1^{st} ORDER O.D.E.

SOLUTIONS BY SUBSTITUTIONS

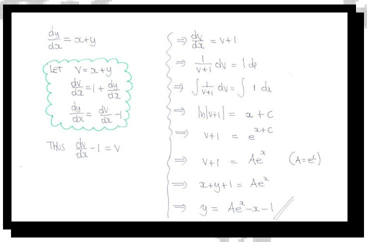
Question 1 (+)**

By using the substitution $u = x + y$ find a general solution of the differential equation

$$\frac{dy}{dx} = x + y,$$

giving the answer in the form $y = f(x)$.

$$\boxed{y = A\mathrm{e}^{x} - x - 1}$$



$\frac{dy}{dx} = x + y$

LET $v = x + y$

$\frac{dv}{dx} = 1 + \frac{dy}{dx}$

$\frac{dy}{dx} = \frac{dv}{dx} - 1$

THUS $\frac{dv}{dx} - 1 = v$

$\Rightarrow \frac{dv}{dx} = v + 1$

$\Rightarrow \frac{1}{v+1}\,dv = 1\,dx$

$\Rightarrow \int \frac{1}{v+1}\,dv = \int 1\,dx$

$\Rightarrow \ln|v+1| = x + C$

$\Rightarrow v + 1 = e^{x+C}$

$\Rightarrow v + 1 = Ae^{x} \quad (A = e^{C})$

$\Rightarrow x + y + 1 = Ae^{x}$

$\Rightarrow y = Ae^{x} - x - 1$

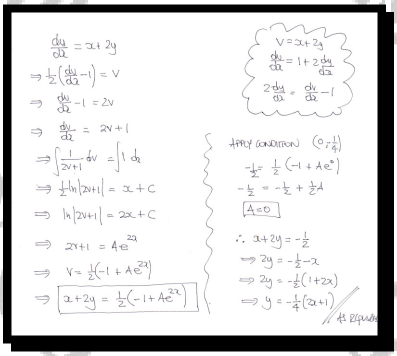
Question 2 (***)

$$\frac{dy}{dx} = x + 2y, \text{ with } y = -\frac{1}{4} \text{ at } x = 0.$$

By using the substitution $v = x + 2y$, show that the solution of the differential equation is given by

$$y = -\frac{1}{4}(2x + 1).$$

$\boxed{\text{proof}}$



$\frac{dy}{dx} = x + 2y$

$v = x + 2y$

$\frac{dv}{dx} = 1 + 2\frac{dy}{dx}$

$2\frac{dy}{dx} = \frac{dv}{dx} - 1$

$\Rightarrow \frac{1}{2}\left(\frac{dv}{dx} - 1\right) = v$

$\Rightarrow \frac{dv}{dx} - 1 = 2v$

$\Rightarrow \frac{dv}{dx} = 2v + 1$

$\Rightarrow \int \frac{1}{2v+1}\,dv = \int 1\,dx$

$\Rightarrow \frac{1}{2}\ln|2v+1| = x + C$

$\Rightarrow \ln|2v+1| = 2x + C$

$\Rightarrow 2v + 1 = Ae^{2x}$

$\Rightarrow v = \frac{1}{2}(-1 + Ae^{2x})$

$\Rightarrow \boxed{x + 2y = \frac{1}{2}(-1 + Ae^{2x})}$

APPLY CONDITION $(0, -\frac{1}{4})$

$-\frac{1}{2} = \frac{1}{2}(-1 + Ae^{0})$

$-\frac{1}{2} = -\frac{1}{2} + \frac{1}{2}A$

$\boxed{A = 0}$

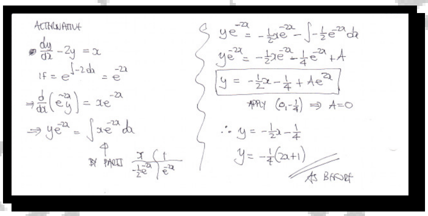
$\therefore x + 2y = -\frac{1}{2}$

$\Rightarrow 2y = -\frac{1}{2} - x$

$\Rightarrow 2y = -\frac{1}{2}(1 + 2x)$

$\Rightarrow y = -\frac{1}{4}(2x + 1)$

AS REQUIRED



ALTERNATIVE

$\frac{dy}{dx} - 2y = x$

$\text{IF} = e^{\int -2\,dx} = e^{-2x}$

$\Rightarrow \frac{d}{dx}\left(e^{-2x}y\right) = xe^{-2x}$

$\Rightarrow ye^{-2x} = \int xe^{-2x}\,dx$

BY PARTS

x	1
$-\frac{1}{2}e^{-2x}$	e^{-2x}

$ye^{-2x} = -\frac{1}{2}xe^{-2x} - \int -\frac{1}{2}e^{-2x}\,dx$

$ye^{-2x} = -\frac{1}{2}xe^{-2x} - \frac{1}{4}e^{-2x} + A$

$\boxed{y = -\frac{1}{2}x - \frac{1}{4} + Ae^{2x}}$

APPLY $(0, -\frac{1}{4}) \Rightarrow A = 0$

$\therefore y = -\frac{1}{2}x - \frac{1}{4}$

$y = -\frac{1}{4}(2x + 1)$

AS BEFORE

Question 3 (***)

By using the substitution $v = \frac{y}{x}$, where $v = f(x)$, solve the differential equation

$$\frac{dy}{dx} = \frac{y}{x} - \left(\frac{y}{x}\right)^2, \ x > 0$$

subject to the condition $y = 1$ at $x = 1$.

$$\boxed{y = \frac{x}{1 + \ln x}}$$

$$\frac{dy}{dx} = \frac{y}{x} - \left(\frac{y}{x}\right)^2$$

$v = \frac{y}{x}$

$y = xv$

$\frac{dy}{dx} = 1 \times v + x \times \frac{dv}{dx}$

$$\Rightarrow v + x\frac{dv}{dx} = v - v^2$$

$$\Rightarrow x\frac{dv}{dx} = -v^2$$

$$\Rightarrow -\frac{1}{v^2}\,dv = \frac{1}{x}\,dx$$

$$\Rightarrow \int -v^{-2}\,dv = \int \frac{1}{x}\,dx$$

$$\Rightarrow v^{-1} = \ln x + C$$

$$\Rightarrow \frac{x}{y} = \ln x + C$$

$$\Rightarrow \frac{x}{\ln x + C} = y$$

$\Rightarrow$ APPLY CONDITION $(1,1)$

$$\frac{1}{\ln 1 + C} = 1$$

$$C = 1$$

$$\therefore y = \frac{x}{1 + \ln x}$$

Question 4 (***)

$$\frac{dy}{dx} = \frac{(4x+y)(x+y)}{x^2}, \quad x > 0.$$

a) Use the substitution $y = xv$, where $v = f(x)$, to show that the above differential equation can be transformed to

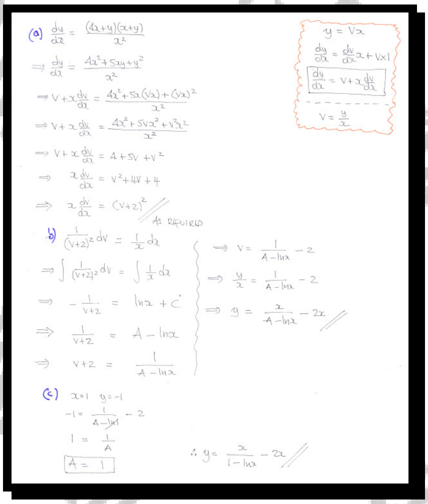
$$x\frac{dv}{dx} = (v+2)^2.$$

b) Hence find the general solution of the original differential equation, giving the answer in the form $y = f(x)$.

c) Use the boundary condition $y = -1$ at $x = 1$, to show that a specific solution of the original differential equation is

$$y = \frac{x}{1 - \ln x} - 2x.$$

$$\boxed{y = \frac{x}{A - \ln x} - 2x}$$



(a) $\frac{dy}{dx} = \frac{(4x+y)(x+y)}{x^2}$

$\Rightarrow \frac{dy}{dx} = \frac{4x^2+5xy+y^2}{x^2}$

$\Rightarrow v + x\frac{dv}{dx} = \frac{4x^2+5x(vx)+(vx)^2}{x^2}$

$\Rightarrow v + x\frac{dv}{dx} = \frac{4x^2+5vx^2+v^2x^2}{x^2}$

$\Rightarrow v + x\frac{dv}{dx} = 4+5v+v^2$

$\Rightarrow x\frac{dv}{dx} = v^2+4v+4$

$\Rightarrow x\frac{dv}{dx} = (v+2)^2$ // AS REQUIRED

$y = vx$

$\frac{dy}{dx} = \frac{dv}{dx}x + v \times 1$

$\boxed{\frac{dy}{dx} = v + x\frac{dv}{dx}}$

$v = \frac{y}{x}$

(b) $\frac{1}{(v+2)^2}\,dv = \frac{1}{x}\,dx$

$\Rightarrow \int \frac{1}{(v+2)^2}\,dv = \int \frac{1}{x}\,dx$

$\Rightarrow -\frac{1}{v+2} = \ln x + C$

$\Rightarrow \frac{1}{v+2} = A - \ln x$

$\Rightarrow v+2 = \frac{1}{A-\ln x}$

$\Rightarrow v = \frac{1}{A-\ln x} - 2$

$\Rightarrow \frac{y}{x} = \frac{1}{A-\ln x} - 2$

$\Rightarrow y = \frac{x}{A-\ln x} - 2x$ //

(c) $x=1 \quad y=-1$

$-1 = \frac{1}{A-\cancel{\ln 1}} - 2$

$1 = \frac{1}{A}$

$\boxed{A = 1}$

$\therefore y = \frac{x}{1-\ln x} - 2x$ //

Question 5 **(***)**

Use the substitution $t = \sqrt{y}$ to solve the following differential equation.

$$\frac{dy}{dx} = y + \sqrt{y}, \quad y > 0, \quad y(0) = 4.$$

Given the answer in the form $y = f(x)$.

$\boxed{}$, $\boxed{y = 9e^{x} - 6e^{\frac{1}{2}x} + 1}$

USING THE SUBSTITUTION GIVEN WE CAN TRANSFORM THE O.D.E.

$$\boxed{\begin{aligned} t &= \sqrt{y} \\ t^2 &= y \\ \frac{dy}{dx} &= 2t\frac{dt}{dx} \end{aligned}}$$

$\Rightarrow \frac{dy}{dx} = y + \sqrt{y}$

$\Rightarrow 2t\frac{dt}{dx} = t^2 + t$

$\Rightarrow 2\frac{dt}{dx} = t + 1 \quad (t \neq 0)$

$\Rightarrow \frac{2}{t+1}\,dt = 1\,dx$

INTEGRATING BOTH SIDES

$\Rightarrow 2\ln|t+1| = x + C$

$\Rightarrow \ln|t+1| = \frac{1}{2}x + D$

$\Rightarrow |t+1| = Ae^{\frac{1}{2}x}$

$\Rightarrow |\sqrt{y}+1| = Ae^{\frac{1}{2}x}$

$\Rightarrow \sqrt{y}+1 = Ae^{\frac{1}{2}x}$

APPLY CONDITION $x=0$ $y=4$ YIELDS $A=3$

$\Rightarrow \sqrt{y}+1 = 3e^{\frac{1}{2}x}$

$\Rightarrow \sqrt{y} = 3e^{\frac{1}{2}x} - 1$

$\Rightarrow y = \left(3e^{\frac{1}{2}x} - 1\right)^2$

$\Rightarrow y = 9e^{x} - 6e^{\frac{1}{2}x} + 1$

Question 6 (***+)

By using the substitution $y = xz$, where $z = f(x)$, solve the differential equation

$$xy\frac{dy}{dx} = x^2 + y^2, \; x > 0$$

subject to the boundary condition $y = 1$ at $x = 1$.

$$\boxed{y = x^2\left(1 + 2\ln x\right)}$$

$$xy\frac{dy}{dx} = x^2+y^2$$

$$\boxed{\begin{aligned} y &= zx \\ \frac{dy}{dx} &= \frac{dz}{dx}x + z \end{aligned}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{x^2+y^2}{xy}$$

$$\Rightarrow x\frac{dz}{dx} + z = \frac{x^2 + z^2x^2}{x(zx)}$$

$$\Rightarrow x\frac{dz}{dx} = \frac{1+z^2}{z} - z$$

$$\Rightarrow x\frac{dz}{dx} = \frac{1}{z} + \cancel{z} - \cancel{z}$$

$$\Rightarrow z\,dz = \frac{1}{x}\,dx$$

$$\Rightarrow \int z\,dz = \int \frac{1}{x}\,dx$$

$$\Rightarrow \frac{1}{2}z^2 = \ln|x| + A$$

$$\Rightarrow z^2 = 2\ln x + A$$

$$\Rightarrow \frac{y^2}{x^2} = A + 2\ln x$$

$$\Rightarrow \boxed{y^2 = Ax^2 + 2x\ln x}$$

- APPLY CONDITION

$1 = A$

$\therefore y^2 = x^2 + 2x\ln x$

Question 7 (***+)

By using the substitution $u = x + y$, or otherwise, solve the differential equation

$$\frac{dy}{dx} = x^2 + 2xy + y^2,$$

subject to the condition $y(0) = 0$.

$$\boxed{y = -x + \tan x}$$

$$\frac{dy}{dx} = x^2+2xy+y^2$$

$$\frac{dy}{dx} = (x+y)^2$$

$$\boxed{\begin{aligned} &u = x+y \quad \text{or} \quad y = u - x \\ &\frac{du}{dx} = 1 + \frac{dy}{dx} \\ &\frac{dy}{dx} = \frac{du}{dx} - 1 \end{aligned}}$$

$$\Rightarrow \frac{du}{dx} - 1 = u^2$$

$$\Rightarrow \frac{du}{dx} = u^2 + 1$$

$$\Rightarrow \frac{1}{u^2+1}\,du = 1\,dx$$

$$\Rightarrow \int \frac{1}{u^2+1}\,du = \int 1\,dx$$

$$\Rightarrow \arctan u = x + C$$

$$\Rightarrow \boxed{\arctan(x+y) = x + C}$$

USING $(0,0)$

$\arctan 0 = 0 + C$

$\boxed{C = 0}$

$$\Rightarrow \arctan(x+y) = x$$

$$\Rightarrow x + y = \tan x$$

$$\Rightarrow y = -x + \tan x$$

Question 8 (***+)

By using the substitution $y = xv$, where $v = f(x)$, solve the differential equation

$$\frac{dy}{dx} = \frac{xy + y^2}{x^2}, \quad x > 0$$

subject to the condition $y = -1$ at $x = 1$.

$\boxed{}$, $\boxed{y = -\dfrac{x}{1 + \ln x}}$

AS THIS IS A FIRST ORDER HOMOGENEOUS EQUATION, USE $y = xV(x)$

$$\frac{dy}{dx} = 1 \times V(x) + x\frac{dV(x)}{dx} = V + x\frac{dv}{dx}$$

SUBSTITUTE INTO THE O.D.E

$$\Rightarrow \frac{dy}{dx} = \frac{xy + y^2}{x^2}$$

$$\Rightarrow V + x\frac{dv}{dx} = \frac{x(xv) + (xv)^2}{x^2}$$

$$\Rightarrow V + x\frac{dv}{dx} = \frac{x^2v + x^2v^2}{x^2}$$

$$\Rightarrow V + x\frac{dv}{dx} = V + v^2$$

$$\Rightarrow x\frac{dv}{dx} = v^2$$

$$\Rightarrow \frac{1}{v^2}\,dv = \frac{1}{x}\,dx$$

$$\Rightarrow \int \frac{1}{v^2}\,dv = \int \frac{1}{x}\,dx$$

$$\Rightarrow -\frac{1}{v} = \ln|x| + C$$

$$\Rightarrow -\frac{1}{y/x} = \ln|x| + C$$

$$\Rightarrow -\frac{x}{y} = \ln|x| + C$$

$$\Rightarrow y = -\frac{x}{\ln|x| + C}$$

$$\Rightarrow y = -\frac{x}{\ln x + C}, \quad x > 0$$

APPLY BOUNDARY CONDITION $(1, -1)$

$$\Rightarrow -1 = -\frac{1}{\cancel{\ln 1} + C}$$

$$\Rightarrow C = 1$$

FINALLY WE HAVE

$$y = -\frac{x}{\ln x + 1}$$

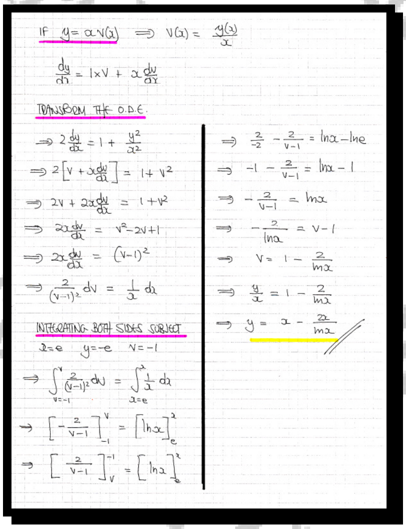
$$y = -\frac{x}{1 + \ln x}$$

Question 9 (***+)

Use the substitution $y = xv$, where $v = v(x)$, to solve the following differential equation

$$2\frac{dy}{dx} = 1 + \frac{y^2}{x^2}, \quad y(\mathrm{e}) = -\mathrm{e}.$$

$\boxed{}$, $\boxed{y = x - \frac{2x}{\ln x}}$



IF $y = xv(x) \Rightarrow v(x) = \frac{y(x)}{x}$

$\frac{dy}{dx} = 1 \times v + x\frac{dv}{dx}$

TRANSFORM THE O.D.E.

$\Rightarrow 2\frac{dy}{dx} = 1 + \frac{y^2}{x^2}$

$\Rightarrow 2\left[v + x\frac{dv}{dx}\right] = 1 + v^2$

$\Rightarrow 2v + 2x\frac{dv}{dx} = 1 + v^2$

$\Rightarrow 2x\frac{dv}{dx} = v^2 - 2v + 1$

$\Rightarrow 2x\frac{dv}{dx} = (v-1)^2$

$\Rightarrow \frac{2}{(v-1)^2}\,dv = \frac{1}{x}\,dx$

INTEGRATING BOTH SIDES SUBJECT

$x = e \quad y = -e \quad v = -1$

$\Rightarrow \int_{v=-1}^{v} \frac{2}{(v-1)^2}\,dv = \int_{x=e}^{x} \frac{1}{x}\,dx$

$\Rightarrow \left[-\frac{2}{v-1}\right]_{-1}^{v} = \left[\ln x\right]_{e}^{x}$

$\Rightarrow \left[\frac{2}{v-1}\right]_{v}^{-1} = \left[\ln x\right]_{e}^{x}$

$\Rightarrow \frac{2}{-2} - \frac{2}{v-1} = \ln x - \ln e$

$\Rightarrow -1 - \frac{2}{v-1} = \ln x - 1$

$\Rightarrow -\frac{2}{v-1} = \ln x$

$\Rightarrow -\frac{2}{\ln x} = v - 1$

$\Rightarrow v = 1 - \frac{2}{\ln x}$

$\Rightarrow \frac{y}{x} = 1 - \frac{2}{\ln x}$

$\Rightarrow y = x - \frac{2x}{\ln x}$

Question 10 (***+)

$$\frac{1}{y}\frac{dy}{dx} = 1 + 2xy^2, \quad y > 0.$$

a) Show that the substitution $z = \dfrac{1}{y^2}$ transforms the above differential equation into the new differential equation

$$\frac{dz}{dx} + 2z = -4x.$$

b) Hence find the general solution of the original differential equation, giving the answer in the form $y^2 = f(x)$.

$\boxed{}$, $\boxed{y^2 = \dfrac{1}{Ae^{-2x} - 2x + 1}}$

a) USING THE SUBSTITUTION GIVEN

$$z = \frac{1}{y^2} \implies \frac{d}{dx}(z) = \frac{d}{dx}\left(\frac{1}{y^2}\right)$$
$$\implies \frac{dz}{dx} = -\frac{2}{y^3}\frac{dy}{dx}$$
$$\implies \frac{dy}{dx} = -\frac{y^3}{2}\frac{dz}{dx}$$

SUBSTITUTE INTO THE O.D.E.

$$\frac{1}{y}\frac{dy}{dx} = 1 + 2xy^2$$
$$\frac{dy}{dx} = y + 2xy^3$$
$$-\frac{y^3}{2}\frac{dz}{dx} = y + 2xy^3 \qquad \times\left(-\frac{2}{y^3}\right)$$
$$\frac{dz}{dx} = -\frac{2}{y^2} - 4x$$
$$\frac{dz}{dx} = -2z - 4x$$
$$\frac{dz}{dx} + 2z = -4x$$

AS REQUIRED

b) LOOKING FOR AN INTEGRATING FACTOR

$$\text{I.F.} = e^{\int 2\,dx} = e^{2x}$$
$$\implies \frac{d}{dx}\left(ze^{2x}\right) = -4xe^{2x}$$
$$\implies ze^{2x} = \int -4xe^{2x}\,dx$$

INTEGRATION BY PARTS ON THE R.H.S.

$$\int -4xe^{2x}\,dx = -2xe^{2x} - \int -2e^{2x}\,dx$$
$$= -2xe^{2x} + \int 2e^{2x}\,dx$$
$$= e^{2x} - 2xe^{2x} + C$$

$-4x$	-4
$\frac{1}{2}e^{2x}$	e^{2x}

RETURNING TO THE O.D.E.

$$ze^{2x} = e^{2x} - 2xe^{2x} + C$$
$$z = 1 - 2x + Ce^{-2x}$$
$$\frac{1}{y^2} = 1 - 2x + Ce^{-2x}$$
$$y^2 = \frac{1}{1 - 2x + Ce^{-2x}}$$

Question 11 (***+)

$$\frac{dy}{dx} = \frac{x^2 + 3y^2}{xy}, \; x > 0, \; y > 0.$$

Use the substitution $y = xv$, where $v = f(x)$, and the boundary condition $y = \frac{1}{\sqrt{2}}$ at $x = 1$, to show that

$$y^2 = x^6 - \frac{1}{2}x^2.$$

$\boxed{}$, $\boxed{\text{proof}}$

USING THE SUBSTITUTION GIVEN $y(x) = xV(x)$

$\frac{dy}{dx} = \frac{d}{dx}\left(xV(x)\right) = 1 \times V(x) + x\frac{dV(x)}{dx}$

I.E $\frac{dy}{dx} = V + x\frac{dV}{dx}$

SUBSTITUTE INTO THE O.D.E.

$\Rightarrow \frac{dy}{dx} = \frac{x^2+3y^2}{xy}$

$\Rightarrow V + x\frac{dV}{dx} = \frac{x^2 + 3(xV)^2}{x(xV)}$

$\Rightarrow V + x\frac{dV}{dx} = \frac{x^2 + 3x^2V^2}{x^2V}$

$\Rightarrow x\frac{dV}{dx} = \frac{x^2(1+3V^2)}{x^2V} - V$

$\Rightarrow x\frac{dV}{dx} = \frac{1+3V^2}{V} - V$

$\Rightarrow x\frac{dV}{dx} = \frac{1+3V^2-V^2}{V}$

$\Rightarrow x\frac{dV}{dx} = \frac{1+2V^2}{V}$

SEPARATING VARIABLES

$\Rightarrow \frac{V}{1+2V^2}\,dV = \frac{1}{x}\,dx$

$\Rightarrow \int \frac{V}{1+2V^2}\,dV = \int \frac{1}{x}\,dx$

$\Rightarrow \frac{1}{4}\ln(1+2V^2) = \ln|x| + \ln A$

$\Rightarrow \ln(1+2V^2) = 4\ln(Ax)$

$\Rightarrow \ln(1+2V^2) = \ln(Bx^4) \quad (B = A^4)$

$\Rightarrow 1+2V^2 = Bx^4$

$\Rightarrow 1+2\left(\frac{y}{x}\right)^2 = Bx^4$

$\Rightarrow x^2 + 2y^2 = Bx^6$

APPLY CONDITION $\left(1, \frac{1}{\sqrt{2}}\right)$

$\Rightarrow 1 + 1 = B$

$\Rightarrow B = 2$

$\therefore x^2 + 2y^2 = 2x^6$

$2y^2 = 2x^6 - x^2$

$y^2 = x^6 - \frac{1}{2}x^2$

AS REQUIRED

Question 12 (****)

By using the substitution $y = xv$, where $v = f(x)$, solve the differential equation

$$\frac{dy}{dx} = \frac{x^3 + y^3}{xy^2}$$

subject to the condition $y = 1$ at $x = 1$.

$$\boxed{y^3 = x^3\left(3\ln x + 1\right)}$$

$$\frac{dy}{dx} = \frac{x^3+y^3}{xy^2}$$

$$y = xv$$
$$\frac{dy}{dx} = 1 \times v + x\frac{dv}{dx}$$
$$v = \frac{y}{x}$$

$$\Rightarrow v + x\frac{dv}{dx} = \frac{x^3 + x^3v^3}{x(x^2v^2)}$$
$$\Rightarrow v + x\frac{dv}{dx} = \frac{1+v^3}{v^2}$$
$$\Rightarrow x\frac{dv}{dx} = \frac{1}{v^2} + \cancel{v - v}$$
$$\Rightarrow v^2\,dv = \frac{1}{x}\,dx$$
$$\Rightarrow \int v^2\,dv = \int \frac{1}{x}\,dx$$
$$\Rightarrow \frac{1}{3}v^3 = \ln|x| + A$$
$$\Rightarrow v^3 = 3\ln|x| + B$$
$$\Rightarrow \frac{y^3}{x^3} = 3\ln|x| + B$$
$$\Rightarrow y^3 = 3x^3\ln|x| + Bx^3$$

$1 = B$ since $x=1$, $y=1$

$$\Rightarrow y^3 = 3x^3\ln x + x^3$$

Question 13 (****)

By using the substitution $y = xz$, where $z = f(x)$, solve the differential equation

$$2x^2\frac{dy}{dx} = x^2 + y^2, \ x > 0$$

subject to the condition $y = 0$ at $x = 1$.

$$\boxed{y = x - \frac{2x}{2 + \ln x}}$$

$$2x^2\frac{dy}{dx} = x^2 + y^2$$

$$y = xz(x)$$
$$\frac{dy}{dx} = 1 \times z + x\frac{dz}{dx}$$
$$z = \frac{y}{x}$$

$$\Rightarrow 2\cancel{x^2}\left[z + x\frac{dz}{dx}\right] = \cancel{x^2} + \cancel{x^2}z^2$$
$$\Rightarrow 2z + 2x\frac{dz}{dx} = 1 + z^2$$
$$\Rightarrow 2x\frac{dz}{dx} = 1 - 2z + z^2$$
$$\Rightarrow 2x\frac{dz}{dx} = (z-1)^2$$
$$\Rightarrow \frac{1}{(z-1)^2}\,dz = \frac{1}{2x}\,dx$$
$$\Rightarrow \int \frac{1}{(z-1)^2}\,dz = \int \frac{1}{2x}\,dx$$
$$\Rightarrow -\frac{1}{z-1} = \frac{1}{2}\ln x + C$$
$$\Rightarrow \frac{1}{z-1} = C - \frac{1}{2}\ln x$$
$$\Rightarrow z - 1 = \frac{1}{C - \frac{1}{2}\ln x}$$
$$\Rightarrow z = \frac{1}{C - \frac{1}{2}\ln x} + 1$$
$$\Rightarrow \frac{y}{x} = \frac{2}{A - \ln x} + 1$$
$$\Rightarrow \boxed{y = \frac{2x}{A - \ln x} + x}$$

Apply $(1,0) \Rightarrow 0 = \frac{2}{A} + 1$

$\Rightarrow A = -$

$$\therefore y = x - \frac{2x}{2 + \ln x}$$

Question 14 (****)

By using the substitution $y = xv$, where $v = f(x)$, solve the differential equation

$$x\frac{dy}{dx} - y = x\cos\left(\frac{y}{x}\right), \; x \neq 0$$

subject to the condition $y = \pi$ at $x = 4$.

The final answer may not involve natural logarithms.

$$\boxed{\sec\left(\frac{y}{x}\right) + \tan\left(\frac{y}{x}\right) = \frac{1}{4}x\left(1+\sqrt{2}\right)}$$

$$y = xv$$
$$\frac{dy}{dx} = 1 \times v + x\frac{dv}{dx}$$
$$v = \frac{y}{x}$$

$$x\frac{dy}{dx} - y = x\cos\left(\frac{y}{x}\right)$$
$$\Rightarrow x\left(v + x\frac{dv}{dx}\right) - xv = x\cos\left(\frac{xv}{x}\right)$$
$$\Rightarrow \cancel{xv} + x^2\frac{dv}{dx} - \cancel{xv} = x\cos v$$
$$\Rightarrow x\frac{dv}{dx} = \cos v$$
$$\Rightarrow \frac{1}{\cos v}\,dv = \frac{1}{x}\,dx$$
$$\Rightarrow \int \sec v \, dv = \int \frac{1}{x}\,dx$$
$$\Rightarrow \ln|\sec v + \tan v| = \ln|x| + \ln A$$
$$\Rightarrow \ln|\sec v + \tan v| = \ln|Ax|$$
$$\Rightarrow \sec v + \tan v = Ax$$
$$\Rightarrow \boxed{\sec\left(\frac{y}{x}\right) + \tan\left(\frac{y}{x}\right) = Ax}$$

APPLY CONDITION $x=4$ $y=\pi$

$$\sec\frac{\pi}{4} + \tan\frac{\pi}{4} = 4A$$
$$\sqrt{2} + 1 = 4A$$
$$A = \frac{1}{4}(1+\sqrt{2})$$

$$\therefore \sec\left(\frac{y}{x}\right) + \tan\left(\frac{y}{x}\right) = \frac{1}{4}x(1+\sqrt{2})$$

Question 15 (****)

By using the substitution $y = \frac{1}{z}$, or otherwise, solve the differential equation

$$x^2 \frac{dy}{dx} + xy = y^2$$

subject to the condition $y = 2$ at $x = \frac{1}{2}$.

$\boxed{}$, $\boxed{y = \frac{2x}{1-2x^2}}$

USING THE SUBSTITUTION GIVEN

$y = \frac{1}{z} \Rightarrow$ diff w.r.t x $\quad \frac{dy}{dx} = \frac{d}{dx}\left(\frac{1}{z}\right)$

$\frac{dy}{dx} = -\frac{1}{z^2}\frac{dz}{dx}$

TRANSFORM THE O.D.E.

$\Rightarrow x^2 \frac{dy}{dx} + xy = y^2$

$\Rightarrow x^2\left[-\frac{1}{z^2}\frac{dz}{dx}\right] + x\left(\frac{1}{z}\right) = \left(\frac{1}{z}\right)^2$

$\Rightarrow -\frac{x^2}{z^2}\frac{dz}{dx} + \frac{x}{z} = \frac{1}{z^2}$

$\Rightarrow -x^2\frac{dz}{dx} + xz = 1$

$\Rightarrow \frac{dz}{dx} - \frac{z}{x} = -\frac{1}{x^2}$

LOOK FOR AN INTEGRATING FACTOR

$e^{\int -\frac{1}{x}dx} = e^{-\ln x} = e^{\ln\left(\frac{1}{x}\right)} = \frac{1}{x}$

HENCE MULTIPLYING BY $\frac{1}{x}$ WILL MAKE THE L.H.S EXACT

$\Rightarrow \frac{1}{x}\frac{dz}{dx} - \frac{z}{x^2} = -\frac{1}{x^3}$

$\Rightarrow \frac{d}{dx}\left(\frac{z}{x}\right) = -\frac{1}{x^3}$

$\Rightarrow \frac{z}{x} = \int -\frac{1}{x^3}\,dx$

$\Rightarrow \frac{z}{x} = \frac{1}{2x^2} + C$

$\Rightarrow z = \frac{1}{2x} + Cx$

$\Rightarrow \frac{1}{y} = \frac{1}{2x} + Cx$

APPLY THE CONDITION $x = \frac{1}{2}$ $y = 2$

$\frac{1}{2} = 1 + \frac{1}{2}C$

$\frac{1}{2}C = -\frac{1}{2}$

$C = -1$

HENCE WE HAVE

$\frac{1}{y} = \frac{1}{2x} - x$

$\frac{1}{y} = \frac{1-2x^2}{2x}$

$y = \frac{2x}{1-2x^2}$

Question 16 **(****)**

Use a suitable substitution to solve the following differential equation.

$$\frac{dy}{dx}+\sqrt{y+1}=y+1, \quad y>-1, \quad y(0)=3.$$

Given the answer in the form $y=f(x)$.

$\boxed{}$, $\boxed{y=e^{x}\pm 2e^{\frac{1}{2}x}}$

USE THE SUBSTITUTION $v=\sqrt{y+1}$

$v^2=y+1$

$2v\frac{dv}{dx}=\frac{dy}{dx}$

$\Rightarrow \frac{dy}{dx}+\sqrt{y+1}=y+1$

$\Rightarrow 2v\frac{dv}{dx}+v=v^2$

$v\neq 0$

$\Rightarrow 2\frac{dv}{dx}+1=v$

$\Rightarrow 2\frac{dv}{dx}=v-1$

$\Rightarrow \frac{2}{v-1}\,dv=1\,dx$

INTEGRATE BOTH SIDES

$\Rightarrow 2\ln|v-1|=x+C$

$\Rightarrow \ln|v-1|=\frac{1}{2}x+D$

$\Rightarrow |v-1|=e^{\frac{1}{2}x+D}$

$\Rightarrow |v-1|=Ae^{\frac{1}{2}x}$

$\Rightarrow |+\sqrt{y+1}-1|=Ae^{\frac{1}{2}x}$ (ORIGINAL SUBSTITUTION)

APPLY CONDITION

$y(0)=3 \Rightarrow |2-1|=A$

$\Rightarrow A=1$

$\Rightarrow |\sqrt{y+1}-1|=e^{\frac{1}{2}x}$

GETTING THE RESULTS AND TIDY UP

$|\sqrt{y+1}-1|=e^{\frac{1}{2}x}$

$\sqrt{y+1}-1=e^{\frac{1}{2}x}$	$-\sqrt{y+1}+1=e^{\frac{1}{2}x}$
$\sqrt{y+1}=e^{\frac{1}{2}x}+1$	$\sqrt{y+1}-1=-e^{\frac{1}{2}x}$
$(\sqrt{y+1})^2=e^{x}+2e^{\frac{1}{2}x}+1$	$\sqrt{y+1}=1-e^{\frac{1}{2}x}$
$y+1=e^{x}+2e^{\frac{1}{2}x}+1$	$y+1=1-2e^{\frac{1}{2}x}+e^{x}$
$y=e^{x}+2e^{\frac{1}{2}x}$	$y=e^{x}-2e^{\frac{1}{2}x}$

$\therefore y=e^{x}\pm 2e^{\frac{1}{2}x}$

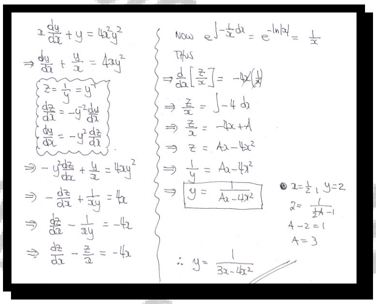
Question 17 (****)

By using the substitution $z = \dfrac{1}{y}$, or otherwise, solve the differential equation

$$x\frac{dy}{dx} + y = 4x^2 y^2$$

subject to the condition $y = 2$ at $x = \dfrac{1}{2}$.

$$\boxed{y = \frac{1}{3x - 4x^2}}$$



$x\dfrac{dy}{dx} + y = 4x^2y^2$

$\Rightarrow \dfrac{dy}{dx} + \dfrac{y}{x} = 4xy^2$

$z = \dfrac{1}{y} = y^{-1}$

$\dfrac{dz}{dx} = -y^{-2}\dfrac{dy}{dx}$

$\dfrac{dy}{dx} = -y^2\dfrac{dz}{dx}$

$\Rightarrow -y^2\dfrac{dz}{dx} + \dfrac{y}{x} = 4xy^2$

$\Rightarrow -\dfrac{dz}{dx} + \dfrac{1}{xy} = 4x$

$\Rightarrow \dfrac{dz}{dx} - \dfrac{1}{xy} = -4x$

$\Rightarrow \dfrac{dz}{dx} - \dfrac{z}{x} = -4x$

Now $e^{\int -\frac{1}{x}dx} = e^{-\ln|x|} = \dfrac{1}{x}$

Thus

$\Rightarrow \dfrac{d}{dx}\left[\dfrac{z}{x}\right] = -4\cancel{x}\left(\dfrac{1}{\cancel{x}}\right)$

$\Rightarrow \dfrac{z}{x} = \int -4\,dx$

$\Rightarrow \dfrac{z}{x} = -4x + A$

$\Rightarrow z = Ax - 4x^2$

$\Rightarrow \dfrac{1}{y} = Ax - 4x^2$

$\Rightarrow \boxed{y = \dfrac{1}{Ax - 4x^2}}$

$x = \frac{1}{2},\ y = 2$

$2 = \dfrac{1}{\frac{1}{2}A - 1}$

$A - 2 = 1$

$A = 3$

$\therefore y = \dfrac{1}{3x - 4x^2}$

Question 18 (****)

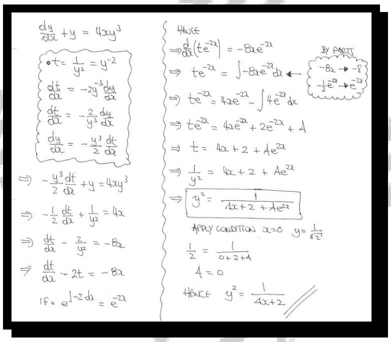
By using the substitution $t = \dfrac{1}{y^2}$, or otherwise, solve the differential equation

$$\frac{dy}{dx} + y = 4xy^3$$

subject to the boundary condition $y = \dfrac{1}{\sqrt{2}}$ at $x = 0$.

Give the answer in the form $y^2 = f(x)$.

$$\boxed{y^2 = \frac{1}{4x+2}}$$



$\frac{dy}{dx} + y = 4xy^3$

- $t = \frac{1}{y^2} = y^{-2}$

$\frac{dt}{dx} = -2y^{-3}\frac{dy}{dx}$

$\frac{dt}{dx} = -\frac{2}{y^3}\frac{dy}{dx}$

$\frac{dy}{dx} = -\frac{y^3}{2}\frac{dt}{dx}$

$\Rightarrow -\frac{y^3}{2}\frac{dt}{dx} + y = 4xy^3$

$\Rightarrow -\frac{1}{2}\frac{dt}{dx} + \frac{1}{y^2} = 4x$

$\Rightarrow \frac{dt}{dx} - \frac{2}{y^2} = -8x$

$\Rightarrow \frac{dt}{dx} - 2t = -8x$

$IF = e^{\int -2\,dx} = e^{-2x}$

HENCE

$\Rightarrow \frac{d}{dx}\left(te^{-2x}\right) = -8xe^{-2x}$

$\Rightarrow te^{-2x} = \int -8xe^{-2x}\,dx$

BY PARTS

$-8x \rightarrow -8$

$-\frac{1}{2}e^{-2x} \rightarrow e^{-2x}$

$\Rightarrow te^{-2x} = 4xe^{-2x} - \int 4e^{-2x}\,dx$

$\Rightarrow te^{-2x} = 4xe^{-2x} + 2e^{-2x} + A$

$\Rightarrow t = 4x + 2 + Ae^{2x}$

$\Rightarrow \frac{1}{y^2} = 4x + 2 + Ae^{2x}$

$\Rightarrow \boxed{y^2 = \frac{1}{4x+2+Ae^{2x}}}$

APPLY CONDITION $x=0$ $y = \frac{1}{\sqrt{2}}$

$\frac{1}{2} = \frac{1}{0+2+A}$

$A = 0$

HENCE $y^2 = \frac{1}{4x+2}$

Question 19 (****)

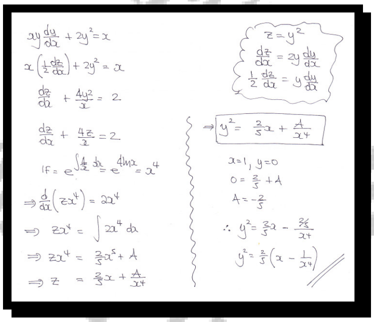
By using the substitution $z = y^2$ or otherwise, solve the differential equation

$$xy\frac{dy}{dx} + 2y^2 = x$$

subject to the boundary condition $y = 0$ at $x = 1$.

Give the answer in the form $y^2 = f(x)$.

$$\boxed{y^2 = \frac{2}{5}\left(x - \frac{1}{x^4}\right)}$$



$z = y^2$

$\frac{dz}{dx} = 2y\frac{dy}{dx}$

$\frac{1}{2}\frac{dz}{dx} = y\frac{dy}{dx}$

$xy\frac{dy}{dx} + 2y^2 = x$

$x\left(\frac{1}{2}\frac{dz}{dx}\right) + 2y^2 = x$

$\frac{dz}{dx} + \frac{4y^2}{x} = 2$

$\frac{dz}{dx} + \frac{4z}{x} = 2$

$\text{IF} = e^{\int \frac{4}{x}\,dx} = e^{4\ln x} = x^4$

$\Rightarrow \frac{d}{dx}\left(zx^4\right) = 2x^4$

$\Rightarrow zx^4 = \int 2x^4\,dx$

$\Rightarrow zx^4 = \frac{2}{5}x^5 + A$

$\Rightarrow z = \frac{2}{5}x + \frac{A}{x^4}$

$\Rightarrow \boxed{y^2 = \frac{2}{5}x + \frac{A}{x^4}}$

$x=1,\ y=0$

$0 = \frac{2}{5} + A$

$A = -\frac{2}{5}$

$\therefore y^2 = \frac{2}{5}x - \frac{2/5}{x^4}$

$y^2 = \frac{2}{5}\left(x - \frac{1}{x^4}\right)$

Question 20 (****)

$$\frac{dy}{dx}+\frac{2y}{x}=y^4,\ x>0,\ y>0.$$

Use the substitution $u=y^{-3}$ and the boundary condition $y=1$ at $x=1$, to show that

$$y^3=\frac{5}{3x+2x^6}.$$

proof

$$\boxed{\frac{dy}{dx}+\frac{2y}{x}=y^4}$$

$u=\frac{1}{y^3}$ OR $u=y^{-3}$

$$\frac{du}{dx}=-3y^{-4}\frac{dy}{dx}$$

$$\frac{du}{dx}=-\frac{3}{y^4}\frac{dy}{dx}$$

$$\boxed{-\frac{y^4}{3}\frac{du}{dx}=\frac{dy}{dx}}$$

THUS TRANSFORMING THE O.D.E

$\Rightarrow -\frac{y^4}{3}\frac{du}{dx}+\frac{2y}{x}=y^4$ (MULTIPLY BY $\frac{-3}{y^4}$)

$\Rightarrow \frac{du}{dx}-\frac{6}{xy^3}=-3$

$\Rightarrow \frac{du}{dx}-\frac{6}{x}\left(\frac{1}{y^3}\right)=-3$

$\Rightarrow \boxed{\frac{du}{dx}-\frac{6u}{x}=-3}$

$$\text{I.F.}=e^{\int -\frac{6}{x}\,dx}=e^{-6\ln x}=e^{\ln x^{-6}}=\frac{1}{x^6}$$

THUS

$\frac{d}{dx}\left(u\times\frac{1}{x^6}\right)=-3\times\frac{1}{x^6}$

$\Rightarrow \frac{u}{x^6}=\int -\frac{3}{x^6}\,dx$

$\Rightarrow \frac{u}{x^6}=\int -3x^{-6}\,dx$

$\Rightarrow \frac{u}{x^6}=\frac{3}{5}x^{-5}+A$

$\Rightarrow u=\frac{3}{5}x+Ax^6$

$\Rightarrow \frac{1}{y^3}=\frac{3}{5}x+Ax^6$

$\Rightarrow y^3=\frac{1}{\frac{3}{5}x+Ax^6}$

$\Rightarrow y^3=\frac{5}{3x+Bx^6}$

NOW $x=1$ $y=1$ $\Rightarrow 1=\frac{5}{3+B}$

$\Rightarrow B+3=5$

$\Rightarrow \boxed{B=2}$

$\therefore y^3=\frac{5}{3x+2x^6}$

AS REQUIRED

Question 21 **(****)**

By using the substitution $t = \dfrac{1}{y^2}$ or otherwise, solve the differential equation

$$\frac{dy}{dx} + \frac{xy}{1+x^2} = y^3$$

subject to the boundary condition $y = 1$ at $x = 0$.

Give the answer in the form $y^2 = f(x)$.

$$\boxed{y^2 = \frac{1}{\left(1+x^2\right)\left(1-2\arctan x\right)}}$$

$$t = \frac{1}{y^2}$$

$$\frac{dt}{dx} = -\frac{2}{y^3}\frac{dy}{dx}$$

$$\frac{dy}{dx} = -\frac{1}{2}y^3\frac{dt}{dx}$$

$$\frac{dy}{dx} + \frac{xy}{1+x^2} = y^3$$

$$\Rightarrow -\frac{1}{2}y^3\frac{dt}{dx} + \frac{xy}{1+x^2} = y^3$$

$$\Rightarrow \frac{dt}{dx} - \frac{2x}{y^2(1+x^2)} = -2$$

$$\Rightarrow \frac{dt}{dx} - \frac{2xt}{1+x^2} = -2$$

$$\text{I.F} = e^{\int \frac{-2x}{1+x^2}\,dx} = e^{-\ln(1+x^2)} = \frac{1}{1+x^2}$$

$$\Rightarrow \frac{d}{dx}\left(t \times \frac{1}{1+x^2}\right) = \frac{-2}{1+x^2}$$

$$\Rightarrow \frac{t}{1+x^2} = \int \frac{-2}{1+x^2}\,dx$$

$$\Rightarrow \frac{t}{1+x^2} = A - 2\arctan x$$

$$\Rightarrow t = A(1+x^2) - 2(1+x^2)\arctan x$$

$$\Rightarrow \frac{1}{y^2} = (1+x^2)(A - 2\arctan x)$$

$$\Rightarrow \boxed{y^2 = \frac{1}{(1+x^2)(A - 2\arctan x)}}$$

when $x=0$ $y=1$ $\quad 1 = \dfrac{1}{1 \times (A-0)}$

$$A = 1$$

$$\therefore y^2 = \frac{1}{(1+x^2)(1-2\arctan x)}$$

Question 22 (****)

By using the substitution $z = x + y$ solve the differential equation

$$\frac{dy}{dx} = \frac{x+y}{4-3(x+y)},$$

subject to the boundary condition $y = 1$ at $x = 0$.

$$\boxed{2\ln|x+y-2| = 3 - x - 3y}$$

$$\frac{dy}{dx} = \frac{x+y}{4-3(x+y)}$$

$z = x+y \iff y = z - x$

$\frac{dz}{dx} = 1 + \frac{dy}{dx}$

$\frac{dy}{dx} = \frac{dz}{dx} - 1$

$\Rightarrow \frac{dz}{dx} - 1 = \frac{z}{4-3z}$

$\Rightarrow \frac{dz}{dx} = \frac{z}{4-3z} + 1$

$\Rightarrow \frac{dz}{dx} = \frac{z+4-3z}{4-3z}$

$\Rightarrow \frac{dz}{dx} = \frac{4-2z}{4-3z}$

$\Rightarrow \int \frac{4-3z}{4-2z}\,dz = \int 1\,dx$

$\Rightarrow \int \frac{8-6z}{4-2z} = \int 2\,dx$

$\Rightarrow \int \frac{3(4-2z)-4}{4-2z}\,dz \int 2\,dx$

$\Rightarrow \int 3 - \frac{4}{4-2z}\,dz = \int 2\,dx$

$\Rightarrow 3z + 2\ln|2-z| = 2x + C$

$x=0,\ y=1 \Rightarrow z=1$

$3 + \cancel{2\ln 1} = 0 + C$

$\boxed{C=3}$

$\Rightarrow 3(x+y) + 2\ln|2-x-y| = 2x+3$

$\Rightarrow -x-3y+3 = 2\ln|2-x-y|$

$\left(\text{OR } 3-x-3y = 2\ln|x+y-2|\right)$

Question 23 (****)

By using the substitution $u = \dfrac{1}{y^4}$, or otherwise, solve the differential equation

$$\frac{dy}{dx} = y\left(1 + xy^4\right)$$

subject to the condition $y = 1$ at $x = 0$.

$$\boxed{\frac{1}{y^4} = \frac{1}{4}\left(1 + 3e^{-4x}\right) - x}$$

$$\frac{dy}{dx} = y(1+xy^4)$$

$$\Rightarrow \frac{dy}{dx} = y + xy^5$$

$$\Rightarrow \frac{dy}{dx} - y = xy^5$$

$$\Rightarrow -\frac{1}{4}y^5\frac{du}{dx} - y = xy^5$$

$$\Rightarrow \frac{du}{dx} + \frac{4}{y^4} = -4x$$

$$\Rightarrow \frac{du}{dx} + 4u = -4x$$

$u = \frac{1}{y^4}$

$\frac{du}{dx} = -\frac{4}{y^5}\frac{dy}{dx}$

$\frac{dy}{dx} = -\frac{1}{4}y^5\frac{du}{dx}$

- I.F. $= e^{\int 4\,dx} = e^{4x}$

$$\Rightarrow \frac{d}{dx}\left(ue^{4x}\right) = -4xe^{4x}$$

$$\Rightarrow ue^{4x} = \int -4xe^{4x} \leftarrow \text{BY PARTS}$$

$-4x$	-4
$\frac{1}{4}e^{4x}$	e^{4x}

$$\Rightarrow ue^{4x} = -xe^{4x} + \int e^{4x}\,dx$$

$$\Rightarrow ue^{4x} = -xe^{4x} + \frac{1}{4}e^{4x} + A$$

$$\Rightarrow u = -x + \frac{1}{4} + Ae^{-4x}$$

$$\Rightarrow \frac{1}{y^4} = \left(\frac{1}{4} - x\right) + Ae^{-4x}$$

APPLY CONDITION

$1 = \left(\frac{1}{4} - 0\right) + Ae^0$

$1 = \frac{1}{4} + A$

$\boxed{A = \frac{3}{4}}$

$$\Rightarrow \frac{1}{y^4} = \left(\frac{1}{4} - x\right) + \frac{3}{4}e^{-4x}$$

$$\Rightarrow \frac{1}{y^4} = \frac{1}{4}\left(1 + e^{-4x}\right) - x$$

OR $y^4 = \dfrac{4}{1 - 4x + e^{-4x}}$

Question 24 (****)

By using the substitution $y = xv$, where $v = f(x)$, or otherwise, solve the differential equation

$$\frac{dy}{dx} = \frac{3x+2y}{3y-2x},$$

subject to the condition $y = 3$ at $x = 1$.

Give the final answer in the form $F(x, y) = 12$

, $\boxed{3y^2 - 4xy - 3x^2 = 12}$

USING THE SUBSTITUTION GIVEN

$\Rightarrow y = xV(x)$

$\Rightarrow \frac{dy}{dx} = 1 \times V(x) + x\frac{dv}{dx}$

$\Rightarrow \frac{dy}{dx} = V + x\frac{dv}{dx}$

SUBSTITUTING INTO THE O.D.E.

$\Rightarrow \frac{dy}{dx} = \frac{3x+2y}{3y-2x}$

$\Rightarrow v + x\frac{dv}{dx} = \frac{3x+2xv}{3xv-2x}$

$\Rightarrow v + x\frac{dv}{dx} = \frac{3+2v}{3v-2}$

$\Rightarrow x\frac{dv}{dx} = \frac{3+2v}{3v-2} - v$

$\Rightarrow x\frac{dv}{dx} = \frac{3+2v-3v^2+2v}{3v-2}$

$\Rightarrow x\frac{dv}{dx} = \frac{3+4v-3v^2}{3v-2}$

SEPARATING VARIABLES

$\Rightarrow \int \frac{3v-2}{3+4v-3v^2}\,dv = \int \frac{1}{x}\,dx$

$\Rightarrow \int \frac{3v-2}{3v^2-4v-3}\,dv = \int -\frac{1}{x}\,dx$

$\Rightarrow \int \frac{6v-4}{3v^2-4v-3}\,dv = \int -\frac{2}{x}\,dx$

$\Rightarrow \ln|3v^2-4v-3| = -2\ln|x| + \ln A$

$\Rightarrow \ln|3v^2-4v-3| = \ln\left(\frac{1}{x^2}\right) + \ln A$

$\Rightarrow \ln|3v^2-4v-3| = \ln\left|\frac{A}{x^2}\right|$

$\Rightarrow 3v^2-4v-3 = \frac{A}{x^2}$

REVERSING THE TRANSFORMATION

$\Rightarrow 3\left(\frac{y}{x}\right)^2 - 4\left(\frac{y}{x}\right) - 3 = \frac{A}{x^2}$

$\Rightarrow \frac{3y^2}{x^2} - \frac{4y}{x} - 3 = \frac{A}{x^2}$

$\Rightarrow 3y^2 - 4xy - 3x^2 = A$

APPLY CONDITION $(1,3)$

$3\times3^2 - 4\times1\times3 - 3\times1^2 = A$

$27 - 12 - 3 = A$

$A = 12$

$\therefore 3y^2 - 4xy - 3x^2 = 12$

ALTERNATIVE SUBSTITUTION

$\Rightarrow \frac{dy}{dx} = \frac{3x+2y}{3y-2x}$

$\Rightarrow 3\frac{dy}{dx} = \frac{9x+6y}{3y-2x}$

$\Rightarrow \frac{dv}{dx} + 2 = \frac{9x+(2v+4x)}{v}$

$\Rightarrow \frac{dv}{dx} + 2 = \frac{2v+13x}{v}$

$\Rightarrow \frac{dv}{dx} + \cancel{2} = \cancel{2} + \frac{13x}{v}$

$\Rightarrow \int v\,dv = \int 13x\,dx$

$\Rightarrow \frac{1}{2}v^2 = \frac{13}{2}x^2 + C$

$\Rightarrow v^2 = 13x^2 + C$

$\Rightarrow (3y-2x)^2 = 13x^2 + C$

LET $V(x) = 3y - 2x$

$\frac{dv}{dx} = 3\frac{dy}{dx} - 2$

$3\frac{dy}{dx} = \frac{dv}{dx} + 2$

ALSO WE HAVE

$3y = v + 2x$

$6y = 2v + 4x$

APPLY $(1,3)$ $\Rightarrow (9-2)^2 = 13\times1^2 + c$

$\Rightarrow 49 = 13 + c$

$\Rightarrow c = 36$

$\Rightarrow (3y-2x)^2 = 13x^2 + 36$

$\Rightarrow 9y^2 - 12xy + 4x^2 = 13x^2 + 36$

$\Rightarrow 9y^2 - 12xy - 9x^2 = 36$

$\Rightarrow 3y^2 - 4xy - 3x^2 = 12$

AS BEFORE

ALTERNATIVE BY MULTIVARIABLE CALCULUS

$\Rightarrow \frac{dy}{dx} = \frac{3x+2y}{3y-2x}$

$\Rightarrow (3y-2x)\,dy = (3x+2y)\,dx$

$\Rightarrow (3x+2y)\,dx + (2x-3y)\,dy = 0$

$\frac{\partial G}{\partial x}\,dx + \frac{\partial G}{\partial y}\,dy = dG$

EVIDENTLY THIS IS EXACT AS $\frac{\partial^2 G}{\partial x \partial y} = \frac{\partial^2 G}{\partial y \partial x} = 2$

HENCE WE HAVE BY DIRECT INTEGRATION

- $dG = 0 \Rightarrow G(x,y) = \text{constant}$
- $\frac{\partial G}{\partial x} = 3x + 2y \Rightarrow G(x,y) = \frac{3}{2}x^2 + 2xy + f(y)$
- $\frac{\partial G}{\partial y} = 2x - 3y \Rightarrow G(x,y) = 2xy - \frac{3}{2}y^2 + g(x)$

$\therefore f(y) = -\frac{3}{2}y^2$ & $g(x) = \frac{3}{2}x^2$

THUS WE OBTAIN

$G(x,y) = \text{constant}$

$\frac{3}{2}x^2 + 2xy - \frac{3}{2}y^2 = \text{constant}$

$3x^2 + 4xy - 3y^2 = \text{constant}$

$(1,3)$ NOW YIELDS

$3 + 12 - 27 = \text{constant}$

$\text{constant} = -12$

$3x^2 + 4xy - 3y^2 = -12$

$3y^2 - 4xy - 3x^2 = 12$

AS BEFORE

Question 25 **(****)**

By using the substitution $y = \mathrm{e}^z$, or otherwise, solve the differential equation

$$x\frac{dy}{dx} + y\ln y = 2xy,$$

subject to the condition $y = \mathrm{e}^2$ at $x = 1$.

$$\boxed{y = \mathrm{e}^{x+\frac{1}{x}}}$$

$$x\frac{dy}{dx} + y\ln y = 2xy$$

$$\boxed{y = e^z \qquad \frac{dy}{dx} = e^z\frac{dz}{dx}}$$

$$\Rightarrow x\left(\cancel{e^z}\frac{dz}{dx}\right) + \cancel{e^z}\left(\ln e^z\right) = 2x\cancel{e^z}$$

$$\Rightarrow x\frac{dz}{dx} + z = 2x$$

BY INTEGRATING FACTOR OR NOTICING LHS IS EXACT

$$\Rightarrow \frac{d}{dx}(xz) = 2x$$

$$\Rightarrow xz = \int 2x \, dx.$$

$$\Rightarrow xz = x^2 + A$$

$$\Rightarrow \boxed{x\ln y = x^2 + A}$$

$$(1, e^2) \Rightarrow 2 = 1 + A$$

$$\Rightarrow \boxed{A = 1}$$

$$\Rightarrow x\ln y = x^2 + 1$$

$$\Rightarrow \ln y = x + \frac{1}{x}$$

$$\Rightarrow y = e^{x+\frac{1}{x}}$$

Question 26 **(****)**

By using the substitution $z = \sin y$, or otherwise, solve the differential equation

$$x\frac{dy}{dx}\cos y - \sin y = x^2 \ln x,$$

subject to the condition $y = 0$ at $x = 1$.

$$\boxed{\sin y = x^2 \ln x - x^2 + x}$$

$$x\frac{dy}{dx}\cos y - \sin y = x^2\ln x$$

$$\boxed{z = \sin y \qquad \frac{dz}{dx} = \cos y\frac{dy}{dx} \qquad \frac{dy}{dx} = \frac{1}{\cos y}\frac{dz}{dx}}$$

$$\frac{x}{\cos y}\frac{dz}{dx}\cos y - z = x^2\ln x$$

$$x\frac{dz}{dx} - z = x^2\ln x$$

$$\boxed{\frac{dz}{dx} - \frac{z}{x} = x\ln x}$$

$$\text{I.F.} = e^{\int -\frac{1}{x}dx} = e^{-\ln x} = \frac{1}{x}$$

$$\therefore \frac{d}{dx}\left(z \times \frac{1}{x}\right) = \ln x$$

$$\Rightarrow \frac{z}{x} = \int \ln x \, dx$$

$$\Rightarrow \frac{z}{x} = x\ln x - x + C$$

STANDARD RESULT OR PARTS

$$\Rightarrow \boxed{z = x^2\ln x - x^2 + Cx}$$

$$\Rightarrow \sin y = x^2\ln x - x^2 + Cx$$

USING $(1, 0)$

$$0 = 0 - 1 + C$$

$$\boxed{C = 1}$$

$$\therefore \sin y = x^2\ln x - x^2 + x$$

Question 27 (****)

a) By using the substitution $z = x^2 + y^2$, solve the following differential equation

$$2xy\frac{dy}{dx} + y^2 = 2x - 3x^2,$$

subject to the condition $y = 1$ at $x = 1$.

b) Verify the answer to part **(a)** by using the substitution $z = y^2$ to solve the same differential equation and subject to the same condition.

$\boxed{\quad}$, $\boxed{y^2 = x - x^2 + \frac{1}{x}}$

a) USING THE SUBSTITUTION GIVEN

$\Rightarrow z = x^2 + y^2$

$\Rightarrow \frac{dz}{dx} = 2x + 2y\frac{dy}{dx}$

$\Rightarrow 2y\frac{dy}{dx} = \frac{dz}{dx} - 2x$

$\Rightarrow 2xy\frac{dy}{dx} = x\frac{dz}{dx} - 2x^2$

SUBSTITUTE INTO THE O.D.E

$\Rightarrow 2xy\frac{dy}{dx} + y^2 = 2x - 3x^2 \quad [x=1,\ y=1]$

$\Rightarrow \left[x\frac{dz}{dx} - 2x^2\right] + y^2 = 2x - 3x^2 \quad [x=1,\ z=2]$

$\Rightarrow x\frac{dz}{dx} - 2x^2 + (z - x^2) = 2x - 3x^2$

$\Rightarrow x\frac{dz}{dx} \cancel{-3x^2} + z = 2x \cancel{-3x^2}$

$\Rightarrow \frac{dz}{dx} + \frac{z}{x} = 2$

INTEGRATING FACTOR NEXT (IN FACT THE ODE WAS EXACT)

$e^{\int \frac{1}{x}dx} = e^{\ln x} = x$

THUS WE FINALLY HAVE

$\Rightarrow \frac{d}{dx}(zx) = 2x$

$\Rightarrow \left[zx\right]_{(1,2)}^{(x,z)} = \left[x^2\right]_1^x$

$\Rightarrow zx - 2 = x^2 - 1$

$\Rightarrow zx = x^2 + 1$

$\Rightarrow z = \frac{x^2+1}{x}$

$\Rightarrow x^2 + y^2 = x + \frac{1}{x}$

$\Rightarrow y^2 = x + \frac{1}{x} - x^2$

b) REWRITE THE O.D.E AS

$\Rightarrow 2xy\frac{dy}{dx} + y^2 = 2x - 3x^2$

$\Rightarrow \frac{dy}{dx} + \frac{y}{2x} = \frac{2x - 3x^2}{2xy}$

$\Rightarrow \frac{dy}{dx} + \frac{y}{2x} = \left(1 - \frac{3}{2}x\right)y^{-1} \leftarrow n = -1$

THIS IS A BERNOULLI TYPE, SO WE USE THE SUBSTITUTION

$z = \frac{1}{y^{n-1}}$ HERE $z = \frac{1}{y^{-2}}$

- $z = y^2$

$\frac{dz}{dx} = 2y\frac{dy}{dx}$

$\frac{dy}{dx} = \frac{1}{2y}\frac{dz}{dx}$

RETURNING TO THE O.D.E

$\Rightarrow \frac{dy}{dx} + \frac{y}{2x} = \left(1 - \frac{3}{2}x\right)y^{-1}$

$\Rightarrow \frac{1}{2y}\frac{dz}{dx} + \frac{y}{2x} = \left(1 - \frac{3}{2}x\right)y^{-1}$

$\Rightarrow \frac{dz}{dx} + \frac{y^2}{x} = 2\left(1 - \frac{3}{2}x\right)$

$\Rightarrow \frac{dz}{dx} + \frac{z}{x} = 2 - 3x$

MULTIPLY THROUGH BY x – OR INTEGRATING FACTOR

$\Rightarrow x\frac{dz}{dx} + z = 2x - 3x^2 \quad [x=1,\ y=1,\ z=1]$

$\Rightarrow \frac{d}{dx}(xz) = 2x - 3x^2$

$\Rightarrow \left[xz\right]_{(1,1)}^{(x,z)} = \int_1^x 2x - 3x^2 \, dx$

$\Rightarrow xz - 1 = \left[x^2 - x^3\right]_1^x$

$\Rightarrow xz - 1 = (x^2 - x^3) - \cancel{(1-1)}$

$\Rightarrow xz = x^2 + 1 - x^3$

$\Rightarrow z = x + \frac{1}{x} - x^2$

$\Rightarrow y^2 = x + \frac{1}{x} - x^2$

AS BEFORE

Question 28 (****+)

A curve C passes through the point $(1,1)$ and satisfies the differential equation

$$\frac{dy}{dx} - \frac{y}{x} = \frac{x^3}{4y^3}, \quad x > 0, \; y > 0,$$

subject to the condition $y = 1$ at $x = 1$.

a) Find an equation of C by using the substitution $z = y^4$.

b) Find an equation of C by using the substitution $v = \dfrac{x}{y}$.

Give the answer in the form $y^4 = f(x)$.

$$\boxed{y^4 = x^4 (1 + \ln x)}$$

(a) $\dfrac{dy}{dx} - \dfrac{y}{x} = \dfrac{x^3}{4y^3}$

$z = \dfrac{1}{y^{-4}} = y^4$

$\dfrac{dz}{dx} = 4y^3 \dfrac{dy}{dx}$

$\dfrac{dy}{dx} = \dfrac{1}{4y^3}\dfrac{dz}{dx}$

$\Rightarrow \dfrac{1}{4y^3}\dfrac{dz}{dx} - \dfrac{y}{x} = \dfrac{x^3}{4y^3}$

$\Rightarrow \dfrac{dz}{dx} - \dfrac{4y^4}{x} = x^3$

$\Rightarrow \dfrac{dz}{dx} - \dfrac{4z}{x} = x^3$

$\text{I.F.} = e^{\int -\frac{4}{x}\,dx} = e^{-4\ln x} = \dfrac{1}{x^4}$

$\Rightarrow \dfrac{d}{dx}\left(\dfrac{z}{x^4}\right) = \dfrac{x^3}{x^4}$

$\Rightarrow \dfrac{z}{x^4} = \int \dfrac{1}{x}\,dx$

$\Rightarrow \dfrac{z}{x^4} = \ln x + A$

$\Rightarrow z = x^4(\ln x + A)$

$\Rightarrow \boxed{y^4 = x^4(\ln x + A)}$

• $x=1 \quad y=1$

$1 = 1(\cancel{\ln 1} + A) \Rightarrow A=1$

$\therefore y^4 = x^4(1 + \ln x)$

(b) $\dfrac{dy}{dx} - \dfrac{y}{x} = \dfrac{x^3}{4y^3}$

$v = \dfrac{x}{y} \Rightarrow y = \dfrac{x}{v}$

$\Rightarrow y = xv^{-1}$

$\Rightarrow \dfrac{dy}{dx} = v^{-1} + x(-v^{-2})\dfrac{dv}{dx}$

$\Rightarrow \dfrac{dy}{dx} = \dfrac{1}{v} - \dfrac{x}{v^2}\dfrac{dv}{dx}$

$\Rightarrow \left(\dfrac{1}{v} - \dfrac{x}{v^2}\dfrac{dv}{dx}\right) - \dfrac{1}{v} = \dfrac{1}{4}v^3$

$\Rightarrow -\dfrac{x}{v^2}\dfrac{dv}{dx} = \dfrac{1}{4}v^3$

$\Rightarrow -\dfrac{4}{v^5}\,dv = \dfrac{1}{x}\,dx$

$\Rightarrow \int -4v^{-5}\,dv = \int \dfrac{1}{x}\,dx$

$\Rightarrow v^{-4} = \ln x + B$

$\Rightarrow \dfrac{1}{v^4} = \ln x + B$

$\Rightarrow \dfrac{y^4}{x^4} = \ln x + B$

$\Rightarrow y^4 = x^4(\ln x + B)$

AS BEFORE IF APPLY SAME CONDITIONS

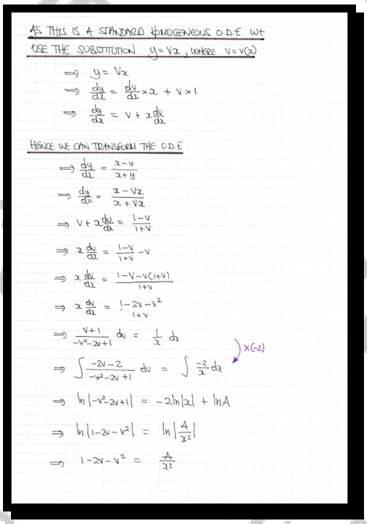
Question 29 (****+)

By using the substitution $y = xv$, where $v = f(x)$, solve the differential equation

$$\frac{dy}{dx} = \frac{x-y}{x+y}$$

subject to the condition $y = 1$ at $x = 1$.

$\boxed{}$, $\boxed{y^2 + 2xy - x^2 = 2}$

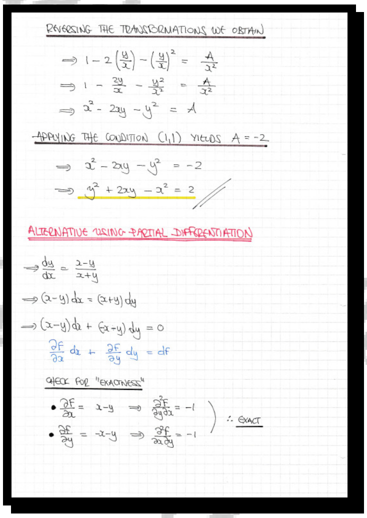


AS THIS IS A STANDARD HOMOGENEOUS O.D.E WE USE THE SUBSTITUTION $y = vx$, WHERE $v = v(x)$

$$\Rightarrow y = vx$$
$$\Rightarrow \frac{dy}{dx} = \frac{dv}{dx} \times x + v \times 1$$
$$\Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$$

HENCE WE CAN TRANSFORM THE O.D.E

$$\Rightarrow \frac{dy}{dx} = \frac{x-y}{x+y}$$
$$\Rightarrow \frac{dy}{dx} = \frac{x - vx}{x + vx}$$
$$\Rightarrow v + x\frac{dv}{dx} = \frac{1-v}{1+v}$$
$$\Rightarrow x\frac{dv}{dx} = \frac{1-v}{1+v} - v$$
$$\Rightarrow x\frac{dv}{dx} = \frac{1 - v - v(1+v)}{1+v}$$
$$\Rightarrow x\frac{dv}{dx} = \frac{1 - 2v - v^2}{1+v}$$
$$\Rightarrow \frac{v+1}{-v^2-2v+1}\,dv = \frac{1}{x}\,dx$$
$$\Rightarrow \int \frac{-2v-2}{-v^2-2v+1}\,dv = \int \frac{-2}{x}\,dx \qquad \times(-2)$$
$$\Rightarrow \ln|-v^2-2v+1| = -2\ln|x| + \ln A$$
$$\Rightarrow \ln|1-2v-v^2| = \ln\left|\frac{A}{x^2}\right|$$
$$\Rightarrow 1 - 2v - v^2 = \frac{A}{x^2}$$



REVERSING THE TRANSFORMATIONS WE OBTAIN

$$\Rightarrow 1 - 2\left(\frac{y}{x}\right) - \left(\frac{y}{x}\right)^2 = \frac{A}{x^2}$$
$$\Rightarrow 1 - \frac{2y}{x} - \frac{y^2}{x^2} = \frac{A}{x^2}$$
$$\Rightarrow x^2 - 2xy - y^2 = A$$

APPLYING THE CONDITION $(1,1)$ YIELDS $A = -2$

$$\Rightarrow x^2 - 2xy - y^2 = -2$$
$$\Rightarrow y^2 + 2xy - x^2 = 2$$

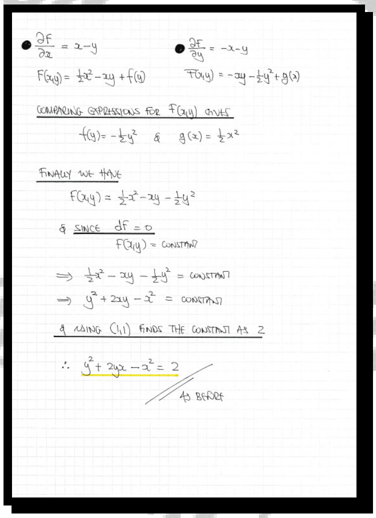
ALTERNATIVE USING PARTIAL DIFFERENTIATION

$$\Rightarrow \frac{dy}{dx} = \frac{x-y}{x+y}$$
$$\Rightarrow (x-y)\,dx = (x+y)\,dy$$
$$\Rightarrow (x-y)\,dx + (-x-y)\,dy = 0$$
$$\frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy = dF$$

CHECK FOR "EXACTNESS"

- $\frac{\partial F}{\partial x} = x - y \Rightarrow \frac{\partial^2 F}{\partial y \partial x} = -1$
- $\frac{\partial F}{\partial y} = -x - y \Rightarrow \frac{\partial^2 F}{\partial x \partial y} = -1$

$\therefore$ EXACT



- $\frac{\partial F}{\partial x} = x - y$ $\quad F(x,y) = \frac{1}{2}x^2 - xy + f(y)$
- $\frac{\partial F}{\partial y} = -x - y$ $\quad F(x,y) = -xy - \frac{1}{2}y^2 + g(x)$

COMPARING EXPRESSIONS FOR $F(x,y)$ GIVES

$$f(y) = -\frac{1}{2}y^2 \quad \& \quad g(x) = \frac{1}{2}x^2$$

FINALLY WE HAVE

$$F(x,y) = \frac{1}{2}x^2 - xy - \frac{1}{2}y^2$$

& SINCE $dF = 0$

$$F(x,y) = \text{CONSTANT}$$
$$\Rightarrow \frac{1}{2}x^2 - xy - \frac{1}{2}y^2 = \text{CONSTANT}$$
$$\Rightarrow y^2 + 2xy - x^2 = \text{CONSTANT}$$

& USING $(1,1)$ FINDS THE CONSTANT AS 2

$$\therefore y^2 + 2yx - x^2 = 2$$

AS BEFORE

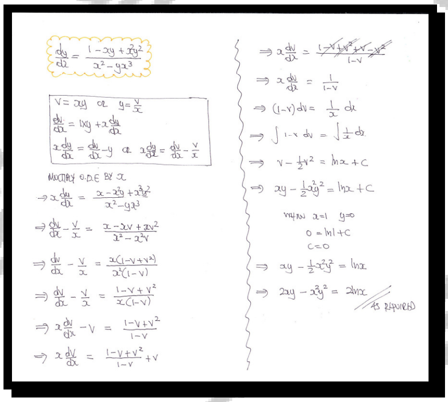
Question 30 (****+)

By using the substitution $v = xy$, where $y = f(x)$, solve the differential equation

$$\frac{dy}{dx} = \frac{1 - xy + x^2y^2}{x^2 - yx^3}, \quad x > 0,$$

subject to the condition $y = 0$ at $x = 1$.

$$\boxed{2xy - x^2y^2 = 2\ln x}$$

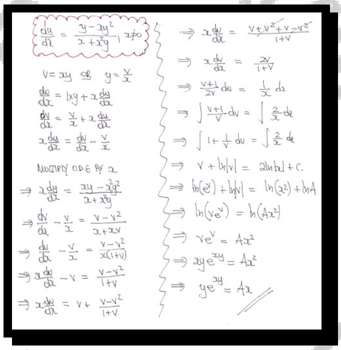


Question 31 (****+)

Use the substitution $v = xy$, where $y = f(x)$, to find a general solution for the differential equation

$$\frac{dy}{dx} = \frac{y - xy^2}{x + yx^2}, \quad x \neq 0.$$

$$\boxed{y\,\mathrm{e}^{xy} = Cx}$$



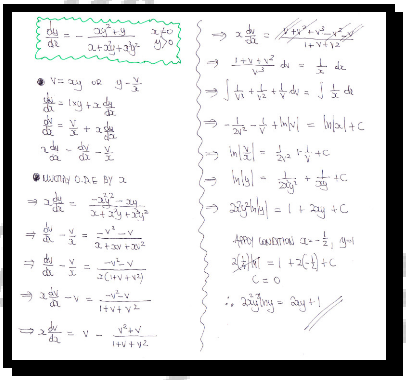
Question 32 (****+)

By using the substitution $v = xy$, where $y = f(x)$, solve the differential equation

$$\frac{dy}{dx} = -\frac{xy^2 + y}{x + yx^2 + x^3y^2}, \quad x \neq 0, \quad y > 0,$$

subject to the condition $y = 1$ at $x = \frac{1}{2}$.

$$\boxed{2x^2y^2 \ln y = 2xy + 1}$$



$\frac{dy}{dx} = -\frac{xy^2+y}{x+x^2y+x^3y^2}$ $\quad x \neq 0$, $y>0$

- $v = xy$ OR $y = \frac{v}{x}$

$\frac{dv}{dx} = 1 \times y + x\frac{dy}{dx}$

$\frac{dv}{dx} = \frac{v}{x} + x\frac{dy}{dx}$

$x\frac{dy}{dx} = \frac{dv}{dx} - \frac{v}{x}$

- MULTIPLY O.D.E BY x

$\Rightarrow x\frac{dy}{dx} = \frac{-x^2y^2 - xy}{x+x^2y+x^3y^2}$

$\Rightarrow \frac{dv}{dx} - \frac{v}{x} = \frac{-v^2-v}{x+xv+xv^2}$

$\Rightarrow \frac{dv}{dx} - \frac{v}{x} = \frac{-v^2-v}{x(1+v+v^2)}$

$\Rightarrow x\frac{dv}{dx} - v = \frac{-v^2-v}{1+v+v^2}$

$\Rightarrow x\frac{dv}{dx} = v - \frac{v^2+v}{1+v+v^2}$

$\Rightarrow x\frac{dv}{dx} = \frac{\cancel{v}+\cancel{v^2}+v^3-\cancel{v^2}-\cancel{v}}{1+v+v^2}$

$\Rightarrow \frac{1+v+v^2}{v^3}\,dv = \frac{1}{x}\,dx$

$\Rightarrow \int \frac{1}{v^3} + \frac{1}{v^2} + \frac{1}{v}\,dv = \int \frac{1}{x}\,dx$

$\Rightarrow -\frac{1}{2v^2} - \frac{1}{v} + \ln|v| = \ln|x| + C$

$\Rightarrow \ln\left|\frac{v}{x}\right| = \frac{1}{2v^2} + \frac{1}{v} + C$

$\Rightarrow \ln|y| = \frac{1}{2x^2y^2} + \frac{1}{xy} + C$

$\Rightarrow 2x^2y^2\ln|y| = 1 + 2xy + C$

APPLY CONDITION $x = -\frac{1}{2}$, $y = 1$

$\cancel{2\left(\frac{1}{4}\right)\ln 1} = 1 + 2\left(-\frac{1}{2}\right) + C$

$C = 0$

$\therefore 2x^2y^2\ln y = 2xy + 1$

Question 33 (****+)

$$\frac{dy}{dx} = \tan\left(x^2 + 2y + \pi\right) - x, \quad y(0) = \tfrac{1}{4}\pi .$$

Solve the above differential equation to show that

$$y = -\tfrac{1}{2}\left[x^2 + \pi + \arcsin\left(e^{2x}\right)\right].$$

$\boxed{}$, $\boxed{\text{proof}}$

USING A SUBSTITUTION

$$t = x^2 + 2y + \pi$$
$$\frac{dt}{dx} = 2x + 2\frac{dy}{dx}$$
$$\frac{dy}{dx} = \frac{1}{2}\frac{dt}{dx} - x$$

TRANSFORMING THE O.D.E

$$\frac{dy}{dx} = \tan(x^2+2y+\pi) - x$$
$$\frac{1}{2}\frac{dt}{dx} - x = \tan t - x$$
$$\frac{dt}{dx} = 2\tan t$$
$$\cot t \, dt = 2 \, dx$$
$$\int \cot t \, dt = \int 2 \, dx$$
$$\ln|\sin t| = 2x + C$$
$$\sin t = e^{2x+C}$$
$$\sin t = Ae^{2x}$$
$$\sin(x^2+2y+\pi) = Ae^{2x}$$

APPLY CONDITION $x=0$, $y=\frac{\pi}{4}$

$$\Rightarrow \sin\left(\frac{3\pi}{2}\right) = Ae^0$$
$$\Rightarrow -1 = A$$

$$\therefore \sin(x^2+2y+\pi) = -e^{2x}$$
$$x^2+2y+\pi = \arcsin(-e^{2x})$$
$$2y = -x^2 - \pi - \arcsin(e^{2x})$$
$$y = -\frac{1}{2}\left[x^2+\pi+\arcsin(e^{2x})\right]$$

Question 34 (****+)

$$\frac{dy}{dx} = \frac{x+y+3}{x+y-1}, \quad y(0) = 0.$$

Use the substitution $z = x + y$, to show that the solution of the above differential equation is

$$y - x - 2\ln(x+y+1) = 0.$$

proof

$$\frac{dy}{dx} = \frac{x+y+3}{x+y-1}$$

$$z = x+y \Rightarrow y = z-x$$
$$\frac{dz}{dx} = 1 + \frac{dy}{dx}$$
$$\frac{dy}{dx} = \frac{dz}{dx} - 1$$

$$\Rightarrow \frac{dz}{dx} - 1 = \frac{z+3}{z-1}$$
$$\Rightarrow \frac{dz}{dx} = \frac{z+3}{z-1} + 1$$
$$\Rightarrow \frac{dz}{dx} = \frac{z+3+z-1}{z-1}$$
$$\Rightarrow \frac{dz}{dx} = \frac{2z+2}{z-1}$$
$$\Rightarrow \frac{z-1}{2(z+1)}\,dz = 1\,dx$$
$$\Rightarrow \int \frac{z-1}{z+1}\,dz = \int 2\,dx$$
$$\Rightarrow \int \frac{(z+1)-2}{(z+1)}\,dz = \int 2\,dx$$
$$\Rightarrow \int 1 - \frac{2}{z+1}\,dx = \int 2\,dx$$
$$\Rightarrow z - 2\ln|z+1| = 2x + C$$
$$\Rightarrow \boxed{x+y-2\ln|x+y+1| = 2x + C}$$

USING $(0,0) \Rightarrow 0+0-2\cancel{\ln 1} = 0 + C$
$$\Rightarrow \boxed{C=0}$$

$$\Rightarrow x+y-2\ln|x+y+1| = 2x$$
$$\Rightarrow y-x-2\ln|x+y+1| = 0$$

AS REQUIRED

Question 35 (****+)

$$\frac{dy}{dx} = \frac{3x - y + 1}{x + y + 1}, \ \ y(1) = 2.$$

a) Show that the transformation equations

$$x = X - \tfrac{1}{2}$$
$$y = Y - \tfrac{1}{2}$$

transform the above differential equation to

$$\frac{dY}{dX} = \frac{3X - Y}{X + Y}.$$

b) Use the substitution $Y = XV$, where $V = f(x)$, to show that the solution of the original differential equation is

$$(y - x)(y + 3x + 2) = 7.$$

proof

(a) $\frac{dy}{dx} = \frac{3x-y+1}{x+y+1}$

$x = X - \frac{1}{2} \Rightarrow 1 = \frac{dX}{dx} \Rightarrow dx = dX$

$y = Y - \frac{1}{2} \Rightarrow \frac{dy}{dx} = \frac{dY}{dx} \Rightarrow \boxed{\frac{dy}{dx} = \frac{dY}{dX}}$

$\frac{dY}{dX} = \frac{3\left(X-\frac{1}{2}\right)-\left(Y-\frac{1}{2}\right)+1}{\left(X-\frac{1}{2}\right)+\left(Y-\frac{1}{2}\right)+1}$

$\frac{dY}{dX} = \frac{3X-Y}{X+Y}$ // AS REQUIRED

(b) $Y = VX$

$\frac{dY}{dX} = 1 \times V + X\frac{dV}{dX}$

HENCE $V + X\frac{dV}{dX} = \frac{3X - VX}{X + VX}$

$\Rightarrow X\frac{dV}{dX} = \frac{3-V}{1+V} - V$

$\Rightarrow X\frac{dV}{dX} = \frac{3-V-V-V^2}{1+V}$

$\Rightarrow X\frac{dV}{dX} = \frac{3-2V-V^2}{1+V}$

$\Rightarrow \frac{V+1}{3-2V-V^2}\,dV = \frac{1}{X}\,dX$

$\Rightarrow \int \frac{-2(V+1)}{3-2V-V^2}\,dV = \int -\frac{2}{X}\,dX$

$\Rightarrow \ln|3-2V-V^2| = -2\ln X + \ln A$

$\Rightarrow \ln|3-2V-V^2| = \ln\left|\frac{A}{X^2}\right|$

$\Rightarrow 3-2V-V^2 = \frac{A}{X^2}$

$\Rightarrow V^2+2V-3 = \frac{B}{X^2}$

$\Rightarrow (V+3)(V-1) = \frac{B}{X^2}$

$\Rightarrow \left(\frac{Y}{X}+3\right)\left(\frac{Y}{X}-1\right) = \frac{B}{X^2}$

$\Rightarrow \frac{1}{X^2}(Y+3X)(Y-X) = \frac{B}{X^2}$

$\Rightarrow (Y+3X)(Y-X) = B$

$\Rightarrow \left[\left(y+\frac{1}{2}\right)+3\left(x+\frac{1}{2}\right)\right]\left[\left(y+\frac{1}{2}\right)-\left(x+\frac{1}{2}\right)\right] = B$

$\Rightarrow (y+3x+2)(y-x) = B$

APPLY CONDITION $x=1$, $y=2$

$(2+3+2)(2-1) = B$

$B = 7$

$\therefore (y+3x+2)(y-x) = 7$ // AS REQUIRED

Question 36 (****+)

$$\frac{dy}{dx}=\frac{2x+5y+3}{4x+y-3}, \quad y(1)=1.$$

a) Show that the transformation equations

$$x = X+1$$
$$y = Y-1$$

transform the above differential equation to

$$\frac{dY}{dX}=\frac{2X+5Y}{4X+Y}.$$

b) Use the substitution $Y = XV$, where $V = f(x)$, to show that the solution of the original differential equation is

$$(y-2x+3)^2 = 2(x+y).$$

$\boxed{}$, $\boxed{\text{proof}}$

a) $\frac{dy}{dx}=\frac{2x+5y+3}{4x+y-3}$

USING THE TRANSFORMATIONS

$x=X+1 \quad dx=dX$

$y=Y-1 \quad dy=dY$

$\Rightarrow \frac{dY}{dX}=\frac{2(X+1)+5(Y-1)+3}{4(X+1)+(Y-1)-3}$

$\Rightarrow \frac{dY}{dX}=\frac{2X+2+5Y-5+3}{4X+4+Y-1-3}$

$\Rightarrow \frac{dY}{dX}=\frac{2X+5Y}{4X+Y}$ // AS REQUIRED

b) USING THE NEXT SUBSTITUTION WE HAVE

$Y=XV \Rightarrow \frac{dY}{dX}=1\times V+X\frac{dV}{dX}$

TRANSFORMING THE O.D.E FURTHER

$\Rightarrow V+X\frac{dV}{dX}=\frac{2X+5XV}{4X+XV}$

$\Rightarrow V+X\frac{dV}{dX}=\frac{2+5V}{4+V}$

$\Rightarrow X\frac{dV}{dX}=\frac{2+5V}{4+V}-V$

$\Rightarrow X\frac{dV}{dX}=\frac{2+5V-4V-V^2}{4+V}$

$\Rightarrow X\frac{dV}{dX}=\frac{-V^2+V+2}{4+V}=-\frac{V^2-V-2}{V+4}=-\frac{(V+1)(V-2)}{V+4}$

SEPARATING VARIABLES

$\Rightarrow \frac{V+4}{(V+1)(V-2)}\,dV=-\frac{1}{X}\,dX$

OBTAIN PARTIAL FRACTIONS & INTEGRATE

$\Rightarrow \int \frac{-1}{V+1}+\frac{2}{V-2}\,dV=\int -\frac{1}{X}\,dX$

$\Rightarrow 2\ln|V-2|-\ln|V+1|=-\ln|X|+\ln A$

$\Rightarrow \ln\left|\frac{(V-2)^2}{V+1}\right|=\ln\left|\frac{A}{X}\right|$

$\Rightarrow \frac{(V-2)^2}{V+1}=\frac{A}{X}$

REVERSING THE TRANSFORMATIONS — $V=\frac{Y}{X}$

$\Rightarrow X(V-2)^2=A(V+1)$

$\Rightarrow X\left(\frac{Y}{X}-2\right)^2=A\left(\frac{Y}{X}+1\right)$

$\Rightarrow X\left(\frac{Y-2X}{X}\right)^2=A\left(\frac{Y+X}{X}\right)$

$\Rightarrow X\frac{(Y-2X)^2}{X^2}=\frac{A}{X}(Y+X)$

$\Rightarrow \frac{(Y-2X)^2}{X}=\frac{A}{X}(Y+X)$

$\Rightarrow (Y-2X)^2=A(Y+X)$

REVERSING THE FINAL TRANSFORMATIONS

$\Rightarrow [(y+1)-2(x-1)]^2=A[(y+1)+(x-1)]$

$\Rightarrow (y-2x+3)^2=A(x+y)$

FINALLY USING THE BOUNDARY CONDITION (1,1)

$(1-2+3)^2=A(1+1)$

$4=2A$

$A=2$

$\therefore (y-2x+3)^2=2(x+y)$ // AS REQUIRED

Question 37 (****+)

$$\frac{dy}{dx}\left(x+y^2\right)=y\,.$$

a) Solve the above differential equation, subject to $y=1$ at $x=1$, by considering $\frac{dx}{dy}$, followed by a suitable substitution.

b) Verify the validity of the answer obtained in part (**a**).

$$\boxed{y^2=x}$$

a) $\boxed{\frac{dy}{dx}(x+y^2)=y}$

$\Rightarrow \frac{dy}{dx}=\frac{y}{x+y^2}$

$\Rightarrow \frac{dx}{dy}=\frac{x+y^2}{y}$

$\Rightarrow \frac{dx}{dy}=\frac{x}{y}+y$

LET $z=\frac{x}{y}$

$x=zy$

DIFF W.R.T y

$\frac{dx}{dy}=\frac{dz}{dy}y+z$

$\Rightarrow \cancel{z}+y\frac{dz}{dy}=\cancel{z}+y$

$\Rightarrow y\frac{dz}{dy}=y$

$\Rightarrow \frac{dz}{dy}=1$

$\Rightarrow dz=dy$

$\Rightarrow z=y+C$

$\Rightarrow \boxed{\frac{x}{y}=y+C}$

APPLY CONDITIONS $(1,1)$ $\therefore \boxed{C=0}$

$\Rightarrow \frac{x}{y}=y$

$\Rightarrow y^2=x$

b) $y^2=x$

Diff w.r.t. x

$2y\frac{dy}{dx}=1$

$\frac{dy}{dx}=\frac{1}{2y}$

MULTIPLY BY $(x+y^2)$

$(x+y^2)\frac{dy}{dx}=\frac{1}{2y}(x+y^2)$

$(x+y^2)\frac{dy}{dx}=\frac{x+y^2}{2y}$

$(x+y^2)\frac{dy}{dx}=\frac{y^2+y^2}{2y}$

$(x+y^2)\frac{dy}{dx}=\frac{2y^2}{2y}$

$(x+y^2)\frac{dy}{dx}=y$

Question 38 (*****)

Use a suitable substitution to solve the following differential equation.

$$\frac{dy}{dx}=(x-y+2)^2, \quad y(0)=4.$$

Given the answer in the form $y=f(x)$.

$$\boxed{}, \quad \boxed{y=\frac{(x+1)e^{2x}\pm 3(x+3)}{e^{2x}\pm 3}=\frac{\pm(x+1)e^{2x}-3(x+3)}{\pm e^{2x}-3}}$$

START WITH THE OBVIOUS SUBSTITUTION

$t=x-y+2$

$\frac{dt}{dx}=1-\frac{dy}{dx}$

$\frac{dy}{dx}=1-\frac{dt}{dx}$

TRANSFORM THE O.D.E

$\Rightarrow \frac{dy}{dx}=(x-y+2)^2$

$\Rightarrow 1-\frac{dt}{dx}=t^2$

$\Rightarrow 1-t^2=\frac{dt}{dx}$

SEPARATE VARIABLES

$\Rightarrow 1\,dx=\frac{1}{1-t^2}\,dt$

$\Rightarrow 1\,dx=\frac{1}{(1+t)(1-t)}\,dt$

PARTIAL FRACTIONS BY INSPECTION

$\Rightarrow 1\,dx=\left[\frac{\frac{1}{2}}{1+t}+\frac{\frac{1}{2}}{1-t}\right]dt$

$\Rightarrow 2\,dx=\left(\frac{1}{1+t}+\frac{1}{1-t}\right)dt$

$\Rightarrow \int 2\,dx=\int \frac{1}{1+t}+\frac{1}{1-t}\,dt$

$\Rightarrow 2x+C=\ln|1+t|-\ln|1-t|$

$\Rightarrow \ln\left|\frac{1+t}{1-t}\right|=2x+C$

$\Rightarrow \left|\frac{1+t}{1-t}\right|=Ae^{2x}$

$\Rightarrow \left|\frac{1+x-y+2}{1-x+y-2}\right|=Ae^{2x}$

$\Rightarrow \left|\frac{x-y+3}{-x+y-1}\right|=Ae^{2x}$

$\Rightarrow \left|\frac{x-y+3}{x-y+1}\right|=Ae^{2x}$

APPLY CONDITION $y(0)=4$

$\left|\frac{0-4+3}{0-4+1}\right|=A$

$A=\left|\frac{-1}{-3}\right|$

$A=\frac{1}{3}$

HENCE IN THE ABSENCE OF ANY OTHER CONSTRAINTS, THERE ARE 2 CASES

$\frac{x-y+3}{x-y+1}=\pm\frac{1}{3}e^{2x}$

CONSIDER EACH CASE SEPARATELY & LET $\pm\frac{1}{3}e^{2x}=E$

$\Rightarrow \frac{x-y+3}{x-y+1}=E$

$\Rightarrow x-y+3=Ex-Ey+E$

$\Rightarrow Ey-y=Ex-x+E-3$

$\Rightarrow y[E-1]=Ex+E-(x+3)$

$\Rightarrow y=\frac{E(x+1)-(x+3)}{E-1}$

$\Rightarrow y=\frac{\pm\frac{1}{3}e^{2x}(x+1)-(x+3)}{\pm\frac{1}{3}e^{2x}-1}$

$\Rightarrow y=\frac{\pm(x+1)e^{2x}-3(x+3)}{\pm e^{2x}-3}$

IN OTHER WORDS WE HAVE

$y=\frac{(x+1)e^{2x}-3(x+3)}{e^{2x}-3}$ OR $y=\frac{-(x+1)e^{2x}-3(x+3)}{-e^{2x}-3}$

$y=\frac{(x+1)e^{2x}+3(x+3)}{e^{2x}+3}$

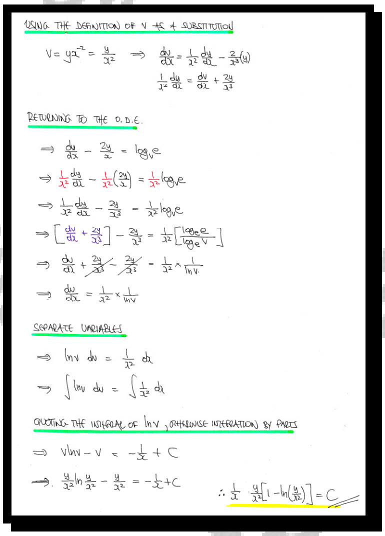
Question 39 (****+)

Given that $v = yx^{-2}$ find a general solution for the following differential equation.

$$\frac{dy}{dx} - \frac{2y}{x} = \log_v \mathrm{e}, \quad u > 0, \quad u \neq 1.$$

Given the answer in the form $f(x, y) = \text{constant}$.

$\boxed{}$, $\boxed{\frac{1}{x} - \frac{y}{x^2}\left[1 - \ln\left(yx^{-2}\right)\right] = \text{constant}}$



USING THE DEFINITION OF V AS A SUBSTITUTION

$V = yx^{-2} = \frac{y}{x^2} \implies \frac{dv}{dx} = \frac{1}{x^2}\frac{dy}{dx} - \frac{2}{x^3}(y)$

$\frac{1}{x^2}\frac{dy}{dx} = \frac{dv}{dx} + \frac{2y}{x^3}$

RETURNING TO THE O.D.E.

$\implies \frac{dy}{dx} - \frac{2y}{x} = \log_v e$

$\implies \frac{1}{x^2}\frac{dy}{dx} - \frac{1}{x^2}\left(\frac{2y}{x}\right) = \frac{1}{x^2}\log_v e$

$\implies \frac{1}{x^2}\frac{dy}{dx} - \frac{2y}{x^3} = \frac{1}{x^2}\log_v e$

$\implies \left[\frac{dv}{dx} + \frac{2y}{x^3}\right] - \frac{2y}{x^3} = \frac{1}{x^2}\left[\frac{\log_e e}{\log_e v}\right]$

$\implies \frac{dv}{dx} + \cancel{\frac{2y}{x^3}} - \cancel{\frac{2y}{x^3}} = \frac{1}{x^2} \times \frac{1}{\ln v}$

$\implies \frac{dv}{dx} = \frac{1}{x^2} \times \frac{1}{\ln v}$

SEPARATE VARIABLES

$\implies \ln v \, dv = \frac{1}{x^2}\, dx$

$\implies \int \ln v \, dv = \int \frac{1}{x^2}\, dx$

QUOTING THE INTEGRAL OF $\ln v$, OTHERWISE INTEGRATION BY PARTS

$\implies v\ln v - v = -\frac{1}{x} + C$

$\implies \frac{y}{x^2}\ln\frac{y}{x^2} - \frac{y}{x^2} = -\frac{1}{x} + C$

$\therefore \frac{1}{x} \ \frac{y}{x^2}\left[1 - \ln\left(\frac{y}{x^2}\right)\right] = C$

Question 40 (*****)

Use a suitable substitution to solve the following differential equation.

$$\frac{dy}{dx}+8xy=y^2+16x^2, \quad y(0)=-6.$$

Given the answer in the form $y=f(x)$.

$\boxed{}$, $\boxed{y=\frac{4x\left(2e^{4x}-1\right)-2\left(2e^{4x}+1\right)}{2e^{4x}-1}}$

REWRITE THE O.D.E

$\Rightarrow \frac{dy}{dx}+8xy=y^2+16x^2$

$\Rightarrow \frac{dy}{dx}=y^2-8xy+16x^2$

$\Rightarrow \frac{dy}{dx}=(y-4x)^2$

NOW A SUBSTITUTION

$\Rightarrow v=y-4x$

$\Rightarrow \frac{dv}{dx}=\frac{dy}{dx}-4$

$\Rightarrow \frac{dy}{dx}=\frac{dv}{dx}+4$

TRANSFORM THE O.D.E

$\Rightarrow \frac{dv}{dx}+4=v^2$

$\Rightarrow \frac{dv}{dx}=v^2-4$

$\Rightarrow \frac{dv}{dx}=(v-2)(v+2)$

SEPARATE VARIABLES

$\Rightarrow \frac{1}{(v-2)(v+2)}\,dv=1\,dx$

$\Rightarrow \int \frac{1}{(v-2)(v+2)}\,dv=\int 1\,dx$

PARTIAL FRACTIONS BY INSPECTION

$\Rightarrow \int \frac{\frac{1}{4}}{v-2}-\frac{\frac{1}{4}}{v+2}\,dv=\int 1\,dx$

$\Rightarrow \int \frac{1}{v-2}-\frac{1}{v+2}\,dv=\int 4\,dx$

$\Rightarrow \ln|v-2|-\ln|v+2|=4x+C$

$\Rightarrow \ln\left|\frac{v-2}{v+2}\right|=4x+C$

$\Rightarrow \frac{v-2}{v+2}=Ae^{4x}$

$\Rightarrow \frac{y-4x-2}{y-4x+2}=Ae^{4x}$

APPLY BOUNDARY CONDITION

$(0,-6) \Rightarrow \frac{-6-2}{-6+2}=A$

$\Rightarrow A=\frac{-8}{-4}=2$

FINALLY MAKE y THE SUBJECT

$\Rightarrow y-4x-2=2e^{4x}(y-4x+2)$

$\Rightarrow y-4x-2=2ye^{4x}-8xe^{4x}+4e^{4x}$

$\Rightarrow 8xe^{4x}-4x-4e^{4x}-2=2ye^{4x}-y$

$\Rightarrow 4x(2e^{4x}-1)-2(2e^{4x}+1)=y(2e^{4x}-1)$

$\Rightarrow y=\frac{4x(2e^{4x}-1)-2(2e^{4x}+1)}{2e^{4x}-1}$

Question 41 (*****)

Sketch the curve which passes through the point with coordinates $(1,2)$ and satisfies

$$\frac{1}{2}\frac{dy}{dx}+\frac{x}{3y^2}=\frac{\sqrt{x^2+y^3}}{y^2}.$$

graph

ESSENTIALLY WE NEED TO SOLVE THE O.D.E. – TRY A SUBSTITUTION

$\Rightarrow v = x^2+y^3$

$\Rightarrow \frac{dv}{dx} = 2x + 3y^2\frac{dy}{dx}$

$\Rightarrow 3y^2\frac{dy}{dx} = \frac{dv}{dx} - 2$

TRANSFORM THE O.D.E.

$\Rightarrow \frac{1}{2}\frac{dy}{dx} + \frac{x}{3y^2} = \frac{\sqrt{x^2+y^3}}{y^2}$ $\quad \times 3y^2$

$\Rightarrow \frac{3}{2}y^2\frac{dy}{dx} + x = 3\sqrt{x^2+y^3}$ $\quad \times 2$

$\Rightarrow 3y^2\frac{dy}{dx} + 2x = 6\sqrt{x^2+y^3}$

$\Rightarrow \left[\frac{dv}{dx} - 2x\right] + 2x = 6v^{\frac{1}{2}}$

$\Rightarrow \frac{dv}{dx} = 6v^{\frac{1}{2}}$

$\Rightarrow \frac{1}{v^{\frac{1}{2}}}\,dv = 6\,dx$

$\Rightarrow \int v^{-\frac{1}{2}}\,dv = \int 6\,dx$

INTEGRATE SUBJECT TO THE CONDITION $x=1, y=2 \Rightarrow v=9$

$\Rightarrow \left[2v^{\frac{1}{2}}\right]_{v=9}^{v} = \left[6x\right]_{x=1}^{x}$

$\Rightarrow 2v^{\frac{1}{2}} - 2\times 3 = 6x - 6$

$\Rightarrow 2v^{\frac{1}{2}} = 6x$

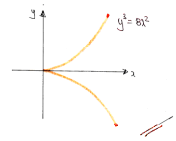
$\Rightarrow v^{\frac{1}{2}} = 3x$

$\Rightarrow v = 9x^2$

$\Rightarrow x^2+y^3 = 9x^2$

$\Rightarrow y^3 = 8x^2$

IT A SEMICUBICAL PARABOLA



Question 42 (*****)

Solve the differential equation

$$\frac{d}{dx}\left(xy^2\right)=\frac{x^4+x^2y^2+y^4}{x^2}, \quad y(\mathrm{e})=\sqrt{2}\,\mathrm{e}.$$

Give the answer in the form $y^2=f(x)$.

$\boxed{}$, $\boxed{y^2=\dfrac{x^2(1+\ln x)}{\ln x}}$

TIDY UP THE O.D.E.

$$\Rightarrow \frac{d}{dx}\left(xy^2\right)=\frac{x^4+x^2y^2+y^4}{x^2}$$

$$\Rightarrow y^2+2xy\frac{dy}{dx}=\frac{x^4+x^2y^2+y^4}{x^2}$$

$$\Rightarrow 2xy\frac{dy}{dx}=\frac{x^4+x^2y^2+y^4}{x^2}-y^2$$

$$\Rightarrow 2xy\frac{dy}{dx}=\frac{x^4\cancel{+x^2y^2}+y^4\cancel{-x^2y^2}}{x^2}$$

$$\Rightarrow \frac{dy}{dx}=\frac{x^4+y^4}{2x^3y}$$

THIS IS A "HOMOGENEOUS" EQUATION — USE A TRANSFORMATION

$$y=xV(x)$$
$$\frac{dy}{dx}=1\times V(x)+x\frac{dV}{dx}$$
$$\frac{dy}{dx}=V+x\frac{dV}{dx}$$

TRANSFORMING THE O.D.E.

$$\Rightarrow V+x\frac{dV}{dx}=\frac{x^4+x^4V^4}{2x^3(xV)}$$

$$\Rightarrow V+x\frac{dV}{dx}=\frac{x^4(1+V^4)}{2x^4V}$$

$$\Rightarrow V+x\frac{dV}{dx}=\frac{1+V^4}{2V}$$

$$\Rightarrow x\frac{dV}{dx}=\frac{1+V^4}{2V}-V$$

$$\Rightarrow x\frac{dV}{dx}=\frac{1+V^4-2V^2}{2V}$$

$$\Rightarrow x\frac{dV}{dx}=\frac{(V^2-1)^2}{2V}$$

SEPARATING VARIABLES & INTEGRATING

$$\Rightarrow \frac{2V}{(V^2-1)^2}\,dV=\frac{1}{x}\,dx$$

$$\Rightarrow \int 2V(V^2-1)^{-2}\,dV=\int\frac{1}{x}\,dx$$

$$\Rightarrow -(V^2-1)^{-1}=\ln x+C$$

$$\Rightarrow \frac{1}{V^2-1}=\ln x+C$$

$$\Rightarrow \frac{1}{\frac{y^2}{x^2}-1}=\ln x+C$$

$$\Rightarrow \frac{x^2}{y^2-x^2}=\ln x+C$$

APPLY CONDITION $(\mathrm{e},\sqrt{2}\mathrm{e})$

$$\Rightarrow \frac{\mathrm{e}^2}{2\mathrm{e}^2-\mathrm{e}^2}=\ln\mathrm{e}+C$$

$$\Rightarrow \frac{\mathrm{e}^2}{\mathrm{e}^2}=1+C$$

$$\Rightarrow C=0$$

$$\therefore \frac{x^2}{y^2-x^2}=\ln x$$

FINALLY REARRANGING

$$\Rightarrow x^2=y^2\ln x-x^2\ln x$$

$$\Rightarrow x^2+x^2\ln x=y^2\ln x$$

$$\Rightarrow y^2\ln x=x^2(1+\ln x)$$

$$\Rightarrow y^2=\frac{x^2(1+\ln x)}{\ln x}$$

Question 43 (*****)

By using a suitable transformation, or otherwise, find a general solution for the following differential equation

$$\frac{dy}{dx} = \frac{1+y^2}{-x+\arctan y}.$$

S X F, $\boxed{x = -1 + \arctan y + A\mathrm{e}^{-\arctan y}}$

LOOKING AT THE TERM "arctany" & $(1+y^2)$ SUGGEST THE SUBSTITUTION

- $y = \tan\theta \quad (\theta = \arctan y)$
- $\frac{dy}{d\theta} = \sec^2\theta$
- $\frac{dy}{dx} \times \frac{dx}{d\theta} = \sec^2\theta$
- $\frac{dy}{dx} = \sec^2\theta \frac{d\theta}{dx}$

TRANSFORMING THE O.D.E

$\Rightarrow \frac{dy}{dx} = \frac{1+y^2}{\arctan y - x}$

$\Rightarrow \sec^2\theta \frac{d\theta}{dx} = \frac{1+\tan^2\theta}{\theta - x}$

$\Rightarrow \sec^2\theta \frac{d\theta}{dx} = \frac{\sec^2\theta}{\theta - x}$

$\Rightarrow \frac{d\theta}{dx} = \frac{1}{\theta - x}$

$\Rightarrow \frac{dx}{d\theta} = \theta - x$

$\Rightarrow \frac{dx}{d\theta} + x = \theta$

LOOK FOR AN INTEGRATING FACTOR

$e^{\int 1 \, d\theta} = e^{\theta}$

$\Rightarrow \frac{d}{d\theta}\left(xe^{\theta}\right) = \theta e^{\theta}$

$\Rightarrow xe^{\theta} = \int \theta e^{\theta} \, d\theta$

INTEGRATION BY PARTS

θ	1
e^{θ}	e^{θ}

$\Rightarrow xe^{\theta} = \theta e^{\theta} - \int e^{\theta} \, d\theta$

$\Rightarrow xe^{\theta} = \theta e^{\theta} - e^{\theta} + A$

$\Rightarrow x = \theta - 1 + Ae^{-\theta}$

$\Rightarrow x = \arctan y - 1 + Ae^{-\arctan y}$

Question 44 (*****)

The curve with equation $y = f(x)$ has as asymptote the line $y = 1$ and satisfies the differential equation

$$x^3 \frac{dy}{dx} - x = xy + 1, \quad x \neq 0 .$$

Solve the above differential equation by using …

a) … the substitution $ux = xy + 1$, where $u = g(x)$.

b) … an integrating factor.

$$\boxed{y = e^{-\frac{1}{x}} - \frac{1}{x}}$$

(a) $x^3 \frac{dy}{dx} - x = xy + 1$

$\Rightarrow x^3\left[\frac{du}{dx} + \frac{1}{x^2}\right] - x = xy + 1$

$\Rightarrow x^3 \frac{du}{dx} + \cancel{x} - \cancel{x} = x\left[u - \frac{1}{x}\right] + 1$

$\Rightarrow x^3 \frac{du}{dx} = xu - 1 + 1$

$\Rightarrow x^3 \frac{du}{dx} = xu$

$\Rightarrow x^2 \frac{du}{dx} = u$

$\Rightarrow \frac{1}{u}\,du = \frac{1}{x^2}\,dx$

$\Rightarrow \int \frac{1}{u}\,du = \int \frac{1}{x^2}\,dx$

$\Rightarrow \ln|u| = -\frac{1}{x} + C$

$\Rightarrow u = e^{-\frac{1}{x} + C}$

$\Rightarrow u = Ae^{-\frac{1}{x}}$

$\Rightarrow y + \frac{1}{x} = Ae^{-\frac{1}{x}}$

$ux = xy + 1$

$ux - 1 = xy$

$y = u - \frac{1}{x}$

$\frac{dy}{dx} = \frac{du}{dx} + \frac{1}{x^2}$

$\Rightarrow y = -\frac{1}{x} + Ae^{-\frac{1}{x}}$

APPLY CONDITION AS $x \to \infty$, $y \to 1$

$1 = 0 + Ae^0$

$\boxed{A = 1}$

$\therefore y = e^{-\frac{1}{x}} - \frac{1}{x}$

(b) $x^3 \frac{dy}{dx} - x = xy + 1$

$\frac{dy}{dx} - \frac{1}{x^2} = \frac{y}{x^2} + \frac{1}{x^3}$

$\frac{dy}{dx} - \frac{y}{x^2} = \frac{1}{x^2} + \frac{1}{x^3}$

$\text{IF} = e^{\int -\frac{1}{x^2}\,dx} = e^{\frac{1}{x}}$

$\frac{d}{dx}\left(ye^{\frac{1}{x}}\right) = \left(\frac{1}{x^2} + \frac{1}{x^3}\right)e^{\frac{1}{x}}$

$ye^{\frac{1}{x}} = \int \left(\frac{1}{x^2} + \frac{1}{x^3}\right)e^{\frac{1}{x}}\,dx$

BY SUBSTITUTION

$\boxed{u = \frac{1}{x}} \Rightarrow \boxed{x = \frac{1}{u}}$

$\frac{du}{dx} = -\frac{1}{x^2}$

$dx = -x^2\,du = -\frac{1}{u^2}\,du$

$ye^{\frac{1}{x}} = \int (u^2 + u^3)e^u \left(-\frac{1}{u^2}\,du\right)$

$ye^{\frac{1}{x}} = -\int (1+u)e^u\,du$

BY PARTS

$1+u$	1
e^u	e^u

$ye^{\frac{1}{x}} = -\left[(1+u)e^u - \int e^u\,du\right]$

$ye^{\frac{1}{x}} = -\left[(1+u)e^u - e^u\right] + C$

$\Rightarrow ye^{\frac{1}{x}} = -\left[\cancel{e^u} + ue^u - \cancel{e^u}\right] + C$

$\Rightarrow ye^{\frac{1}{x}} = -ue^u + C$

$\Rightarrow ye^{\frac{1}{x}} = -\frac{1}{x}e^{\frac{1}{x}} + C$

$\Rightarrow \boxed{y = -\frac{1}{x} + Ce^{-\frac{1}{x}}}$

IF $y = 1$ IS AN ASYMPTOTE THEN

AS $x \mapsto \infty$, $y \mapsto 1$

$\therefore C = 1$

$y = -\frac{1}{x} + e^{-\frac{1}{x}}$

$y = e^{-\frac{1}{x}} - \frac{1}{x}$

Question 45 (*****)

Use the substitution $v = \dfrac{y-x}{y+x}$, $y+x \neq 0$, to solve the following differential equation

$$x\frac{dy}{dx} - y = \frac{(1-x)\left(x^2-y^2\right)}{x^3+x^2+x+1}, \quad y(0)=1.$$

Give the answer in the form $y = f(x)$.

$\boxed{}$, $\boxed{y = x^2+x+1}$

$$\boxed{x\frac{dy}{dx} - y = \frac{(1-x)(x^2-y^2)}{x^3+x^2+x+1} \qquad y(0)=1}$$

START WITH THE GIVEN SUBSTITUTION

$$v = \frac{y-x}{y+x} \Rightarrow yv + xv = y - x$$
$$\Rightarrow xv + x = y - yv$$
$$\Rightarrow x(v+1) = y(1-v)$$
$$\Rightarrow \boxed{y = \frac{x(v+1)}{1-v}}$$

NEXT DIFFERENTIATE THE ORIGINAL SUBSTITUTION W.R.T x

$$\Rightarrow v = \frac{y-x}{y+x} = \frac{y+x-2x}{y+x} = 1 - \frac{2x}{y+x}$$
$$\Rightarrow \frac{dv}{dx} = -\frac{(y+x)\times 2 - 2x\left(\frac{dy}{dx}+1\right)}{(y+x)^2}$$
$$\Rightarrow \frac{dv}{dx} = -\frac{2y + \cancel{2x} - 2x\frac{dy}{dx} - \cancel{2x}}{(y+x)^2}$$
$$\Rightarrow \boxed{\frac{dv}{dx} = \frac{2\left(x\frac{dy}{dx} - y\right)}{(y+x)^2}} \Leftrightarrow \boxed{x\frac{dy}{dx} - y = \frac{(y+x)^2}{2}\frac{dv}{dx}}$$

SUBSTITUTE INTO THE O.D.E.

$$\Rightarrow \frac{(y+x)^2}{2}\frac{dv}{dx} = \frac{(1-x)(x^2-y^2)}{x^3+x^2+x+1}$$
$$\Rightarrow \frac{dv}{dx} = \frac{2(1-x)(x^2-y^2)}{(x^3+x^2+x+1)(y+x)^2}$$
$$\Rightarrow \frac{dv}{dx} = \frac{2(1-x)(x-y)\cancel{(x+y)}}{(x^3+x^2+x+1)(x+y)^{\cancel{2}}} \qquad (x+y \neq 0)$$
$$\Rightarrow \frac{dv}{dx} = \frac{2(1-x)}{x^3+x^2+x+1} \times \frac{x-y}{x+y}$$
$$\Rightarrow \frac{dv}{dx} = \frac{2(1-x)}{x^2(x+1)+(x+1)} \times (-v)$$
$$\Rightarrow \frac{dv}{dx} = \frac{2v(x-1)}{(x+1)(x^2+1)}$$
$$\Rightarrow \int \frac{1}{v}\,dv = \int \frac{2(x-1)}{(x+1)(x^2+1)}\,dx$$

PROCEED BY OBTAINING THE PARTIAL FRACTIONS

$$\frac{2(x-1)}{(x+1)(x^2+1)} \equiv \frac{A}{x+1} + \frac{Bx+C}{x^2+1}$$
$$\boxed{2(x-1) \equiv A(x^2+1) + (Bx+C)(x+1)}$$

- IF $x=-1$: $-4 = 2A$, $A=-2$
- IF $x=0$: $-2 = A+C$, $-2 = -2+C$, $C=0$
- IF $x=2$: $2 = 5A+6B$, $2 = -10+6B$, $12 = 6B$, $B=2$

RETURNING TO THE O.D.E

$$\Rightarrow \int \frac{1}{v}\,dv = \int \frac{2x}{x^2+1} - \frac{2}{x+1}\,dx$$
$$\Rightarrow \ln|v| = \ln|x^2+1| - 2\ln|x+1| + \ln A$$
$$\Rightarrow \ln|v| = \ln\left|\frac{A(x^2+1)}{(x+1)^2}\right|$$
$$\Rightarrow v = \frac{A(x^2+1)}{(x+1)^2}$$
$$\Rightarrow \boxed{\frac{y-x}{y+x} = \frac{A(x^2+1)}{(x+1)^2}}$$

APPLY BOUNDARY CONDITION $y(0)=1$

$$\Rightarrow \frac{1-0}{1+0} = \frac{A\times 1}{(0+1)^2}$$
$$\Rightarrow A = 1$$
$$\Rightarrow \boxed{\frac{y-x}{y+x} = \frac{x^2+1}{(x+1)^2}}$$

REARRANGE THE SOLUTION IN THE FORM $y=f(x)$

$$\Rightarrow \frac{y-x}{y+x} = F(x) \qquad \left[F(x) = \frac{x^2+1}{(x+1)^2}\right]$$
$$\Rightarrow y - x = yF(x) + xF(x)$$
$$\Rightarrow y - F(x)y = xF(x) + x$$
$$\Rightarrow y(1-F(x)) = x(F(x)+1)$$
$$\Rightarrow y = \frac{x(1+F(x))}{1-F(x)}$$
$$\Rightarrow y = \frac{x\left[1 + \frac{x^2+1}{(x+1)^2}\right]}{\left[1 - \frac{x^2+1}{(x+1)^2}\right]}$$

MULTIPLY "TOP & BOTTOM" OF THE FRACTION IN THE R.H.S BY $(x+1)^2$

$$\Rightarrow y = \frac{x\left[(x+1)^2 + (x^2+1)\right]}{(x+1)^2 - (x^2+1)}$$
$$\Rightarrow y = \frac{x(x^2+2x+1+x^2+1)}{\cancel{x^2}+2x\cancel{+1}\cancel{-x^2}\cancel{-1}}$$
$$\Rightarrow y = \frac{\cancel{x}(2x^2+2x+2)}{2\cancel{x}}$$
$$y = x^2+x+1$$