THE SQUARE (OR CALCULUS)

$\Rightarrow |\overrightarrow{AB}| = \sqrt{2t^2 - 8t + 17}$

$\Rightarrow |\overrightarrow{AB}| = \sqrt{2\left(t^2 - 4t + \frac{17}{2}\right)}$

$\Rightarrow |\overrightarrow{AB}| = \sqrt{2\left[(t-2)^2 - 4 + \frac{17}{2}\right]}$

$\Rightarrow |\overrightarrow{AB}| = \sqrt{2(t-2)^2 - 8 + 17}$

$\Rightarrow |\overrightarrow{AB}| = \sqrt{2(t-2)^2 + 9}$

HENCE $|\overrightarrow{AB}|_{\text{MIN}} = 3$ $\leftarrow \sqrt{9}$

WHICH OCCURS WHEN $t=2$

Question 12 (***)

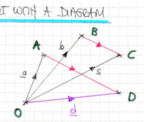
The points $A(5,-1,0)$, $B(3,5,-4)$, $C(12,2,8)$ are referred relative to a fixed origin O.

The point D is such so that $\overrightarrow{AD} = 2\overrightarrow{BC}$.

Determine the distance CD.

$\boxed{}$, $\boxed{|CD| = \sqrt{458} \approx 21.40}$

START WITH A DIAGRAM



$A(5,-1,0)$
$B(3,5,-4)$
$C(12,2,8)$
$\overrightarrow{AD} = 2\overrightarrow{BC}$

FORMING A VECTOR EQUATION

$\Rightarrow \overrightarrow{OD} = \overrightarrow{OA} + \overrightarrow{AD}$

$\Rightarrow \overrightarrow{OD} = \overrightarrow{OA} + 2(\overrightarrow{BC})$

$\Rightarrow \underline{d} = \underline{a} + 2(\underline{c} - \underline{b})$

$\Rightarrow \underline{d} = \underline{a} + 2\underline{c} - 2\underline{b}$

$\Rightarrow \underline{d} = (5,-1,0) + 2(12,2,8) - 2(3,5,-4)$

$\Rightarrow \underline{d} = (23,-7,24)$

FINALLY THE DISTANCE CD CAN BE FOUND

$\Rightarrow |\overrightarrow{CD}| = |\underline{d} - \underline{c}|$

$= |(23,-7,24) - (12,2,8)|$

$= |11,-9,16|$

$= \sqrt{121 + 81 + 256}$

$= \sqrt{458}$

$\simeq 21.40$

Question 13 (***)

The points $A(t,3,2)$ and $B(5,2,2t)$, where t is a scalar constant, are referred relative to a fixed origin O.

Given that $|\overrightarrow{AB}| = \sqrt{21}$, find the possible values of t.

$\boxed{}$, $\boxed{t = 3,\ t = \frac{3}{5}}$

$A(t,3,2) \quad B(5,2,2t) \quad |\overrightarrow{AB}| = \sqrt{21}$

$\Rightarrow |\overrightarrow{AB}| = \sqrt{21}$ (GIVEN)

$\Rightarrow |\underline{b} - \underline{a}| = \sqrt{21}$

$\Rightarrow |(5,2,2t)-(t,3,2)| = \sqrt{21}$

$\Rightarrow |5-t, -1, 2t-2| = \sqrt{21}$

$\Rightarrow \sqrt{(5-t)^2+(-1)^2+(2t-2)^2} = \sqrt{21}$ (DEFINITION OF THE MODULUS OF A VECTOR)

$\Rightarrow \sqrt{25-10t+t^2+1+4t^2-8t+4} = \sqrt{21}$

$\Rightarrow \sqrt{5t^2-18t+30} = \sqrt{21}$

$\Rightarrow 5t^2-18t+30 = 21$

$\Rightarrow 5t^2-18t+9 = 0$

$\Rightarrow (5t-3)(t-3) = 0$

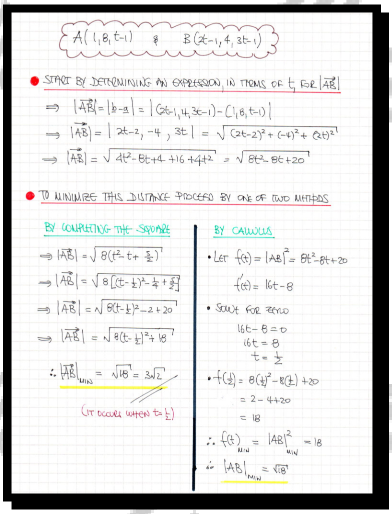
$\Rightarrow t = \frac{3}{5},\ 3$

Question 14 (***+)

The variable points $A(1,8,t-1)$ and $B(2t-1,4,3t-1)$, where t is a scalar variable, are referred relative to a fixed origin O.

Find the shortest distance between A and B, as t varies.

$\boxed{}$, $\boxed{\left|\overrightarrow{AB}\right|_{\min} = \sqrt{18}}$



$A(1,8,t-1)$ & $B(2t-1,4,3t-1)$

- START BY DETERMINING AN EXPRESSION, IN TERMS OF t, FOR $|\overrightarrow{AB}|$

$\Rightarrow |\overrightarrow{AB}| = |\underline{b}-\underline{a}| = |(2t-1,4,3t-1)-(1,8,t-1)|$

$\Rightarrow |\overrightarrow{AB}| = |2t-2, -4, 3t| = \sqrt{(2t-2)^2+(-4)^2+(2t)^2}$

$\Rightarrow |\overrightarrow{AB}| = \sqrt{4t^2-8t+4+16+4t^2} = \sqrt{8t^2-8t+20}$

- TO MINIMIZE THIS DISTANCE PROCEED BY ONE OF TWO METHODS

BY COMPLETING THE SQUARE

$\Rightarrow |\overrightarrow{AB}| = \sqrt{8(t^2-t+\frac{5}{2})}$

$\Rightarrow |\overrightarrow{AB}| = \sqrt{8[(t-\frac{1}{2})^2-\frac{1}{4}+\frac{5}{2}]}$

$\Rightarrow |\overrightarrow{AB}| = \sqrt{8(t-\frac{1}{2})^2-2+20}$

$\Rightarrow |\overrightarrow{AB}| = \sqrt{8(t-\frac{1}{2})^2+18}$

$\therefore |\overrightarrow{AB}|_{\text{MIN}} = \sqrt{18} = 3\sqrt{2}$

(IT OCCURS WHEN $t=\frac{1}{2}$)

BY CALCULUS

- LET $f(t) = |AB|^2 = 8t^2-8t+20$

$f'(t) = 16t-8$

- SOLVE FOR ZERO

$16t-8=0$

$16t=8$

$t=\frac{1}{2}$

- $f(\frac{1}{2}) = 8(\frac{1}{2})^2-8(\frac{1}{2})+20$

$= 2-4+20$

$= 18$

$\therefore f(t)_{\text{MIN}} = |AB|^2_{\text{MIN}} = 18$

$\therefore |AB|_{\text{MIN}} = \sqrt{18}$

Question 15 (***+)

The points $A(1,1,2)$, $B(2,1,5)$, $C(4,0,1)$ and D form the parallelogram $ABCD$, where the above coordinates are measured relative to a fixed origin.

a) Find the coordinates of D.

The points E, B and D are collinear, so that B is the midpoint of ED.

b) Determine the coordinates of E.

The point F is such so that $ABEF$ is also a parallelogram.

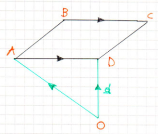
c) Find the coordinates of F.

d) Show that B is the midpoint of FC.

e) Prove that $ADBF$ is another parallelogram.

$\boxed{}$, $\boxed{D(3,0,-2)}$, $\boxed{E(1,2,12)}$, $\boxed{F(0,2,9)}$

a) LOOKING AT THE DIAGRAM



$\Rightarrow \overrightarrow{OD} = \overrightarrow{OA} + \overrightarrow{AD}$) PARALLEL
$\Rightarrow \overrightarrow{OD} = \overrightarrow{OA} + \overrightarrow{BC}$
$\Rightarrow \underline{d} = \underline{a} + (\underline{c} - \underline{b})$
$\Rightarrow \underline{d} = \underline{a} + \underline{c} - \underline{b}$
$\Rightarrow \underline{d} = (1,1,2)+(4,0,1)-(2,1,5)$
$\Rightarrow \underline{d} = (3,0,-2)$

$\therefore D(3,0,-2)$

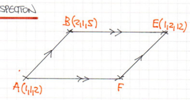
A(1,1,2)
B(2,1,5)
C(4,0,1)

b) E(1,2,12)   B(2,1,5)   D(3,0,-2)

BY SEQUENCING / INSPECTION / MIDPOINT FORMULA

x: 1 $\xleftarrow{-1}$ 2 $\xleftarrow{-1}$ 3
y: 2 $\xleftarrow{+1}$ 1 $\xleftarrow{+1}$ 0
z: 12 $\xleftarrow{+7}$ 5 $\xleftarrow{+7}$ −2
E   B   D

c) AS IN PART (a) OR INSPECTION



(B) to (E)
$2 \xrightarrow{-1} 1$
$1 \xrightarrow{+1} 2$
$5 \xrightarrow{+7} 12$

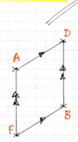
(A) to (F)
$1 \xrightarrow{-1} 0$
$1 \xrightarrow{+1} 2$
$2 \xrightarrow{+7} 9$

$\therefore F(0,2,9)$

d) F(0,2,9)   C(4,0,1)   B(2,1,5)

MIDPOINT OF FC $= \left(\frac{0+4}{2}, \frac{2+0}{2}, \frac{9+1}{2}\right) = (2,1,5)$ WHICH IS B

e)



$\overrightarrow{AD} = \underline{d} - \underline{a} = (3,0,-2)-(1,1,2) = (2,-1,-4)$
$\overrightarrow{FB} = \underline{b} - \underline{f} = (2,1,5)-(0,2,9) = (2,-1,-4)$
$\overrightarrow{FA} = \underline{a} - \underline{f} = (1,1,2)-(0,2,9) = (1,-1,-7)$
$\overrightarrow{BD} = \underline{d} - \underline{b} = (3,0,-2)-(2,1,5) = (1,-1,-7)$

REFERRING TO THE CALCULATIONS & THE DIAGRAM

OPPOSITE SIDES ARE PARALLEL (AND EQUAL)

INDEED ADBF IS A PARALLELOGRAM

Question 16 (***+)

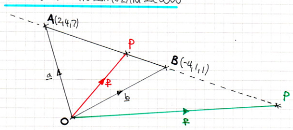
With respect to a fixed origin, the points A and B have position vectors $2\mathbf{i}+4\mathbf{j}+7\mathbf{k}$ and $-4\mathbf{i}+\mathbf{j}+\mathbf{k}$, respectively.

The point P lies on the straight line through A and B.

Find the possible position vectors of P if $\left|\overrightarrow{AP}\right| = 2\left|\overrightarrow{PB}\right|$.

$\boxed{\overrightarrow{OP} = \mathbf{p} = -2\mathbf{i}+2\mathbf{j}+3\mathbf{k}}$, $\boxed{\overrightarrow{OP} = \mathbf{p} = -10\mathbf{i}-2\mathbf{j}-5\mathbf{k}}$

LOOKING AT THE DIAGRAM BELOW



$\overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{AP}$

$\Rightarrow \overrightarrow{OP} = \overrightarrow{OA} + \frac{2}{3}\overrightarrow{AB}$

$\Rightarrow \underline{p} = \underline{a} + \frac{2}{3}(\underline{b}-\underline{a})$

$\Rightarrow \underline{p} = (2,4,7) + \frac{2}{3}[(-4,1,1)-(2,4,7)]$

$\Rightarrow \underline{p} = (2,4,7) + \frac{2}{3}(-6,-3,-6)$

$\Rightarrow \underline{p} = (2,4,7) + (-4,-2,-4)$

$\Rightarrow \underline{p} = (-2,2,3)$

$\therefore \overrightarrow{OP} = -2\underline{i} + 2\underline{j} + 3\underline{k}$

$\overrightarrow{OP} = \overrightarrow{OB} + \overrightarrow{BP}$

$\Rightarrow \overrightarrow{OP} = \overrightarrow{OB} + \overrightarrow{AB}$

$\Rightarrow \underline{p} = \underline{b} + (\underline{b}-\underline{a})$

$\Rightarrow \underline{p} = 2\underline{b} - \underline{a}$

$\Rightarrow \underline{p} = 2(-4,1,1) - (2,4,7)$

$\Rightarrow \underline{p} = (-10,-2,-5)$

$\therefore \overrightarrow{OP} = -10\underline{i} - 2\underline{j} - 5\underline{k}$

Question 17 (***+)

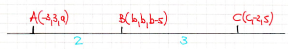
The points $A(-3,3,a)$, $B(b,b,b-5)$ and $C(c,-2,5)$, where a, b and c are scalar constants, are referred relative to a fixed origin O.

Ii is further given that A, B and C are collinear and the ratio $|\overrightarrow{AB}| : |\overrightarrow{BC}| = 2:3$.

Use vector algebra to find the value of a, the value of b and the value of c.

$\boxed{}$, $\boxed{[a,b,c]=[-10,1,7]}$

PUTTING THE INFORMATION IN A DIAGRAM



"CALCULATE" THE VECTORS $\overrightarrow{AB}$ & $\overrightarrow{BC}$

$\overrightarrow{AB} = \underline{b} - \underline{a} = (b,b,b-5)-(-3,3,a) = (b+3,\ b-3,\ b-a-5)$

$\overrightarrow{BC} = \underline{c} - \underline{b} = (c,-2,5) - (b,b,b-5) = (c-b,-2-b,10-b)$

LOOKING AT $\underline{j}$

$\frac{b-3}{-2-b} = \frac{2}{3} \implies 3b-9 = -4-2b$
$\implies 5b = 5$
$\implies b=1$

LOOKING AT $\underline{i}$

$\frac{b+3}{c-b} = \frac{2}{3} \implies 3b+9 = 2c-2b$
$\implies 3+9 = 2c-2$
$\implies 14 = 2c$
$\implies c=7$

LOOKING AT $\underline{k}$

$\frac{b-a-5}{10-b} = \frac{2}{3} \implies 3b-3a-15 = 20-2b$
$\implies 3-3a-15 = 20-2$
$\implies -30 = 3a$
$\implies a=-10$

Question 18 (***+)

The points $A(7,4,3)$, B and $C(1,2,-1)$ form the parallelogram $OABC$, where the above coordinates are measured relative to a fixed origin O.

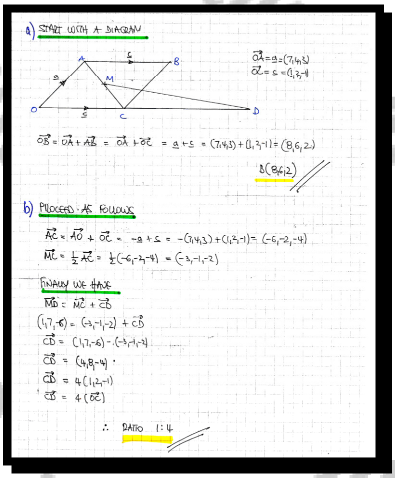
a) Find the coordinates of B.

The side OC is extended in the $\overrightarrow{OC}$ direction to a point D.

The point M is the midpoint of AC.

b) Given further that $\overrightarrow{MD} = \mathbf{i} + 7\mathbf{j} - 6\mathbf{k}$, determine $|\overrightarrow{OC}| : |\overrightarrow{CD}|$.

$\boxed{}$, $\boxed{B(8,6,2)}$, $\boxed{|\overrightarrow{OC}| : |\overrightarrow{CD}| = 1 : 4}$


a) START WITH A DIAGRAM

$\overrightarrow{OA} = \underline{a} = (7,4,3)$
$\overrightarrow{OC} = \underline{c} = (1,2,-1)$

$\overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AB} = \overrightarrow{OA} + \overrightarrow{OC} = \underline{a} + \underline{c} = (7,4,3) + (1,2,-1) = (8,6,2)$

$B(8,6,2)$

b) PROCEED AS FOLLOWS

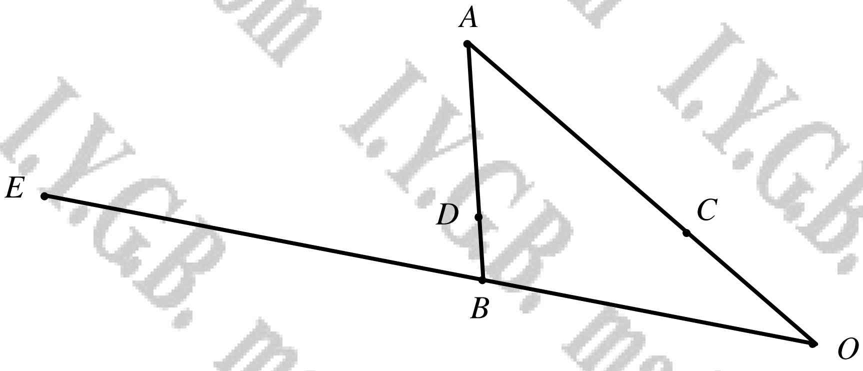
$\overrightarrow{AC} = \overrightarrow{AO} + \overrightarrow{OC} = -\underline{a} + \underline{c} = -(7,4,3) + (1,2,-1) = (-6,-2,-4)$
$\overrightarrow{MC} = \frac{1}{2}\overrightarrow{AC} = \frac{1}{2}(-6,-2,-4) = (-3,-1,-2)$

FINALLY WE HAVE

$\overrightarrow{MD} = \overrightarrow{MC} + \overrightarrow{CD}$
$(1,7,-6) = (-3,-1,-2) + \overrightarrow{CD}$
$\overrightarrow{CD} = (1,7,-6) - (-3,-1,-2)$
$\overrightarrow{CD} = (4,8,-4)$
$\overrightarrow{CD} = 4(1,2,-1)$
$\overrightarrow{CD} = 4(\overrightarrow{OC})$

$\therefore$ RATIO $1:4$

Question 19 (***+)



The figure above shows the triangle OAB, where O is the origin and the position vectors of A and B relative to O, are $-6\mathbf{i}+27\mathbf{j}-9\mathbf{k}$ and $4\mathbf{i}+6\mathbf{j}-6\mathbf{k}$, respectively.

The point E is such so that O, B and E are collinear with $OB:BE=1:2$

The point C is such so that O, C and A are collinear with $OC:CA=1:2$

The point D is such so that B, D and A are collinear with $BD:DA=1:3$

a) Determine the coordinates of C, D and E, relative to O.

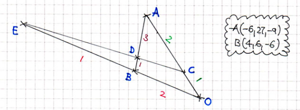
b) Show that the points C, D and E are collinear, and find the ratio $CD:DE$.

c) Show further that BC is parallel to EA, and find the ratio $BC:EA$.

$\boxed{\text{MP2}}$, $\boxed{C(-2,9,-3)}$, $\boxed{D\left(\frac{3}{2},\frac{45}{4},-\frac{27}{4}\right)}$, $\boxed{E(12,18,-18)}$, $\boxed{CD:DE=1:3}$,

$\boxed{BC:EA=1:3}$

a) START BY FINDING THE POSITION VECTORS OF C, D & E



- $\vec{OE} = 3\vec{OB} = 3(4,6,-6) = (12,18,-18)$ IE $E(12,18,-18)$
- $\vec{OC} = \frac{1}{3}\vec{OA} = \frac{1}{3}(-6,27,-9) = (-2,9,-3)$ IE $C(-2,9,-3)$
- $\vec{OD} = \vec{OB} + \vec{BD} = \vec{OB} + \frac{1}{4}\vec{BA}$
$= \vec{OB} + \frac{1}{4}(\vec{BO}+\vec{OA}) = \vec{OB} + \frac{1}{4}\vec{BO} + \frac{1}{4}\vec{OA}$
$= \vec{OB} - \frac{1}{4}\vec{OB} + \frac{1}{4}\vec{OA} = \frac{3}{4}\vec{OB} + \frac{1}{4}\vec{OA}$
$= \frac{3}{4}(4,6,-6) + \frac{1}{4}(-6,27,-9) = \left(\frac{3}{2},\frac{45}{4},-\frac{27}{4}\right)$ IE $D\left(\frac{3}{2},\frac{45}{4},-\frac{27}{4}\right)$

b) DETERMINE THE VECTORS $\vec{CD}$ & $\vec{DE}$

$$\begin{cases} \vec{CD} = \underline{d} - \underline{c} = \left(\frac{3}{2},\frac{45}{4},-\frac{27}{4}\right) - (-2,9,-3) = \left(\frac{7}{2},\frac{9}{4},-\frac{15}{4}\right) \\ \vec{DE} = \underline{e} - \underline{d} = (12,18,-18) - \left(\frac{3}{2},\frac{45}{4},-\frac{27}{4}\right) = \left(\frac{21}{2},\frac{27}{4},-\frac{45}{4}\right) \end{cases}$$

$$\begin{cases} \vec{CD} = \frac{1}{4}(14,9,-15) \\ \vec{DE} = \frac{3}{4}(14,9,-15) \end{cases}$$

AS BOTH $\vec{CD}$ & $\vec{DE}$ ARE IN THE SAME DIRECTION & SHARE THE POINT D, IMPLIES THAT C, D & E ARE COLLINEAR

$|\vec{CD}| : |\vec{DE}|$
$\frac{1}{4} : \frac{3}{4}$
$1 : 3$

c) SIMILARLY COMPARE $\vec{CB}$ & $\vec{AE}$

$$\begin{cases} \vec{CB} = \underline{b} - \underline{c} = (4,6,-6) - (-2,9,-3) = (6,-3,-3) \\ \vec{AE} = \underline{e} - \underline{a} = (12,18,-18) - (-6,27,-9) = (18,-9,-9) \end{cases}$$

$\vec{CB} = (6,-3,-3) = 3(2,-1,-1)$
$\vec{AE} = (18,-9,-9) = 9(2,-1,-1)$

AS $\vec{CB}$ & $\vec{AE}$ ARE IN THE SAME DIRECTION, CB IS PARALLEL TO AE

$|\vec{CB}| : |\vec{AE}|$
$3 : 9$
$1 : 3$

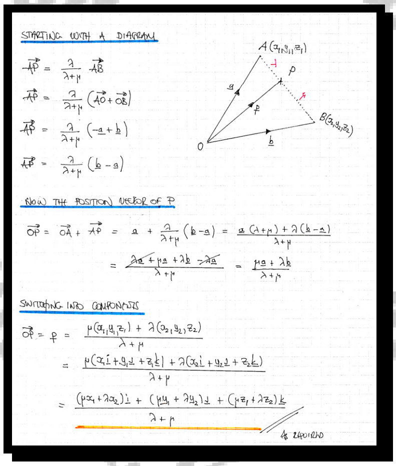
Question 20 (***+)

The points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ are referred relative to a fixed origin O.

If the point P is such so that $\overrightarrow{AP} : \overrightarrow{PB} = \lambda : \mu$, use vector algebra to show that

$$\overrightarrow{OP} = \frac{(\mu x_1 + \lambda x_2)\mathbf{i} + (\mu y_1 + \lambda y_2)\mathbf{j} + (\mu z_1 + \lambda z_2)\mathbf{k}}{\lambda + \mu}.$$

$\boxed{}$, $\boxed{\text{proof}}$



STARTING WITH A DIAGRAM

$\overrightarrow{AP} = \frac{\lambda}{\lambda+\mu}\overrightarrow{AB}$

$\overrightarrow{AP} = \frac{\lambda}{\lambda+\mu}(\overrightarrow{AO} + \overrightarrow{OB})$

$\overrightarrow{AP} = \frac{\lambda}{\lambda+\mu}(-\underline{a} + \underline{b})$

$\overrightarrow{AP} = \frac{\lambda}{\lambda+\mu}(\underline{b} - \underline{a})$

NOW THE POSITION VECTOR OF P

$\overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{AP} = \underline{a} + \frac{\lambda}{\lambda+\mu}(\underline{b} - \underline{a}) = \frac{\underline{a}(\lambda+\mu) + \lambda(\underline{b} - \underline{a})}{\lambda+\mu}$

$= \frac{\cancel{\lambda\underline{a}} + \mu\underline{a} + \lambda\underline{b} - \cancel{\lambda\underline{a}}}{\lambda+\mu} = \frac{\mu\underline{a} + \lambda\underline{b}}{\lambda+\mu}$

SWITCHING INTO COMPONENTS

$\overrightarrow{OP} = \underline{p} = \frac{\mu(x_1, y_1, z_1) + \lambda(x_2, y_2, z_2)}{\lambda+\mu}$

$= \frac{\mu(x_1\underline{i} + y_1\underline{j} + z_1\underline{k}) + \lambda(x_2\underline{i} + y_2\underline{j} + z_2\underline{k})}{\lambda+\mu}$

$= \frac{(\mu x_1 + \lambda x_2)\underline{i} + (\mu y_1 + \lambda y_2)\underline{j} + (\mu z_1 + \lambda z_2)\underline{k}}{\lambda+\mu}$

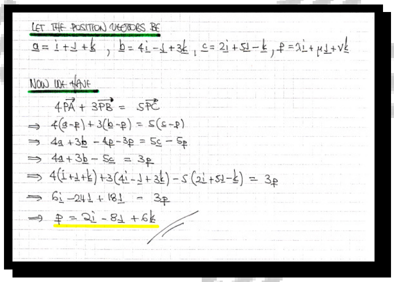
As REQUIRED

Question 21 (***+)

Relative to a fixed origin, the coordinates of three points $A(1,1,1)$, $B(4,-1,3)$ and $C(2,5,-1)$, are given.

Find the position vector of the point P if $4\overrightarrow{PA} + 3\overrightarrow{PB} = 5\overrightarrow{PC}$.

$\boxed{}$, $\boxed{\mathbf{p} = 2\mathbf{i} - 8\mathbf{j} + 6\mathbf{k}}$



LET THE POSITION VECTORS BE

$\mathbf{a} = \mathbf{i} + \mathbf{j} + \mathbf{k}$, $\mathbf{b} = 4\mathbf{i} - \mathbf{j} + 3\mathbf{k}$, $\mathbf{c} = 2\mathbf{i} + 5\mathbf{j} - \mathbf{k}$, $\mathbf{p} = \lambda\mathbf{i} + \mu\mathbf{j} + \nu\mathbf{k}$

NOW WE HAVE

$4\overrightarrow{PA} + 3\overrightarrow{PB} = 5\overrightarrow{PC}$

$\Rightarrow 4(\mathbf{a}-\mathbf{p}) + 3(\mathbf{b}-\mathbf{p}) = 5(\mathbf{c}-\mathbf{p})$

$\Rightarrow 4\mathbf{a} + 3\mathbf{b} - 4\mathbf{p} - 3\mathbf{p} = 5\mathbf{c} - 5\mathbf{p}$

$\Rightarrow 4\mathbf{a} + 3\mathbf{b} - 5\mathbf{c} = 3\mathbf{p}$

$\Rightarrow 4(\mathbf{i}+\mathbf{j}+\mathbf{k}) + 3(4\mathbf{i}-\mathbf{j}+3\mathbf{k}) - 5(2\mathbf{i}+5\mathbf{j}-\mathbf{k}) = 3\mathbf{p}$

$\Rightarrow 6\mathbf{i} - 24\mathbf{j} + 18\mathbf{j} = 3\mathbf{p}$

$\Rightarrow \mathbf{p} = 2\mathbf{i} - 8\mathbf{j} + 6\mathbf{k}$

Question 22 (****)

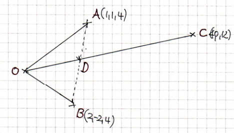
Relative to a fixed origin O, the positions vectors of the points A, B and C are defined below.

$$\overrightarrow{OA} = \mathbf{i} + \mathbf{j} + 4\mathbf{k}, \qquad \overrightarrow{OB} = 2\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}, \qquad \overrightarrow{OC} = 4\mathbf{i} + 12\mathbf{k}.$$

If $\overrightarrow{OD} = \frac{1}{3}\overrightarrow{OC}$ prove that the point D lies on the straight line AB.

$\boxed{}$, $\boxed{\text{proof}}$

DRAWING A DIAGRAM



DETERMINE THE POSITION VECTOR OF D

$\overrightarrow{OD} = \frac{1}{3}\overrightarrow{OC} = \frac{1}{3}(4,0,12) = (\frac{4}{3},0,4)$ I.E $D(\frac{4}{3},0,4)$

DETERMINE THE VECTORS $\overrightarrow{AD}$ & $\overrightarrow{DB}$

$\overrightarrow{AD} = \underline{d} - \underline{a} = (\frac{4}{3},0,4) - (1,1,4) = (\frac{1}{3},-1,0) = 1(\frac{1}{3},-1,0)$

$\overrightarrow{DB} = \underline{b} - \underline{d} = (2,-2,4) - (\frac{4}{3},0,4) = (\frac{2}{3},-2,0) = 2(\frac{1}{3},-1,0)$

AS $\overrightarrow{AD}$ IS IN THE SAME DIRECTION AS $\overrightarrow{DB}$, AND SHARE A POINT A, D & B ARE COLLINEAR, SO D LIES ON THE LINE AB

Question 23 (****)

Relative to a fixed origin O, the position vectors of three points A, B and C are

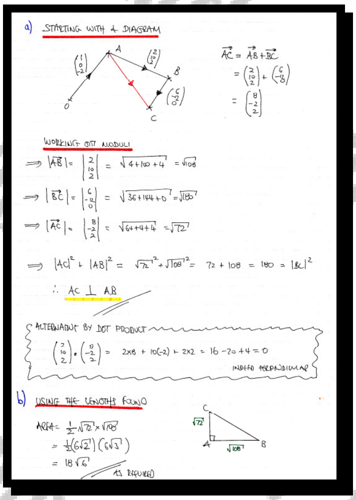
$$\overrightarrow{OA} = \mathbf{i} - 2\mathbf{k}, \quad \overrightarrow{AB} = 2\mathbf{i} + 10\mathbf{j} + 2\mathbf{k} \quad \text{and} \quad \overrightarrow{BC} = 6\mathbf{i} - 12\mathbf{j}.$$

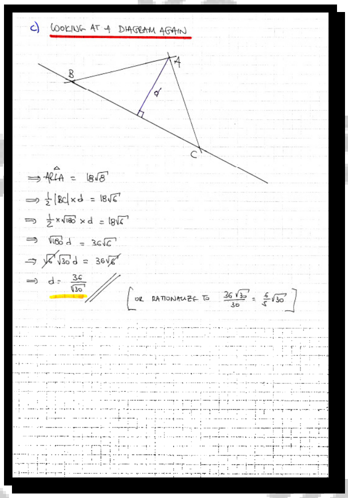
a) Show that $\overrightarrow{AC}$ is perpendicular to $\overrightarrow{AB}$.

b) Show further that the area of the triangle ABC is $18\sqrt{6}$.

c) Hence, or otherwise, determine the shortest distance of A from the straight line through B and C.

$\boxed{\quad}$, $\boxed{\text{distance} = \frac{6}{5}\sqrt{30}}$





Question 24 (****)

The points $A(2,-1,4)$, $B(0,-5,10)$, $C(3,1,3)$ and $D(6,7,-8)$ are referred relative to a fixed origin O.

a) Use vector algebra to show that three of the above four points are collinear.

A triangle is drawn using three of the above four points as its vertices.

b) Given further that the triangle has the largest possible area, determine, in exact surd form, the length of its shortest side.

$\boxed{}$, $\boxed{\sqrt{94}}$

a) $A(2,-1,4)$ $B(0,-5,10)$ $C(3,1,3)$ $D(6,7,-8)$

- PICK A POINT AT RANDOM AND CALCULATE ALL OTHER VECTORS TO THE OTHER 3 POINTS

$\overrightarrow{AB} = \underline{b} - \underline{a} = (0,-5,10)-(2,-1,4) = (-2,-4,6) = 2(-1,-2,3)$

$\overrightarrow{AC} = \underline{c} - \underline{a} = (3,1,3)-(2,-1,4) = (1,2,-1) = 1(1,2,-1)$

$\overrightarrow{AD} = \underline{d} - \underline{a} = (6,7,-8)-(2,-1,4) = (4,8,-12) = 4(1,2,-3)$

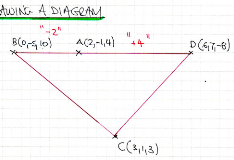
- HENCE WE HAVE $\overrightarrow{AB}$ & $\overrightarrow{AD}$ IN "PARALLEL CONFIGURATION"

$\overrightarrow{AB} = 2(-1,-2,3) = -2(1,2,-3)$

$\overrightarrow{AD} = 4(1,2,-3)$

$\therefore$ A, B & D ARE COLLINEAR

b) - DRAWING A DIAGRAM



- THE LENGTH OF BD IS $6|1,2,-3|$ (OR COMPUTE $|\underline{d}-\underline{b}|$)

$\Rightarrow 6\sqrt{1+4+9} = 6\sqrt{14}$

- ALSO WE HAVE

• $|\overrightarrow{BC}| = |\underline{c}-\underline{b}| = |(3,1,3)-(0,-5,10)| = |3,6,-7|$
$= \sqrt{9+36+49} = \sqrt{94}$

• $|\overrightarrow{DC}| = |\underline{c}-\underline{d}| = |(3,1,3)-(6,7,-8)| = |-3,-6,11|$
$= \sqrt{9+36+121} = \sqrt{166}$

$\therefore$ THE SHORTEST SIDE OF THE TRIANGLE WHICH HAS THE LARGEST AREA IS $\sqrt{94}$

Question 125 (****)

Relative to a fixed origin, the points P and Q have position vectors $9\mathbf{j}-2\mathbf{k}$ and $7\mathbf{i}-8\mathbf{j}+11\mathbf{k}$, respectively.

a) Determine the distance between the points P and Q.

b) Find the position vector of the point M, where M is the midpoint of PQ.

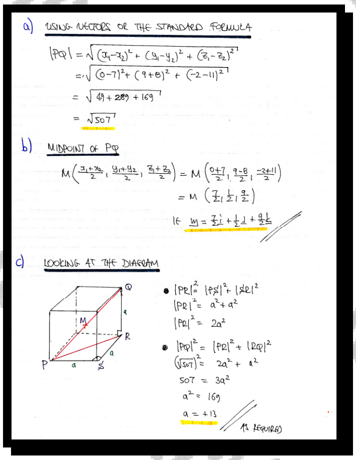
The points P and Q are vertices of a cube, so that PQ is one of the longest diagonals of the cube.

c) Show that the length of one of the sides of the cube is 13 units.

d) Calculate the distance of the point M from the origin O.

e) Show that the origin O lies inside the cube.

$\boxed{|PQ|=\sqrt{507}}$, $\boxed{\overrightarrow{OM}=\frac{7}{2}\mathbf{i}+\frac{1}{2}\mathbf{j}+\frac{9}{2}\mathbf{k}}$, $\boxed{|OM|=\frac{1}{2}\sqrt{131}}$



a) USING VECTORS OR THE STANDARD FORMULA

$|PQ| = \sqrt{(x_1-x_2)^2+(y_1-y_2)^2+(z_1-z_2)^2}$
$= \sqrt{(0-7)^2+(9+8)^2+(-2-11)^2}$
$= \sqrt{49+289+169}$
$= \sqrt{507}$

b) MIDPOINT OF PQ

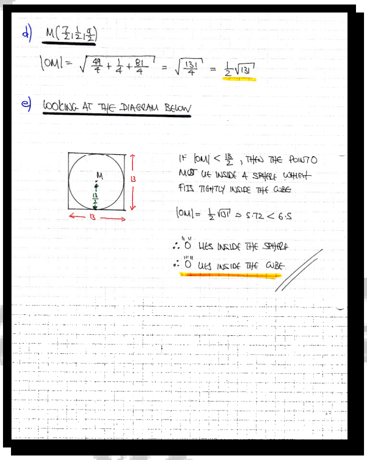
$M\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2}\right) = M\left(\frac{0+7}{2}, \frac{9-8}{2}, \frac{-2+11}{2}\right)$
$= M\left(\frac{7}{2}, \frac{1}{2}, \frac{9}{2}\right)$

IE $\underline{m} = \frac{7}{2}\underline{i} + \frac{1}{2}\underline{j} + \frac{9}{2}\underline{k}$

c) LOOKING AT THE DIAGRAM

- $|PR|^2 = |PS|^2 + |SR|^2$
  $|PR|^2 = a^2 + a^2$
  $|PR|^2 = 2a^2$
- $|PQ|^2 = |PR|^2 + |RQ|^2$
  $(\sqrt{507})^2 = 2a^2 + a^2$
  $507 = 3a^2$
  $a^2 = 169$
  $a = +13$

AS REQUIRED



d) $M\left(\frac{7}{2}, \frac{1}{2}, \frac{9}{2}\right)$

$|OM| = \sqrt{\frac{49}{4}+\frac{1}{4}+\frac{81}{4}} = \sqrt{\frac{131}{4}} = \frac{1}{2}\sqrt{131}$

e) LOOKING AT THE DIAGRAM BELOW

IF $|OM| < \frac{13}{2}$, THEN THE POINT O MUST LIE INSIDE A SPHERE WHICH FITS TIGHTLY INSIDE THE CUBE

$|OM| = \frac{1}{2}\sqrt{131} \approx 5.72 < 6.5$

$\therefore$ "O" LIES INSIDE THE SPHERE
$\therefore$ "O" LIES INSIDE THE CUBE

Question 26 (****+)

The points $A(3,2,14)$, $B(0,1,13)$ and $C(5,6,8)$ are defined with respect to a fixed origin O.

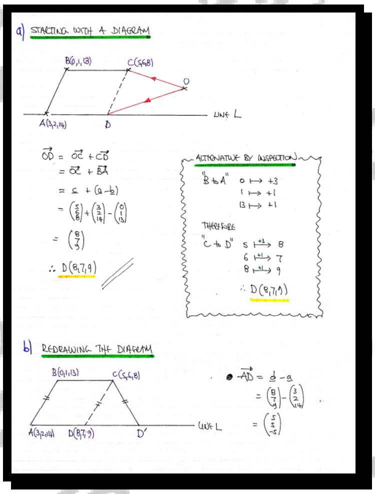
The straight line L passes through A and it is parallel to the vector $\overrightarrow{BC}$.

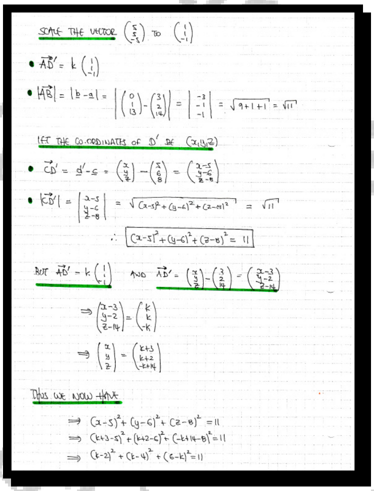
The point D lies on L so that $ABCD$ is a parallelogram.

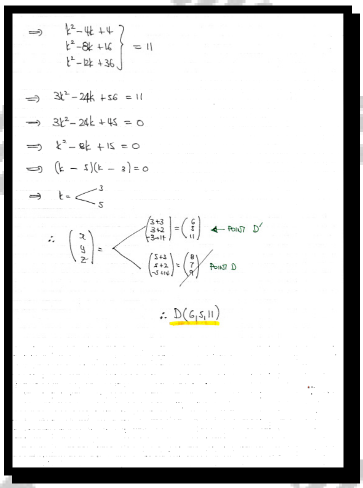
a) Find the coordinates of D.

b) If instead $ABCD$ is an isosceles trapezium and the point D still lies on L, determine the new coordinates of D.

$\boxed{}$, $\boxed{D(8,7,9)}$, $\boxed{D(6,5,11)}$







Question 27 (*****)

With respect to a fixed origin, the points A and B have position vectors $10\mathbf{i}+9\mathbf{j}-6\mathbf{k}$ and $6\mathbf{i}-3\mathbf{j}+10\mathbf{k}$, respectively.

The position vector of the point C has $\mathbf{i}$ component equal to 2.

The distance of C from both A and B is 12 units.

Show that one of the two possible position vectors of C is $2\mathbf{i}+5\mathbf{j}+2\mathbf{k}$ and determine the other.

, $\boxed{\mathbf{c} = 2\mathbf{i} + \frac{61}{25}\mathbf{j} + \frac{2}{25}\mathbf{k}}$

$A(10,9,-6) \quad B(6,-3,10) \quad C(2,y,z)$

START BY FINDING $\overrightarrow{AC}$ & $\overrightarrow{BC}$

$\overrightarrow{AC} = \underline{c} - \underline{a} = (2,y,z)-(10,9,-6) = (-8,y-9,z+6)$

$\overrightarrow{BC} = \underline{c} - \underline{b} = (2,y,z)-(6,-3,10) = (-4,y+3,z-10)$

NEXT SET SIMPLIFIED EXPRESSIONS FOR EACH OF THE MODULI

$\Rightarrow |-8,y-9,z+6| = 12$

$\Rightarrow \sqrt{64+(y-9)^2+(z+6)^2} = 12$

$\Rightarrow 64+(y-9)^2+(z+6)^2 = 144$

$\Rightarrow (y-9)^2+(z+6)^2 = 80$

$\Rightarrow y^2-18y+81+z^2+12z+36 = 80$

$\Rightarrow y^2+z^2-18y+12z = -37$

$\Rightarrow |-4,y+3,z-10| = 12$

$\Rightarrow \sqrt{16+(y+3)^2+(z-10)^2} = 12$

$\Rightarrow 16+(y+3)^2+(z-10)^2 = 144$

$\Rightarrow (y+3)^2+(z-10)^2 = 128$

$\Rightarrow y^2+6y+9+z^2-20z+100 = 128$

$\Rightarrow y^2+z^2+6y-20z = 19$

SOLVING SIMULTANEOUSLY BY SUBTRACTING THE EQUATIONS

$\Rightarrow \left\{ \begin{array}{l} \cancel{y^2}+\cancel{z^2}+6y-20z = 19 \\ \cancel{y^2}+\cancel{z^2}-18y+12z = -37 \end{array} \right\} \Rightarrow 24y-32z = 56$

$\Rightarrow 3y-4z = 7$

$\Rightarrow 3y = 4z+7$

TAKE ONE OF THE EQUATIONS SUCH AS

$\Rightarrow y^2+z^2+6y-20z = 19 \quad \times 9$

$\Rightarrow 9y^2+9z^2+54y-180z = 171$

$\Rightarrow (3y)^2+9z^2+18(3y)-180z = 171$

$\Rightarrow (4z+7)^2+9z^2+18(4z+7)-180z = 171$

$\Rightarrow 16z^2+56z+49+9z^2+72z+126-180z-171 = 0$

$\Rightarrow 25z^2-52z+4 = 0$

$\Rightarrow (z-2)(25z-2) = 0$

$\Rightarrow z = \begin{cases} 2 \\ \frac{2}{25} \end{cases}$

FINALLY USING $3y = 4z+7$

- IF $z=2$: $3y = 15$, $y = 5$
- IF $z = \frac{2}{25}$: $3y = \frac{8}{25}+7$, $3y = \frac{183}{25}$, $y = \frac{61}{25}$

$\therefore (2,5,2)$ & $\left(2,\frac{61}{25},\frac{2}{25}\right)$

Question 28 (*****)

The vertices of the triangle OAB have coordinates $A(6,-18,-6)$, $B(7,-1,3)$, where O is a fixed origin.

The point N lies on OA so that $ON : NA = 1:2$.

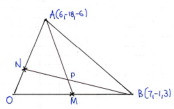
The point M is the midpoint of OB.

The point P is the intersection of AM and BN.

By using vector methods, or otherwise, determine the coordinates of P.

$\boxed{\text{MP2}}$, $\boxed{P(4,-4,0)}$

STARTING WITH A DIAGRAM



- BY INSPECTION

$N(2,-6,-2)$

$M\left(\frac{7}{2},-\frac{1}{2},\frac{3}{2}\right)$

WORK AS FOLLOWS

$\vec{NP} = k\,\vec{NB}$, $0<k<1$

$\vec{NP} = k(\underline{b}-\underline{n}) = k\left[(7,-1,3)-(2,-6,-2)\right] = k(5,5,5)$

$\vec{NP} = (5k,5k,5k)$

NEXT WE WORK AN EXPRESSION FOR $\vec{MP}$

$\vec{MP} = \vec{MO} + \vec{ON} + \vec{NP}$

$\vec{MP} = -\underline{m} + \underline{n} + (5k,5k,5k)$

$\vec{MP} = -\left(\frac{7}{2},-\frac{1}{2},\frac{3}{2}\right) + (2,-6,-2) + (5k,5k,5k)$

$\vec{MP} = \left(5k-\frac{3}{2},\ 5k-\frac{11}{2},\ 5k-\frac{7}{2}\right)$

NEXT A SIMILAR EXPRESSION FOR $\vec{PA}$

$\vec{PA} = \vec{PN} + \vec{NA}$

$\vec{PA} = -\vec{NP} + 2\vec{ON}$

$\vec{PA} = (-5k,-5k,-5k) + (4,-12,-4)$

$\vec{PA} = (-5k+4,-5k-12,-5k-4)$

BUT P, M & A ARE COLLINEAR

$\Rightarrow \vec{MP} = \lambda\,\vec{MA}$ FOR SOME SCALAR λ

$\Rightarrow \left(5k-\frac{3}{2}, 5k-\frac{11}{2}, 5k-\frac{7}{2}\right) = \lambda(-5k+4,-5k-12,-5k-4)$

EQUATE ANY TWO COMPONENTS

$\left.\begin{array}{l} 5k-\frac{3}{2} = -5\lambda k+4\lambda \\ 5k-\frac{11}{2} = -5\lambda k-12\lambda \end{array}\right\} \Rightarrow \left.\begin{array}{l} 5k+5\lambda k = 4\lambda+\frac{3}{2} \\ 5k+5\lambda k = \frac{11}{2}-12\lambda \end{array}\right\} \Rightarrow$

$\Rightarrow 4\lambda+\frac{3}{2} = \frac{11}{2}-12\lambda$

$\Rightarrow 16\lambda = 4$

$\Rightarrow \lambda = \frac{1}{4}$

USING $5k-\frac{3}{2} = -5\lambda k+4\lambda$

$5k-\frac{3}{2} = -\frac{5}{4}k+1$

$\frac{25}{4}k = \frac{5}{2}$

$k = \frac{2}{5}$

CHECKING FOR CONSISTENCY THE THIRD COMPONENT (NOT USED ABOVE)

$5k-\frac{7}{2} = 5\times\frac{2}{5}-\frac{7}{2} = 2-\frac{7}{2} = -\frac{3}{2}$

$-5\lambda k-4\lambda = -5\times\frac{1}{4}\times\frac{2}{5}-4\times\frac{1}{4} = -\frac{1}{2}-1 = -\frac{3}{2}$

(ALL FINE!)

FINALLY WE HAVE

$\vec{OP} = \vec{ON} + \vec{NP}$

$= (2,-6,2) + (5k,5k,5k)$

$= (2,-6,2) + (2,2,2)$

$= (4,-4,0)$

$\therefore P(4,-4,0)$