NUMERICAL METHODS for O.D.E.s

1^{st} order O.D.E.s

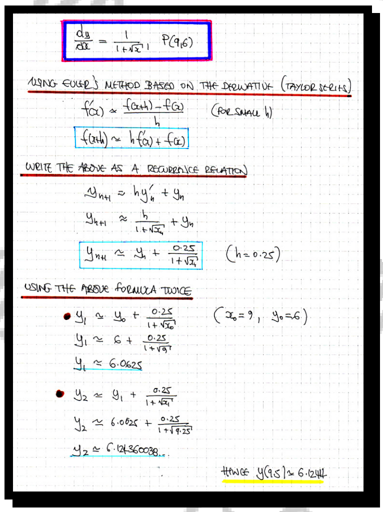
Question 1 (**)

The curve with equation $y = f(x)$, passes through the point $(9,6)$ and satisfies

$$\frac{dy}{dx} = \frac{1}{1+\sqrt{x}}, \quad x \geq 0 .$$

Use Euler's method, with a step of 0.25, to find, correct to 4 decimal places, the value of y at $x = 9.5$.

$\boxed{}$, $\boxed{y(9.5) \approx 6.1244}$



$$\boxed{\frac{dy}{dx} = \frac{1}{1+\sqrt{x}}, \quad P(9,6)}$$

USING EULER'S METHOD BASED ON THE DERIVATIVE (TAYLOR SERIES)

$$f'(x) \approx \frac{f(x+h) - f(x)}{h} \quad \text{(FOR SMALL } h\text{)}$$

$$\boxed{f(x+h) \approx h f'(x) + f(x)}$$

WRITE THE ABOVE AS A RECURRENCE RELATION

$$y_{n+1} \approx h y'_n + y_n$$

$$y_{n+1} \approx \frac{h}{1+\sqrt{x_n}} + y_n$$

$$\boxed{y_{n+1} \approx y_n + \frac{0.25}{1+\sqrt{x_n}}} \quad (h = 0.25)$$

USING THE ABOVE FORMULA TWICE

- $y_1 \approx y_0 + \frac{0.25}{1+\sqrt{x_0}}$ $\quad (x_0 = 9, \; y_0 = 6)$

  $y_1 \approx 6 + \frac{0.25}{1+\sqrt{9}}$

  $y_1 \approx 6.0625$

- $y_2 \approx y_1 + \frac{0.25}{1+\sqrt{x_1}}$

  $y_2 \approx 6.0625 + \frac{0.25}{1+\sqrt{9.25}}$

  $y_2 \approx 6.124360038...$

HENCE $y(9.5) \approx 6.1244$

Question 2 (**)

The curve with equation $y = f(x)$, passes through the point $(1,1)$ and satisfies the following differential equation.

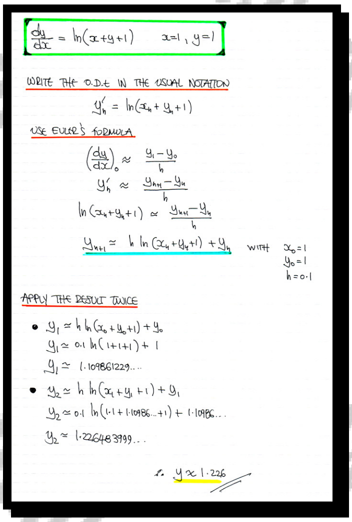
$$\frac{dy}{dx} = \ln(x+y+1), \quad x+y > -1.$$

Use the approximation

$$\left(\frac{dy}{dx}\right)_0 \approx \frac{y_1 - y_0}{h},$$

with $h = 0.1$, to find, correct to 3 decimal places, the value of y at $x = 1.2$.

$\boxed{}$, $\boxed{y(1.2) \approx 1.226}$



$\boxed{\frac{dy}{dx} = \ln(x+y+1) \qquad x=1,\ y=1}$

WRITE THE O.D.E IN THE USUAL NOTATION

$$y_n' = \ln(x_n + y_n + 1)$$

USE EULER'S FORMULA

$$\left(\frac{dy}{dx}\right)_0 \approx \frac{y_1 - y_0}{h}$$

$$y_n' \approx \frac{y_{n+1} - y_n}{h}$$

$$\ln(x_n + y_n + 1) \approx \frac{y_{n+1} - y_n}{h}$$

$$y_{n+1} \approx h\ln(x_n + y_n + 1) + y_n \quad \text{WITH} \quad x_0 = 1,\ y_0 = 1,\ h = 0.1$$

APPLY THE RESULT TWICE

- $y_1 \approx h\ln(x_0 + y_0 + 1) + y_0$
  $y_1 \approx 0.1\ln(1+1+1) + 1$
  $y_1 \approx 1.109861229\ldots$
- $y_2 \approx h\ln(x_1 + y_1 + 1) + y_1$
  $y_2 \approx 0.1\ln(1.1 + 1.10986\ldots + 1) + 1.10986\ldots$
  $y_2 \approx 1.226483999\ldots$

$\therefore y \approx 1.226$

Question 3 (**)

The curve with equation $y = f(x)$, passes through the point $(1,4)$ and satisfies

$$\frac{dy}{dx} = \frac{1}{2x+\sqrt{x}}, \quad x \geq 0 .$$

Use Euler's method, with a step of 0.2, to find, correct to 4 decimal places, the value of y at $x = 1.6$.

$\boxed{}$, $\boxed{y(1.6) \approx 1.1741}$

USING EULER'S STEP BY STEP METHOD

$$f(x+h) \approx f(x) + hf'(x)$$

IN OUR CASE WE HAVE

$$y_{n+1} = y_n + hy'_n$$
$$y_{n+1} = y_n + h\left(\frac{1}{2x_n+\sqrt{x_n}}\right)$$

STARTING WITH $x_1 = 1$, $y_1 = 4$ & $h = 0.2$

$$y_2 = y_1 + h\left(\frac{1}{2x_1+\sqrt{x_1}}\right)$$
$$y_2 = 4 + \frac{0.2}{2\times 1+\sqrt{1}} \simeq 4.06667$$

REPEAT WITH $x_2 = 1.2$, $y_2 \simeq 4.06667\ldots$, $h = 0.2$

$$y_3 = y_2 + \frac{h}{2x_2+\sqrt{x_2}}$$
$$y_3 = 4.06667\ldots + \frac{0.2}{2\times 1.2+\sqrt{1.2}} \simeq 4.12338\ldots$$

FINAL ITERATION WITH $x_3 = 1.4$, $y_3 \simeq 4.12338\ldots$, $h = 0.2$

$$y_4 = y_3 + \frac{h}{2x_3+\sqrt{x_3}}$$
$$y_4 = 4.12338\ldots + \frac{0.2}{2\times 1.4+\sqrt{1.4}} \simeq 4.17409\ldots$$

$$\therefore \ y(1.6) \simeq 4.1741$$

Question 4 (**)

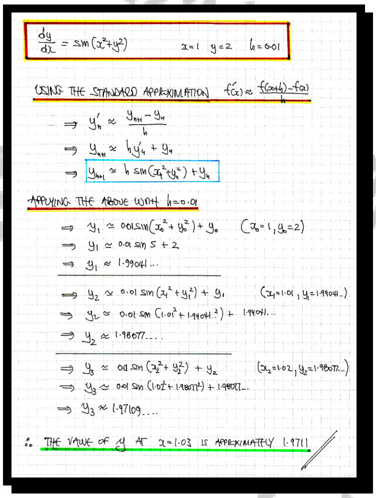
$$\frac{dy}{dx} = \sin\left(x^2 + y^2\right), \quad y(1) = 2.$$

Use, in the standard notation, the approximation

$$y_n' \approx \frac{y_{n+1} - y_n}{h},$$

with $h = 0.01$, to find, correct to 4 decimal places, the value of y at $x = 1.03$.

FP3-K, $\boxed{y(1.03) \approx 1.9711}$



$\frac{dy}{dx} = \sin(x^2+y^2)$ $\quad x=1 \quad y=2 \quad h=0.01$

USING THE STANDARD APPROXIMATION $f'(x) \approx \frac{f(x+h)-f(x)}{h}$

$\Rightarrow y_n' \approx \frac{y_{n+1}-y_n}{h}$

$\Rightarrow y_{n+1} \approx h y_n' + y_n$

$\Rightarrow \boxed{y_{n+1} \approx h \sin(x_n^2+y_n^2) + y_n}$

APPLYING THE ABOVE WITH $h=0.01$

$\Rightarrow y_1 \approx 0.01\sin(x_0^2+y_0^2) + y_0 \quad (x_0=1, y_0=2)$

$\Rightarrow y_1 \approx 0.01 \sin 5 + 2$

$\Rightarrow y_1 \approx 1.99041\ldots$

$\Rightarrow y_2 \approx 0.01 \sin(x_1^2+y_1^2) + y_1 \quad (x_1=1.01, y_1=1.99041\ldots)$

$\Rightarrow y_2 \approx 0.01 \sin(1.01^2 + 1.99041\ldots^2) + 1.99041\ldots$

$\Rightarrow y_2 \approx 1.98077\ldots$

$\Rightarrow y_3 \approx 0.01 \sin(x_2^2+y_2^2) + y_2 \quad (x_2=1.02, y_2=1.98077\ldots)$

$\Rightarrow y_3 \approx 0.01 \sin(1.02^2 + 1.98077^2) + 1.98077\ldots$

$\Rightarrow y_3 \approx 1.97109\ldots$

$\therefore$ THE VALUE OF y AT $x=1.03$ IS APPROXIMATELY 1.9711

Question 5 (**)

$$\frac{dy}{dx} = 4x^2 - y^2, \quad y(1) = 0.$$

Use the approximation

$$f'(x) \approx \frac{f(x+h) - f(x)}{h},$$

with $h = 0.05$, to find, correct to 4 decimal places, the value of y at $x = 1.2$.

No credit will be given for solving the differential equation analytically.

$\boxed{}$, $\boxed{y(1.2) \approx 0.8080}$

$\frac{dy}{dx} = 4x^2 - y^2$ SUBJECT TO $y = 0$ AT $x = 1$

USING EULER'S METHOD BASED ON THE DERIVATIVE (TAYLOR SERIES)

$f'(x) \simeq \frac{f(x+h) - f(x)}{h}$ (FOR SMALL h)

$\boxed{f(x+h) \simeq h f'(x) + f(x)}$

WRITE THE ABOVE RESULT AS A RECURRENCE RELATION

$y_{n+1} \simeq h y'_n + y_n$

$y_{n+1} \simeq h(4x_n^2 - y_n^2) + y_n$

$y_{n+1} \simeq 4h x_n^2 + y_n - 4h y_n^2$

$y_{n+1} \simeq \frac{1}{5} x_n^2 + y_n\left(1 - \frac{1}{5} y_n\right)$ $(h = 0.05)$

$\boxed{y_{n+1} \simeq \frac{1}{5}\left[x_n^2 + y_n(5 - y_n)\right]}$

APPLYING THE ABOVE FORMULA, STARTING WITH $x_0 = 1$ $y_0 = 0$

$y_1 \simeq \frac{1}{5}\left[x_0^2 + y_0(5 - y_0)\right] = \frac{1}{5}\left[1^2 + 0\right] = 0.2$

$y_2 \simeq \frac{1}{5}\left[x_1^2 + y_1(5 - y_1)\right] = \frac{1}{5}\left[(1.05)^2 + 0.2(4.8)\right] = 0.4125$

$y_3 \simeq \frac{1}{5}\left[x_2^2 + y_2(5 - y_2)\right] = \frac{1}{5}\left[1.1^2 + 0.4125(4.5875)\right] = 0.62046825$

$y_4 \simeq \frac{1}{5}\left[x_3^2 + y_3(5 - y_3)\right] = \frac{1}{5}\left[1.15^2 + 0.62046\ldots(5 - 0.62046\ldots)\right] \simeq 0.80797\ldots$

$\therefore$ AT $x = 1.2$, $y \approx 0.8080$

Question 6 (**+)

The curve with equation $y = f(x)$, passes through the point $(1,0)$ and satisfies the following differential equation.

$$\frac{dy}{dx} = x + \ln x, \quad x > 0.$$

Use the approximation

$$\left(\frac{dy}{dx}\right)_0 \approx \frac{y_1 - y_0}{h}, \quad h = 0.1,$$

to find the value of y at $x = 1.1$, and use this answer with the approximation

$$\left(\frac{dy}{dx}\right)_0 \approx \frac{y_1 - y_{-1}}{2h}, \quad h = 0.1,$$

to find, correct to 3 decimal places, the value of y at $x = 1.2$, $x = 1.3$ and $x = 1.4$.

$\boxed{}$, $\boxed{y(1.1) = 0.1}$, $\boxed{y(1.2) \approx 0.2391}$, $\boxed{y(1.3) \approx 0.3765}$, $\boxed{y(1.4) \approx 0.5515}$

$\boxed{\frac{dy}{dx} = x + \ln x \qquad x = 1, \ y = 0}$

USING THE RESULT $\left(\frac{dy}{dx}\right)_0 = \frac{y_1 - y_0}{h}$ FIRST

$$y'_0 = \frac{y_1 - y_0}{h}$$
$$y_1 = hy'_0 + y_0$$

HERE WE HAVE $x_0 = 1$, $y_0 = 0$, $h = 0.1$

$$y_1 = h[x_0 + \ln x_0] + y_0$$
$$y_1 = 0.1[1 + \ln 1] + 0$$
$$y_1 = 0.1$$

NEXT WE ARE USING $\left(\frac{dy}{dx}\right)_0 = \frac{y_1 - y_{-1}}{2h}$

REWRITE AS

$$y'_0 = \frac{y_1 - y_{-1}}{2h}$$
$$y'_n = \frac{y_{n+1} - y_{n-1}}{2h}$$
$$2hy'_n = y_{n+1} - y_{n-1}$$
$$y_{n+1} = 2hy'_n + y_{n-1}$$
$$y_{n+2} = 2hy'_{n+1} + y_n$$
$$\boxed{y_{n+2} = 2h[x_{n+1} + \ln x_{n+1}] + y_n}$$

WITH $x_0 = 1$ $y_0 = 0$
$x_1 = 1.1$ $y_1 = 0.1$
$x_2 = 1.2$
$x_3 = 1.3$

APPLYING THE FORMULA IN SUCCESSION

- $y_2 \approx 2h[x_1 + \ln x_1] + y_0$
  $y_2 \approx 2 \times 0.1[1.1 + \ln(1.1)] + 0$
  $y_2 \approx 0.239062\ldots \approx 0.2391$
- $y_3 \approx 2h[x_2 + \ln x_2] + y_1$
  $y_3 \approx 2(0.1)[1.2 + \ln(1.2)] + 0.1$
  $y_3 \approx 0.376464\ldots \approx 0.3765$
- $y_4 \approx 2h(x_3 + \ln x_3) + y_2$
  $y_4 \approx 2(0.1)[1.3 + \ln(1.3)] + 0.239062\ldots$
  $y_4 \approx 0.551534\ldots \approx 0.5515$

Question 7 (**+)

The curve with equation $y = f(x)$ satisfies the differential equation

$$\frac{dy}{dx} = x + y + y^2, \quad y(0.9) = 3.75, \quad y(1) = 4$$

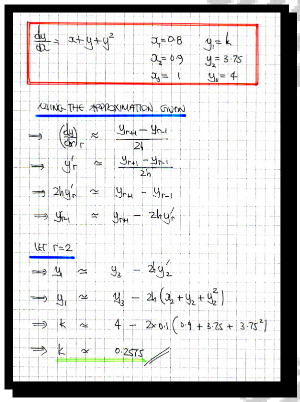
Using, in the standard notation, the approximation

$$\left(\frac{dy}{dx}\right)_r \approx \frac{y_{r+1} - y_{r-1}}{2h},$$

with $h = 0.1$, the value of y at $x = 0.8$ was estimated to k.

Determine the value of k.

$\boxed{}$, $\boxed{k \approx 0.2575}$



$\frac{dy}{dx} = x + y + y^2$ $\quad x_1 = 0.8 \quad y_1 = k$

$x_2 = 0.9 \quad y_2 = 3.75$

$x_3 = 1 \quad y_3 = 4$

USING THE APPROXIMATION GIVEN

$\Rightarrow \left(\frac{dy}{dx}\right)_r \approx \frac{y_{r+1} - y_{r-1}}{2h}$

$\Rightarrow y'_r \approx \frac{y_{r+1} - y_{r-1}}{2h}$

$\Rightarrow 2hy'_r \approx y_{r+1} - y_{r-1}$

$\Rightarrow y_{r-1} \approx y_{r+1} - 2hy'_r$

LET $r=2$

$\Rightarrow y_1 \approx y_3 - 2hy'_2$

$\Rightarrow y_1 \approx y_3 - 2h(x_2 + y_2 + y_2^2)$

$\Rightarrow k \approx 4 - 2 \times 0.1(0.9 + 3.75 + 3.75^2)$

$\Rightarrow k \approx 0.2575$

Question 8 (**+)

$$\frac{dy}{dx} = x^2 - y^2, \quad y(3) = 2.$$

Use the approximation

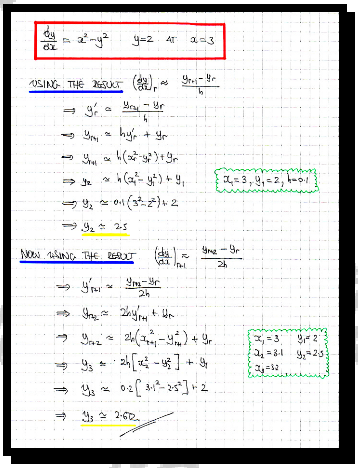
$$\left(\frac{dy}{dx}\right)_r \approx \frac{y_{r+1} - y_r}{h}, \quad h = 0.1,$$

to find the value of y at $x = 2.1$, and use this answer with the approximation

$$\left(\frac{dy}{dx}\right)_{r+1} \approx \frac{y_{r+2} - y_r}{2h}, \quad h = 0.1,$$

to find, correct to 3 decimal places, the value of y at $x = 2.2$, $x = 1.3$ and $x = 1.4$.

$\boxed{}$, $\boxed{y(2.2) = 2.672}$



$\boxed{\frac{dy}{dx} = x^2 - y^2 \qquad y = 2 \text{ AT } x = 3}$

USING THE RESULT $\left(\frac{dy}{dx}\right)_r \approx \frac{y_{r+1} - y_r}{h}$

$\Rightarrow y'_r \approx \frac{y_{r+1} - y_r}{h}$

$\Rightarrow y_{r+1} \approx hy'_r + y_r$

$\Rightarrow y_{r+1} \approx h(x_r^2 - y_r^2) + y_r$

$\Rightarrow y_2 \approx h(x_1^2 - y_1^2) + y_1$ $\quad x_1 = 3, y_1 = 2, h = 0.1$

$\Rightarrow y_2 \approx 0.1(3^2 - 2^2) + 2$

$\Rightarrow y_2 \simeq 2.5$

NOW USING THE RESULT $\left(\frac{dy}{dx}\right)_{r+1} \approx \frac{y_{r+2} - y_r}{2h}$

$\Rightarrow y'_{r+1} \approx \frac{y_{r+2} - y_r}{2h}$

$\Rightarrow y_{r+2} \approx 2hy'_{r+1} + y_r$

$\Rightarrow y_{r+2} \approx 2h(x_{r+1}^2 - y_{r+1}^2) + y_r$ $\quad x_1 = 3 \quad y_1 = 2 \quad x_2 = 3.1 \quad y_2 = 2.5 \quad x_3 = 3.2$

$\Rightarrow y_3 \approx 2h[x_2^2 - y_2^2] + y_1$

$\Rightarrow y_3 \approx 0.2[3.1^2 - 2.5^2] + 2$

$\Rightarrow y_3 \simeq 2.6$[illegible]

Question 9 (**+)

$$\frac{dy}{dx} = xy, \quad y(0) = 2.$$

Use the approximation

$$f'(x) \approx \frac{f(x+h) - f(x)}{h}, \quad h = 0.1$$

to find the value of y at $x = 0.1$, and use this answer with the approximation

$$f'(x) \approx \frac{f(x+h) - f(x-h)}{2h}, \quad h = 0.1,$$

to find the value of y at $x = 0.4$.

No credit will be given for solving the differential equation analytically.

$\boxed{\quad}$, $\boxed{y(0.1) = 2}$, $\boxed{y(0.4) \approx 2.1649}$

$\frac{dy}{dx} = xy$ SUBJECT TO $x=0, y=2$

USING THE RESULT $f'(x) \approx \frac{f(x+h) - f(x)}{h}$

$\Rightarrow y'_n \simeq \frac{y_{n+1} - y_n}{h}$

$\Rightarrow y_{n+1} \simeq h y'_n + y_n$

$\Rightarrow y_{n+1} \simeq h(x_n y_n) + y_n$

$\Rightarrow y_{n+1} \simeq y_n(h x_n + 1)$

$\Rightarrow y_1 \simeq y_0[0.1 \times x_0 + 1] = 2[0.1 \times 0 + 1]$

$\Rightarrow y_1 \simeq 2$

NEXT WE USE THE RESULT $f'(x) \simeq \frac{f(x+h) - f(x-h)}{2h}$

$\Rightarrow y'_{n+1} \simeq \frac{y_{n+2} - y_n}{2h}$

$\Rightarrow y_{n+2} \approx 2h y'_{n+1} + y_n$

$\Rightarrow y_{n+2} \simeq 2h(x_{n+1} y_{n+1}) + y_n$

$\Rightarrow \boxed{y_{n+2} \simeq 2h x_{n+1} y_{n+1} + y_n}$

$h = 0.1$, $x_0 = 0$, $y_0 = 2$
$x_1 = 0.1$, $y_1 = 2$

HENCE WE OBTAIN

- $y_2 \simeq 2h x_1 y_1 + y_0 = 2 \times 0.1 \times 0.1 \times 2 + 2 = 2.04$ ← AT $x_2 = 0.2$
- $y_3 \approx 2h x_2 y_2 + y_1 = 2 \times 0.1 \times 0.2 \times 2.04 + 2 = 2.0816$ ← AT $x_3 = 0.3$
- $y_4 \approx 2h x_3 y_3 + y_2 = 2 \times 0.1 \times 0.3 \times 2.0813 + 2.04 = 2.164896$

Question 10 (+)**

The curve with equation $y = f(x)$ satisfies the differential equation

$$\frac{dy}{dx} = \frac{e^{x+y}}{3x+y+k}, \quad y(0) = 0,$$

where k is a positive constant.

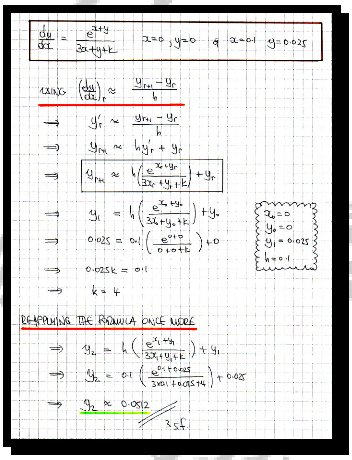
Using, in the standard notation, the approximation

$$\left(\frac{dy}{dx}\right)_r \approx \frac{y_{r+1} - y_r}{h},$$

with $h = 0.1$, the value of y at $x = 0.1$ was estimated to 0.025.

Use the approximation formula given above to find, correct to 3 significant figures, the value of y at $x = 0.2$.

$\boxed{}$, $\boxed{y(1.2) \approx 0.0512}$



$\frac{dy}{dx} = \frac{e^{x+y}}{3x+y+k}$ $x=0, y=0$ & $x=0.1$ $y=0.025$

USING $\left(\frac{dy}{dx}\right)_r \approx \frac{y_{r+1}-y_r}{h}$

$\Rightarrow y'_r \approx \frac{y_{r+1}-y_r}{h}$

$\Rightarrow y_{r+1} \approx h y'_r + y_r$

$\Rightarrow \boxed{y_{r+1} \approx h\left(\frac{e^{x_r+y_r}}{3x_r+y_r+k}\right)+y_r}$

$\Rightarrow y_1 = h\left(\frac{e^{x_0+y_0}}{3x_0+y_0+k}\right)+y_0$

$\Rightarrow 0.025 = 0.1\left(\frac{e^{0+0}}{0+0+k}\right)+0$

$\Rightarrow 0.025k = 0.1$

$\Rightarrow k = 4$

$x_0 = 0$, $y_0 = 0$, $y_1 = 0.025$, $h = 0.1$

REAPPLYING THE FORMULA ONCE MORE

$\Rightarrow y_2 = h\left(\frac{e^{x_1+y_1}}{3x_1+y_1+k}\right)+y_1$

$\Rightarrow y_2 = 0.1\left(\frac{e^{0.1+0.025}}{3\times0.1+0.025+4}\right)+0.025$

$\Rightarrow y_2 \approx 0.0512$ // 3 s.f.

Question 11 (+)**

$$\frac{dy}{dx} = \frac{4x^2 + y^2}{x + y}, \quad y(1) = 4.$$

Use the approximation

$$f'(x) \approx \frac{f(x+h) - f(x)}{h},$$

with h to be found, given further that $f(1+h) \approx 4.8$.

$\boxed{}$, $\boxed{h = 0.2}$

$$\boxed{\frac{dy}{dx} = \frac{4x^2+y^2}{x+y} \qquad y(1) = 4 \qquad y(1+h) \approx 4.8}$$

$\Rightarrow f'(x) \approx \dfrac{f(x+h) - f(x)}{h}$, if h is small

$\Rightarrow h f'(x) \approx f(x+h) - f(x)$

$\Rightarrow h\left[\dfrac{4x + [f(x)]^2}{x + f(x)}\right] \approx f(x+h) - f(x)$

use $x = 1$ $y = 4$ $f(1+h) = 4.8$

$\Rightarrow h\left[\dfrac{4 \times 1 + 4^2}{1 + 4}\right] \simeq 4.8 - 4$

$\Rightarrow h \times 4 \simeq 0.8$

$\Rightarrow h \simeq 0.2$

Question 12 (***)

$$\frac{dy}{dx} = e^{x} - y^{2}, \quad y(0) = 0.$$

a) Use, in the standard notation, the approximation

$$y_{n+1} \approx h\, y_n' + y_n,$$

with $h = 0.1$, to find the approximate value of y at $x = 0.1$.

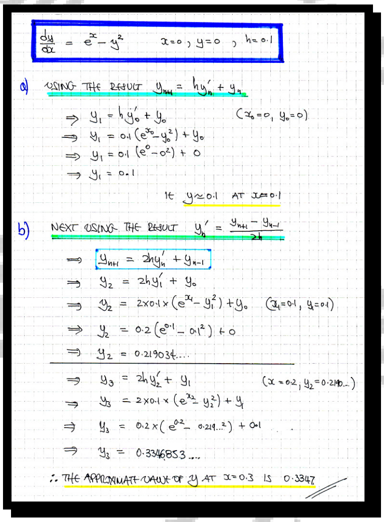
b) Use the answer of part **(a)** and the approximation

$$y_n' \approx \frac{y_{n+1} - y_{n-1}}{2h},$$

with $h = 0.1$, to find, correct to 4 decimal places, the approximate value of y at $x = 0.3$.

c) By differentiating the differential equation given, determine the first four **non zero** terms in the infinite series expansion of y in ascending powers of x, and use it to find, correct to 4 decimal places, another approximation for the value of y at $x = 0.3$.

$\boxed{}$, $\boxed{y(0.1) \approx 0.1}$, $\boxed{y(0.3) \approx 0.3347}$, $\boxed{y(0.3) \approx 0.3388}$



$\boxed{\frac{dy}{dx} = e^{x} - y^{2} \qquad x=0,\ y=0,\ h=0.1}$

a) USING THE RESULT $y_{n+1} = hy_n' + y_n$

$\Rightarrow y_1 = hy_0' + y_0 \qquad (x_0=0,\ y_0=0)$
$\Rightarrow y_1 = 0.1(e^{x_0} - y_0^2) + y_0$
$\Rightarrow y_1 = 0.1(e^{0} - 0^2) + 0$
$\Rightarrow y_1 = 0.1$

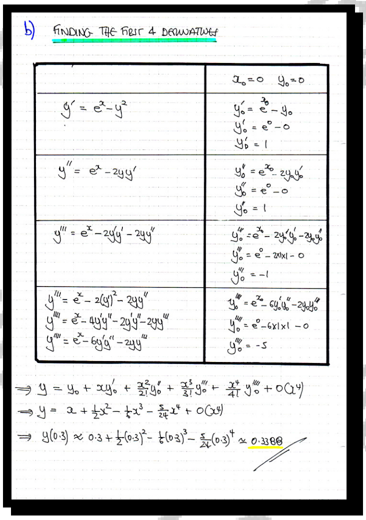
IE $y \simeq 0.1$ AT $x = 0.1$

b) NEXT USING THE RESULT $y_n' = \frac{y_{n+1} - y_{n-1}}{2h}$

$\Rightarrow \boxed{y_{n+1} = 2hy_n' + y_{n-1}}$
$\Rightarrow y_2 = 2hy_1' + y_0$
$\Rightarrow y_2 = 2\times0.1\times(e^{x_1} - y_1^2) + y_0 \qquad (x_1=0.1,\ y_1=0.1)$
$\Rightarrow y_2 = 0.2(e^{0.1} - 0.1^2) + 0$
$\Rightarrow y_2 = 0.219034\ldots$

$\Rightarrow y_3 = 2hy_2' + y_1 \qquad (x=0.2,\ y_2 = 0.2190\ldots)$
$\Rightarrow y_3 = 2\times0.1\times(e^{x_2} - y_2^2) + y_1$
$\Rightarrow y_3 = 0.2\times(e^{0.2} - 0.219\ldots^2) + 0.1$
$\Rightarrow y_3 = 0.3346853\ldots$

$\therefore$ THE APPROXIMATE VALUE OF y AT $x=0.3$ IS 0.3347



b) FINDING THE FIRST 4 DERIVATIVES

	$x_0 = 0 \quad y_0 = 0$
$y' = e^{x} - y^2$	$y_0' = e^{x_0} - y_0$ $y_0' = e^{0} - 0$ $y_0' = 1$
$y'' = e^{x} - 2yy'$	$y_0'' = e^{x_0} - 2y_0y_0'$ $y_0'' = e^{0} - 0$ $y_0'' = 1$
$y''' = e^{x} - 2y'y' - 2yy''$	$y_0''' = e^{x_0} - 2y_0'y_0' - 2y_0y_0''$ $y_0''' = e^{0} - 2\times1\times1 - 0$ $y_0''' = -1$
$y'''' = e^{x} - 2(y')^2 - 2yy''$ $y'''' = e^{x} - 4y'y'' - 2y'y'' - 2yy'''$ $y'''' = e^{x} - 6y'y'' - 2yy'''$	$y_0'''' = e^{x_0} - 6y_0'y_0'' - 2y_0y_0'''$ $y_0'''' = e^{0} - 6\times1\times1 - 0$ $y_0'''' = -5$

$\Rightarrow y = y_0 + xy_0' + \frac{x^2}{2!}y_0'' + \frac{x^3}{3!}y_0''' + \frac{x^4}{4!}y_0'''' + O(x^4)$
$\Rightarrow y = x + \frac{1}{2}x^2 - \frac{1}{6}x^3 - \frac{5}{24}x^4 + O(x^4)$
$\Rightarrow y(0.3) \approx 0.3 + \frac{1}{2}(0.3)^2 - \frac{1}{6}(0.3)^3 - \frac{5}{24}(0.3)^4 \approx 0.3388$

Question 13 (***)

$$\frac{dy}{dx} = \frac{3x^2 - y^2}{2xy}, \quad y(k) = 2, \quad k > 0.$$

a) Use, in the standard notation, the approximation

$$y_n' \approx \frac{y_{n+1} - y_n}{h}, \quad h = 0.1,$$

to find the value of k, given further that $y(k+h) \approx 2.275$.

b) Use the answer of part **(a)** and the approximation

$$y_n' \approx \frac{y_{n+1} - y_{n-1}}{2h}, \quad h = 0.1,$$

with, to find, correct to 3 decimal places, the approximate value of $y(k+2h)$.

$\boxed{}$, $\boxed{k = 4}$, $\boxed{y(k+2h) \approx 2.485}$

$\boxed{\dfrac{dy}{dx} = \dfrac{3x^2 - y^2}{2xy} \qquad y(k) = 2 \qquad h = 0.1}$

a) USING $y_n' \simeq \dfrac{y_{n+1} - y_n}{h}$

$\Rightarrow y_{n+1} \simeq h y_n' + y_n$

$\Rightarrow y_{n+1} \simeq h\left[\dfrac{3x_n^2 - y_n^2}{2x_n y_n}\right] + y_n$

$\Rightarrow y_1 \simeq h\left[\dfrac{3x_0^2 - y_0^2}{2x_0 y_0}\right] + y_0$

$\Rightarrow 2.275 \approx 0.1\left[\dfrac{3k^2 - 2^2}{2 \times k \times 2}\right] + 2$

$\Rightarrow 2.275 \simeq 0.1\left(\dfrac{3k^2 - 4}{4k}\right) + 2$

$\Rightarrow 0.275 \simeq 0.1\left(\dfrac{3k^2 - 4}{4k}\right)$

$\Rightarrow 2.75 \simeq \dfrac{3k^2 - 4}{4k}$

$\Rightarrow 11k \simeq 3k^2 - 4$

$\Rightarrow 0 \simeq 3k^2 - 11k - 4$

$\Rightarrow 0 \simeq (3k + 1)(k - 4)$

$\Rightarrow k = \begin{cases} 4 \\ \cancel{-\frac{1}{3}} \end{cases}$

b) NOW USING $y_n' \simeq \dfrac{y_{n+1} - y_{n-1}}{2h}$

$x_0 = 4 \quad y_0 = 2$
$x_1 = 4.1 \quad y_1 = 2.275$
$x_2 = 4.2 \quad y_2 = ?$

$\Rightarrow y_{n+1} \approx 2h y_n' + y_{n-1}$

$\Rightarrow y_{n+1} \approx 2h\left[\dfrac{3x_n^2 - y_n^2}{2x_n y_n}\right] + y_{n-1}$

$\Rightarrow y_2 \sim 2h\left[\dfrac{3x_1^2 - y_1^2}{2x_1 y_1}\right] + y_0$

$\Rightarrow y_2 \simeq 2 \times 0.1 \times \left[\dfrac{3 \times 4.1^2 - 2.275^2}{2 \times 4.1 \times 2.275}\right] + 2$

$\Rightarrow y_2 \simeq 2.485$

Question 14 (*+)**

The curve with equation $y = f(x)$, passes through the point $(0,1)$ and satisfies the following differential equation.

$$\frac{dy}{dx} = 3x^2 y + x^5 .$$

a) Use the approximation

$$\left(\frac{dy}{dx}\right)_n \approx \frac{y_{n+1} - y_n}{h},$$

with $h = 0.1$, to find, correct to 6 decimal places, the value of y at $x = 0.2$.

b) Find the solution of the differential equation, and use it to obtain the value of y at $x = 0.2$.

$\boxed{}$, $\boxed{y(0.2) \approx 1.003001}$, $\boxed{y(0.2) \approx 1.008}$

$\frac{dy}{dx} = 3x^2y + x^5 \quad x=0, \ y=1$

a) WRITE INFORMATION IN THE USUAL NOTATION

$\Rightarrow y_{n+1} \approx h y'_n + y_n \qquad x_0 = 0, \ y_0 = 1, \ h = 0.1$

$\Rightarrow \boxed{y_{n+1} \approx h(3x_n^2 y_n + x_n^5) + y_n}$

USING THE ABOVE FORMULA TWICE

$\Rightarrow y_1 \approx h[3 \times x_0^2 \times y_0 + x_0^5] + y_0$

$y_1 \approx 0.1[3 \times 0^2 \times 1 + 0^5] + 1$

$y_1 \approx 1$

$\Rightarrow y_2 \approx h[3 \times x_1^2 \times y_1 + x_1^5] + y_1$

$y_2 \approx 0.1[3 \times 0.1^2 \times 1 + 0.1^5] + 1$

$y_2 \approx 1.003001$

b) WRITE THE O.D.E IN THE USUAL ORDER

$\Rightarrow \frac{dy}{dx} - 3x^2 y = x^5$

INTEGRATING FACTOR $= e^{\int -3x^2 \, dx} = e^{-x^3}$

$\Rightarrow \frac{d}{dx}\left(y e^{-x^3}\right) = x^5 e^{-x^3}$

$\Rightarrow y e^{-x^3} = \int x^5 e^{-x^3} \, dx$

$\Rightarrow y e^{-x^3} = \int x^3 (x^2 e^{-x^3}) \, dx$

INTEGRATION BY PARTS

$\Rightarrow y e^{-x^3} = -\frac{1}{3}x^3 e^{-x^3} - \int -x^2 e^{-x^3} \, dx$

$\Rightarrow y e^{-x^3} = -\frac{1}{3}x^3 e^{-x^3} + \int x^2 e^{-x^3} \, dx$

$\Rightarrow y e^{-x^3} = -\frac{1}{3}x^3 e^{-x^3} - \frac{1}{3}e^{-x^3} + A$

$\Rightarrow \boxed{y = Ae^{x^3} - \frac{1}{3}x^3 - \frac{1}{3}}$

x^3	$3x^2$
$-\frac{1}{3}e^{-x^3}$	$x^2 e^{-x^3}$

APPLY CONDITIONS $x=0, \ y=1$

$\Rightarrow 1 = A - \frac{1}{3}$

$\Rightarrow A = \frac{4}{3}$

$\Rightarrow y = \frac{1}{3}\left[4e^{x^3} - x^3 - 1\right]$

FINALLY APPLY THE SOLUTION AT $x = 0.2$

$y\big|_{x=0.2} = \frac{1}{3}\left[4 \times e^{0.2^3} - 0.2^3 - 1\right] = 1.008042781\ldots$

≈ 1.008

2^{nd} order O.D.E.s

Question 1 (**+)

The differential equation

$$\frac{d^2y}{dx^2} = \frac{x}{y^2} + \frac{1}{y}, \quad y \neq 0,$$

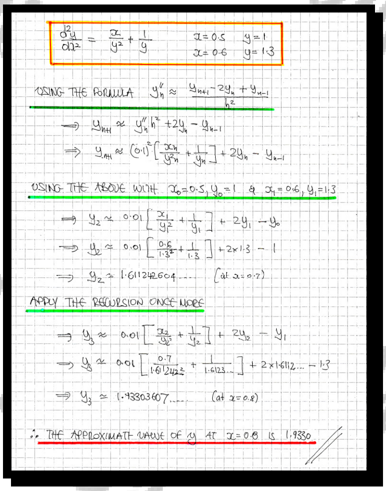
is to be solved numerically subject to the conditions $y(0.5) = 1$ and $y(0.6) = 1.3$.

Use the approximation

$$y_n' \approx \frac{y_{n+1} - 2y_n + y_{n-1}}{h^2}, \quad h = 0.1,$$

to find, correct to 4 decimal places the value of y at $x = 0.8$.

$\boxed{}$, $\boxed{y(0.8) \approx 1.9330}$



$$\boxed{\frac{d^2y}{dx^2} = \frac{x}{y^2} + \frac{1}{y} \qquad \begin{matrix} x = 0.5 & y = 1 \\ x = 0.6 & y = 1.3 \end{matrix}}$$

USING THE FORMULA $y_n'' \approx \frac{y_{n+1} - 2y_n + y_{n-1}}{h^2}$

$\Rightarrow y_{n+1} \approx y_n'' h^2 + 2y_n - y_{n-1}$

$\Rightarrow y_{n+1} \approx (0.1)^2 \left[\frac{x_n}{y_n^2} + \frac{1}{y_n}\right] + 2y_n - y_{n-1}$

USING THE ABOVE WITH $x_0 = 0.5, y_0 = 1$ & $x_1 = 0.6, y_1 = 1.3$

$\Rightarrow y_2 \approx 0.01 \left[\frac{x_1}{y_1^2} + \frac{1}{y_1}\right] + 2y_1 - y_0$

$\Rightarrow y_2 \approx 0.01 \left[\frac{0.6}{1.3^2} + \frac{1}{1.3}\right] + 2 \times 1.3 - 1$

$\Rightarrow y_2 \approx 1.611242604\ldots$ (at $x = 0.7$)

APPLY THE RECURSION ONCE MORE

$\Rightarrow y_3 \approx 0.01 \left[\frac{x_2}{y_2^2} + \frac{1}{y_2}\right] + 2y_2 - y_1$

$\Rightarrow y_3 \approx 0.01 \left[\frac{0.7}{1.611242\ldots^2} + \frac{1}{1.6123\ldots}\right] + 2 \times 1.6112\ldots - 1.3$

$\Rightarrow y_3 \approx 1.93303607\ldots$ (at $x = 0.8$)

$\therefore$ THE APPROXIMATE VALUE OF y AT $x = 0.8$ IS 1.9330

Question 2 (***)

The differential equation

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} + y^3 = 0, \quad y \neq 0,$$

is to be solved numerically subject to the conditions $y(2) = 3$ and $y(2.1) = 4$.

Use the following approximations

$$\left(\frac{d^2y}{dx^2}\right)_{n+1} \approx \frac{y_{n+2} - 2y_{n+1} + y_n}{h^2}, \quad \left(\frac{dy}{dx}\right)_{n+1} \approx \frac{y_{n+2} - y_n}{2h}, \qquad h = 0.1,$$

to find, correct to 2 decimal places the value of y at $x = 2.2$.

$\boxed{}$, $\boxed{y(2.2) \approx 4.30}$

$$\boxed{\frac{d^2y}{dx^2} + \frac{dy}{dx} + y^3 = 0 \qquad y(2) = 3, \quad y(2.1) = 4}$$

USING THE FORMULAS GIVEN

$$\left.\begin{aligned} \left(\frac{d^2y}{dx^2}\right)_{n+1} &\approx \frac{y_{n+2} - 2y_{n+1} + y_n}{h^2} \\ \left(\frac{dy}{dx}\right)_{n+1} &\approx \frac{y_{n+2} - y_n}{2h} \end{aligned}\right\}$$

SUB INTO THE O.D.E

$$\frac{y_{n+2} - 2y_{n+1} + y_n}{h^2} + \frac{y_{n+2} - y_n}{2h} + y_{n+1}^3 = 0$$

HERE $x_0 = 2$, $y_0 = 3$, $x_1 = 2.1$ $y_1 = 4$, $h = 0.1$

$$\Rightarrow \frac{y_2 - 2y_1 + y_0}{h^2} + \frac{y_2 - y_0}{2h} + y_1^3 = 0$$

$$\Rightarrow \frac{y_2 - 8 + 3}{0.01} + \frac{y_2 - 3}{0.2} + 64 = 0$$

$$\Rightarrow (y_2 - 5) \times 100 + (y_2 - 3) \times 5 + 64 = 0$$

$$\Rightarrow 105 y_2 = 451$$

$$\Rightarrow y_2 \simeq 4.295238\ldots$$

$\therefore$ THE APPROXIMATE VALUE OF y AT $x = 2.2$ IS 4.30

Question 3 (***+)

The curve with equation $y = f(x)$, satisfies

$$\frac{d^2y}{dx^2} = 1 + x\sin y,$$

subject to the boundary conditions $y = 1$, $\frac{dy}{dx} = 2$, at $x = 1$.

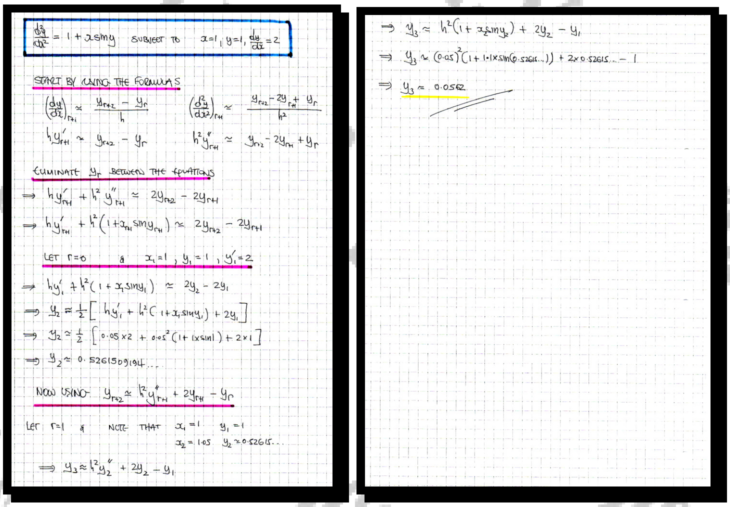
Use the approximations

$$\left(\frac{d^2y}{dx^2}\right)_{r+1} \approx \frac{y_{r+2} - 2y_{r+1} + y_r}{h^2} \quad \text{and} \quad \left(\frac{dy}{dx}\right)_{r+1} \approx \frac{y_{r+2} - y_r}{2h},$$

to determine, correct to 4 decimal places, the value of y at $x = 1.1$.

Use $h = 0.05$ throughout this question.

$\boxed{}$, $\boxed{y(1.6) \approx 2.85}$



$\boxed{\frac{d^2y}{dx^2} = 1 + x\sin y \quad \text{SUBJECT TO} \quad x=1, y=1, \frac{dy}{dx}=2}$

START BY USING THE FORMULAS

$\left(\frac{dy}{dx}\right)_{r+1} \approx \frac{y_{r+2} - y_r}{h}$ $\qquad$ $\left(\frac{d^2y}{dx^2}\right)_{r+1} \approx \frac{y_{r+2} - 2y_{r+1} + y_r}{h^2}$

$hy'_{r+1} \approx y_{r+2} - y_r$ $\qquad$ $h^2y''_{r+1} \approx y_{r+2} - 2y_{r+1} + y_r$

ELIMINATE y_r BETWEEN THE EQUATIONS

$\Rightarrow hy'_{r+1} + h^2y''_{r+1} \approx 2y_{r+2} - 2y_{r+1}$

$\Rightarrow hy'_{r+1} + h^2(1 + x_{r+1}\sin y_{r+1}) \approx 2y_{r+2} - 2y_{r+1}$

LET $r=0$ & $x_1 = 1$, $y_1 = 1$, $y'_1 = 2$

$\Rightarrow hy'_1 + h^2(1 + x_1\sin y_1) \approx 2y_2 - 2y_1$

$\Rightarrow y_2 \approx \frac{1}{2}\left[hy'_1 + h^2(1 + x_1\sin y_1) + 2y_1\right]$

$\Rightarrow y_2 \approx \frac{1}{2}\left[0.05 \times 2 + 0.05^2(1 + 1\times\sin 1) + 2\times 1\right]$

$\Rightarrow y_2 \approx 0.5261509194\ldots$

NOW USING $y_{r+2} \approx h^2y''_{r+1} + 2y_{r+1} - y_r$

LET $r=1$ & NOTE THAT $x_1 = 1$ $\quad y_1 = 1$

$x_2 = 1.05$ $\quad y_2 \approx 0.52615\ldots$

$\Rightarrow y_3 \approx h^2y''_2 + 2y_2 - y_1$

$\Rightarrow y_3 \approx h^2(1 + x_2\sin y_2) + 2y_2 - y_1$

$\Rightarrow y_3 \approx (0.05)^2(1 + 1.1\times\sin(0.52615\ldots)) + 2\times 0.52615\ldots - 1$

$\Rightarrow y_3 \approx 0.0562$

Question 4 (***+)

The curve with equation $y = f(x)$, satisfies

$$\frac{d^2y}{dx^2} = 4 + \sinh x \sinh y\,, \qquad y(1) = 1, \qquad \frac{dy}{dx}(1) = 1.$$

Use the approximations

$$\left(\frac{d^2y}{dx^2}\right)_n \approx \frac{y_{n+1} - 2y_n + y_{n-1}}{h^2} \qquad \text{and} \qquad \left(\frac{dy}{dx}\right)_n \approx \frac{y_{n+1} - y_{n-1}}{2h},$$

to determine, correct to 2 decimal places, the value of y at $x = 1.6$.

Use $h = 0.2$ throughout this question.

$\boxed{}$, $\boxed{y(1.6) \approx 2.85}$

$$\boxed{\frac{d^2y}{dx^2} = 4 + \sinh x \sinh y \text{, SUBJECT TO } y=1,\ \frac{dy}{dx}=1 \text{ AT } x=1}$$

START BY REARRANGING THE RESULTS FOR y_{n+1}

$$\Rightarrow y_n'' \simeq \frac{y_{n+1} - 2y_n + y_{n-1}}{h^2}$$
$$\Rightarrow h^2 y_n'' \simeq y_{n+1} - 2y_n + y_{n-1}$$
$$\Rightarrow y_{n+1} \approx h^2 y_n'' + 2y_n - y_{n-1}$$

$$\Rightarrow y_n' \simeq \frac{y_{n+1} - y_{n-1}}{2h}$$
$$\Rightarrow 2h y_n' \approx y_{n+1} - y_{n-1}$$
$$\Rightarrow y_{n+1} \simeq 2h y_n' + y_{n-1}$$

ADDING THE ABOVE EXPRESSION YIELDS

$$\Rightarrow 2y_{n+1} \simeq h^2 y_n'' + 2h y_n' + 2y_n$$
$$\Rightarrow 2y_1 \simeq h^2 y_0'' + 2h y_0' + 2y_0$$

$n=0$

$$\Rightarrow 2y_1 \simeq h^2(4 + \sinh x_0 \sinh y_0) + 2h y_0' + 2y_0$$

NOW AT $x_0 = 1$, $y_0 = 1$ & $y_0' = 1$ (GIVEN)

$$\Rightarrow 2y_1 \simeq 0.2^2(4 + \sinh 1 \sinh 1) + 2 \times 0.2 \times 1 + 2 \times 1$$
$$\Rightarrow 2y_1 \simeq 0.2152439\ldots + 0.4 + 2$$
$$\Rightarrow 2y_1 \simeq 2.6152439\ldots$$
$$\Rightarrow y_1 \approx 1.3076621957\ldots$$

VALUE OF y AT $x_1 = 1.2$

FINALLY USING $y_{n+1} \simeq h^2 y_n'' + 2y_n - y_{n-1}$

$$\Rightarrow y_{n+1} \simeq h^2(4 + \sinh x_n \sinh y_n) + 2y_n - y_{n-1}$$
$$\Rightarrow y_2 \simeq h^2[4 + \sinh x_1 \sinh y_1] + 2y_1 - y_0$$
$$\simeq 0.2^2[4 + \sinh(1.2)\sin(1.307\ldots)] + 2 \times 1.307\ldots - 1$$
$$\approx 1.878784924\ldots$$
$$\Rightarrow y_3 \simeq h^2[4 + \sinh(x_2)\sinh(y_2)] + 2y_2 - y_1$$
$$\simeq 0.2^2[4 + \sinh(1.4)\sinh(1.8787\ldots)] + 2 \times 1.8787\ldots - 1.307\ldots$$
$$\simeq 2.853382923$$

$\therefore$ THE VALUE OF y AT $x = 1.6$ IS APPROX 2.85

Question 5 (***+)

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} + x^3 = 0, \qquad y(0) = 1, \qquad \frac{dy}{dx}(0) = 2.$$

a) Use the approximation formulae

$$\left(\frac{d^2y}{dx^2}\right)_{n+1} \approx \frac{y_{n+2} - 2y_{n+1} + y_n}{h^2} \quad \text{and} \quad \left(\frac{dy}{dx}\right)_{n+1} \approx \frac{y_{n+2} - y_n}{2h},$$

to determine, correct to 2 decimal places, the value of y at $x = 0.1$.

Use $h = 0.1$ throughout this part of the question.

b) By differentiating the differential equation given above, find the first 4 terms of the infinite convergent series expansion of y, in ascending powers of x, and use it to find, correct to 2 decimal places, another approximation for the value of y at $x = 0.1$.

$\boxed{}$, $\boxed{y(0.1) \approx 1.19}$

$$\boxed{\frac{d^2y}{dx^2} + \frac{dy}{dx} + x^3 = 0 \quad \text{SUBJECT TO } y=1,\ \frac{dy}{dx} = 2 \text{ AT } x = 0}$$

a) USING THE FORMULAS GIVEN

$$\left(\frac{dy}{dx}\right)_{n+1} \approx \frac{y_{n+2} - y_n}{2h} \qquad \left(\frac{d^2y}{dx^2}\right)_{n+1} \approx \frac{y_{n+2} - 2y_{n+1} + y_n}{h^2}$$

$$y'_{n+1} \approx \frac{y_{n+2} - y_n}{2h} \qquad y''_{n+1} \approx \frac{y_{n+2} - 2y_{n+1} + y_n}{h^2}$$

$$y_{n+2} - y_n \approx 2hy'_{n+1} \qquad y_{n+2} - 2y_{n+1} + y_n \approx h^2 y''_{n+1}$$

ADDING THE TWO EQUATIONS ABOVE GIVES

$\Rightarrow 2y_{n+2} - 2y_{n+1} \approx 2hy'_{n+1} + h^2 y''_{n+1}$

$\Rightarrow 2y_{n+2} = 2y_{n+1} + 2hy'_{n+1} + h^2 y''_{n+1}$

LET $n=0$ & NOTE $x_1 = 0$, $y_1 = 1$, $y'_1 = 2$

$\Rightarrow 2y_2 = 2y_1 + 2hy'_1 + h^2 y''_1$

$\Rightarrow 2y_2 = 2y_1 + 2hy'_1 + h^2[-y'_1 - x_1^3]$ ← FROM THE O.D.E DIRECTLY

$\Rightarrow 2y_2 = 2y_1 + 2hy'_1 - h^2 y'_1 - h^2 x_1^3$

$\Rightarrow 2y_2 = 2 \times 1 + 2 \times 0.1 \times 2 - (0.1)^2 \times 2 - (0.1)^2 \times 0^3$

$\Rightarrow 2y_2 = 2 + 0.4 - 0.02 - 0$

$\Rightarrow 2y_2 = 2.38$

$\Rightarrow y_2 = 1.19$

b) WRITE O.D.E MORE COMPACTLY

$$y'' + y' + x^3 = 0$$

$x_0 = 0$
$y_0 = 1$
$y'_0 = 2$
$y''_0 = -2$

DIRECTLY FROM THE O.D.E

DIFFERENTIATE W.R.T x

$\Rightarrow y''' + y'' + 3x^2 = 0$

$\Rightarrow y'''_0 + y''_0 + 3x_0^2 = 0$

$\Rightarrow y'''_0 - 2 + 3 \times 0^2 = 0$

$\Rightarrow y'''_0 = 2$

HENCE WE HAVE

$\Rightarrow y = y_0 + xy'_0 + \frac{x^2}{2!}y''_0 + \frac{x^3}{3!}y'''_0 + O(x^4)$

$\Rightarrow y = 1 + 2x - x^2 + \frac{1}{3}x^3 + O(x^4)$

$\Rightarrow y(0.1) = 1 + 2(0.1) - (0.1)^2 + \frac{1}{3}(0.1)^3 + \dots$

$\Rightarrow y(0.1) \approx 1.190333\dots$

INDEED AGREES TO 2 d.p TO THE ANSWER OF (a)

Question 6 (****)

The curve with equation $y = f(x)$, satisfies

$$\frac{d^2y}{dx^2} = x + y + 2, \quad y(0) = 0, \quad \frac{dy}{dx}(0) = 1.$$

a) Use Taylor expansions to justify the validity of the following approximations.

$$\left(\frac{d^2y}{dx^2}\right)_n \approx \frac{y_{n+1} - 2y_n + y_{n-1}}{h^2} \quad \text{and} \quad \left(\frac{dy}{dx}\right)_n \approx \frac{y_{n+1} - y_{n-1}}{2h}.$$

b) Hence show that $y(0.1) \approx 0.11$

c) Determine, correct to 4 decimal places, the value of $y(0.2)$ and $y(0.3)$.

$\boxed{}$, $\boxed{y(0.2) \approx 0.2421...}$, $\boxed{y(0.3) \approx 0.3986...}$

$\boxed{\dfrac{d^2y}{dx^2} = x + y + 2, \quad x = 0,\ y = 0,\ \dfrac{dy}{dx} = 1}$

a) BY CONSIDERING TAYLOR EXPANSIONS WE OBTAIN

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + O(h^3)$$
$$f(x-h) = f(x) - hf'(x) + \frac{h^2}{2!}f''(x) + O(h^3)$$

ADDING THE EXPRESSIONS

$$f(x+h) + f(x-h) = 2f(x) + h^2f''(x) + O(h^4)$$
$$f''(x) \approx \frac{f(x+h) - 2f(x) + f(x-h)}{h^2}$$
$$y''_n \approx \frac{y_{n+1} - 2y_n - y_{n-1}}{h^2}$$

AS REQUIRED

SUBTRACTING THE EXPRESSIONS

$$f(x+h) - f(x-h) = 2hf'(x) + O(h^3)$$
$$f'(x) \approx \frac{f(x+h) - f(x-h)}{2h}$$
$$y'_n \approx \frac{y_{n+1} - y_{n-1}}{2h}$$

AS REQUIRED

b) PROCEED AS FOLLOWS

$$\left.\begin{aligned} y_{n+1} + y_{n-1} &\approx 2y_n + h^2y''_n \\ y_{n+1} - y_{n-1} &\approx 2hy'_n \end{aligned}\right\} \text{ADDING}$$

$$\Rightarrow 2y_{n+1} = 2y_n + h^2y''_n + 2hy'_n \qquad n=0$$
$$\Rightarrow 2y_1 = 2y_0 + h^2y''_0 + 2hy'_0$$
$$\Rightarrow 2y_1 = 2y_0 + h^2[x_0 + y_0 + 2] + 2hy'_0$$

NOW $x_0 = 0,\ y_0 = 0,\ y'_0 = 1,\ h = 0.1$

$$\Rightarrow 2y_1 = 2\times 0 + (0.1)^2[0+0+2] + 2(0.1)\times 1$$
$$\Rightarrow 2y_1 = 0.22$$
$$\Rightarrow y_1 = 0.11$$

c) FINALLY USING

$$y_{n+1} \approx 2y_n + h^2y''_n - y_{n-1}$$
$$\rightarrow y_{n+1} \approx 2y_n - y_{n-1} + h^2[x_n + y_n + 2]$$

$x_0 = 0 \quad y_0 = 0$
$x_1 = 0.1 \quad y_1 = 0.11$
$x_2 = 0.2$

ETC

- $y_2 \approx 2y_1 - y_0 + h^2(x_1 + y_1 + 2)$
  $y_2 \approx 2(0.11) - 0 + (0.1)^2[0.1 + 0.11 + 2]$
  $y_2 \approx 0.2421$

- $y_3 \approx 2y_2 - y_1 + h^2(x_2 + y_2 + 2)$
  $y_3 \approx 2(0.2421) - 0.11 + (0.1)^2[0.2 + 0.2421 + 2]$
  $y_3 \approx 0.398621 \approx 0.3986$

Question 7 (***+)

$$\frac{d^2y}{dx^2}=2+x^2y+y^2=0\,,\qquad y(0)=1\,,\qquad \frac{dy}{dx}(0)=1\,.$$

a) Use the approximation formulae

$$y_{n+2}\approx 2y_{n+1}+h^2y''_{n+1}-y_n \qquad \text{and} \qquad y_{n+2}\approx 2hy'_{n+1}+y_n\,,$$

to find, correct to 3 decimal places, the value of y at $x=0.1$ and $x=0.2$.

Use $h=0.1$ throughout this part of the question.

b) By differentiating the differential equation given above, determine the first 5 terms of the infinite convergent series expansion of y, in ascending powers of x, and use it to find, correct to 3 decimal places, approximations for the value of y at $x=0.1$ and $x=0.2$.

[ ] , $\boxed{y(0.1)\approx 1.115}$, $\boxed{y(0.2)\approx 1.263}$

$\frac{d^2y}{dx^2}=2+x^2y+y^2$ SUBJECT TO $x=0,\ y=1,\ \frac{dy}{dx}=1$

a) $y_{n+2}\approx 2y_{n+1}+h^2y''_{n+1}-y_n$ & $y_{n+2}\approx 2hy'_{n+1}+y_n$

HERE $x_1=0,\ y_1=1,\ y'_1=1$

$y_2\approx 2hy'_1+y_0$

$y_2\approx 2y_1+h^2y''_1-y_0$

ADDING TO ELIMINATE y_0

$\Rightarrow 2y_2\approx 2hy'_1+2y_1+h^2y''_1$

$\Rightarrow 2y_2\approx 2hy'_1+2y_1+h^2\left[2+x_1^2y_1+y_1^2\right]$

$\Rightarrow 2y_2\approx 2\times 0.1\times 1+2\times 1+(0.1)^2\left[2+0+1^2\right]$

$\Rightarrow 2y_2\approx 0.2+2+0.03$

$\Rightarrow 2y_2\approx 2.23$

$\Rightarrow y_2\approx 1.115$

NOW $x_1=0 \quad y_1=1$

$x_2=0.1 \quad y_2=1.115$

$\Rightarrow y_3\approx 2y_2+h^2y''_2-y_1$

$\Rightarrow y_3\approx 2y_2+h^2\left[2+x_2^2y_2+y_2^2\right]-y_1$

$\Rightarrow y_3\approx 2\times 1.115+(0.1)^2\left[2+0.1^2\times 1.115+1.115^2\right]-1$

$\Rightarrow y_3\approx 2.23+0.03254375-1$

$\Rightarrow y_3\approx 1.26254375$

$\Rightarrow y_3\approx 1.263$

b) WRITE THE O.D.E IN "COMPACT" NOTATION FOR DIFFERENTIATION

$y''=2+x^2y+y^2$

$x_0=0$

$y_0=1$

$y'_0=1$

$y''_0=3$ (FROM THE O.D.E.)

DIFFERENTIATE WITH RESPECT TO x

$y'''=0+2xy+x^2y'+2yy'$

$y'''_0=2x_0y_0+x_0^2y'_0+2y_0y'_0$

$y'''_0=0+0+2\times 1\times 1$

$y'''_0=2$

DIFFERENTIATE W.R.T x AGAIN

$y''''=2y+2xy'+2xy'+x^2y''+2y'y'+2yy''$

$y''''=2y+4xy'+x^2y''+2(y')^2+2yy''$

$y''''_0=2y_0+4x_0y'_0+x_0^2y''_0+2(y'_0)^2+2y_0y''_0$

$y''''_0=2\times 1+0+0+2\times 1^2+2\times 1\times 3$

$y''''_0=10$

FINALLY WE HAVE

$y=y_0+xy'_0+\frac{x^2}{2!}y''_0+\frac{x^3}{3!}y'''_0+\frac{x^4}{4!}y''''_0+O(x^5)$

$y=1+x+\frac{3}{2}x^2+\frac{1}{3}x^3+\frac{5}{12}x^4+O(x^5)$

$y(0.1)=1+0.1+\frac{3}{2}(0.1)^2+\frac{1}{3}(0.1)^3+\frac{5}{12}(0.1)^4+\dots\approx 1.115375$

$y(0.2)=1+0.2+\frac{3}{2}(0.2)^2+\frac{1}{3}(0.2)^3+\frac{5}{12}(0.2)^4+\dots\approx 1.263333\dots$

INDEED THE SAME AT 3 D.P.

Question 8 (****)

$$\frac{d^2y}{dx^2} = 1 + y\frac{dy}{dx}, \qquad y(0) = 1, \qquad \frac{dy}{dx}(0.1) = 1.1.$$

a) Use the approximation formulae

$$\left(\frac{d^2y}{dx^2}\right)_{r+1} \approx \frac{y_{r+2} - 2y_{r+1} + y_r}{h^2} \quad \text{and} \quad \left(\frac{dy}{dx}\right)_{r+1} \approx \frac{y_{r+2} - y_r}{2h},$$

to show that

$$y_{r+2} \approx \frac{(4 - hy_r)\,y_{r+1} - 2\left(y_r - h^2\right)}{2 - hy_{r+1}}.$$

determine, correct to 2 decimal places, the value of y at $x = 0.1$.

b) Use the result shown in part **(a)**, with $h = 0.1$, to find the value of y at $x = 0.3$, correct to 3 decimal places.

$\boxed{}$, $\boxed{y(0.3) \approx 1.371}$

$$\boxed{\frac{d^2y}{dx^2} = 1 + y\frac{dy}{dx} \quad \text{WITH} \quad y(0) = 1 \quad \& \quad y(0.1) = 1.1}$$

a) USING THE APPROXIMATIONS

$$\left(\frac{d^2y}{dx^2}\right)_{r+1} \approx \frac{y_{r+2} - 2y_{r+1} + y_r}{h^2} \quad \& \quad \left(\frac{dy}{dx}\right)_{r+1} \approx \frac{y_{r+2} - y_r}{2h}$$

$$y''_{r+1} \approx \frac{y_{r+2} - 2y_{r+1} + y_r}{h^2} \quad \& \quad y'_{r+1} \approx \frac{y_{r+2} - y_r}{2h}$$

SUBSTITUTE INTO THE O.D.E

$$\frac{y_{r+2} - 2y_{r+1} + y_r}{h^2} = 1 + y_{r+1}\left[\frac{y_{r+2} - y_r}{2h}\right]$$

MULTIPLY THROUGH BY $2h^2$ AND TIDY

$$2(y_{r+2} - 2y_{r+1} + y_r) = 2h^2 + hy_{r+1}\left[y_{r+2} - y_r\right]$$

$$2y_{r+2} - 4y_{r+1} + 2y_r = 2h^2 + hy_{r+1}y_{r+2} - hy_r y_{r+1}$$

$$2y_{r+2} - hy_{r+1}y_{r+2} = 2h^2 - hy_r y_{r+1} + 4y_{r+1} - 2y_r$$

$$y_{r+2}(2 - hy_{r+1}) = y_{r+1}(4 - hy_r) - 2y_r + 2h^2$$

$$y_{r+2} = \frac{y_{r+1}(4 - hy_r) - 2(y_r - h^2)}{2 - hy_{r+1}}$$

b) NOW USING THE FORMULA

$x_1 = 0 \quad y_1 = 1$
$x_2 = 0.1 \quad y_2 = 1.1$
$x_3 = 0.2$

- $r=1$

$$\Rightarrow y_3 = \frac{y_2(4 - hy_1) - 2(y_1 - h^2)}{2 - hy_2}$$

$$\Rightarrow y_3 = \frac{1.1(4 - 0.1 \times 1) - 2(1 - 0.1^2)}{2 - 0.1 \times 1.1}$$

$$\Rightarrow y_3 = \frac{4.29 - 1.98}{1.89}$$

$$\Rightarrow y_3 \approx 1.222\ldots$$

- $r=2$

$$\Rightarrow y_4 = \frac{y_3(4 - hy_2) - 2(y_2 - h^2)}{2 - hy_3}$$

$$\Rightarrow y_4 = \frac{1.222\ldots(4 - 0.1 \times 1.1) - 2(1.1 - 0.1^2)}{2 - 0.1 \times 1.222\ldots}$$

$$\Rightarrow y_4 = \frac{4.75444\ldots - 2.18}{1.87777\ldots}$$

$$\Rightarrow y_4 \approx 1.371$$

Question 9 (****)

The curve with equation $x = f(t)$, satisfies

$$\frac{d^2x}{dt^2} = -x, \qquad x(0) = 0, \qquad \frac{dx}{dt}(0) = 1.$$

Use Euler's method, with a step of 0.1, to find the approximate value of x at $t = 0.5$.

$\boxed{}$, $\boxed{x(0.5) \approx 0.480...}$

$$\boxed{\frac{d^2x}{dt^2} = -x \text{, SUBJECT TO THE CONDITIONS } \begin{array}{l} t_0 = 0 \\ x_0 = 0 \\ \dot{x}_0 = x_0' = 1 \end{array}}$$

USE THE TWO FORMULAS

$$\boxed{\left(\frac{dy}{dx}\right)_0 \approx \frac{y_1 - y_{-1}}{2h}} \quad \& \quad \boxed{\frac{d^2y}{dx^2} \approx \frac{y_1 - 2y_0 + y_{-1}}{h^2}}$$

HERE WITH OUR VARIABLES, SINCE $x = x(t)$

$$\boxed{x_0' \approx \frac{x_1 - x_{-1}}{2h}} \quad \& \quad \boxed{x_0'' \approx \frac{x_1 - 2x_0 + x_{-1}}{h^2}}$$

FROM THE O.D.E ITSELF $x_0'' = -x_0 = 0$

HENCE WE OBTAIN FROM EACH FORMULA, WITH $h = 0.1$

$$1 = \frac{x_1 - x_{-1}}{2} \quad \& \quad 0 = \frac{x_1 - 0 + x_{-1}}{(0.1)^2}$$

$$x_1 - x_{-1} = 0.2 \qquad\qquad x_1 + x_{-1} = 0$$

$$2x_1 = 0.2$$

$$x_1 = 0.1 \quad \& \quad x_{-1} = -0.1$$

HENCE THE SECOND DERIVATIVE APPROXIMATION FORMULA YIELDS

$$\Rightarrow x_0'' = \frac{x_1 - 2x_0 + x_{-1}}{h^2}$$

$$\Rightarrow -x_0 = \frac{x_1 - 2x_0 + x_{-1}}{h^2}$$

$$\Rightarrow -h^2 x_0 = x_1 - 2x_0 + x_{-1}$$

$$\Rightarrow x_1 = 2x_0 - h^2 x_0 - x_{-1}$$

$$\Rightarrow \boxed{x_1 = x_0(2 - h^2) - x_{-1}}$$

OR AS A RECURRENCE RELATION

$$\Rightarrow x_{n+2} = x_{n+1}(2 - h^2) - x_n, \text{ WITH } \begin{array}{l} x_0 = 0 \\ x_1 = 0.1 \\ h = 0.1 \end{array}$$

$$\Rightarrow \boxed{x_{n+2} = 1.99x_{n+1} - x_n}$$

APPLY THE RESULT RECURSIVELY

$$\Rightarrow x_2 = 1.99x_1 - x_0$$
$$\Rightarrow x_2 = 0.199 - 0$$
$$\Rightarrow x_2 = 0.199$$

$$\Rightarrow x_3 = 1.99x_2 - x_1$$
$$\Rightarrow x_3 = 1.99(0.199) - 0.1$$
$$\Rightarrow x_3 = 0.29601$$

$$\Rightarrow x_4 = 1.99x_3 - x_2$$
$$\Rightarrow x_4 = 1.99(0.29601) - 0.199$$
$$\Rightarrow x_4 = 0.3900599...$$

$$\Rightarrow x_5 = 1.99x_4 - x_3$$
$$\Rightarrow x_5 = 1.99(0.3900599...) - 0.29601...$$
$$\Rightarrow x_5 = 0.4802090...$$

$\therefore$ AT $t = 0.5$ $\quad x \simeq 0.480$