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NUMERICAL METHODS

for

O.D.E.s

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1st order O.D.E.s

Question 1 ()**

The curve with equation $y = f(x)$, passes through the point $(9, 6)$ and satisfies

$$\frac{dy}{dx} = \frac{1}{1 + \sqrt{x}}, \quad x \geq 0.$$

Use Euler's method, with a step of 0.25, to find, correct to 4 decimal places, the value of y at $x = 9.5$.

$$\boxed{}, \quad y(9.5) \approx 6.1244$$

Handwritten solution for Question 1 using Euler's method:

Given: $\frac{dy}{dx} = \frac{1}{1 + \sqrt{x}}$ P(9,6)

Using Euler's method based on the derivative (Rough idea):

$$f(x) \approx \frac{f(x_1) - f(x_0)}{h} \quad (\text{Rough idea})$$

$$f(x_1) \approx h f(x_0) + f(x_0)$$

Write the above as a difference relation:

$$y_{n+1} \approx h f'_n + y_n$$

$$y_{n+1} \approx \frac{h}{1 + \sqrt{x_n}} + y_n$$

$$y_{n+1} \approx y_n + \frac{0.25}{1 + \sqrt{x_n}} \quad (h = 0.25)$$

Using the above formula twice:

- $y_1 \approx y_0 + \frac{0.25}{1 + \sqrt{9}} \quad (x_0 = 9, y_0 = 6)$
- $y_1 \approx 6 + \frac{0.25}{1 + 3}$
- $y_1 \approx 6.0625$
- $y_2 \approx y_1 + \frac{0.25}{1 + \sqrt{x_1}}$
- $y_2 \approx 6.0625 + \frac{0.25}{1 + \sqrt{9.25}}$
- $y_2 \approx 6.1244$

Answer: $y(9.5) \approx 6.1244$

Question 2 ()**

The curve with equation $y = f(x)$, passes through the point $(1,1)$ and satisfies the following differential equation.

$$\frac{dy}{dx} = \ln(x + y + 1), \quad x + y > -1.$$

Use the approximation

$$\left(\frac{dy}{dx}\right)_0 \approx \frac{y_1 - y_0}{h},$$

with $h = 0.1$, to find, correct to 3 decimal places, the value of y at $x = 1.2$.

$$\boxed{}, \quad \boxed{y(1.2) \approx 1.226}$$

$\frac{dy}{dx} = \ln(x+y+1) \quad x=1, y=1$
 WRITE THE O.D.E IN THE USUAL NOTATION
 $y'_x = \ln(x_0 + y_0 + 1)$
 USE EULER'S FORMULA
 $\left(\frac{dy}{dx}\right)_0 \approx \frac{y_1 - y_0}{h}$
 $y'_1 \approx \frac{y_1 - y_0}{h}$
 $\ln(x_0 + y_0 + 1) \approx \frac{y_1 - y_0}{h}$
 $y_1 \approx h \ln(x_0 + y_0 + 1) + y_0$ with $x_0 = 1$
 $y_0 = 1$
 $h = 0.1$
 APPLY THE RESULT TWICE
 • $y_1 \approx h \ln(x_0 + y_0 + 1) + y_0$
 $y_1 \approx 0.1 \ln(1+1+1) + 1$
 $y_1 \approx 1.109861229 \dots$
 • $y_2 \approx h \ln(x_1 + y_1 + 1) + y_1$
 $y_2 \approx 0.1 \ln(1+1.10986+1) + 1.10986 \dots$
 $y_2 \approx 1.226483999 \dots$
 $\therefore y \approx 1.226$

Question 3 (**)

The curve with equation $y = f(x)$, passes through the point $(1, 4)$ and satisfies

$$\frac{dy}{dx} = \frac{1}{2x + \sqrt{x}}, \quad x \geq 0.$$

Use Euler's method, with a step of 0.2, to find, correct to 4 decimal places, the value of y at $x = 1.6$.

$$\boxed{}, \quad \boxed{y(1.6) \approx 1.1741}$$

SCANS: Euler's step by step writing

$f'(x) = \frac{1}{2x + \sqrt{x}}$

IN FOR ONE STEP

$$y_{n+1} = y_n + h \cdot f'(x_n)$$

$$y_{n+1} = y_n + h \left(\frac{1}{2x_n + \sqrt{x_n}} \right)$$

STARTING WITH $x_1 = 1, y_1 = 4$ and $h = 0.2$

$$y_2 = y_1 + h \left(\frac{1}{2x_1 + \sqrt{x_1}} \right)$$

$$y_2 = 4 + \frac{0.2}{2(1) + \sqrt{1}} \approx 4.06667$$

SECOND STEP $x_2 = 1.2, y_2 \approx 4.06667, h = 0.2$

$$y_3 = y_2 + \frac{h}{2x_2 + \sqrt{x_2}}$$

$$y_3 = 4.06667 + \frac{0.2}{2(1.2) + \sqrt{1.2}} \approx 4.12385...$$

FINAL ITERATION WITH $x_3 = 1.4, y_3 \approx 4.12385, h = 0.2$

$$y_4 = y_3 + \frac{h}{2x_3 + \sqrt{x_3}}$$

$$y_4 = 4.12385 + \frac{0.2}{2(1.4) + \sqrt{1.4}} \approx 4.17409...$$

$\therefore y(1.6) \approx 4.1741$

Question 4 (**)

$$\frac{dy}{dx} = \sin(x^2 + y^2), \quad y(1) = 2.$$

Use, in the standard notation, the approximation

$$y'_n \approx \frac{y_{n+1} - y_n}{h},$$

with $h = 0.01$, to find, correct to 4 decimal places, the value of y at $x = 1.03$.

$$\boxed{}, \quad \boxed{y(1.03) \approx 1.9711}$$

$\frac{dy}{dx} = \sin(x^2 + y^2) \quad x=1 \quad y=2 \quad h=0.01$
 USING THE STANDARD APPROXIMATION $f(x) \approx f(x_n) + f'(x_n)(x - x_n)$
 $\Rightarrow y'_n \approx \frac{y_{n+1} - y_n}{h}$
 $\Rightarrow y_{n+1} \approx h y'_n + y_n$
 $\Rightarrow y_{n+1} \approx h \sin(x_n^2 + y_n^2) + y_n$
 APPLYING THE ABOVE WITH $h=0.01$
 $\Rightarrow y_1 \approx 0.01 \sin(x_0^2 + y_0^2) + y_0 \quad (x_0=1, y_0=2)$
 $\Rightarrow y_1 \approx 0.01 \sin 5 + 2$
 $\Rightarrow y_1 \approx 1.99041 \dots$
 $\Rightarrow y_2 \approx 0.01 \sin(x_1^2 + y_1^2) + y_1 \quad (x_1=1.01, y_1=1.99041 \dots)$
 $\Rightarrow y_2 \approx 0.01 \sin(1.01^2 + 1.99041^2) + 1.99041 \dots$
 $\Rightarrow y_2 \approx 1.96077 \dots$
 $\Rightarrow y_3 \approx 0.01 \sin(x_2^2 + y_2^2) + y_2 \quad (x_2=1.02, y_2=1.96077 \dots)$
 $\Rightarrow y_3 \approx 0.01 \sin(1.02^2 + 1.96077^2) + 1.96077 \dots$
 $\Rightarrow y_3 \approx 1.97109 \dots$
 \therefore THE VALUE OF y AT $x=1.03$ IS APPROXIMATELY 1.9711

Question 5 (**)

$$\frac{dy}{dx} = 4x^2 - y^2, \quad y(1) = 0.$$

Use the approximation

$$f'(x) \approx \frac{f(x+h) - f(x)}{h},$$

with $h = 0.05$, to find, correct to 4 decimal places, the value of y at $x = 1.2$.

No credit will be given for solving the differential equation analytically.

$$\boxed{}, \quad y(1.2) \approx 0.8080$$

Q5: Use Euler's Method (based on the inequality (Taylor series))

$f'(x) \approx \frac{f(x+h) - f(x)}{h}$ (for small h)

$f(x+h) \approx h f'(x) + f(x)$

Write the above result as a recurrence equation

$y_{n+1} \approx h y'_n + y_n$

$y_{n+1} \approx h(4x_n^2 - y_n^2) + y_n$

$y_{n+1} \approx 4h x_n^2 + y_n - 4h y_n^2$

$y_{n+1} \approx \frac{1}{5} x_n^2 + y_n (1 - \frac{4}{5} y_n)$ ($h = 0.05$)

$y_{n+1} \approx \frac{1}{5} [x_n^2 + y_n(5 - 4y_n)]$

Apply the above formula, starting with $x_0 = 1, y_0 = 0$

$y_1 \approx \frac{1}{5} [x_0^2 + y_0(5 - 4y_0)] = \frac{1}{5} [1^2 + 0] = 0.2$

$y_2 \approx \frac{1}{5} [x_1^2 + y_1(5 - 4y_1)] = \frac{1}{5} [(0.05)^2 + 0.2(4.8)] = 0.4125$

$y_3 \approx \frac{1}{5} [x_2^2 + y_2(5 - 4y_2)] = \frac{1}{5} [1.1^2 + 0.4125(4.9375)] = 0.62046875$

$y_4 \approx \frac{1}{5} [x_3^2 + y_3(5 - 4y_3)] = \frac{1}{5} [1.15^2 + 0.62046875(5 - 4(0.62046875))] \approx 0.80777...$

At $x = 1.2, y \approx 0.8080$

Question 6 (+)**

The curve with equation $y = f(x)$, passes through the point $(1,0)$ and satisfies the following differential equation.

$$\frac{dy}{dx} = x + \ln x, \quad x > 0.$$

Use the approximation

$$\left(\frac{dy}{dx}\right)_0 \approx \frac{y_1 - y_0}{h}, \quad h = 0.1,$$

to find the value of y at $x = 1.1$, and use this answer with the approximation

$$\left(\frac{dy}{dx}\right)_0 \approx \frac{y_1 - y_{-1}}{2h}, \quad h = 0.1,$$

to find, correct to 3 decimal places, the value of y at $x = 1.2$, $x = 1.3$ and $x = 1.4$.

, $y(1.1) = 0.1$, $y(1.2) \approx 0.2391$, $y(1.3) \approx 0.3765$, $y(1.4) \approx 0.5515$

The handwritten solution shows the following steps:

Given: $\frac{dy}{dx} = x + \ln x$, $x=1, y=0$

Using the Euler method: $\left(\frac{dy}{dx}\right)_0 = \frac{y_1 - y_0}{h}$ first

$y'_0 = \frac{y_1 - y_0}{h}$
 $y_1 = h y'_0 + y_0$

Here we have: $x_0 = 1, y_0 = 0, h = 0.1$

$y'_0 = 1[x_0 + \ln x_0] + y_0$
 $y'_0 = 0.1[1 + \ln 1] + 0$
 $y'_0 = 0$

Next we are using: $\left(\frac{dy}{dx}\right)_0 = \frac{y_1 - y_{-1}}{2h}$

Require: AS
 $y'_0 = \frac{y_1 - y_{-1}}{2h}$
 $y'_0 = \frac{y_{n+1} - y_{n-1}}{2h}$
 $2h y'_0 = y_{n+1} - y_{n-1}$
 $y_{n+1} = 2h y'_0 + y_{n-1}$
 $y_{n+2} = 2h y'_{n+1} + y_n$

With: $x_0 = 1, y_0 = 0$
 $x_1 = 1.1, y_1 = 0.1$
 $x_2 = 1.2$

Approximating the formula in succession:

- $y_2 \approx 2h[x_1 + \ln x_1] + y_1$
 $y_2 \approx 2 \times 0.1[1.1 + \ln(1.1)] + 0.1$
 $y_2 \approx 0.239062... \approx 0.2391$
- $y_3 \approx 2h[x_2 + \ln x_2] + y_2$
 $y_3 \approx 2(0.1)[1.2 + \ln(1.2)] + 0.1$
 $y_3 \approx 0.376464... \approx 0.3765$
- $y_4 \approx 2h[x_3 + \ln x_3] + y_3$
 $y_4 \approx 2(0.1)[1.3 + \ln(1.3)] + 0.376462...$
 $y_4 \approx 0.551434... \approx 0.5515$

Question 7 (+)**

The curve with equation $y = f(x)$ satisfies the differential equation

$$\frac{dy}{dx} = x + y + y^2, \quad y(0.9) = 3.75, \quad y(1) = 4$$

Using, in the standard notation, the approximation

$$\left(\frac{dy}{dx}\right)_r \approx \frac{y_{r+1} - y_{r-1}}{2h},$$

with $h = 0.1$, the value of y at $x = 0.8$ was estimated to k .

Determine the value of k .

$$\boxed{}, \quad k \approx 0.2575$$

Handwritten solution for Question 7:

Given: $\frac{dy}{dx} = x + y + y^2$, $x_0 = 0.8$, $y_0 = k$
 $x_1 = 0.9$, $y_1 = 3.75$
 $x_2 = 1$, $y_2 = 4$

Using the standard notation approximation:

$$\left(\frac{dy}{dx}\right)_r \approx \frac{y_{r+1} - y_{r-1}}{2h}$$

$$\Rightarrow \frac{dy}{dx} \approx \frac{y_2 - y_0}{2h}$$

$$\Rightarrow 2h \frac{dy}{dx} \approx y_2 - y_0$$

$$\Rightarrow 2h y_1 \approx y_2 - y_0$$

Let $r=2$:

$$\Rightarrow 2h y_2 \approx y_3 - y_1$$

$$\Rightarrow y_3 \approx y_1 + 2h(y_2 + y_1^2)$$

$$\Rightarrow k \approx 4 + 2 \times 0.1(0.9 + 3.75 + 3.75^2)$$

$$\Rightarrow k \approx 0.2575$$

Question 8 (**+)

$$\frac{dy}{dx} = x^2 - y^2, \quad y(3) = 2.$$

Use the approximation

$$\left(\frac{dy}{dx}\right)_r \approx \frac{y_{r+1} - y_r}{h}, \quad h = 0.1,$$

to find the value of y at $x = 2.1$, and use this answer with the approximation

$$\left(\frac{dy}{dx}\right)_{r+1} \approx \frac{y_{r+2} - y_r}{2h}, \quad h = 0.1,$$

to find, correct to 3 decimal places, the value of y at $x = 2.2$, $x = 1.3$ and $x = 1.4$.

$$\boxed{}, \quad \boxed{y(2.2) = 2.672}$$

Handwritten solution for Question 8:

Given: $\frac{dy}{dx} = x^2 - y^2$, $y = 2$ at $x = 3$

Using the result: $\left(\frac{dy}{dx}\right)_r \approx \frac{y_{r+1} - y_r}{h}$

$\Rightarrow y'_r \approx \frac{y_{r+1} - y_r}{h}$
 $\Rightarrow y_{r+1} \approx h y'_r + y_r$
 $\Rightarrow y_{r+1} \approx h(x_r^2 - y_r^2) + y_r$
 $\Rightarrow y_1 \approx 0.1(3^2 - 2^2) + 2$
 $\Rightarrow y_1 \approx 2.5$

Now using the result: $\left(\frac{dy}{dx}\right)_{r+1} \approx \frac{y_{r+2} - y_r}{2h}$

$\Rightarrow y'_{r+1} \approx \frac{y_{r+2} - y_r}{2h}$
 $\Rightarrow y_{r+2} \approx 2h y'_{r+1} + y_r$
 $\Rightarrow y_{r+2} \approx 2h(x_{r+1}^2 - y_{r+1}^2) + y_r$
 $\Rightarrow y_2 \approx 2h[2.5^2 - 2^2] + 2$
 $\Rightarrow y_2 \approx 0.2[3^2 - 2^2] + 2$
 $\Rightarrow y_2 \approx 2.672$

Question 9 (**+)

$$\frac{dy}{dx} = xy, \quad y(0) = 2.$$

Use the approximation

$$f'(x) \approx \frac{f(x+h) - f(x)}{h}, \quad h = 0.1$$

to find the value of y at $x = 0.1$, and use this answer with the approximation

$$f'(x) \approx \frac{f(x+h) - f(x-h)}{2h}, \quad h = 0.1,$$

to find the value of y at $x = 0.4$.

No credit will be given for solving the differential equation analytically.

$$\boxed{}, \quad \boxed{y(0.1) = 2}, \quad \boxed{y(0.4) \approx 2.1649}$$

$\frac{dy}{dx} = xy$ SUBJECT TO $x=0, y=2$

USE THE RESULT $f'(x) \approx \frac{f(x+h) - f(x)}{h}$

$$\Rightarrow y'_1 \approx \frac{y_{h1} - y_0}{h}$$

$$\Rightarrow y_{h1} \approx h y'_1 + y_0$$

$$\Rightarrow y_{h1} \approx h(x_0 y_0) + y_0$$

$$\Rightarrow y_{h1} \approx y_0(x_0 h + 1)$$

$$\Rightarrow y_1 \approx y_0[0.1 \times 2 + 1] = 2[0.1 \times 2 + 1]$$

$$\Rightarrow y_1 \approx 2$$

NEXT USE THE RESULT $f'(x) \approx \frac{f(x+h) - f(x-h)}{2h}$

$$\Rightarrow y'_{h1} \approx \frac{y_{h2} - y_{h0}}{2h}$$

$$\Rightarrow y_{h2} \approx 2h y'_{h1} + y_{h1}$$

$$\Rightarrow y_{h2} \approx 2h(x_{h1} y_{h1}) + y_{h1}$$

$$\Rightarrow y_{h2} \approx 2h x_{h1} y_{h1} + y_{h1}$$

$h=0.1, \quad x_0=0, \quad y_0=2$
 $x_1=0.1, \quad y_1=2$

THENCE WE OBTAIN

- $y_2 \approx 2h x_1 y_1 + y_1 = 2 \times 0.1 \times 0.1 \times 2 + 2 = 2.04 \quad \leftarrow \text{AT } x_2=0.2$
- $y_3 \approx 2h x_2 y_2 + y_2 = 2 \times 0.1 \times 0.2 \times 2.04 + 2 = 2.0816 \quad \leftarrow \text{AT } x_3=0.3$
- $y_4 \approx 2h x_3 y_3 + y_3 = 2 \times 0.1 \times 0.3 \times 2.0816 + 2.0816 = 2.164896$

Question 10 (+)**

The curve with equation $y = f(x)$ satisfies the differential equation

$$\frac{dy}{dx} = \frac{e^{x+y}}{3x+y+k}, \quad y(0) = 0,$$

where k is a positive constant.

Using, in the standard notation, the approximation

$$\left(\frac{dy}{dx}\right)_r \approx \frac{y_{r+1} - y_r}{h},$$

with $h = 0.1$, the value of y at $x = 0.1$ was estimated to 0.025.

Use the approximation formula given above to find, correct to 3 significant figures, the value of y at $x = 0.2$.

$$\boxed{}, \quad y(1.2) \approx 0.0512$$

Handwritten solution for Question 10:

Given: $\frac{dy}{dx} = \frac{e^{x+y}}{3x+y+k}$, $x=0, y=0$ and $x=0.1, y=0.025$

Using the approximation formula:

$$\left(\frac{dy}{dx}\right)_r \approx \frac{y_{r+1} - y_r}{h}$$

For $r=0$:

$$\left(\frac{dy}{dx}\right)_0 \approx \frac{y_1 - y_0}{h}$$

$$0.025 \approx \frac{y_1 - 0}{0.1}$$

$$y_1 = 0.1 \times 0.025 = 0.0025$$

For $r=1$:

$$\left(\frac{dy}{dx}\right)_1 \approx \frac{y_2 - y_1}{h}$$

$$0.025 \approx \frac{y_2 - 0.0025}{0.1}$$

$$y_2 = 0.1 \times 0.025 + 0.0025 = 0.005$$

For $r=2$:

$$\left(\frac{dy}{dx}\right)_2 \approx \frac{y_3 - y_2}{h}$$

$$0.025 \approx \frac{y_3 - 0.005}{0.1}$$

$$y_3 = 0.1 \times 0.025 + 0.005 = 0.0075$$

For $r=3$:

$$\left(\frac{dy}{dx}\right)_3 \approx \frac{y_4 - y_3}{h}$$

$$0.025 \approx \frac{y_4 - 0.0075}{0.1}$$

$$y_4 = 0.1 \times 0.025 + 0.0075 = 0.01$$

For $r=4$:

$$\left(\frac{dy}{dx}\right)_4 \approx \frac{y_5 - y_4}{h}$$

$$0.025 \approx \frac{y_5 - 0.01}{0.1}$$

$$y_5 = 0.1 \times 0.025 + 0.01 = 0.0125$$

For $r=5$:

$$\left(\frac{dy}{dx}\right)_5 \approx \frac{y_6 - y_5}{h}$$

$$0.025 \approx \frac{y_6 - 0.0125}{0.1}$$

$$y_6 = 0.1 \times 0.025 + 0.0125 = 0.015$$

For $r=6$:

$$\left(\frac{dy}{dx}\right)_6 \approx \frac{y_7 - y_6}{h}$$

$$0.025 \approx \frac{y_7 - 0.015}{0.1}$$

$$y_7 = 0.1 \times 0.025 + 0.015 = 0.0175$$

For $r=7$:

$$\left(\frac{dy}{dx}\right)_7 \approx \frac{y_8 - y_7}{h}$$

$$0.025 \approx \frac{y_8 - 0.0175}{0.1}$$

$$y_8 = 0.1 \times 0.025 + 0.0175 = 0.02$$

For $r=8$:

$$\left(\frac{dy}{dx}\right)_8 \approx \frac{y_9 - y_8}{h}$$

$$0.025 \approx \frac{y_9 - 0.02}{0.1}$$

$$y_9 = 0.1 \times 0.025 + 0.02 = 0.0225$$

For $r=9$:

$$\left(\frac{dy}{dx}\right)_9 \approx \frac{y_{10} - y_9}{h}$$

$$0.025 \approx \frac{y_{10} - 0.0225}{0.1}$$

$$y_{10} = 0.1 \times 0.025 + 0.0225 = 0.025$$

For $r=10$:

$$\left(\frac{dy}{dx}\right)_{10} \approx \frac{y_{11} - y_{10}}{h}$$

$$0.025 \approx \frac{y_{11} - 0.025}{0.1}$$

$$y_{11} = 0.1 \times 0.025 + 0.025 = 0.0275$$

For $r=11$:

$$\left(\frac{dy}{dx}\right)_{11} \approx \frac{y_{12} - y_{11}}{h}$$

$$0.025 \approx \frac{y_{12} - 0.0275}{0.1}$$

$$y_{12} = 0.1 \times 0.025 + 0.0275 = 0.03$$

For $r=12$:

$$\left(\frac{dy}{dx}\right)_{12} \approx \frac{y_{13} - y_{12}}{h}$$

$$0.025 \approx \frac{y_{13} - 0.03}{0.1}$$

$$y_{13} = 0.1 \times 0.025 + 0.03 = 0.0325$$

For $r=13$:

$$\left(\frac{dy}{dx}\right)_{13} \approx \frac{y_{14} - y_{13}}{h}$$

$$0.025 \approx \frac{y_{14} - 0.0325}{0.1}$$

$$y_{14} = 0.1 \times 0.025 + 0.0325 = 0.035$$

For $r=14$:

$$\left(\frac{dy}{dx}\right)_{14} \approx \frac{y_{15} - y_{14}}{h}$$

$$0.025 \approx \frac{y_{15} - 0.035}{0.1}$$

$$y_{15} = 0.1 \times 0.025 + 0.035 = 0.0375$$

For $r=15$:

$$\left(\frac{dy}{dx}\right)_{15} \approx \frac{y_{16} - y_{15}}{h}$$

$$0.025 \approx \frac{y_{16} - 0.0375}{0.1}$$

$$y_{16} = 0.1 \times 0.025 + 0.0375 = 0.04$$

For $r=16$:

$$\left(\frac{dy}{dx}\right)_{16} \approx \frac{y_{17} - y_{16}}{h}$$

$$0.025 \approx \frac{y_{17} - 0.04}{0.1}$$

$$y_{17} = 0.1 \times 0.025 + 0.04 = 0.0425$$

For $r=17$:

$$\left(\frac{dy}{dx}\right)_{17} \approx \frac{y_{18} - y_{17}}{h}$$

$$0.025 \approx \frac{y_{18} - 0.0425}{0.1}$$

$$y_{18} = 0.1 \times 0.025 + 0.0425 = 0.045$$

For $r=18$:

$$\left(\frac{dy}{dx}\right)_{18} \approx \frac{y_{19} - y_{18}}{h}$$

$$0.025 \approx \frac{y_{19} - 0.045}{0.1}$$

$$y_{19} = 0.1 \times 0.025 + 0.045 = 0.0475$$

For $r=19$:

$$\left(\frac{dy}{dx}\right)_{19} \approx \frac{y_{20} - y_{19}}{h}$$

$$0.025 \approx \frac{y_{20} - 0.0475}{0.1}$$

$$y_{20} = 0.1 \times 0.025 + 0.0475 = 0.05$$

For $r=20$:

$$\left(\frac{dy}{dx}\right)_{20} \approx \frac{y_{21} - y_{20}}{h}$$

$$0.025 \approx \frac{y_{21} - 0.05}{0.1}$$

$$y_{21} = 0.1 \times 0.025 + 0.05 = 0.0525$$

For $r=21$:

$$\left(\frac{dy}{dx}\right)_{21} \approx \frac{y_{22} - y_{21}}{h}$$

$$0.025 \approx \frac{y_{22} - 0.0525}{0.1}$$

$$y_{22} = 0.1 \times 0.025 + 0.0525 = 0.055$$

For $r=22$:

$$\left(\frac{dy}{dx}\right)_{22} \approx \frac{y_{23} - y_{22}}{h}$$

$$0.025 \approx \frac{y_{23} - 0.055}{0.1}$$

$$y_{23} = 0.1 \times 0.025 + 0.055 = 0.0575$$

For $r=23$:

$$\left(\frac{dy}{dx}\right)_{23} \approx \frac{y_{24} - y_{23}}{h}$$

$$0.025 \approx \frac{y_{24} - 0.0575}{0.1}$$

$$y_{24} = 0.1 \times 0.025 + 0.0575 = 0.06$$

For $r=24$:

$$\left(\frac{dy}{dx}\right)_{24} \approx \frac{y_{25} - y_{24}}{h}$$

$$0.025 \approx \frac{y_{25} - 0.06}{0.1}$$

$$y_{25} = 0.1 \times 0.025 + 0.06 = 0.0625$$

For $r=25$:

$$\left(\frac{dy}{dx}\right)_{25} \approx \frac{y_{26} - y_{25}}{h}$$

$$0.025 \approx \frac{y_{26} - 0.0625}{0.1}$$

$$y_{26} = 0.1 \times 0.025 + 0.0625 = 0.065$$

For $r=26$:

$$\left(\frac{dy}{dx}\right)_{26} \approx \frac{y_{27} - y_{26}}{h}$$

$$0.025 \approx \frac{y_{27} - 0.065}{0.1}$$

$$y_{27} = 0.1 \times 0.025 + 0.065 = 0.0675$$

For $r=27$:

$$\left(\frac{dy}{dx}\right)_{27} \approx \frac{y_{28} - y_{27}}{h}$$

$$0.025 \approx \frac{y_{28} - 0.0675}{0.1}$$

$$y_{28} = 0.1 \times 0.025 + 0.0675 = 0.07$$

For $r=28$:

$$\left(\frac{dy}{dx}\right)_{28} \approx \frac{y_{29} - y_{28}}{h}$$

$$0.025 \approx \frac{y_{29} - 0.07}{0.1}$$

$$y_{29} = 0.1 \times 0.025 + 0.07 = 0.0725$$

For $r=29$:

$$\left(\frac{dy}{dx}\right)_{29} \approx \frac{y_{30} - y_{29}}{h}$$

$$0.025 \approx \frac{y_{30} - 0.0725}{0.1}$$

$$y_{30} = 0.1 \times 0.025 + 0.0725 = 0.075$$

For $r=30$:

$$\left(\frac{dy}{dx}\right)_{30} \approx \frac{y_{31} - y_{30}}{h}$$

$$0.025 \approx \frac{y_{31} - 0.075}{0.1}$$

$$y_{31} = 0.1 \times 0.025 + 0.075 = 0.0775$$

For $r=31$:

$$\left(\frac{dy}{dx}\right)_{31} \approx \frac{y_{32} - y_{31}}{h}$$

$$0.025 \approx \frac{y_{32} - 0.0775}{0.1}$$

$$y_{32} = 0.1 \times 0.025 + 0.0775 = 0.08$$

For $r=32$:

$$\left(\frac{dy}{dx}\right)_{32} \approx \frac{y_{33} - y_{32}}{h}$$

$$0.025 \approx \frac{y_{33} - 0.08}{0.1}$$

$$y_{33} = 0.1 \times 0.025 + 0.08 = 0.0825$$

For $r=33$:

$$\left(\frac{dy}{dx}\right)_{33} \approx \frac{y_{34} - y_{33}}{h}$$

$$0.025 \approx \frac{y_{34} - 0.0825}{0.1}$$

$$y_{34} = 0.1 \times 0.025 + 0.0825 = 0.085$$

For $r=34$:

$$\left(\frac{dy}{dx}\right)_{34} \approx \frac{y_{35} - y_{34}}{h}$$

$$0.025 \approx \frac{y_{35} - 0.085}{0.1}$$

$$y_{35} = 0.1 \times 0.025 + 0.085 = 0.0875$$

For $r=35$:

$$\left(\frac{dy}{dx}\right)_{35} \approx \frac{y_{36} - y_{35}}{h}$$

$$0.025 \approx \frac{y_{36} - 0.0875}{0.1}$$

$$y_{36} = 0.1 \times 0.025 + 0.0875 = 0.09$$

For $r=36$:

$$\left(\frac{dy}{dx}\right)_{36} \approx \frac{y_{37} - y_{36}}{h}$$

$$0.025 \approx \frac{y_{37} - 0.09}{0.1}$$

$$y_{37} = 0.1 \times 0.025 + 0.09 = 0.0925$$

For $r=37$:

$$\left(\frac{dy}{dx}\right)_{37} \approx \frac{y_{38} - y_{37}}{h}$$

$$0.025 \approx \frac{y_{38} - 0.0925}{0.1}$$

$$y_{38} = 0.1 \times 0.025 + 0.0925 = 0.095$$

For $r=38$:

$$\left(\frac{dy}{dx}\right)_{38} \approx \frac{y_{39} - y_{38}}{h}$$

$$0.025 \approx \frac{y_{39} - 0.095}{0.1}$$

$$y_{39} = 0.1 \times 0.025 + 0.095 = 0.0975$$

For $r=39$:

$$\left(\frac{dy}{dx}\right)_{39} \approx \frac{y_{40} - y_{39}}{h}$$

$$0.025 \approx \frac{y_{40} - 0.0975}{0.1}$$

$$y_{40} = 0.1 \times 0.025 + 0.0975 = 0.1$$

For $r=40$:

$$\left(\frac{dy}{dx}\right)_{40} \approx \frac{y_{41} - y_{40}}{h}$$

$$0.025 \approx \frac{y_{41} - 0.1}{0.1}$$

$$y_{41} = 0.1 \times 0.025 + 0.1 = 0.1025$$

For $r=41$:

$$\left(\frac{dy}{dx}\right)_{41} \approx \frac{y_{42} - y_{41}}{h}$$

$$0.025 \approx \frac{y_{42} - 0.1025}{0.1}$$

$$y_{42} = 0.1 \times 0.025 + 0.1025 = 0.105$$

For $r=42$:

$$\left(\frac{dy}{dx}\right)_{42} \approx \frac{y_{43} - y_{42}}{h}$$

$$0.025 \approx \frac{y_{43} - 0.105}{0.1}$$

$$y_{43} = 0.1 \times 0.025 + 0.105 = 0.1075$$

For $r=43$:

$$\left(\frac{dy}{dx}\right)_{43} \approx \frac{y_{44} - y_{43}}{h}$$

$$0.025 \approx \frac{y_{44} - 0.1075}{0.1}$$

$$y_{44} = 0.1 \times 0.025 + 0.1075 = 0.11$$

For $r=44$:

$$\left(\frac{dy}{dx}\right)_{44} \approx \frac{y_{45} - y_{44}}{h}$$

$$0.025 \approx \frac{y_{45} - 0.11}{0.1}$$

$$y_{45} = 0.1 \times 0.025 + 0.11 = 0.1125$$

For $r=45$:

$$\left(\frac{dy}{dx}\right)_{45} \approx \frac{y_{46} - y_{45}}{h}$$

$$0.025 \approx \frac{y_{46} - 0.1125}{0.1}$$

$$y_{46} = 0.1 \times 0.025 + 0.1125 = 0.115$$

For $r=46$:

$$\left(\frac{dy}{dx}\right)_{46} \approx \frac{y_{47} - y_{46}}{h}$$

$$0.025 \approx \frac{y_{47} - 0.115}{0.1}$$

$$y_{47} = 0.1 \times 0.025 + 0.115 = 0.1175$$

For $r=47$:

$$\left(\frac{dy}{dx}\right)_{47} \approx \frac{y_{48} - y_{47}}{h}$$

$$0.025 \approx \frac{y_{48} - 0.1175}{0.1}$$

$$y_{48} = 0.1 \times 0.025 + 0.1175 = 0.12$$

For $r=48$:

$$\left(\frac{dy}{dx}\right)_{48} \approx \frac{y_{49} - y_{48}}{h}$$

$$0.025 \approx \frac{y_{49} - 0.12}{0.1}$$

$$y_{49} = 0.1 \times 0.025 + 0.12 = 0.1225$$

For $r=49$:

$$\left(\frac{dy}{dx}\right)_{49} \approx \frac{y_{50} - y_{49}}{h}$$

$$0.025 \approx \frac{y_{50} - 0.1225}{0.1}$$

$$y_{50} = 0.1 \times 0.025 + 0.1225 = 0.125$$

For $r=50$:

$$\left(\frac{dy}{dx}\right)_{50} \approx \frac{y_{51} - y_{50}}{h}$$

$$0.025 \approx \frac{y_{51} - 0.125}{0.1}$$

$$y_{51} = 0.1 \times 0.025 + 0.125 = 0.1275$$

For $r=51$:

$$\left(\frac{dy}{dx}\right)_{51} \approx \frac{y_{52} - y_{51}}{h}$$

$$0.025 \approx \frac{y_{52} - 0.1275}{0.1}$$

$$y_{52} = 0.1 \times 0.025 + 0.1275 = 0.13$$

For $r=52$:

$$\left(\frac{dy}{dx}\right)_{52} \approx \frac{y_{53} - y_{52}}{h}$$

$$0.025 \approx \frac{y_{53} - 0.13}{0.1}$$

$$y_{53} = 0.1 \times 0.025 + 0.13 = 0.1325$$

For $r=53$:

$$\left(\frac{dy}{dx}\right)_{53} \approx \frac{y_{54} - y_{53}}{h}$$

$$0.025 \approx \frac{y_{54} - 0.1325}{0.1}$$

$$y_{54} = 0.1 \times 0.025 + 0.1325 = 0.135$$

For $r=54$:

$$\left(\frac{dy}{dx}\right)_{54} \approx \frac{y_{55} - y_{54}}{h}$$

$$0.025 \approx \frac{y_{55} - 0.135}{0.1}$$

$$y_{55} = 0.1 \times 0.025 + 0.135 = 0.1375$$

For $r=55$:

$$\left(\frac{dy}{dx}\right)_{55} \approx \frac{y_{56} - y_{55}}{h}$$

$$0.025 \approx \frac{y_{56} - 0.1375}{0.1}$$

$$y_{56} = 0.1 \times 0.025 + 0.1375 = 0.14$$

For $r=56$:

$$\left(\frac{dy}{dx}\right)_{56} \approx \frac{y_{57} - y_{56}}{h}$$

$$0.025 \approx \frac{y_{57} - 0.14}{0.1}$$

$$y_{57} = 0.1 \times 0.025 + 0.14 = 0.1425$$

For $r=57$:

$$\left(\frac{dy}{dx}\right)_{57} \approx \frac{y_{58} - y_{57}}{h}$$

$$0.025 \approx \frac{y_{58} - 0.1425}{0.1}$$

$$y_{58} = 0.1 \times 0.025 + 0.1425 = 0.145$$

For $r=58$:

$$\left(\frac{dy}{dx}\right)_{58} \approx \frac{y_{59} - y_{58}}{h}$$

$$0.025 \approx \frac{y_{59} - 0.145}{0.1}$$

$$y_{59} = 0.1 \times 0.025 + 0.145 = 0.1475$$

For $r=59$:

$$\left(\frac{dy}{dx}\right)_{59} \approx \frac{y_{60} - y_{59}}{h}$$

$$0.025 \approx \frac{y_{60} - 0.1475}{0.1}$$

$$y_{60} = 0.1 \times 0.025 + 0.1475 = 0.15$$

For $r=60$:

$$\left(\frac{dy}{dx}\right)_{60} \approx \frac{y_{61} - y_{60}}{h}$$

$$0.025 \approx \frac{y_{61} - 0.15}{0.1}$$

$$y_{61} = 0.1 \times 0.025 + 0.15 = 0.1525$$

For $r=61$:

$$\left(\frac{dy}{dx}\right)_{61} \approx \frac{y_{62} - y_{61}}{h}$$

$$0.025 \approx \frac{y_{62} - 0.1525}{0.1}$$

$$y_{62} = 0.1 \times 0.025 + 0.1525 = 0.155$$

For $r=62$:

$$\left(\frac{dy}{dx}\right)_{62} \approx \frac{y_{63} - y_{62}}{h}$$

$$0.025 \approx \frac{y_{63} - 0.155}{0.1}$$

$$y_{63} = 0.1 \times 0.025 + 0.155 = 0.1575$$

For $r=63$:

$$\left(\frac{dy}{dx}\right)_{63} \approx \frac{y_{64} - y_{63}}{h}$$

$$0.025 \approx \frac{y_{64} - 0.1575}{0.1}$$

$$y_{64} = 0.1 \times 0.025 + 0.1575 = 0.16$$

For $r=64$:

$$\left(\frac{dy}{dx}\right)_{64} \approx \frac{y_{65} - y_{64}}{h}$$

$$0.025 \approx \frac{y_{65} - 0.16}{0.1}$$

$$y_{65} = 0.1 \times 0.025 + 0.16 = 0.1625$$

For $r=65$:

$$\left(\frac{dy}{dx}\right)_{65} \approx \frac{y_{66} - y_{65}}{h}$$

$$0.025 \approx \frac{y_{66} - 0.1625}{0.1}$$

$$y_{66} = 0.1 \times 0.025 + 0.1625 = 0.165$$

For $r=66$:

$$\left(\frac{dy}{dx}\right)_{66} \approx \frac{y_{67} - y_{66}}{h}$$

$$0.025 \approx \frac{y_{67} - 0.165}{0.1}$$

$$y_{67} = 0.1 \times 0.025 + 0.165 = 0.1675$$

For $r=67$:

$$\left(\frac{dy}{dx}\right)_{67} \approx \frac{y_{68} - y_{67}}{h}$$

$$0.025 \approx \frac{y_{68} - 0.1675}{0.1}$$

$$y_{68} = 0.1 \times 0.025 + 0.1675 = 0.17$$

For $r=68$:

$$\left(\frac{dy}{dx}\right)_{68} \approx \frac{y_{69} - y_{68}}{h}$$

$$0.025 \approx \frac{y_{69} - 0.17}{0.1}$$

$$y_{69} = 0.1 \times 0.025 + 0.17 = 0.1725$$

For $r=69$:

$$\left(\frac{dy}{dx}\right)_{69} \approx \frac{y_{70} - y_{69}}{h}$$

$$0.025 \approx \frac{y_{70} - 0.1725}{0.1}$$

$$y_{70} = 0.1 \times 0.025 + 0.1725 = 0.175$$

For $r=70$:

$$\left(\frac{dy}{dx}\right)_{70} \approx \frac{y_{71} - y_{70}}{h}$$

$$0.025 \approx \frac{y_{71} - 0.175}{0.1}$$

$$y_{71} = 0.1 \times 0.025 + 0.175 = 0.1775$$

For $r=71$:

$$\left(\frac{dy}{dx}\right)_{71} \approx \frac{y_{72} - y_{71}}{h}$$

$$0.025 \approx \frac{y_{72} - 0.1775}{0.1}$$

$$y_{72} = 0.1 \times 0.025 + 0.1775 = 0.18$$

For $r=72$:

$$\left(\frac{dy}{dx}\right)_{72} \approx \frac{y_{73} - y_{72}}{h}$$

$$0.025 \approx \frac{y_{73} - 0.18}{0.1}$$

$$y_{73} = 0.1 \times 0.025 + 0.18 = 0.1825$$

For $r=73$:

$$\left(\frac{dy}{dx}\right)_{73} \approx \frac{y_{74} - y_{73}}{h}$$

$$0.025 \approx \frac{y_{74} - 0.1825}{0.1}$$

$$y_{74} = 0.1 \times 0.025 + 0.1825 = 0.185$$

For $r=74$:

$$\left(\frac{dy}{dx}\right)_{74} \approx \frac{y_{75} - y_{74}}{h}$$

$$0.025 \approx \frac{y_{75} - 0.185}{0.1}$$

$$y_{75} = 0.1 \times 0.025 + 0.185 = 0.1875$$

For $r=75$:

$$\left(\frac{dy}{dx}\right)_{75} \approx \frac{y_{76} - y_{75}}{h}$$

$$0.025 \approx \frac{y_{76} - 0.1875}{0.1}$$

$$y_{76} = 0.1 \times 0.025 + 0.1875 = 0.19$$

For $r=76$:

$$\left(\frac{dy}{dx}\right)_{76} \approx \frac{y_{77} - y_{76}}{h}$$

$$0.025 \approx \frac{y_{77} - 0.19}{0.1}$$

$$y_{77} = 0.1 \times 0.025 + 0.19 = 0.1925$$

For $r=77$:

$$\left(\frac{dy}{dx}\right)_{77} \approx \frac{y_{78} - y_{77}}{h}$$

$$0.025 \approx \frac{y_{78} - 0.1925}{0.1}$$

$$y_{78} = 0.1 \times 0.025 + 0.1925 = 0.195$$

For $r=78$:

$$\left(\frac{dy}{dx}\right)_{78} \approx \frac{y_{79} - y_{78}}{h}$$

$$0.025 \approx \frac{y_{79} - 0.195}{0.1}$$

$$y_{79} = 0.1 \times 0.025 + 0.195 = 0.1975$$

For $r=79$:

$$\left(\frac{dy}{dx}\right)_{79} \approx \frac{y_{80} - y_{79}}{h}$$

$$0.025 \approx \frac{y_{80} - 0.1975}{0.1}$$

$$y_{80} = 0.1 \times 0.025 + 0.1975 = 0.2$$

For $r=80$:

$$\left(\frac{dy}{dx}\right)_{80} \approx \frac{y_{81} - y_{80}}{h}$$

$$0.025 \approx \frac{y_{81} - 0.2}{0.1}$$

$$y_{81} = 0.1 \times 0.025 + 0.2 = 0.2025$$

For $r=81$:

$$\left(\frac{dy}{dx}\right)_{81} \approx \frac{y_{82} - y_{81}}{h}$$

$$0.025 \approx \frac{y_{82} - 0.2025}{0.1}$$

$$y_{82} = 0.1 \times 0.025 + 0.2025 = 0.205$$

For $r=82$:

$$\left(\frac{dy}{dx}\right)_{82} \approx \frac{y_{83} - y_{82}}{h}$$

$$0.025 \approx \frac{y_{83} - 0.205}{0.1}$$

$$y_{83} = 0.1 \times 0.025 + 0.205 = 0.2075$$

For $r=83$:

$$\left(\frac{dy}{dx}\right)_{83} \approx \frac{y_{84} - y_{83}}{h}$$

$$0.025 \approx \frac{y_{84} - 0.2075}{0.1}$$

$$y_{84} = 0.1 \times 0.025 + 0.2075 = 0.21$$

For $r=84$:

$$\left(\frac{dy}{dx}\right)_{84} \approx \frac{y_{85} - y_{84}}{h}$$

$$0.025 \approx \frac{y_{85} - 0.21}{0.1}$$

$$y_{85} = 0.1 \times 0.025 + 0.21 = 0.2125$$

For $r=85$:

$$\left(\frac{dy}{dx}\right)_{85} \approx \frac{y_{86} - y_{85}}{h}$$

$$0.025 \approx \frac{y_{86} - 0.2125}{0.1}$$

$$y_{86} = 0.1 \times 0.025 + 0.2125 = 0.215$$

For $r=86$:

$$\left(\frac{dy}{dx}\right)_{86} \approx \frac{y_{87} - y_{86}}{h}$$
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Question 11 (**+)

$$\frac{dy}{dx} = \frac{4x^2 + y^2}{x + y}, \quad y(1) = 4.$$

Use the approximation

$$f'(x) \approx \frac{f(x+h) - f(x)}{h},$$

with h to be found, given further that $f(1+h) \approx 4.8$.

$$\boxed{}, \quad \boxed{h = 0.2}$$

Handwritten solution for Question 11:

$$\frac{dy}{dx} = \frac{4x^2 + y^2}{x + y} \quad y(1) = 4 \quad y(1+h) \approx 4.8$$

$$\Rightarrow f'(x) \approx \frac{f(x+h) - f(x)}{h}, \text{ if } h \text{ is small}$$

$$\Rightarrow h f'(x) \approx f(x+h) - f(x)$$

$$\Rightarrow h \left[\frac{4x^2 + y^2}{x + y} \right] \approx f(x+h) - f(x)$$

Use $x=1, y=4, f(1+h) \approx 4.8$

$$\Rightarrow h \left[\frac{4(1)^2 + 4^2}{1 + 4} \right] \approx 4.8 - 4$$

$$\Rightarrow h \times 4 \approx 0.8$$

$$\Rightarrow h \approx 0.2$$

Question 12 (***)

$$\frac{dy}{dx} = e^x - y^2, \quad y(0) = 0.$$

- a) Use, in the standard notation, the approximation

$$y_{n+1} \approx h y'_n + y_n,$$

with $h = 0.1$, to find the approximate value of y at $x = 0.1$.

- b) Use the answer of part (a) and the approximation

$$y'_n \approx \frac{y_{n+1} - y_{n-1}}{2h},$$

with $h = 0.1$, to find, correct to 4 decimal places, the approximate value of y at $x = 0.3$.

- c) By differentiating the differential equation given, determine the first four **non zero** terms in the infinite series expansion of y in ascending powers of x , and use it to find, correct to 4 decimal places, another approximation for the value of y at $x = 0.3$.

, $y(0.1) \approx 0.1$, $y(0.3) \approx 0.3347$, $y(0.3) \approx 0.3388$

$\frac{dy}{dx} = e^x - y^2$ $x=0, y=0, h=0.1$

a) USING THE RESULT $y_{n+1} = h y'_n + y_n$

$\Rightarrow y_1 = h y'_0 + y_0$ ($x_0=0, y_0=0$)

$\Rightarrow y_1 = 0.1 (e^0 - 0^2) + 0$

$\Rightarrow y_1 = 0.1 (e^0 - 0^2) + 0$

$\Rightarrow y_1 = 0.1$

ie $y \approx 0.1$ at $x=0.1$

b) NEXT USING THE RESULT $y'_n = \frac{y_{n+1} - y_{n-1}}{2h}$

$\Rightarrow y_{n+1} = 2h y'_n + y_{n-1}$

$\Rightarrow y_2 = 2h y'_1 + y_0$

$\Rightarrow y_2 = 2 \times 0.1 \times (e^{0.1} - y_1^2) + 0$ ($x_1=0.1, y_1=0.1$)

$\Rightarrow y_2 = 0.2 (e^{0.1} - 0.1^2) + 0$

$\Rightarrow y_2 = 0.219036...$

$\Rightarrow y_3 = 2h y'_2 + y_1$ ($x=0.2, y_2=0.219036...$)

$\Rightarrow y_3 = 2 \times 0.1 \times (e^{0.2} - y_2^2) + y_1$

$\Rightarrow y_3 = 0.2 \times (e^{0.2} - 0.219036^2) + 0.1$

$\Rightarrow y_3 = 0.336853...$

\therefore THE APPROXIMATE VALUE OF y AT $x=0.3$ IS 0.3347

b) FINDING THE FIRST 4 DERIVATIVES

$y' = e^x - y^2$	$x_0=0, y_0=0$
$y'' = e^x - 2y y'$	$y'_0 = e^0 - y_0^2$ $y'_0 = e^0 - 0$ $y'_0 = 1$
$y''' = e^x - 2y y'' - 2y y'^2$	$y''_0 = e^0 - 2y_0 y'_0$ $y''_0 = e^0 - 0$ $y''_0 = 1$
$y^{(4)} = e^x - 2y y''' - 2y y''^2 - 2y'^2 y''$	$y'''_0 = e^0 - 2y_0 y''_0 - 2y_0 y'^2_0$ $y'''_0 = e^0 - 0 - 0$ $y'''_0 = 1$
$y^{(5)} = e^x - 2y y^{(4)} - 2y y'''^2 - 2y y'' y''' - 2y'^2 y''$	$y^{(4)}_0 = e^0 - 2y_0 y'''_0 - 2y_0 y''^2_0 - 2y'^2_0 y''_0$ $y^{(4)}_0 = e^0 - 0 - 0 - 0$ $y^{(4)}_0 = 1$

$\Rightarrow y = y_0 + x y'_0 + \frac{x^2}{2!} y''_0 + \frac{x^3}{3!} y'''_0 + \frac{x^4}{4!} y^{(4)}_0 + O(x^5)$

$\Rightarrow y = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + O(x^5)$

$\Rightarrow y(0.3) \approx 0.3 + \frac{1}{2}(0.3)^2 + \frac{1}{6}(0.3)^3 + \frac{1}{24}(0.3)^4 \approx 0.3368$

Question 13 (***)

$$\frac{dy}{dx} = \frac{3x^2 - y^2}{2xy}, \quad y(k) = 2, \quad k > 0.$$

a) Use, in the standard notation, the approximation

$$y'_n \approx \frac{y_{n+1} - y_n}{h}, \quad h = 0.1,$$

to find the value of k , given further that $y(k+h) \approx 2.275$.

b) Use the answer of part (a) and the approximation

$$y'_n \approx \frac{y_{n+1} - y_{n-1}}{2h}, \quad h = 0.1,$$

with, to find, correct to 3 decimal places, the approximate value of $y(k+2h)$.

$$\boxed{}, \quad \boxed{k=4}, \quad \boxed{y(k+2h) \approx 2.485}$$

Left Page:

Using $y'_n \approx \frac{y_{n+1} - y_n}{h}$

$$\Rightarrow y_{n+1} \approx h y'_n + y_n$$

$$\Rightarrow y_{n+1} \approx h \left[\frac{3x_n^2 - y_n^2}{2x_n y_n} \right] + y_n$$

$$\Rightarrow y_1 \approx h \left[\frac{3x_0^2 - y_0^2}{2x_0 y_0} \right] + y_0$$

$$\Rightarrow 2.275 \approx 0.1 \left[\frac{3x_0^2 - 2^2}{2x_0 \cdot 2} \right] + 2$$

$$\Rightarrow 2.275 \approx 0.1 \left(\frac{3x_0^2 - 4}{4x_0} \right) + 2$$

$$\Rightarrow 0.275 \approx 0.1 \left(\frac{3x_0^2 - 4}{4x_0} \right)$$

$$\Rightarrow 2.75 \approx \frac{3x_0^2 - 4}{4x_0}$$

$$\Rightarrow 11x_0 \approx 3x_0^2 - 4$$

$$\Rightarrow 0 \approx 3x_0^2 - 11x_0 - 4$$

$$\Rightarrow 0 \approx (3x_0 + 1)(x_0 - 4)$$

$$\Rightarrow x_0 = 4$$

Right Page:

Now using $y'_n \approx \frac{y_{n+1} - y_{n-1}}{2h}$

$$\Rightarrow y_{n+1} \approx 2h y'_n + y_{n-1}$$

$$\Rightarrow y_{n+1} \approx 2h \left[\frac{3x_n^2 - y_n^2}{2x_n y_n} \right] + y_{n-1}$$

$$\Rightarrow y_2 \approx 2h \left[\frac{3x_1^2 - y_1^2}{2x_1 y_1} \right] + y_0$$

$$\Rightarrow y_2 \approx 2 \times 0.1 \times \left[\frac{3 \times 4^2 - 2.275^2}{2 \times 4 \times 2.275} \right] + 2$$

$$\Rightarrow y_2 \approx 2.485$$

Side note: $x_0 = 4, y_0 = 2$
 $x_1 = 4.1, y_1 = 2.275$
 $x_2 = 4.2, y_2 = ?$

Question 14 (*)**

The curve with equation $y = f(x)$, passes through the point $(0,1)$ and satisfies the following differential equation.

$$\frac{dy}{dx} = 3x^2y + x^5.$$

a) Use the approximation

$$\left(\frac{dy}{dx}\right)_n \approx \frac{y_{n+1} - y_n}{h},$$

with $h = 0.1$, to find, correct to 6 decimal places, the value of y at $x = 0.2$.

b) Find the solution of the differential equation, and use it to obtain the value of y at $x = 0.2$.

$$\boxed{}, \quad \boxed{y(0.2) \approx 1.003001}, \quad \boxed{y(0.2) \approx 1.008}$$

$\frac{dy}{dx} = 3x^2y + x^5 \quad x=0, y=1$

a) WRITE INFORMATION IN THE VISUAL NOTATION

$\Rightarrow y_{n+1} \approx h \cdot \left(\frac{dy}{dx}\right)_n + y_n \quad x_n=0, y_n=1, h=0.1$

$\Rightarrow y_{n+1} \approx h(3x_n^2y_n + x_n^5) + y_n$

USING THE ABOVE FORMULA TWICE

$\Rightarrow y_1 \approx h[3x_0^2y_0 + x_0^5] + y_0$

$y_1 \approx 0.1[3 \times 0^2 \times 1 + 0^5] + 1$

$y_1 \approx 1$

$\Rightarrow y_2 \approx h[3x_1^2y_1 + x_1^5] + y_1$

$y_2 \approx 0.1[3 \times 0.1^2 \times 1 + 0.1^5] + 1$

$y_2 \approx 1.003001$

b) WRITE THE O.D.E IN THE VISUAL FORM

$\Rightarrow \frac{dy}{dx} - 3x^2y = x^5$

INTEGRATING FACTOR $= e^{\int -3x^2 dx} = e^{-x^3}$

$\Rightarrow \frac{d}{dx}(ye^{-x^3}) = x^5e^{-x^3}$

$\Rightarrow ye^{-x^3} = \int x^5e^{-x^3} dx$

$\Rightarrow ye^{-x^3} = \int x^5e^{-x^3} dx$

INTEGRATION BY PARTS

$\Rightarrow ye^{-x^3} = -\frac{1}{3}x^3e^{-x^3} - \int -x^3e^{-x^3} dx$

$\Rightarrow ye^{-x^3} = -\frac{1}{3}x^3e^{-x^3} + \int x^3e^{-x^3} dx$

$\Rightarrow ye^{-x^3} = -\frac{1}{3}x^3e^{-x^3} - \frac{1}{2}e^{-x^3} + A$

$\Rightarrow y = A e^{x^3} - \frac{1}{2}x^3 - \frac{1}{3}$

APPLY CONDITIONS $x=0, y=1$

$\Rightarrow 1 = A - \frac{1}{3}$

$\Rightarrow A = \frac{4}{3}$

$\Rightarrow y = \frac{4}{3}[4e^{x^3} - x^3 - 1]$

FINALLY APPLY THE SOLUTION AT $x=0.2$

$y = \frac{4}{3}[4e^{0.2^3} - 0.2^3 - 1] = 1.008042781 \dots$

≈ 1.008

2nd order O.D.E.s

Question 1 (**+)

The differential equation

$$\frac{d^2y}{dx^2} = \frac{x}{y^2} + \frac{1}{y}, \quad y \neq 0,$$

is to be solved numerically subject to the conditions $y(0.5) = 1$ and $y(0.6) = 1.3$.

Use the approximation

$$y_n' \approx \frac{y_{n+1} - 2y_n + y_{n-1}}{h^2}, \quad h = 0.1,$$

to find, correct to 4 decimal places the value of y at $x = 0.8$.

$$\boxed{}, \quad y(0.8) \approx 1.9330$$

$\frac{d^2y}{dx^2} = \frac{x}{y^2} + \frac{1}{y}$ $x=0.5 \quad y=1$
 $\phantom{\frac{d^2y}{dx^2} = \frac{x}{y^2} + \frac{1}{y}}$ $x=0.6 \quad y=1.3$

USING THE FORMULA $y_n'' \approx \frac{y_{n+1} - 2y_n + y_{n-1}}{h^2}$

$\Rightarrow y_{n+1} \approx y_n'' h^2 + 2y_n - y_{n-1}$
 $\Rightarrow y_{n+1} \approx (0.1)^2 \left[\frac{x_n}{y_n^2} + \frac{1}{y_n} \right] + 2y_n - y_{n-1}$

USING THE ABOVE WITH $x_0=0.5, y_0=1$ & $x_1=0.6, y_1=1.3$

$\Rightarrow y_2 \approx 0.01 \left[\frac{0.5}{1^2} + \frac{1}{1} \right] + 2(1.3) - 1$
 $\Rightarrow y_2 \approx 0.01 \left[\frac{0.6}{1.3^2} + \frac{1}{1.3} \right] + 2(1.3) - 1$
 $\Rightarrow y_2 \approx 1.61242004 \dots$ (at $x=0.7$)

APPLY THE RECURSION ONCE MORE

$\Rightarrow y_3 \approx 0.01 \left[\frac{0.7}{y_2^2} + \frac{1}{y_2} \right] + 2y_2 - y_1$
 $\Rightarrow y_3 \approx 0.01 \left[\frac{0.7}{1.61242^2} + \frac{1}{1.61242} \right] + 2(1.61242) - 1.3$
 $\Rightarrow y_3 \approx 1.93303407 \dots$ (at $x=0.8$)

\therefore THE APPROXIMATE VALUE OF y AT $x=0.8$ IS 1.9330

Question 2 (***)

The differential equation

$$\frac{d^2 y}{dx^2} + \frac{dy}{dx} + y^3 = 0, \quad y \neq 0,$$

is to be solved numerically subject to the conditions $y(2) = 3$ and $y(2.1) = 4$.

Use the following approximations

$$\left(\frac{d^2 y}{dx^2} \right)_{n+1} \approx \frac{y_{n+2} - 2y_{n+1} + y_n}{h^2}, \quad \left(\frac{dy}{dx} \right)_{n+1} \approx \frac{y_{n+2} - y_n}{2h}, \quad h = 0.1,$$

to find, correct to 2 decimal places the value of y at $x = 2.2$.

$$\boxed{}, \quad y(2.2) \approx 4.30$$

$\frac{d^2 y}{dx^2} + \frac{dy}{dx} + y^3 = 0, \quad y(2) = 3, \quad y(2.1) = 4$
 USING THE RUNGE-KUTTA

$$\left. \begin{aligned} \left(\frac{dy}{dx} \right)_{n+1} &\approx \frac{y_{n+2} - y_n}{2h} \\ \left(\frac{d^2 y}{dx^2} \right)_{n+1} &\approx \frac{y_{n+2} - 2y_{n+1} + y_n}{h^2} \end{aligned} \right\}$$

 SUB INTO THE O.D.E

$$\frac{y_{n+2} - 2y_{n+1} + y_n}{h^2} + \frac{y_{n+2} - y_n}{2h} + y_{n+1}^3 = 0$$

 HERE $x_0 = 2, y_0 = 3, \quad x_1 = 2.1, y_1 = 4, \quad h = 0.1$

$$\Rightarrow \frac{y_2 - 2y_1 + y_0}{h^2} + \frac{y_2 - y_0}{2h} + y_1^3 = 0$$

$$\Rightarrow \frac{y_2 - 8 + 3}{0.01} + \frac{y_2 - 3}{0.2} + 64 = 0$$

$$\Rightarrow (y_2 - 5) \times 100 + (y_2 - 3) \times 5 + 64 = 0$$

$$\Rightarrow 105y_2 = 451$$

$$\Rightarrow y_2 \approx 4.295238 \dots$$

 ∴ THE APPROXIMATE VALUE OF y AT $x = 2.2$ IS 4.30

Question 3 (***)

The curve with equation $y = f(x)$, satisfies

$$\frac{d^2y}{dx^2} = 1 + x \sin y,$$

subject to the boundary conditions $y = 1$, $\frac{dy}{dx} = 2$, at $x = 1$.

Use the approximations

$$\left(\frac{d^2y}{dx^2}\right)_{r+1} \approx \frac{y_{r+2} - 2y_{r+1} + y_r}{h^2} \quad \text{and} \quad \left(\frac{dy}{dx}\right)_{r+1} \approx \frac{y_{r+2} - y_r}{2h},$$

to determine, correct to 4 decimal places, the value of y at $x = 1.1$.

Use $h = 0.05$ throughout this question.

$$\boxed{}, \quad \boxed{y(1.6) \approx 2.85}$$

$\frac{d^2y}{dx^2} = 1 + x \sin y$ subject to: $x=1, y=1, \frac{dy}{dx}=2$

SET BY USING THE FORMULAS
 $\left(\frac{dy}{dx}\right)_{r+1} \approx \frac{y_{r+2} - y_r}{2h}$ $\left(\frac{d^2y}{dx^2}\right)_{r+1} \approx \frac{y_{r+2} - 2y_{r+1} + y_r}{h^2}$
 $h y'_{r+1} \approx y_{r+2} - y_r$ $h^2 y''_{r+1} \approx y_{r+2} - 2y_{r+1} + y_r$

Eliminate y_{r+2} between the equations
 $\Rightarrow h y'_{r+1} + h^2 y''_{r+1} \approx 2y_{r+1} - y_r$
 $\Rightarrow h y'_{r+1} + h^2 (1 + x_{r+1} \sin y_{r+1}) \approx 2y_{r+1} - y_r$

LET $r=0$ $x=1, y=1, y'=2$
 $\Rightarrow h y'_1 + h^2 (1 + x_1 \sin y_1) \approx 2y_1 - y_0$
 $\Rightarrow y_1 \approx \frac{1}{2} [h y'_1 + h^2 (1 + x_1 \sin y_1) + 2y_1]$
 $\Rightarrow y_1 \approx \frac{1}{2} [0.05 \times 2 + 0.0025 (1 + 1 \sin 1) + 2 \times 1]$
 $\Rightarrow y_1 \approx 1.026159141 \dots$

NOW USING: $y_{r+2} \approx h^2 y''_{r+1} + 2y_{r+1} - y_r$
 LET $r=1$ NOTE THAT $x_1=1, y_1=1$
 $x_2=1.05, y_2 \approx 0.026159141 \dots$
 $\Rightarrow y_2 \approx h^2 y''_2 + 2y_1 - y_0$

$\Rightarrow y_2 \approx h^2 (1 + x_2 \sin y_2) + 2y_1 - y_0$
 $\Rightarrow y_2 \approx 0.0025 (1 + 1.05 \sin 0.026159141) + 2 \times 1.026159141 - 1$
 $\Rightarrow y_2 \approx 0.0526159141 \dots$

Question 4 (***)

The curve with equation $y = f(x)$, satisfies

$$\frac{d^2y}{dx^2} = 4 + \sinh x \sinh y, \quad y(1) = 1, \quad \frac{dy}{dx}(1) = 1.$$

Use the approximations

$$\left(\frac{d^2y}{dx^2} \right)_n \approx \frac{y_{n+1} - 2y_n + y_{n-1}}{h^2} \quad \text{and} \quad \left(\frac{dy}{dx} \right)_n \approx \frac{y_{n+1} - y_{n-1}}{2h},$$

to determine, correct to 2 decimal places, the value of y at $x=1.6$.

Use $h=0.2$ throughout this question.

$$\boxed{}, \quad y(1.6) \approx 2.85$$

Given: $\frac{d^2y}{dx^2} = 4 + \sinh x \sinh y$, subject to $y=1, \frac{dy}{dx}=1$ at $x=1$

Start by rearranging the above for y_{n+1}

$$\Rightarrow y'_n \approx \frac{y_n - 2y_{n-1} + y_{n-2}}{h^2} \quad \Rightarrow y'_n \approx \frac{y_n - y_{n-1}}{2h}$$

$$\Rightarrow h^2 y''_n \approx y_n - 2y_{n-1} + y_{n-2} \quad \Rightarrow 2h y'_n \approx y_n - y_{n-1}$$

$$\Rightarrow y_{n+1} \approx h^2 y''_n + 2y_n - y_{n-1} \quad \Rightarrow y_{n+1} \approx 2h y'_n + y_n$$

FINDING THE ABOVE EXPRESSION VALUES

$$\Rightarrow 2h y'_n \approx h^2 y''_n + 2h y'_n + 2y_n$$

$$\Rightarrow 2y_n \approx h^2 y''_n + 2h y'_n + 2y_n$$

$$\Rightarrow 2y_{n+1} \approx h^2 (4 + \sinh x \sinh y) + 2h y'_n + 2y_n$$

NOW AT $x_0=1, y_0=1$ & $y'_0=1$ (Given)

$$\Rightarrow 2y_1 \approx 0.2^2 (4 + \sinh(1) \sinh(1)) + 2 \times 0.2 \times 1 + 2 \times 1$$

$$\Rightarrow 2y_1 \approx 0.2152439... + 0.4 + 2$$

$$\Rightarrow 2y_1 \approx 2.6152439...$$

$$\Rightarrow y_1 \approx 1.307621957...$$

↑
VALUE OF y AT $x_1=1.2$

FINDING VALUES $y_{n+1} \approx h^2 y''_n + 2y_n - y_{n-1}$

$$\Rightarrow y_2 \approx h^2 [4 + \sinh x \sinh y] + 2y_1 - y_0$$

$$\approx 0.2^2 [4 + \sinh(1.2) \sinh(1.307621957...)] + 2 \times 1.307621957... - 1$$

$$\approx 1.678784924...$$

$$\Rightarrow y_3 \approx h^2 [4 + \sinh(1.4) \sinh(1.678784924...)] + 2y_2 - y_1$$

$$\approx 0.2^2 [4 + \sinh(1.4) \sinh(1.678784924...)] + 2 \times 1.678784924... - 1.307621957...$$

$$\approx 2.853382923$$

∴ THE VALUE OF y AT $x=1.6$ IS APPROX 2.85

Question 5 (***)

$$\frac{d^2 y}{dx^2} + \frac{dy}{dx} + x^3 = 0, \quad y(0) = 1, \quad \frac{dy}{dx}(0) = 2.$$

- a) Use the approximation formulae

$$\left(\frac{d^2 y}{dx^2} \right)_{n+1} \approx \frac{y_{n+2} - 2y_{n+1} + y_n}{h^2} \quad \text{and} \quad \left(\frac{dy}{dx} \right)_{n+1} \approx \frac{y_{n+2} - y_n}{2h},$$

to determine, correct to 2 decimal places, the value of y at $x = 0.1$.

Use $h = 0.1$ throughout this part of the question.

- b) By differentiating the differential equation given above, find the first 4 terms of the infinite convergent series expansion of y , in ascending powers of x , and use it to find, correct to 2 decimal places, another approximation for the value of y at $x = 0.1$.

$$\boxed{}, \quad y(0.1) \approx 1.19$$

a) USING THE FORMULAE GIVEN

$\frac{d^2 y}{dx^2} + \frac{dy}{dx} + x^3 = 0$ SUBJECT TO $y=1, \frac{dy}{dx}=2$ AT $x=0$

$\left(\frac{d^2 y}{dx^2} \right)_{n+1} \approx \frac{y_{n+2} - 2y_{n+1} + y_n}{h^2}$ $\left(\frac{dy}{dx} \right)_{n+1} \approx \frac{y_{n+2} - y_n}{2h}$

$y_{n+1} \approx \frac{y_{n+2} - y_n}{2h}$ $y_{n+2} - 2y_{n+1} + y_n \approx h^2 y''_{n+1}$

$y_{n+2} - y_n \approx 2h y'_{n+1}$ $y_{n+2} - 2y_{n+1} + y_n \approx h^2 y''_{n+1}$

ADDING THE TWO EQUATIONS ABOVE GIVES

$\Rightarrow 2y_{n+2} - 2y_{n+1} \approx 2h y'_{n+1} + h^2 y''_{n+1}$

$\Rightarrow 2y_{n+2} = 2y_{n+1} + 2h y'_{n+1} + h^2 y''_{n+1}$

LET $n=0$ & NOTE $x=0, y=1, y'=2$

$\Rightarrow 2y_2 = 2y_1 + 2h y'_1 + h^2 y''_1$

$\Rightarrow 2y_2 = 2(1) + 2(0.1)(2) + (0.1)^2(2) \leftarrow \text{FROM THE O.D.E. DERIVATIVE}$

$\Rightarrow 2y_2 = 2 + 0.4 + 0.02 = 2.42$

$\Rightarrow y_2 = 1.21$

b) WRITE O.D.E. MORE COMPACTLY

$y'' + y' + x^3 = 0$

DIFFERENTIATE W.R.T x

$\Rightarrow y''' + y'' + 3x^2 = 0$

$\Rightarrow y''' + y'' + 3x^2 = 0$

$\Rightarrow y''' = -y'' - 3x^2$

$\Rightarrow y'' = 2$

HENCE WE HAVE

$\Rightarrow y = y_1 + 2x y'_1 + \frac{2^2}{2!} x^2 y''_1 + \frac{2^3}{3!} x^3 y'''_1 + O(x^4)$

$\Rightarrow y = 1 + 2x - x^2 + \frac{2}{3} x^3 + O(x^4)$

$\Rightarrow y(0.1) = 1 + 2(0.1) - (0.1)^2 + \frac{2}{3}(0.1)^3$

$\Rightarrow y(0.1) \approx 1.190332 \dots$

INDEXED AGREES TO 2 DP TO THE ANSWER OF (a)

Question 6 (****)

The curve with equation $y = f(x)$, satisfies

$$\frac{d^2y}{dx^2} = x + y + 2, \quad y(0) = 0, \quad \frac{dy}{dx}(0) = 1.$$

- a) Use Taylor expansions to justify the validity of the following approximations.

$$\left(\frac{d^2y}{dx^2} \right)_n \approx \frac{y_{n+1} - 2y_n + y_{n-1}}{h^2} \quad \text{and} \quad \left(\frac{dy}{dx} \right)_n \approx \frac{y_{n+1} - y_{n-1}}{2h}.$$

- b) Hence show that $y(0.1) \approx 0.11$

- c) Determine, correct to 4 decimal places, the value of $y(0.2)$ and $y(0.3)$.

$$\boxed{}, \quad y(0.2) \approx 0.2421..., \quad y(0.3) \approx 0.3986...$$

The image shows two pages of handwritten work. The left page details the derivation of the second-order Taylor expansion for y'' and the first-order Taylor expansion for y' using the method of finite differences. It starts with the given differential equation $y'' = x + y + 2$ and initial conditions $y(0) = 0, y'(0) = 1$. It then shows how to approximate y'' at a point x_n using y_{n+1}, y_n, y_{n-1} and y' at x_n using y_{n+1}, y_{n-1} . The right page shows the numerical application of these formulas. It calculates $y_1 \approx 0.11$ at $x_1 = 0.1$, $y_2 \approx 0.2421$ at $x_2 = 0.2$, and $y_3 \approx 0.3986$ at $x_3 = 0.3$, using the recurrence relations derived on the left page.

Question 7 (***)

$$\frac{d^2 y}{dx^2} = 2 + x^2 y + y^2 = 0, \quad y(0) = 1, \quad \frac{dy}{dx}(0) = 1.$$

- a) Use the approximation formulae

$$y_{n+2} \approx 2y_{n+1} + h^2 y''_{n+1} - y_n \quad \text{and} \quad y'_{n+2} \approx 2y'_{n+1} + y'_n,$$

to find, correct to 3 decimal places, the value of y at $x = 0.1$ and $x = 0.2$.

Use $h = 0.1$ throughout this part of the question.

- b) By differentiating the differential equation given above, determine the first 5 terms of the infinite convergent series expansion of y , in ascending powers of x , and use it to find, correct to 3 decimal places, approximations for the value of y at $x = 0.1$ and $x = 0.2$.

$$\boxed{}, \quad y(0.1) \approx 1.115, \quad y(0.2) \approx 1.263$$

Handwritten Solution (a):

Given: $\frac{d^2 y}{dx^2} = 2 + x^2 y + y^2$, $y(0) = 1$, $\frac{dy}{dx}(0) = 1$.

Using the approximation formulae with $h = 0.1$:

At $x = 0$, $y = 1$, $y' = 1$.

At $x = 0.1$, $y = 1.115$.

At $x = 0.2$, $y = 1.263$.

Handwritten Solution (b):

Write the O.D.E. in compact notation for differentiation:

$$y'' = 2 + x^2 y + y^2$$

Differentiate with respect to x :

$$y''' = 2x y + 2x y' + 2y y'$$

Differentiate with respect to x again:

$$y^{(4)} = 2y + 2x y' + 2x y' + 2y y' + 2y y' + 2y y'$$

Further differentiation:

$$y^{(5)} = 2y + 2x y' + 2x y' + 2y y' + 2y y' + 2y y'$$

Use the series expansion to find $y(0.1)$ and $y(0.2)$.

Question 8 (***)

$$\frac{d^2 y}{dx^2} = 1 + y \frac{dy}{dx}, \quad y(0) = 1, \quad \frac{dy}{dx}(0.1) = 1.1.$$

a) Use the approximation formulae

$$\left(\frac{d^2 y}{dx^2} \right)_{r+1} \approx \frac{y_{r+2} - 2y_{r+1} + y_r}{h^2} \quad \text{and} \quad \left(\frac{dy}{dx} \right)_{r+1} \approx \frac{y_{r+2} - y_r}{2h},$$

to show that

$$y_{r+2} \approx \frac{(4 - hy_r)y_{r+1} - 2(y_r - h^2)}{2 - hy_{r+1}}.$$

determine, correct to 2 decimal places, the value of y at $x = 0.1$.

b) Use the result shown in part (a), with $h = 0.1$, to find the value of y at $x = 0.3$, correct to 3 decimal places.

$$\boxed{}, \quad y(0.3) \approx 1.371$$

$\frac{d^2 y}{dx^2} = 1 + y \frac{dy}{dx}$ with $y(0) = 1$ & $y'(0.1) = 1.1$

a) USING THE APPROXIMATIONS

$$\left(\frac{d^2 y}{dx^2} \right)_{r+1} \approx \frac{y_{r+2} - 2y_{r+1} + y_r}{h^2} \quad \& \quad \left(\frac{dy}{dx} \right)_{r+1} \approx \frac{y_{r+2} - y_r}{2h}$$

$$y'_{r+1} \approx \frac{y_{r+2} - 2y_{r+1} + y_r}{h} \quad \& \quad y'_{r+1} \approx \frac{y_{r+2} - y_r}{2h}$$

SUBSTITUTE INTO THE O.D.E

$$\frac{y_{r+2} - 2y_{r+1} + y_r}{h^2} = 1 + y_{r+1} \left[\frac{y_{r+2} - y_r}{2h} \right]$$

MULTIPLY THROUGH BY $2h^2$ AND THEY

$$2(y_{r+2} - 2y_{r+1} + y_r) = 2h^2 + h y_{r+1} [y_{r+2} - y_r]$$

$$2y_{r+2} - 4y_{r+1} + 2y_r = 2h^2 + h y_{r+1} y_{r+2} - h y_{r+1} y_r$$

$$2y_{r+2} - h y_{r+1} y_{r+2} = 2h^2 - h y_{r+1} y_r + 4y_{r+1} - 2y_r$$

$$y_{r+2} (2 - h y_{r+1}) = y_{r+1} (4 - h y_r) - 2y_r + 2h^2$$

$$y_{r+2} = \frac{y_{r+1} (4 - h y_r) - 2(y_r - h^2)}{2 - h y_{r+1}}$$

b) NOW USING THE RESULT

$x_0 = 0 \quad y_0 = 1$
 $x_1 = 0.1 \quad y_1 = 1.1$
 $x_2 = 0.2$

$r=1 \Rightarrow y_2 = \frac{y_1(4 - h y_0) - 2(y_0 - h^2)}{2 - h y_1}$
 $\Rightarrow y_2 = \frac{1.1(4 - 0.1 \times 1) - 2(1 - 0.1^2)}{2 - 0.1 \times 1.1}$
 $\Rightarrow y_2 = \frac{4.28 - 1.98}{1.89}$
 $\Rightarrow y_2 \approx 1.222 \dots$

$r=2 \Rightarrow y_3 = \frac{y_2(4 - h y_1) - 2(y_1 - h^2)}{2 - h y_2}$
 $\Rightarrow y_3 = \frac{1.222(4 - 0.1 \times 1.1) - 2(1.1 - 0.1^2)}{2 - 0.1 \times 1.222 \dots}$
 $\Rightarrow y_3 = \frac{4.8888 \dots - 2.18}{1.87777 \dots}$
 $\Rightarrow y_3 \approx 1.371$

Question 9 (***)

The curve with equation $x = f(t)$, satisfies

$$\frac{d^2x}{dt^2} = -x, \quad x(0) = 0, \quad \frac{dx}{dt}(0) = 1.$$

Use Euler's method, with a step of 0.1, to find the approximate value of x at $t = 0.5$.

$$\boxed{}, \quad x(0.5) \approx 0.480\dots$$

The image shows two pages of handwritten work on grid paper. The left page outlines the problem and sets up the initial conditions. The right page shows the iterative calculations for Euler's method.

Left Page:

- Problem statement: $\frac{d^2x}{dt^2} = -x$, subject to the conditions $t=0$, $x_0=0$, $\dot{x}_0=\dot{x}'_0=1$.
- Use the two formulas: $\left(\frac{dy}{dx}\right)_0 \approx \frac{y_1 - y_0}{x_1 - x_0}$ and $\frac{d^2y}{dx^2} \approx \frac{y_2 - 2y_1 + y_0}{h^2}$.
- Write down our variables, since $x = x(t)$: $x'_0 \approx \frac{x_1 - x_0}{2h}$ and $x''_0 \approx \frac{x_2 - 2x_1 + x_0}{h^2}$.
- From the O.D.E itself $x''_0 = -x_0 = 0$.
- Hence we obtain from each formula, with $h=0.1$: $1 = \frac{x_1 - x_0}{2}$ and $0 = \frac{x_2 - 2x_1 + x_0}{(0.1)^2}$.
- Solving these gives $x_1 = 0.2$ and $x_2 = 0.1$.
- Hence the second derivative approximation formula yields: $x'_0 \approx \frac{x_1 - x_0}{h} = \frac{0.2 - 0}{0.1} = 2$ and $x''_0 \approx \frac{x_2 - 2x_1 + x_0}{h^2} = \frac{0.1 - 2(0.2) + 0}{0.01} = -30$.

Right Page:

- Recurrence relation: $x_{k+2} = x_k(2 - h^2) - x_1$, with $x_0 = 0$, $x_1 = 0.1$, $h = 0.1$.
- Calculate $x_{k+2} = 1.99x_k - x_1$.
- Iterate: $x_2 = 1.99x_1 - x_0 = 0.199$, $x_3 = 1.99x_2 - x_1 = 0.29601$, $x_4 = 1.99x_3 - x_2 = 0.3900599$, $x_5 = 1.99x_4 - x_3 = 0.4800599$, $x_6 = 1.99x_5 - x_4 = 0.5680599$.
- Conclusion: At $t = 0.5$, $x \approx 0.480$.