2nd ORDER O.D.E.s SUBSTITUTIONS

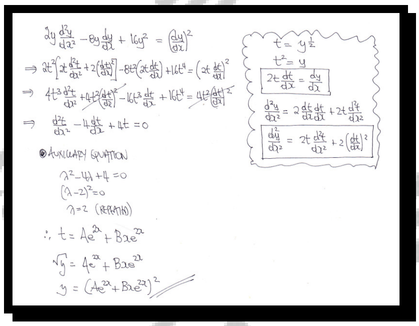
Question 1 (***+)

$$2y\frac{d^2y}{dx^2}-8y\frac{dy}{dx}+16y^2=\left(\frac{dy}{dx}\right)^2,\ \ y\neq 0,$$

Find the general solution of the above differential equation by using the transformation equation $t=\sqrt{y}$.

Give the answer in the form $y=f(x)$.

$$\boxed{y=\left(Ae^{2x}+Bxe^{2x}\right)^2}$$



$$2y\frac{d^2y}{dx^2}-8y\frac{dy}{dx}+16y^2=\left(\frac{dy}{dx}\right)^2$$

$$\Rightarrow 2t^2\left[2t\frac{d^2t}{dx^2}+2\left(\frac{dt}{dx}\right)^2\right]-8t^2\left(2t\frac{dt}{dx}\right)+16t^4=\left(2t\frac{dt}{dx}\right)^2$$

$$\Rightarrow 4t^3\frac{d^2t}{dx^2}+4t^2\left(\frac{dt}{dx}\right)^2-16t^3\frac{dt}{dx}+16t^4=4t^2\left(\frac{dt}{dx}\right)^2$$

$$\Rightarrow \frac{d^2t}{dx^2}-4\frac{dt}{dx}+4t=0$$

$t=y^{\frac{1}{2}}$

$t^2=y$

$2t\frac{dt}{dx}=\frac{dy}{dx}$

$\frac{d^2y}{dx^2}=2\frac{dt}{dx}\frac{dt}{dx}+2t\frac{d^2t}{dx^2}$

$\frac{d^2y}{dx^2}=2t\frac{d^2t}{dx^2}+2\left(\frac{dt}{dx}\right)^2$

• AUXILIARY EQUATION

$\lambda^2-4\lambda+4=0$

$(\lambda-2)^2=0$

$\lambda=2$ (REPEATED)

$\therefore t=Ae^{2x}+Bxe^{2x}$

$\sqrt{y}=Ae^{2x}+Bxe^{2x}$

$y=\left(Ae^{2x}+Bxe^{2x}\right)^2$

Question 2 (***+)

The differential equation

$$x\frac{d^2y}{dx^2}+2\frac{dy}{dx}=3x\,,\quad x\neq 0\,,$$

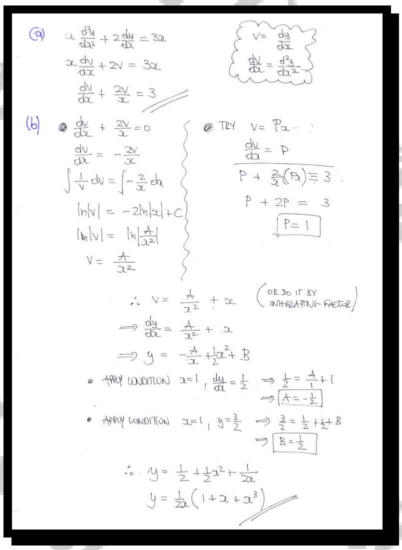
is to be solved subject to the boundary conditions $y=\frac{3}{2}$, $\frac{dy}{dx}=\frac{1}{2}$ at $x=1$.

a) Show that the substitution $v=\frac{dy}{dx}$, transforms the above differential equation into

$$\frac{dv}{dx}+\frac{2v}{x}=3\,.$$

b) Hence find the solution of the original differential equation, giving the answer in the form $y=f(x)$.

$$\boxed{y=\frac{1}{2}\left(x^2+\frac{1}{x}+1\right)}$$



(a) $x\frac{d^2y}{dx^2}+2\frac{dy}{dx}=3x$

$x\frac{dv}{dx}+2v=3x$

$\frac{dv}{dx}+\frac{2v}{x}=3$

$v=\frac{dy}{dx}$, $\frac{dv}{dx}=\frac{d^2y}{dx^2}$

(b) • $\frac{dv}{dx}+\frac{2v}{x}=0$

$\frac{dv}{dx}=-\frac{2v}{x}$

$\int\frac{1}{v}\,dv=\int-\frac{2}{x}\,dx$

$\ln|v|=-2\ln|x|+C$

$\ln|v|=\ln\left|\frac{A}{x^2}\right|$

$v=\frac{A}{x^2}$

• TRY $v=Px$

$\frac{dv}{dx}=P$

$P+\frac{2}{x}(Px)\equiv 3$

$P+2P=3$

$\boxed{P=1}$

$\therefore\ v=\frac{A}{x^2}+x$ (OR DO IT BY INTEGRATING FACTOR)

$\Rightarrow \frac{dy}{dx}=\frac{A}{x^2}+x$

$\Rightarrow y=-\frac{A}{x}+\frac{1}{2}x^2+B$

• APPLY CONDITION $x=1$, $\frac{dy}{dx}=\frac{1}{2}$ $\Rightarrow \frac{1}{2}=\frac{A}{1}+1$ $\Rightarrow \boxed{A=-\frac{1}{2}}$

• APPLY CONDITION $x=1$, $y=\frac{3}{2}$ $\Rightarrow \frac{3}{2}=\frac{1}{2}+\frac{1}{2}+B$ $\Rightarrow \boxed{B=\frac{1}{2}}$

$\therefore\ y=\frac{1}{2}+\frac{1}{2}x^2+\frac{1}{2x}$

$y=\frac{1}{2x}(1+x+x^3)$

Question 3 (***+)

The curve C has equation $y = f(x)$ and satisfies the differential equation

$$x^2 \frac{d^2y}{dx^2} - 2x\frac{dy}{dx} - 2y\left(2x^2 - 1\right) = 3x^3 e^x, \quad x \neq 0$$

is to be solved subject to the boundary conditions $y = \frac{3}{2}$, $\frac{dy}{dx} = \frac{1}{2}$ at $x = 1$.

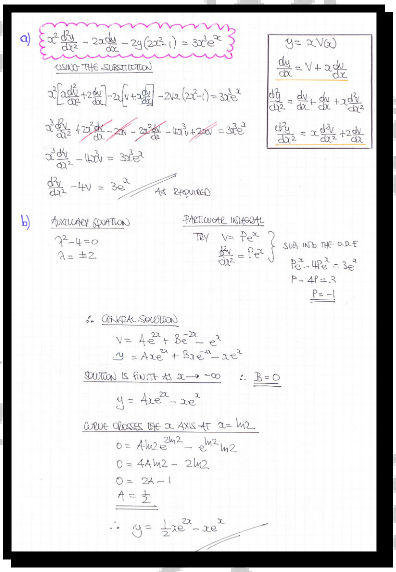
a) Show that the substitution $y = xv$, where v is a function of x transforms the above differential equation into

$$\frac{d^2v}{dx^2} - 4v = 3e^x.$$

It is further given that C meets the x axis at $x = \ln 2$ and has a finite value for y as x gets infinitely negatively large.

b) Express the equation of C in the form $y = f(x)$.

$$\boxed{y = \frac{1}{2}xe^{2x} - xe^x}$$



a) $x^2\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} - 2y(2x^2-1) = 3x^3e^x$

USING THE SUBSTITUTION

$y = xv(x)$

$\frac{dy}{dx} = v + x\frac{dv}{dx}$

$\frac{d^2y}{dx^2} = \frac{dv}{dx} + \frac{dv}{dx} + x\frac{d^2v}{dx^2}$

$\frac{d^2y}{dx^2} = x\frac{d^2v}{dx^2} + 2\frac{dv}{dx}$

$x^2\left[x\frac{d^2v}{dx^2} + 2\frac{dv}{dx}\right] - 2x\left[v + x\frac{dv}{dx}\right] - 2vx(2x^2-1) = 3x^3e^x$

$x^3\frac{d^2v}{dx^2} + 2x^2\frac{dv}{dx} - 2xv - 2x^2\frac{dv}{dx} - 4x^3v + 2xv = 3x^3e^x$

$x^3\frac{d^2v}{dx^2} - 4x^3v = 3x^3e^x$

$\frac{d^2v}{dx^2} - 4v = 3e^x$ AS REQUIRED

b) AUXILIARY EQUATION

$\lambda^2 - 4 = 0$

$\lambda = \pm 2$

PARTICULAR INTEGRAL

TRY $v = Pe^x$, $\frac{d^2v}{dx^2} = Pe^x$ SUB INTO THE O.D.E

$Pe^x - 4Pe^x = 3e^x$

$P - 4P = 3$

$P = -1$

∴ GENERAL SOLUTION

$v = Ae^{2x} + Be^{-2x} - e^x$

$y = Axe^{2x} + Bxe^{-2x} - xe^x$

SOLUTION IS FINITE AS $x \to -\infty$ ∴ $B = 0$

$y = Axe^{2x} - xe^x$

CURVE CROSSES THE x AXIS AT $x = \ln 2$

$0 = A\ln 2\, e^{2\ln 2} - e^{\ln 2}\ln 2$

$0 = 4A\ln 2 - 2\ln 2$

$0 = 2A - 1$

$A = \frac{1}{2}$

∴ $y = \frac{1}{2}xe^{2x} - xe^x$

Question 4 (***+)

Given that if $x = e^t$ and $y = f(x)$, show clearly that …

a) … $x\dfrac{dy}{dx} = \dfrac{dy}{dt}$.

b) … $x^2\dfrac{d^2y}{dx^2} = \dfrac{d^2y}{dt^2} - \dfrac{dy}{dt}$.

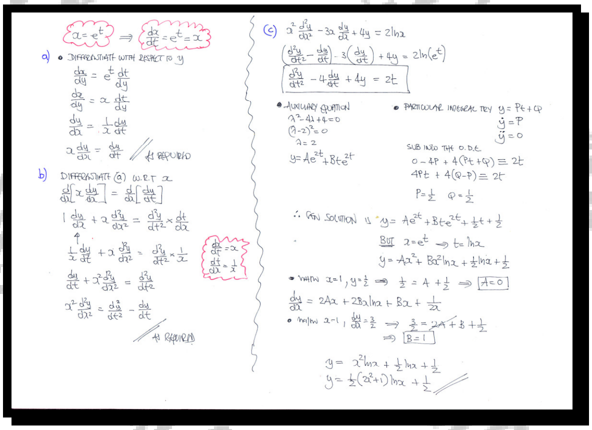
The following differential equation is to be solved

$$x^2\frac{d^2y}{dx^2} - 3x\frac{dy}{dx} + 4y = 2\ln x$$

subject to the boundary conditions $y = \dfrac{1}{2}, \dfrac{dy}{dx} = \dfrac{3}{2}$ at $x = 1$.

c) Use the substitution $x = e^t$ to solve the above differential equation.

$$\boxed{y = \frac{1}{2} + \frac{1}{2}\left(2x^2 + 1\right)\ln x}$$



$x = e^t \Rightarrow \dfrac{dx}{dt} = e^t = x$

a) • DIFFERENTIATE WITH RESPECT TO y

$\dfrac{dx}{dy} = e^t\dfrac{dt}{dy}$

$\dfrac{dx}{dy} = x\dfrac{dt}{dy}$

$\dfrac{dy}{dx} = \dfrac{1}{x}\dfrac{dy}{dt}$

$x\dfrac{dy}{dx} = \dfrac{dy}{dt}$ // AS REQUIRED

b) DIFFERENTIATE (a) W.R.T x

$\dfrac{d}{dx}\left[x\dfrac{dy}{dx}\right] = \dfrac{d}{dx}\left[\dfrac{dy}{dt}\right]$

$1\dfrac{dy}{dx} + x\dfrac{d^2y}{dx^2} = \dfrac{d^2y}{dt^2}\times\dfrac{dt}{dx}$

$\dfrac{1}{x}\dfrac{dy}{dt} + x\dfrac{d^2y}{dx^2} = \dfrac{d^2y}{dt^2}\times\dfrac{1}{x}$ $\quad\left(\dfrac{dx}{dt} = x,\ \dfrac{dt}{dx} = \dfrac{1}{x}\right)$

$\dfrac{dy}{dt} + x^2\dfrac{d^2y}{dx^2} = \dfrac{d^2y}{dt^2}$

$x^2\dfrac{d^2y}{dx^2} = \dfrac{d^2y}{dt^2} - \dfrac{dy}{dt}$ // AS REQUIRED

(c) $x^2\dfrac{d^2y}{dx^2} - 3x\dfrac{dy}{dx} + 4y = 2\ln x$

$\left(\dfrac{d^2y}{dt^2} - \dfrac{dy}{dt}\right) - 3\left(\dfrac{dy}{dt}\right) + 4y = 2\ln(e^t)$

$\boxed{\dfrac{d^2y}{dt^2} - 4\dfrac{dy}{dt} + 4y = 2t}$

• AUXILIARY EQUATION

$\lambda^2 - 4\lambda + 4 = 0$

$(\lambda - 2)^2 = 0$

$\lambda = 2$

$y = Ae^{2t} + Bte^{2t}$

• PARTICULAR INTEGRAL TRY $y = Pt + Q$

$\dot{y} = P$

$\ddot{y} = 0$

SUB INTO THE O.D.E.

$0 - 4P + 4(Pt + Q) \equiv 2t$

$4Pt + 4(Q - P) \equiv 2t$

$P = \frac{1}{2} \quad Q = \frac{1}{2}$

∴ GEN SOLUTION IS $y = Ae^{2t} + Bte^{2t} + \frac{1}{2}t + \frac{1}{2}$

BUT $x = e^t \Rightarrow t = \ln x$

$y = Ax^2 + Bx^2\ln x + \frac{1}{2}\ln x + \frac{1}{2}$

• WHEN $x = 1, y = \frac{1}{2} \Rightarrow \frac{1}{2} = A + \frac{1}{2} \Rightarrow \boxed{A = 0}$

$\dfrac{dy}{dx} = 2Ax + 2Bx\ln x + Bx + \dfrac{1}{2x}$

• WHEN $x = 1, \frac{dy}{dx} = \frac{3}{2} \Rightarrow \frac{3}{2} = 2A + B + \frac{1}{2}$

$\Rightarrow \boxed{B = 1}$

$y = x^2\ln x + \frac{1}{2}\ln x + \frac{1}{2}$

$y = \frac{1}{2}(2x^2 + 1)\ln x + \frac{1}{2}$ //

Question 5 (****)

The differential equation

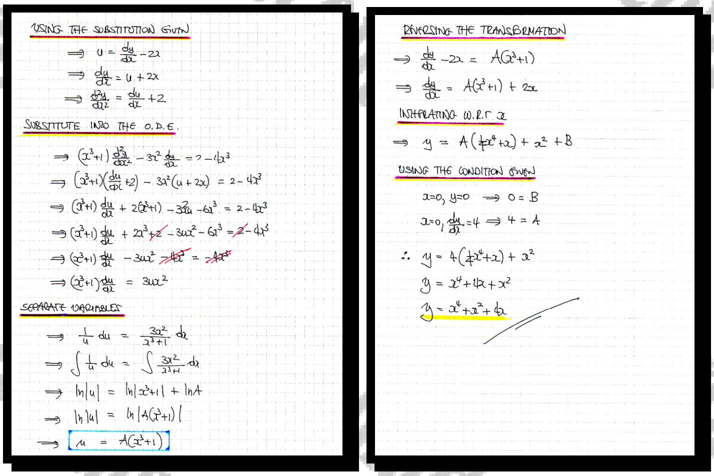
$$\left(x^3+1\right)\frac{d^2y}{dx^2} - 3x^2\frac{dy}{dx} = 2 - 4x^3,$$

is to be solved subject to the boundary conditions $y = 0$, $\frac{dy}{dx} = 4$ at $x = 0$.

Use the substitution $u = \frac{dy}{dx} - 2x$, where u is a function of x, to show that the solution of the above differential equation is

$$y = x^4 + x^2 + 4x.$$

proof


USING THE SUBSTITUTION GIVEN

$\Rightarrow u = \frac{dy}{dx} - 2x$

$\Rightarrow \frac{dy}{dx} = u + 2x$

$\Rightarrow \frac{d^2y}{dx^2} = \frac{du}{dx} + 2$

SUBSTITUTE INTO THE O.D.E.

$\Rightarrow (x^3+1)\frac{d^2y}{dx^2} - 3x^2\frac{dy}{dx} = 2 - 4x^3$

$\Rightarrow (x^3+1)\left(\frac{du}{dx}+2\right) - 3x^2(u+2x) = 2 - 4x^3$

$\Rightarrow (x^3+1)\frac{du}{dx} + 2(x^3+1) - 3x^2u - 6x^3 = 2 - 4x^3$

$\Rightarrow (x^3+1)\frac{du}{dx} + 2x^3 + 2 - 3ux^2 - 6x^3 = 2 - 4x^3$

$\Rightarrow (x^3+1)\frac{du}{dx} - 3ux^2 - 4x^3 = -4x^3$

$\Rightarrow (x^3+1)\frac{du}{dx} = 3ux^2$

SEPARATE VARIABLES

$\Rightarrow \frac{1}{u}\,du = \frac{3x^2}{x^3+1}\,dx$

$\Rightarrow \int \frac{1}{u}\,du = \int \frac{3x^2}{x^3+1}\,dx$

$\Rightarrow \ln|u| = \ln|x^3+1| + \ln A$

$\Rightarrow \ln|u| = \ln|A(x^3+1)|$

$\Rightarrow u = A(x^3+1)$

REVERSING THE TRANSFORMATION

$\Rightarrow \frac{dy}{dx} - 2x = A(x^3+1)$

$\Rightarrow \frac{dy}{dx} = A(x^3+1) + 2x$

INTEGRATING W.R.T x

$\Rightarrow y = A\left(\frac{1}{4}x^4 + x\right) + x^2 + B$

USING THE CONDITION GIVEN

$x=0, y=0 \Rightarrow 0 = B$

$x=0, \frac{dy}{dx}=4 \Rightarrow 4 = A$

$\therefore y = 4\left(\frac{1}{4}x^4+x\right) + x^2$

$y = x^4 + 4x + x^2$

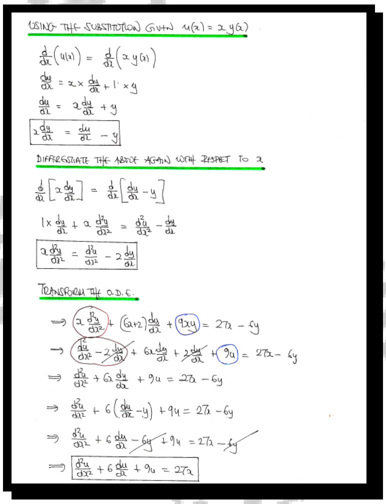
$y = x^4 + x^2 + 4x$

Question 6 (****)

$$x\frac{d^2y}{dx^2}+(6x+2)\frac{dy}{dx}+9xy=27x-6y.$$

Use the substitution $u=xy$, where u is a function of x, to find a general solution of the above differential equation.

$$\boxed{}, \quad \boxed{y=\frac{A}{x}e^{-3x}+Be^{-3x}+3-\frac{2}{x}}$$



USING THE SUBSTITUTION GIVEN $u(x)=xy(x)$

$\frac{d}{dx}(u(x))=\frac{d}{dx}(xy(x))$

$\frac{du}{dx}=x\times\frac{dy}{dx}+1\times y$

$\frac{du}{dx}=x\frac{dy}{dx}+y$

$\boxed{x\frac{dy}{dx}=\frac{du}{dx}-y}$

DIFFERENTIATE THE ABOVE AGAIN WITH RESPECT TO x

$\frac{d}{dx}\left[x\frac{dy}{dx}\right]=\frac{d}{dx}\left[\frac{du}{dx}-y\right]$

$1\times\frac{dy}{dx}+x\frac{d^2y}{dx^2}=\frac{d^2u}{dx^2}-\frac{dy}{dx}$

$\boxed{x\frac{d^2y}{dx^2}=\frac{d^2u}{dx^2}-2\frac{dy}{dx}}$

TRANSFORM THE O.D.E.

$\Rightarrow x\frac{d^2y}{dx^2}+(6x+2)\frac{dy}{dx}+9xy=27x-6y$

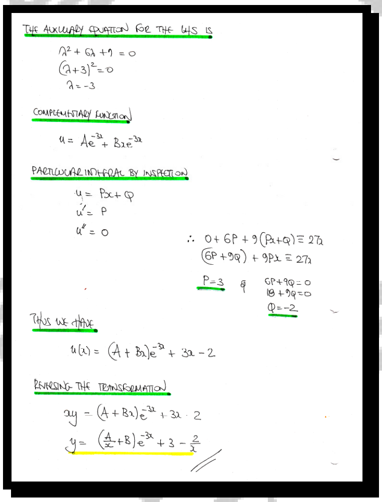
$\Rightarrow \frac{d^2u}{dx^2}-2\frac{dy}{dx}+6x\frac{dy}{dx}+2\frac{dy}{dx}+9u=27x-6y$

$\Rightarrow \frac{d^2u}{dx^2}+6x\frac{dy}{dx}+9u=27x-6y$

$\Rightarrow \frac{d^2u}{dx^2}+6\left(\frac{du}{dx}-y\right)+9u=27x-6y$

$\Rightarrow \frac{d^2u}{dx^2}+6\frac{du}{dx}-6y+9u=27x-6y$

$\Rightarrow \boxed{\frac{d^2u}{dx^2}+6\frac{du}{dx}+9u=27x}$



THE AUXILIARY EQUATION FOR THE LHS IS

$\lambda^2+6\lambda+9=0$

$(\lambda+3)^2=0$

$\lambda=-3$

COMPLEMENTARY FUNCTION

$u=Ae^{-3x}+Bxe^{-3x}$

PARTICULAR INTEGRAL BY INSPECTION

$u=Px+Q$

$u'=P$

$u''=0$

$\therefore\ 0+6P+9(Px+Q)\equiv 27x$

$(6P+9Q)+9Px\equiv 27x$

$P=3$ & $6P+9Q=0$, $18+9Q=0$, $Q=-2$

THUS WE HAVE

$u(x)=(A+Bx)e^{-3x}+3x-2$

REVERSING THE TRANSFORMATION

$xy=(A+Bx)e^{-3x}+3x-2$

$y=\left(\frac{A}{x}+B\right)e^{-3x}+3-\frac{2}{x}$

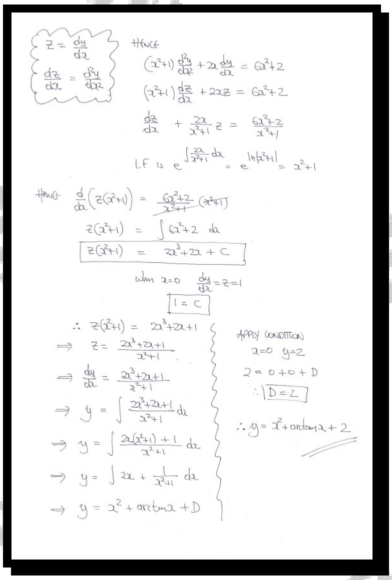
Question 7 (****)

By using the substitution $z = \dfrac{dy}{dx}$, or otherwise, solve the differential equation

$$\left(x^2+1\right)\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} = 6x^2 + 2 ,$$

subject to the conditions $x = 0$, $y = 2$, $\dfrac{dy}{dx} = 1$

$$\boxed{y = x^2 + 2 + \arctan x}$$



Question 8 (****)

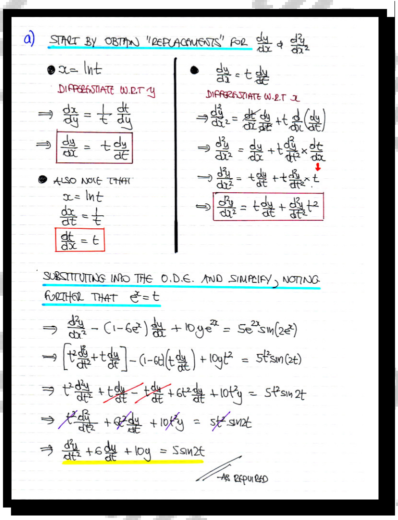
$$\frac{d^2y}{dx^2}-\left(1-6e^x\right)\frac{dy}{dx}+10y\,e^{2x}=5e^{2x}\sin\left(2e^x\right).$$

a) By using the substitution $x=\ln t$ or otherwise, show that the above differential equation can be transformed to

$$\frac{d^2y}{dt^2}+6\frac{dy}{dt}+10y=5\sin 2t\,.$$

b) Hence find a general solution for the original differential equation.

$$y=e^{-3e^x}\left[A\cos\left(e^x\right)+B\sin\left(e^x\right)\right]+\tfrac{1}{6}\sin\left(2e^x\right)-\tfrac{1}{3}\cos\left(2e^x\right)$$



a) START BY OBTAIN "REPLACEMENTS" FOR $\frac{dy}{dx}$ & $\frac{d^2y}{dx^2}$

- $x=\ln t$, DIFFERENTIATE W.R.T y
$\Rightarrow \frac{dx}{dy}=\frac{1}{t}\frac{dt}{dy}$
$\Rightarrow \frac{dy}{dx}=t\frac{dy}{dt}$

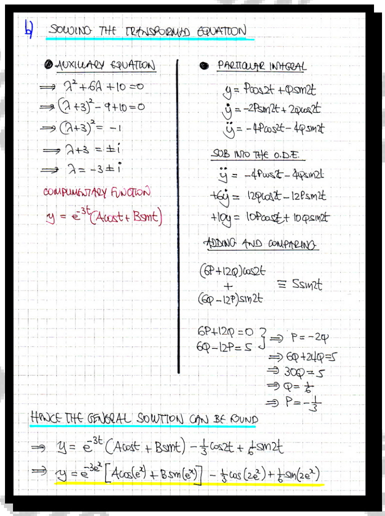
- ALSO NOTE THAT $x=\ln t$, $\frac{dx}{dt}=\frac{1}{t}$, $\frac{dt}{dx}=t$

- $\frac{dy}{dx}=t\frac{dy}{dt}$, DIFFERENTIATE W.R.T x
$\Rightarrow \frac{d^2y}{dx^2}=\frac{dt}{dx}\frac{dy}{dt}+t\frac{d}{dx}\left(\frac{dy}{dt}\right)$
$\Rightarrow \frac{d^2y}{dx^2}=\frac{dy}{dx}+t\frac{d^2y}{dt^2}\times\frac{dt}{dx}$
$\Rightarrow \frac{d^2y}{dx^2}=t\frac{dy}{dt}+t\frac{d^2y}{dt^2}\times t$
$\Rightarrow \frac{d^2y}{dx^2}=t\frac{dy}{dt}+\frac{d^2y}{dt^2}t^2$

SUBSTITUTING INTO THE O.D.E. AND SIMPLIFY, NOTING FURTHER THAT $e^x=t$

$\Rightarrow \frac{d^2y}{dx^2}-(1-6e^x)\frac{dy}{dx}+10ye^{2x}=5e^{2x}\sin(2e^x)$
$\Rightarrow \left[t^2\frac{d^2y}{dt^2}+t\frac{dy}{dt}\right]-(1-6t)\left(t\frac{dy}{dt}\right)+10yt^2=5t^2\sin(2t)$
$\Rightarrow t^2\frac{d^2y}{dt^2}+t\frac{dy}{dt}-t\frac{dy}{dt}+6t^2\frac{dy}{dt}+10t^2y=5t^2\sin 2t$
$\Rightarrow t^2\frac{d^2y}{dt^2}+6t^2\frac{dy}{dt}+10t^2y=5t^2\sin 2t$
$\Rightarrow \frac{d^2y}{dt^2}+6\frac{dy}{dt}+10y=5\sin 2t$

// AS REQUIRED



b) SOLVING THE TRANSFORMED EQUATION

- AUXILIARY EQUATION
$\Rightarrow \lambda^2+6\lambda+10=0$
$\Rightarrow (\lambda+3)^2-9+10=0$
$\Rightarrow (\lambda+3)^2=-1$
$\Rightarrow \lambda+3=\pm i$
$\Rightarrow \lambda=-3\pm i$

COMPLEMENTARY FUNCTION
$y=e^{-3t}(A\cos t+B\sin t)$

- PARTICULAR INTEGRAL
$y=P\cos 2t+Q\sin 2t$
$\dot{y}=-2P\sin 2t+2Q\cos 2t$
$\ddot{y}=-4P\cos 2t-4Q\sin 2t$

SUB INTO THE O.D.E.
$\ddot{y}=-4P\cos 2t-4Q\sin 2t$
$+6\dot{y}=12Q\cos 2t-12P\sin 2t$
$+10y=10P\cos 2t+10Q\sin 2t$

ADDING AND COMPARING
$(6P+12Q)\cos 2t+(6Q-12P)\sin 2t\equiv 5\sin 2t$

$6P+12Q=0$, $6Q-12P=5$ $\Rightarrow P=-2Q$
$\Rightarrow 6Q+24Q=5$
$\Rightarrow 30Q=5$
$\Rightarrow Q=\frac{1}{6}$
$\Rightarrow P=-\frac{1}{3}$

HENCE THE GENERAL SOLUTION CAN BE FOUND
$\Rightarrow y=e^{-3t}(A\cos t+B\sin t)-\frac{1}{3}\cos 2t+\frac{1}{6}\sin 2t$
$\Rightarrow y=e^{-3e^x}\left[A\cos(e^x)+B\sin(e^x)\right]-\frac{1}{3}\cos(2e^x)+\frac{1}{6}\sin(2e^x)$

Question 9 (****)

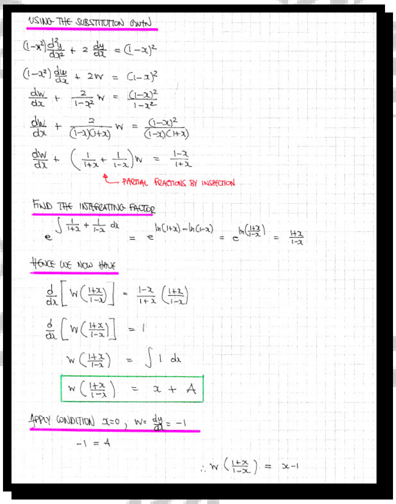
Use the substitution $w = \frac{dy}{dx}$ to solve the following differential equation

$$\left(1-x^2\right)\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = (1-x)^2, \ |x| < 1$$

subject to the boundary conditions $y = -4.5$ and $\frac{dy}{dx} = -1$ at $x = 0$.

Give the answer in the form $y = \alpha(x-3)^2 + \beta\ln(x+1)$, where α and β are constants to be found.

$\boxed{}$, $\boxed{y = -\frac{1}{4}(x-3)^2 - 4\ln(x+1)}$



USING THE SUBSTITUTION GIVEN

$(1-x^2)\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = (1-x)^2$

$(1-x^2)\frac{dw}{dx} + 2w = (1-x)^2$

$\frac{dw}{dx} + \frac{2}{1-x^2}w = \frac{(1-x)^2}{1-x^2}$

$\frac{dw}{dx} + \frac{2}{(1-x)(1+x)}w = \frac{(1-x)^2}{(1-x)(1+x)}$

$\frac{dw}{dx} + \left(\frac{1}{1+x} + \frac{1}{1-x}\right)w = \frac{1-x}{1+x}$

← PARTIAL FRACTIONS BY INSPECTION

FIND THE INTEGRATING FACTOR

$e^{\int \frac{1}{1+x} + \frac{1}{1-x}\,dx} = e^{\ln(1+x) - \ln(1-x)} = e^{\ln\left(\frac{1+x}{1-x}\right)} = \frac{1+x}{1-x}$

HENCE WE NOW HAVE

$\frac{d}{dx}\left[w\left(\frac{1+x}{1-x}\right)\right] = \frac{1-x}{1+x}\left(\frac{1+x}{1-x}\right)$

$\frac{d}{dx}\left[w\left(\frac{1+x}{1-x}\right)\right] = 1$

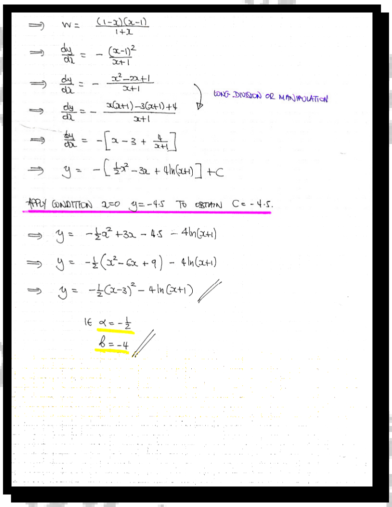
$w\left(\frac{1+x}{1-x}\right) = \int 1\,dx$

$\boxed{w\left(\frac{1+x}{1-x}\right) = x + A}$

APPLY CONDITION $x=0$, $w = \frac{dy}{dx} = -1$

$-1 = A$

$\therefore w\left(\frac{1+x}{1-x}\right) = x-1$



$\Rightarrow w = \frac{(1-x)(x-1)}{1+x}$

$\Rightarrow \frac{dy}{dx} = -\frac{(x-1)^2}{x+1}$

$\Rightarrow \frac{dy}{dx} = -\frac{x^2-2x+1}{x+1}$

$\Rightarrow \frac{dy}{dx} = -\frac{x(x+1)-3(x+1)+4}{x+1}$ LONG DIVISION OR MANIPULATION

$\Rightarrow \frac{dy}{dx} = -\left[x - 3 + \frac{4}{x+1}\right]$

$\Rightarrow y = -\left[\frac{1}{2}x^2 - 3x + 4\ln(x+1)\right] + C$

APPLY CONDITION $x=0$ $y=-4.5$ TO OBTAIN $C = -4.5$.

$\Rightarrow y = -\frac{1}{2}x^2 + 3x - 4.5 - 4\ln(x+1)$

$\Rightarrow y = -\frac{1}{2}(x^2 - 6x + 9) - 4\ln(x+1)$

$\Rightarrow y = -\frac{1}{2}(x-3)^2 - 4\ln(x+1)$

IE $\alpha = -\frac{1}{2}$

$\beta = -4$

Question 10 (****)

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx}\tan x - y\sec^4 x = 0.$$

The above differential equation is to be solved by a substitution.

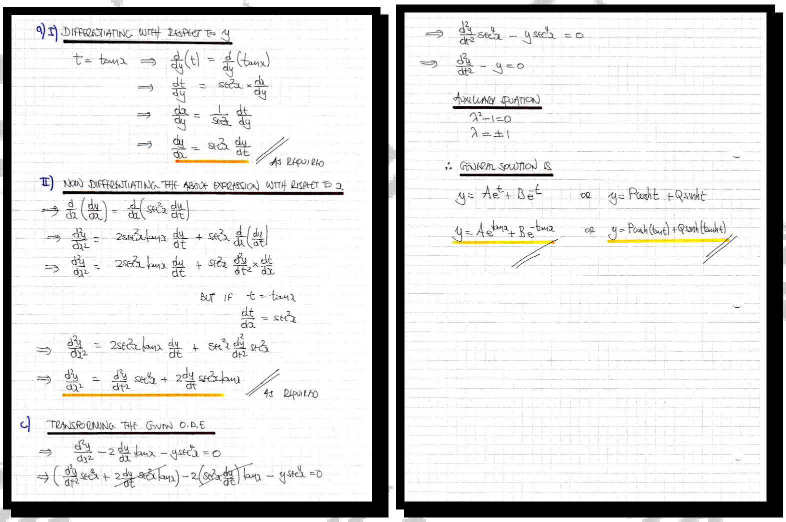
a) If $t = \tan x$ show that …

i. … $\dfrac{dy}{dx} = \dfrac{dy}{dt}\sec^2 x$

ii. … $\dfrac{d^2y}{dx^2} = \dfrac{d^2y}{dt^2}\sec^4 x + 2\dfrac{dy}{dt}\sec^2 x \tan x$

b) Use the results obtained in part **(a)** to find a general solution of the differential equation in the form $y = f(x)$.

FP2-Q , $y = Ae^{\tan x} + Be^{-\tan x}$



a) I) DIFFERENTIATING WITH RESPECT TO y

$t = \tan x \Rightarrow \frac{d}{dy}(t) = \frac{d}{dy}(\tan x)$

$\Rightarrow \frac{dt}{dy} = \sec^2 x \times \frac{dx}{dy}$

$\Rightarrow \frac{dx}{dy} = \frac{1}{\sec^2 x}\frac{dt}{dy}$

$\Rightarrow \frac{dy}{dx} = \sec^2 x \frac{dy}{dt}$ // AS REQUIRED

II) NOW DIFFERENTIATING THE ABOVE EXPRESSION WITH RESPECT TO x

$\Rightarrow \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\sec^2 x \frac{dy}{dt}\right)$

$\Rightarrow \frac{d^2y}{dx^2} = 2\sec^2 x \tan x \frac{dy}{dt} + \sec^2 x \frac{d}{dx}\left(\frac{dy}{dt}\right)$

$\Rightarrow \frac{d^2y}{dx^2} = 2\sec^2 x \tan x \frac{dy}{dt} + \sec^2 x \frac{d^2y}{dt^2} \times \frac{dt}{dx}$

BUT IF $t = \tan x$, $\frac{dt}{dx} = \sec^2 x$

$\Rightarrow \frac{d^2y}{dx^2} = 2\sec^2 x \tan x \frac{dy}{dt} + \sec^2 x \frac{d^2y}{dt^2}\sec^2 x$

$\Rightarrow \frac{d^2y}{dx^2} = \frac{d^2y}{dt^2}\sec^4 x + 2\frac{dy}{dt}\sec^2 x \tan x$ // AS REQUIRED

c) TRANSFORMING THE GIVEN O.D.E

$\Rightarrow \frac{d^2y}{dx^2} - 2\frac{dy}{dx}\tan x - y\sec^4 x = 0$

$\Rightarrow \left(\frac{d^2y}{dt^2}\sec^4 x + 2\frac{dy}{dt}\sec^2 x \tan x\right) - 2\left(\sec^2 x \frac{dy}{dt}\right)\tan x - y\sec^4 x = 0$

$\Rightarrow \frac{d^2y}{dt^2}\sec^4 x - y\sec^4 x = 0$

$\Rightarrow \frac{d^2y}{dt^2} - y = 0$

AUXILIARY EQUATION

$\lambda^2 - 1 = 0$

$\lambda = \pm 1$

$\therefore$ GENERAL SOLUTION IS

$y = Ae^{t} + Be^{-t}$ OR $y = P\cosh t + Q\sinh t$

$y = Ae^{\tan x} + Be^{-\tan x}$ // OR $y = P\cosh(\tan x) + Q\sinh(\tan x)$ //

Question 11 (****)

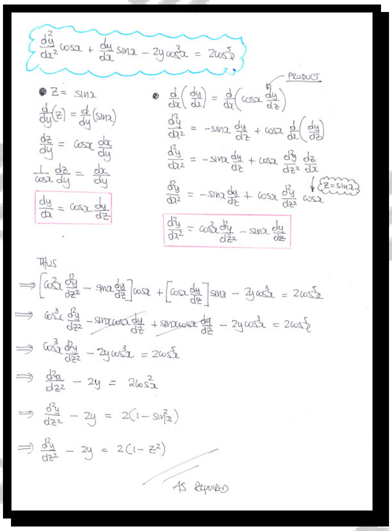
Show clearly that the substitution $z = \sin x$, transforms the differential equation

$$\frac{d^2y}{dx^2}\cos x + \frac{dy}{dx}\sin x - 2y\cos^3 x = 2\cos^5 x,$$

into the differential equation

$$\frac{d^2y}{dz^2} - 2y = 2\left(1 - z^2\right)$$

proof



$$\frac{d^2y}{dx^2}\cos x + \frac{dy}{dx}\sin x - 2y\cos^3 x = 2\cos^5 x$$

- $z = \sin x$

$$\frac{d}{dy}(z) = \frac{d}{dy}(\sin x)$$

$$\frac{dz}{dy} = \cos x \frac{dx}{dy}$$

$$\frac{1}{\cos x}\frac{dz}{dy} = \frac{dx}{dy}$$

$$\frac{dy}{dx} = \cos x \frac{dy}{dz}$$

- $\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\cos x \frac{dy}{dz}\right)$ PRODUCT

$$\frac{d^2y}{dx^2} = -\sin x \frac{dy}{dz} + \cos x \frac{d}{dx}\left(\frac{dy}{dz}\right)$$

$$\frac{d^2y}{dx^2} = -\sin x \frac{dy}{dz} + \cos x \frac{d^2y}{dz^2}\frac{dz}{dx}$$

$$\frac{d^2y}{dx^2} = -\sin x \frac{dy}{dz} + \cos x \frac{d^2y}{dz^2}\cos x \quad (z = \sin x)$$

$$\frac{d^2y}{dx^2} = \cos^2 x \frac{d^2y}{dz^2} - \sin x \frac{dy}{dz}$$

THUS

$$\Rightarrow \left[\cos^2 x \frac{d^2y}{dz^2} - \sin x \frac{dy}{dz}\right]\cos x + \left[\cos x \frac{dy}{dz}\right]\sin x - 2y\cos^3 x = 2\cos^5 x$$

$$\Rightarrow \cos^3 x \frac{d^2y}{dz^2} - \sin x\cos x \frac{dy}{dz} + \sin x\cos x \frac{dy}{dz} - 2y\cos^3 x = 2\cos^5 x$$

$$\Rightarrow \cos^3 x \frac{d^2y}{dz^2} - 2y\cos^3 x = 2\cos^5 x$$

$$\Rightarrow \frac{d^2y}{dz^2} - 2y = 2\cos^2 x$$

$$\Rightarrow \frac{d^2y}{dz^2} - 2y = 2(1 - \sin^2 x)$$

$$\Rightarrow \frac{d^2y}{dz^2} - 2y = 2(1 - z^2)$$

AS REQUIRED

Question 12 (****+)

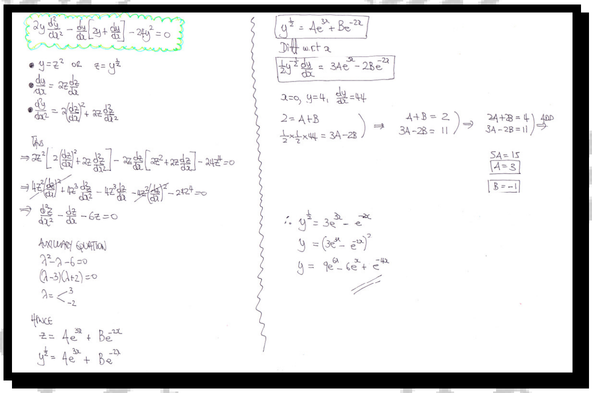
Use the substitution $z=\sqrt{y}$, where $y=f(x)$, to solve the differential equation

$$\frac{d^2y}{dx^2}+\frac{1}{y}\left(\frac{dy}{dx}\right)^2-5\frac{dy}{dx}+2y=0,$$

subject to the boundary conditions $y=4$, $\frac{dy}{dx}=44$ at $x=0$.

Give the answer in the form $y=f(x)$.

$$\boxed{y=9e^{6x}-6e^{x}+e^{-4x}}$$



Question 13 (****+)

$$2x\frac{d^2y}{dx^2}+\left(1-3x^{\frac{1}{2}}\right)\frac{dy}{dx}+y=0.$$

The above differential equation is to be solved by a substitution.

a) Given that $y=f(x)$ and $t=x^{\frac{1}{2}}$, show clearly that …

i. … $\frac{dy}{dx}=\frac{1}{2t}\frac{dy}{dt}$.

ii. … $\frac{d^2y}{dx^2}=\frac{1}{4t^2}\frac{d^2y}{dt^2}-\frac{1}{4t^3}\frac{dy}{dt}$.

b) Hence show further that the differential equation

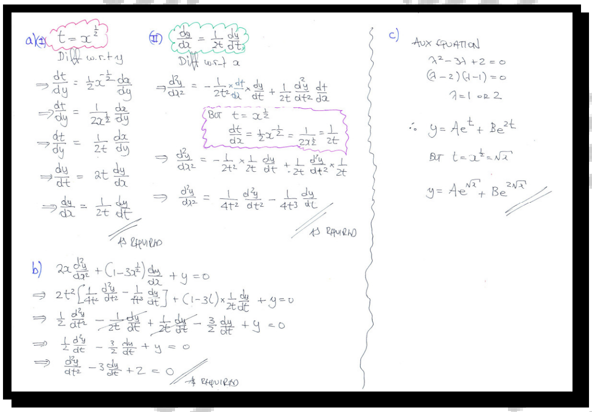
$$2x\frac{d^2y}{dx^2}+\left(1-3x^{\frac{1}{2}}\right)\frac{dy}{dx}+y=0,$$

can be transformed to the differential equation

$$\frac{d^2y}{dt^2}-3\frac{dy}{dt}+2y=0.$$

c) Find a general solution of the **original** differential equation, giving the answer in the form $y=f(x)$.

$$\boxed{y=Ae^{\sqrt{x}}+Be^{2\sqrt{x}}}$$



Question 14 **(****+)**

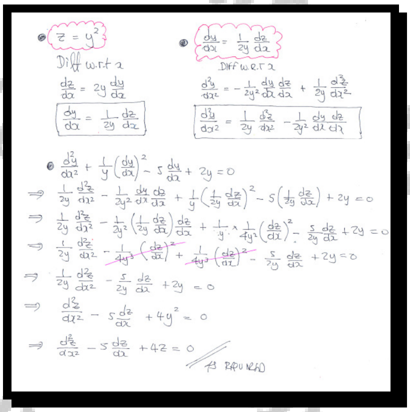
Show clearly that the substitution $z = y^2$, where $y = f(x)$, transforms the differential equation

$$\frac{d^2y}{dx^2} + \frac{1}{y}\left(\frac{dy}{dx}\right)^2 - 5\frac{dy}{dx} + 2y = 0,$$

into the differential equation

$$\frac{d^2z}{dx^2} - 5\frac{dz}{dx} + 4z = 0$$

proof



- $z = y^2$

Diff w.r.t x

$$\frac{dz}{dx} = 2y\frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{1}{2y}\frac{dz}{dx}$$

- $\frac{dy}{dx} = \frac{1}{2y}\frac{dz}{dx}$

Diff w.r.t x

$$\frac{d^2y}{dx^2} = -\frac{1}{2y^2}\frac{dy}{dx}\frac{dz}{dx} + \frac{1}{2y}\frac{d^2z}{dx^2}$$

$$\frac{d^2y}{dx^2} = \frac{1}{2y}\frac{d^2z}{dx^2} - \frac{1}{2y^2}\frac{dy}{dx}\frac{dz}{dx}$$

- $\frac{d^2y}{dx^2} + \frac{1}{y}\left(\frac{dy}{dx}\right)^2 - 5\frac{dy}{dx} + 2y = 0$

$$\Rightarrow \frac{1}{2y}\frac{d^2z}{dx^2} - \frac{1}{2y^2}\frac{dy}{dx}\frac{dz}{dx} + \frac{1}{y}\left(\frac{1}{2y}\frac{dz}{dx}\right)^2 - 5\left(\frac{1}{2y}\frac{dz}{dx}\right) + 2y = 0$$

$$\Rightarrow \frac{1}{2y}\frac{d^2z}{dx^2} - \frac{1}{2y^2}\left(\frac{1}{2y}\frac{dz}{dx}\right)\frac{dz}{dx} + \frac{1}{y}\times\frac{1}{4y^2}\left(\frac{dz}{dx}\right)^2 - \frac{5}{2y}\frac{dz}{dx} + 2y = 0$$

$$\Rightarrow \frac{1}{2y}\frac{d^2z}{dx^2} - \frac{1}{4y^3}\left(\frac{dz}{dx}\right)^2 + \frac{1}{4y^3}\left(\frac{dz}{dx}\right)^2 - \frac{5}{2y}\frac{dz}{dx} + 2y = 0$$

$$\Rightarrow \frac{1}{2y}\frac{d^2z}{dx^2} - \frac{5}{2y}\frac{dz}{dx} + 2y = 0$$

$$\Rightarrow \frac{d^2z}{dx^2} - 5\frac{dz}{dx} + 4y^2 = 0$$

$$\Rightarrow \frac{d^2z}{dx^2} - 5\frac{dz}{dx} + 4z = 0$$

AS REQUIRED

Question 15 (****+)

Given that if $x = t^{\frac{1}{2}}$, where $y = f(x)$, show clearly that

a) $\dfrac{dy}{dx} = 2t^{\frac{1}{2}}\dfrac{dy}{dt}$.

b) $\dfrac{d^2y}{dx^2} = 4t\dfrac{d^2y}{dt^2} + 2\dfrac{dy}{dt}$.

The following differential equation is to be solved

$$x\frac{d^2y}{dx^2} - \left(8x^2+1\right)\frac{dy}{dx} + 12x^3y = 12x^5,$$

subject to the boundary conditions $y = \dfrac{10}{3}$, $\dfrac{d^2y}{dx^2} = 10$ at $x = 0$.

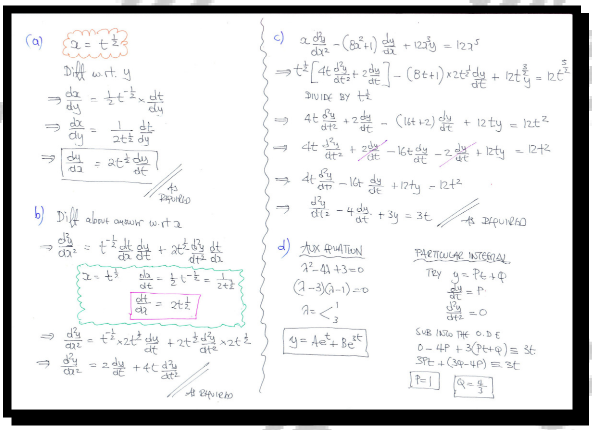
c) Show further that the substitution $x = t^{\frac{1}{2}}$, where $y = f(x)$, transforms the above differential equation into the differential equation

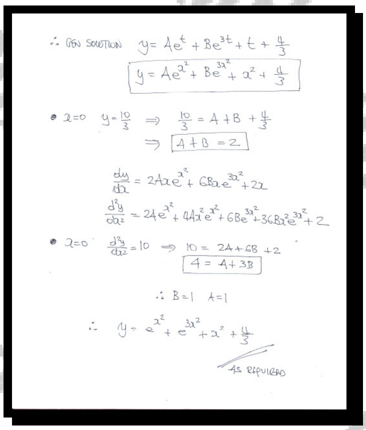
$$\frac{d^2y}{dt^2} - 4\frac{dy}{dt} + 3y = 3t .$$

d) Show that a solution of the **original** differential equation is

$$y = e^{3x^2} + e^{x^2} + x^2 + \frac{4}{3}.$$

proof





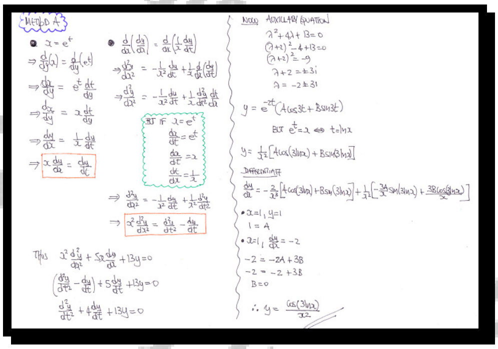
Question 16 **(****+)**

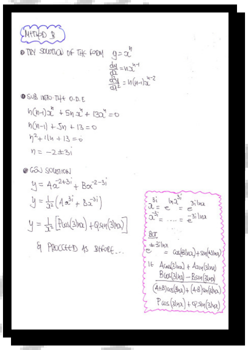
The curve with equation $y = f(x)$ satisfies

$$x^2 \frac{d^2y}{dx^2} + 5x\frac{dy}{dx} + 13y = 0, \;\; x > 0.$$

By using the substitution $x = e^t$, or otherwise, determine an equation for $y = f(x)$, given further that $y = 1$ and $\frac{dy}{dx} = -2$ at $x = 1$.

$$\boxed{y = \frac{\cos(3\ln x)}{x^2}}$$





Question 17 (****+)

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx}\cot x + 2y\,\text{cosec}^2 x = 2\cos x - 2\cos^3 x.$$

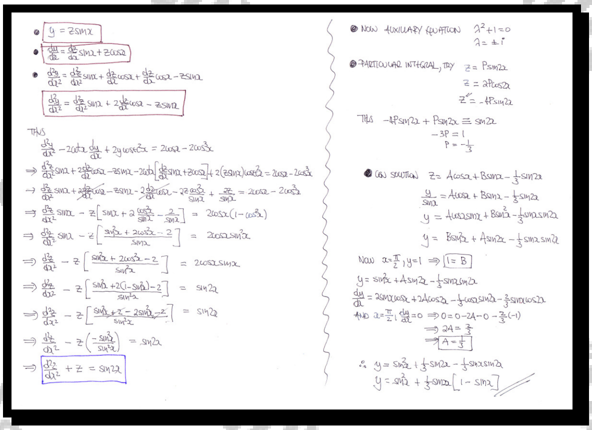
Use the substitution $y = z\sin x$, where z is a function of x, to solve the above differential equation subject to the boundary conditions $y = 1$, $\frac{dy}{dx} = 0$ at $x = \frac{\pi}{2}$.

Give the answer in the form

$$y = a\sin^2 x + b(1 - \sin x)\sin 2x,$$

where a and b are constants to be found.

$\boxed{a = 1}$, $\boxed{b = \frac{1}{3}}$



Question 18 **(****+)**

The function $y = f(x)$ satisfies the following relationship.

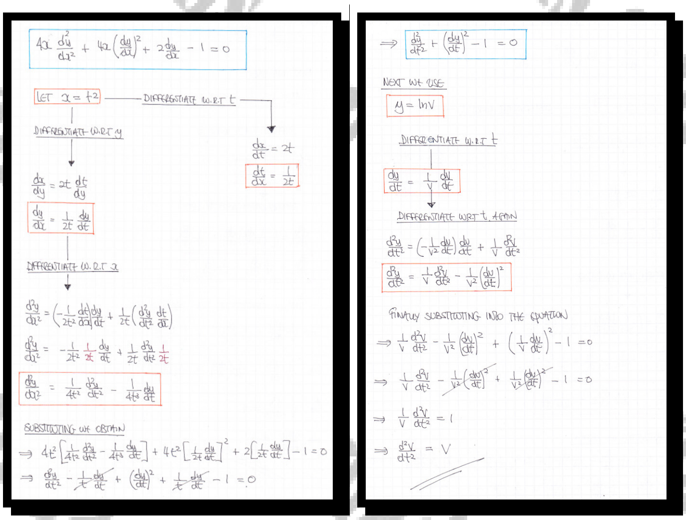
$$4x\frac{d^2y}{dx^2} + 4x\left(\frac{dy}{dx}\right)^2 + 2\frac{dy}{dx} - 1 = 0.$$

It is further given that $x = t^2$ and $y = \ln v$.

Show that

$$\frac{d^2v}{dt^2} = v.$$

proof



$$4x\frac{d^2y}{dx^2} + 4x\left(\frac{dy}{dx}\right)^2 + 2\frac{dy}{dx} - 1 = 0$$

LET $x = t^2$ — DIFFERENTIATE W.R.T t

$\frac{dx}{dt} = 2t$

$\frac{dt}{dx} = \frac{1}{2t}$

DIFFERENTIATE W.R.T y

$\frac{dx}{dy} = 2t\frac{dt}{dy}$

$\frac{dy}{dx} = \frac{1}{2t}\frac{dy}{dt}$

DIFFERENTIATE W.R.T x

$\frac{d^2y}{dx^2} = \left(-\frac{1}{2t^2}\frac{dt}{dx}\right)\frac{dy}{dt} + \frac{1}{2t}\left(\frac{d^2y}{dt^2}\frac{dt}{dx}\right)$

$\frac{d^2y}{dx^2} = -\frac{1}{2t^2}\frac{1}{2t}\frac{dy}{dt} + \frac{1}{2t}\frac{d^2y}{dt^2}\frac{1}{2t}$

$\frac{d^2y}{dx^2} = \frac{1}{4t^2}\frac{d^2y}{dt^2} - \frac{1}{4t^3}\frac{dy}{dt}$

SUBSTITUTING WE OBTAIN

$\Rightarrow 4t^2\left[\frac{1}{4t^2}\frac{d^2y}{dt^2} - \frac{1}{4t^3}\frac{dy}{dt}\right] + 4t^2\left[\frac{1}{2t}\frac{dy}{dt}\right]^2 + 2\left[\frac{1}{2t}\frac{dy}{dt}\right] - 1 = 0$

$\Rightarrow \frac{d^2y}{dt^2} - \frac{1}{t}\frac{dy}{dt} + \left(\frac{dy}{dt}\right)^2 + \frac{1}{t}\frac{dy}{dt} - 1 = 0$

$\Rightarrow \frac{d^2y}{dt^2} + \left(\frac{dy}{dt}\right)^2 - 1 = 0$

NEXT WE USE

$y = \ln v$

DIFFERENTIATE W.R.T t

$\frac{dy}{dt} = \frac{1}{v}\frac{dv}{dt}$

DIFFERENTIATE W.R.T t, AGAIN

$\frac{d^2y}{dt^2} = \left(-\frac{1}{v^2}\frac{dv}{dt}\right)\frac{dv}{dt} + \frac{1}{v}\frac{d^2v}{dt^2}$

$\frac{d^2y}{dt^2} = \frac{1}{v}\frac{d^2v}{dt^2} - \frac{1}{v^2}\left(\frac{dv}{dt}\right)^2$

FINALLY SUBSTITUTING INTO THE EQUATION

$\Rightarrow \frac{1}{v}\frac{d^2v}{dt^2} - \frac{1}{v^2}\left(\frac{dv}{dt}\right)^2 + \left(\frac{1}{v}\frac{dv}{dt}\right)^2 - 1 = 0$

$\Rightarrow \frac{1}{v}\frac{d^2v}{dt^2} - \frac{1}{v^2}\left(\frac{dv}{dt}\right)^2 + \frac{1}{v^2}\left(\frac{dv}{dt}\right)^2 - 1 = 0$

$\Rightarrow \frac{1}{v}\frac{d^2v}{dt^2} = 1$

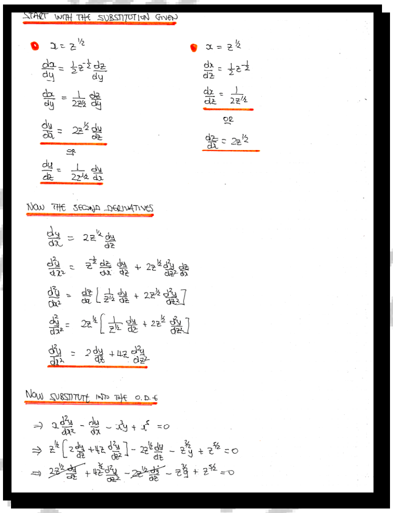
$\Rightarrow \frac{d^2v}{dt^2} = v$

Question 19 (****+)

$$x\frac{d^2y}{dx^2}-\frac{dy}{dx}-x^3y+x^5=0.$$

Use the substitution $x=z^{\frac{1}{2}}$, where $y=f(x)$, to find a general solution of the above differential equation.

V, , $\boxed{y=A\mathrm{e}^{\frac{1}{2}x^2}+B\mathrm{e}^{-\frac{1}{2}x^2}+x^2}$



START WITH THE SUBSTITUTION GIVEN

- $x=z^{1/2}$

$\frac{dx}{dy}=\frac{1}{2}z^{-\frac{1}{2}}\frac{dz}{dy}$

$\frac{dx}{dy}=\frac{1}{2z^{1/2}}\frac{dz}{dy}$

$\frac{dy}{dx}=2z^{1/2}\frac{dy}{dz}$

or

$\frac{dy}{dz}=\frac{1}{2z^{1/2}}\frac{dy}{dx}$

- $x=z^{1/2}$

$\frac{dx}{dz}=\frac{1}{2}z^{-\frac{1}{2}}$

$\frac{dx}{dz}=\frac{1}{2z^{1/2}}$

or

$\frac{dz}{dx}=2z^{1/2}$

NOW THE SECOND DERIVATIVES

$\frac{dy}{dx}=2z^{1/2}\frac{dy}{dz}$

$\frac{d^2y}{dx^2}=z^{-\frac{1}{2}}\frac{dz}{dx}\frac{dy}{dz}+2z^{1/2}\frac{d^2y}{dz^2}\frac{dz}{dx}$

$\frac{d^2y}{dx^2}=\frac{dz}{dx}\left[\frac{1}{z^{1/2}}\frac{dy}{dz}+2z^{1/2}\frac{d^2y}{dz^2}\right]$

$\frac{d^2y}{dx^2}=2z^{1/2}\left[\frac{1}{z^{1/2}}\frac{dy}{dz}+2z^{1/2}\frac{d^2y}{dz^2}\right]$

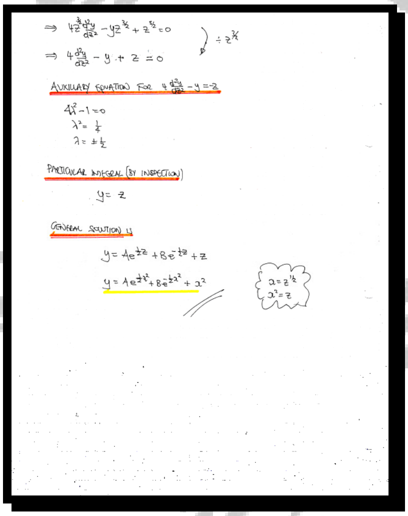
$\frac{d^2y}{dx^2}=2\frac{dy}{dz}+4z\frac{d^2y}{dz^2}$

NOW SUBSTITUTE INTO THE O.D.E

$\Rightarrow x\frac{d^2y}{dx^2}-\frac{dy}{dx}-x^3y+x^5=0$

$\Rightarrow z^{1/2}\left[2\frac{dy}{dz}+4z\frac{d^2y}{dz^2}\right]-2z^{1/2}\frac{dy}{dz}-z^{3/2}y+z^{5/2}=0$

$\Rightarrow 2z^{1/2}\frac{dy}{dz}+4z^{3/2}\frac{d^2y}{dz^2}-2z^{1/2}\frac{dy}{dz}-z^{3/2}y+z^{5/2}=0$



$\Rightarrow 4z^{3/2}\frac{d^2y}{dz^2}-yz^{3/2}+z^{5/2}=0 \quad \div z^{3/2}$

$\Rightarrow 4\frac{d^2y}{dz^2}-y+z=0$

AUXILIARY EQUATION FOR $4\frac{d^2y}{dz^2}-y=-z$

$4\lambda^2-1=0$

$\lambda^2=\frac{1}{4}$

$\lambda=\pm\frac{1}{2}$

PARTICULAR INTEGRAL (BY INSPECTION)

$y=z$

GENERAL SOLUTION IS

$y=Ae^{\frac{1}{2}z}+Be^{-\frac{1}{2}z}+z$

$y=Ae^{\frac{1}{2}x^2}+Be^{-\frac{1}{2}x^2}+x^2$

$x=z^{1/2}$
$x^2=z$

Question 20 (*****)

Find the solution of following differential equation

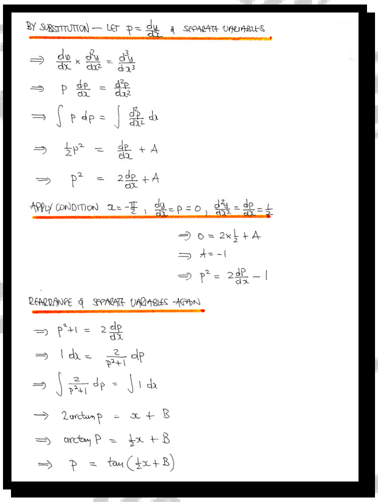
$$\left(\frac{dy}{dx}\right)\left(\frac{d^2y}{dx^2}\right) = \frac{d^3y}{dx^3},$$

subject to the boundary conditions.

$$y\left(-\tfrac{1}{2}\pi\right) = y'\left(-\tfrac{1}{2}\pi\right) = 0, \qquad y''\left(-\tfrac{1}{2}\pi\right) = \tfrac{1}{2}.$$

Given the answer in the form $y = f(x)$.

$\boxed{}$, $\boxed{y = 2\ln\left|\sec\left(\frac{1}{2}x + \frac{1}{4}\pi\right)\right|}$


BY SUBSTITUTION — LET $p = \frac{dy}{dx}$ & SEPARATE VARIABLES

$\Rightarrow \frac{dy}{dx} \times \frac{d^2y}{dx^2} = \frac{d^3y}{dx^3}$

$\Rightarrow p\frac{dp}{dx} = \frac{d^2p}{dx^2}$

$\Rightarrow \int p\,dp = \int \frac{d^2p}{dx^2}\,dx$

$\Rightarrow \frac{1}{2}p^2 = \frac{dp}{dx} + A$

$\Rightarrow p^2 = 2\frac{dp}{dx} + A$

APPLY CONDITION $x = -\frac{\pi}{2}$, $\frac{dy}{dx} = p = 0$, $\frac{d^2y}{dx^2} = \frac{dp}{dx} = \frac{1}{2}$

$\Rightarrow 0 = 2 \times \frac{1}{2} + A$

$\Rightarrow A = -1$

$\Rightarrow p^2 = 2\frac{dp}{dx} - 1$

REARRANGE & SEPARATE VARIABLES AGAIN

$\Rightarrow p^2 + 1 = 2\frac{dp}{dx}$

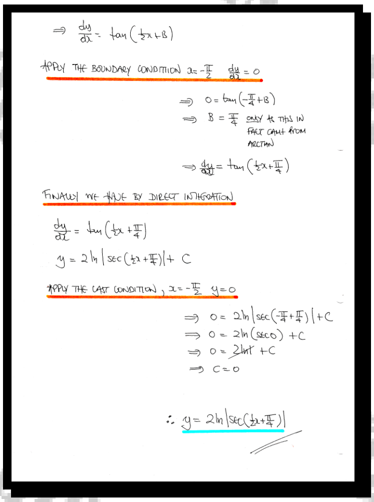
$\Rightarrow 1\,dx = \frac{2}{p^2+1}\,dp$

$\Rightarrow \int \frac{2}{p^2+1}\,dp = \int 1\,dx$

$\Rightarrow 2\arctan p = x + B$

$\Rightarrow \arctan p = \frac{1}{2}x + B$

$\Rightarrow p = \tan\left(\frac{1}{2}x + B\right)$


$\Rightarrow \frac{dy}{dx} = \tan\left(\frac{1}{2}x + B\right)$

APPLY THE BOUNDARY CONDITION $x = -\frac{\pi}{2}$ $\frac{dy}{dx} = 0$

$\Rightarrow 0 = \tan\left(-\frac{\pi}{4} + B\right)$

$\Rightarrow B = \frac{\pi}{4}$ ONLY AS THIS IN FACT CAME FROM ARCTAN

$\Rightarrow \frac{dy}{dx} = \tan\left(\frac{1}{2}x + \frac{\pi}{4}\right)$

FINALLY WE HAVE BY DIRECT INTEGRATION

$\frac{dy}{dx} = \tan\left(\frac{1}{2}x + \frac{\pi}{4}\right)$

$y = 2\ln\left|\sec\left(\frac{1}{2}x + \frac{\pi}{4}\right)\right| + C$

APPLY THE LAST CONDITION, $x = -\frac{\pi}{2}$ $y = 0$

$\Rightarrow 0 = 2\ln\left|\sec\left(-\frac{\pi}{4} + \frac{\pi}{4}\right)\right| + C$

$\Rightarrow 0 = 2\ln(\sec 0) + C$

$\Rightarrow 0 = 2\ln 1 + C$

$\Rightarrow C = 0$

$\therefore y = 2\ln\left|\sec\left(\frac{1}{2}x + \frac{\pi}{4}\right)\right|$

Question 21 (*****)

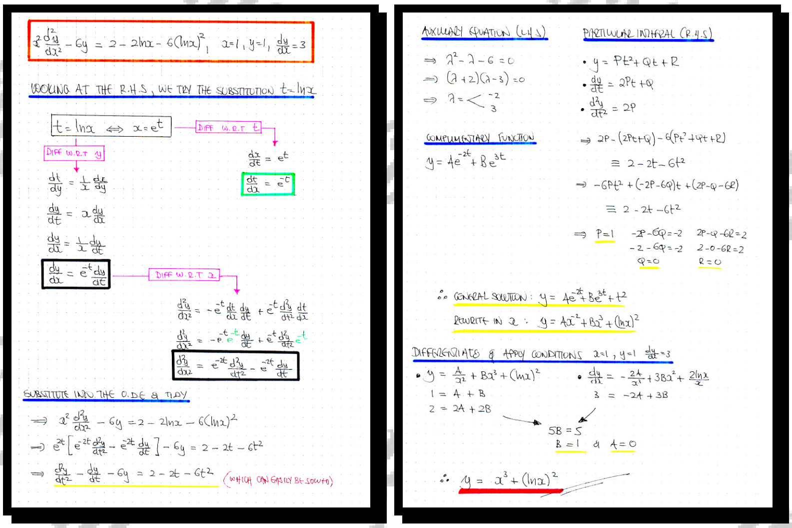
Use a suitable substitution to solve the differential equation

$$x^2\frac{d^2y}{dx^2}-6y=2-2\ln x-6(\ln x)^2,$$

subject to the boundary conditions $y(1)=1$, $\dfrac{dy}{dx}(1)=3$

Give a simplified answer in the form $y=f(x)$.

F.P.S, $\boxed{y=x^3+(\ln x)^2}$



Question 22 (*****)

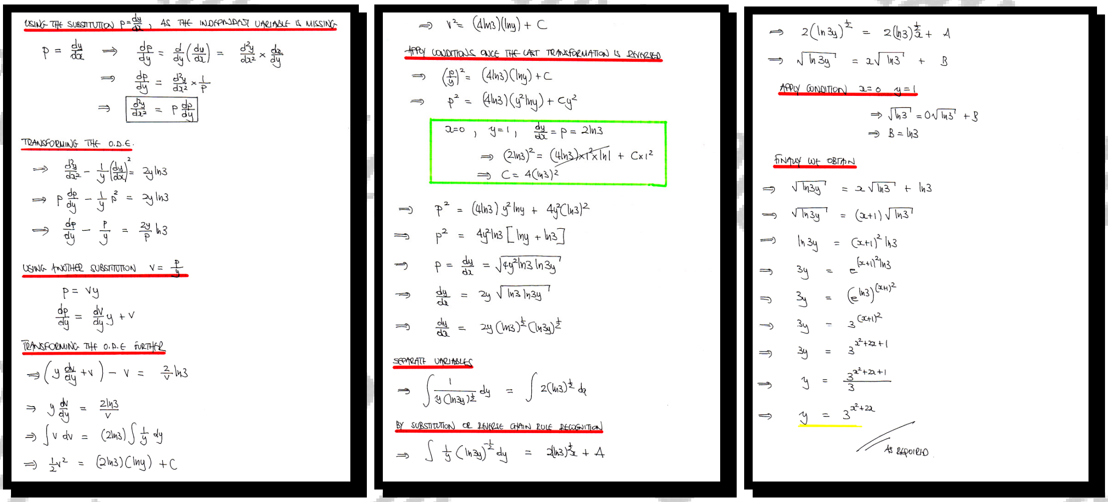
The function with equation $y = f(x)$ satisfies the differential equation

$$\frac{d^2y}{dx^2} - \frac{1}{y}\left(\frac{dy}{dx}\right)^2 = 2y\ln 3, \quad y(0)=1, \quad \frac{dy}{dx}(0) = 2\ln 3.$$

Solve the above differential equation by using the substitution $p = \frac{dy}{dx}$, to show that

$$y = 3^{x^2+2x}.$$

[ ] , [proof]



USING THE SUBSTITUTION $p=\frac{dy}{dx}$, AS THE INDEPENDENT VARIABLE IS MISSING

$p = \frac{dy}{dx} \Rightarrow \frac{dp}{dy} = \frac{d}{dy}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2} \times \frac{dx}{dy}$

$\Rightarrow \frac{dp}{dy} = \frac{d^2y}{dx^2} \times \frac{1}{p}$

$\Rightarrow \boxed{\frac{d^2y}{dx^2} = p\frac{dp}{dy}}$

TRANSFORMING THE O.D.E.

$\Rightarrow \frac{d^2y}{dx^2} - \frac{1}{y}\left(\frac{dy}{dx}\right)^2 = 2y\ln 3$

$\Rightarrow p\frac{dp}{dy} - \frac{1}{y}p^2 = 2y\ln 3$

$\Rightarrow \frac{dp}{dy} - \frac{p}{y} = \frac{2y}{p}\ln 3$

USING ANOTHER SUBSTITUTION $v = \frac{p}{y}$

$p = vy$

$\frac{dp}{dy} = \frac{dv}{dy}y + v$

TRANSFORMING THE O.D.E FURTHER

$\Rightarrow \left(y\frac{dv}{dy} + v\right) - v = \frac{2}{v}\ln 3$

$\Rightarrow y\frac{dv}{dy} = \frac{2\ln 3}{v}$

$\Rightarrow \int v\,dv = (2\ln 3)\int \frac{1}{y}\,dy$

$\Rightarrow \frac{1}{2}v^2 = (2\ln 3)(\ln y) + C$

$\Rightarrow v^2 = (4\ln 3)(\ln y) + C$

APPLY CONDITIONS ONCE THE LAST TRANSFORMATION IS REVERSED

$\Rightarrow \left(\frac{p}{y}\right)^2 = (4\ln 3)(\ln y) + C$

$\Rightarrow p^2 = (4\ln 3)(y^2\ln y) + Cy^2$

$x=0,\ y=1,\ \frac{dy}{dx} = p = 2\ln 3$

$\Rightarrow (2\ln 3)^2 = (4\ln 3)\times 1^2 \times \ln 1 + C\times 1^2$

$\Rightarrow C = 4(\ln 3)^2$

$\Rightarrow p^2 = (4\ln 3)y^2\ln y + 4y^2(\ln 3)^2$

$\Rightarrow p^2 = 4y^2\ln 3\left[\ln y + \ln 3\right]$

$\Rightarrow p = \frac{dy}{dx} = \sqrt{4y^2\ln 3\ln 3y}$

$\Rightarrow \frac{dy}{dx} = 2y\sqrt{\ln 3\ln 3y}$

$\Rightarrow \frac{dy}{dx} = 2y(\ln 3)^{\frac{1}{2}}(\ln 3y)^{\frac{1}{2}}$

SEPARATE VARIABLES

$\Rightarrow \int \frac{1}{y(\ln 3y)^{\frac{1}{2}}}\,dy = \int 2(\ln 3)^{\frac{1}{2}}\,dx$

BY SUBSTITUTION OR REVERSE CHAIN RULE RECOGNITION

$\Rightarrow \int \frac{1}{y}(\ln 3y)^{-\frac{1}{2}}\,dy = 2(\ln 3)^{\frac{1}{2}}x + A$

$\Rightarrow 2(\ln 3y)^{\frac{1}{2}} = 2(\ln 3)^{\frac{1}{2}}x + A$

$\Rightarrow \sqrt{\ln 3y} = x\sqrt{\ln 3} + B$

APPLY CONDITION $x=0\ \ y=1$

$\Rightarrow \sqrt{\ln 3} = 0\sqrt{\ln 3} + B$

$\Rightarrow B = \ln 3$

FINALLY WE OBTAIN

$\Rightarrow \sqrt{\ln 3y} = x\sqrt{\ln 3} + \ln 3$

$\Rightarrow \sqrt{\ln 3y} = (x+1)\sqrt{\ln 3}$

$\Rightarrow \ln 3y = (x+1)^2\ln 3$

$\Rightarrow 3y = e^{(x+1)^2\ln 3}$

$\Rightarrow 3y = \left(e^{\ln 3}\right)^{(x+1)^2}$

$\Rightarrow 3y = 3^{(x+1)^2}$

$\Rightarrow 3y = 3^{x^2+2x+1}$

$\Rightarrow y = \frac{3^{x^2+2x+1}}{3}$

$\Rightarrow y = 3^{x^2+2x}$

AS REQUIRED

Question 23 (*****)

$$4x\frac{d^2y}{dx^2}+4x\left(\frac{dy}{dx}\right)^2+2\frac{dy}{dx}=1.$$

By using the substitution $t=\sqrt{x}$, or otherwise, show that the general solution of the above differential equation is

$$y=A-\sqrt{x}+\ln\left[1+B\mathrm{e}^{2\sqrt{x}}\right],$$

where A and B are arbitrary constants.

, proof