2^{nd} ORDER O.D.E.s PRACTICE

Question 1

Find a general solution for each of the following differential equations.

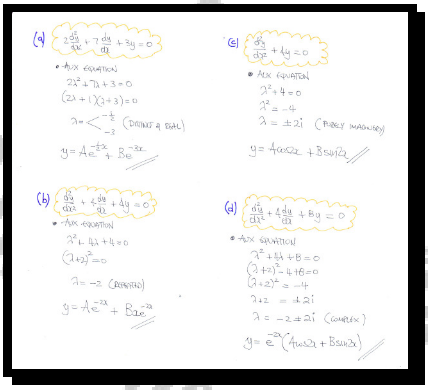
a) $2\frac{d^2y}{dx^2}+7\frac{dy}{dx}+3y=0$

b) $\frac{d^2y}{dx^2}+4\frac{dy}{dx}+4y=0$

c) $\frac{d^2y}{dx^2}+4y=0$

d) $\frac{d^2y}{dx^2}+4\frac{dy}{dx}+8y=0$

$\boxed{y=Ae^{-3x}+Be^{-\frac{1}{2}x}}$, $\boxed{y=Ae^{-2x}+Bxe^{-2x}}$, $\boxed{y=A\cos 2x+B\sin 2x}$, $\boxed{y=e^{-2x}\left(A\cos 2x+B\sin 2x\right)}$



Question 2

Find a general solution for each of the following differential equations.

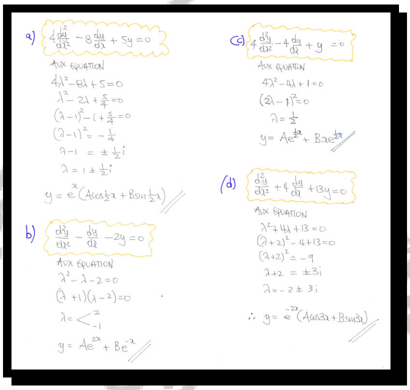
a) $4\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 5y = 0$

b) $\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y = 0$

c) $4\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + y = 0$

d) $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 13y = 0$

$\boxed{y = \mathrm{e}^{x}\left(A\cos\frac{1}{2}x + B\sin\frac{1}{2}x\right)}$, $\boxed{y = A\mathrm{e}^{-x} + B\mathrm{e}^{2x}}$, $\boxed{y = A\mathrm{e}^{\frac{1}{2}x} + Bx\mathrm{e}^{\frac{1}{2}x}}$, $\boxed{y = \mathrm{e}^{-2x}\left(A\cos 3x + B\sin 3x\right)}$



Question 3

Find a general solution for each of the following differential equations.

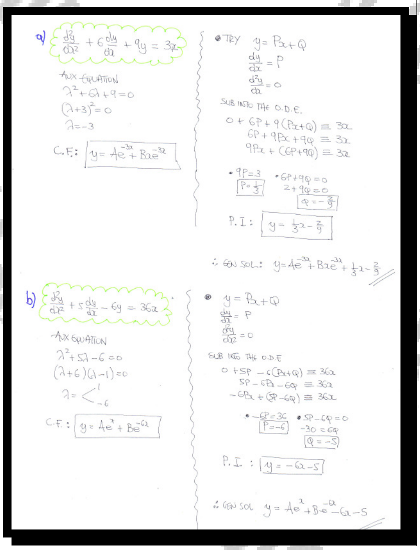
a) $\dfrac{d^2y}{dx^2}+6\dfrac{dy}{dx}+9y=3x$

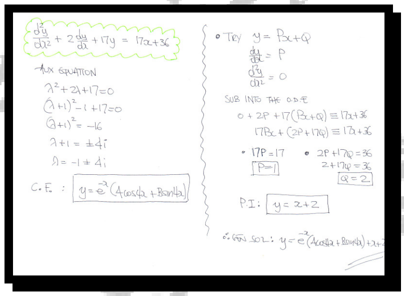
b) $\dfrac{d^2y}{dx^2}+5\dfrac{dy}{dx}-6y=36x$

c) $\dfrac{d^2y}{dx^2}+2\dfrac{dy}{dx}+17y=17x+36$

$\boxed{y=A\mathrm{e}^{-3x}+Bx\mathrm{e}^{-3x}+\frac{1}{3}x-\frac{2}{9}}$, $\boxed{y=A\mathrm{e}^{x}+B\mathrm{e}^{-6x}-6x-5}$,

$\boxed{y=\mathrm{e}^{-x}\left(A\cos 4x+B\sin 4x\right)+x+2}$





Question 4

Find a general solution for each of the following differential equations.

a) $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 5y = 10\sin x$

b) $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 17y = 17x + 36$

c) $\frac{d^2y}{dx^2} + 4y = 12\cos x$

$\boxed{y = e^{-x}\left(A\cos 2x + B\sin 2x\right) + 2\sin x - \cos x}$, $\boxed{y = e^{-x}\left(A\cos 4x + B\sin 4x\right) + x + 2}$, $\boxed{y = A\cos 2x + B\sin 2x + 4\cos x}$

Question 5

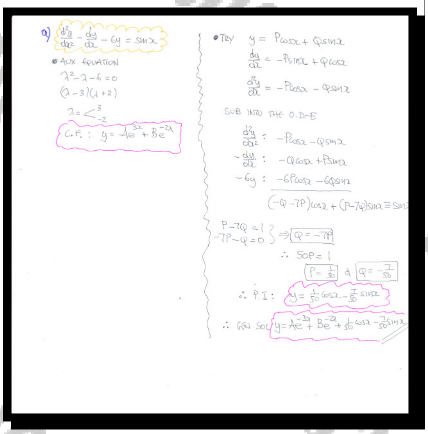
Find a general solution for each of the following differential equations.

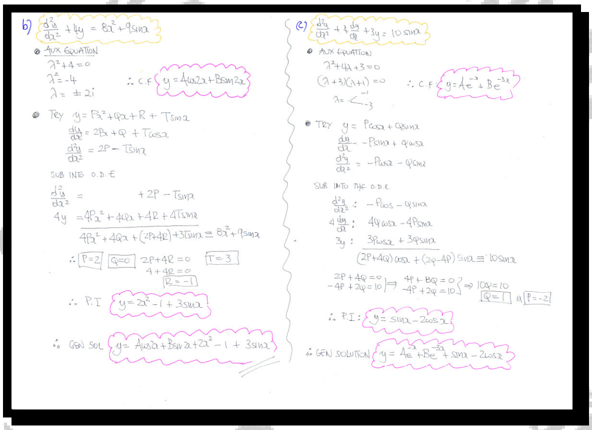
a) $\dfrac{d^2y}{dx^2} - \dfrac{dy}{dx} - 6y = \sin x$

b) $\dfrac{d^2y}{dx^2} + 4y = 8x^2 + 9\sin x$

c) $\dfrac{d^2y}{dx^2} + 4\dfrac{dy}{dx} + 3y = 10\sin x$

$\boxed{y = A\mathrm{e}^{3x} + B\mathrm{e}^{-2x} + \frac{1}{50}(\cos x - 7\sin x)}$, $\boxed{y = A\cos 2x + B\sin 2x + 3\sin x + 2x^2 - 1}$, $\boxed{y = A\mathrm{e}^{-x} + B\mathrm{e}^{-3x} + \sin x - 2\cos x}$





Question 6

Find the solution of each of the following differential equations.

a) $$\frac{d^2y}{dx^2}+6\frac{dy}{dx}+9y=\cos 3x,$$

subject to the conditions $y=\frac{1}{2}$, $\frac{dy}{dx}=0$ at $x=0$.

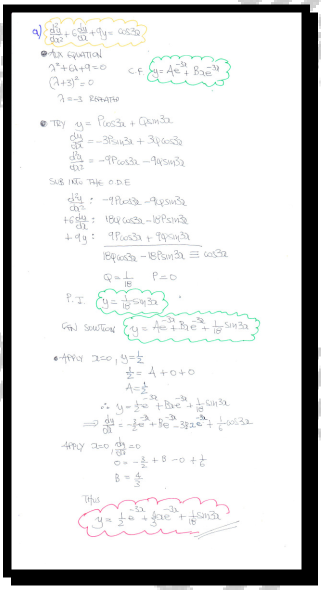
b) $$\frac{d^2y}{dx^2}+2\frac{dy}{dx}+5y=10,$$

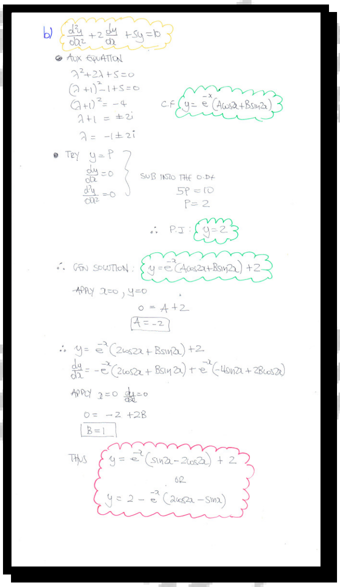
subject to the conditions $y=0$, $\frac{dy}{dx}=0$ at $x=0$.

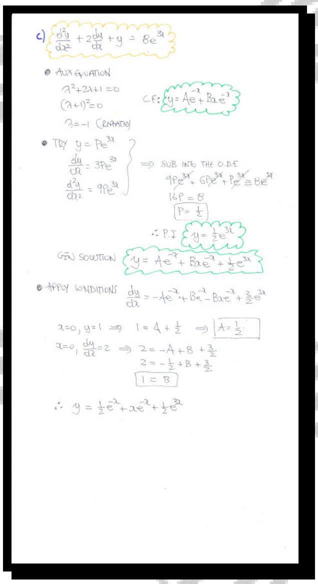
c) $$\frac{d^2y}{dx^2}+2\frac{dy}{dx}+y=8e^{3x},$$

subject to the conditions $y=1$, $\frac{dy}{dx}=2$ at $x=0$.

$\boxed{y=\frac{1}{2}e^{-3x}+\frac{4}{3}xe^{-3x}+\frac{1}{18}\sin 3x}$, $\boxed{y=2-e^{-x}\left(2\cos 2x-\sin 2x\right)}$, $\boxed{y=\frac{1}{2}e^{3x}+\frac{1}{2}e^{-x}+xe^{-x}}$







Question 7

Find the solution of each of the following differential equations.

a) $\frac{d^2y}{dx^2} - 4y = 10e^{3x}$,

subject to the conditions $y = 3$, $\frac{dy}{dx} = 8$ at $x = 0$.

b) $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 3y = 15e^{2x}$,

subject to the conditions $y = 9$, $\frac{dy}{dx} = 4$ at $x = 0$.

c) $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = 8e^{3x}$,

subject to the conditions $x = 0$, $y = 1$, $\frac{dy}{dx} = 2$

$\boxed{y = e^{2x} + 2e^{3x}}$, $\boxed{y = 4e^{x} + 2e^{-3x} + 3e^{2x}}$, $\boxed{y = 2e^{3x} - e^{-x} - 5xe^{-x}}$

Question 8

Find the solution of each of the following differential equations.

a) $2\frac{d^2y}{dx^2}+5\frac{dy}{dx}+2y=2x+9$,

subject to the conditions $y=3$, $\frac{dy}{dx}=-1$ at $x=0$.

b) $\frac{d^2y}{dx^2}-6\frac{dy}{dx}+9y=27x$,

subject to the conditions $y=2$, $\frac{dy}{dx}=6$ at $x=0$.

c) $\frac{d^2y}{dx^2}+\frac{dy}{dx}=2x+3$,

subject to the conditions $y=2$, $\frac{dy}{dx}=-5$ at $x=0$.

$\boxed{y=e^{-2x}+x+2}$, $\boxed{y=3x\left(1+e^{3x}\right)+2}$, $\boxed{y=x^2+x-4+6e^{-x}}$

Question 9

Find the solution of each of the following differential equations.

a) $2\frac{d^2y}{dx^2}-7\frac{dy}{dx}-4y=8\sin x-19\cos x$,

subject to the conditions $y=0$, $\frac{dy}{dx}=11$ at $x=0$.

b) $2\frac{d^2y}{dx^2}-7\frac{dy}{dx}-4y=8\sin x-19\cos x$,

subject to the conditions $y=0$, $\frac{dy}{dx}=11$ at $x=0$.

c) $2\frac{d^2y}{dx^2}-7\frac{dy}{dx}-4y=8\sin x-19\cos x$,

subject to the conditions $y=0$, $\frac{dy}{dx}=11$ at $x=0$.

$\boxed{y=e^{-2x}+x+2}$, $\boxed{y=3x\left(1+e^{3x}\right)+2}$, $\boxed{y=2e^{4x}-4e^{\frac{1}{2}x}+\sin x+2\cos x}$

Question 10 (harder PIs)

Find the solution of each of the following differential equations.

a) $\dfrac{d^2y}{dx^2}+3\dfrac{dy}{dx}+2y=2e^{-2x}$,

subject to the conditions $y=2$, $\dfrac{dy}{dx}=0$ at $x=0$.

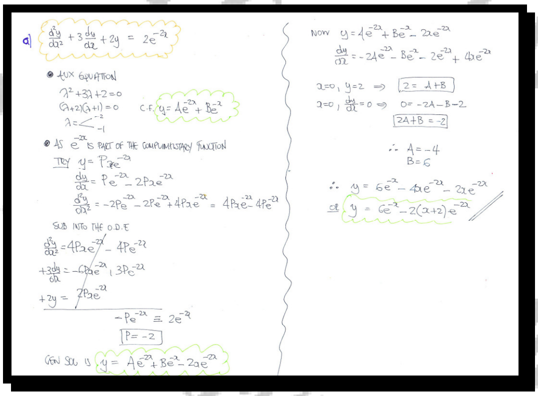
b) $\dfrac{d^2y}{dx^2}+9y=\cos 3x$,

subject to the conditions $y=0$, $\dfrac{dy}{dx}=3$ at $x=0$.

c) $\dfrac{d^2y}{dx^2}+\dfrac{dy}{dx}=x$,

subject to the conditions $y=1$, $\dfrac{dy}{dx}=3$ at $x=0$.

$\boxed{y=6e^{-x}-2(x+2)e^{-2x}}$, $\boxed{y=\sin 3x+\frac{1}{6}x\sin 3x}$, $\boxed{y=\frac{1}{2}x^2-x+5-4e^{-x}}$



a) $\dfrac{d^2y}{dx^2}+3\dfrac{dy}{dx}+2y=2e^{-2x}$

- AUX EQUATION

$\lambda^2+3\lambda+2=0$

$(\lambda+2)(\lambda+1)=0$

$\lambda = -2, -1$ C.F. $y=Ae^{-2x}+Be^{-x}$

- AS e^{-2x} IS PART OF THE COMPLEMENTARY FUNCTION

TRY $y=Pxe^{-2x}$

$\dfrac{dy}{dx}=Pe^{-2x}-2Pxe^{-2x}$

$\dfrac{d^2y}{dx^2}=-2Pe^{-2x}-2Pe^{-2x}+4Pxe^{-2x}=4Pxe^{-2x}-4Pe^{-2x}$

SUB INTO THE O.D.E

$\dfrac{d^2y}{dx^2}=4Pxe^{-2x}-4Pe^{-2x}$

$+3\dfrac{dy}{dx}=-6Pxe^{-2x}+3Pe^{-2x}$

$+2y=2Pxe^{-2x}$

$-Pe^{-2x}\equiv 2e^{-2x}$

$\boxed{P=-2}$

GEN SOL IS $y=Ae^{-2x}+Be^{-x}-2xe^{-2x}$

NOW $y=Ae^{-2x}+Be^{-x}-2xe^{-2x}$

$\dfrac{dy}{dx}=-2Ae^{-2x}-Be^{-x}-2e^{-2x}+4xe^{-2x}$

$x=0,\ y=2 \Rightarrow \boxed{2=A+B}$

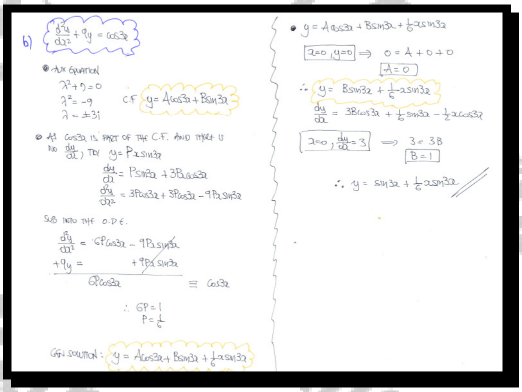
$x=0,\ \dfrac{dy}{dx}=0 \Rightarrow 0=-2A-B-2$

$\boxed{2A+B=-2}$

$\therefore A=-4$, $B=6$

$\therefore y=6e^{-x}-4e^{-2x}-2xe^{-2x}$

or $y=6e^{-x}-2(x+2)e^{-2x}$



b) $\dfrac{d^2y}{dx^2}+9y=\cos 3x$

- AUX EQUATION

$\lambda^2+9=0$

$\lambda^2=-9$

$\lambda=\pm 3i$ C.F $y=A\cos 3x+B\sin 3x$

- AS $\cos 3x$ IS PART OF THE C.F. AND THERE IS NO $\frac{dy}{dx}$, TRY $y=Px\sin 3x$

$\dfrac{dy}{dx}=P\sin 3x+3Px\cos 3x$

$\dfrac{d^2y}{dx^2}=3P\cos 3x+3P\cos 3x-9Px\sin 3x$

SUB INTO THE O.D.E.

$\dfrac{d^2y}{dx^2}=6P\cos 3x-9Px\sin 3x$

$+9y=\quad +9Px\sin 3x$

$6P\cos 3x\equiv\cos 3x$

$\therefore 6P=1$

$P=\frac{1}{6}$

GEN SOLUTION: $y=A\cos 3x+B\sin 3x+\frac{1}{6}x\sin 3x$

- $y=A\cos 3x+B\sin 3x+\frac{1}{6}x\sin 3x$

$\boxed{x=0,\ y=0} \Rightarrow 0=A+0+0$

$\boxed{A=0}$

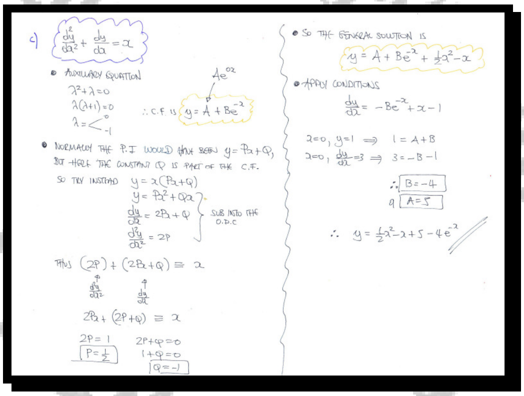
$\therefore y=B\sin 3x+\frac{1}{6}x\sin 3x$

$\dfrac{dy}{dx}=3B\cos 3x+\frac{1}{6}\sin 3x+\frac{1}{2}x\cos 3x$

$\boxed{x=0,\ \frac{dy}{dx}=3} \Rightarrow 3=3B$

$\boxed{B=1}$

$\therefore y=\sin 3x+\frac{1}{6}x\sin 3x$



c) $\dfrac{d^2y}{dx^2}+\dfrac{dy}{dx}=x$

- AUXILIARY EQUATION

$\lambda^2+\lambda=0$

$\lambda(\lambda+1)=0$

$\lambda = 0, -1$ $\therefore$ C.F. IS $y=A+Be^{-x}$ (Ae^{0x})

- NORMALLY THE P.I WOULD HAVE BEEN $y=Px+Q$, BUT HERE THE CONSTANT Q IS PART OF THE C.F.

SO TRY INSTEAD $y=x(Px+Q)$

$y=Px^2+Qx$

$\dfrac{dy}{dx}=2Px+Q$

$\dfrac{d^2y}{dx^2}=2P$

SUB INTO THE O.D.E

THUS $(2P)+(2Px+Q)\equiv x$

$2Px+(2P+Q)\equiv x$

$2P=1$, $\boxed{P=\frac{1}{2}}$

$2P+Q=0$, $1+Q=0$, $\boxed{Q=-1}$

- SO THE GENERAL SOLUTION IS

$y=A+Be^{-x}+\frac{1}{2}x^2-x$

- APPLY CONDITIONS

$\dfrac{dy}{dx}=-Be^{-x}+x-1$

$x=0,\ y=1 \Rightarrow 1=A+B$

$x=0,\ \dfrac{dy}{dx}=3 \Rightarrow 3=-B-1$

$\therefore \boxed{B=-4}$

$\boxed{A=5}$

$\therefore y=\frac{1}{2}x^2-x+5-4e^{-x}$

Question 11 (harder PIs)

Find the solution of each of the following differential equations.

a) $\dfrac{d^2y}{dx^2}+3\dfrac{dy}{dx}-4y=10e^x$,

subject to the conditions $y=1$, $\dfrac{dy}{dx}=3$ at $x=0$.

b) $\dfrac{d^2y}{dx^2}+16y=8\cos 4x$,

subject to the conditions $y=2$, $\dfrac{dy}{dx}=0$ at $x=0$.

c) $\dfrac{d^2y}{dx^2}+100y=2\cos 10x$,

subject to the conditions $y=0$, $\dfrac{dy}{dx}=0$ at $x=0$.

$\boxed{y=e^x(2x+1)}$, $\boxed{y=2\cos 4x+x\sin 4x}$, $\boxed{y=\frac{1}{10}x\sin 10x}$

Question 12 (harder PIs)

Find the solution of each of the following differential equations.

a) $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = e^x$,

subject to the conditions $y = 1$, $\frac{dy}{dx} = 2$ at $x = 0$.

b) $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = e^{-x}$,

subject to the conditions $y = 1$, $\frac{dy}{dx} = -1$ at $x = 0$.

c) $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} = 100x$,

subject to the conditions $y = 2$, $\frac{dy}{dx} = 1$ at $x = 0$.

$\boxed{y = \left(\frac{1}{2}x^2 + x + 1\right)e^x}$, $\boxed{y = \left(\frac{1}{2}x^2 + 1\right)e^{-x}}$, $\boxed{y = e^{5x} + 1 - 4x - 10x^2}$

2^{nd} ORDER O.D.E. EULER TYPE

Question 1

Find the solution of each of the following differential equations.

a) $x^2 \frac{d^2 y}{dx^2} - 6x\frac{dy}{dx} + 12y = 0$,

subject to the conditions $x = 1$, $y = 1$, $\frac{dy}{dx} = 0$

b) $x^2 \frac{d^2 y}{dx^2} - 2x\frac{dy}{dx} - 4y = 0$,

subject to the conditions $x = 1$, $y = 3$, $\frac{dy}{dx} = 7$

c) $x^2 \frac{d^2 y}{dx^2} - 4x\frac{dy}{dx} + 4y = 0$,

subject to the conditions $x = 1$, $y = 5$, $\frac{dy}{dx} = 8$

$\boxed{y = 4x^3 - 3x^4}$, $\boxed{y = \frac{1}{x} + 2x^4}$, $\boxed{y = 4x + x^4}$