INTEGRATION MIX

1. $\displaystyle\int 6(4x+3)^{\frac{1}{2}}\,dx = (4x+3)^{\frac{3}{2}} + C$

$\int 6(4x+3)^{\frac{1}{2}}\,dx =$ BY INSPECTION $= \frac{6}{6}(4x+3)^{\frac{3}{2}} + C$
(LINEAR ADJUSTMENT)
$= (4x+3)^{\frac{3}{2}} + C$

2. $\displaystyle\int \frac{12}{\sqrt{3x+1}}\,dx = 8(3x+1)^{\frac{1}{2}} + C$

$\int \frac{12}{\sqrt{3x+1}}\,dx = \int 12(3x+1)^{-\frac{1}{2}}\,dx =$ BY INSPECTION
(LINEAR ADJUSTMENT)
$= \frac{12}{\frac{3}{2}}(3x+1)^{\frac{1}{2}} + C = 8(3x+1)^{\frac{1}{2}} + C$

3. $\displaystyle\int \frac{12x}{\sqrt{4x-1}}\,dx = \frac{1}{2}(4x-1)^{\frac{3}{2}} + \frac{3}{2}(4x-1)^{\frac{1}{2}} + C$

$\int \frac{12x}{\sqrt{4x-1}}\,dx = \dots$ BY SUBSTITUTION ...

$u = 4x-1$
$\frac{du}{dx} = 4$
$dx = \frac{du}{4}$
$4x = u+1$

$= \int \frac{12x}{u^{\frac{1}{2}}}\left(\frac{du}{4}\right) = \int \frac{12x}{4u^{\frac{1}{2}}}\,du = \int \frac{3(4x)}{4u^{\frac{1}{2}}}\,du$

$= \int \frac{3(u+1)}{4u^{\frac{1}{2}}}\,du = \frac{3}{4}\int \frac{u+1}{u^{\frac{1}{2}}}\,du$

$= \frac{3}{4}\int \frac{u}{u^{\frac{1}{2}}} + \frac{1}{u^{\frac{1}{2}}}\,du = \frac{3}{4}\int u^{\frac{1}{2}} + u^{-\frac{1}{2}}\,du$

$= \frac{3}{4}\left[\frac{2}{3}u^{\frac{3}{2}} + 2u^{\frac{1}{2}}\right] + C = \frac{1}{2}u^{\frac{3}{2}} + \frac{3}{2}u^{\frac{1}{2}} + C$

$= \frac{1}{2}(4x-1)^{\frac{3}{2}} + \frac{3}{2}(4x-1)^{\frac{1}{2}} + C$

ALTERNATIVE SUBSTITUTION

$u = (4x-1)^{\frac{1}{2}}$
$u^2 = 4x-1$
$2u\frac{du}{dx} = 4$
$2u\,du = 4dx$
$dx = \frac{u\,du}{2}$
$4x = u^2+1$

$\int \frac{12x}{\sqrt{4x-1}}\,dx = \dots = \int \frac{12x}{u}\left(\frac{u\,du}{2}\right)$

$= \int \frac{12xu}{2u}\,du = \int \frac{3(4x)}{2}\,du$

$= \frac{3}{2}\int \underbrace{(u^2+1)}_{4x}\,du = \frac{3}{2}\left[\frac{1}{3}u^3 + u\right] + C$

$= \frac{1}{2}u^3 + \frac{3}{2}u + C$

$= \frac{1}{2}(4x-1)^{\frac{3}{2}} + \frac{3}{2}(4x-1)^{\frac{1}{2}} + C$

4. $\displaystyle\int \frac{12}{4x+1}\,dx = 3\ln|4x+1| + C$

$\int \frac{12}{4x+1}\,dx = \dots$ RECOGNIZING A STANDARD LOG DIFFERENTIATION
(LINEAR ADJUSTMENT)
$= \frac{12}{4}\ln|4x+1| + C = 3\ln|4x+1| + C$

5. $\displaystyle\int \frac{12x}{4x+1}\,dx = 3x - \frac{3}{4}\ln|4x+1| + C$

$$\int \frac{12x}{4x+1}\,dx = \dots \text{SUBSTITUTION} \dots = \int \frac{12x}{u}\left(\frac{du}{4}\right)$$

$u = 4x+1$

$\frac{du}{dx} = 4$

$dx = \frac{du}{4}$

$4x = u-1$

$$= \int \frac{12x}{4u}\,du = \int \frac{3(4x)}{4u}\,du$$

$$= \int \frac{3(u-1)}{4u}\,du = \frac{3}{4}\int \frac{u-1}{u}\,du$$

$$= \frac{3}{4}\int \frac{u}{u} - \frac{1}{u}\,du = \frac{3}{4}\int 1 - \frac{1}{u}\,du = \frac{3}{4}\Big[u - \ln|u|\Big] + C$$

$$= \frac{3}{4}u - \frac{3}{4}\ln|u| + C = \frac{3}{4}(4x+1) - \frac{3}{4}\ln|4x+1| + C$$

ALTERNATIVE BY MANIPULATION

$$\int \frac{12x}{4x+1}\,dx = \int \frac{3(4x+1)-3}{4x+1}\,dx$$

$$= \int \frac{3\cancel{(4x+1)}}{\cancel{4x+1}} - \frac{3}{4x+1}\,dx$$

$$= \int 3 - \frac{3}{4x+1}\,dx$$

$$= 3x - \frac{3}{4}\ln|4x+1| + C$$

6. $\displaystyle\int (\sin x + \cos x)^2\,dx = x - \frac{1}{2}\cos 2x + C$

$$\int (\sin x + \cos x)^2\,dx = \int \sin^2 x + 2\sin x\cos x + \cos^2 x\,dx$$

$$= \int 1 + \sin 2x\,dx$$

$$= x - \frac{1}{2}\cos 2x + C$$

7. $\displaystyle\int (2\sin x + \operatorname{cosec} x)^2\,dx = 6x - \sin 2x - \cot x + C$

$$\int (2\sin x + \operatorname{cosec} x)^2\,dx = \int 4\sin^2 x + 4\sin x\operatorname{cosec} x + \operatorname{cosec}^2 x\,dx$$

$$= \int 4\left(\frac{1}{2} - \frac{1}{2}\cos 2x\right) + 4\cancel{\sin x}\left(\frac{1}{\cancel{\sin x}}\right) + \operatorname{cosec}^2 x\,dx$$

$$= \int 2 - 2\cos 2x + 4 + \operatorname{cosec}^2 x$$

$$= \int 6 - 2\cos 2x + \operatorname{cosec}^2 x$$

$$= 6x - \sin 2x - \cot x + C$$

8. $\displaystyle\int \frac{5}{(3x-1)(2x+1)}\,dx = \ln\left|\frac{3x-1}{2x+1}\right| + C$

$\displaystyle\int \frac{5}{(3x-1)(2x+1)}\,dx = \dots$ BY PARTIAL FRACTIONS ...

$$\frac{5}{(3x-1)(2x+1)} \equiv \frac{A}{3x-1} + \frac{B}{2x+1}$$
$$5 \equiv A(2x+1) + B(3x-1)$$

- IF $x = \frac{1}{3} \Rightarrow 5 = \frac{5}{3}A$
  $\Rightarrow \underline{A = 3}$
- IF $x = -\frac{1}{2} \Rightarrow 5 = -\frac{5}{2}B$
  $\Rightarrow \underline{B = -2}$

$$\dots = \int \frac{3}{3x-1} - \frac{2}{2x+1}\,dx = \ln|3x-1| - \ln|2x+1| + C$$
$$= \ln\left|\frac{3x-1}{2x+1}\right| + C$$

9. $\displaystyle\int 4x\sin 2x\,dx = -2x\cos 2x + \sin 2x + C$

$\displaystyle\int 4x\sin 2x\,dx$ ... INTEGRATION BY PARTS ...

$4x$	4
$-\frac{1}{2}\cos 2x$	$\sin 2x$

$$= -2x\cos 2x - \int -2\cos 2x\,dx$$
$$= -2x\cos 2x + \int 2\cos 2x\,dx$$
$$= -2x\cos 2x + \sin 2x + C$$

10. $\displaystyle\int \frac{7}{4x}\,dx = \frac{7}{4}\ln|x| + C$

$\displaystyle\int \frac{7}{4x}\,dx = \int \frac{7}{4}\times\frac{1}{x}\,dx = \dots$ BY INSPECTION ... $= \frac{7}{4}\ln|x| + C$

11. $\displaystyle\int \left(x + \frac{2}{x}\right)^2 dx = \frac{1}{3}x^3 + 4x - \frac{4}{x} + C$

$$\int \left(x + \frac{2}{x}\right)^2 dx = \int x^2 + 2\times \cancel{x} \times \frac{2}{\cancel{x}} + \frac{4}{x^2}\,dx$$
$$= \int x^2 + 4 + 4x^{-2}\,dx$$
$$= \frac{1}{3}x^3 + 4x - 4x^{-1} + C$$
$$= \frac{1}{3}x^3 + 4x - \frac{4}{x} + C$$

12. $\displaystyle\int \frac{10}{(1-4x)^{\frac{7}{2}}}\,dx = \frac{1}{(1-4x)^{\frac{5}{2}}} + C$

$$\int \frac{10}{(1-4x)^{\frac{7}{2}}}\,dx = \int 10(1-4x)^{-\frac{7}{2}} = \text{BY RECOGNITION}$$
$$= \frac{10}{10}(1-4x)^{-\frac{5}{2}} + C = (1-4x)^{-\frac{5}{2}} + C$$
$$= \frac{1}{(1-4x)^{5/2}} + C$$

13. $\displaystyle\int (1+2\cos x)\sin x\,dx = \left[\begin{array}{c} -\cos x - \frac{1}{2}\cos 2x + C \\ -\cos x + \sin^2 x + C \\ -\cos x - \cos^2 x + C \\ -\frac{1}{4}(1+2\cos x)^2 + C \end{array}\right]$

$$\int (1+2\cos x)\sin x\,dx = \int \sin x + 2\cos x\sin x\,dx$$
$$= \int \sin x + \sin 2x\,dx$$
$$= -\cos x - \frac{1}{2}\cos 2x + C$$

OR $\quad \dots = -\cos x + \sin^2 x + C$

OR $\quad \dots = -\cos x - \cos^2 x + C$

ALTERNATIVE BY REVERSE RULE RECOGNITION, OR SUBSTITUTION

$$\int (1+2\cos x)\sin x\,dx = \dots$$
$$= \int u\cancel{\sin x}\left(-\frac{du}{2\cancel{\sin x}}\right)$$
$$= \int -\frac{1}{2}u\,du$$
$$= -\frac{1}{4}u^2 + C$$
$$= -\frac{1}{4}(1+2\cos x)^2 + C$$

SUBSTITUTION

$u = 1 + 2\cos x$

$\frac{du}{dx} = -2\sin x$

$dx = -\frac{du}{2\sin x}$

14. $\displaystyle\int 2\cos^2 x - \cos\left(\tfrac{1}{2}x\right)\,dx = x + \tfrac{1}{2}\sin 2x - 2\sin\left(\tfrac{1}{2}x\right) + C$

$$\int 2\cos^2 x - \cos\tfrac{1}{2}x\,dx = \int 2\left(\tfrac{1}{2} + \tfrac{1}{2}\cos 2x\right) - \cos\tfrac{1}{2}x\,dx$$
$$= \int 1 + \cos 2x - \cos\tfrac{1}{2}x\,dx$$
$$= x + \tfrac{1}{2}\sin 2x - 2\sin\tfrac{1}{2}x + C$$

15. $\displaystyle\int \frac{14x}{x^2-4}\,dx = 7\ln\left|x^2-4\right| + C$

$$\int \frac{14x}{x^2-4}\,dx = 7\int \frac{2x}{x^2-4}\,dx = 7\ln|x^2-4| + C$$

$$\boxed{\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + C}$$

ALTERNATIVE BY SUBSTITUTION

$$\int \frac{14x}{x^2-4}\,dx = \dots \text{SUBSTITUTION} \dots = \int \frac{14x}{u}\left(\frac{du}{2x}\right)$$

$$= \int \frac{7}{u}\,du = 7\ln|u| + C$$

$$= 7\ln|x^2-4| + C$$

$$u = x^2-4 \qquad \frac{du}{dx} = 2x \qquad dx = \frac{du}{2x}$$

ALTERNATIVE BY PARTIAL FRACTIONS

FIRSTLY WE HAVE: $\displaystyle\frac{14x}{x^2-4} \equiv \frac{14x}{(x-2)(x+2)} \equiv \frac{A}{x-2} + \frac{B}{x+2}$

$$\boxed{14x \equiv A(x+2) + B(x-2)}$$

If $x=2$, $28 = 4A \Rightarrow A = 7$
If $x=-2$, $-28 = -4B \Rightarrow B = 7$

HENCE THE INTEGRAL BECOMES

$$\int \frac{14x}{x^2-4}\,dx = \int \frac{7}{x-2} + \frac{7}{x+2}\,dx = 7\ln|x-2| + 7\ln|x+2| + C$$

$$= 7\left[\ln|x-2| + \ln|x+2|\right] + C$$

$$= 7\ln|x^2-4| + C$$

16. $\displaystyle\int \frac{9}{2x^2+x-1}\,dx = 3\ln\left|\frac{2x-1}{x+1}\right| + C$

$$\int \frac{9}{2x^2+x-1}\,dx = \int \frac{9}{(2x-1)(x+1)}\,dx \dots \text{BY PARTIAL FRACTIONS}$$

$$\frac{9}{(2x-1)(x+1)} \equiv \frac{A}{2x-1} + \frac{B}{x+1}$$

$$\boxed{9 \equiv A(x+1) + B(2x-1)}$$

- If $x = -1$: $9 = -3B$, $B = -3$
- If $x = \frac{1}{2}$: $9 = \frac{3}{2}A$, $A = 6$

$$\dots = \int \frac{6}{2x-1} - \frac{3}{x+1}\,dx = 3\ln|2x-1| - 3\ln|x+1| + C$$

$$= 3\ln\left|\frac{2x-1}{x+1}\right| + C$$

17. $\displaystyle\int (8x+1)\mathrm{e}^{-2x}\,dx = -\tfrac{1}{2}(8x+1)\mathrm{e}^{-2x} - 2\mathrm{e}^{-2x} + C$

$$\int (8x+1)e^{-2x}\,dx = \dots \text{INTEGRATION BY PARTS}$$

$8x+1$	8
$-\frac{1}{2}e^{-2x}$	e^{-2x}

$$\dots = -\tfrac{1}{2}(8x+1)e^{-2x} - \int -4e^{-2x}\,dx$$

$$= -\tfrac{1}{2}(8x+1)e^{-2x} + \int 4e^{-2x}\,dx$$

$$= -\tfrac{1}{2}(8x+1)e^{-2x} - 2e^{-2x} + C$$

18. $\displaystyle\int \left(e^{2x}+e^{-x}\right)^2\,dx = \frac{1}{4}e^{4x}+2e^{x}-\frac{1}{2}e^{-2x}+C$

$$\int \left(e^{2x}+e^{-x}\right)^2 dx = \int \left(e^{2x}\right)^2+2\left(e^{2x}\right)\left(e^{-x}\right)+\left(e^{-x}\right)^2 dx$$
$$= \int e^{4x}+2e^{x}+e^{-2x}\,dx$$
$$= \frac{1}{4}e^{4x}+2e^{x}-\frac{1}{2}e^{-2x}+C$$

19. $\displaystyle\int \frac{6}{(3x+2)^3}\,dx = -\frac{1}{(3x+2)^2}+C$

$$\int \frac{6}{(3x+2)^3}\,dx = \int 6(3x+2)^{-3}\,dx = \text{BY INSPECTION}$$
$$= \frac{6}{-6}(3x+2)^{-2}+C = -\frac{1}{(3x+2)^2}+C$$

20. $\displaystyle\int \frac{6x}{(3x+2)^3}\,dx = \frac{2}{3}\left[\frac{1}{(3x+2)^2}-\frac{1}{3x+2}\right]+C$

$$\int \frac{6x}{(3x+2)^3}\,dx = \dots$$

BY SUBSTITUTION

$u = 3x+2$

$\frac{du}{dx} = 3$

$dx = \frac{du}{3}$

$3x = u-2$

$6x = 2u-4$

$$= \int \frac{6x}{u^3}\left(\frac{du}{3}\right)$$
$$= \int \frac{2u-4}{3u^3}\,du$$
$$= \int \frac{2u}{3u^3}-\frac{4}{3u^3}\,du$$
$$= \int \frac{2}{3}u^{-2}-\frac{4}{3}u^{-3}\,du$$
$$= -\frac{2}{3}u^{-1}+\frac{2}{3}u^{-2}+C$$
$$= \frac{2}{3}\left[\frac{1}{u^2}-\frac{1}{u}\right]+C$$
$$= \frac{2}{3}\left[\frac{1}{(3x+2)^2}-\frac{1}{3x+2}\right]+C$$

ALTERNATIVE BY PARTIAL FRACTIONS

$$\frac{6x}{(3x+2)^3} \equiv \frac{A}{(3x+2)^3}+\frac{B}{(3x+2)^2}+\frac{C}{3x+2}$$
$$\boxed{6x \equiv A + B(3x+2) + C(3x+2)^2}$$

- If $x=-\frac{2}{3}$: $-4 = A$
- If $x=0$: $0 = A+2B+4C$, $0=-4+2B+4C$, $4 = 2B+4C$, $2 = B+4C$
- If $x=-1$: $-6 = A-B+C$, $-6=-4-B+C$, $-2 = -B+C$

ADDING

$5C=0$

$C=0$

&

$B=2$

$$\int \frac{6x}{(3x+2)^3}\,dx = \int \frac{2}{(3x+2)^2}-\frac{4}{(3x+2)^3}\,dx$$
$$= \int 2(3x+2)^{-2}-4(3x+2)^{-3}$$
$$= \frac{2}{-3}(3x+2)^{-1}-\frac{4}{-6}(3x+2)^{-2}+C$$
$$= -\frac{2}{3}\left[\frac{1}{3x+2}-\frac{1}{(3x+2)^2}\right]+C$$
$$= \frac{2}{3}\left[\frac{1}{(3x+2)^2}-\frac{1}{3x+2}\right]+C$$

21. $$\int (\sin x - 2\cos x)\sin x \, dx = \begin{bmatrix} \frac{1}{2}x - \frac{1}{4}\sin 2x + \frac{1}{2}\cos 2x + C \\ \frac{1}{2}x - \frac{1}{4}\sin 2x + \cos^2 x + C \\ \frac{1}{2}x - \frac{1}{4}\sin 2x - \sin^2 x + C \end{bmatrix}$$

$$\int (\sin x - 2\cos x)\sin x \, dx = \int \sin^2 x \underbrace{- 2\cos x \sin x}_{} \, dx$$

$$= \int \left(\tfrac{1}{2} - \tfrac{1}{2}\cos 2x\right) - \sin 2x \, dx$$

$$= \tfrac{1}{2}x - \tfrac{1}{4}\sin 2x + \tfrac{1}{2}\cos 2x + C$$

This can also be integrated directly to either $+\cos^2 x$ or $-\sin^2 x$

22. $$\int \frac{(2+\cos x)\sin 2x}{2\cos x} \, dx = \begin{bmatrix} -\frac{1}{2}(2+\cos x)^2 + C \\ -2\cos x + \frac{1}{2}\sin^2 x + C \\ -2\cos x - \frac{1}{2}\cos^2 x + C \\ -2\cos x - \frac{1}{4}\cos 2x + C \end{bmatrix}$$

$$\int \frac{(2+\cos x)\sin 2x}{2\cos x} \, dx = \int \frac{(2+\cos x)(\cancel{2}\sin x \cancel{\cos x})}{\cancel{2}\cancel{\cos x}} \, dx$$

$$= \int (2+\cos x)\sin x \, dx = -\tfrac{1}{2}(2+\cos x)^2 + C$$

(BY REVERSE CHAIN RULE)

OR CONTINUING WITH TRIGONOMETRIC IDENTITIES

$$= \int 2\sin x + \cos x \sin x \, dx = \begin{cases} -2\cos x + \tfrac{1}{2}\sin^2 x + C \\ -2\cos x - \tfrac{1}{2}\cos^2 x + C \end{cases}$$

(BOTH BY RECOGNITION)

OR ANOTHER VARIATION

$$= \int 2\sin x + \tfrac{1}{2}(2\sin x\cos x) \, dx$$

$$= \int 2\sin x + \tfrac{1}{2}\sin 2x \, dx$$

$$= -2\cos x - \tfrac{1}{4}\cos 2x + C$$

23. $$\int \frac{8x-3}{4x} \, dx = 2x - \frac{3}{4}\ln|x| + C$$

$$\int \frac{8x-3}{4x} \, dx = \int \frac{8x}{4x} - \frac{3}{4x} \, dx = \int 2 - \tfrac{3}{4}\left(\tfrac{1}{x}\right) \, dx$$

$$= 2x - \tfrac{3}{4}\ln|x| + C$$

24. $\displaystyle\int \frac{6x^2}{x^3+8}\,dx = 2\ln\left|x^3+8\right| + C$

$$\int \frac{6x^2}{x^3+8}\,dx = 2\int \frac{3x^2}{x^3+8}\,dx \longleftarrow \boxed{\int \frac{f'(x)}{f(x)}\,dx = \ln|x|+C}$$

$$= 2\ln|x^3+8|+C$$

ALTERNATIVE BY SUBSTITUTION

$$\int \frac{6x^2}{x^3+8}\,dx = \dots \text{SUBSTITUTION} \dots = \int \frac{\cancel{6x^2}}{u}\left(\frac{du}{\cancel{3x^2}}\right)$$

$$= \int \frac{2}{u}\,du = 2\ln|u|+C$$

$$= 2\ln|x^3+8|+C$$

$$\boxed{\begin{aligned} u &= x^3+8 \\ \frac{du}{dx} &= 3x^2 \\ dx &= \frac{du}{3x^2} \end{aligned}}$$

25. $\displaystyle\int (1+2\cos x)^3 \sin x\,dx = -\tfrac{1}{8}(1+2\cos x)^4 + C$

$$\int (1+2\cos x)^3\sin x\,dx = \frac{1}{-8}(1+2\cos x)^4 + C$$

(BY REVERSE CHAIN RULE)

ALTERNATIVE BY SUBSTITUTION

$$\int (1+2\cos x)^3 \sin x\,dx \;(\text{BY SUBSTITUTION})\dots$$

$$= \int u^3 \cancel{\sin x}\left(\frac{du}{-2\cancel{\sin x}}\right) = \int -\frac{1}{2}u^3\,du$$

$$= -\frac{1}{8}u^4 + C = -\frac{1}{8}(1+2\cos x)^4 + C$$

$$\boxed{\begin{aligned} u &= 1+2\cos x \\ \frac{du}{dx} &= -2\sin x \\ dx &= \frac{du}{-2\sin x} \end{aligned}}$$

[THE SUBSTITUTION $u = 2\cos x$ OR $u = \cos x$ ALSO "WORK"]

26. $\displaystyle\int \cos\sqrt{x}\,dx = 2\sqrt{x}\sin\sqrt{x} + 2\cos\sqrt{x} + C$

$$\int \cos\sqrt{x}\,dx \dots \text{BY SUBSTITUTION} \dots$$

$$= \int \cos u\,(2u\,du) = \int 2u\cos u\,du$$

$$\boxed{\begin{aligned} u &= \sqrt{x} \\ u^2 &= x \\ 2u\frac{du}{dx} &= 1 \\ 2u\,du &= dx \end{aligned}}$$

INTEGRATION BY PARTS NEXT

$$= 2u\sin u - \int 2\sin u\,du$$

$$= 2u\sin u + 2\cos u + C$$

$$= 2\sqrt{x}\sin\sqrt{x} + 2\cos\sqrt{x} + C$$

$2u$	2
$\sin u$	$\cos u$

27. $\displaystyle\int \frac{3}{2x-1}\,dx = \tfrac{3}{2}\ln|2x-1| + C$

$$\int \frac{3}{2x-1}\,dx = \dots \text{BY INSPECTION} \dots = \frac{3}{2}\ln|2x-1| + C$$

28. $\displaystyle\int \frac{10}{(2x+1)^6}\,dx = -\frac{1}{(2x+1)^5} + C$

$\displaystyle\int \frac{10}{(2x+1)^6}\,dx = \int 10(2x+1)^{-6}\,dx \ldots$ BY INSPECTION $\ldots = \frac{10}{-5}(2x+1)^{-5} + C$

$\displaystyle = -(2x+1)^{-5} + C = -\frac{1}{(2x+1)^5} + C$

29. $\displaystyle\int \frac{3x-1}{(2x+1)(x-2)}\,dx = \frac{1}{2}\ln|2x+1| + \ln|x-2| + C$

$\displaystyle\int \frac{3x-1}{(2x+1)(x-2)}\,dx = \ldots$ BY PARTIAL FRACTIONS

$\displaystyle \frac{3x-1}{(2x+1)(x-2)} \equiv \frac{A}{2x+1} + \frac{B}{x-2}$

$\boxed{3x-1 \equiv A(x-2) + B(2x+1)}$

- If $x=2$, $5 = 5B$, $\underline{B=1}$
- If $x=-\frac{1}{2}$, $-\frac{5}{2} = -\frac{5}{2}A$, $\underline{A=1}$

$\displaystyle = \int \frac{1}{2x+1} + \frac{1}{x-2}\,dx = \frac{1}{2}\ln|2x+1| + \ln|x-2| + C$

30. $\displaystyle\int (2+\sin x)^2\,dx = \frac{9}{2}x - 4\cos x - \frac{1}{4}\sin 2x + C$

$\displaystyle\int (2+\sin x)^2\,dx = \int 4 + 4\sin x + \sin^2 x\,dx$

$\displaystyle = \int 4 + 4\sin x + \left(\frac{1}{2} - \frac{1}{2}\cos 2x\right)dx$

$\displaystyle = \int \frac{9}{2} + 4\sin x - \frac{1}{2}\cos 2x\,dx$

$\displaystyle = \frac{9}{2}x - 4\cos x - \frac{1}{4}\sin 2x + C$

31. $\displaystyle\int 5(2x-3)^{\frac{1}{4}}\,dx = 2(2x-3)^{\frac{5}{4}} + C$

$\displaystyle\int 5(2x-3)^{\frac{1}{4}}\,dx = \ldots$ BY INSPECTION $\ldots = \frac{5}{\frac{5}{2}}(2x-3)^{\frac{5}{4}} + C$

$\displaystyle = 2(2x-3)^{\frac{5}{4}} + C$

32. $$\int \frac{e^{4x}+3}{e^{3x}}\,dx = e^{x} - e^{-3x} + C$$

$$\int \frac{e^{4x}+3}{e^{3x}}\,dx = \int \frac{e^{4x}}{e^{3x}} + \frac{3}{e^{3x}}\,dx = \int e^{x} + 3e^{-3x}$$
$$= e^{x} - e^{-3x} + C$$

33. $$\int x e^{5x}\,dx = \frac{1}{5}x e^{5x} - \frac{1}{25}e^{5x} + C$$

$\int x e^{5x}\,dx = \ldots$ INTEGRATION BY PARTS

$$= \frac{1}{5}xe^{5x} - \int \frac{1}{5}e^{5x}\,dx$$
$$= \frac{1}{5}xe^{5x} - \frac{1}{25}e^{5x} + C$$

x	1
$\frac{1}{5}e^{5x}$	e^{5x}

34. $$\int \frac{x^3}{x^4+2}\,dx = \frac{1}{4}\ln\left(x^4+2\right) + C$$

$\int \frac{x^3}{x^4+2}\,dx = \frac{1}{4}\int \frac{4x^3}{x^4+2}\,dx = \ldots$ OF THE FORM $\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + C$

$$= \frac{1}{4}\ln(x^4+2) + C$$

ALTERNATIVE BY SUBSTITUTION

$$\int \frac{x^3}{x^4+2}\,dx = \int \frac{\cancel{x^3}}{u}\left(\frac{du}{4\cancel{x^3}}\right) = \int \frac{1}{4u}\,du$$
$$= \int \frac{1}{4} \times \frac{1}{u}\,du = \frac{1}{4}\ln|u| + C$$
$$= \frac{1}{4}\ln(x^4+2) + C$$

$u = x^4+2$

$\frac{du}{dx} = 4x^3$

$dx = \frac{du}{4x^3}$

35. $$\int \frac{2}{(x-2)(x-4)}\,dx = \ln\left|\frac{x-4}{x-2}\right| + C$$

$\int \frac{2}{(x-2)(x-4)}\,dx = \ldots$ BY PARTIAL FRACTIONS ...

$$= \int \frac{1}{x-4} - \frac{1}{x-2}\,dx$$
$$= \ln|x-4| - \ln|x-2| + C$$
$$= \ln\left|\frac{x-4}{x-2}\right| + C$$

$$\frac{2}{(x-2)(x-4)} \equiv \frac{A}{x-2} + \frac{B}{x-4}$$
$$\boxed{2 \equiv A(x-4) + B(x-2)}$$

- If $x=4$, $2 = 2B$, $B = 1$
- If $x=2$, $2 = -2A$, $A = -1$

36. $\displaystyle\int \frac{3}{4x+1}\,dx = \frac{3}{4}\ln|4x+1| + C$

$\displaystyle\int \frac{3}{4x+1}\,dx = \ldots$ BY INSPECTION $\ldots = \underline{\frac{3}{4}\ln|4x+1| + C}$

37. $\displaystyle\int \left(1+\frac{1}{x}\right)^2 dx = x + 2\ln|x| - \frac{1}{x} + C$

$$\int \left(1+\frac{1}{x}\right)^2 dx = \int 1^2 + 2\times1\times\frac{1}{x} + \left(\frac{1}{x}\right)^2 dx = \int 1 + \frac{2}{x} + \frac{1}{x^2}\,dx$$
$$= \int 1 + \frac{2}{x} + x^{-2}\,dx = x + 2\ln|x| - x^{-1} + C$$
$$= \underline{x + 2\ln|x| - \frac{1}{x} + C}$$

38. $\displaystyle\int \frac{x}{\left(x^2-1\right)^3}\,dx = -\frac{1}{4\left(x^2-1\right)^2} + C$

$\displaystyle\int \frac{x}{(x^2-1)^3}\,dx = \ldots$ BY REVERSE CHAIN RULE $\ldots = \displaystyle\int x(x^2-1)^{-3}\,dx$

$\frac{d}{dx}(x^2-1) = 2x$

$$= -\frac{1}{4}(x^2-1)^{-2} + C = \underline{-\frac{1}{4(x^2-1)^2} + C}$$

ALTERNATIVE BY SUBSTITUTION

$$\int \frac{x}{(x^2-1)^3}\,dx = \int \frac{\cancel{x}}{u^3}\left(\frac{du}{2\cancel{x}}\right) = \int \frac{1}{2}u^{-3}\,du$$
$$= \frac{\frac{1}{2}}{-2}u^{-2} + C = -\frac{1}{4}u^{-2}$$
$$= \underline{-\frac{1}{4(x^2-1)^2} + C}$$

$$\boxed{\begin{aligned} u &= x^2-1 \\ \frac{du}{dx} &= 2x \\ dx &= \frac{du}{2x} \end{aligned}}$$

39. $\displaystyle\int \cos x - \sin x\,dx = \sin x + \cos x + C$

$\displaystyle\int \cos x - \sin x\,dx = \underline{\sin x + \cos x + C}$

40. $\displaystyle\int \sin x - \cos x \, dx = -\cos x - \sin x + C$

$\int \sin x - \cos x \, dx = \underline{-\cos x - \sin x + C}$

41. $\displaystyle\int \sin(4x+3) \, dx = -\tfrac{1}{4}\cos(4x+3) + C$

$\int \sin(4x+3) \, dx = \dots$ BY INSPECTION $\dots = \underline{-\tfrac{1}{4}\cos(4x+3) + C}$

42. $\displaystyle\int \frac{x}{\sqrt{x+1}} \, dx = \tfrac{2}{3}(x+1)^{\frac{3}{2}} - 2(x+1)^{\frac{1}{2}} + C$

$\int \frac{x}{\sqrt{x+1}} \, dx = \dots$ BY SUBSTITUTION $\dots$

$= \int \frac{x}{u^{\frac{1}{2}}}(du) = \int \frac{u-1}{u^{\frac{1}{2}}} \, du = \int \frac{u}{u^{\frac{1}{2}}} - \frac{1}{u^{\frac{1}{2}}} \, du$

$= \int u^{\frac{1}{2}} - u^{-\frac{1}{2}} \, du = \frac{1}{\frac{3}{2}}u^{\frac{3}{2}} - \frac{1}{\frac{1}{2}}u^{\frac{1}{2}} + C$

$= \underline{\tfrac{2}{3}(x+1)^{\frac{3}{2}} - 2(x+1)^{\frac{1}{2}} + C}$

$u = x+1$

$\frac{du}{dx} = 1$

$dx = du$

$x = u-1$

ALTERNATIVE SUBSTITUTION

$\int \frac{x}{\sqrt{x+1}} \, dx = \int \frac{x}{\cancel{u}}(2\cancel{u} \, du) = \int 2x \, du$

$= \int 2(u^2-1) \, du = \int 2u^2 - 2 \, du$

$= \tfrac{2}{3}u^3 - 2u + C = \underline{\tfrac{2}{3}(x+1)^{\frac{3}{2}} - 2(x+1)^{\frac{1}{2}} + C}$

$u = \sqrt{x+1}$

$u^2 = x+1$

$2u\frac{du}{dx} = 1$

$dx = 2u \, du$

$x = u^2 - 1$

43. $\displaystyle\int \cos(5-2x) \, dx = -\tfrac{1}{2}\sin(5-2x) + C$

$\int \cos(5-2x) \, dx = \underline{-\tfrac{1}{2}\sin(5-2x) + C}$

44. $\displaystyle\int 3\sin 2x \, dx = -\frac{3}{2}\cos 2x + C$

$$\int 3\sin 2x \, dx = \underline{-\tfrac{3}{2}\cos 2x + C}$$

45. $\displaystyle\int \left(1+\sec^2 x\right)\sin x \, dx = -\cos x + \sec x + C$

$$\begin{aligned}\int \sin x\left(1+\sec^2 x\right) dx &= \int \sin x + \sin x\sec^2 x \, dx\\ &= \int \sin x + \frac{\sin x}{\cos x}\sec x \, dx\\ &= \int \sin x + \tan x\sec x \, dx \quad \boxed{\tfrac{d}{dx}(\sec x) = \sec x\tan x}\\ &= \underline{-\cos x + \sec x + C}\end{aligned}$$

46. $\displaystyle\int (1-2\cos x)^2 \, dx = 3x - 4\sin x + \sin 2x + C$

$$\begin{aligned}\int (1-2\cos x)^2 dx &= \int 1 - 4\cos x + 4\cos^2 x \, dx\\ &= \int 1 - 4\cos x + 4\left(\tfrac{1}{2}+\tfrac{1}{2}\cos 2x\right) dx\\ &= \int 1 - 4\cos x + 2 + 2\cos 2x \, dx\\ &= \int 3 - 4\cos x + 2\cos 2x \, dx\\ &= \underline{3x - 4\sin x + \sin 2x + C}\end{aligned}$$

47. $\displaystyle\int \left(1+\cot^2 x\right)\sec^2 x \, dx = \tan x - \cot x + C$

$$\begin{aligned}\int \sec^2 x\left(1+\cot^2 x\right) dx &= \text{ALTHOUGH } 1+\cot^2 x \equiv \operatorname{cosec}^2 x\text{, MULTIPLY OUT}\\ &= \int \sec^2 x + \sec^2 x\cot^2 x \, dx\\ &= \int \sec^2 x + \frac{1}{\cancel{\cos^2 x}} \times \frac{\cancel{\cos^2 x}}{\sin^2 x} \, dx\\ &= \int \sec^2 x + \operatorname{cosec}^2 x \, dx\\ &= \underline{\tan x - \cot x + C}\end{aligned}$$

$$\boxed{\begin{aligned}&\tfrac{d}{dx}(\tan x) = \sec^2 x\\ &\tfrac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x\end{aligned}}$$

48. $\displaystyle\int 2x\cos 3x \, dx = \frac{2}{3}x\sin 3x + \frac{2}{9}\cos 3x + C$

$\displaystyle\int 2x\cos 3x \, dx = \ldots$ INTEGRATION BY PARTS

x	2
$\frac{1}{3}\sin 3x$	$\cos 3x$

$= (2x)\left(\frac{1}{3}\sin 3x\right) - \displaystyle\int \frac{2}{3}\sin 3x \, dx$

$= \frac{2}{3}x\sin 3x - \left[-\frac{2}{9}\cos 3x\right] + C$

$= \frac{2}{3}x\sin 3x + \frac{2}{9}\cos 3x + C$

49. $\displaystyle\int \frac{3}{(2+x)(1-x)} \, dx = \ln\left|\frac{x+2}{x-1}\right| + C$

$\displaystyle\int \frac{3}{(2+x)(1-x)} \, dx = \ldots$ BY PARTIAL FRACTIONS ...

$\frac{3}{(2+x)(1-x)} \equiv \frac{A}{2+x} + \frac{B}{1-x}$

$3 \equiv A(1-x) + B(2+x)$

- If $x=1 \Rightarrow 3 = 3B \Rightarrow B = 1$
- If $x=-2$, $3 = 3A$, $A = 1$

$\ldots = \displaystyle\int \frac{1}{2+x} + \frac{1}{1-x} \, dx = \ln|2+x| - \ln|1-x| + C$

$= \ln\left|\frac{2+x}{1-x}\right| + C$

$= \ln\left|\frac{x+2}{x-1}\right| + C$

50. $\displaystyle\int 10(3x+1)^4 \, dx = \frac{2}{3}(3x+1)^5 + C$

$\displaystyle\int 10(3x+1)^4 \, dx = \ldots$ BY INSPECTION ... $= \frac{10}{15}(3x+1)^5 + C$

$= \frac{2}{3}(3x+1)^5 + C$

51. $\displaystyle\int 6(2x+1)^{\frac{1}{2}} \, dx = 2(2x+1)^{\frac{3}{2}} + C$

$\displaystyle\int \frac{1}{\cos^2 x \tan^2 x} \, dx = \int \frac{1}{\cancel{\cos^2 x} \times \frac{\sin^2 x}{\cancel{\cos^2 x}}} \, dx = \int \operatorname{cosec}^2 x \, dx$

$= -\cot x + C$

ALTERNATIVE

$\displaystyle\int \sec^2 x (\tan x)^{-2} \, dx = -(\tan x)^{-1} + C$

$\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$

52. $\displaystyle\int \frac{1}{\cos^2 x \tan^2 x}\,dx = -\cot x + C$

$$\int \frac{1}{\cos^2 x \tan^2 x}\,dx = \int \frac{1}{\cancel{\cos^2 x} \times \frac{\sin^2 x}{\cancel{\cos^2 x}}}\,dx = \int \operatorname{cosec}^2 x\,dx$$

$$= \underline{-\cot x + C}$$

ALTERNATIVE

$$\boxed{\int \sec^2 x(\tan x)^{-2}\,dx = -(\tan x)^{-1} + C}$$

$$\boxed{\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x}$$

53. $\displaystyle\int \cos x \sin x\,dx = \begin{bmatrix} -\frac{1}{4}\cos 2x + C \\ -\frac{1}{2}\cos^2 x + C \\ \frac{1}{2}\sin^2 x + C \end{bmatrix}$

$$\int \cos x \sin x\,dx = \int \tfrac{1}{2}(2\sin x\cos x)\,dx = \int \tfrac{1}{2}\sin 2x\,dx$$

$$= \underline{-\tfrac{1}{4}\cos 2x + C}$$

$$\int \cos x \sin x\,dx = \underline{\tfrac{1}{2}\sin^2 x + C} \qquad \text{SINCE } \frac{d}{dx}(\sin^2 x) = 2\sin x\cos x$$

$$\int \cos x \sin x\,dx = \underline{-\tfrac{1}{2}\cos^2 x + C} \qquad \text{SINCE } \frac{d}{dx}(\cos^2 x) = 2\cos x(-\sin x)$$

54. $\displaystyle\int \frac{2}{\cos^2 x}\,dx = 2\tan x + C$

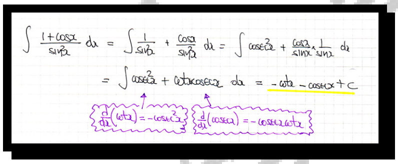
$$\int \frac{2}{\cos^2 x}\,dx = \int 2\sec^2 x\,dx \;\dots \text{ BY INSPECTION } \dots = \underline{2\tan x + C}$$

55. $\displaystyle\int 2 + 2\tan^2 x\,dx = 2\tan x + C$

$$\int 2 + 2\tan^2 x\,dx = \int 2(1 + \tan^2 x)\,dx = \int 2\sec^2 x\,dx$$

$$= \dots \text{ BY INSPECTION } \dots = \underline{2\tan x + C}$$

56. $\displaystyle\int \frac{1+\cos x}{\sin^2 x}\, dx = -\cot x - \operatorname{cosec} x + C$

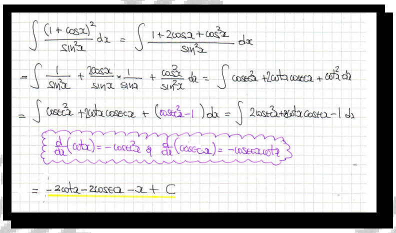


$$\int \frac{1+\cos x}{\sin^2 x}\,dx = \int \frac{1}{\sin^2 x} + \frac{\cos x}{\sin^2 x}\,dx = \int \operatorname{cosec}^2 x + \frac{\cos x}{\sin x}\times\frac{1}{\sin x}\,dx$$

$$= \int \operatorname{cosec}^2 x + \cot x \operatorname{cosec} x\,dx = -\cot x - \operatorname{cosec} x + C$$

$\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$ $\quad$ $\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x\cot x$

57. $\displaystyle\int \frac{(1+\cos x)^2}{\sin^2 x}\, dx = -x - 2\cot x - 2\operatorname{cosec} x + C$



$$\int \frac{(1+\cos x)^2}{\sin^2 x}\,dx = \int \frac{1+2\cos x+\cos^2 x}{\sin^2 x}\,dx$$

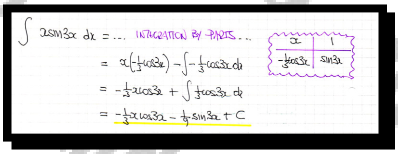
$$= \int \frac{1}{\sin^2 x} + \frac{2\cos x}{\sin x}\times\frac{1}{\sin x} + \frac{\cos^2 x}{\sin^2 x}\,dx = \int \operatorname{cosec}^2 x + 2\cot x\operatorname{cosec} x + \cot^2 x\,dx$$

$$= \int \operatorname{cosec}^2 x + 2\cot x\operatorname{cosec} x + (\operatorname{cosec}^2 x - 1)\,dx = \int 2\operatorname{cosec}^2 x + 2\cot x\operatorname{cosec} x - 1\,dx$$

$\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$ & $\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x\cot x$

$$= -2\cot x - 2\operatorname{cosec} x - x + C$$

58. $\displaystyle\int x\sin 3x\, dx = -\tfrac{1}{3}x\cos 3x - -\tfrac{1}{9}\sin 3x + C$



$\int x\sin 3x\,dx = \ldots$ INTEGRATION BY PARTS ...

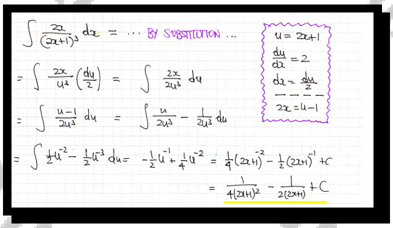
x	1
$-\frac{1}{3}\cos 3x$	$\sin 3x$

$$= x\left(-\tfrac{1}{3}\cos 3x\right) - \int -\tfrac{1}{3}\cos 3x\,dx$$

$$= -\tfrac{1}{3}x\cos 3x + \int \tfrac{1}{3}\cos 3x\,dx$$

$$= -\tfrac{1}{3}x\cos 3x - \tfrac{1}{9}\sin 3x + C$$

59. $\displaystyle\int \frac{2x}{(2x+1)^3}\, dx = \frac{1}{4(2x+1)^2} - \frac{1}{2(2x+1)} + C$



$\int \frac{2x}{(2x+1)^3}\,dx = \ldots$ BY SUBSTITUTION ...

$u = 2x+1$, $\frac{du}{dx} = 2$, $dx = \frac{du}{2}$, $2x = u-1$

$$= \int \frac{2x}{u^3}\left(\frac{du}{2}\right) = \int \frac{2x}{2u^3}\,du$$

$$= \int \frac{u-1}{2u^3}\,du = \int \frac{u}{2u^3} - \frac{1}{2u^3}\,du$$

$$= \int \tfrac{1}{2}u^{-2} - \tfrac{1}{2}u^{-3}\,du = -\tfrac{1}{2}u^{-1} + \tfrac{1}{4}u^{-2} = \tfrac{1}{4}(2x+1)^{-2} - \tfrac{1}{2}(2x+1)^{-1} + C$$

$$= \frac{1}{4(2x+1)^2} - \frac{1}{2(2x+1)} + C$$

60. $\int (4-5x)^{-1}\, dx = -\frac{1}{5}\ln|4-5x| + C$

$$\int (4-5x)^{-1}\,dx = \int \frac{1}{4-5x}\,dx = -\frac{1}{5}\ln|4-5x| + C$$

61. $\int \frac{1}{4x}\, dx = \frac{1}{4}\ln|x| + C$

$$\int \frac{1}{4x}\,dx = \int \frac{1}{4}\times\frac{1}{x}\,dx = \frac{1}{4}\ln|x| + C$$

62. $\int \frac{1}{(x+1)(x+2)}\, dx = \ln\left|\frac{x+1}{x-2}\right| + C$

$$\int \frac{1}{(x+1)(x+2)}\,dx = \dots$$ BY PARTIAL FRACTIONS …

$$\frac{1}{(x+1)(x+2)} \equiv \frac{A}{x+1} + \frac{B}{x+2}$$

$$\boxed{1 \equiv A(x+2) + B(x+1)}$$

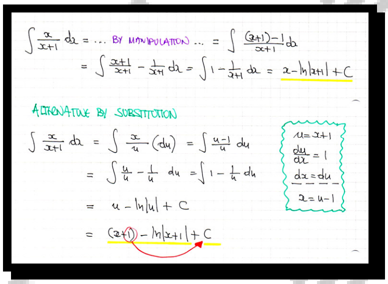
- If $x=-1 \Rightarrow 1 = A \Rightarrow A=1$
- If $x=-2 \Rightarrow 1 = -B \Rightarrow B=-1$

$$\dots = \int \frac{1}{x+1} - \frac{1}{x-2}\,dx = \ln|x+1| - \ln|x-2| + C = \ln\left|\frac{x+1}{x-2}\right| + C$$

63. $\int \frac{x+1}{x}\, dx = x + \ln|x| + C$

$$\int \frac{x+1}{x}\,dx = \int \frac{x}{x} + \frac{1}{x}\,dx = \int 1 + \frac{1}{x}\,dx = x + \ln|x| + C$$

64. $\displaystyle\int \frac{x}{x+1}\,dx = x-\ln|x+1|+C$



$$\int \frac{x}{x+1}\,dx = \dots \text{ BY MANIPULATION } \dots = \int \frac{(x+1)-1}{x+1}\,dx$$

$$= \int \frac{x+1}{x+1} - \frac{1}{x+1}\,dx = \int 1 - \frac{1}{x+1}\,dx = x - \ln|x+1| + C$$

ALTERNATIVE BY SUBSTITUTION

$$\int \frac{x}{x+1}\,dx = \int \frac{x}{u}(du) = \int \frac{u-1}{u}\,du$$

$$= \int \frac{u}{u} - \frac{1}{u}\,du = \int 1 - \frac{1}{u}\,du$$

$$= u - \ln|u| + C$$

$$= (x+1) - \ln|x+1| + C$$

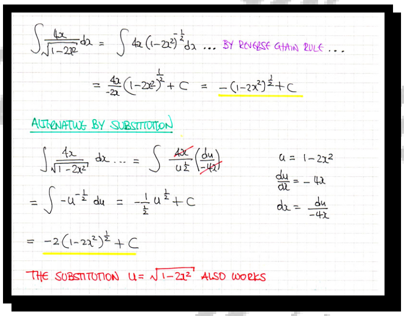
$u = x+1$

$\frac{du}{dx} = 1$

$dx = du$

$x = u-1$

65. $\displaystyle\int \frac{4x}{\sqrt{1-2x^2}}\,dx = -\sqrt{1-2x^2}+C$



$$\int \frac{4x}{\sqrt{1-2x^2}}\,dx = \int 4x(1-2x^2)^{-\frac{1}{2}}\,dx \dots \text{ BY REVERSE CHAIN RULE } \dots$$

$$= \frac{4x}{-2x}(1-2x^2)^{\frac{1}{2}} + C = -(1-2x^2)^{\frac{1}{2}} + C$$

ALTERNATIVE BY SUBSTITUTION

$$\int \frac{4x}{\sqrt{1-2x^2}}\,dx \dots = \int \frac{\cancel{4x}}{u^{\frac{1}{2}}}\left(\frac{du}{\cancel{-4x}}\right)$$

$$= \int -u^{-\frac{1}{2}}\,du = -\frac{1}{\frac{1}{2}}u^{\frac{1}{2}} + C$$

$$= -2(1-2x^2)^{\frac{1}{2}} + C$$

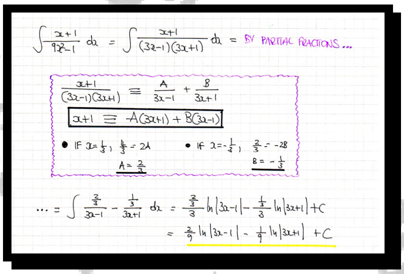
$u = 1-2x^2$

$\frac{du}{dx} = -4x$

$dx = \frac{du}{-4x}$

THE SUBSTITUTION $u = \sqrt{1-2x^2}$ ALSO WORKS

66. $\displaystyle\int \frac{x+1}{9x^2-1}\,dx = \tfrac{2}{9}\ln|3x-1| - \tfrac{1}{9}\ln|3x+1|+C$



$$\int \frac{x+1}{9x^2-1}\,dx = \int \frac{x+1}{(3x-1)(3x+1)}\,dx = \text{ BY PARTIAL FRACTIONS} \dots$$

$$\frac{x+1}{(3x-1)(3x+1)} \equiv \frac{A}{3x-1} + \frac{B}{3x+1}$$

$$\boxed{x+1 \equiv A(3x+1) + B(3x-1)}$$

- IF $x = \frac{1}{3}$, $\frac{4}{3} = 2A$, $A = \frac{2}{3}$
- IF $x = -\frac{1}{3}$, $\frac{2}{3} = -2B$, $B = -\frac{1}{3}$

$$\dots = \int \frac{\frac{2}{3}}{3x-1} - \frac{\frac{1}{3}}{3x+1}\,dx = \frac{\frac{2}{3}}{3}\ln|3x-1| - \frac{\frac{1}{3}}{3}\ln|3x+1| + C$$

$$= \frac{2}{9}\ln|3x-1| - \frac{1}{9}\ln|3x+1| + C$$

67. $\displaystyle\int x\sin 4x \, dx = \frac{1}{16}\sin 4x - \frac{1}{4}x\cos 4x + C$

$\int x\sin 4x \, dx = \ldots$ INTEGRATION BY PARTS ...

x	1
$-\frac{1}{4}\cos 4x$	$\sin 4x$

$= x\left(-\frac{1}{4}\cos 4x\right) - \int -\frac{1}{4}\cos 4x \, dx$

$= -\frac{1}{4}x\cos 4x + \int \frac{1}{4}\cos 4x \, dx$

$= -\frac{1}{4}x\cos 4x + \frac{1}{16}\sin 4x + C$

68. $\displaystyle\int \ln x \, dx = x\ln x - x + C$

$\int \ln x \, dx = \int 1 \times \ln x \, dx = \ldots$ INTEGRATION BY PARTS

$\ln x$	$\frac{1}{x}$
x	1

$= x\ln x - \int x\left(\frac{1}{x}\right) dx$

$= x\ln x - \int 1 \, dx$

$= x\ln x - x + C$

(THE SUBSTITUTION $u = \ln x \Rightarrow x = e^u$, FOLLOWED BY INTEGRATION BY PARTS ALSO WORKS HERE)

69. $\displaystyle\int \frac{4}{(2x-7)^2} \, dx = -\frac{2}{2x-7} + C$

$\int \frac{4}{(2x-7)^2} \, dx = \int 4(2x-7)^{-2} \, dx = \ldots$ BY INSPECTION ...

$= \frac{4}{-2}(2x-7)^{-1} + C = -2(2x-7)^{-1} + C = -\frac{2}{2x-7} + C$

70. $\displaystyle\int 4\cos^2 x \, dx = 2x + \sin 2x + C$

$\int 4\cos^2 x \, dx = \int 4\left(\frac{1}{2} + \frac{1}{2}\cos 2x\right) dx = \int 2 + 2\cos 2x \, dx$

$= 2x + \sin 2x + C$

71. $\displaystyle\int \left(1+\tan^2 x\right)\sec^2 x\,dx = \tan x + \tfrac{1}{3}\tan^3 x + C$

$\int \sec^2 x(1+\tan^2 x)\,dx = \int \sec^2 x + \sec^2 x\tan^2 x\,dx = \dots$ BY RECOGNITION...

$= \underline{\tan x + \frac{1}{3}\tan^3 x + C}$

$\frac{d}{dx}(\tan^3 x) = 3\times\tan^2 x\times\sec^2 x$

72. $\displaystyle\int (1+\tan x)\sec^2 x\,dx = \begin{bmatrix} \tfrac{1}{2}(1+\tan x)^2 + C \\ \tan x + \tfrac{1}{2}\tan^2 x + C \\ \tan x + \tfrac{1}{2}\sec^2 x + C \end{bmatrix}$

$\int \sec^2 x(1+\tan x)\,dx = \dots$ BY REVERSE CHAIN RULE/RECOGNITION

$= \underline{\frac{1}{2}(1+\tan x)^2 + C}$

ALTERNATIVE

$\int \sec^2 x(1+\tan x)\,dx = \int \sec^2 x + \sec^2 x\tan x\,dx \dots$ BY RECOGNITION..

$= \underline{\tan x + \frac{1}{2}\tan^2 x + C}$

$\frac{d}{dx}(\tan^2 x) = 2\tan x\times\sec^2 x$

ANOTHER VARIANT

$\int \sec^2 x(1+\tan x)\,dx = \int \sec^2 x + \sec^2 x\tan x\,dx$

$= \int \sec^2 x + \sec x(\sec x\tan x)\,dx$

$= \underline{\tan x + \frac{1}{2}\sec^2 x + C}$

$\frac{d}{dx}(\sec^2 x) = 2\times\sec x\times\sec x\tan x$

73. $\displaystyle\int \operatorname{cosec}^2(3x+1)\,dx = -\tfrac{1}{3}\cot(3x+1) + C$

$\int \operatorname{cosec}^2(3x+1)\,dx = \underline{-\frac{1}{3}\cot(3x+1) + C}$

STANDARD DIFFERENTIATION $\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$

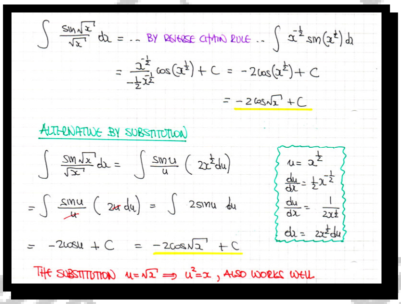
74. $\displaystyle\int 12\sec^2(2x+3)\,dx = 6\tan(2x+3) + C$

$\int 12\sec^2(2x+3)\,dx = \underline{6\tan(2x+3) + C}$

STANDARD DIFFERENTIATION RESULT

$\frac{d}{dx}(\tan x) = \sec^2 x$

75. $\displaystyle\int \frac{\sin\sqrt{x}}{\sqrt{x}}\,dx = -2\cos\sqrt{x} + C$



$\displaystyle\int \frac{\sin\sqrt{x}}{\sqrt{x}}\,dx = \dots$ BY REVERSE CHAIN RULE $\dots \displaystyle\int x^{-\frac{1}{2}}\sin\left(x^{\frac{1}{2}}\right)dx$

$\displaystyle = \frac{x^{-\frac{1}{2}}}{-\frac{1}{2}x^{-\frac{1}{2}}}\cos\left(x^{\frac{1}{2}}\right) + C = -2\cos\left(x^{\frac{1}{2}}\right) + C$

$= -2\cos\sqrt{x} + C$

ALTERNATIVE BY SUBSTITUTION

$\displaystyle\int \frac{\sin\sqrt{x}}{\sqrt{x}}\,dx = \int \frac{\sin u}{u}\left(2x^{\frac{1}{2}}du\right)$

$\displaystyle = \int \frac{\sin u}{\cancel{u}}\left(2\cancel{u}\,du\right) = \int 2\sin u\,du$

$= -2\cos u + C = -2\cos\sqrt{x} + C$

$u = x^{\frac{1}{2}}$

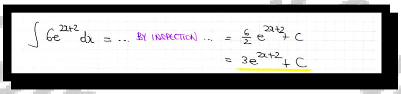
$\frac{du}{dx} = \frac{1}{2}x^{-\frac{1}{2}}$

$\frac{du}{dx} = \frac{1}{2x^{\frac{1}{2}}}$

$dx = 2x^{\frac{1}{2}}du$

THE SUBSTITUTION $u = \sqrt{x} \Rightarrow u^2 = x$, ALSO WORKS WELL

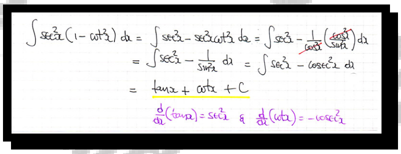
76. $\displaystyle\int 6e^{2x+2}\,dx = 3e^{2x+2} + C$



$\displaystyle\int 6e^{2x+2}\,dx = \dots$ BY INSPECTION $\dots = \frac{6}{2}e^{2x+2} + C$

$= 3e^{2x+2} + C$

77. $\displaystyle\int \left(1-\cot^2 x\right)\sec^2 x\,dx = \tan x + \cot x + C$



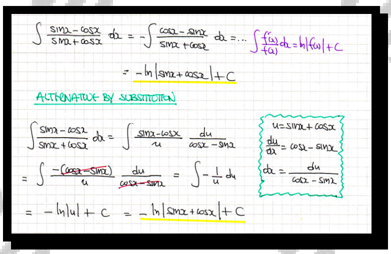
$\displaystyle\int \sec^2 x\left(1-\cot^2 x\right)dx = \int \sec^2 x - \sec^2 x\cot^2 x\,dx = \int \sec^2 x - \frac{1}{\cancel{\cos^2 x}}\left(\frac{\cancel{\cos^2 x}}{\sin^2 x}\right)dx$

$\displaystyle = \int \sec^2 x - \frac{1}{\sin^2 x}\,dx = \int \sec^2 x - \operatorname{cosec}^2 x\,dx$

$= \tan x + \cot x + C$

$\frac{d}{dx}(\tan x) = \sec^2 x$ & $\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$

78. $\displaystyle\int \frac{\sin x - \cos x}{\sin x + \cos x}\,dx = -\ln\left|\sin x + \cos x\right| + C$



$\displaystyle\int \frac{\sin x - \cos x}{\sin x + \cos x}\,dx = -\int \frac{\cos x - \sin x}{\sin x + \cos x}\,dx = \dots \int \frac{f'(x)}{f(x)}dx = \ln|f(x)| + C$

$= -\ln|\sin x + \cos x| + C$

ALTERNATIVE BY SUBSTITUTION

$\displaystyle\int \frac{\sin x - \cos x}{\sin x + \cos x}\,dx = \int \frac{\sin x - \cos x}{u}\,\frac{du}{\cos x - \sin x}$

$\displaystyle = \int \frac{-\cancel{(\cos x - \sin x)}}{u}\,\frac{du}{\cancel{\cos x - \sin x}} = \int -\frac{1}{u}\,du$

$= -\ln|u| + C = -\ln|\sin x + \cos x| + C$

$u = \sin x + \cos x$

$\frac{du}{dx} = \cos x - \sin x$

$dx = \frac{du}{\cos x - \sin x}$

79. $\displaystyle\int \sec x \tan x \sqrt{1+\sec x} \; dx = \frac{2}{3}(1+\sec x)^{\frac{3}{2}} + C$

$\int \sec x \tan x \sqrt{1+\sec x} \, dx = \int (\sec x + 1)^{\frac{1}{2}} (\sec x \tan x) \, dx$

BY REVERSE CHAIN RULE AS $\frac{d}{dx}(\sec x) = \sec x \tan x$

$= \frac{1}{\frac{3}{2}}(\sec x + 1)^{\frac{3}{2}} + C$

$= \frac{2}{3}(1+\sec x)^{\frac{3}{2}} + C$

ALTERNATIVE BY SUBSTITUTION

$\int \sec x \tan x \sqrt{1+\sec x} \, dx = \ldots$

$u = 1 + \sec x$

$\frac{du}{dx} = \sec x \tan x$

$dx = \frac{du}{\sec x \tan x}$

$= \int \cancel{(\sec x \tan x)} \, u^{\frac{1}{2}} \frac{du}{\cancel{\sec x \tan x}} = \int u^{\frac{1}{2}} \, du$

$= \frac{2}{3}u^{\frac{3}{2}} + C = \frac{2}{3}(1+\sec x)^{\frac{3}{2}} + C$

(THE SUBSTITUTIONS $u = \sec x$ OR $u = (1+\sec x)^{\frac{1}{2}}$ ALSO WORK)

80. $\displaystyle\int \tan 2x \sec 2x \; dx = \frac{1}{2}\sec 2x + C$

$\int \tan 2x \sec 2x \, dx \ldots$ STANDARD DIFFERENTIAL $= \frac{1}{2}\sec 2x + C$

$\frac{d}{dx}(\sec x) = \sec x \tan x$

81. $\displaystyle\int x^2 \ln x \; dx = \frac{1}{3}x^3 \ln|x| - \frac{1}{9}x^3 + C$

$\int x^2 \ln x \, dx = \ldots$ INTEGRATION BY PARTS

$\ln x$	$\frac{1}{x}$
$\frac{1}{3}x^3$	x^2

$= \left(\frac{1}{3}x^3\right)(\ln|x|) - \int \left(\frac{1}{x}\right)\left(\frac{1}{3}x^3\right) dx$

$= \frac{1}{3}x^3 \ln|x| - \int \frac{1}{3}x^2 \, dx$

$= \frac{1}{3}x^3 \ln|x| - \frac{1}{9}x^3 + C$

82. $\displaystyle\int \frac{6}{x^2-2x-8} \; dx = \ln\left|\frac{x-4}{x+2}\right| + C$

$\int \frac{6}{x^2-2x-8} \, dx = \int \frac{6}{(x+2)(x-4)} \, dx = \ldots$ BY PARTIAL FRACTIONS

$\frac{6}{(x+2)(x-4)} \equiv \frac{A}{x+2} + \frac{B}{x-4}$

$\boxed{6 = A(x-4) + B(x+2)}$

- IF $x = 4 \Rightarrow 6 = 6B \Rightarrow B = 1$
- IF $x = -2 \Rightarrow 6 = -6A \Rightarrow A = -1$

$= \int \frac{1}{x-4} - \frac{1}{x+2} \, dx = \ln|x-4| - \ln|x+2| + C = \ln\left|\frac{x-4}{x+2}\right| + C$

83. $\displaystyle\int 3\cot^2 x \; dx = -3x - 3\cot x + C$

$$\int 3\cot^2 x \, dx = \int 3(\operatorname{cosec}^2 x - 1)\, dx = \int 3\operatorname{cosec}^2 x - 3 \, dx$$

$\{1+\cot^2 x \equiv \operatorname{cosec}^2 x\}$

$$= -3\cot x - 3x + C \quad \leftarrow \text{SINCE } \frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$$

84. $$\int \cos 2x \sin x \; dx = \begin{bmatrix} -\frac{2}{3}\cos^3 x + \cos x + C \\ -\frac{1}{6}\cos 3x + \frac{1}{2}\cos x + C \\ \frac{4}{3}\cos^3 x - \cos 2x \cos x + C \\ \frac{1}{3}\cos 2x \cos x + \frac{2}{3}\sin 2x \sin x + C \\ \frac{1}{3}\sin x \sin 2x + \frac{1}{3}\cos x + C \\ \frac{2}{3}\sin^2 x \cos x + \frac{1}{3}\cos x + C \\ \frac{1}{3}\left(1 + 2\sin^2 x\right)\cos x + C \\ -\frac{1}{4}\cos 3x + \frac{5}{4}\cos x + C \end{bmatrix}$$

$$\int \cos 2x \sin x \, dx = \int (2\cos^2 x - 1)\sin x \, dx$$

$$= \int \underbrace{2\cos^2 x \sin x}_{} - \sin x \, dx = \dots = -\frac{2}{3}\cos^3 x + \cos x + C$$

BY SUBSTITUTION $u = \cos x$

OR REVERSE CHAIN RULE SINCE $\frac{d}{dx}(\cos^3 x) = 3\cos^2 x(-\sin x)$

STANDARD ALTERNATIVE FOR THIS TYPE OF INTEGRAL

- $\sin(2x+x) = \sin 2x \cos x + \cos 2x \sin x$
- $\sin(2x-x) = \sin 2x \cos x - \cos 2x \sin x$

SUBTRACT

$$\sin(2x+x) - \sin(2x-x) = 2\cos 2x \sin x$$

$$\cos 2x \sin x = \frac{1}{2}\sin 3x - \frac{1}{2}\sin x$$

$$\int \cos 2x \sin x \, dx = \int \frac{1}{2}\sin 3x - \frac{1}{2}\sin x \, dx = -\frac{1}{6}\cos 3x + \frac{1}{2}\cos x + C$$

BY PARTS ONCE & RECOGNITION

$$\int \cos 2x \sin x \, dx = \dots$$

$\cos 2x$	$-2\sin 2x$
$-\cos x$	$\sin x$

$$\dots = -\cos 2x \cos x - \int (-2\sin 2x)(-\cos x)\, dx$$

$$= -\cos 2x \cos x - \int 2\sin 2x \cos x \, dx$$

$$= -\cos 2x \cos x - \int 2(2\sin x \cos x)\cos x \, dx$$

$$= -\cos 2x \cos x - \int 4\sin x \cos^2 x \, dx$$

$$= -\cos 2x \cos x + \frac{4}{3}\cos^3 x + C$$

SINCE $\frac{d}{dx}\left(\frac{4}{3}\cos^3 x\right) = \frac{4}{3} \times 3\cos^2 x \times (-\sin x)$

BY PARTS TWICE

$$\int \cos 2x \sin x \, dx = \dots$$

$\cos 2x$	$-2\sin 2x$
$-\cos x$	$\sin x$

$$\dots = -\cos x \cos 2x - \int (-2\sin 2x)(-\cos x)\, dx$$

$$= -\cos x \cos 2x - \int 2\sin 2x \cos x \, dx$$

$2\sin 2x$	$4\cos 2x$
$\sin x$	$\cos x$

$$= -\cos x \cos 2x - \left[2\sin 2x \sin x - \int 4\cos 2x \sin x \, dx\right]$$

$$= -\cos x \cos 2x - 2\sin 2x \sin x + 4\int \cos 2x \sin x \, dx$$

IF $\displaystyle\int \cos 2x \sin x \, dx = -\cos x \cos 2x - 2\sin 2x \sin x + 4\int \cos 2x \sin x \, dx$

$$\cos x \cos 2x + 2\sin 2x \sin x = 3\int \cos 2x \sin x \, dx$$

$$\therefore \int \cos 2x \sin x \, dx = \frac{1}{3}\cos x \cos 2x + \frac{2}{3}\sin 2x \sin x + C$$

$$\vdots$$

$$= \frac{1}{3}\left[\cos 2x \cos x + \sin 2x \sin x\right] + \frac{1}{3}\sin 2x \sin x + C$$

$$= \frac{1}{3}\cos(2x - x) + \frac{1}{3}\sin 2x \sin x + C$$

$$= \frac{1}{3}\cos x + \frac{1}{3}\sin x \sin 2x + C$$

$$= \frac{1}{3}\cos x + \frac{2}{3}\sin^2 x \cos x + C$$

$$= \frac{1}{3}\cos x\left(1 + 2\sin^2 x\right) + C$$

USING THE IDENTITY $\sin 3x \equiv 3\sin x - 4\sin^3 x$

$$\int \cos 2x \sin x \, dx = \int (1 - 2\sin^2 x)\sin x \, dx = \int \sin x - 3\sin^3 x \, dx$$

$$= \int \sin x - 3\left(\frac{3}{4}\sin x - \frac{1}{4}\sin 3x\right) dx = \int \frac{3}{4}\sin 3x - \frac{5}{4}\sin x \, dx$$

$$= -\frac{1}{4}\cos 3x + \frac{5}{4}\cos x + C$$

85. $$\int \frac{\sqrt{x^2+4}}{x}\,dx = \sqrt{x^2+4} + \ln\left|\frac{\sqrt{x^2+4}-2}{\sqrt{x^2+4}+2}\right| + C$$

$$\int \frac{\sqrt{x^2+4}}{x}\,dx = \dots \text{ SUBSTITUTION} \dots$$

$u = (x^2+4)^{\frac{1}{2}}$

$u^2 = x^2+4$

$2u\frac{du}{dx} = 2x$

$u\,du = x\,dx$

$dx = \frac{u}{x}\,du$

$x^2 = u^2-4$

$$= \int \frac{u}{x}\left(\frac{u}{x}\,du\right) = \int \frac{u^2}{x^2}\,du$$

$$= \int \frac{u^2}{u^2-4}\,du = \int \frac{(u^2-4)+4}{u^2-4}\,du$$

IMPROPER FRACTION — LONG DIVIDE OR MANIPULATE

$$= \int 1 + \frac{4}{u^2-4}\,du = \int 1 + \frac{4}{(u-2)(u+2)}\,du$$

BY PARTIAL FRACTIONS

$$\frac{4}{(u-2)(u+2)} \equiv \frac{A}{u-2} + \frac{B}{u+2}$$

$$\boxed{4 \equiv A(u+2) + B(u-2)}$$

- IF $u=2$, $4=4A$, $A=1$
- IF $u=-2$, $4=-4B$, $B=-1$

$$= \int 1 + \frac{1}{u-2} - \frac{1}{u+2}\,du = u + \ln|u-2| - \ln|u+2| + C$$

$$= u + \ln\left|\frac{u-2}{u+2}\right| + C = \sqrt{x^2+4} + \ln\left|\frac{\sqrt{x^2+4}-2}{\sqrt{x^2+4}+2}\right| + C$$

86. $$\int 7(2x-3)^{\frac{5}{2}}\,dx = (2x-3)^{\frac{7}{2}} + C$$

$$\int 7(2x-3)^{\frac{5}{2}}\,dx = \dots \text{ BY INSPECTION } \dots = \frac{7}{7}(2x-3)^{\frac{7}{2}} + C$$

$$= (2x-3)^{\frac{7}{2}} + C$$

87. $$\int \frac{x^2}{4-x^3}\,dx = -\tfrac{1}{3}\ln\left|4-x^3\right| + C$$

$$\int \frac{x^2}{4-x^3}\,dx = -\frac{1}{3}\int \frac{-3x^2}{4-x^3}\,dx = \dots = -\frac{1}{3}\ln|4-x^3| + C$$

OF THE FORM $\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + C$

ALTERNATIVE BY SUBSTITUTION

$u = 4-x^3$

$\frac{du}{dx} = -3x^2$

$dx = \frac{du}{-3x^2}$

$$\int \frac{x^2}{4-x^3}\,dx = \int \frac{\cancel{x^2}}{u}\left(-\frac{du}{3\cancel{x^2}}\right) = \int -\frac{1}{3u}\,du$$

$$= \int -\frac{1}{3} \times \frac{1}{u}\,du = -\frac{1}{3}\ln|u| + C$$

$$= -\frac{1}{3}\ln|4-x^3| + C$$

88. $\displaystyle\int x\sin\left(\tfrac{1}{2}x\right)\,dx = -2x\cos\left(\tfrac{1}{2}x\right) + 4\sin\left(\tfrac{1}{2}x\right) + C$

$\displaystyle\int x\sin\left(\tfrac{1}{2}x\right)dx = \dots$ INTEGRATION BY PARTS

$\displaystyle = -2x\cos\left(\tfrac{1}{2}x\right) - \int -2\cos\left(\tfrac{1}{2}x\right)dx$

$\displaystyle = -2x\cos\left(\tfrac{1}{2}x\right) + \int 2\cos\left(\tfrac{1}{2}x\right)dx$

$\displaystyle = -2x\cos\left(\tfrac{1}{2}x\right) + 4\sin\left(\tfrac{1}{2}x\right) + C$

x	1
$-2\cos\left(\tfrac{1}{2}x\right)$	$\sin\left(\tfrac{1}{2}x\right)$

89. $\displaystyle\int \frac{4}{4x^2+4x+1}\,dx = -\frac{2}{2x+1} + C$

$\displaystyle\int \frac{4}{4x^2+4x+1}dx = \dots$ THIS IS A PERFECT SQUARE $\displaystyle\dots = \int \frac{4}{(2x+1)^2}dx$

$\displaystyle = \int 4(2x+1)^{-2}dx = -\frac{4}{2}(2x+1)^{-1} + C$

$\displaystyle = -\frac{2}{2x+1} + C$

90. $\displaystyle\int \frac{3}{\sqrt{4x+1}}\,dx = \frac{3}{2}\sqrt{4x+1} + C$

$\displaystyle\int \frac{3}{\sqrt{4x+1}}dx = \int 3(4x+1)^{-\frac{1}{2}} = \dots$ BY INSPECTION $\dots$

$\displaystyle = \frac{3}{2}(4x+1)^{\frac{1}{2}} + C = \frac{3}{2}\sqrt{4x+1} + C$

91. $\displaystyle\int \frac{x}{\sqrt{x-1}}\,dx = \frac{2}{3}(x-1)^{\frac{3}{2}} + 2(x-1)^{\frac{1}{2}} + C$

$\displaystyle\int \frac{x}{\sqrt{x-1}}dx \dots$ BY SUBSTITUTION $\dots$

$\displaystyle = \int \frac{u+1}{u^{\frac{1}{2}}}du = \int \frac{u}{u^{\frac{1}{2}}} + \frac{1}{u^{\frac{1}{2}}}du$

$\displaystyle = \int u^{\frac{1}{2}} + u^{-\frac{1}{2}}du = \frac{2}{3}u^{\frac{3}{2}} + 2u^{\frac{1}{2}} + C$

$\displaystyle = \frac{2}{3}(x-1)^{\frac{3}{2}} + 2(x-1)^{\frac{1}{2}} + C$

$u = x-1$

$\frac{du}{dx} = 1$

$dx = du$

$x = u+1$

ALTERNATIVE SUBSTITUTION

$\displaystyle\int \frac{x}{\sqrt{x-1}}dx = \int \frac{x}{\cancel{u}}(2\cancel{u}\,du)$

$\displaystyle = \int 2x\,du = \int 2(u^2+1)\,du$

$\displaystyle = \int 2u^2 + 2\,du = \frac{2}{3}u^3 + 2u + C$

$\displaystyle = \frac{2}{3}(x-1)^{\frac{3}{2}} + 2(x-1)^{\frac{1}{2}} + C$

$u = (x-1)^{\frac{1}{2}}$

$u^2 = x-1$

$2u\frac{du}{dx} = 1$

$dx = 2u\,du$

$x = u^2+1$

92. $\displaystyle\int \frac{1}{3(x-2)^{\frac{1}{2}}}\,dx = \frac{2}{3}(x-2)^{\frac{1}{2}} + C$

$\displaystyle\int \frac{1}{3(x-2)^{\frac{1}{2}}}\,dx = \ldots$ BY INSPECTION $\ldots = \displaystyle\int \frac{1}{3}(x-2)^{-\frac{1}{2}}\,dx$

$= \dfrac{\frac{1}{3}}{\frac{1}{2}}(x-2)^{\frac{1}{2}} + C = \dfrac{2}{3}(x-2)^{\frac{1}{2}} + C$

93. $\displaystyle\int \frac{6x+3}{2x}\,dx = 3x + \frac{3}{2}\ln|x| + C$

$\displaystyle\int \frac{6x+3}{2x}\,dx = \int \frac{6x}{2x} + \frac{3}{2x}\,dx = \int 3 + \frac{3}{2}\times\frac{1}{x}\,dx$

$= 3x + \frac{3}{2}\ln|x| + C$

94. $\displaystyle\int \frac{4x+1}{2x-5}\,dx = 2x + \frac{11}{2}\ln|2x-5| + C$

$\displaystyle\int \frac{4x+1}{2x-5}\,dx = \ldots$ BY MANIPULATION TO SPLIT THE FRACTION

$\displaystyle= \int \frac{2(2x-5)+11}{2x-5}\,dx = \int \frac{2(2x-5)}{2x-5} + \frac{11}{2x-5}\,dx$

$\displaystyle= \int 2 + \frac{11}{2x-5}\,dx = 2x + \frac{11}{2}\ln|2x-5| + C$

ALTERNATIVE BY SUBSTITUTION

$\displaystyle\int \frac{4x+1}{2x-5}\,dx = \int \frac{4x+1}{u}\,\frac{du}{2}$

$\displaystyle= \int \frac{2(u+5)+1}{2u}\,du = \int \frac{2u+11}{2u}\,du$

$\displaystyle= \int \frac{2u}{2u} + \frac{11}{2u}\,du = \int 1 + \frac{11}{2}\times\frac{1}{u}\,du$

$= u + \frac{11}{2}\ln|u| + C = (2x-5) + \frac{11}{2}\ln|2x-5| + C$

$= 2x + \frac{11}{2}\ln|2x-5| + C$

$u = 2x-5$
$\frac{du}{dx} = 2$
$dx = \frac{du}{2}$
$2x = u+5$

95. $\displaystyle\int \frac{4x}{x^2-1}\,dx = 2\ln\left|x^2-1\right| + C$

$\displaystyle\int \frac{4x}{x^2-1}\,dx = 2\int \frac{2x}{x^2-1}\,dx = \ldots = 2\ln|x^2-1| + C$

OF THE FORM $\displaystyle\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + C$

ALTERNATIVE BY SUBSTITUTION

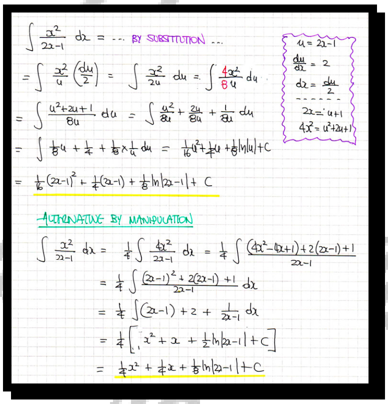
$\displaystyle\int \frac{4x}{x^2-1}\,dx = \int \frac{\overset{2}{\cancel{4x}}}{u}\left(\frac{du}{\cancel{2x}}\right) = \int \frac{2}{u}\,du$

$= 2\ln|u| + C = 2\ln|x^2-1| + C$

$u = x^2-1$
$\frac{du}{dx} = 2x$
$dx = \frac{du}{2x}$

PARTIAL FRACTION ALSO WORK HERE AFTER FACTORIZATION OF THE DENOMINATOR

96. $$\int \frac{x^2}{2x-1}\,dx = \left[\begin{array}{c} \frac{1}{16}(2x-1)^2+\frac{1}{4}(2x-1)+\frac{1}{8}\ln|2x-1|+C \\ \frac{1}{4}x^2+\frac{1}{4}x+\frac{1}{8}\ln|2x-1|+C \end{array}\right]$$



$\int \frac{x^2}{2x-1}\,dx = \ldots$ BY SUBSTITUTION ...

$u = 2x-1$

$\frac{du}{dx} = 2$

$dx = \frac{du}{2}$

$2x = u+1$

$4x^2 = u^2+2u+1$

$= \int \frac{x^2}{u}\left(\frac{du}{2}\right) = \int \frac{x^2}{2u}\,du = \int \frac{4x^2}{8u}\,du$

$= \int \frac{u^2+2u+1}{8u}\,du = \int \frac{u^2}{8u}+\frac{2u}{8u}+\frac{1}{8u}\,du$

$= \int \frac{1}{8}u+\frac{1}{4}+\frac{1}{8}\times\frac{1}{u}\,du = \frac{1}{16}u^2+\frac{1}{4}u+\frac{1}{8}\ln|u|+C$

$= \frac{1}{16}(2x-1)^2+\frac{1}{4}(2x-1)+\frac{1}{8}\ln|2x-1|+C$

ALTERNATIVE BY MANIPULATION

$\int \frac{x^2}{2x-1}\,dx = \frac{1}{4}\int \frac{4x^2}{2x-1}\,dx = \frac{1}{4}\int \frac{(4x^2-4x+1)+2(2x-1)+1}{2x-1}$

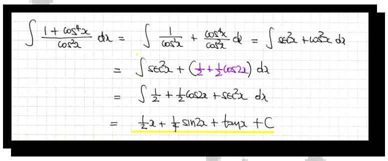
$= \frac{1}{4}\int \frac{(2x-1)^2+2(2x-1)+1}{2x-1}\,dx$

$= \frac{1}{4}\int (2x-1)+2+\frac{1}{2x-1}\,dx$

$= \frac{1}{4}\left[x^2+x+\frac{1}{2}\ln|2x-1|+C\right]$

$= \frac{1}{4}x^2+\frac{1}{4}x+\frac{1}{8}\ln|2x-1|+C$

97. $$\int \frac{1+\cos^4 x}{\cos^2 x}\,dx = \frac{1}{2}x+\frac{1}{4}\sin 2x+\tan x+C$$



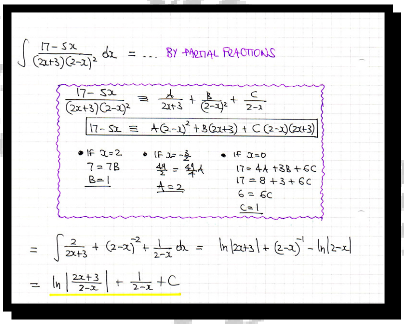
$\int \frac{1+\cos^4 x}{\cos^2 x}\,dx = \int \frac{1}{\cos^2 x}+\frac{\cos^4 x}{\cos^2 x}\,dx = \int \sec^2 x+\cos^2 x\,dx$

$= \int \sec^2 x+\left(\frac{1}{2}+\frac{1}{2}\cos 2x\right)dx$

$= \int \frac{1}{2}+\frac{1}{2}\cos 2x+\sec^2 x\,dx$

$= \frac{1}{2}x+\frac{1}{4}\sin 2x+\tan x+C$

98. $$\int \frac{17-5x}{(2x+3)(2-x)^2}\,dx = \ln\left|\frac{2x+3}{2-x}\right|+\frac{1}{2-x}+C$$



$\int \frac{17-5x}{(2x+3)(2-x)^2}\,dx = \ldots$ BY PARTIAL FRACTIONS

$\frac{17-5x}{(2x+3)(2-x)^2} \equiv \frac{A}{2x+3}+\frac{B}{(2-x)^2}+\frac{C}{2-x}$

$\boxed{17-5x \equiv A(2-x)^2+B(2x+3)+C(2-x)(2x+3)}$

• IF $x=2$: $7=7B$, $B=1$

• IF $x=-\frac{3}{2}$: $\frac{49}{2}=\frac{49}{4}A$, $A=2$

• IF $x=0$: $17=4A+3B+6C$, $17=8+3+6C$, $6=6C$, $C=1$

$= \int \frac{2}{2x+3}+(2-x)^{-2}+\frac{1}{2-x}\,dx = \ln|2x+3|+(2-x)^{-1}-\ln|2-x|$

$= \ln\left|\frac{2x+3}{2-x}\right|+\frac{1}{2-x}+C$

99. $\int x\sin(2x-1)\ dx = -\frac{1}{2}x\cos(2x-1)+\frac{1}{4}\sin(2x-1)+C$

$\int x\sin(2x-1)\,dx = \ldots$ INTEGRATION BY PARTS ...

x	1
$-\frac{1}{2}\cos(2x-1)$	$\sin(2x-1)$

$= -\frac{1}{2}x\cos(2x-1) - \int -\frac{1}{2}\cos(2x-1)\,dx$

$= -\frac{1}{2}x\cos(2x-1) + \int \frac{1}{2}\cos(2x-1)\,dx$

$= -\frac{1}{2}x\cos(2x-1) + \frac{1}{4}\sin(2x-1) + C$

100. $\int 4(3x-2)^3\ dx = \frac{1}{3}(3x-2)^4 + C$

$\int 4(3x-2)^3\,dx = \ldots$ BY INSPECTION ... $= \frac{4}{12}(3x-2)^4 + C$

$= \frac{1}{3}(3x-2)^4 + C$

101. $\int \sqrt{x\sqrt{x}}\ dx = \frac{4}{7}x^{\frac{7}{4}} + C$

$\int \sqrt{x\sqrt{x}}\,dx = \int \left(x \cdot x^{\frac{1}{2}}\right)^{\frac{1}{2}} dx = \int \left(x^{\frac{3}{2}}\right)^{\frac{1}{2}} dx$

$= \int x^{\frac{3}{4}}\,dx = \frac{4}{7}x^{\frac{7}{4}} + C$

102. $\int \frac{1}{x^2\sqrt[3]{x^2}}\ dx = -\frac{3}{5}x^{-\frac{5}{3}} + C$

$\int \frac{1}{x^2\sqrt[3]{x^2}}\,dx = \int \frac{1}{x^2 \cdot x^{\frac{2}{3}}}\,dx = \int \frac{1}{x^{\frac{8}{3}}}\,dx$

$= \int x^{-\frac{8}{3}}\,dx = -\frac{3}{5}x^{-\frac{5}{3}} + C$

103. $$\int \frac{3}{\sqrt{2-4x}}\,dx = -\tfrac{3}{2}(2-4x)^{\frac{1}{2}} + C$$

$$\int \frac{3}{\sqrt{2-4x}}\,dx = \int 3(2-4x)^{-\frac{1}{2}}\,dx = \dots \text{ BY INSPECTION}$$
$$= -\frac{3}{2}(2-4x)^{\frac{1}{2}} + C$$

104. $$\int \frac{1}{1+\cos 2x}\,dx = \left[\begin{array}{c} \frac{1}{2}\tan x + C \\ \frac{1}{2}\operatorname{cosec} 2x - \frac{1}{2}\cot 2x + C \end{array}\right]$$

$$\int \frac{1}{1+\cos 2x}\,dx = \int \frac{1}{1+(2\cos^2 x-1)}\,dx = \int \frac{1}{2\cos^2 x}\,dx$$
$$= \int \frac{1}{2}\sec^2 x\,dx = \frac{1}{2}\tan x + C$$

ALTERNATIVE

$$\int \frac{1}{1+\cos 2x}\,dx = \int \frac{1(1-\cos 2x)}{(1+\cos 2x)(1-\cos 2x)}\,dx = \int \frac{1-\cos 2x}{1-\cos^2 2x}\,dx$$
$$= \int \frac{1-\cos 2x}{\sin^2 2x}\,dx = \int \frac{1}{\sin^2 2x} - \frac{\cos 2x}{\sin^2 2x}\,dx = \int \operatorname{cosec}^2 2x - \frac{\cos 2x}{\sin 2x}\frac{1}{\sin 2x}\,dx$$
$$= \int \operatorname{cosec}^2 2x - \cot 2x \operatorname{cosec} 2x\,dx = -\frac{1}{2}\cot 2x + \frac{1}{2}\operatorname{cosec} 2x + C$$

$\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$ & $\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$

105. $$\int \frac{2}{2x-x^2}\,dx = \ln\left|\frac{x}{2-x}\right| + C$$

$$\int \frac{2}{2x-x^2}\,dx = \int \frac{2}{x(2-x)}\,dx = \dots \text{ BY PARTIAL FRACTIONS}$$

$$\frac{2}{x(2-x)} \equiv \frac{A}{x} + \frac{B}{2-x}$$
$$\boxed{2 \equiv A(2-x) + Bx}$$

- If $x=2$: $2 = 2B$, $B=1$
- If $x=0$: $2 = 2A$, $A=1$

$$= \int \frac{1}{x} + \frac{1}{2-x}\,dx = \ln|x| - \ln|2-x| + C = \ln\left|\frac{x}{2-x}\right| + C$$

106. $$\int \frac{\ln x}{x^2}\,dx = -\frac{1}{x} - \frac{\ln|x|}{x} + C$$

$$\int \frac{\ln x}{x^2}\,dx = \int x^{-2}\ln x\,dx = \ldots$$ INTEGRATION BY PARTS

$\ln x$	$\frac{1}{x}$
$-x^{-1}$	x^{-2}

$$= -x^{-1}\ln|x| - \int -x^{-1}\left(\tfrac{1}{x}\right)dx$$
$$= -\frac{1}{x}\ln|x| + \int x^{-2}\,dx$$
$$= -\frac{1}{x}\ln|x| - x^{-1} + C$$
$$= -\frac{1}{x}\ln|x| - \frac{1}{x} + C$$

107. $$\int \frac{3x^2}{x^3+1}\,dx = \ln\left|x^3+1\right| + C$$

$$\int \frac{3x^2}{x^3+1}\,dx = \ldots$$ OF THE FORM $\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + C$

$$= \ln|x^3+1| + C$$

THE SUBSTITUTION $u = x^3+1$ ALSO WORKS WELL

108. $$\int \frac{2x+1}{2x-1}\,dx = x + \ln|2x-1| + C$$

$$\int \frac{2x+1}{2x-1}\,dx =$$ BY MANIPULATION & SPLITTING

$$= \int \frac{(2x-1)+2}{(2x-1)}\,dx = \int \frac{2x-1}{2x-1} + \frac{2}{2x-1}\,dx$$
$$= \int 1 + \frac{2}{2x-1}\,dx = x + \ln|2x-1| + C$$

ALTERNATIVE BY SUBSTITUTION

$u = 2x-1$

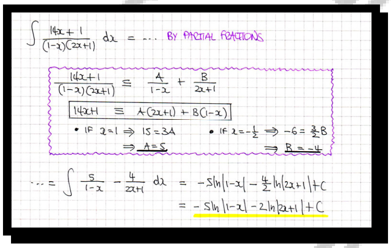
$\frac{du}{dx} = 2$

$dx = \frac{du}{2}$

$2x = u+1$

$$\int \frac{2x+1}{2x-1}\,dx = \int \frac{2x+1}{u}\,\frac{du}{2}$$
$$= \int \frac{2x+1}{2u}\,du = \int \frac{u+2}{2u}\,du$$
$$= \int \frac{u}{2u} + \frac{2}{2u}\,du = \int \frac{1}{2} + \frac{1}{u}\,du$$
$$= \frac{1}{2}u + \ln|u| + C = \frac{1}{2}(2x-1) + \ln|2x-1| + C$$
$$= x + \ln|2x-1| + C$$

109. $$\int \frac{14x+1}{(1-x)(2x+1)}\,dx = -5\ln|1-x| - 2\ln|2x+1| + C$$



$\int \frac{14x+1}{(1-x)(2x+1)}\,dx = \dots$ BY PARTIAL FRACTIONS

$\frac{14x+1}{(1-x)(2x+1)} \equiv \frac{A}{1-x} + \frac{B}{2x+1}$

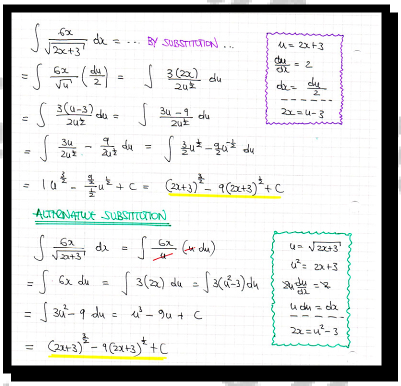
$\boxed{14x+1 \equiv A(2x+1) + B(1-x)}$

- IF $x=1 \Rightarrow 15 = 3A \Rightarrow A=5$
- IF $x=-\frac{1}{2} \Rightarrow -6 = \frac{3}{2}B \Rightarrow B=-4$

$\dots = \int \frac{5}{1-x} - \frac{4}{2x+1}\,dx = -5\ln|1-x| - \frac{4}{2}\ln|2x+1| + C$

$= -5\ln|1-x| - 2\ln|2x+1| + C$

110. $$\int \frac{6x}{\sqrt{2x+3}}\,dx = (2x+3)^{\frac{3}{2}} - 9(2x+3)^{\frac{1}{2}} + C$$



$\int \frac{6x}{\sqrt{2x+3}}\,dx = \dots$ BY SUBSTITUTION …

$u = 2x+3$, $\frac{du}{dx} = 2$, $dx = \frac{du}{2}$, $2x = u-3$

$= \int \frac{6x}{\sqrt{u}}\left(\frac{du}{2}\right) = \int \frac{3(2x)}{2u^{\frac{1}{2}}}\,du$

$= \int \frac{3(u-3)}{2u^{\frac{1}{2}}}\,du = \int \frac{3u-9}{2u^{\frac{1}{2}}}\,du$

$= \int \frac{3u}{2u^{\frac{1}{2}}} - \frac{9}{2u^{\frac{1}{2}}}\,du = \int \frac{3}{2}u^{\frac{1}{2}} - \frac{9}{2}u^{-\frac{1}{2}}\,du$

$= 1u^{\frac{3}{2}} - \frac{\frac{9}{2}}{\frac{1}{2}}u^{\frac{1}{2}} + C = (2x+3)^{\frac{3}{2}} - 9(2x+3)^{\frac{1}{2}} + C$

ALTERNATIVE SUBSTITUTION

$u = \sqrt{2x+3}$, $u^2 = 2x+3$, $\cancel{2}u\frac{du}{dx} = \cancel{2}$, $u\,du = dx$, $2x = u^2-3$

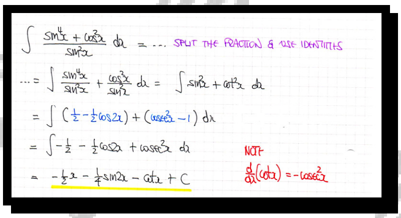
$\int \frac{6x}{\sqrt{2x+3}}\,dx = \int \frac{6x}{\cancel{u}}(\cancel{u}\,du)$

$= \int 6x\,du = \int 3(2x)\,du = \int 3(u^2-3)\,du$

$= \int 3u^2 - 9\,du = u^3 - 9u + C$

$= (2x+3)^{\frac{3}{2}} - 9(2x+3)^{\frac{1}{2}} + C$

111. $$\int \frac{\sin^4 x + \cos^2 x}{\sin^2 x}\,dx = -\frac{1}{2}x - \frac{1}{4}\sin 2x - \cot x + C$$



$\int \frac{\sin^4 x + \cos^2 x}{\sin^2 x}\,dx = \dots$ SPLIT THE FRACTION & USE IDENTITIES

$\dots = \int \frac{\sin^4 x}{\sin^2 x} + \frac{\cos^2 x}{\sin^2 x}\,dx = \int \sin^2 x + \cot^2 x\,dx$

$= \int \left(\frac{1}{2} - \frac{1}{2}\cos 2x\right) + \left(\operatorname{cosec}^2 x - 1\right)dx$

$= \int -\frac{1}{2} - \frac{1}{2}\cos 2x + \operatorname{cosec}^2 x\,dx$

$= -\frac{1}{2}x - \frac{1}{4}\sin 2x - \cot x + C$

NOTE $\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$

112. $\displaystyle\int \frac{2}{(x-4)\sqrt{x}}\,dx = \ln\left|\frac{\sqrt{x}-2}{\sqrt{x}+2}\right| + C$

$\displaystyle\int \frac{2}{(x-4)\sqrt{x}}\,dx = \dots$ BY SUBSTITUTION

$u=\sqrt{x}$
$u^2 = x$
$2u\frac{du}{dx}=1$
$2u\,du = dx$

$\displaystyle = \int \frac{2}{(u^2-4)\cancel{u}}\,(2\cancel{u}\,du) = \int \frac{4}{u^2-4}\,du$

$\displaystyle = \int \frac{4}{(u-2)(u+2)}\,du = \dots$ BY PARTIAL FRACTIONS

$\displaystyle \frac{4}{(u-2)(u+2)} \equiv \frac{A}{u-2} + \frac{B}{u+2}$

$4 \equiv A(u+2) + B(u-2)$

- IF $u=2 \Rightarrow 4 = 4A \Rightarrow \underline{A=1}$
- IF $u=-2 \Rightarrow 4 = -4B \Rightarrow \underline{B=-1}$

$\displaystyle \dots = \int \frac{1}{u-2} - \frac{1}{u+2}\,du = \ln|u-2| + \ln|u+2| + C$

$\displaystyle = \ln\left|\frac{u-2}{u+2}\right| + C = \ln\left|\frac{\sqrt{x}-2}{\sqrt{x}+2}\right| + C$

113. $\displaystyle\int \frac{1}{2+\sqrt{x-1}}\,dx = 2\sqrt{x-1} - 4\ln\left|2+\sqrt{x-1}\right| + C$

$\displaystyle\int \frac{1}{2+\sqrt{x-1}}\,dx = \dots$ BY SUBSTITUTION

$u = 2+\sqrt{x-1}$
$u-2 = \sqrt{x-1}$
$x-1 = (u-2)^2$
$x = 1+(u-2)^2$
$\frac{dx}{du} = 2(u-2)$
$dx = 2(u-2)\,du$

$\displaystyle = \int \frac{1}{u}\,(2(u-2)\,du) = \int \frac{2u-4}{u}\,du$

$\displaystyle = \int \frac{2u}{u} - \frac{4}{u}\,du = \int 2 - \frac{4}{u}\,du$

$= 2u - 4\ln|u| + C$

$= 2\left[2+\sqrt{x-1}\right] - 4\ln\left|2+\sqrt{x-1}\right| + C$

$= 2\sqrt{x-1} - 4\ln\left|2+\sqrt{x-1}\right| + C$

ALTERNATIVE SUBSTITUTION

$u = \sqrt{x-1}$
$u^2 = x-1$
$2u\frac{du}{dx} = 1$
$dx = 2u\,du$

$\displaystyle\int \frac{1}{2+\sqrt{x-1}}\,dx = \int \frac{1}{2+u}\,(2u\,du)$

$\displaystyle = \int \frac{2u}{u+2}\,du$ ← BY ANOTHER SUBSTITUTION $v=u+2$, IN ORDER TO SPLIT THE FRACTION, OR "MANIPULATION TO SPLIT"

$\displaystyle = \int \frac{2(u+2)-4}{u+2}\,du = \int \frac{2(u+2)}{u+2} - \frac{4}{u+2}\,du$

$\displaystyle = \int 2 - \frac{4}{u+2}\,du = 2u - 4\ln|u+2| + C$

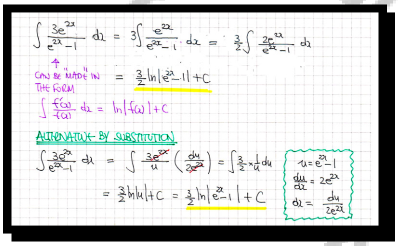
$= 2\sqrt{x-1} - 4\ln\left|\sqrt{x-1}+2\right| + C$

114. $\displaystyle\int \frac{4}{\sqrt{6x-1}}\,dx = \frac{4}{3}\sqrt{6x-1} + C$

$\displaystyle\int \frac{4}{\sqrt{6x-1}}\,dx = \int 4(6x-1)^{-\frac{1}{2}}\,dx = \dots$ RECOGNITION

$= \frac{4}{3}(6x-1)^{\frac{1}{2}} + C = \frac{4}{3}\sqrt{6x-1} + C$

115. $\displaystyle\int \frac{3e^{2x}}{e^{2x}-1}\,dx = \frac{3}{2}\ln\left|e^{2x}-1\right| + C$



$\displaystyle\int \frac{3e^{2x}}{e^{2x}-1}\,dx = 3\int \frac{e^{2x}}{e^{2x}-1}\,dx = \frac{3}{2}\int \frac{2e^{2x}}{e^{2x}-1}\,dx$

CAN BE "MADE" IN THE FORM $\displaystyle\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)|+C$

$= \frac{3}{2}\ln|e^{2x}-1|+C$

ALTERNATIVE BY SUBSTITUTION

$\displaystyle\int \frac{3e^{2x}}{e^{2x}-1}\,dx = \int \frac{3\cancel{e^{2x}}}{u}\left(\frac{du}{2\cancel{e^{2x}}}\right) = \int \frac{3}{2}\times\frac{1}{u}\,du$

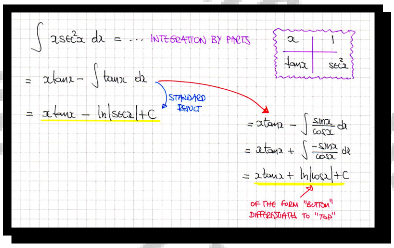
$= \frac{3}{2}\ln|u|+C = \frac{3}{2}\ln|e^{2x}-1|+C$

$u = e^{2x}-1$

$\frac{du}{dx} = 2e^{2x}$

$dx = \frac{du}{2e^{2x}}$

116. $\displaystyle\int x\sec^2 x\,dx = \begin{bmatrix} x\tan x - \ln|\sec x| + C \\ x\tan x + \ln|\cos x| + C \end{bmatrix}$



$\displaystyle\int x\sec^2 x\,dx = \ldots$ INTEGRATION BY PARTS

x	1
$\tan x$	$\sec^2 x$

$\displaystyle = x\tan x - \int \tan x\,dx$

STANDARD RESULT

$= x\tan x - \ln|\sec x| + C$

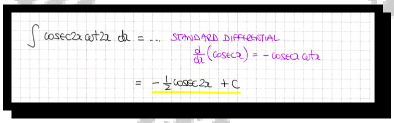
$\displaystyle = x\tan x - \int \frac{\sin x}{\cos x}\,dx$

$\displaystyle = x\tan x + \int \frac{-\sin x}{\cos x}\,dx$

$= x\tan x + \ln|\cos x| + C$

OF THE FORM "BOTTOM" DIFFERENTIATE TO "TOP"

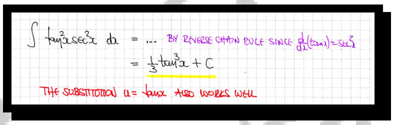
117. $\displaystyle\int \operatorname{cosec} 2x\cot 2x\,dx = -\frac{1}{2}\operatorname{cosec} 2x + C$



$\displaystyle\int \operatorname{cosec}2x\cot 2x\,dx = \ldots$ STANDARD DIFFERENTIAL $\frac{d}{dx}(\operatorname{cosec}x) = -\operatorname{cosec}x\cot x$

$= -\frac{1}{2}\operatorname{cosec}2x + C$

118. $\displaystyle\int \tan^2 x\sec^2 x\,dx = \frac{1}{3}\tan^3 x + C$



$\displaystyle\int \tan^2 x\sec^2 x\,dx = \ldots$ BY REVERSE CHAIN RULE SINCE $\frac{d}{dx}(\tan x) = \sec^2 x$

$= \frac{1}{3}\tan^3 x + C$

THE SUBSTITUTION $u = \tan x$ ALSO WORKS WELL

119. $\displaystyle\int \sin 2x \operatorname{cosec} x \ dx = 2\sin x + C$

$\int \sin 2x \operatorname{cosec} x \, dx = \ldots$ SIMPLIFY & INTEGRATE

$= \int (2\cancel{\sin x}\cos x)\left(\frac{1}{\cancel{\sin x}}\right) dx$

$= \int 2\cos x \, dx$

$= \underline{2\sin x + C}$

120. $\displaystyle\int (2\cos x - 3\sin x)^2 \ dx = \frac{13}{2}x - \frac{5}{4}\sin 2x + 3\cos 2x + C$

$\int (2\cos x - 3\sin x)^2 \, dx = \ldots$ EXPAND & USE IDENTITIES …

$= \int 4\cos^2 x - 12\cos x \sin x + 9\sin^2 x \, dx$

$= \int 4\left(\frac{1}{2} + \frac{1}{2}\cos 2x\right) - 6(2\sin x \cos x) + 9\left(\frac{1}{2} - \frac{1}{2}\cos 2x\right) dx$

$= \int 2 + 2\cos 2x - 6\sin 2x + \frac{9}{2} - \frac{9}{2}\cos 2x \, dx$

$= \int \frac{13}{2} - \frac{5}{2}\cos 2x - 6\sin 2x \, dx = \underline{\frac{13}{2}x - \frac{5}{4}\sin 2x + 3\cos 2x + C}$

121. $\displaystyle\int \frac{\sin x + \tan x}{\cos x} \ dx = \begin{bmatrix} \sec x + \ln|\sec x| + C \\ \sec x - \ln|\cos x| + C \end{bmatrix}$

$\int \frac{\sin x + \tan x}{\cos x} \, dx = \ldots$ SPLIT & USE TRIGONOMETRIC RESULTS

$= \int \frac{\sin x}{\cos x} + \frac{\tan x}{\cos x} \, dx = \int \tan x + \tan x \sec x \, dx$

$= \underline{\ln|\sec x| + \sec x + C}$

STANDARD RESULT $\qquad \frac{d}{dx}(\sec x) = \sec x \tan x$

[OR $\underline{-\ln|\cos x| + \sec x + C}$]

$\int \tan x \, dx = \int \frac{\sin x}{\cos x} dx = -\int \frac{-\sin x}{\cos x} dx$

$= -\ln|\cos x| + C$

AS IT IS OF THE FORM "BOTTOM" DIFFERENTIATES TO "TOP"

122. $\displaystyle\int \frac{1 + \sin x}{\cos^2 x} \ dx = \tan x + \sec x + C$

$\int \frac{1 + \sin x}{\cos^2 x} \, dx = \ldots$ SPLIT THE FRACTION & USE TRIG RESULTS

$= \int \frac{1}{\cos^2 x} + \frac{\sin x}{\cos^2 x} \, dx = \int \sec^2 x + \frac{\sin x}{\cos x} \times \frac{1}{\cos x} dx$

$= \int \sec^2 x + \tan x \sec x \, dx = \underline{\tan x + \sec x + C}$

$\frac{d}{dx}(\tan x) = \sec^2 x$ & $\frac{d}{dx}(\sec x) = \sec x \tan x$

123. $\displaystyle\int e^{\sin x}\cos x \; dx = e^{\sin x} + C$

$\displaystyle\int e^{\sin x}\cos x \, dx = \ldots$ BY REVERSE CHAIN RULE SINCE $\frac{d}{dx}\left(e^{\sin x}\right) = e^{\sin x}\cos x$

$= e^{\sin x} + C$

THE SUBSTITUTION $u = \sin x$ ALSO WORKS WELL

124. $\displaystyle\int x^2 \sin x \; dx = -x^2\cos x + 2x\sin x + 2\cos x + C$

$\displaystyle\int x^2 \sin x \, dx = \ldots$ INTEGRATION BY PARTS TWICE

$\begin{array}{c|c} x^2 & 2x \\ \hline -\cos x & \sin x \end{array}$

$= -x^2\cos x - \displaystyle\int -2x\cos x \, dx$

$= -x^2\cos x + \displaystyle\int 2x\cos x \, dx$

REPEATING THE PROCESS

$\begin{array}{c|c} 2x & 2 \\ \hline \sin x & \cos x \end{array}$

$= -x^2\cos x + \left[2x\sin x - \displaystyle\int 2\sin x \, dx\right]$

$= -x^2\cos x + 2x\sin x - \displaystyle\int 2\sin x \, dx$

$= -x^2\cos x + 2x\sin x + 2\cos x + C$

125. $\displaystyle\int (2x+1)^3 \; dx = \frac{1}{8}(2x+1)^4 + C$

$\displaystyle\int (2x+1)^3 \, dx = \ldots$ BY RECOGNITION $\ldots = \frac{1}{8}(2x+1)^4 + C$

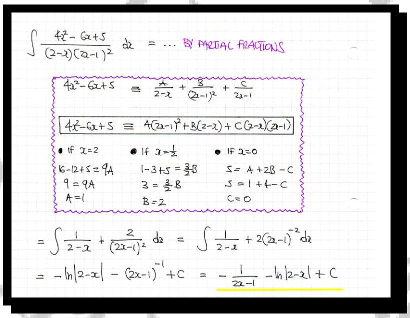
126. $\displaystyle\int \frac{\tan^4 x}{\cos^2 x} \; dx = \frac{1}{5}\tan^5 x + C$

$\displaystyle\int \frac{\tan^4 x}{\cos^2 x} \, dx = \int \tan^4 x \sec^2 x \, dx = \ldots$ BY REVERSE CHAIN RULE AS $\frac{d}{dx}(\tan x) = \sec^2 x$

$= \frac{1}{5}\tan^5 x + C$

THE SUBSTITUTION $u = \tan x$ ALSO WORKS WELL

127. $$\int \frac{4x^2-6x+5}{(2-x)(2x-1)^2}\,dx = -\frac{1}{2x-1} - \ln|2-x| + C$$



$$\int \frac{4x^2-6x+5}{(2-x)(2x-1)^2}\,dx = \ldots \text{ BY PARTIAL FRACTIONS}$$

$$\frac{4x^2-6x+5}{(2-x)(2x-1)^2} \equiv \frac{A}{2-x} + \frac{B}{(2x-1)^2} + \frac{C}{2x-1}$$

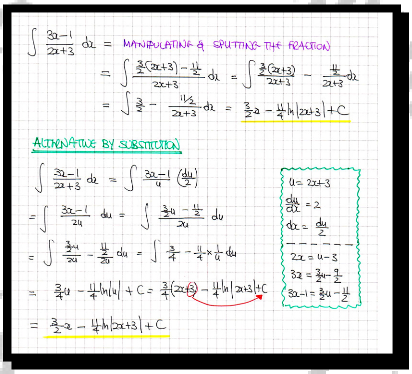
$$\boxed{4x^2-6x+5 \equiv A(2x-1)^2 + B(2-x) + C(2-x)(2x-1)}$$

- If $x=2$: $16-12+5=9A$, $9=9A$, $A=1$
- If $x=\frac{1}{2}$: $1-3+5=\frac{3}{2}B$, $3=\frac{3}{2}B$, $B=2$
- If $x=0$: $5=A+2B-C$, $5=1+4-C$, $C=0$

$$= \int \frac{1}{2-x} + \frac{2}{(2x-1)^2}\,dx = \int \frac{1}{2-x} + 2(2x-1)^{-2}\,dx$$

$$= -\ln|2-x| - (2x-1)^{-1} + C = -\frac{1}{2x-1} - \ln|2-x| + C$$

128. $$\int \frac{3x-1}{2x+3}\,dx = \frac{3}{2}x - \frac{11}{4}\ln|2x+3| + C$$



$$\int \frac{3x-1}{2x+3}\,dx = \text{MANIPULATING \& SPLITTING THE FRACTION}$$

$$= \int \frac{\frac{3}{2}(2x+3)-\frac{11}{2}}{2x+3}\,dx = \int \frac{\frac{3}{2}(2x+3)}{2x+3} - \frac{\frac{11}{2}}{2x+3}\,dx$$

$$= \int \frac{3}{2} - \frac{11/2}{2x+3}\,dx = \frac{3}{2}x - \frac{11}{4}\ln|2x+3| + C$$

ALTERNATIVE BY SUBSTITUTION

$u = 2x+3$, $\frac{du}{dx} = 2$, $dx = \frac{du}{2}$

$2x = u-3$, $3x = \frac{3}{2}u - \frac{9}{2}$, $3x-1 = \frac{3}{2}u - \frac{11}{2}$

$$\int \frac{3x-1}{2x+3}\,dx = \int \frac{3x-1}{u}\left(\frac{du}{2}\right)$$

$$= \int \frac{3x-1}{2u}\,du = \int \frac{\frac{3}{2}u - \frac{11}{2}}{2u}\,du$$

$$= \int \frac{\frac{3}{2}u}{2u} - \frac{\frac{11}{2}}{2u}\,du = \int \frac{3}{4} - \frac{11}{4}\times\frac{1}{u}\,du$$

$$= \frac{3}{4}u - \frac{11}{4}\ln|u| + C = \frac{3}{4}(2x+3) - \frac{11}{4}\ln|2x+3| + C$$

$$= \frac{3}{2}x - \frac{11}{4}\ln|2x+3| + C$$

129. $$\int \frac{8x^2}{1-2x}\,dx = \begin{bmatrix} -2x^2-2x-\ln|1-2x|+C \\ 2(1-2x)-\frac{1}{2}(1-2x)^2-\ln|1-2x|+C \end{bmatrix}$$

$$\int \frac{8x^2}{1-2x}\,dx = \dots \text{ BY MANIPULATION & SPLITTING THE FRACTION}$$
$$= \int \frac{2(1-2x)^2-4(1-2x)+2}{1-2x}\,dx$$
$$= \int \frac{2(1-2x)^2}{1-2x}-\frac{4(1-2x)}{1-2x}+\frac{2}{1-2x}\,dx$$
$$= \int 2(1-2x)-4+\frac{2}{1-2x}\,dx$$
$$= \int -4x-2+\frac{2}{1-2x}\,dx$$
$$= -2x^2-2x-\ln|1-2x|+C$$

ALTERNATIVE MANIPULATION

$$8x^2 = -4x(1-2x)-2(1-2x)+2$$

ALTERNATIVE BY SUBSTITUTION

$$\int \frac{8x^2}{1-2x}\,dx = \int \frac{8x^2}{u}\left(-\frac{du}{2}\right)$$
$$= \int -\frac{4x^2}{u}\,du = \int -\frac{1-2u+u^2}{u}\,du$$
$$= \int -\frac{1}{u}+\frac{2u}{u}-\frac{u^2}{u}\,du$$
$$= \int -\frac{1}{u}+2-u\,du$$
$$= -\ln|u|+2u-\frac{1}{2}u^2+C$$
$$= -\ln|1-2x|+2(1-2x)-\frac{1}{2}(1-2x)^2+C$$
$$\dots = -\ln|1-2x|+2-4x-\frac{1}{2}+2x-2x^2+C$$
$$= -\ln|1-2x|-2x-2x^2+C \quad \text{(AS BEFORE)}$$

$$u = 1-2x$$
$$\frac{du}{dx} = -2$$
$$dx = -\frac{du}{2}$$
$$2x = 1-u$$
$$4x^2 = 1-2u+u^2$$

130. $$\int \frac{10}{(3x+1)^{\frac{3}{2}}}\,dx = -\frac{20}{3\sqrt{3x+1}}+C$$

$$\int \frac{10}{(3x+1)^{\frac{3}{2}}}\,dx = \int 10(3x+1)^{-\frac{3}{2}}\,dx = \dots \text{ BY RECOGNITION}$$
$$= \frac{10}{-\frac{3}{2}}(3x+1)^{-\frac{1}{2}}+C = -\frac{20}{3}(3x+1)^{-\frac{1}{2}}+C$$
$$= -\frac{20}{3\sqrt{3x+1}}+C$$

131. $$\int 5^x\,dx = \frac{5^x}{\ln 5}+C$$

$$\int 5^x\,dx = \dots \text{ BY RECOGNITION SINCE } \frac{d}{dx}(5^x) = 5^x\ln 5$$
$$= \frac{1}{\ln 5}5^x+C = \frac{5^x}{\ln 5}+C$$

132. $$\int \sqrt{\sin x \cos^2 x}\ dx = \tfrac{2}{3}(\sin x)^{\frac{3}{2}} + C$$

$$\int \sqrt{\sin x \cos^2 x}\ dx \ dx = \int (\sin x)^{\frac{1}{2}} \cos x \ dx$$
$= \dots$ BY REVERSE CHAIN RULE
$$= \frac{2}{3}(\sin x)^{\frac{3}{2}} + C$$

THE SUBSTITUTION $u = \sin x$, OR $u = \sqrt{\sin x}$ ALSO WORKS

133. $$\int (2x+1)\sin\left(x^2+x+1\right)\ dx = -\cos\left(x^2+x+1\right) + C$$

$\int (2x+1)\sin(x^2+x+1)\ dx = \dots$ BY REVERSE CHAIN RULE SINCE $\frac{d}{dx}(x^2+x+1) = 2x+1$
$$= -\cos(x^2+x+1) + C$$

(THE SUBSTITUTION $u = x^2+x+1$ ALSO WORKS)

134. $$\int (2x+1)\left(x^2+x+1\right)\ dx = \begin{bmatrix} \tfrac{1}{2}\left(x^2+x+1\right)^2 + C \\ \tfrac{1}{2}x^4 + x^3 + \tfrac{3}{2}x^2 + x + C \end{bmatrix}$$

$\int (2x+1)(x^2+x+1)\ dx =$ BY REVERSE CHAIN RULE SINCE $\frac{d}{dx}(x^2+x+1) = 2x+1$
$$= \frac{1}{2}(x^2+x+1)^2 + C$$

(THE SUBSTITUTION $u = x^2+x+1$, ALSO WORKS)

ALTERNATIVE BY EXPANSION ... $\frac{1}{2}x^4 + x^3 + \frac{3}{2}x^2 + x + C$

135. $$\int (x+1)\cos\left(x^2+2x+1\right)\ dx = \tfrac{1}{2}\sin\left(x^2+2x+1\right) + C$$

$\int (x+1)\cos(x^2+2x+1) = \dots$ BY REVERSE CHAIN RULE SINCE $\frac{d}{dx}(x^2+2x+1) = 2x+2 = 2(x+1)$
$$= \frac{1}{2}\sin(x^2+2x+1) + C$$

(THE SUBSTITUTION $u = x^2+2x+1$, ALSO WORKS)

136. $\displaystyle\int \frac{1}{2+\sqrt{x}}\,dx = 2\sqrt{x} - 4\ln\left(2+\sqrt{x}\right) + C$

$$\int \frac{1}{2+\sqrt{x}}\,dx = \ldots \quad \text{BY SUBSTITUTION}$$

$$u = 2+\sqrt{x}, \quad u-2 = \sqrt{x}, \quad x = (u-2)^2, \quad \frac{dx}{du} = 2(u-2), \quad dx = 2(u-2)\,du$$

$$= \int \frac{1}{u}\left(2(u-2)\,du\right) = \int \frac{2u-4}{u}\,du$$

$$= \int \frac{2u}{u} - \frac{4}{u}\,du = \int 2 - \frac{4}{u}\,du$$

$$= 2u - 4\ln|u| + C = 2\left[2+\sqrt{x}\right] - 4\ln\left(2+\sqrt{x}\right) + C$$

$$= 2\sqrt{x} - 4\ln\left(2+\sqrt{x}\right) + C$$

(THE SUBSTITUTION $u=\sqrt{x}$ ALSO WORKS)

137. $\displaystyle\int \frac{1}{\mathrm{e}^{x}+\mathrm{e}^{-x}+2}\,dx = -\frac{1}{\mathrm{e}^{x}+1} + C$

$$\int \frac{1}{e^{x}+e^{-x}+2}\,dx = \ldots \quad \text{BY SUBSTITUTION}$$

$$u = e^{x}, \quad \frac{du}{dx} = e^{x}, \quad \frac{du}{dx} = u, \quad dx = \frac{du}{u}$$

$$= \int \frac{1}{u + \frac{1}{u} + 2}\left(\frac{du}{u}\right) = \int \frac{1}{u^2+1+2u}\,du$$

$$= \int \frac{1}{u^2+2u+1}\,du = \int \frac{1}{(u+1)^2}\,du$$

$$= \int (u+1)^{-2}\,du = -(u+1)^{-1} + C = -\frac{1}{e^{x}+1} + C$$

138. $\displaystyle\int x^2\tan\left(x^3+1\right)\,dx = \tfrac{1}{3}\ln\left|\sec\left(x^3+1\right)\right| + C$

$$\int x^2\tan(x^3+1) = \ldots \quad \text{BY REVERSE CHAIN RULE SINCE } \frac{d}{dx}(x^3+1) = x^2$$

$$= \frac{1}{3}\ln\left|\sec(x^3+1)\right| + C$$

$$\text{SINCE } \int \tan x\,dx = \ln|\sec x| + C$$

(THE SUBSTITUTION $u = x^3+1$, ALSO WORKS)

139. $$\int x^3 \ln\left(x^2+1\right)\, dx = \begin{bmatrix} \frac{1}{4}\left(x^2+1\right)\left(x^2-1\right)\ln\left(x^2+1\right)-\frac{1}{8}\left(x^2+1\right)\left(x^2-3\right)+C \\ \frac{1}{4}\left(x^4-1\right)\ln\left(x^2+1\right)-\frac{1}{8}x^2\left(x^2-2\right)+C \end{bmatrix}$$

$\int x^3 \ln(x^2+1)\, dx = \dots$ BY SUBSTITUTION FIRST

$u = x^2+1$
$\frac{du}{dx} = 2x$
$dx = \frac{du}{2x}$
$x^2 = u-1$

$= \int x^3 \ln u \left(\frac{du}{2x}\right) = \int \frac{1}{2}x^2 \ln u\, du$

$= \int \frac{1}{2}(u-1)\ln u\, du$

PROCEED BY INTEGRATION BY PARTS

$\ln u$	$\frac{1}{u}$
$\frac{1}{4}(u-1)^2$	$\frac{1}{2}(u-1)$

$= \frac{1}{4}(u-1)^2\ln|u| - \int \frac{1}{4}\frac{(u-1)^2}{u}\, du$

$= \frac{1}{4}(u-1)^2\ln|u| - \int \frac{u^2-2u+1}{4u}\, du$

$= \frac{1}{4}(u-1)^2\ln|u| - \int \frac{1}{4}u - \frac{1}{2} + \frac{1}{4}\times\frac{1}{u}\, du$

$= \frac{1}{4}(u-1)^2\ln|u| - \frac{1}{8}u^2 + \frac{1}{2}u - \frac{1}{4}\ln|u| + C$

$= \frac{1}{4}\left[(u-1)^2-1\right]\ln|u| - \frac{1}{8}u^2 + \frac{1}{2}u + C$

$= \frac{1}{4}(u^2-2u)\ln|u| - \frac{u}{8}(u-4) + C$

$= \frac{1}{4}u(u-2)\ln|u| - \frac{1}{8}u(u-4) + C$

$= \frac{1}{4}(x^2+1)(x^2-1)\ln(x^2+1) - \frac{1}{8}(x^2+1)(x^2-3) + C$

$= \frac{1}{4}(x^4-1)\ln(x^2+1) - \frac{1}{8}(x^4-2x^2-3) + C$

$= \frac{1}{4}(x^4-1)\ln(x^2+1) - \frac{1}{8}x^2(x^2-2) + C$

140. $$\int \sin^2 x \sec^2 x\ \, dx = -x + \tan x + C$$

$\int \sin^2 x \sec^2 x\, dx = \int \sin^2 x \times \frac{1}{\cos^2 x}\, dx = \int \frac{\sin^2 x}{\cos^2 x}\, dx$

$= \int \tan^2 x\, dx = \int \sec^2 x - 1\, dx$

$= \tan x - x + C$

SINCE $\frac{d}{dx}(\tan x) = \sec^2 x$

141. $$\int 3\sec^2 x \sin x\ \, dx = 3\sec x + C$$

$\int 3\sec^2 x \sin x\, dx = \int 3\left(\frac{1}{\cos^2 x}\right)\sin x\, dx = \int \frac{3\sin x}{\cos x}\times\frac{1}{\cos x}\, dx$

$= \int 3\tan x \sec x\, dx = 3\sec x + C$

SINCE $\frac{d}{dx}(\sec x) = \sec x \tan x$

142. $\displaystyle\int \frac{1}{x(1+\ln x)^3}\,dx = -\frac{1}{2(1+\ln x)^2} + C$

$\displaystyle\int \frac{1}{x(1+\ln x)^3}\,dx = \ldots$ BY REVERSE CHAIN RULE SINCE $\frac{d}{dx}(\ln x) = \frac{1}{x}$

$\displaystyle = \int \frac{1}{x}(1+\ln x)^{-3}\,dx$

$\displaystyle = \frac{1}{-2}(1+\ln x)^{-2} + C$

$\displaystyle = -\frac{1}{2(1+\ln x)^2} + C$

ALTERNATIVE BY SUBSTITUTION

$u = 1+\ln x$

$\frac{du}{dx} = \frac{1}{x}$

$dx = x\,du$

$\displaystyle\int \frac{1}{x(1+\ln x)^3}\,dx = \int \frac{1}{\cancel{x}u^3}(\cancel{x}\,du)$

$\displaystyle = \int \frac{1}{u^3}\,du = \int u^{-3}\,du = -\frac{1}{2}u^{-2} + C$

$\displaystyle = -\frac{1}{2u^2} + C = -\frac{1}{2(1+\ln x)^2} + C$

143. $\displaystyle\int x^3 \ln x\,dx = \frac{1}{4}x^4 \ln|x| - \frac{1}{16}x^4 + C$

$\displaystyle\int x^3 \ln x\,dx =$ INTEGRATION BY PARTS

$\ln x$	$\frac{1}{x}$
$\frac{1}{4}x^4$	x^3

$\displaystyle = \frac{1}{4}x^4\ln|x| - \int \frac{1}{4}x^3\,dx$

$\displaystyle = \frac{1}{4}x^4\ln|x| - \frac{1}{16}x^4 + C$

144. $\displaystyle\int x\ln\left(2x^3\right)\,dx = \frac{1}{2}x^2\ln\left|2x^3\right| - \frac{3}{4}x^2 + C$

$\displaystyle\int x\ln(2x^3)\,dx = \ldots$ INTEGRATION BY PARTS

$\ln(2x^3)$	$\frac{6x^2}{2x^3} = \frac{3}{x}$
$\frac{1}{2}x^2$	x

$\displaystyle = \frac{1}{2}x^2\ln(2x^3) - \int \frac{1}{2}x^2\left(\frac{3}{x}\right)\,dx$

$\displaystyle = \frac{1}{2}x^2\ln(2x^3) - \int \frac{3}{2}x\,dx$

$\displaystyle = \frac{1}{2}x^2\ln(2x^3) - \frac{3}{4}x^2 + C$

145. $\displaystyle\int 4 - \cos^4 x \sin x\,dx = 4x + \frac{1}{5}\cos^5 x + C$

$\displaystyle\int 4 - \cos^4 x\sin x\,dx = \ldots$ BY REVERSE CHAIN RULE ...

$\displaystyle = 4x + \frac{1}{5}\cos^5 x + C$

(THE SUBSTITUTION $u = \cos x$ ALSO WORKS WELL)

146. $$\int \frac{\cos x}{\sin^3 x}\,dx = \begin{bmatrix} -\frac{1}{2}\cot^2 x + C \\ -\frac{1}{2}\operatorname{cosec}^2 x + C \end{bmatrix}$$

$$\int \frac{\cos x}{\sin^3 x}\,dx = \int \cos x(\sin x)^{-3}\,dx = \dots$$ BY REVERSE CHAIN RULE

$$= -\frac{1}{2}(\sin x)^{-2} + C = -\frac{1}{2}\operatorname{cosec}^2 x + C$$

VARIATIONS BY REVERSE CHAIN RULE

$$\int \frac{\cos x}{\sin^3 x}\,dx = \int \frac{\cos x}{\sin x} \times \frac{1}{\sin^2 x}\,dx = \int \cot x \operatorname{cosec}^2 x\,dx$$

$$= \begin{cases} -\frac{1}{2}\cot^2 x + C & \leftarrow \frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x \\ -\frac{1}{2}\operatorname{cosec}^2 x + C & \leftarrow \frac{d}{dx}(\operatorname{cosec}^2 x) = 2\operatorname{cosec} x(-\operatorname{cosec} x \cot x) \end{cases}$$

THE SUBSTITUTIONS, $u = \sin x$, $u = \cot x$, $u = \operatorname{cosec} x$ ALL WORK WELL

147. $$\int \frac{4\sec^2 x}{\tan x}\,dx = \begin{bmatrix} 4\ln|\tan x| + C \\ 4\ln|\sec x| + 4\ln|\sin x| + C \end{bmatrix}$$

$$\int \frac{4\sec^2 x}{\tan x}\,dx = \dots$$ BY REVERSE CHAIN RULE

$$= 4\ln|\tan x| + C$$

ALTERNATIVE BY SUBSTITUTION

$$\int \frac{4\sec^2 x}{\tan x}\,dx = \int \frac{4\cancel{\sec^2 x}}{u}\left(\frac{du}{\cancel{\sec^2 x}}\right)$$

$$= \int \frac{4}{u}\,du = 4\ln|u| + C$$

$$= 4\ln|\tan x| + C$$

$u = \tan x$
$\frac{du}{dx} = \sec^2 x$
$dx = \frac{du}{\sec^2 x}$

ALTERNATIVE BY TRIGONOMETRIC MANIPULATIONS

$$\int \frac{4\sec^2 x}{\tan x}\,dx = \int \frac{4(\tan^2 x + 1)}{\tan x}\,dx = \int \frac{4\tan^2 x}{\tan x} + \frac{4}{\tan x}\,dx$$

$$= \int 4\tan x + 4\cot x\,dx = 4\ln|\sec x| + 4\ln|\sin x| + C$$

$$\dots = \left[4\ln|\sec x \sin x| + C = 4\ln|\tan x| + C\right]$$

148. $$\int \sec^2 x \tan x\sqrt{1+\tan x}\,dx = \frac{2}{5}(1+\tan x)^{\frac{5}{2}} - \frac{2}{3}(1+\tan x)^{\frac{3}{2}} + C$$

$$\int \sec^2 x \tan x\sqrt{1+\tan x}\,dx = \dots$$ BY SUBSTITUTION ...

$$= \int \cancel{\sec^2 x}\tan x\, u \frac{2u}{\cancel{\sec^2 x}}\,du = \int 2u^2 \tan x\,du$$

$$= \int 2u^2(u^2-1)\,du = \int 2u^4 - 2u^2\,du$$

$$= \frac{2}{5}u^5 - \frac{2}{3}u^3 + C$$

$$= \frac{2}{5}(1+\tan x)^{\frac{5}{2}} - \frac{2}{3}(1+\tan x)^{\frac{3}{2}} + C$$

$u = \sqrt{1+\tan x}$
$u^2 = 1 + \tan x$
$2u\frac{du}{dx} = \sec^2 x$
$dx = \frac{2u}{\sec^2 x}\,du$
$\tan x = u^2 - 1$

THE SUBSTITUTION: $u = 1 + \tan x$ ALSO WORKS

149. $$\int \frac{\sqrt{1+2\tan x}}{\cos^2 x}\,dx = \frac{1}{3}\left(1+2\tan x\right)^{\frac{3}{2}} + C$$

$$\int \frac{\sqrt{1+2\tan x}}{\cos^2 x}\,dx = \dots$$ BY REVERSE CHAIN RULE

$$= \int (1+2\tan x)^{\frac{1}{2}}\sec^2 x\,dx = \underline{\frac{1}{3}(1+2\tan x)^{\frac{3}{2}}+C}$$

ALTERNATIVE BY SUBSTITUTION

$$\int \frac{u}{\cos^2 x}\left(\frac{u}{\sec^2 x}\,du\right) = \int u^2\,du$$

$$= \frac{1}{3}u^3 + C = \underline{\frac{1}{3}(1+2\tan x)^{\frac{3}{2}}+C}$$

$u=\sqrt{1+2\tan x}$

$u^2 = 1+2\tan x$

$2u\frac{du}{dx} = 2\sec^2 x$

$dx = \frac{u}{\sec^2 x}\,du$

THE SUBSTITUTION $u = 1+2\tan x$ ALSO WORKS

150. $$\int \tan^2 x\;dx = -x+\tan x + C$$

$$\int \tan^2 x\,dx = \int \sec^2 x - 1\,dx = \underline{\tan x - x + C}$$

151. $$\int \frac{(1+\sin x)^2}{\cos^2 x}\,dx = -x+2\tan x+2\sec x + C$$

$$\int \frac{(1+\sin x)^2}{\cos^2 x}\,dx = \int \frac{1+2\sin x+\sin^2 x}{\cos^2 x}\,dx$$

$$= \int \frac{1}{\cos^2 x} + \frac{2\sin x}{\cos^2 x} + \frac{\sin^2 x}{\cos^2 x}\,dx$$

$$= \int \frac{1}{\cos^2 x} + \frac{2\sin x}{\cos x}\times\frac{1}{\cos x} + \frac{\sin^2 x}{\cos^2 x}\,dx$$

$$= \int \sec^2 x + 2\tan x\sec x + \tan^2 x\,dx$$

$$= \int \sec^2 x + 2\tan x\sec x + (\sec^2 x - 1)\,dx$$

$$= \int 2\sec^2 x + 2\tan x\sec x - 1\,dx$$

$$= \underline{2\tan x + 2\sec x - x + C}$$

152. $$\int \frac{\cos^2 x}{1+\sin x}\,dx = x+\cos x + C$$

$$\int \frac{\cos^2 x}{1+\sin x}\,dx = \int \frac{1-\sin^2 x}{1+\sin x}\,dx = \int \frac{(1-\sin x)\cancel{(1+\sin x)}}{\cancel{(1+\sin x)}}\,dx$$

$$= \int 1-\sin x\,dx = \underline{x+\cos x + C}$$

VARIATION

$$\int \frac{\cos^2 x}{1+\sin x}\,dx = \int \frac{\cos^2 x(1-\sin x)}{(1+\sin x)(1-\sin x)}\,dx$$

$$= \int \frac{\cos^2 x(1-\sin x)}{1-\sin^2 x}\,dx = \int \frac{\cancel{\cos^2 x}(1-\sin x)}{\cancel{\cos^2 x}}\,dx$$

$$= \int 1-\sin x\,dx = \underline{x+\cos x+C}$$

153. $$\int \frac{1}{1+\cos x}\,dx = \begin{bmatrix} \operatorname{cosec} x - \cot x + C \\ \tan\left(\frac{1}{2}x\right) + C \end{bmatrix}$$

$$\int \frac{1}{1+\cos x}\,dx = \int \frac{1-\cos x}{(1+\cos x)(1-\cos x)}\,dx = \int \frac{1-\cos x}{1-\cos^2 x}\,dx$$

$$= \int \frac{1-\cos x}{\sin^2 x}\,dx = \int \frac{1}{\sin^2 x} - \frac{\cos x}{\sin^2 x}\,dx = \int \operatorname{cosec}^2 x - \frac{\cos x}{\sin x}\frac{1}{\sin x}\,dx$$

$$= \int \operatorname{cosec}^2 x - \cot x \operatorname{cosec} x\,dx = -\cot x + \operatorname{cosec} x + C$$

ALTERNATIVE USING THE DOUBLE ANGLE IDENTITIES

$$\int \frac{1}{1+\cos x}\,dx = \int \frac{1}{1+\left(2\cos^2\frac{x}{2}-1\right)}\,dx$$

$$\boxed{\begin{aligned}\cos 2\theta &\equiv 2\cos^2\theta - 1\\ \cos\theta &\equiv 2\cos^2\tfrac{\theta}{2} - 1\end{aligned}}$$

$$= \int \frac{1}{2\cos^2\frac{x}{2}}\,dx = \int \frac{1}{2}\sec^2\frac{x}{2}\,dx$$

$$= \tan\frac{x}{2} + C$$

NOTE: $\operatorname{cosec} x - \cot x = \frac{1}{\sin x} - \frac{\cos x}{\sin x} = \frac{1-\cos x}{\sin x} = \frac{1-\left(1-2\sin^2\frac{x}{2}\right)}{2\sin\frac{x}{2}\cos\frac{x}{2}}$

$$= \frac{2\sin^2\frac{x}{2}}{2\sin\frac{x}{2}\cos\frac{x}{2}} = \frac{\sin\frac{x}{2}}{\cos\frac{x}{2}} = \tan\frac{x}{2}$$

154. $$\int \frac{\cos x}{\sqrt{\sin x}}\,dx = 2\sqrt{\sin x} + C$$

$$\int \frac{\cos x}{\sqrt{\sin x}}\,dx = \int \cos x(\sin x)^{-\frac{1}{2}}\,dx = \ldots \text{ BY REVERSE CHAIN RULE...}$$

$$= 2(\sin x)^{\frac{1}{2}} + C$$

THE SUBSTITUTION $u = \sin x$ OR $u = \sqrt{\sin x}$ BOTH WORK WELL

155. $$\int \frac{10x^4}{2x^{\frac{5}{2}}+1}\,dx = 2x^{\frac{5}{2}} - \ln\left(2x^{\frac{5}{2}}+1\right) + C$$

$$\int \frac{10x^4}{2x^{\frac{5}{2}}+1}\,dx = \ldots \text{ BY SUBSTITUTION}$$

$$\boxed{\begin{aligned} u &= 2x^{\frac{5}{2}}+1\\ \frac{du}{dx} &= 5x^{\frac{3}{2}}\\ dx &= \frac{du}{5x^{\frac{3}{2}}}\\ 2x^{\frac{5}{2}} &= u-1\end{aligned}}$$

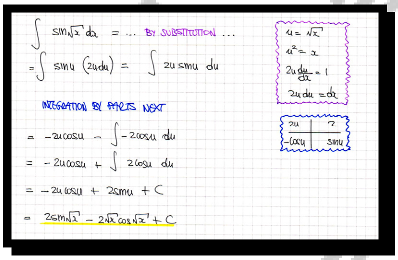
$$= \int \frac{10x^4}{u}\left(\frac{du}{5x^{\frac{3}{2}}}\right) = \int \frac{2x^{\frac{5}{2}}}{u}\,du$$

$$= \int \frac{u-1}{u}\,du = \int 1 - \frac{1}{u}\,du$$

$$= u - \ln|u| = \left(2x^{\frac{5}{2}}+1\right) - \ln\left(2x^{\frac{5}{2}}+1\right) + C$$

$$= 2x^{\frac{5}{2}} - \ln\left(2x^{\frac{5}{2}}+1\right) + C$$

156. $\displaystyle\int \sin\sqrt{x}\ dx = 2\sin\sqrt{x} - 2\sqrt{x}\ \cos\sqrt{x} + C$



$\int \sin\sqrt{x}\, dx = \ldots$ BY SUBSTITUTION ...

$= \int \sin u\,(2u\,du) = \int 2u\sin u\, du$

$u = \sqrt{x}$, $u^2 = x$, $2u\frac{du}{dx} = 1$, $2u\,du = dx$

INTEGRATION BY PARTS NEXT

$2u$	2
$-\cos u$	$\sin u$

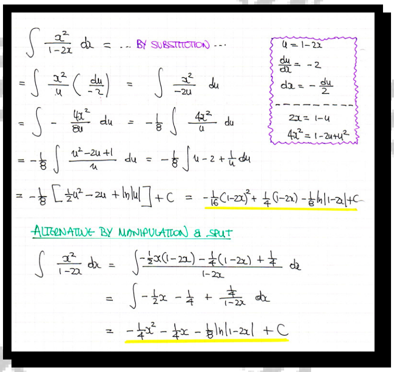
$= -2u\cos u - \int -2\cos u\, du$

$= -2u\cos u + \int 2\cos u\, du$

$= -2u\cos u + 2\sin u + C$

$= 2\sin\sqrt{x} - 2\sqrt{x}\cos\sqrt{x} + C$

157. $$\int \frac{x^2}{1-2x}\ dx = \begin{bmatrix} -\frac{1}{16}(1-2x)^2 + \frac{1}{4}(1-2x) - \frac{1}{8}\ln|1-2x| + C \\ -\frac{1}{4}x^2 - \frac{1}{4}x - \frac{1}{8}\ln|1-2x| + C \end{bmatrix}$$



$\int \frac{x^2}{1-2x}\, dx = \ldots$ BY SUBSTITUTION ...

$u = 1-2x$, $\frac{du}{dx} = -2$, $dx = -\frac{du}{2}$

$2x = 1-u$, $4x^2 = 1-2u+u^2$

$= \int \frac{x^2}{u}\left(\frac{du}{-2}\right) = \int \frac{x^2}{-2u}\, du$

$= \int -\frac{4x^2}{8u}\, du = -\frac{1}{8}\int \frac{4x^2}{u}\, du$

$= -\frac{1}{8}\int \frac{u^2-2u+1}{u}\, du = -\frac{1}{8}\int u - 2 + \frac{1}{u}\, du$

$= -\frac{1}{8}\left[\frac{1}{2}u^2 - 2u + \ln|u|\right] + C = -\frac{1}{16}(1-2x)^2 + \frac{1}{4}(1-2x) - \frac{1}{8}\ln|1-2x| + C$

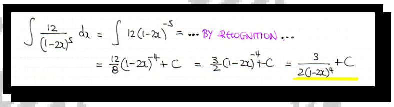
ALTERNATIVE BY MANIPULATION & SPLIT

$\int \frac{x^2}{1-2x}\, dx = \int \frac{-\frac{1}{2}x(1-2x) - \frac{1}{4}(1-2x) + \frac{1}{4}}{1-2x}\, dx$

$= \int -\frac{1}{2}x - \frac{1}{4} + \frac{\frac{1}{4}}{1-2x}\, dx$

$= -\frac{1}{4}x^2 - \frac{1}{4}x - \frac{1}{8}\ln|1-2x| + C$

158. $$\int \frac{12}{(1-2x)^5}\ dx = \frac{3}{2(1-2x)^4} + C$$



$\int \frac{12}{(1-2x)^5}\, dx = \int 12(1-2x)^{-5} = \ldots$ BY RECOGNITION ...

$= \frac{12}{8}(1-2x)^{-4} + C = \frac{3}{2}(1-2x)^{-4} + C = \frac{3}{2(1-2x)^4} + C$

159. $$\int \frac{x^4+2x}{x^5+5x^2+8}\,dx = \frac{1}{5}\ln\left|x^5+5x^2+8\right| + C$$

$$\int \frac{x^4+2x}{x^5+5x^2+8}\,dx = \frac{1}{5}\int \frac{5x^4+10x}{x^5+5x^2+8}\,dx = \ldots$$

...OF THE FORM $\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + C$ ...

$$= \frac{1}{5}\ln\left|x^5+5x^2+8\right| + C$$

ALTERNATIVE METHOD BY THE SUBSTITUTION, $u = x^5+5x^2+8$

160. $$\int \frac{x}{x-1}\,dx = x + \ln|x-1| + C$$

$$\int \frac{x}{x-1}\,dx = \ldots \text{ BY SUBSTITUTION } \ldots$$

$$u = x-1,\quad \frac{du}{dx} = 1,\quad du = dx,\quad x = u+1$$

$$= \int \frac{x}{u}\,(du) = \int \frac{u+1}{u}\,du = \int 1 + \frac{1}{u}\,du$$

$$= u + \ln|u| + C = (x-1) + \ln|x-1| + C$$

$$= x + \ln|x-1| + C$$

ALTERNATIVE BY MANIPULATIONS

$$\int \frac{x}{x-1}\,dx = \int \frac{(x-1)+1}{x-1}\,dx = \int \frac{x-1}{x-1} + \frac{1}{x-1}\,dx$$

$$= \int 1 + \frac{1}{x-1}\,dx = x + \ln|x-1| + C$$

161. $$\int \frac{1}{\left(1+\sqrt{x}\right)\sqrt{x}}\,dx = 2\ln\left(1+\sqrt{x}\right) + C$$

$$\int \frac{1}{(1+\sqrt{x})\sqrt{x}}\,dx = \ldots \text{ BY SUBSTITUTION } \ldots$$

$$u = \sqrt{x},\quad u^2 = x,\quad 2u\frac{du}{dx} = 1,\quad 2u\,du = dx$$

$$= \int \frac{1}{(1+u)\cancel{u}}\,(2\cancel{u}\,du) = \int \frac{2}{1+u}\,du$$

$$= 2\ln|u+1| + C = 2\ln(\sqrt{x}+1) + C$$

CAN ALSO BY DONE BY "REVERSE CHAIN RULE" BY DIRECT RECOGNITION OF THE FUNCTION & ITS DERIVATIVE

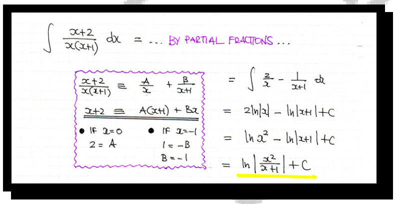
162. $$\int \frac{2x^3+1}{x^4+2x}\,dx = \frac{1}{2}\ln\left|x^4+2x\right| + C$$

$$\int \frac{2x^3+1}{x^4+2x}\,dx = \frac{1}{2}\int \frac{4x^3+2}{x^4+2x}\,dx = \frac{1}{2}\ln\left|x^4+2x\right| + C$$

(THIS IS OF THE FORM $\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + C$)

THE SUBSTITUTION $u = x^4+2x$ ALSO WORKS WELL

163. $$\int \frac{x+2}{x(x+1)}\,dx = \ln\left|\frac{x^2}{x+1}\right| + C$$



$$\int \frac{x+2}{x(x+1)}\,dx = \dots \text{ BY PARTIAL FRACTIONS } \dots$$

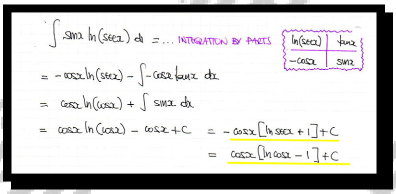
$$\frac{x+2}{x(x+1)} \equiv \frac{A}{x} + \frac{B}{x+1}$$

$$x+2 \equiv A(x+1) + Bx$$

- IF $x=0$: $2 = A$
- IF $x=-1$: $1 = -B$, $B = -1$

$$= \int \frac{2}{x} - \frac{1}{x+1}\,dx = 2\ln|x| - \ln|x+1| + C = \ln x^2 - \ln|x+1| + C = \ln\left|\frac{x^2}{x+1}\right| + C$$

164. $$\int \sin x \ln(\sec x)\,dx = \begin{bmatrix} -\left[1+\ln|\sec x|\right]\cos x \\ \left[-1+\ln|\cos x|\right]\cos x \end{bmatrix} + C$$



$$\int \sin x \ln(\sec x)\,dx = \dots \text{ INTEGRATION BY PARTS}$$

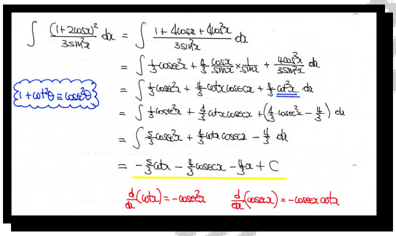
$\ln(\sec x)$	$\tan x$
$-\cos x$	$\sin x$

$$= -\cos x \ln(\sec x) - \int -\cos x \tan x\,dx$$

$$= \cos x \ln(\cos x) + \int \sin x\,dx$$

$$= \cos x \ln(\cos x) - \cos x + C = -\cos x\left[\ln\sec x + 1\right] + C = \cos x\left[\ln\cos x - 1\right] + C$$

165. $$\int \frac{(1+2\cos x)^2}{3\sin^2 x}\,dx = -\frac{5}{3}\cot x - \frac{4}{3}\operatorname{cosec} x - \frac{4}{3}x + C$$



$$\int \frac{(1+2\cos x)^2}{3\sin^2 x}\,dx = \int \frac{1+4\cos x+4\cos^2 x}{3\sin^2 x}\,dx$$

$$= \int \frac{1}{3}\operatorname{cosec}^2 x + \frac{4}{3}\frac{\cos x}{\sin x}\times\frac{1}{\sin x} + \frac{4\cos^2 x}{3\sin^2 x}\,dx$$

$1+\cot^2\theta \equiv \operatorname{cosec}^2\theta$

$$= \int \frac{1}{3}\operatorname{cosec}^2 x + \frac{4}{3}\cot x\operatorname{cosec} x + \frac{4}{3}\cot^2 x\,dx$$

$$= \int \frac{1}{3}\operatorname{cosec}^2 x + \frac{4}{3}\cot x\operatorname{cosec} x + \left(\frac{4}{3}\operatorname{cosec}^2 x - \frac{4}{3}\right)dx$$

$$= \int \frac{5}{3}\operatorname{cosec}^2 x + \frac{4}{3}\cot x\operatorname{cosec} x - \frac{4}{3}\,dx$$

$$= -\frac{5}{3}\cot x - \frac{4}{3}\operatorname{cosec} x - \frac{4}{3}x + C$$

$\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x \qquad \frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x\cot x$

166. $\displaystyle\int \frac{1}{x\ln x}\,dx = \ln\left|\ln|x|\right| + C$

$$\int \frac{1}{x\ln x}\,dx = \int \frac{1}{x}\times\frac{1}{\ln x}\,dx = \dots \text{ BY REVERSE CHAIN RULE} \dots$$
$$= \ln\left|\ln|x|\right| + C$$

ALTERNATIVE BY SUBSTITUTION

$$\int \frac{1}{x\ln x}\,dx = \int \frac{1}{\cancel{x}u}(\cancel{x}\,du) = \int \frac{1}{u}\,du$$
$$= \ln|u| + C$$
$$= \ln\left|\ln|x|\right| + C$$

$$\boxed{\begin{aligned} u &= \ln x \\ \frac{du}{dx} &= \frac{1}{x} \\ dx &= x\,du \end{aligned}}$$

167. $\displaystyle\int (2-3x)^{-2}\,dx = \frac{1}{3(2-3x)} + C$

$$\int (2-3x)^{-2}\,dx = \dots \text{ BY RECOGNITION} \dots$$
$$= \frac{1}{3}(2-3x)^{-1} + C = \frac{1}{3(2-3x)} + C$$

168. $\displaystyle\int 2\sec^2 x + \frac{1}{2}\sin 2x\,dx = 2\tan x - \frac{1}{4}\cos 2x + C$

$$\int 2\sec^2 x + \frac{1}{2}\sin 2x\,dx = \dots \text{ BY RECOGNITION} \dots$$
$$= 2\tan x - \frac{1}{4}\cos 2x + C$$

169. $\displaystyle\int \frac{\ln x}{\sqrt{x}}\,dx = 2\sqrt{x}\left[\ln|x| - 2\right] + C$

$$\int \frac{\ln x}{\sqrt{x}}\,dx = \int x^{-\frac{1}{2}}\ln x\,dx = \dots \text{ INTEGRATION BY PARTS} \dots$$
$$= 2x^{\frac{1}{2}}\ln|x| - \int 2x^{\frac{1}{2}}\left(\frac{1}{x}\right)dx$$
$$= 2x^{\frac{1}{2}}\ln|x| - \int 2x^{-\frac{1}{2}}\,dx$$
$$= 2x^{\frac{1}{2}}\ln|x| - 4x^{\frac{1}{2}} + C$$
$$= 2x^{\frac{1}{2}}\left[\ln|x| - 2\right] + C$$
$$= 2\sqrt{x}\left[\ln|x| - 2\right] + C$$

$\ln x$	$\frac{1}{x}$
$2x^{\frac{1}{2}}$	$x^{-\frac{1}{2}}$

170. $$\int \frac{\cos^4 x}{\sin x}\,dx = \ln\left|\tan\left(\tfrac{1}{2}x\right)\right| + \cos x + \tfrac{1}{3}\cos^3 x + C$$

$$\int \frac{\cos^4 x}{\sin x}\,dx = \int \frac{(1-\sin^2 x)^2}{\sin x}\,dx = \int \frac{1-2\sin^2 x+\sin^4 x}{\sin x}\,dx$$

$$= \int \operatorname{cosec} x - 2\sin x + \sin^3 x \,dx$$

MANIPULATE THIS FURTHER

$$= \int \operatorname{cosec} x - 2\sin x + \sin x(1-\cos^2 x)\,dx$$

$$= \int \operatorname{cosec} x - \sin x - \sin x\cos^2 x \,dx$$

STANDARD RESULTS

BY REVERSE CHAIN RULE

$$\frac{d}{dx}(\cos^3 x) = 3\cos^2 x(-\sin x)$$

$$= \ln\left|\tan\tfrac{x}{2}\right| + \cos x + \tfrac{1}{3}\cos^3 x + C$$

171. $$\int 6\tan^2 x - \sec^2 x \;dx = 5\tan x - 6x + C$$

$$\int 6\tan^2 x - \sec^2 x\,dx = \int 6(\sec^2 x - 1) - \sec^2 x\,dx$$

$$= \int 5\sec^2 x - 6\,dx$$

$$= 5\tan x - 6x + C$$

172. $$\int 6\cos^4 x - 2\sin^2 x \;dx = \tfrac{1}{2}x + \tfrac{3}{2}\sin 2x + \tfrac{1}{8}\sin 4x + C$$

$$\int 4\cos^4 x - 2\sin^2 x\,dx = \int 4(\cos^2 x)^2 - 2\left(\tfrac{1}{2} - \tfrac{1}{2}\cos 2x\right)dx$$

$$= \int 4\left(\tfrac{1}{2} + \tfrac{1}{2}\cos 2x\right)^2 - 1 + \cos 2x\,dx$$

$$= \int \cancel{1} + 2\cos 2x + \cos^2 2x - \cancel{1} + \cos 2x\,dx$$

$$= \int 3\cos 2x + \cos^2 2x\,dx$$

$$= \int 3\cos 2x + \left(\tfrac{1}{2} + \tfrac{1}{2}\cos 4x\right)dx$$

$$= \int \tfrac{1}{2} + 3\cos 2x + \tfrac{1}{2}\cos 4x\,dx$$

$$= \tfrac{1}{2}x + \tfrac{3}{2}\sin 2x + \tfrac{1}{8}\sin 4x + C$$

173. $$\int \sin 4x \cos 4x \; dx = \begin{bmatrix} -\frac{1}{16}\cos 8x + C \\ \frac{1}{8}\sin^2 4x + C \\ -\frac{1}{8}\cos^2 4x + C \end{bmatrix}$$

$$\int \sin 4x \cos 4x \, dx = \int \tfrac{1}{2}(2\sin 4x \cos 4x)\, dx = \int \tfrac{1}{2}\sin 8x \, dx$$

$$= -\tfrac{1}{16}\cos 8x + C$$

ALTERNATIVE BY REVERSE CHAIN RULE

$$\int \sin 4x \cos 4x \, dx = \tfrac{1}{8}\sin^2 4x + C, \text{ SINCE } \tfrac{d}{dx}(\sin^2 4x) = 2\sin 4x(4\cos 4x)$$

$$\int \sin 4x \cos 4x \, dx = -\tfrac{1}{8}\cos^2 4x + C, \text{ SINCE } \tfrac{d}{dx}(\cos^2 4x) = 2\cos 4x(-4\sin 4x)$$

174. $$\int \frac{1}{\operatorname{cosec} x - \cot x} \; dx = \begin{bmatrix} \ln\left|\dfrac{\sin x}{\operatorname{cosec} x + \cot x}\right| + C \\ \ln|1-\cos x| + C \end{bmatrix}$$

$$\int \frac{1}{\operatorname{cosec} x - \cot x}\, dx = \int \frac{\operatorname{cosec} x + \cot x}{(\operatorname{cosec} x - \cot x)(\operatorname{cosec} x + \cot x)}\, dx$$

$$= \int \frac{\operatorname{cosec} x + \cot x}{\underbrace{\operatorname{cosec}^2 x - \cot^2 x}_{1+\cot^2 x \equiv \operatorname{cosec}^2 x}}\, dx = \int \operatorname{cosec} x + \cot x \, dx$$

Both are standard results

$$= -\ln|\operatorname{cosec} x + \cot x| + \ln|\sin x| + C$$

$$= \ln\left|\frac{\sin x}{\operatorname{cosec} x + \cot x}\right| + C$$

$$= \ln\left|\frac{\sin x \sin x}{\operatorname{cosec} x \sin x + \cot x \sin x}\right| + C$$

$$= \ln\left|\frac{\sin^2 x}{1+\cos x}\right| + C = \ln\left|\frac{1-\cos^2 x}{1+\cos x}\right| + C$$

$$= \ln\left|\frac{(1-\cos x)\cancel{(1+\cos x)}}{\cancel{1+\cos x}}\right| + C = \ln|1-\cos x| + C$$

ALTERNATIVE | VARIATION

$$\int \frac{1}{\operatorname{cosec} x - \cot x}\, dx = \int \frac{1 \sin x}{\operatorname{cosec} x \sin x - \cot x \sin x}\, dx = \int \frac{\sin x}{1-\cos x}\, dx$$

OF THE FORM $\int \frac{f'(x)}{f(x)}\, dx = \ln|f(x)| + C$

$$= \ln|1-\cos x| + C$$

175. $$\int \frac{x^2}{\sqrt{x-1}} \; dx = \frac{2}{5}(x-1)^{\frac{5}{2}} + \frac{4}{3}(x-1)^{\frac{3}{2}} + 2(x-1)^{\frac{1}{2}} + C$$

$$\int \frac{x^2}{\sqrt{x-1}}\, dx = \dots \text{ BY SUBSTITUTION } \dots$$

$$= \int \frac{x^2}{\cancel{u}}(2\cancel{u}\, du) = \int 2x^2 \, du$$

$$= \int 2u^4 + 4u^2 + 2 \, du = \tfrac{2}{5}u^5 + \tfrac{4}{3}u^3 + 2u + C$$

$$= \tfrac{2}{5}(x-1)^{\frac{5}{2}} + \tfrac{4}{3}(x-1)^{\frac{3}{2}} + 2(x-1)^{\frac{1}{2}} + C$$

$u = \sqrt{x-1}$

$u^2 = x-1$

$2u\frac{du}{dx} = 1$

$dx = 2u\, du$

$x = u^2+1$

$x^2 = u^4 + 2u^2 + 1$

$2x^2 = 2u^4 + 4u^2 + 2$

[THE SUBSTITUTION $u = x-1$ ALSO WORKS WELL]

176. $$\int \frac{3e^{2x}}{\sqrt{e^x-1}}\,dx = 2\left(e^x-1\right)^{\frac{3}{2}} + 6\left(e^x-1\right)^{\frac{1}{2}} + C$$

$\int \frac{3e^{2x}}{\sqrt{e^x-1}}\,dx = \dots$ BY SUBSTITUTION ...

$u = \sqrt{e^x-1}$
$u^2 = e^x-1$
$2u\frac{du}{dx} = e^x$
$dx = \frac{2u}{e^x}\,du$
$e^x = u^2+1$

$= \int \frac{3e^{2x}}{\cancel{u}}\left(\frac{2\cancel{u}}{e^x}\,du\right) = \int \frac{6e^{2x}}{e^x}\,du$

$= \int 6e^x\,du = \int 6u^2+6\,du = 2u^3+6u+C$

$= 2(e^x-1)^{\frac{3}{2}} + 6(e^x-1)^{\frac{1}{2}} + C$

(THE SUBSTITUTION $u=e^x-1$ ALSO WORKS WELL)

177. $$\int \frac{1}{\left(2+\sqrt[3]{x}\right)\sqrt[3]{x^2}}\,dx = 3\ln\left|2+\sqrt[3]{x}\right| + C$$

$\int \frac{1}{\sqrt[3]{x^2}\,(2+\sqrt[3]{x})}\,dx = \dots$ BY REVERSE CHAIN RULE RECOGNITION OR SUBSTITUTION

$u = \sqrt[3]{x}$
$u^3 = x$
$3u^2\frac{du}{dx} = 1$
$3u^2\,du = dx$
$u^2 = (\sqrt[3]{x})^2$
$u^2 = \sqrt[3]{x^2}$

$= \int \frac{1}{\cancel{u^2}(2+u)}(3\cancel{u^2}\,du) = \int \frac{3}{u+2}\,du$

$= 3\ln|u+2| + C = 3\ln|\sqrt[3]{x}+2| + C$

178. $$\int \frac{4x^7}{x^4+1}\,dx = x^4 - \ln\left(x^4+1\right) + C$$

$\int \frac{4x^7}{x^4+1}\,dx = \dots$ BY SUBSTITUTION

$u = x^4+1$
$\frac{du}{dx} = 4x^3$
$dx = \frac{du}{4x^3}$
$x^4 = u-1$

$= \int \frac{4x^7}{u}\times\frac{du}{4x^3} = \int \frac{x^4}{u}\,du$

$= \int \frac{u-1}{u}\,du = \int 1-\frac{1}{u}\,du$

$= u - \ln|u| + C = (x^4+1) - \ln(x^4+1) + C$

$= x^4 - \ln(x^4+1) + C$

ALTERNATIVE BY MANIPULATION / LONG DIVISION

$\int \frac{4x^7}{x^4+1}\,dx = \int \frac{4x^3(x^4+1) - 4x^3}{x^4+1}\,dx = \int 4x^3 - \frac{4x^3}{x^4+1}\,dx$

$\int \frac{f'(x)}{f(x)}\,dx$

$= x^4 - \ln(x^4+1) + C$

179. $$\int \frac{x}{9x^2+1}\,dx = \frac{1}{18}\ln\left(9x^2+1\right) + C$$

$\int \frac{x}{9x^2+1}\,dx = \frac{1}{18}\int \frac{18x}{9x^2+1}\,dx = \frac{1}{18}\ln(9x^2+1) + C$

$f'(x)$
$f(x)$

[THE SUBSTITUTION $u=9x^2+1$ ALSO WORKS WELL]

180. $\displaystyle\int \frac{2}{x+\sqrt[3]{x}}\,dx = 3\ln\left|x+\sqrt[3]{x}\right| + C$

$\displaystyle\int \frac{2}{x+\sqrt[3]{x}}\,dx = \dots$ BY SUBSTITUTION ...

$u = x^{\frac{1}{3}}$

$u^3 = x$

$3u^2\,du = 1\,dx$

$\displaystyle = \int \frac{2}{u^3+u}\left(3u^2\,du\right) = \int \frac{6u^2}{u^3+u}\,du = \int \frac{6u}{u^2+1}\,du$

$\displaystyle = 3\int \frac{\overset{f'(u)}{2u}}{\underset{f(u)}{u^2+1}}\,du = 3\ln\left(u^2+1\right) + C$

$= 3\ln\left(x^{\frac{2}{3}}+1\right) + C$

181. $\displaystyle\int \frac{3x}{1+\sqrt{x}}\,dx = \begin{bmatrix} 2x^{\frac{3}{2}} - 3x + 6x^{\frac{1}{2}} - 6\ln\left|1+\sqrt{x}\right| + C \\ 2\left(1+\sqrt{x}\right)^3 - 9\left(1+\sqrt{x}\right)^2 + 18\left(1+\sqrt{x}\right) - 6\ln\left|1+\sqrt{x}\right| + C \end{bmatrix}$

$\displaystyle\int \frac{3x}{1+\sqrt{x}}\,dx = \dots$ BY SUBSTITUTION ...

$u = \sqrt{x}$

$x = u^2$

$\frac{dx}{du} = 2u$

$dx = 2u\,du$

$\displaystyle = \int \frac{3u^2}{1+u}\left(2u\,du\right) = \int \frac{6u^3}{u+1}\,du$

BY LONG DIVISION / MANIPULATION

$\displaystyle = 6\int \frac{u^2(u+1) - u(u+1) + (u+1) - 1}{u+1}\,du = 6\int u^2 - u + 1 - \frac{1}{u+1}\,du$

$= 6\left[\frac{1}{3}u^3 - \frac{1}{2}u^2 + u - \ln|u+1|\right] + C$

$= 2x^{\frac{3}{2}} - 3x + 6x^{\frac{1}{2}} - 6\ln(\sqrt{x}+1) + C$

ALTERNATIVE SUBSTITUTION

$u = 1+\sqrt{x}$

$u-1 = \sqrt{x}$

$x = (u-1)^2$

$\frac{dx}{du} = 2(u-1)$

$dx = 2(u-1)\,du$

$\displaystyle\int \frac{3x}{1+\sqrt{x}}\,dx = \int \frac{3(u-1)^2}{u}\left[2(u-1)\,du\right]$

$\displaystyle = \int \frac{6(u-1)^3}{u}\,du = \int \frac{6u^3 - 18u^2 + 18u - 6}{u}\,du$

$\displaystyle = \int 6u^2 - 18u + 18 - \frac{6}{u}\,du$

$= 2u^3 - 9u^2 + 18u - 6\ln|u| + C$

$= 2(1+\sqrt{x})^3 - 9(1+\sqrt{x})^2 + 18(1+\sqrt{x}) - 6\ln(1+\sqrt{x}) + C$

OR EXPAND IGNORING CONSTANTS

$= 2\left[x^{\frac{3}{2}} + 3x + 3x^{\frac{1}{2}}\right] - 9\left[x + 2\sqrt{x}\right] + 18\sqrt{x} - 6\ln(1+\sqrt{x}) + C$

$= 2x^{\frac{3}{2}} + 6x + \cancel{6x^{\frac{1}{2}}} - 9x - \cancel{18x^{\frac{1}{2}}} + 18x^{\frac{1}{2}} - 6\ln(1+\sqrt{x}) + C$

$= 2x^{\frac{3}{2}} - 3x + 6x^{\frac{1}{2}} - 6\ln(1+\sqrt{x}) + C$

182. $\displaystyle\int \frac{3x^2+2}{4x+1}\,dx = \begin{bmatrix} \frac{3}{8}x^2 - \frac{3}{16}x + \frac{35}{64}\ln|4x+1| + C \\ \frac{3}{128}(4x+1)^2 - \frac{3}{32}(4x+1) + \frac{35}{64}\ln|4x+1| + C \end{bmatrix}$

$\displaystyle\int \frac{3x^2+2}{4x+1}\,dx = \dots$ BY SUBSTITUTION

$u = 4x+1$

$\frac{du}{dx} = 4$

$dx = \frac{du}{4}$

$4x = u-1$

$16x^2 = u^2 - 2u + 1$

$\displaystyle = \frac{1}{16}\int \frac{48x^2+32}{u}\left(\frac{du}{4}\right) = \frac{1}{64}\int \frac{3(16x^2)+32}{u}\,du$

$\displaystyle = \frac{1}{64}\int \frac{3(u^2-2u+1)+32}{u}\,du = \frac{1}{64}\int \frac{3u^2-6u+35}{u}\,du$

$\displaystyle = \frac{1}{64}\int 3u - 6 + \frac{35}{u}\,du = \frac{1}{64}\left[\frac{3}{2}u^2 - 6u + 35\ln|u|\right] + C$

$= \frac{3}{128}u^2 - \frac{3}{32}u + \frac{35}{64}\ln|u| + C = \frac{3}{128}(4x+1)^2 - \frac{3}{32}(4x+1) + \frac{35}{64}\ln|4x+1| + C$

ALTERNATIVE BY MANIPULATION / DIVISION

$\displaystyle\int \frac{3x^2+2}{4x+1}\,dx = \frac{1}{4}\int \frac{3x^2+2}{x+\frac{1}{4}}\,dx = \frac{1}{4}\int \frac{3x\left(x+\frac{1}{4}\right) - \frac{3}{4}\left(x+\frac{1}{4}\right) + \frac{3}{16} + 2}{x+\frac{1}{4}}\,dx$

$\displaystyle = \frac{1}{4}\int 3x - \frac{3}{4} + \frac{\frac{35}{16}}{x+\frac{1}{4}}\,dx = \int \frac{3}{4}x - \frac{3}{16} + \frac{35/16}{4x+1}\,dx$

$= \frac{3}{8}x^2 - \frac{3}{16}x + \frac{35}{64}\ln|4x+1| + C$

183. $\displaystyle\int \frac{3}{x}+\frac{4}{x^{2}}-\frac{2}{x^{3}}\,dx=\frac{1}{x^{2}}-\frac{4}{x}+3\ln|x|+C$

$\int \frac{3}{x}+\frac{4}{x^2}-\frac{2}{x^3}\,dx = \dots$ BY INSPECTION $\dots = \int \frac{3}{x}+4x^{-2}-2x^{-3}\,dx$

$= 3\ln|x|-4x^{-1}+x^{-2}+C = \frac{1}{x^2}-\frac{4}{x}+3\ln|x|+C$

184. $\displaystyle\int 2\sin 2x\cos^{2}x\,dx=-\cos^{4}x+C$

$\int 2\sin 2x\cos^2 x\,dx = \int 2(2\sin x\cos x)\cos^2 x\,dx$

$= \int 4\sin x\cos^3 x\,dx$

$= \dots$ BY REVERSE CHAIN RULE (RECOGNITION)

$= -\cos^4 x + C$

THE SUBSTITUTION $u=\cos x$ ALSO WORKS WELL

185. $\displaystyle\int \frac{10x^{2}-23x+11}{(2-3x)(2x-1)^{2}}\,dx=-\frac{1}{3}\ln|3x-2|-\frac{1}{2}\ln|2x-1|-\frac{2}{2x-1}+C$

$\int \frac{10x^2-23x+11}{(2-3x)(2x-1)^2}\,dx = \dots$ BY PARTIAL FRACTIONS

$\frac{10x^2-23x+11}{(2-3x)(2x-1)^2} \equiv \frac{A}{2-3x}+\frac{B}{(2x-1)^2}+\frac{C}{2x-1}$

$\boxed{10x^2-23x+11 \equiv A(2x-1)^2+B(2-3x)+C(2x-1)(2-3x)}$

- IF $x=\frac{1}{2}$: $\frac{5}{2}-\frac{23}{2}+11=\frac{1}{2}B$, $5-23+22=B$, $B=4$
- IF $x=\frac{2}{3}$: $\frac{40}{9}-\frac{46}{3}+11=\frac{1}{9}A$, $40-138+99=A$, $A=1$
- IF $x=0$: $11=A+2B-2C$, $11=1+8-2C$, $2C=-2$, $C=-1$

$= \int \frac{1}{2-3x}-\frac{1}{2x-1}+4(2x-1)^{-2}\,dx = -\frac{1}{3}\ln|2-3x|-\frac{1}{2}\ln|2x-1|-2(2x-1)^{-2}$

$= -\frac{1}{3}\ln|3x-2|-\frac{1}{2}\ln|2x-1|-\frac{2}{2x-1}+C$

186. $\displaystyle\int \frac{\sec x}{\sec x-\tan x}\,dx=\tan x+\sec x+C$

$\int \frac{\sec x}{\sec x-\tan x}\,dx = \int \frac{\sec x(\sec x+\tan x)}{(\sec x-\tan x)(\sec x+\tan x)}\,dx$

$= \int \frac{\sec^2 x+\sec x\tan x}{\sec^2 x-\tan^2 x}\,dx = \int \sec^2 x+\sec x\tan x\,dx$

$1+\tan^2 x \equiv \sec^2 x$

$= \tan x+\sec x+C$

187. $\displaystyle\int x\cos 6x \ dx = \frac{1}{6}x\sin 6x + \frac{1}{36}\cos 6x + C$

$\displaystyle\int x\cos 6x \, dx = \ldots$ INTEGRATION BY PARTS

x	1
$\frac{1}{6}\sin 6x$	$\cos 6x$

$= \frac{1}{6}x\sin 6x - \int \frac{1}{6}\sin 6x \, dx$

$= \frac{1}{6}x\sin 6x + \frac{1}{36}\cos 6x + C$

188. $\displaystyle\int \sin x \sin 3x \ dx = \begin{bmatrix} \frac{1}{4}\sin 2x - \frac{1}{8}\sin 4x + C \\ \frac{1}{8}\cos x\sin 3x - \frac{3}{8}\sin x\cos 3x + C \end{bmatrix}$

$\displaystyle\int \sin x \sin 3x \, dx = \ldots$ BY TRIG IDENTITIES

$= \int \frac{1}{2}\cos 2x - \frac{1}{2}\cos 4x \, dx$ (SEE BELOW)

$= \frac{1}{4}\sin 2x - \frac{1}{8}\sin 4x + C$

$\cos(3x+x) \equiv \cos 3x\cos x - \sin 3x\sin x$
$\cos(3x-x) \equiv \cos 3x\cos x + \sin 3x\sin x$ } SUBTRACT

$\cos 4x - \cos 2x \equiv -2\sin 3x\sin x$

$\sin 3x\sin x \equiv \frac{1}{2}\cos 2x - \frac{1}{2}\cos 4x$

ALTERNATIVE BY USING THE TRIPLE ANGLE FOR SINE

$\sin 3x = \sin(2x+x) = \sin 2x\cos x + \cos 2x\sin x$
$= (2\sin x\cos x)\cos x + (1-2\sin^2 x)\sin x$
$= 2\sin x\cos^2 x + \sin x - 2\sin^3 x$
$= 2\sin x(1-\sin^2 x) + \sin x - 2\sin^3 x$
$= 2\sin x - 2\sin^3 x + \sin x - 2\sin^3 x$
$= 3\sin x - 4\sin^3 x$

$\displaystyle\int \sin x\sin 3x \, dx = \int \sin x(3\sin x - 4\sin^3 x) \, dx$
$= \int 3\sin^2 x - 4\sin^4 x \, dx$
$= \int 3\left(\frac{1}{2} - \frac{1}{2}\cos 2x\right) - 4\left(\frac{1}{2} - \frac{1}{2}\cos 2x\right)^2 dx$
$= \int \frac{3}{2} - \frac{3}{2}\cos 2x - 4\left(\frac{1}{4} - \frac{1}{2}\cos 2x + \frac{1}{4}\cos^2 2x\right) dx$
$= \int \frac{3}{2} - \frac{3}{2}\cos 2x - 1 + 2\cos 2x - \cos^2 2x \, dx$
$= \int \cancel{\frac{1}{2}} + \frac{1}{2}\cos 2x - \left(\cancel{\frac{1}{2}} + \frac{1}{2}\cos 4x\right) dx$
$= \int \frac{1}{2}\cos 2x - \frac{1}{2}\cos 4x \, dx$
$= \frac{1}{4}\sin 2x - \frac{1}{8}\sin 4x + C$

ALTERNATIVE BY DOUBLE INTEGRATION BY PARTS

$\displaystyle\int \sin x\sin 3x \, dx = \ldots$ by parts

$\sin 3x$	$3\cos 3x$
$-\cos x$	$\sin x$

$= -\cos x\sin 3x - \int -\cos x\cos 3x \, dx$
$= -\cos x\sin 3x + \int 3\cos 3x\cos x \, dx$

By parts again

$3\cos 3x$	$-9\sin 3x$
$\sin x$	$\cos x$

$= -\cos x\sin 3x + \left[3\cos 3x\sin x - \int -9\sin 3x\sin x \, dx\right]$
$= -\cos x\sin 3x + 3\cos 3x\sin x + 9\int \sin 3x\sin x \, dx$

COLLECTING THE RESULTS

$\Rightarrow \int \sin x\sin 3x \, dx = -\cos x\sin 3x + 3\cos 3x\sin x + 9\int \sin 3x\sin x \, dx$

$\Rightarrow \cos x\sin 3x - 3\cos 3x\sin x = 8\int \sin 3x\sin x \, dx$

$\Rightarrow \int \sin x\sin 3x \, dx = \frac{1}{8}\cos x\sin 3x - \frac{3}{8}\cos 3x\sin x + C$
$= \frac{1}{8}(\cos x\sin 3x - \cos 3x\sin x) - \frac{1}{4}\cos 3x\sin x + C$
$= \frac{1}{8}\sin(3x-x) - \frac{1}{8}(2\cos 3x\sin x) + C$
$= \frac{1}{8}\sin 2x - \frac{1}{8}[\sin 4x - \sin 2x] + C$
$= \frac{1}{4}\sin 2x - \frac{1}{8}\sin 4x + C$ (AS BEFORE)

189. $\displaystyle\int 4\cos 3x + \frac{1}{2}\sin 3x \ dx = \frac{4}{3}\sin 3x - \frac{1}{6}\cos 3x + C$

$\displaystyle\int 4\cos 3x + \frac{1}{2}\sin 3x \, dx = \ldots$ BY INSPECTION …

$= \frac{4}{3}\sin 3x - \frac{1}{6}\cos 3x + C$

190. $\displaystyle\int \sin^2 6x \; dx = \frac{1}{2}x - \frac{1}{24}\sin 12x + C$

$$\int \sin^2 6x \, dx = \int \frac{1}{2} - \frac{1}{2}\cos 12x \, dx = \frac{1}{2}x - \frac{1}{24}\sin 12x + C$$

$\cos 2\theta \equiv 1 - 2\sin^2\theta$
$\cos 12\theta \equiv 1 - 2\sin^2 6\theta$
$2\sin^2 6\theta \equiv 1 - \cos 12\theta$
$\sin^2 6\theta \equiv \frac{1}{2} - \frac{1}{2}\cos 12\theta$

191. $\displaystyle\int \frac{\cos 2x}{1-\cos^2 2x} \; dx = -\frac{1}{2}\operatorname{cosec} 2x + C$

$$\int \frac{\cos 2x}{1-\cos^2 2x}\,dx = \int \frac{\cos 2x}{\sin^2 2x}\,dx = \int \frac{\cos 2x}{\sin 2x} \times \frac{1}{\sin 2x}\,dx$$
$$= \int \cot 2x \operatorname{cosec} 2x \, dx = -\frac{1}{2}\operatorname{cosec} 2x + C$$

$\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$

ALTERNATIVE/VARIATION

$$\int \frac{\cos 2x}{1-\cos^2 2x}\,dx = \int \frac{\cos 2x}{\sin^2 2x}\,dx = \int (\sin 2x)^{-2}\cos 2x \, dx$$

... BY REVERSE CHAIN RULE OR THE SUBSTITUTION $u = \sin 2x$

$$= -\frac{1}{2}(\sin 2x)^{-1} + C = -\frac{1}{2}\operatorname{cosec} 2x + C$$

192. $\displaystyle\int \frac{1}{\cos^2 x \tan^2 x} \; dx = -\frac{1}{3}\cot^3 x + C$

$$\int \frac{1}{\cos^2 x \tan^4 x}\,dx = \int \frac{1}{\cos^2 x}(\tan x)^{-4}\,dx = \int (\tan x)^{-4}\sec^2 x \, dx$$

... BY REVERSE CHAIN RULE AS $\frac{d}{dx}(\tan x) = \sec^2 x$

$$= -\frac{1}{3}(\tan x)^{-3} + C = -\frac{1}{3}\cot^3 x + C$$

[THE SUBSTITUTION $u = \tan x$ WORK WELL ALSO]

193. $\displaystyle\int 3\cos^3 3x \; dx = \sin 3x - \frac{1}{3}\sin^3 3x + C$

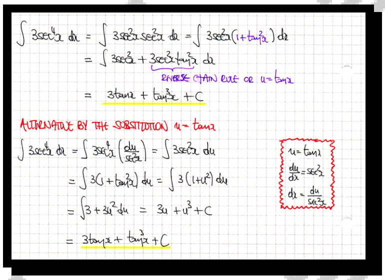
$$\int 3\cos^3 3x \, dx = \int 3\cos 3x \cos^2 3x \, dx = \int 3\cos 3x (1 - \sin^2 3x)\,dx$$
$$= \int 3\cos 3x - \underbrace{3\cos 3x \sin^2 3x}_{\text{REVERSE CHAIN RULE (INSPECTION)}}\,dx$$
$$= \sin 3x - \frac{1}{3}\sin^3 3x + C$$

ALTERNATIVE BY SUBSTITUTION $u = \sin 3x$

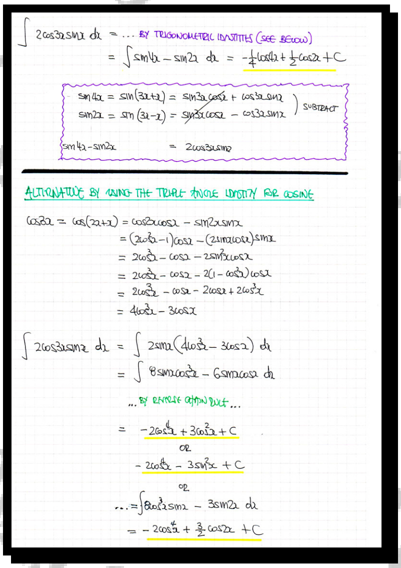
$$\int 3\cos^3 3x \, dx = \int 3\cos^3 3x \left(\frac{du}{3\cos 3x}\right)$$
$$= \int \cos^2 3x \, du$$
$$= \int 1 - \sin^2 3x \, du$$
$$= \int 1 - u^2 \, du$$
$$= u - \frac{1}{3}u^3 + C = \sin 3x - \frac{1}{3}\sin^3 3x + C$$

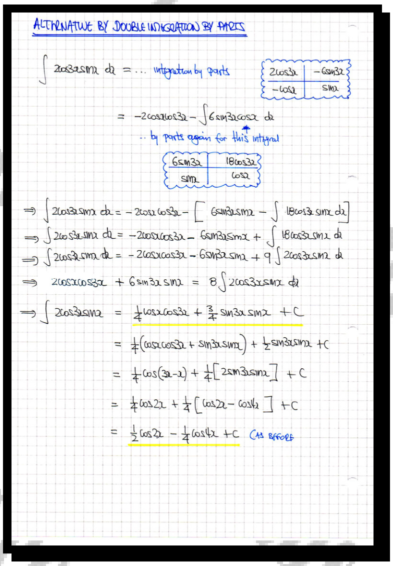
$u = \sin 3x$
$\frac{du}{dx} = 3\cos 3x$
$dx = \frac{du}{3\cos 3x}$

194. $$\int 3\sec^4 x \ dx = 3\tan x + \tan^3 x + C$$



195. $$\int 2\sin x \cos 3x \ dx = \begin{bmatrix} \frac{1}{2}\cos 2x - \frac{1}{4}\cos 4x + C \\ -2\cos^4 x + 3\cos^2 x + C \\ -2\cos^4 x - 3\sin^2 x + C \\ -2\cos^4 x + \frac{3}{2}\cos 2x + C \\ \frac{1}{4}\cos x \cos 3x + \frac{3}{4}\sin x \sin 3x + C \end{bmatrix}$$





196. $$\int \frac{2\sin x}{\cos x+\sin x}\,dx = \left[\begin{array}{c} x-\ln\left|\cos x+\sin x\right| + C \\ x+\frac{1}{2}\ln\left|\sec 2x\right|-\frac{1}{2}\ln\left|\sec 2x+\tan 2x\right| + C \end{array}\right]$$

$$\int \frac{2\sin x}{\cos x+\sin x}\,dx = \int \frac{(\cos x+\sin x)+(\sin x-\cos x)}{\cos x+\sin x}\,dx$$
$$= \int 1+\frac{\sin x-\cos x}{\cos x+\sin x}\,dx = \int 1-\frac{-\sin x+\cos x}{\cos x+\sin x}\,dx$$

THIS IS OF THE FORM $\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)|+C$

$$= \underline{x-\ln|\cos x+\sin x|+C}$$

ALTERNATIVE BY TRIGONOMETRIC IDENTITIES

$$\int \frac{2\sin x}{\cos x+\sin x}\,dx = \int \frac{2\sin x(\cos x-\sin x)}{(\cos x+\sin x)(\cos x-\sin x)}\,dx$$
$$= \int \frac{2\sin x\cos x-2\sin^2 x}{\cos^2 x-\sin^2 x}\,dx = \int \frac{\sin 2x-2\sin^2 x}{\cos 2x}\,dx$$
$$= \int \frac{\sin 2x-2\left(\frac{1}{2}-\frac{1}{2}\cos 2x\right)}{\cos 2x}\,dx$$
$$= \int \frac{\sin 2x+\cos 2x-1}{\cos 2x}\,dx$$
$$= \int \tan 2x+1-\sec 2x\,dx$$
$$= x+\frac{1}{2}\ln|\sec 2x|-\frac{1}{2}\ln|\sec 2x+\tan 2x|+C$$
$$= x+\frac{1}{2}\ln\left|\frac{\sec 2x}{\sec 2x+\tan 2x}\right|+C$$
$$= x+\frac{1}{2}\ln\left|\frac{\sec 2x\cos 2x}{\sec 2x\cos 2x+\tan 2x\cos 2x}\right|+C$$
$$= x+\frac{1}{2}\ln\left|\frac{1}{1+\sin 2x}\right|+C$$
$$= x+\frac{1}{2}\ln\left|\frac{1}{\cos^2 x+\sin^2 x+2\sin x\cos x}\right|+C$$
$$= x+\frac{1}{2}\ln\left|\frac{1}{(\cos x+\sin x)^2}\right|+C = x+\frac{1}{2}\ln|\cos x+\sin x|^{-2}+C$$
$$= \underline{x-\ln|\cos x+\sin x|+C} \quad \text{(AS ABOVE)}$$

197. $$\int \frac{4}{2x-1}+\frac{1}{3-4x}\,dx = 2\ln\left|2x-1\right|-\frac{1}{4}\ln\left|3-4x\right| + C$$

$$\int \frac{4}{2x-1}+\frac{1}{3-4x}\,dx = \ldots \text{BY INSPECTION} \ldots$$
$$= \underline{2\ln|2x-1|-\frac{1}{4}\ln|3-4x|+C}$$

198. $$\int x^2\sin 3x\;dx = -\frac{1}{3}x^2\cos 3x+\frac{2}{9}x\sin 3x+\frac{2}{27}\cos 3x + C$$

$$\int x^2\sin 3x\,dx = \ldots \text{INTEGRATION BY PARTS}$$

x^2	$2x$
$-\frac{1}{3}\cos 3x$	$\sin 3x$

$$= -\frac{1}{3}x^2\cos 3x-\int -\frac{2}{3}x\cos 3x\,dx$$
$$= -\frac{1}{3}x^2\cos 3x+\int \underbrace{\frac{2}{3}x\cos 3x}_{\text{INTEGRATION BY PARTS AGAIN}}\,dx$$

$\frac{2}{3}x$	$\frac{2}{3}$
$\frac{1}{3}\sin 3x$	$\cos 3x$

$$= -\frac{1}{3}x^2\cos 3x+\left[\frac{2}{9}x\sin 3x-\int \frac{2}{9}\sin 3x\,dx\right]$$
$$= \underline{-\frac{1}{3}x^2\cos 3x+\frac{2}{9}x\sin 3x+\frac{2}{27}\cos 3x+C}$$

199. $\displaystyle\int \operatorname{cosec}^2 2x \ dx = -\frac{1}{2}\cot 2x + C$

$\displaystyle\int \operatorname{cosec}^2 2x \, dx = \ldots$ BY INSPECTION AS $\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$

$= -\frac{1}{2}\cot 2x + C$

200. $\displaystyle\int \sin^3 2x \cos 2x \ dx = \frac{1}{8}\sin^4 2x + C$

$\displaystyle\int \sin^3 2x \cos 2x \, dx = \ldots$ BY REVERSE CHAIN RULE / INSPECTION

$= \frac{1}{8}\sin^4 2x + C$

[THE SUBSTITUTION $u = \sin 2x$ ALSO WORKS WELL]

201. $\displaystyle\int \cot^2 3x \ dx = -x - \frac{1}{3}\cot 3x + C$

$\displaystyle\int \cot^2 3x \, dx = \int (\operatorname{cosec}^2 3x - 1) \, dx = \ldots$

BY INSPECTION SINCE $\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$

$= -\frac{1}{3}\cot 3x - x + C$

202. $\displaystyle\int \frac{7}{3x} \ dx = \frac{7}{3}\ln|x| + C$

$\displaystyle\int \frac{7}{3x} \, dx = \int \frac{7}{3} \times \frac{1}{x} \, dx = \ldots$ BY INSPECTION $\ldots = \frac{7}{3}\ln|x| + C$

203. $$\int \frac{\sin x \cos x}{\sqrt{1-\cos 2x}}\, dx = \begin{bmatrix} \frac{1}{2}\sqrt{1-\cos 2x} + C \\ \frac{1}{\sqrt{2}}\sin x + C \end{bmatrix}$$

$$\int \frac{\sin x \cos x}{\sqrt{1-\cos 2x}}\, dx = \frac{1}{2}\int \frac{2\sin x \cos x}{(1-\cos 2x)^{\frac{1}{2}}}\, dx = \frac{1}{2}\int (1-\cos 2x)^{-\frac{1}{2}} \sin 2x \, dx$$

BY REVERSE CHAIN RULE (INSPECTION)

$$= \frac{1}{2}(1-\cos 2x)^{\frac{1}{2}} + C$$

[THE SUBSTITUTIONS $u=\sqrt{1-\cos 2x}$, $u = 1-\cos 2x$, $u = \cos 2x$ ALL WORK]

ALTERNATIVE BY TRIGONOMETRIC IDENTITIES

$$\int \frac{\sin x \cos x}{\sqrt{1-\cos 2x}}\, dx = \int \frac{\sin x \cos x}{\sqrt{1-(1-2\sin^2 x)}}\, dx$$

$$= \int \frac{\sin x \cos x}{\sqrt{2\sin^2 x}}\, dx = \int \frac{\sin x \cos x}{\sqrt{2}\,|\sin x|}\, dx \quad \text{\& IF } \sin x \geqslant 0$$

$$= \int \frac{1}{\sqrt{2}}\cos x \, dx = \frac{1}{\sqrt{2}}\sin x + C$$

204. $$\int \frac{\sin x \cos x}{1+\cos 2x}\, dx = \begin{bmatrix} -\frac{1}{4}\ln|1+\cos 2x| + C \\ -\frac{1}{2}\ln|\cos x| + C \\ \frac{1}{2}\ln|\sec x| + C \end{bmatrix}$$

$$\int \frac{\sin x \cos x}{1+\cos 2x}\, dx = \int \frac{\frac{1}{2}(2\sin x \cos x)}{1+\cos 2x}\, dx = \frac{1}{2}\int \frac{\sin 2x}{1+\cos 2x}\, dx$$

$$= -\frac{1}{4}\int \frac{-2\sin 2x}{1+\cos 2x}\, dx$$

WHICH IS OF THE FORM $\int \frac{f'(x)}{f(x)}\, dx = \ln|f(x)| + C$

$$= -\frac{1}{4}\ln|1+\cos 2x| + C$$

[THE SUBSTITUTIONS $u = 1+\cos 2x$, $u = \cos 2x$ ALSO WORK]

ALTERNATIVE BY TRIG IDENTITIES

$$\int \frac{\sin x \cos x}{1+\cos 2x}\, dx = \int \frac{\sin x \cos x}{1+(2\cos^2 x - 1)}\, dx = \int \frac{\sin x \cos x}{2\cos^2 x}\, dx$$

$$= \int \frac{1}{2}\frac{\sin x}{\cos x}\, dx = -\frac{1}{2}\int \frac{-\sin x}{\cos x}\, dx = -\frac{1}{2}\ln|\cos x| + C = \frac{1}{2}\ln|\sec x| + C$$

205. $$\int 4(3-2x)^5 \, dx = -\tfrac{1}{3}(3-2x)^6 + C$$

$$\int 4(3-2x)^5 \, dx = \ldots \text{ BY INSPECTION } \ldots = -\frac{4}{12}(3-2x)^6 + C$$

$$= -\frac{1}{3}(3-2x)^6 + C$$

206. $\displaystyle\int \frac{1+\sin x}{\cos x}\,dx = \left[\begin{array}{c} \ln\left|\sec^2 x+\sec x\tan x\right| + C \\ -\ln\left|1-\sin x\right| + C \end{array}\right]$

$$\int \frac{1+\sin x}{\cos x}\,dx = \int \frac{1}{\cos x}+\frac{\sin x}{\cos x}\,dx = \int \sec x+\tan x\,dx$$

... "STANDARD RESULTS" ...

$$= \ln|\sec x+\tan x| + \ln|\sec x| + C$$
$$= \ln|\sec^2 x+\sec x\tan x| + C$$

OR TIDY FURTHER

$$= \ln\left|\frac{1}{\cos^2 x}+\frac{\sin x}{\cos^2 x}\right| + C = \ln\left|\frac{1+\sin x}{\cos^2 x}\right| + C$$
$$= \ln\left|\frac{1+\sin x}{1-\sin^2 x}\right| + C = \ln\left|\frac{\cancel{1+\sin x}}{(1-\sin x)\cancel{(1+\sin x)}}\right| + C$$
$$= \ln\left|\frac{1}{1-\sin x}\right| + C = -\ln|1-\sin x| + C$$

ALTERNATIVE BY TRIGONOMETRIC MANIPULATIONS

$$\int \frac{1+\sin x}{\cos x}\,dx = \int \frac{(1+\sin x)(1-\sin x)}{\cos x(1-\sin x)}\,dx = \int \frac{1-\sin^2 x}{\cos x(1-\sin x)}\,dx$$
$$= \int \frac{\cos^2 x}{\cos x(1-\sin x)}\,dx = \int \frac{\cos x}{1-\sin x}\,dx$$
$$= -\int \frac{-\cos x}{1-\sin x}\,dx = -\ln|1-\sin x| + C$$

[WHICH IS OF THE FORM $\displaystyle\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + C$]

207. $\displaystyle\int \sin 2x \sec x\,dx = -2\cos x + C$

$$\int \sin 2x\sec x\,dx = \int (2\sin x\cos x)\times\frac{1}{\cos x}\,dx = \int 2\sin x\,dx$$
$$= -2\cos x + C$$

208. $\displaystyle\int \sin 2x \sin x \ dx = \begin{bmatrix} \frac{2}{3}\sin^3 x + C \\ \frac{1}{2}\sin x - \frac{1}{6}\sin 3x + C \\ \frac{1}{2}\cos x \sin 2x - \frac{2}{3}\sin x \cos 2x + C \end{bmatrix}$

$\displaystyle\int \sin 2x \sin x \, dx = \int (2\sin x \cos x)\sin x \, dx = \int 2\sin^2 x \cos x \, dx$

... BY REVERSE CHAIN RULE (INSPECTION) OR THE SUBSTITUTION $u = \sin x$ ...

$= \frac{2}{3}\sin^3 x + C$

ALTERNATIVE BY COMPOUND ANGLE IDENTITIES

$$\left.\begin{aligned} \cos 3x &\equiv \cos(2x+x) = \cos 2x \cos x - \sin 2x \sin x \\ \cos x &\equiv \cos(2x-x) = \cos 2x \cos x + \sin 2x \sin x \end{aligned}\right\} \text{SUBTRACT}$$

$\Rightarrow \cos 3x - \cos x \equiv -2\sin 2x \sin x$

$\Rightarrow \sin 2x \sin x \equiv -\frac{1}{2}\cos 3x + \frac{1}{2}\cos x$

$\Rightarrow \displaystyle\int \sin 2x \sin x \, dx = \int \frac{1}{2}\cos x - \frac{1}{2}\cos 3x \, dx = \frac{1}{2}\sin x - \frac{1}{6}\sin 3x + C$

ALTERNATIVE BY INTEGRATION BY PARTS

$\displaystyle\int \sin 2x \sin x \, dx = -\cos x \sin 2x - \int -2\cos 2x \cos x \, dx$

$\displaystyle= -\cos x \sin 2x + \int 2\cos 2x \cos x \, dx$

$\sin 2x$	$2\cos 2x$
$-\cos x$	$\sin x$

BY PARTS AGAIN ON THIS SECTION

$2\cos 2x$	$-4\sin 2x$
$\sin x$	$\cos x$

$\displaystyle\int \sin 2x \sin x \, dx = -\cos x \sin 2x + \left[2\sin x \cos 2x - \int -4\sin x \sin 2x \, dx\right]$

$\displaystyle\int \sin 2x \sin x \, dx = -\cos x \sin 2x + 2\sin x \cos 2x + 4\int \sin 2x \sin x \, dx$

$\displaystyle\cos x \sin 2x - 2\sin x \cos 2x = 3\int \sin 2x \sin x \, dx$

$\displaystyle\int \sin 2x \sin x \, dx = \frac{1}{3}\cos x \sin 2x - \frac{2}{3}\sin x \cos 2x + C$

209. $\displaystyle\int x\cos^2 x \ dx = \frac{1}{4}x^2 + \frac{1}{4}x\sin 2x + \frac{1}{8}\cos 2x + C$

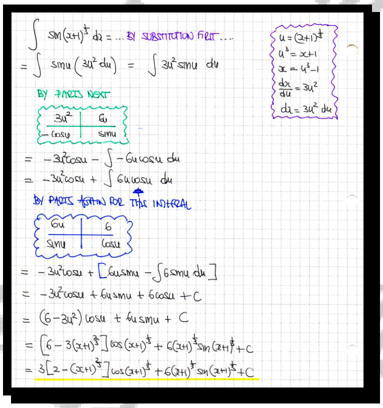
$\displaystyle\int x\cos^2 x \, dx = \int x\left(\frac{1}{2} + \frac{1}{2}\cos 2x\right) dx = \int \frac{1}{2}x + \underbrace{\frac{1}{2}x\cos 2x \, dx}_{\text{BY PARTS}}$

$\frac{1}{2}x$	$\frac{1}{2}$
$\frac{1}{2}\sin 2x$	$\cos 2x$

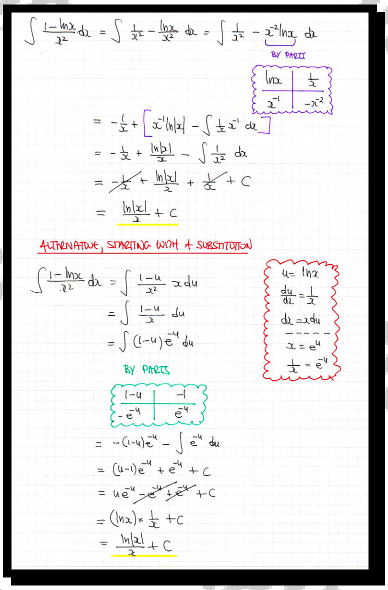
$\displaystyle= \frac{1}{4}x^2 + \left[\frac{1}{4}x\sin 2x - \int \frac{1}{4}\sin 2x \, dx\right]$

$\displaystyle= \frac{1}{4}x^2 + \frac{1}{4}x\sin 2x + \frac{1}{8}\cos 2x + C$

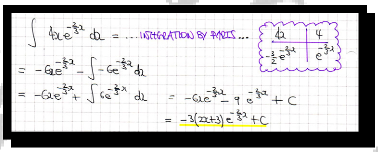
210. $$\int \sin(x+1)^{\frac{1}{3}}\ dx = 3\left[2-(x+1)^{\frac{2}{3}}\right]\cos(x+1)^{\frac{1}{3}} + 6(x+1)^{\frac{1}{3}}\sin(x+1)^{\frac{1}{3}} + C$$



211. $$\int \frac{1-\ln x}{x^2}\ dx = \frac{\ln|x|}{x} + C$$



212. $$\int 4xe^{-\frac{2}{3}x}\ dx = -3(2x+3)e^{-\frac{2}{3}x} + C$$



213. $\displaystyle\int \left(e^{x}+x\right)^{2}\ dx = \frac{1}{2}e^{2x}+2xe^{x}-2e^{x}+\frac{1}{3}x^{3}+C$

$\int (e^{x}+x)^{2}\,dx = \int e^{2x}+\underbrace{2xe^{x}}+x^{2}\,dx$ BY PARTS

$\begin{array}{c|c} 2x & 2 \\ \hline e^{x} & e^{x} \end{array}$

$= \frac{1}{2}e^{2x}+\left[2xe^{x}-\int 2e^{x}\,dx\right]+\frac{1}{3}x^{3}$

$= \frac{1}{2}e^{2x}+2xe^{x}-2e^{x}+\frac{1}{3}x^{3}+C$

214. $\displaystyle\int \frac{e^{4x}-e^{-x}}{e^{2x}}\ dx = \frac{1}{2}e^{2x}+\frac{1}{3}e^{-3x}+C$

$\int \frac{e^{4x}-e^{-x}}{e^{2x}}\,dx = \int \frac{e^{4x}}{e^{2x}}-\frac{e^{-x}}{e^{2x}}\,dx = \int e^{2x}-e^{-3x}\,dx$

$= \frac{1}{2}e^{2x}+\frac{1}{3}e^{-3x}+C$

215. $\displaystyle\int \frac{e^{\ln x}}{x}\ dx = x+C$

$\int \frac{e^{\ln x}}{x}\,dx = \int \frac{x}{x}\,dx = \int 1\,dx = x+C$

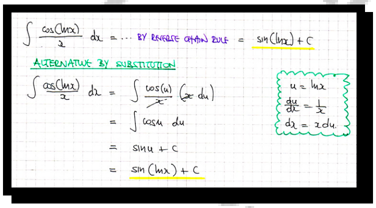
216. $\displaystyle\int \frac{1}{2(3x+1)^{4}}\ dx = -\frac{1}{18(3x+1)^{3}}+C$

$\int \frac{1}{2(3x+1)^{4}}\,dx = \int \frac{1}{2}(3x+1)^{-4}\,dx = \ldots$ BY INSPECTION

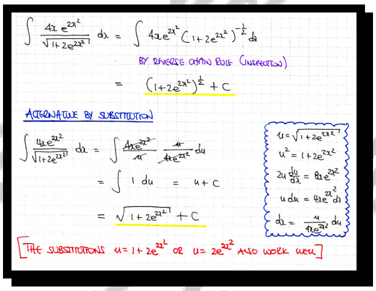
$= \frac{\frac{1}{2}}{-9}(3x+1)^{-3}+C = -\frac{1}{18}(3x+1)^{-3}+C$

$= -\frac{1}{18(3x+1)^{3}}+C$

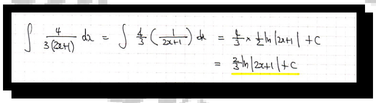
217. $\displaystyle\int \frac{\cos(\ln x)}{x}\, dx = \sin(\ln x) + C$



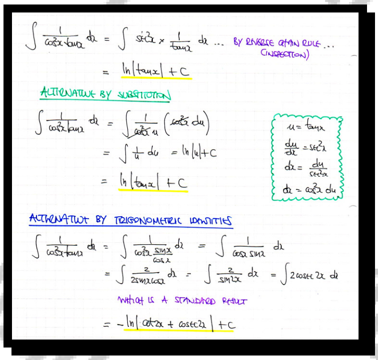
218. $\displaystyle\int \frac{4x\,\mathrm{e}^{2x^2}}{\sqrt{1+2\,\mathrm{e}^{2x^2}}}\, dx = \sqrt{1+2\,\mathrm{e}^{2x^2}} + C$



219. $\displaystyle\int \frac{4}{3(2x+1)}\, dx = \frac{2}{3}\ln|2x+1| + C$



220. $$\int \frac{1}{\cos^2 x \tan x}\,dx = \begin{bmatrix} \ln|\tan x| + C \\ -\ln|\cot 2x + \operatorname{cosec} 2x| + C \end{bmatrix}$$



$$\int \frac{1}{\cos^2 x \tan x}\,dx = \int \sec^2 x \times \frac{1}{\tan x}\,dx \;\dots$$ BY REVERSE CHAIN RULE … (INSPECTION)

$$= \ln|\tan x| + C$$

ALTERNATIVE BY SUBSTITUTION

$$\int \frac{1}{\cos^2 x \tan x}\,dx = \int \frac{1}{\cos^2 x\, u}\left(\cos^2 x\,du\right) = \int \frac{1}{u}\,du = \ln|u| + C = \ln|\tan x| + C$$

$u = \tan x$, $\frac{du}{dx} = \sec^2 x$, $dx = \frac{du}{\sec^2 x}$, $dx = \cos^2 x\,du$

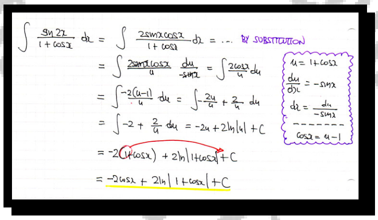
ALTERNATIVE BY TRIGONOMETRIC IDENTITIES

$$\int \frac{1}{\cos^2 x \tan x}\,dx = \int \frac{1}{\cos^2 x \frac{\sin x}{\cos x}}\,dx = \int \frac{1}{\cos x \sin x}\,dx = \int \frac{2}{2\sin x \cos x}\,dx = \int \frac{2}{\sin 2x}\,dx = \int 2\operatorname{cosec} 2x\,dx$$

WHICH IS A STANDARD RESULT

$$= -\ln|\cot 2x + \operatorname{cosec} 2x| + C$$

221. $$\int \frac{\sin 2x}{1+\cos x}\,dx = -2\cos x + 2\ln|1+\cos x| + C$$



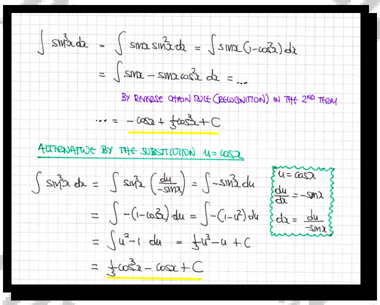
$$\int \frac{\sin 2x}{1+\cos x}\,dx = \int \frac{2\sin x\cos x}{1+\cos x}\,dx = \dots$$ BY SUBSTITUTION

$$= \int \frac{2\sin x\cos x}{u}\,\frac{du}{-\sin x} = \int -\frac{2\cos x}{u}\,du = \int -\frac{2(u-1)}{u}\,du = \int -\frac{2u}{u} + \frac{2}{u}\,du$$

$$= \int -2 + \frac{2}{u}\,du = -2u + 2\ln|u| + C = -2(1+\cos x) + 2\ln|1+\cos x| + C = -2\cos x + 2\ln|1+\cos x| + C$$

$u = 1+\cos x$, $\frac{du}{dx} = -\sin x$, $dx = \frac{du}{-\sin x}$, $\cos x = u-1$

222. $$\int \sin^3 x\; dx = \frac{1}{3}\cos^2 x - \cos x + C$$



$$\int \sin^3 x\,dx = \int \sin x \sin^2 x\,dx = \int \sin x(1-\cos^2 x)\,dx = \int \sin x - \sin x\cos^2 x\,dx = \dots$$

BY REVERSE CHAIN RULE (RECOGNITION) IN THE 2ND TERM

$$\dots = -\cos x + \frac{1}{3}\cos^3 x + C$$

ALTERNATIVE BY THE SUBSTITUTION $u = \cos x$

$$\int \sin^3 x\,dx = \int \sin^3 x\left(\frac{du}{-\sin x}\right) = \int -\sin^2 x\,du = \int -(1-\cos^2 x)\,du = \int -(1-u^2)\,du = \int u^2 - 1\,du = \frac{1}{3}u^3 - u + C = \frac{1}{3}\cos^3 x - \cos x + C$$

$u = \cos x$, $\frac{du}{dx} = -\sin x$, $dx = \frac{du}{-\sin x}$

223. $$\int \tfrac{1}{3}\sin 2x - \tfrac{1}{2}\cos 3x \; dx = -\tfrac{1}{6}\cos 2x - \tfrac{1}{6}\sin 3x + C$$

$\int \frac{1}{3}\sin 2x - \frac{1}{2}\cos 3x \, dx = \dots$ STANDARD ANTIDERIVATIVES

$= -\frac{1}{6}\cos 2x - \frac{1}{6}\sin 3x + C$

224. $$\int \frac{3x}{\sqrt{4-2x^2}} \; dx = -\tfrac{3}{2}\left(4-2x^2\right)^{\frac{1}{2}} + C$$

$\int \frac{3x}{\sqrt{4-2x^2}}\,dx = \int 3x(4-2x^2)^{-\frac{1}{2}}\,dx = \dots$ BY REVERSE CHAIN RULE RECOGNITION

$= -\frac{3}{2}(4-2x^2)^{\frac{1}{2}} + C$

ALTERNATIVE BY SUBSTITUTION

$\int \frac{3x}{\sqrt{4-2x^2}}\,dx = \int \frac{3x}{u}\left(-\frac{u}{2x}\,du\right) = \int -\frac{3}{2}\,du$

$= -\frac{3}{2}u + C = -\frac{3}{2}(4-2x^2)^{\frac{1}{2}} + C$

$u = \sqrt{4-2x^2}$

$u^2 = 4-2x^2$

$2u\frac{du}{dx} = -4x$

$dx = -\frac{u}{2x}\,du$

THE SUBSTITUTION $u = 4-2x^2$ ALSO WORKS WELL

225. $$\int \frac{1}{\sin x \cos^2 x} \; dx = \begin{bmatrix} \ln\left|\tan\left(\frac{1}{2}x\right)\right| + \sec x + C \\ \sec x - \ln\left|\operatorname{cosec} x + \cot x\right| + C \\ \frac{1}{2}\ln\left|\frac{\cos x - 1}{\cos x + 1}\right| + \sec x + C \end{bmatrix}$$

$\int \frac{1}{\sin x\cos^2 x}\,dx = \int \frac{\cos^2 x + \sin^2 x}{\sin x\cos^2 x}\,dx = \int \frac{\cos^2 x}{\sin x\cos^2 x} + \frac{\sin^2 x}{\sin x\cos^2 x}\,dx$

$= \int \operatorname{cosec} x + \tan x\sec x = \dots$ STANDARD RESULTS

$= \ln\left|\tan\frac{x}{2}\right| + \sec x + C$

OR $\quad \sec x - \ln\left|\operatorname{cosec} x + \cot x\right| + C$

ALTERNATIVE BY SUBSTITUTION & PARTIAL FRACTIONS

$\int \frac{1}{\sin x\cos^2 x}\,dx = \int \frac{1}{u^2\sin x}\left(-\frac{du}{\sin x}\right)$

$= \int -\frac{1}{u^2\sin^2 x}\,du = \int -\frac{1}{u^2(1-\cos^2 x)}\,du$

$= \int -\frac{1}{u^2(1-u^2)}\,du = \int \frac{1}{u^2(u+1)(u-1)}\,du$

$u = \cos x$

$\frac{du}{dx} = -\sin x$

$dx = \frac{du}{-\sin x}$

BY PARTIAL FRACTIONS

$\frac{1}{u^2(u+1)(u-1)} \equiv \frac{A}{u} + \frac{B}{u-1} + \frac{C}{u+1} + \frac{D}{u^2}$

$\boxed{1 \equiv Au(u-1)(u+1) + Bu^2(u+1) + Cu^2(u-1) + D(u+1)(u-1)}$

- IF $u=0$: $1 = -D$, $D = -1$
- IF $u=1$: $1 = 2B$, $B = \frac{1}{2}$
- IF $u=-1$: $1 = -2C$, $C = -\frac{1}{2}$
- IF $u=2 \Rightarrow 1 = 6A + 12B + 4C + 3D$, $1 = 6A + 6 - 2 - 3$, $A = 0$

$= \int \frac{\frac{1}{2}}{u-1} - \frac{\frac{1}{2}}{u+1} - \frac{1}{u^2}\,du = \frac{1}{2}\ln\left|\frac{u-1}{u+1}\right| + \frac{1}{u} + C$

$= \frac{1}{2}\ln\left|\frac{\cos x - 1}{\cos x + 1}\right| + \frac{1}{\cos x} + C = \frac{1}{2}\ln\left|\frac{\cos x - 1}{\cos x + 1}\right| + \sec x + C$

$= \frac{1}{2}\ln\left|\frac{1 - 2\sin^2 x/2 - 1}{2\cos^2 x/2 - 1 + 1}\right| + \sec x + C = \frac{1}{2}\ln\left|-\tan^2\frac{x}{2}\right| + \sec x + C$

$= \frac{1}{2}\ln\left|\tan^2\frac{x}{2}\right| + \sec x + C = \ln\left|\tan\frac{x}{2}\right| + \sec x + C$

AS ABOVE

226. $\displaystyle\int \frac{3x}{4-2x^2}\,dx = -\frac{3}{4}\ln\left|4-2x^2\right| + C$

$$\int \frac{3x}{4-2x^2}\,dx = -\frac{3}{4}\int \frac{-4x}{4-2x^2}\,dx = \dots \quad \int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + C$$

$$= -\frac{3}{4}\ln|4-2x^2| + C$$

THE SUBSTITUTION $u = 4-2x^2$ ALSO WORKS WELL

227. $\displaystyle\int \frac{1-4x}{x(4x-\ln x)^{\frac{3}{2}}}\,dx = \frac{2}{\sqrt{4x-\ln x}} + C$

$$\int \frac{1-4x}{x(4x-\ln x)^{\frac{3}{2}}}\,dx = \int \left(\frac{1-4x}{x}\right)(4x-\ln x)^{-\frac{3}{2}}\,dx = \int -\left(4-\frac{1}{x}\right)(4x-\ln x)^{-\frac{3}{2}}\,dx$$

BY REVERSE CHAIN RULE / RECOGNITION

$$= \frac{-1}{-\frac{1}{2}}(4x-\ln x)^{-\frac{1}{2}} + C = \frac{2}{\sqrt{4x-\ln x}} + C$$

ALTERNATIVE BY SUBSTITUTION

$$\int \frac{1-4x}{x(4x-\ln x)^{\frac{3}{2}}}\,dx = \int \frac{1-4x}{x\,u^{3/2}}\left(\frac{x}{4x-1}\,du\right)$$

$$= \int -\frac{1}{u^{3/2}}\,du = \int -u^{-\frac{3}{2}}\,du = 2u^{-\frac{1}{2}} + C$$

$$= 2(4x-\ln x)^{-\frac{1}{2}} + C = \frac{2}{\sqrt{4x-\ln x}} + C$$

$u = 4x - \ln x$

$\frac{du}{dx} = 4 - \frac{1}{x}$

$\frac{du}{dx} = \frac{4x-1}{x}$

$dx = \frac{x}{4x-1}\,du$

THE SUBSTITUTION $u = \sqrt{4x-\ln x}$ ALSO WORKS WELL

228. $\displaystyle\int \frac{\sin x}{\cos^4 x}\,dx = \frac{1}{3}\sec^3 x + C$

$$\int \frac{\sin x}{\cos^4 x}\,dx = \int (\cos x)^{-4}\sin x\,dx = \dots \text{ BY REVERSE CHAIN RULE} \dots$$

$$= \frac{1}{3}(\cos x)^{-3} + C = \frac{1}{3\cos^3 x} + C = \frac{1}{3}\sec^3 x + C$$

ALTERNATIVE VARIATION OF REVERSE CHAIN RULE

$$\int \frac{\sin x}{\cos^4 x}\,dx = \int \frac{\sin x}{\cos x}\,\frac{1}{\cos^3 x}\,dx = \int \sec^3 x \tan x\,dx$$

$$= \int \sec^2 x(\sec x \tan x)\,dx = \frac{1}{3}\sec^3 x + C$$

$\frac{d}{dx}(\sec x) = \sec x \tan x$

THE SUBSTITUTIONS $u = \cos x$ OR $u = \sec x$ ALSO WORK WELL

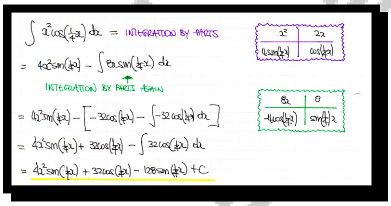
229. $\displaystyle\int \frac{\operatorname{cosec}^2 x}{1+\cot x}\,dx = -\ln\left|1+\cot x\right| + C$

$$\int \frac{\operatorname{cosec}^2 x}{1+\cot x}\,dx = -\int \frac{-\operatorname{cosec}^2 x}{1+\cot x}\,dx = \dots \quad \int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + C$$

$$= -\ln|1+\cot x| + C$$

THE SUBSTITUTION $u = 1+\cot x$ ALSO WORKS WELL

230. $\displaystyle\int x^2\cos\left(\tfrac{1}{4}x\right)\,dx = 4x^2\sin\left(\tfrac{1}{4}x\right) + 32\cos\left(\tfrac{1}{4}x\right) - 128\sin\left(\tfrac{1}{4}x\right) + C$



$\int x^2\cos(\frac{1}{4}x)\,dx$ = INTEGRATION BY PARTS

x^2	$2x$
$4\sin(\frac{1}{4}x)$	$\cos(\frac{1}{4}x)$

$= 4x^2\sin(\frac{1}{4}x) - \int 8x\sin(\frac{1}{4}x)\,dx$

INTEGRATION BY PARTS AGAIN

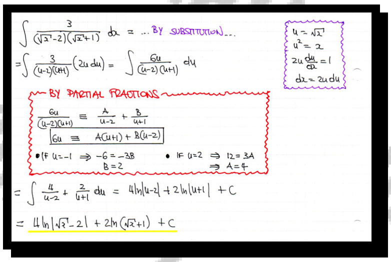
$8x$	8
$-4\cos(\frac{1}{4}x)$	$\sin(\frac{1}{4}x)$

$= 4x^2\sin(\frac{1}{4}x) - \left[-32x\cos(\frac{1}{4}x) - \int -32\cos(\frac{1}{4}x)\,dx\right]$

$= 4x^2\sin(\frac{1}{4}x) + 32x\cos(\frac{1}{4}x) - \int 32\cos(\frac{1}{4}x)\,dx$

$= 4x^2\sin(\frac{1}{4}x) + 32x\cos(\frac{1}{4}x) - 128\sin(\frac{1}{4}x) + C$

231. $\displaystyle\int \frac{3}{\left(\sqrt{x}-2\right)\left(\sqrt{x}+1\right)}\,dx = 4\ln\left|\sqrt{x}-2\right| + 2\ln\left(\sqrt{x}+1\right) + C$



$\int \frac{3}{(\sqrt{x}-2)(\sqrt{x}+1)}\,dx = \dots$ BY SUBSTITUTION...

$u = \sqrt{x}$
$u^2 = x$
$2u\frac{du}{dx} = 1$
$dx = 2u\,du$

$= \int \frac{3}{(u-2)(u+1)}(2u\,du) = \int \frac{6u}{(u-2)(u+1)}\,du$

BY PARTIAL FRACTIONS

$\frac{6u}{(u-2)(u+1)} \equiv \frac{A}{u-2} + \frac{B}{u+1}$

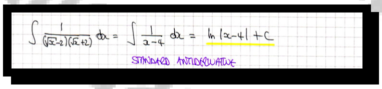
$\boxed{6u \equiv A(u+1) + B(u-2)}$

• IF $u=-1 \Rightarrow -6 = -3B$, $B = 2$ • IF $u=2 \Rightarrow 12 = 3A \Rightarrow A = 4$

$= \int \frac{4}{u-2} + \frac{2}{u+1}\,du = 4\ln|u-2| + 2\ln|u+1| + C$

$= 4\ln|\sqrt{x}-2| + 2\ln(\sqrt{x}+1) + C$

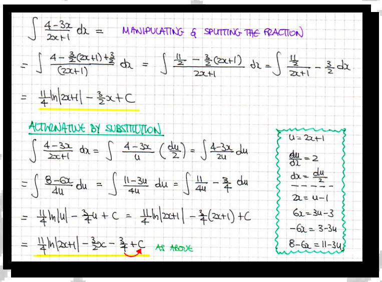
232. $\displaystyle\int \frac{1}{\left(\sqrt{x}-2\right)\left(\sqrt{x}+2\right)}\,dx = \ln|x-4| + C$



$\int \frac{1}{(\sqrt{x}-2)(\sqrt{x}+2)}\,dx = \int \frac{1}{x-4}\,dx = \ln|x-4| + C$

STANDARD ANTIDERIVATIVE

233. $\displaystyle\int \frac{4-3x}{2x+1}\,dx = \frac{11}{4}\ln|2x+1| - \frac{3}{2}x + C$



$\int \frac{4-3x}{2x+1}\,dx$ = MANIPULATING & SPLITTING THE FRACTION

$= \int \frac{4 - \frac{3}{2}(2x+1) + \frac{3}{2}}{(2x+1)}\,dx = \int \frac{\frac{11}{2} - \frac{3}{2}(2x+1)}{2x+1}\,dx = \int \frac{\frac{11}{2}}{2x+1} - \frac{3}{2}\,dx$

$= \frac{11}{4}\ln|2x+1| - \frac{3}{2}x + C$

ALTERNATIVE BY SUBSTITUTION

$u = 2x+1$
$\frac{du}{dx} = 2$
$dx = \frac{du}{2}$
$2x = u-1$
$6x = 3u-3$
$-6x = 3-3u$
$8-6x = 11-3u$

$\int \frac{4-3x}{2x+1}\,dx = \int \frac{4-3x}{u}\left(\frac{du}{2}\right) = \int \frac{4-3x}{2u}\,du$

$= \int \frac{8-6x}{4u}\,du = \int \frac{11-3u}{4u}\,du = \int \frac{11}{4u} - \frac{3}{4}\,du$

$= \frac{11}{4}\ln|u| - \frac{3}{4}u + C = \frac{11}{4}\ln|2x+1| - \frac{3}{4}(2x+1) + C$

$= \frac{11}{4}\ln|2x+1| - \frac{3}{2}x - \frac{3}{4} + C$ AS ABOVE

234. $\displaystyle\int \left(1+\frac{1}{x}\right)\sqrt{x}\ dx = \frac{2}{3}x^{\frac{3}{2}} + 2x^{\frac{1}{2}} + C$

$$\int \sqrt{x}\left(1+\frac{1}{x}\right)dx = \int x^{\frac{1}{2}}\left(1+x^{-1}\right)dx = \int x^{\frac{1}{2}} + x^{-\frac{1}{2}}\,dx$$
$$= \frac{2}{3}x^{\frac{3}{2}} + 2x^{\frac{1}{2}} + C$$

235. $\displaystyle\int \frac{\sec x}{\cos x - \sin x}\ dx = \ln\left|1-\tan x\right| + C$

$$\int \frac{\sec x}{\cos x - \sin x}\,dx = \int \frac{\sec x\sec x}{\cos x\sec x - \sin x\sec x}\,dx$$
$$= \int \frac{\sec^2 x}{1-\tan x}\,dx$$
$$\dots \int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + C$$
$$= \ln|1-\tan x| + C$$

236. $\displaystyle\int \frac{x^2-2}{x^2-1}\ dx = x + \frac{1}{2}\ln\left|\frac{x+1}{x-1}\right| + C$

$$\int \frac{x^2-2}{x^2-1}\,dx = \int \frac{(x^2-1)-1}{x^2-1}\,dx = \int 1 - \frac{1}{x^2-1}\,dx$$

! IMPROPER !

$$= \int 1 - \frac{1}{(x-1)(x+1)}\,dx = \dots$$

PARTIAL FRACTIONS BY INSPECTION

$$= \int 1 - \left(\frac{\frac{1}{2}}{x-1} + \frac{-\frac{1}{2}}{x+1}\right)dx = \int 1 + \frac{\frac{1}{2}}{x+1} - \frac{\frac{1}{2}}{x-1}\,dx$$
$$= x + \frac{1}{2}\ln|x+1| - \frac{1}{2}\ln|x-1| + C = x + \frac{1}{2}\ln\left|\frac{x+1}{x-1}\right| + C$$

237. $\displaystyle\int \frac{1}{x^3-x^2}\ dx = \frac{1}{x} + \ln\left|\frac{x-1}{x}\right| + C$

$$\int \frac{1}{x^3-x^2}\,dx = \int \frac{1}{x^2(x-1)}\,dx = \dots$$

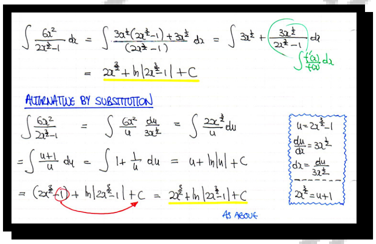
BY PARTIAL FRACTIONS

$$\frac{1}{x^2(x-1)} \equiv \frac{A}{x^2} + \frac{B}{x-1} + \frac{C}{x}$$
$$\boxed{1 \equiv A(x-1) + Bx^2 + Cx(x-1)}$$

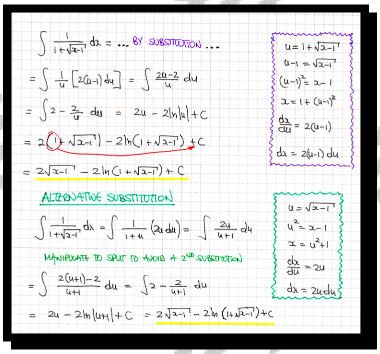
- IF $x=1$: $1 = B$
- IF $x=0$: $1 = -A$, $A = -1$
- IF $x=2$: $1 = A + 4B + 2C$, $1 = -1 + 4 + 2C$, $C = -1$

$$= \int \frac{1}{x-1} - \frac{1}{x} - \frac{1}{x^2}\,dx = \ln|x-1| - \ln|x| + \frac{1}{x} + C = \frac{1}{x} + \ln\left|\frac{x-1}{x}\right| + C$$

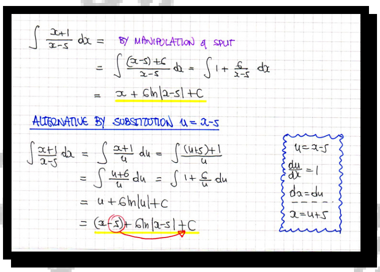
238. $$\int \frac{6x^2}{2x^{\frac{3}{2}}-1}\,dx = 2x^{\frac{3}{2}} + \ln\left|2x^{\frac{3}{2}}-1\right| + C$$



239. $$\int \frac{1}{1+\sqrt{x-1}}\,dx = 2\sqrt{x-1} - 2\ln\left(1+\sqrt{x-1}\right) + C$$



240. $$\int \frac{x+1}{x-5}\,dx = x + 6\ln|x-5| + C$$



241. $\displaystyle\int \frac{(x+2)^2}{3x}\ dx = \tfrac{1}{6}x^2 + \tfrac{4}{3}x + \tfrac{4}{3}\ln|x| + C$

$\displaystyle\int \frac{(x+2)^2}{3x}\,dx = \int \frac{x^2+4x+4}{3x}\,dx = \int \tfrac{1}{3}x + \tfrac{4}{3} + \tfrac{4}{3x}\,dx$

... STANDARD ANTIDERIVATIVES ...

$= \tfrac{1}{6}x^2 + \tfrac{4}{3}x + \tfrac{4}{3}\ln|x| + C$

242. $\displaystyle\int 4e^{-2x} - \tfrac{1}{3}\sin 3x\ \ dx = -2e^{-2x} + \tfrac{1}{9}\cos 3x + C$

$\displaystyle\int 4e^{-2x} - \tfrac{1}{3}\sin 3x\,dx = \ldots$ STANDARD ANTIDERIVATIVES ...

$= -2e^{-2x} + \tfrac{1}{9}\cos 3x + C$

243. $\displaystyle\int 2x\sec^2 2x\ \ dx = \begin{bmatrix} x\tan x + \tfrac{1}{2}\ln|\cos 2x| + C \\ x\tan x - \tfrac{1}{2}\ln|\sec 2x| + C \end{bmatrix}$

$\displaystyle\int 2x\sec^2 2x\,dx = \ldots$ INTEGRATION BY PARTS ...

$2x$	2
$\tfrac{1}{2}\tan 2x$	$\sec^2 2x$

$\displaystyle = x\tan 2x - \int \tan 2x\,dx$

STANDARD RESULT WHICH CAN EASILY BE PROVEN

$\displaystyle\int \tan x\,dx = \ln|\sec x| + C = -\ln|\cos x| + C$

$= x\tan x + \tfrac{1}{2}\ln|\cos 2x| + C = x\tan x - \tfrac{1}{2}\ln|\sec 2x| + C$

244. $\displaystyle\int \cos x\sin^8 x\ \ dx = \tfrac{1}{9}\sin^9 x + C$

$\displaystyle\int \cos x\sin^8 x\,dx = \ldots$ BY REVERSE CHAIN RULE (RECOGNITION)

$= \tfrac{1}{9}\sin^9 x + C$

[THE SUBSTITUTION $u = \sin x$ ALSO WORKS WELL]

245. $\displaystyle\int \frac{\sin^5 x}{\cos^7 x}\, dx = \frac{1}{6}\tan^6 x + C$

$$\int \frac{\sin^5 x}{\cos^7 x}\,dx = \int \frac{\sin^5 x}{\cos^5 x} \times \frac{1}{\cos^2 x}\,dx = \int \tan^5 x \sec^2 x\,dx$$

$= \ldots$ BY REVERSE CHAIN RULE (RECOGNITION)

$= \frac{1}{6}\tan^6 x + C$

[THE SUBSTITUTION $u = \tan x$, ALSO WORKS WELL]

246. $\displaystyle\int \tan 3x\ dx = \begin{bmatrix} \frac{1}{3}\ln|\sec x| + C \\ -\frac{1}{3}\ln|\cos x| + C \end{bmatrix}$

$$\int \tan 3x\,dx = \int \frac{\sin 3x}{\cos 3x}\,dx = -\frac{1}{3}\int \frac{-3\sin 3x}{\cos 3x}\,dx$$

$\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + C$

$= -\frac{1}{3}\ln|\cos 3x| + C = \frac{1}{3}\ln|\sec 3x| + C$

[THE SUBSTITUTION $u = \cos 3x$ OR $u = \sec 3x$ ALSO WORKS]

247. $\displaystyle\int \frac{1}{\sec x - 1}\, dx = -x - \cot x - \operatorname{cosec} x + C$

$$\int \frac{1}{\sec x - 1}\,dx = \int \frac{\sec x + 1}{(\sec x - 1)(\sec x + 1)}\,dx = \int \frac{\sec x + 1}{\sec^2 x - 1}\,dx$$

$= \int \frac{\sec x + 1}{\tan^2 x}$ [$1 + \tan^2 x \equiv \sec^2 x$]

$= \int \frac{\sec x}{\tan^2 x} + \frac{1}{\tan^2 x}\,dx = \int \sec x \cot^2 x + \cot^2 x\,dx$

$= \int \frac{1}{\cos x}\frac{\cos^2 x}{\sin^2 x} + (\operatorname{cosec}^2 x - 1)\,dx$

$= \int \frac{\cos x}{\sin x}\frac{1}{\sin x} + \operatorname{cosec}^2 x - 1\,dx$

$= \int \cot x \operatorname{cosec} x + \operatorname{cosec}^2 x - 1\,dx$

$= -\operatorname{cosec} x - \cot x - x + C$

$\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$ & $\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$

248. $\displaystyle\int \frac{\sin^2 x}{\cos^4 x}\, dx = \frac{1}{3}\tan^3 x + C$

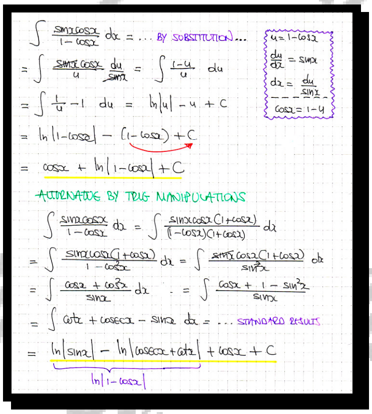
$$\int \frac{\sin^2 x}{\cos^4 x}\,dx = \int \frac{\sin^2 x}{\cos^2 x}\frac{1}{\cos^2 x}\,dx = \int \tan^2 x \sec^2 x\,dx$$

BY REVERSE CHAIN RULE SINCE $\frac{d}{dx}(\tan x) = \sec^2 x$

$= \frac{1}{3}\tan^3 x + C$

[THE SUBSTITUTION $u = \tan x$ ALSO WORKS WELL]

249. $\displaystyle\int \frac{\sin x \cos x}{1-\cos x}\,dx = \left[\begin{array}{c} \cos x + \ln|1-\cos x| + C \\ \cos x - \ln|\cot x + \operatorname{cosec} x| + C \end{array}\right]$



$\displaystyle\int \frac{\sin x\cos x}{1-\cos x}\,dx = \ldots$ BY SUBSTITUTION...

$u = 1-\cos x$
$\frac{du}{dx} = \sin x$
$dx = \frac{du}{\sin x}$
$\cos x = 1-u$

$\displaystyle = \int \frac{\cancel{\sin x}\cos x}{u}\,\frac{du}{\cancel{\sin x}} = \int \frac{1-u}{u}\,du$

$\displaystyle = \int \frac{1}{u} - 1\,du = \ln|u| - u + C$

$= \ln|1-\cos x| - (1-\cos x) + C$

$= \cos x + \ln|1-\cos x| + C$

ALTERNATIVE BY TRIG MANIPULATIONS

$\displaystyle \int \frac{\sin x\cos x}{1-\cos x}\,dx = \int \frac{\sin x\cos x(1+\cos x)}{(1-\cos x)(1+\cos x)}\,dx$

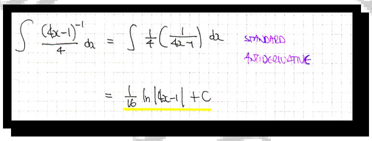
$\displaystyle = \int \frac{\sin x\cos x(1+\cos x)}{1-\cos^2 x}\,dx = \int \frac{\cancel{\sin x}\cos x(1+\cos x)}{\sin^{\cancel{2}} x}\,dx$

$\displaystyle = \int \frac{\cos x + \cos^2 x}{\sin x}\,dx = \int \frac{\cos x + 1 - \sin^2 x}{\sin x}$

$\displaystyle = \int \cot x + \operatorname{cosec} x - \sin x\,dx = \ldots$ STANDARD RESULTS

$= \underbrace{\ln|\sin x| - \ln|\operatorname{cosec} x + \cot x|}_{\ln|1-\cos x|} + \cos x + C$

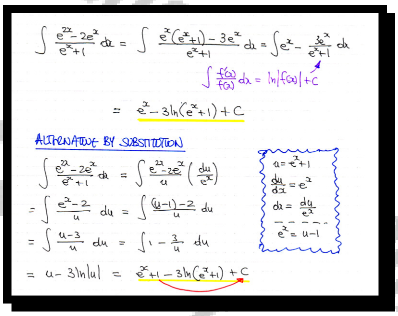
250. $\displaystyle\int \frac{(4x-1)^{-1}}{4}\,dx = -\frac{1}{16}\ln|4x-1| + C$



$\displaystyle\int \frac{(4x-1)^{-1}}{4}\,dx = \int \frac{1}{4}\left(\frac{1}{4x-1}\right)dx$ STANDARD ANTIDERIVATIVE

$= \frac{1}{16}\ln|4x-1| + C$

251. $\displaystyle\int \frac{e^{2x}-2e^{x}}{e^{x}+1}\,dx = e^{x} - 3\ln\left(e^{x}+1\right) + C$



$\displaystyle\int \frac{e^{2x}-2e^{x}}{e^{x}+1}\,dx = \int \frac{e^{x}(e^{x}+1)-3e^{x}}{e^{x}+1}\,dx = \int e^{x} - \frac{3e^{x}}{e^{x}+1}\,dx$

$\displaystyle\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + C$

$= e^{x} - 3\ln(e^{x}+1) + C$

ALTERNATIVE BY SUBSTITUTION

$u = e^{x}+1$
$\frac{du}{dx} = e^{x}$
$dx = \frac{du}{e^{x}}$
$e^{x} = u-1$

$\displaystyle\int \frac{e^{2x}-2e^{x}}{e^{x}+1}\,dx = \int \frac{e^{2x}-2e^{x}}{u}\left(\frac{du}{e^{x}}\right)$

$\displaystyle = \int \frac{e^{x}-2}{u}\,du = \int \frac{(u-1)-2}{u}\,du$

$\displaystyle = \int \frac{u-3}{u}\,du = \int 1 - \frac{3}{u}\,du$

$= u - 3\ln|u| = e^{x}+1 - 3\ln(e^{x}+1) + C$

252. $$\int \frac{2x^2 - 3x + 2}{x - 1}\, dx = x^2 - x + \ln|x-1| + C$$

$$\int \frac{2x^2-3x+2}{x-1}\,dx = \dots$$ BY LONG DIVISION OR MANIPULATION IN ORDER TO "SPLIT" THE FRACTION

$$= \int \frac{2x(x-1)-(x-1)+1}{x-1}\,dx = \int 2x - 1 + \frac{1}{x-1}\,dx$$

$$= x^2 - x + \ln|x-1| + C$$

[THE SUBSTITUTION $u = x-1$ ALSO WORKS WELL]

253. $$\int 1 - \cot^2 x \; dx = 2x + \cot x + C$$

$$\int 1-\cot^2 x\,dx = \int 1-(\operatorname{cosec}^2 x - 1)\,dx = \int 2 - \operatorname{cosec}^2 x\,dx$$

$$= 2x + \cot x + C$$

$$\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$$

254. $$\int \frac{x^2 + 1}{x^4 - x^2}\, dx = \frac{1}{x} + \ln\left|\frac{x-1}{x+1}\right| + C$$

$$\int \frac{x^2+1}{x^4-x^2}\,dx = \int \frac{x^2+1}{x^2(x^2-1)}\,dx = \int \frac{x^2+1}{x^2(x+1)(x-1)}\,dx$$

BY PARTIAL FRACTIONS

$$\frac{x^2+1}{x^2(x+1)(x-1)} \equiv \frac{A}{x+1} + \frac{B}{x-1} + \frac{C}{x^2} + \frac{D}{x}$$

$$\boxed{x^2+1 \equiv Ax^2(x-1) + Bx^2(x+1) + C(x+1)(x-1) + Dx(x-1)(x+1)}$$

- IF $x=0$: $1 = -C$, $C = -1$
- IF $x=1$: $2 = 2B$, $B = 1$
- IF $x=-1$: $2 = -2A$, $A = -1$
- IF $x=2$: $5 = 4A + 12B + 3C + 6D$, $5 = -4 + 12 - 3 + 6D$, $5 = 5 + 6D$, $D = 0$

$$\dots = \int \frac{1}{x-1} - \frac{1}{x+1} - \frac{1}{x^2}\,dx = \ln|x-1| - \ln|x+1| + \frac{1}{x} + C$$

$$= \frac{1}{x} + \ln\left|\frac{x-1}{x+1}\right| + C$$

255. $\displaystyle\int \operatorname{cosec}^4 x \, dx = -\cot x - \frac{1}{3}\cot^3 x + C$

$$\int \operatorname{cosec}^4 x \, dx = \int \operatorname{cosec}^2 x \operatorname{cosec}^2 x \, dx = \int \operatorname{cosec}^2 x \left(1 + \cot^2 x\right) dx$$
$$= \int \operatorname{cosec}^2 x + \operatorname{cosec}^2 x \cot^2 x \, dx$$

BY REVERSE CHAIN RULE (INSPECTION) AS $\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$

$$= -\cot x - \frac{1}{3}\cot^3 x + C$$

ALTERNATIVE BY SUBSTITUTION

$$\int \operatorname{cosec}^4 x \, dx = \int \operatorname{cosec}^4 x \left(\frac{du}{-\operatorname{cosec}^2 x}\right) = \int -\operatorname{cosec}^2 x \, du$$
$$= \int -\left(1 + \cot^2 x\right) du = \int -1 - u^2 \, du$$
$$= -u - \frac{1}{3}u^3 + C = -\cot x - \frac{1}{3}\cot^3 x + C$$

$u = \cot x$

$\frac{du}{dx} = -\operatorname{cosec}^2 x$

$dx = \frac{du}{-\operatorname{cosec}^2 x}$

256. $\displaystyle\int x^2 e^{\frac{1}{2}x} \, dx = 2x^2 e^{\frac{1}{2}x} - 8x e^{\frac{1}{2}x} + 16 e^{\frac{1}{2}x} + C$

$$\int x^2 e^{\frac{1}{2}x} \, dx = \dots \text{ INTEGRATION BY PARTS}$$

x^2	$2x$
$2e^{\frac{1}{2}x}$	$e^{\frac{1}{2}x}$

$$= 2x^2 e^{\frac{1}{2}x} - \underbrace{\int 4x e^{\frac{1}{2}x} \, dx}_{\text{INTEGRATION BY PARTS AGAIN}}$$

$4x$	4
$2e^{\frac{1}{2}x}$	$e^{\frac{1}{2}x}$

$$= 2x^2 e^{\frac{1}{2}x} - \left[8x e^{\frac{1}{2}x} - \int 8 e^{\frac{1}{2}x} \, dx\right]$$
$$= 2x^2 e^{\frac{1}{2}x} - 8x e^{\frac{1}{2}x} + \int 8 e^{\frac{1}{2}x} \, dx$$
$$= 2x^2 e^{\frac{1}{2}x} - 8x e^{\frac{1}{2}x} + 16 e^{\frac{1}{2}x} + C = \left(2x^2 - 8x + 16\right) e^{\frac{1}{2}x} + C$$

257. $\displaystyle\int \left(e^x + 2e^{-x}\right)^2 \, dx = \frac{1}{2}e^{2x} + 4x - 2e^{-2x} + C$

$$\int \left(e^x + 2e^{-x}\right)^2 dx = \int \left(e^x\right)^2 + 2\left(e^x\right)\left(2e^{-x}\right) + \left(2e^{-x}\right)^2 dx$$
$$= \int e^{2x} + 4 + 4e^{-2x} \, dx$$
$$= \frac{1}{2}e^{2x} + 4x - 2e^{-2x} + C$$

258. $\displaystyle\int e^x \sin\left(e^x\right) \, dx = -\cos\left(e^x\right) + C$

$$\int e^x \sin\left(e^x\right) dx = \dots \text{ BY REVERSE CHAIN RULE (INSPECTION)} \dots$$
$$= -\cos\left(e^x\right) + C$$

[THE SUBSTITUTION $u = e^x$ WORKS WELL ALSO]

259. $$\int x\mathrm{e}^{2}\ dx = \frac{1}{2}\mathrm{e}^{2}x^{2} + C$$

$$\int x e^{2}\,dx = \tfrac{1}{2}x^{2}e^{2} + C = \underline{\tfrac{1}{2}e^{2}x^{2} + C} \quad \left[e^{2} \text{ IS A CONSTANT}\right]$$

260. $$\int (2\cos x - 3)^{2}\ dx = 11x + \sin 2x - 12\sin x + C$$

$$\begin{aligned}\int (2\cos x - 3)^{2}\,dx &= \int 4\cos^{2}x - 12\cos x + 9\,dx\\ &= \int 4\left(\tfrac{1}{2} + \tfrac{1}{2}\cos 2x\right) - 12\cos x + 9\,dx\\ &= \int 11 + 2\cos 2x - 12\cos x\,dx\\ &= \underline{11x + \sin 2x - 12\sin x + C}\end{aligned}$$

261. $$\int \frac{x^{2}}{x-2}\ dx = \begin{bmatrix}\frac{1}{2}x^{2} + 2x + 4\ln|x-2| + C\\ \frac{1}{2}(x-2)^{2} + 4(x-2) + 4\ln|x-2| + C\end{bmatrix}$$

$\int \frac{x^{2}}{x-2}\,dx = \ldots$ BY DIVISION/MANIPULATION IN ORDER TO SPLIT THE FRACTION

$$= \int \frac{x(x-2) + 2(x-2) + 4}{x-2}\,dx = \int x + 2 + \frac{4}{x-2}\,dx$$

$$= \underline{\tfrac{1}{2}x^{2} + 2x + 4\ln|x-2| + C}$$

ALTERNATIVE BY SUBSTITUTION

$$\int \frac{x^{2}}{x-2}\,dx = \int \frac{u^{2}+4u+4}{u}\,du = \int u + 4 + \frac{4}{u}\,du$$

$$= \tfrac{1}{2}u^{2} + 4u + 4\ln|u| + C$$

$$= \underline{\tfrac{1}{2}(x-2)^{2} + 4(x-2) + 4\ln|x-2| + C}$$

$u = x - 2$
$\frac{du}{dx} = 1$
$dx = du$
$x = u + 2$
$x^{2} = (u+2)^{2}$
$x^{2} = u^{2} + 4u + 4$

262. $$\int \frac{(x+1)\mathrm{e}^{\frac{1}{x}}}{x^{3}}\ dx = -\frac{1}{x}\mathrm{e}^{\frac{1}{x}} + C$$

$\int \frac{(x+1)e^{\frac{1}{x}}}{x^{3}}\,dx = \ldots$ BY SUBSTITUTION ...

$u = \frac{1}{x}$
$\frac{du}{dx} = -\frac{1}{x^{2}}$
$dx = -x^{2}\,du$

$$= \int \frac{x+1}{x^{3}} e^{u}\left(-x^{2}\,du\right) = \int -\left(\frac{x+1}{x}\right)e^{u}\,du$$

$$= \int -\left(1 + \tfrac{1}{x}\right)e^{u}\,du = \int -(1+u)e^{u}\,du$$

INTEGRATION BY PARTS TO FOLLOW

$-(1+u)$	-1
e^{u}	e^{u}

$$= -(1+u)e^{u} - \int -e^{u}\,du$$

$$= -(1+u)e^{u} + \int e^{u}\,du$$

$$= -(1+u)e^{u} + e^{u} + C$$

$$= e^{u}(-1-u+1) + C$$

$$= -ue^{u} + C = \underline{-\tfrac{1}{x}e^{\frac{1}{x}} + C}$$

263. $$\int \frac{\sqrt{4x+1}}{x}\,dx = 2(4x-1)^{\frac{1}{2}} + \ln\left|\frac{\sqrt{4x+1}-1}{\sqrt{4x+1}+1}\right| + C$$

$\int \frac{\sqrt{4x+1}}{x}\,dx = \dots$ BY SUBSTITUTION

$u = \sqrt{4x+1}$
$u^2 = 4x+1$
$2u\frac{du}{dx} = 4$
$dx = \frac{1}{2}u\,du$
$4x = u^2-1$

$= \int \frac{u}{x}\left(\frac{1}{2}u\,du\right) = \int \frac{u^2}{2x}\,du$

$= \int \frac{2u^2}{4x}\,du = \int \frac{2u^2}{u^2-1}\,du$

AS THE FRACTIONAL INTEGRAND IS IMPROPER DIVIDE OR MANIPULATE

$= \int \frac{2(u^2-1)+2}{u^2-1}\,du = \int 2 + \frac{2}{u^2-1}\,du$

$= \int 2 + \frac{2}{(u-1)(u+1)}\,du = \dots$ PARTIAL FRACTIONS BY INSPECTION ...

$= \int 2 + \frac{1}{u-1} - \frac{1}{u+1}\,du = 2u + \ln|u-1| + \ln|u+1| + C$

$= 2u + \ln\left|\frac{u-1}{u+1}\right| + C = 2(4x+1)^{\frac{1}{2}} + \ln\left|\frac{\sqrt{4x+1}-1}{\sqrt{4x+1}+1}\right| + C$

264. $$\int \frac{1}{(1-x^2)^{\frac{3}{2}}}\,dx = \frac{x}{\sqrt{1-x^2}} + C$$

$\int \frac{1}{(1-x^2)^{\frac{3}{2}}}\,dx = \dots$ BY A TRIGONOMETRIC SUBSTITUTION

$x = \sin\theta$
$\frac{dx}{d\theta} = \cos\theta$
$dx = \cos\theta\,d\theta$

$= \int \frac{1}{(1-\sin^2\theta)^{\frac{3}{2}}}(\cos\theta\,d\theta) = \int \frac{\cos\theta}{(\cos^2\theta)^{\frac{3}{2}}}\,d\theta$

$= \int \frac{\cos\theta}{\cos^3\theta}\,d\theta = \int \frac{1}{\cos^2\theta}\,d\theta = \int \sec^2\theta\,d\theta$

$= \tan\theta + C = \frac{\sin\theta}{\cos\theta} + C$

NOW THE SUBSTITUTION IS IN FACT $\theta = \arcsin x$, SO

$= \frac{\sin\theta}{+\sqrt{1-\sin^2\theta}} + C = \frac{x}{\sqrt{1-x^2}} + C$

[THE SUBSTITUTION $x = \cos\theta$ / $\theta = \arccos x$ ALSO WORKS]

265. $$\int \frac{2^x}{2^x+1}\,dx = \frac{\ln\left(2^x+1\right)}{\ln 2} + C$$

$\int \frac{2^x}{2^x+1}\,dx = \frac{1}{\ln 2}\int \frac{2^x \ln 2}{2^x+1}\,dx$

$\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + C$

$= \frac{1}{\ln 2}\ln(2^x+1) + C = \frac{\ln(2^x+1)}{\ln 2} + C$

[THE SUBSTITUTION $u = 2^x+1$ ALSO WORKS WELL]

266. $$\int (2x-1)\sqrt{2x-3}\;dx = \left[\begin{array}{c} \frac{1}{5}(2x-3)^{\frac{5}{2}} + \frac{2}{3}(2x-3)^{\frac{3}{2}} + C \\ \frac{1}{3}(2x-1)(2x-3)^{\frac{3}{2}} - \frac{2}{15}(2x-3)^{\frac{5}{2}} + C \end{array}\right]$$

$\int (2x-1)\sqrt{2x-3}\,dx =$ BY SUBSTITUTION

$u=\sqrt{2x-3}$, $u^2=2x-3$, $2u\frac{du}{dx}=2$, $dx=u\,du$, $2x=u^2+3$

$= \int (2x-1)u\,(u\,du) = \int u^2(u^2+3-1)\,du$

$= \int u^2(u^2+2)\,du = \int u^4+2u^2\,du$

$= \frac{1}{5}u^5 + \frac{2}{3}u^3 + C$

$= \frac{1}{5}(2x-3)^{\frac{5}{2}} + \frac{2}{3}(2x-3)^{\frac{3}{2}} + C$

[THE SUBSTITUTION $u=2x-3$ ALSO WORKS]

ALTERNATIVE USING INTEGRATION BY PARTS

$2x-1$	2
$\frac{1}{3}(2x-3)^{\frac{3}{2}}$	$(2x-3)^{\frac{1}{2}}$

$\int (2x-1)\sqrt{2x-3}\,dx = \frac{1}{3}(2x-1)(2x-3)^{\frac{3}{2}} - \int \frac{2}{3}(2x-3)^{\frac{3}{2}}\,dx$

$= \frac{1}{3}(2x-1)(2x-3)^{\frac{3}{2}} - \frac{2}{15}(2x-3)^{\frac{5}{2}} + C$

ALTERNATIVE BY MANIPULATION

$\int (2x-1)\sqrt{2x-3}\,dx = \int [(2x-3)+2](2x-3)^{\frac{1}{2}}\,dx$

$= \int (2x-3)^{\frac{3}{2}} + 2(2x-3)^{\frac{1}{2}}\,dx = \frac{1}{5}(2x-3)^{\frac{5}{2}} + \frac{2}{3}(2x-3)^{\frac{3}{2}} + C$

AS ABOVE

267. $$\int \frac{9x^5}{\sqrt{x^3+1}}\,dx = \left[\begin{array}{c} 2\left(x^3+1\right)^{\frac{3}{2}} - 6\left(x^3+1\right)^{\frac{1}{2}} + C \\ 2x^3\left(x^3+1\right)^{\frac{1}{2}} - 4\left(x^3+1\right)^{\frac{3}{2}} + C \\ 2\left(x^3-2\right)\left(x^3+1\right)^{\frac{1}{2}} + C \end{array}\right]$$

$\int \frac{9x^5}{\sqrt{x^3+1}}\,dx$ ... BY SUBSTITUTION

$u=\sqrt{x^3+1}$, $u^2=x^3+1$, $2u\frac{du}{dx}=3x^2$, $dx=\frac{2u}{3x^2}\,du$, $x^3=u^2-1$, $6x^3=6u^2-6$

$= \int \frac{9x^5}{\cancel{u}}\left(\frac{2\cancel{u}}{3x^2}\,du\right) = \int 6x^3\,du$

$= \int 6u^2-6\,du = 2u^3-6u+C$

$= 2(x^3+1)^{\frac{3}{2}} - 6(x^3+1)^{\frac{1}{2}} + C$

$= 2(x^3+1)^{\frac{1}{2}}\left[(x^3+1)-3\right] + C$

$= 2(x^3-2)(x^3+1)^{\frac{1}{2}} + C$

ALTERNATIVE BY INTEGRATION BY PARTS

$9x^3$	$27x^2$
$\frac{2}{3}(x^3+1)^{\frac{1}{2}}$	$x^2(x^3+1)^{-\frac{1}{2}}$

$\int \frac{9x^5}{\sqrt{x^3+1}}\,dx = \int (9x^3)\left(x^2(x^3+1)^{-\frac{1}{2}}\right)dx$

$= 6x^3(x^3+1)^{\frac{1}{2}} - \int 18x^2(x^3+1)^{\frac{1}{2}}\,dx$

$= 6x^3(x^3+1)^{\frac{1}{2}} - 4(x^3+1)^{\frac{3}{2}} + C$

$= 2(x^3+1)^{\frac{1}{2}}\left[3x^3-2(x^3+1)\right] + C$

$= 2(x^3+1)^{\frac{1}{2}}(x^3-2) + C$

ALTERNATIVE BY MANIPULATION & RECOGNITION

$\int \frac{9x^5}{\sqrt{x^3+1}}\,dx = \int \frac{9x^2(x^3+1)-9x^2}{(x^3+1)^{\frac{1}{2}}}\,dx = \int 9x^2(x^3+1)^{\frac{1}{2}} - 9x^2(x^3+1)^{-\frac{1}{2}}\,dx$

$= 2(x^3+1)^{\frac{3}{2}} - 6(x^3+1)^{\frac{1}{2}} + C$ AS BY THE SUBSTITUTION METHOD

268. $\displaystyle\int (3\sin x - \cos x)^2 \, dx = \begin{bmatrix} 5x - \sin 2x + \frac{3}{2}\cos 2x + C \\ 5x - \sin 2x - 3\sin^2 x + C \\ 5x - \sin 2x + 3\cos^2 x + C \end{bmatrix}$

$\int (3\sin x - \cos x)^2 \, dx = \int 9\sin^2 x - 6\sin x\cos x + \cos^2 x \, dx$

$= \int 9\left(\frac{1}{2} - \frac{1}{2}\cos 2x\right) - 3\sin 2x + \left(\frac{1}{2} + \frac{1}{2}\cos 2x\right) dx$

$= \int 5 - 4\cos 2x - 3\sin 2x \, dx = \underline{5x - 2\sin 2x + \frac{3}{2}\cos 2x + C}$

VARIATION IF $\int -6\sin x\cos x \, dx = \begin{cases} -3\sin^2 x \\ 3\cos^2 x \end{cases}$

$\ldots = \begin{cases} \underline{5x - 2\sin 2x - 3\sin^2 x + C} \\ \underline{5x - 2\sin 2x + 3\cos^2 x + C} \end{cases}$

269. $\displaystyle\int \frac{4x}{x^2 - 10} \, dx = 2\ln\left|x^2 - 10\right| + C$

$\int \frac{4x}{x^2-10} \, dx = 2\int \frac{2x}{x^2-10} \, dx = \underline{2\ln\left|x^2-10\right| + C}$

$\int \frac{f'(x)}{f(x)} \, dx = \ln|f(x)| + C$

[THE SUBSTITUTION $u = x^2 - 10$ (OR EVEN PARTIAL FRACTIONS!) WORK IN THIS QUESTION]

270. $\displaystyle\int \frac{4e^{3x}}{1 - e^{3x}} \, dx = -\frac{4}{3}\ln\left|1 - e^{3x}\right| + C$

$\int \frac{4e^{3x}}{1-e^{3x}} \, dx = \frac{4}{-3}\int \frac{-3e^{3x}}{1-e^{3x}} \, dx = \underline{-\frac{4}{3}\ln\left|1-e^{3x}\right| + C}$

$\int \frac{f'(x)}{f(x)} \, dx = \ln|f(x)| + C$

[THE SUBSTITUTION $u = 1 - e^{3x}$ ALSO WORKS WELL]

271. $\displaystyle\int (1 - \cos x)\sin x \cos x \, dx = \begin{bmatrix} \cos^3 x - \frac{1}{4}\cos 2x + C \\ \cos^3 x - \frac{1}{2}\cos^2 x + C \\ \cos^3 x + \frac{1}{2}\sin^2 x + C \end{bmatrix}$

$\int \sin x\cos x(1 - \cos x) \, dx = \int \sin x\cos x - \sin x\cos^2 x \, dx.$

OR $\frac{1}{2}\sin 2x$

BY RECOGNITION (REVERSE CHAIN RULE)

$= \underline{\frac{1}{2}\sin^2 x + \cos^3 x + C} = \underline{-\frac{1}{2}\cos^2 x + \cos^3 x + C} = \underline{-\frac{1}{4}\cos 2x + \cos^3 x + C}$

272. $$\int (\tan x - 1)^2 \ dx = \begin{bmatrix} \tan x - 2\ln|\sec x| + C \\ \tan x + 2\ln|\cos x| + C \end{bmatrix}$$

$$\int (\tan x - 1)^2 \, dx = \int \tan^2 x - 2\tan x + 1 \, dx$$

$$= \int \sec^2 x - 2\tan x \, dx = \dots$$ BOTH ARE STANDARD RESULTS

$$= \tan x - 2\ln|\sec x| + C = \tan x + 2\ln|\cos x| + C$$

273. $$\int \frac{1+e^x}{1-e^x} \ dx = x - 2\ln\left|1-e^x\right| + C$$

$$\int \frac{1+e^x}{1-e^x} \, dx = \dots$$ BY SUBSTITUTION

$u = 1 - e^x$

$\frac{du}{dx} = -e^x$

$dx = \frac{du}{-e^x}$

$e^x = 1 - u$

$1 + e^x = 2 - u$

$$\int \frac{2-u}{u} \frac{du}{-e^x} = \int \frac{2-u}{u} \frac{du}{u-1}$$

$$\int \frac{2-u}{u(u-1)} \, du$$ ... PARTIAL FRACTIONS BY INSPECTION (COVER UP)

$$\int \frac{1}{u-1} - \frac{2}{u} \, du = \ln|u-1| - 2\ln|u| + C$$

$$= \ln|\cancel{1} - e^x \cancel{-1}| - 2\ln|1-e^x| + C = x - 2\ln|1-e^x| + C$$

ALTERNATIVE BY MANIPULATION

$$\int \frac{1+e^x}{1-e^x} \, dx = \int \frac{(1-e^x) + 2e^x}{1-e^x} \, dx = \int 1 + \frac{2e^x}{1-e^x} \, dx$$

$$= \int 1 - 2\left(\frac{-e^x}{1-e^x}\right) \, dx$$

$$\int \frac{f'(x)}{f(x)} \, dx = \ln|f(x)| + C$$

$$= x - 2\ln|1-e^x| + C$$

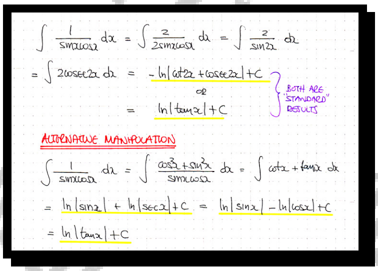
274. $$\int \sin 2x \cos^4 2x \ dx = -\frac{1}{10}\cos^5 2x + C$$

$$\int \sin 2x \cos^4 2x \, dx = \dots$$ BY REVERSE CHAIN RULE (INSPECTION) ...

$$= -\frac{1}{10}\cos^5 2x + C$$

[THE SUBSTITUTION $u = \cos 2x$ ALSO WORKS WELL]

275. $$\int \frac{1}{\sin x \cos x}\, dx = \begin{bmatrix} \ln|\sin x| - \ln|\sec x| + C \\ \ln|\sin x| + \ln|\cos x| + C \\ \ln|\tan x| + C \\ -\ln|\cot 2x + \operatorname{cosec} 2x| + C \end{bmatrix}$$



$$\int \frac{1}{\sin x\cos x}\,dx = \int \frac{2}{2\sin x\cos x}\,dx = \int \frac{2}{\sin 2x}\,dx$$

$$= \int 2\operatorname{cosec}2x\,dx = -\ln|\cot 2x + \operatorname{cosec}2x| + C$$

OR

$$= \ln|\tan x| + C$$

BOTH ARE "STANDARD" RESULTS

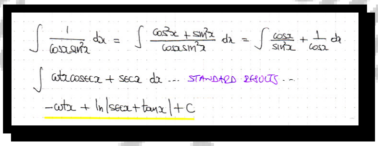
ALTERNATIVE MANIPULATION

$$\int \frac{1}{\sin x\cos x}\,dx = \int \frac{\cos^2 x + \sin^2 x}{\sin x\cos x}\,dx = \int \cot x + \tan x\,dx$$

$$= \ln|\sin x| + \ln|\sec x| + C = \ln|\sin x| - \ln|\cos x| + C$$

$$= \ln|\tan x| + C$$

276. $$\int \frac{1}{\sin^2 x \cos x}\, dx = -\cot x + \ln|\sec x + \tan x| + C$$

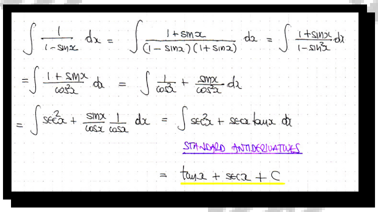


$$\int \frac{1}{\cos x\sin^2 x}\,dx = \int \frac{\cos^2 x + \sin^2 x}{\cos x\sin^2 x}\,dx = \int \frac{\cos x}{\sin^2 x} + \frac{1}{\cos x}\,dx$$

$$\int \cot x\operatorname{cosec}x + \sec x\,dx$$ ... STANDARD RESULTS ...

$$-\cot x + \ln|\sec x + \tan x| + C$$

277. $$\int \frac{1}{1-\sin x}\, dx = \tan x + \sec x + C$$



$$\int \frac{1}{1-\sin x}\,dx = \int \frac{1+\sin x}{(1-\sin x)(1+\sin x)}\,dx = \int \frac{1+\sin x}{1-\sin^2 x}\,dx$$

$$= \int \frac{1+\sin x}{\cos^2 x}\,dx = \int \frac{1}{\cos^2 x} + \frac{\sin x}{\cos^2 x}\,dx$$

$$= \int \sec^2 x + \frac{\sin x}{\cos x}\frac{1}{\cos x}\,dx = \int \sec^2 x + \sec x\tan x\,dx$$

STANDARD ANTIDERIVATIVES

$$= \tan x + \sec x + C$$

278. $\displaystyle\int \frac{1}{e^x+1}\,dx = x - \ln\left(e^x+1\right) + C$

$$\int \frac{1}{e^x+1}\,dx = \dots \text{ BY SUBSTITUTION}\dots$$

$$u = e^x+1, \quad \frac{du}{dx} = e^x, \quad dx = \frac{du}{e^x}, \quad e^x = u-1$$

$$= \int \frac{1}{u}\left(\frac{du}{u-1}\right) = \int \frac{1}{u(u-1)}\,du$$

PARTIAL FRACTIONS BY INSPECTION (COVER UP)

$$= \int \frac{1}{u-1} - \frac{1}{u}\,du = \ln|u-1| - \ln|u| + C$$

$$= \ln\left(e^x\right) - \ln\left(e^x+1\right) + C = x - \ln\left(e^x+1\right) + C$$

279. $\displaystyle\int (1+\sin x)\sin 2x\,dx = \begin{bmatrix} \frac{2}{3}\sin^3 x - \cos^2 x + C \\ \frac{2}{3}\sin^3 x + \sin^2 x + C \\ \frac{2}{3}\sin^3 x - \frac{1}{2}\cos 2x + C \end{bmatrix}$

$$\int \sin 2x(1+\sin x)\,dx = \int 2\sin x\cos x(1+\sin x)\,dx$$

$$= \int 2\sin x\cos x + 2\sin^2 x\cos x\,dx$$

BY REVERSE CHAIN RULE (INSPECTION)

$$= \sin^2 x + \frac{2}{3}\sin^3 x + C = -\cos^2 x + \frac{2}{3}\sin^3 x + C$$

VARIATION

$$\dots \int 2\sin x\cos x + 2\sin^2 x\cos x\,dx = \int \sin 2x + 2\sin^2 x\cos x\,dx$$

$$= -\frac{1}{2}\cos 2x + \frac{2}{3}\sin^3 x + C$$

280. $\displaystyle\int x\sec x\tan x\,dx = x\sec x - \ln\left|\sec x + \tan x\right| + C$

$$\int x\sec x\tan x\,dx = \dots$$

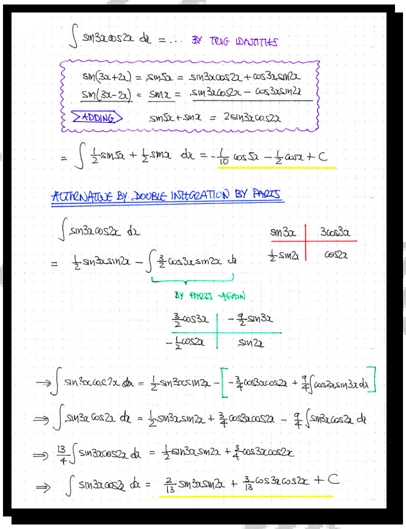
INTEGRATION BY PARTS

x	1
$\sec x$	$\sec x\tan x$

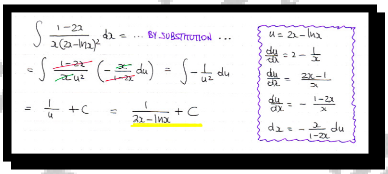
$$= x\sec x - \int \sec x\,dx \leftarrow \text{STANDARD RESULT}$$

$$= x\sec x - \ln\left|\sec x + \tan x\right| + C$$

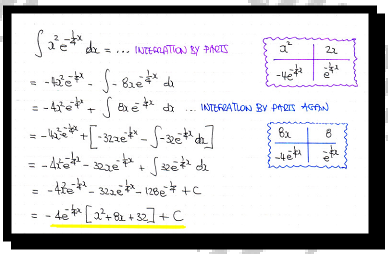
281. $\displaystyle\int \sin 3x \cos 2x \ dx = \begin{bmatrix} -\frac{1}{10}\cos 5x - \frac{1}{2}\cos x + C \\ \frac{2}{13}\sin 3x \sin 2x + \frac{3}{13}\cos 3x \cos 2x + C \end{bmatrix}$



282. $\displaystyle\int \frac{1-2x}{x(2x-\ln x)^2} \ dx = \frac{1}{2x-\ln x} + C$



283. $\displaystyle\int x^2 e^{-\frac{1}{4}x} \ dx = -4e^{-\frac{1}{4}x}\left[x^2+8x+32\right] + C$



284. $\displaystyle\int \frac{x+4}{x-4}\,dx = x + 8\ln|x-4| + C$

$$\int \frac{x+4}{x-4}\,dx = \int \frac{(x-4)+8}{(x-4)}\,dx = \int 1 + \frac{8}{x-4}\,dx$$
$$= \underline{x + 8\ln|x-4| + C}$$

[THE SUBSTITUTION $u = x-4$ ALSO WORKS WELL]

285. $\displaystyle\int 3x^2\left(4-2x^3\right)^{\frac{3}{2}}\,dx = -\frac{1}{5}\left(4-2x^3\right)^{\frac{5}{2}} + C$

$$\int 3x^2(4-2x^3)^{\frac{3}{2}}\,dx = \text{BY REVERSE CHAIN RULE (INSPECTION)} \ldots$$
$$= \underline{-\frac{1}{5}(4-2x^3)^{5/2} + C}$$

[THE SUBSTITUTION $u = 4-2x^3$ OR $u = (4-2x^3)^{\frac{1}{2}}$ ALSO WORK WELL]

286. $\displaystyle\int x\sqrt{x+1}\,dx = \begin{bmatrix} \frac{2}{5}(x+1)^{\frac{5}{2}} - \frac{2}{3}(x+1)^{\frac{3}{2}} + C \\ \frac{2}{3}x(x+1)^{\frac{3}{2}} - \frac{4}{15}(x+1)^{\frac{5}{2}} + C \end{bmatrix}$

$$\int x\sqrt{x+1}\,dx = \ldots \text{ BY SUBSTITUTION } \ldots$$
$$= \int xu\,(2u\,du) = \int 2u^2x\,du$$
$$= \int 2u^2(u^2-1)\,du = \int 2u^4 - 2u^2\,du$$
$$= \frac{2}{5}u^5 - \frac{2}{3}u^3 + C = \underline{\frac{2}{5}(x+1)^{\frac{5}{2}} - \frac{2}{3}(x+1)^{\frac{3}{2}} + C}$$

$u = \sqrt{x+1}$
$u^2 = x+1$
$x = u^2 - 1$
$\frac{dx}{du} = 2u$
$dx = 2u\,du$

ALTERNATIVE BY INTEGRATION BY PARTS

$$\int x\sqrt{x+1}\,dx = \frac{2}{3}x(x+1)^{\frac{3}{2}} - \int \frac{2}{3}(x+1)^{\frac{3}{2}}\,dx$$
$$= \underline{\frac{2}{3}x(x+1)^{\frac{3}{2}} - \frac{4}{15}(x+1)^{\frac{5}{2}} + C}$$

x	1
$\frac{2}{3}(x+1)^{\frac{3}{2}}$	$(x+1)^{\frac{1}{2}}$

287. $\displaystyle\int \frac{x+1}{\sqrt[3]{x^2+2x+3}}\,dx = \frac{3}{4}\left(x^2+2x+3\right)^{\frac{2}{3}} + C$

$$\int \frac{x+1}{\sqrt[3]{x^2+2x+3}}\,dx = \int (x+1)(x^2+2x+3)^{-\frac{1}{3}}\,dx$$
$$= \frac{1}{2}\int (2x+2)(x^2+2x+3)^{-\frac{1}{3}}\,dx = \ldots \text{ BY REVERSE CHAIN RULE (INSPECTION)}$$
$$= \frac{1}{2}\times\frac{3}{2}(x^2+2x+3)^{\frac{2}{3}} + C = \underline{\frac{3}{4}(x^2+2x+3)^{\frac{2}{3}} + C}$$

ALTERNATIVE BY SUBSTITUTION

$$\int \frac{x+1}{\sqrt[3]{x^2+2x+3}}\,dx = \int \frac{\cancel{x+1}}{u}\left(\frac{3u^2}{2\cancel{(x+1)}}\,du\right)$$
$$= \int \frac{3}{2}u\,du = \frac{3}{4}u^2 + C$$
$$= \underline{\frac{3}{4}(x^2+2x+3)^{\frac{2}{3}} + C}$$

$u = (x^2+2x+3)^{\frac{1}{3}}$
$u^3 = x^2+2x+3$
$3u^2\frac{du}{dx} = 2x+2$
$dx = \frac{3u^2}{2x+2}\,du$
$dx = \frac{3u^2}{2(x+1)}\,du$

[THE SUBSTITUTION $u = x^2+2x+3$ ALSO WORKS WELL]

288. $$\int \sin 2x \cos x \, dx = \begin{bmatrix} -\frac{2}{3}\cos^3 x + C \\ -\frac{1}{6}\cos 3x - \frac{1}{2}\cos x + C \\ -\frac{1}{3}\left(\sin 2x \sin x + 2\cos 2x \cos x\right) + C \end{bmatrix}$$

$$\int \sin 2x \cos x \, dx = \int (2\sin x \cos x)\cos x \, dx = \int 2\sin x \cos^2 x \, dx$$

BY REVERSE CHAIN RULE (INSPECTION) $\ldots = \underline{-\frac{2}{3}\cos^3 x + C}$

[THE SUBSTITUTION $u = \cos x$ ALSO WORKS WELL]

ALTERNATIVE BY TRIG IDENTITIES

$$\sin(2x + x) = \sin 3x = \sin 2x \cos x + \cos 2x \sin x$$
$$\sin(2x - x) = \sin x = \sin 2x \cos x - \cos 2x \sin x$$

ADDING $\quad \sin 3x + \sin x = 2\sin 2x \cos x$

OR $\quad \sin 2x \cos x = \frac{1}{2}\sin 3x + \frac{1}{2}\sin x$

$$\int \sin 2x \cos x \, dx = \int \frac{1}{2}\sin 3x + \frac{1}{2}\sin x \, dx$$
$$= \underline{-\frac{1}{6}\cos 3x - \frac{1}{2}\cos x + C}$$

ALTERNATIVE USING INTEGRATION BY PARTS TWICE

$$\int \sin 2x \cos x \, dx = \sin 2x \sin x - \int 2\cos 2x \sin x \, dx$$

$\sin 2x$	$2\cos 2x$
$\sin x$	$\cos x$

$2\cos 2x$	$-4\sin 2x$
$-\cos x$	$\sin x$

$$\Rightarrow \int \sin 2x \cos x \, dx = \sin 2x \sin x - \left[-2\cos 2x \cos x - \int 4\sin 2x \cos x \, dx\right]$$
$$\Rightarrow \int \sin 2x \cos x \, dx = \sin 2x \sin x + 2\cos 2x \cos x + 4\int \sin 2x \cos x \, dx$$
$$\Rightarrow -3\int \sin 2x \cos x \, dx = \sin 2x \sin x + 2\cos 2x \cos x$$
$$\Rightarrow \int \sin 2x \cos x \, dx = \underline{-\frac{1}{3}\left[\sin 2x \sin x + 2\cos 2x \cos x\right]}$$

289. $$\int \sin 2x \cos 2x \, dx = \begin{bmatrix} \frac{1}{4}\sin^2 2x + C \\ -\frac{1}{4}\cos^2 2x + C \\ -\frac{1}{8}\cos 4x + C \end{bmatrix}$$

$\int \sin 2x \cos 2x \, dx = \ldots$ BY REVERSE CHAIN RULE (INSPECTION)
OR THE SUBSTITUTION $u = \sin 2x$
OR THE SUBSTITUTION $u = \cos 2x$

$\ldots = \underline{\frac{1}{4}\sin^2 2x + C}$ OR $\underline{-\frac{1}{4}\cos^2 2x + C}$

ALTERNATIVE BY DOUBLE ANGLES

$$\int \sin 2x \cos 2x \, dx = \int \frac{1}{2}(2\sin 2x \cos 2x)\, dx = \int \frac{1}{2}\sin 4x \, dx$$
$$= \underline{-\frac{1}{8}\cos 4x + C}$$

290. $$\int \frac{3^x}{3^x + 1} \, dx = \frac{\ln\left(3^x + 1\right)}{\ln 3} + C$$

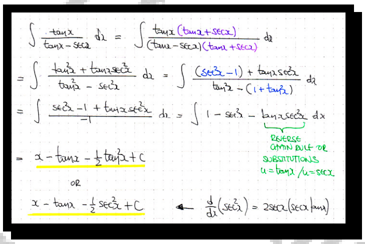
$$\int \frac{3^x}{3^x + 1} \, dx = \frac{1}{\ln 3}\int \frac{3^x \ln 3}{3^x + 1} \, dx = \frac{1}{\ln 3}\ln(3^x + 1) + C$$

OF THE FORM $\int \frac{f'(x)}{f(x)} \, dx = \ln|f(x)| + C$

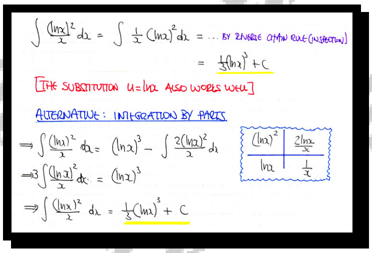
$$= \underline{\frac{\ln(3^x + 1)}{\ln 3} + C}$$

[THE SUBSTITUTION $u = 3^x + 1$ ALSO WORKS WELL]

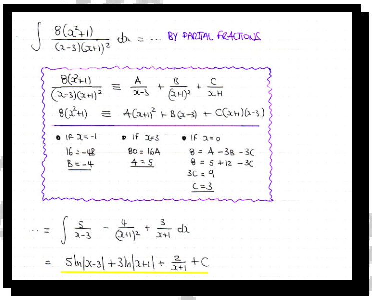
291. $$\int \frac{\tan x}{\tan x - \sec x}\, dx = \begin{bmatrix} x - \tan x - \frac{1}{2}\tan^2 x + C \\ x - \tan x - \frac{1}{2}\sec^2 x + C \end{bmatrix}$$



292. $$\int \frac{(\ln x)^2}{x}\, dx = \frac{1}{3}(\ln x)^3 + C$$



293. $$\int \frac{8\left(x^2+1\right)}{(x-3)(x+1)^2}\, dx = 5\ln|x-3| + 3\ln|x+1| + \frac{2}{x+1} + C$$



294. $\displaystyle\int \sqrt[3]{x\sqrt{\frac{1}{x}}}\ dx = \frac{6}{7}x^{\frac{7}{6}} + C$

$$\int \sqrt[3]{x\sqrt{\frac{1}{x}}}\,dx = \int \left[x\,x^{-\frac{1}{2}}\right]^{\frac{1}{3}}dx = \int \left(x^{\frac{1}{2}}\right)^{\frac{1}{3}}dx$$
$$= \int x^{\frac{1}{6}}\,dx = \frac{6}{7}x^{\frac{7}{6}} + C$$

295. $\displaystyle\int (x+1)\left(x^2+2x-1\right)^4\ dx = \frac{1}{10}\left(x^2+2x-1\right)^5 + C$

$$\int (x+1)(x^2+2x-1)^4\,dx = \frac{1}{2}\int (2x+2)(x^2+2x-1)^4\,dx$$

BY REVERSE CHAIN RULE (INSPECTION)

$$= \frac{1}{10}(x^2+2x-1)^5 + C$$

ALTERNATIVE BY SUBSTITUTION

$$\int (x+1)(x^2+2x-1)^4\,dx = \int \cancel{(x+1)}\,u^4\left(\frac{du}{2\cancel{(x+1)}}\right)$$
$$= \int \frac{1}{2}u^4\,du = \frac{1}{10}u^5 + C$$
$$= \frac{1}{10}(x^2+2x-1)^5 + C$$

$u = x^2+2x-1$
$\frac{du}{dx} = 2x+2$
$dx = \frac{du}{2(x+1)}$

296. $\displaystyle\int \sin x\cos^4 x\ dx = -\frac{1}{5}\cos^5 x + C$

$$\int \sin x\cos^4 x\,dx = \ldots \text{ BY REVERSE CHAIN RULE (INSPECTION)}$$
$$= -\frac{1}{5}\cos^5 x + C$$

[THE SUBSTITUTION $u = \cos x$ ALSO WORKS WELL]

297. $\displaystyle\int \frac{2x+6}{x^2+6x+1}\ dx = \ln\left|x^2+6x+1\right| + C$

$$\int \frac{2x+6}{x^2+6x+1}\,dx = \ldots \text{ OF THE FORM } \int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + C$$
$$= \ln|x^2+6x+1| + C$$

298. $\displaystyle\int \frac{1}{x(x-4)}\,dx = \frac{1}{4}\ln\left|\frac{x-4}{x}\right| + C$

$\int \frac{1}{x(x-4)}\,dx = \ldots$ PARTIAL FRACTIONS BY INSPECTION (COVER UP)

$= \int \frac{\frac{1}{4}}{x-4} - \frac{\frac{1}{4}}{x}\,dx = \frac{1}{4}\ln|x-4| - \frac{1}{4}\ln|x| + C$

$= \frac{1}{4}\ln\left|\frac{x-4}{x}\right| + C$

299. $\displaystyle\int \frac{\tan x}{\sqrt{1+\cos 2x}}\,dx = \pm\frac{1}{\sqrt{2}}\sec x + C$

$\int \frac{\tan x}{\sqrt{1+\cos 2x}}\,dx = \int \frac{\tan x}{\sqrt{1+2\cos^2 x-1}}\,dx = \int \frac{\tan x}{\sqrt{2\cos^2 x}}\,dx$

NOW IF $\cos x > 0$

$= \int \frac{\tan x}{\sqrt{2}\cos x}\,dx = \int \frac{1}{\sqrt{2}}\tan x\sec x\,dx = \frac{1}{\sqrt{2}}\sec x + C$

AND IF $\cos x < 0$

$= \int \frac{\tan x}{-\sqrt{2}\cos x}\,dx = \int -\frac{1}{\sqrt{2}}\tan x\sec x\,dx = -\frac{1}{\sqrt{2}}\sec x + C$

COLLECTING RESULTS

$\ldots = \pm\frac{1}{\sqrt{2}}\sec x + C$

300. $\displaystyle\int \cos^2 x\sin^2 x\,dx = \frac{1}{8}x - \frac{1}{32}\sin 4x + C$

$\int \cos^2 x\sin^2 x\,dx = \int\left(\frac{1}{2}+\frac{1}{2}\cos 2x\right)\left(\frac{1}{2}-\frac{1}{2}\cos 2x\right)dx$

$= \int \frac{1}{4} - \frac{1}{4}\cos^2 2x\,dx = \int \frac{1}{4} - \frac{1}{4}\left(\frac{1}{2}+\frac{1}{2}\cos 4x\right)dx$

$= \int \frac{1}{4} - \frac{1}{8} - \frac{1}{8}\cos 4x\,dx = \int \frac{1}{8} - \frac{1}{8}\cos 4x\,dx$

$= \frac{1}{8}x - \frac{1}{32}\sin 4x + C$

ALTERNATIVE METHOD

$\int \cos^2 x\sin^2 x\,dx = \int \frac{1}{4}(4\cos^2 x\sin^2 x)\,dx = \int \frac{1}{4}(2\cos x\sin x)^2\,dx$

$= \int \frac{1}{4}(\sin 2x)^2\,dx = \int \frac{1}{4}\sin^2 2x\,dx$

$= \int \frac{1}{4}\left(\frac{1}{2}-\frac{1}{2}\cos 4x\right)dx = \int \frac{1}{8} - \frac{1}{8}\cos 4x\,dx$

$= \frac{1}{8}x - \frac{1}{32}\sin 4x + C$

301. $\displaystyle\int \frac{3x^3+5x}{x^2+1}\,dx = \frac{3}{2}x^2 + \ln\left(x^2+1\right) + C$

$\displaystyle\int \frac{3x^3+5x}{x^2+1}\,dx = \dots$ BY ALGEBRAIC DIVISION OR MANIPULATION ...

$\displaystyle= \int \frac{3x(x^2+1)+2x}{x^2+1}\,dx = \int 3x + \frac{2x}{x^2+1}\,dx$

OF THE FORM $\displaystyle\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)|+C$

$= \frac{3}{2}x^2 + \ln(x^2+1) + C$

ALTERNATIVE BY DIRECT SUBSTITUTION

$\displaystyle\int \frac{3x^3+5x}{x^2+1}\,dx = \int \frac{3x^3+5x}{u}\left(\frac{du}{2x}\right)$

$\displaystyle= \int \frac{x(3x^2+5)}{2xu}\,du = \int \frac{3x^2+5}{2u}\,du$

$\displaystyle= \int \frac{3u+2}{2u}\,du = \int \frac{3}{2} + \frac{1}{u}\,du$

$= \frac{3}{2}u + \ln|u| + C = \frac{3}{2}(x^2+1) + \ln(x^2+1) + C$

$u = x^2+1$
$\frac{du}{dx} = 2x$
$dx = \frac{du}{2x}$
$x^2 = u-1$
$3x^2 = 3u-3$
$3x^2+5 = 3u+2$

302. $\displaystyle\int \sin^2 2x\,dx = \frac{1}{2}x - \frac{1}{8}\sin 4x + C$

$\displaystyle\int \sin^2 2x\,dx = \int \frac{1}{2} - \frac{1}{2}\cos 4x\,dx = \frac{1}{2}x - \frac{1}{8}\sin 4x + C$

$\left[\begin{array}{l}\cos 2\theta = 1 - 2\sin^2\theta \\ \cos 4\theta = 1 - 2\sin^2 2\theta\end{array}\right]$

303. $\displaystyle\int \frac{\cos 2x}{\cos^2 x}\,dx = 2x - \tan x + C$

$\displaystyle\int \frac{\cos 2x}{\cos^2 x}\,dx = \int \frac{2\cos^2 x - 1}{\cos^2 x}\,dx = \int 2 - \frac{1}{\cos^2 x}\,dx$

$\displaystyle= \int 2 - \sec^2 x\,dx = 2x - \tan x + C$

304. $\displaystyle\int \sec^3 x \tan x\,dx = \frac{1}{3}\sec^3 x + C$

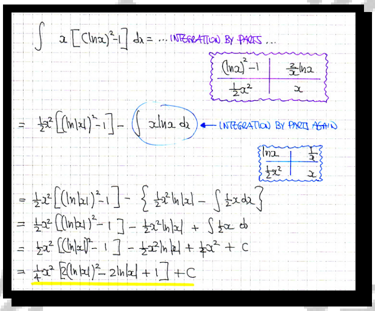
$\displaystyle\int \sec^3 x \tan x\,dx = \dots$ BY REVERSE CHAIN RULE (INSPECTION), SINCE

$\frac{d}{dx}(\sec^3 x) = 3\sec^2 x(\sec x \tan x) = 3\sec^3 x \tan x$

$= \frac{1}{3}\sec^3 x + C$

[THE SUBSTITUTION $u = \sec x$ ALSO WORKS WELL]

305. $\displaystyle\int x\left[(\ln x)^2 - 1\right]\,dx = \frac{1}{4}x^2\left[2\left(\ln|x|\right)^2 - 2\ln|x| + 1\right] + C$


$\int x\left[(\ln x)^2-1\right]dx = \dots$ INTEGRATION BY PARTS ...

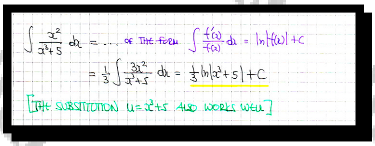
$(\ln x)^2-1$	$\frac{2}{x}\ln x$
$\frac{1}{2}x^2$	x

$= \frac{1}{2}x^2\left[(\ln|x|)^2-1\right] - \int x\ln x\,dx \longleftarrow$ INTEGRATION BY PARTS AGAIN

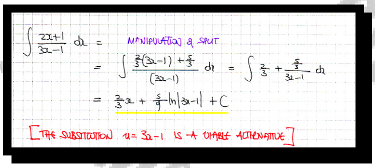
$\ln x$	$\frac{1}{x}$
$\frac{1}{2}x^2$	x

$= \frac{1}{2}x^2\left[(\ln|x|)^2-1\right] - \left\{\frac{1}{2}x^2\ln|x| - \int\frac{1}{2}x\,dx\right\}$
$= \frac{1}{2}x^2\left[(\ln|x|)^2-1\right] - \frac{1}{2}x^2\ln|x| + \int\frac{1}{2}x\,dx$
$= \frac{1}{2}x^2\left[(\ln|x|)^2-1\right] - \frac{1}{2}x^2\ln|x| + \frac{1}{4}x^2 + C$
$= \frac{1}{4}x^2\left[2(\ln|x|)^2 - 2\ln|x| + 1\right] + C$

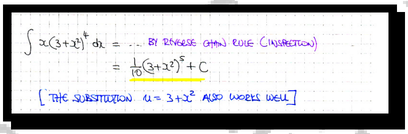
306. $\displaystyle\int \frac{x^2}{x^3+5}\,dx = \frac{1}{3}\ln\left|x^3+5\right| + C$


$\int\frac{x^2}{x^3+5}\,dx = \dots$ OF THE FORM $\int\frac{f'(x)}{f(x)}\,dx = \ln|f(x)|+C$
$= \frac{1}{3}\int\frac{3x^2}{x^3+5}\,dx = \frac{1}{3}\ln|x^3+5|+C$
[THE SUBSTITUTION $u=x^3+5$ ALSO WORKS WELL]

307. $\displaystyle\int \frac{2x+1}{3x-1}\,dx = \frac{2}{3}x + \frac{5}{9}\ln|3x-1| + C$


$\int\frac{2x+1}{3x-1}\,dx =$ MANIPULATION & SPLIT
$= \int\frac{\frac{2}{3}(3x-1)+\frac{5}{3}}{(3x-1)}\,dx = \int\frac{2}{3}+\frac{\frac{5}{3}}{3x-1}\,dx$
$= \frac{2}{3}x + \frac{5}{9}\ln|3x-1| + C$
[THE SUBSTITUTION $u=3x-1$ IS A VIABLE ALTERNATIVE]

308. $\displaystyle\int x\left(3+x^2\right)^4\,dx = \frac{1}{10}\left(3+x^2\right)^5 + C$


$\int x(3+x^2)^4\,dx = \dots$ BY REVERSE CHAIN RULE (INSPECTION)
$= \frac{1}{10}(3+x^2)^5 + C$
[THE SUBSTITUTION $u=3+x^2$ ALSO WORKS WELL]

309. $$\int \frac{9}{x^2\sqrt{9-x^2}}\,dx = -\frac{\sqrt{9-x^2}}{x} + C$$

$\int \frac{9}{x^2\sqrt{9-x^2}}\,dx = \ldots$ BY A TRIGONOMETRIC SUBSTITUTION

$\left[\theta = \arcsin\frac{x}{3}\right]$
$x = 3\sin\theta$
$dx = 3\cos\theta\,d\theta$

$= \int \frac{9}{(3\sin\theta)^2\sqrt{9-(3\sin\theta)^2}}(3\cos\theta\,d\theta)$

$= \int \frac{27\cos\theta}{9\sin^2\theta\sqrt{9-9\sin^2\theta}}\,d\theta$

$= \int \frac{3\cos\theta}{\sin^2\theta\sqrt{9(1-\sin^2\theta)}}\,d\theta = \int \frac{3\cos\theta}{\sin^2\theta\sqrt{9\cos^2\theta}}\,d\theta$

$= \int \frac{\cancel{3\cos\theta}}{\sin^2\theta\,(\cancel{3\cos\theta})}\,d\theta = \int \operatorname{cosec}^2\theta\,d\theta = -\cot\theta + C$

$= -\frac{\cos\theta}{\sin\theta} + C = -\frac{\sqrt{1-\sin^2\theta}}{\sin\theta} + C$

$= -\frac{\sqrt{1-\frac{x^2}{9}}}{\frac{x}{3}} + C = -\frac{\sqrt{\frac{9-x^2}{9}}}{\frac{x}{3}} + C$

$= -\frac{\frac{\sqrt{9-x^2}}{\cancel{3}}}{\frac{x}{\cancel{3}}} + C = -\frac{\sqrt{9-x^2}}{x} + C$ //

310. $$\int \frac{\cos x}{\sqrt{\sin^3 x}}\,dx = -\frac{2}{\sqrt{\sin x}} + C$$

$\int \frac{\cos x}{\sqrt{\sin^3 x}}\,dx = \int (\cos x)(\sin x)^{-\frac{3}{2}}\,dx$

BY REVERSE CHAIN RULE (INSPECTION)

$= -2(\sin x)^{-\frac{1}{2}} + C = -\frac{2}{\sqrt{\sin x}} + C$

[THE SUBSTITUTION $u = \sin x$ OR $u = \sqrt{\sin x}$ ALSO WORK]

311. $$\int \frac{x-3}{\sqrt{x+1}-2}\,dx = 2x + \tfrac{2}{3}(x+1)^{\frac{3}{2}} + C$$

$\int \frac{x-3}{\sqrt{x+1}-2}\,dx = \ldots$ BY MANIPULATION SINCE $(x+1)-4 = x-3$

$= \int \frac{(x-3)(\sqrt{x+1}+2)}{(\sqrt{x+1}-2)(\sqrt{x+1}+2)}\,dx = \int \frac{(x-3)(\sqrt{x+1}+2)}{(x+1)-4}\,dx$

(DIFFERENCE OF SQUARES

$= \int \frac{\cancel{(x-3)}(\sqrt{x+1}+2)}{\cancel{(x-3)}}\,dx = \int (x+1)^{\frac{1}{2}} + 2\,dx$

$= 2x + \frac{2}{3}(x+1)^{\frac{3}{2}} + C$

ALTERNATIVE BY SUBSTITUTION

$u = \sqrt{x+1}$
$u^2 = x+1$
$2u\frac{du}{dx} = 1$
$dx = 2u\,du$
$x = u^2-1$
$x-3 = u^2-4$

$\int \frac{x-3}{\sqrt{x+1}-2}\,dx = \int \frac{u^2-4}{u-2}(2u\,du)$

$= \int \frac{2u\cancel{(u-2)}(u+2)}{\cancel{u-2}}\,du = \int 2u^2+4u\,du$

$= \frac{2}{3}u^3 + 2u^2 + C = \frac{2}{3}(x+1)^{\frac{3}{2}} + 2(x+1) + C$

$= \frac{2}{3}(x+1)^{\frac{3}{2}} + 2x + C$

[THE SUBSTITUTION $u = \sqrt{x+1}-2$ ALSO WORKS]

312. $$\int \frac{x-4}{x^2-4}\,dx = \begin{bmatrix} \frac{1}{2}\ln\left|x^2-4\right| + \ln\left|\frac{x+2}{x-2}\right| + C \\ \frac{3}{2}\ln\left|x+2\right| - \frac{1}{2}\ln\left|x-2\right| + C \end{bmatrix}$$

$$\int \frac{x-4}{x^2-4}\,dx = \int \frac{x}{x^2-4} - \frac{4}{x^2-4}\,dx$$ OR DIRECTLY INTO PARTIAL FRACTIONS

$$= \frac{1}{2}\int \frac{2x}{x^2-4}\,dx - \int \frac{4}{(x-2)(x+2)}\,dx$$ (PARTIAL FRACTIONS BY INSPECTION

$$= \frac{1}{2}\ln\left|x^2-4\right| - \int \frac{1}{x-2} - \frac{1}{x+2}\,dx = \frac{1}{2}\ln\left|x^2-4\right| - \ln\left|x-2\right| + \ln\left|x+2\right| + C$$

OR DIRECTLY INTO PARTIAL FRACTIONS BY INSPECTION

$$\int \frac{x-4}{x^2-4}\,dx = \int \frac{x-4}{(x-2)(x+2)}\,dx = \int \frac{-\frac{1}{2}}{x-2} + \frac{\frac{3}{2}}{x+2}\,dx$$

$$= \frac{3}{2}\ln\left|x+2\right| - \frac{1}{2}\ln\left|x-2\right| + C$$

313. $$\int \frac{2\sin x}{\cos x + \sin x}\,dx = \begin{bmatrix} x - \ln\left|\cos x + \sin x\right| + C \\ x + \frac{1}{2}\ln\left|\sec 2x\right| - \frac{1}{2}\ln\left|\sec 2x + \tan 2x\right| + C \end{bmatrix}$$

$$\int \frac{2\sin x}{\cos x + \sin x}\,dx = \dots$$ BY MANIPULATION

$$= \int \frac{\cos x + \sin x + \sin x - \cos x}{\cos x + \sin x}\,dx$$

$$= \int \frac{\cancel{\cos x + \sin x}}{\cancel{\cos x + \sin x}} - \frac{\sin x + \cos x}{\cos x + \sin x}\,dx \longrightarrow \int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + C$$

$$= x - \ln\left|\cos x + \sin x\right| + C$$

ALTERNATIVE BY TRIGONOMETRIC MANIPULATIONS

$$\int \frac{2\sin x}{\cos x + \sin x}\,dx = \int \frac{2\sin x(\cos x - \sin x)}{(\cos x + \sin x)(\cos x - \sin x)}\,dx$$

$$= \int \frac{2\sin x\cos x - 2\sin^2 x}{\cos^2 x - \sin^2 x}\,dx = \int \frac{\sin 2x + \overbrace{(1 - 2\sin^2 x)}^{\cos 2x} - 1}{\cos 2x}\,dx$$

$$= \int \frac{\sin 2x + \cos 2x - 1}{\cos 2x}\,dx = \int \tan 2x + 1 - \sec 2x\,dx$$

NOW THESE ARE STANDARD QUOTABLE INTEGRALS, WHICH OF COURSE CAN EASILY BE PROVEN (NOT HERE)

$$= \frac{1}{2}\ln\left|\sec 2x\right| + x - \frac{1}{2}\ln\left|\sec 2x + \tan 2x\right| + C$$

$$= x + \frac{1}{2}\ln\left|\frac{\sec 2x}{\sec 2x + \tan 2x}\right| + C = x - \frac{1}{2}\ln\left|\frac{\sec 2x + \tan 2x}{\sec 2x}\right| + C$$

$$= x - \frac{1}{2}\ln\left|1 + \tan 2x\cos 2x\right| + C$$

$$= x - \frac{1}{2}\ln\left|1 + \frac{\sin 2x}{\cancel{\cos 2x}}\cancel{\cos 2x}\right| + C = x - \frac{1}{2}\ln\left|1 + \sin 2x\right| + C$$

$$= x - \frac{1}{2}\ln\left|\cos^2 x + \sin^2 x + 2\sin x\cos x\right| + C$$

$$= x - \frac{1}{2}\ln\left|(\cos x + \sin x)^2\right| + C$$

$$= x - \ln\left|\cos x + \sin x\right| + C$$ (AS ABOVE)

314. $$\int \left(1 - x^{-2}\right)^2\,dx = 2 + \frac{2}{x} - \frac{1}{3x^3} + C$$

$$\int (1 - x^{-2})^2\,dx = \int 1 - 2x^{-2} + x^{-4}\,dx = x + 2x^{-1} - \frac{1}{3}x^{-3} + C$$

$$= x + \frac{2}{x} - \frac{1}{3x^3} + C$$

315. $$\int \frac{\sqrt{\tan x}}{\cos^2 x}\,dx = \frac{2}{3}\sqrt{\tan^3 x} + C$$

$$\int \frac{\sqrt{\tan x}}{\cos^2 x}\,dx = \int (\tan x)^{\frac{1}{2}}\sec^2 x\,dx$$ BY REVERSE CHAIN RULE (INSPECTION)

$$= \frac{2}{3}(\tan x)^{\frac{3}{2}} + C = \frac{2}{3}\sqrt{\tan^3 x} + C$$

[THE SUBSTITUTION $u = \tan x$ OR $u = \sqrt{\tan x}$ ALSO WORK WELL]

316. $$\int (3\sin x + \cos x)^2\,dx = \begin{bmatrix} 5x - 2\sin 2x - \frac{3}{2}\cos 2x + C \\ 5x - 2\sin 2x + 3\sin^2 x + C \\ 5x - 2\sin 2x - 3\cos^2 x + C \end{bmatrix}$$

$$\int (3\sin x + \cos x)^2\,dx = \int 9\sin^2 x + 6\sin x\cos x + \cos^2 x\,dx$$

$$= \int 9\left(\tfrac{1}{2} - \tfrac{1}{2}\cos 2x\right) + 3\sin 2x + \left(\tfrac{1}{2} + \tfrac{1}{2}\cos 2x\right)dx$$

$$= \int 5 - 4\cos 2x + 3\sin 2x\,dx = 5x - 2\sin 2x - \frac{3}{2}\cos 2x + C$$

OR IF $6\sin x\cos x$ IS INTEGRATED TO $3\sin^2 x$ OR $-3\cos^2 x$

$$= 5x - 2\sin 2x + 3\sin^2 x + C$$

$$= 5x - 2\sin 2x - 3\cos^2 x + C$$

317. $$\int \frac{\sec^2 x}{(1+\tan x)^3}\,dx = -\frac{1}{2(1+\tan x)^2} + C$$

$$\int \frac{\sec^2 x}{(1+\tan x)^3}\,dx = \int (1+\tan x)^{-3}\sec^2 x\,dx$$ ... BY REVERSE CHAIN RULE (INSPECTION)

$$= -\frac{1}{2}(1+\tan x)^{-2} + C$$

$$= -\frac{1}{2(1+\tan x)^2} + C$$

[THE SUBSTITUTION $u = 1+\tan x$, OR $u = \tan x$ ALSO WORK WELL]

318. $$\int \frac{1}{\cos^2 x \sin^2 x}\, dx = \begin{bmatrix} -2\cot 2x + C \\ -\cot x + \tan x + C \end{bmatrix}$$

$$\int \frac{1}{\cos^2 x \sin^2 x}\, dx = \int \frac{1}{\left(\frac{1}{2}+\frac{1}{2}\cos 2x\right)\left(\frac{1}{2}-\frac{1}{2}\cos 2x\right)}\, dx$$

$$= \int \frac{1}{\frac{1}{4}-\frac{1}{4}\cos^2 2x}\, dx = \int \frac{4}{1-\cos^2 2x}\, dx = \int \frac{4}{\sin^2 2x}\, dx$$

$$= \int 4\operatorname{cosec}^2 2x\, dx = -2\cot 2x + C$$

ALTERNATIVE (VARIATION)

$$\int \frac{1}{\cos^2 x \sin^2 x}\, dx = \int \frac{4}{4\cos^2 x \sin^2 x}\, dx = \int \frac{4}{(2\cos x \sin x)^2}\, dx$$

$$= \int \frac{4}{(\sin 2x)^2}\, dx = \int 4\operatorname{cosec}^2 2x\, dx = -2\cot 2x + C$$

ANOTHER ALTERNATIVE

$$\int \frac{1}{\cos^2 x \sin^2 x}\, dx = \int \frac{\cos^2 x + \sin^2 x}{\cos^2 x \sin^2 x}\, dx = \ldots$$ SPLITTING THE FRACTION ...

$$= \int \frac{1}{\sin^2 x} + \frac{1}{\cos^2 x}\, dx = \int \operatorname{cosec}^2 x + \sec^2 x\, dx$$

$$= -\cot x + \tan x + C$$

319. $$\int \cot 2x\, dx = \frac{1}{2}\ln|\sin 2x| + C$$

$$\int \cot 2x\, dx = \int \frac{\cos 2x}{\sin 2x}\, dx = \frac{1}{2}\int \frac{2\cos 2x}{\sin 2x}\, dx$$

OF THE FORM $\int \frac{f'(x)}{f(x)}\, dx$

$$= \frac{1}{2}\ln|\sin 2x| + C$$

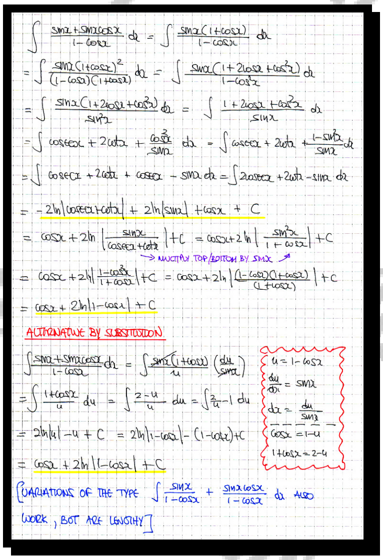
[THE SUBSTITUTION $u = \sin 2x$ ALSO WORKS WELL]

320. $$\int 2^x\, dx = \frac{2^x}{\ln 2} + C$$

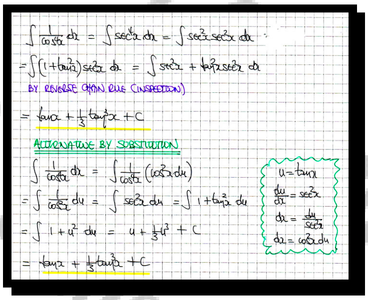
$$\int 2^x\, dx = \frac{1}{\ln 2}\int 2^x \ln 2\, dx = \frac{2^x}{\ln 2} + C$$

$$\frac{d}{dx}\left(2^x\right) = 2^x \ln 2$$

321. $$\int \frac{\sin x + \sin x \cos x}{1-\cos x}\, dx = \begin{bmatrix} \cos x + 2\ln\left|1-\cos x\right| + C \\ \cos x + 2\ln\left|\sin x\right| - 2\ln\left|\cot x + \operatorname{cosec} x\right| + C \end{bmatrix}$$



322. $$\int \frac{1}{\cos^4 x}\, dx = \tan x + \frac{1}{3}\tan^3 x + C$$



323. $$\int \frac{\sin x \cos x}{\sqrt{1+\cos 2x}}\, dx = \left[\begin{array}{c} -\frac{1}{2}\sqrt{1+\cos 2x} + C \\ \left[\begin{array}{l} -\dfrac{\cos x}{\sqrt{2}} + C \quad \text{if} \quad \cos x > 0 \\ \dfrac{\cos x}{\sqrt{2}} + C \quad \text{if} \quad \cos x < 0 \end{array} \right] \end{array} \right]$$

$$\int \frac{\sin x \cos x}{\sqrt{1+\cos 2x}}\, dx = \int \frac{\frac{1}{2}(2\sin x \cos x)}{(1+\cos 2x)^{\frac{1}{2}}}\, dx$$

$$= \int \tfrac{1}{2}\sin 2x\,(1+\cos 2x)^{-\frac{1}{2}} = \ldots$$ BY REVERSE CHAIN RULE (RECOGNITION)

$$= -\tfrac{1}{2}(1+\cos 2x)^{\frac{1}{2}} + C = -\tfrac{1}{2}\sqrt{1+\cos 2x} + C$$

ALTERNATIVE

$$\int \frac{\sin x \cos x}{\sqrt{1+\cos 2x}}\, dx = \int \frac{\sin x \cos x}{\sqrt{1+2\cos^2 x-1}}\, dx = \int \frac{\sin x \cos x}{\sqrt{2\cos^2 x}}\, dx$$

$$= \frac{1}{\sqrt{2}} \int \frac{\sin x \cos x}{|\cos x|}\, dx = \begin{cases} \frac{1}{\sqrt{2}} \int \sin x\, dx & \text{IF } \cos x > 0 \\ -\frac{1}{\sqrt{2}} \int \sin x\, dx & \text{IF } \cos x < 0 \end{cases}$$

$$= \begin{cases} -\frac{1}{\sqrt{2}}\cos x & \text{IF } \cos x > 0 \\ \frac{1}{\sqrt{2}}\cos x & \text{IF } \sin x < 0 \end{cases}$$

[THE SUBSTITUTION $u = \cos 2x$, $u = 1+\cos 2x$, $u = (1+\cos 2x)^{\frac{1}{2}}$ ALSO WORK]

324. $$\int \frac{\ln x^2}{x}\, dx = \left(\ln|x|\right)^2 + C$$

$$\int \frac{\ln x^2}{x}\, dx = \int \frac{2\ln x}{x}\, dx = \int 2 \cdot \frac{1}{x}(\ln x)\, dx$$

BY REVERSE CHAIN RULE (INSPECTION)

$$= (\ln x)^2 + C$$

[THE SUBSTITUTION $u = \ln x$ WORKS WELL AFTER WRITING $\ln x^2$ AS $2\ln x$, OR EVEN INTEGRATION BY PARTS]

325. $$\int \sin x \cos^2\left(\tfrac{1}{2}x\right)\, dx = -\cos^4\left(\tfrac{1}{2}x\right) + C$$

$$\int \sin x \cos^2(\tfrac{1}{2}x)\, dx = \int \left[2\sin(\tfrac{1}{2}x)\cos(\tfrac{1}{2}x)\right]\cos^2(\tfrac{1}{2}x)\, dx$$

$$= \int 2\sin(\tfrac{1}{2}x)\cos^3(\tfrac{1}{2}x)\, dx$$ BY REVERSE CHAIN RULE (INSPECTION) OR USING THE SUBSTITUTION $u = \cos\frac{1}{2}x$

$$= \ldots -\cos^4(\tfrac{1}{2}x) + C$$

326. $$\int \frac{e^{3x}+1}{e^{x}+1}\,dx = \frac{1}{2}e^{2x} - e^{x} + x + C$$

$$\int \frac{e^{3x}+1}{e^{x}+1}\,dx = \dots \qquad \boxed{A^3+B^3 \equiv (A+B)(A^2-AB+B^2)}$$

$$= \int \frac{\cancel{(e^x+1)}(e^{2x}-e^{x}+1)}{\cancel{e^x+1}}\,dx = \int e^{2x}-e^{x}+1\,dx \qquad (e^x+1)\neq 0$$

$$= \frac{1}{2}e^{2x} - e^{x} + x + C$$

ALTERNATIVE (VARIATION) WITH SUBSTITUTION

$$\boxed{u = e^x \quad \frac{du}{dx} = e^x \quad \frac{du}{dx} = u \quad dx = \frac{du}{u}}$$

$$\int \frac{e^{3x}+1}{e^{x}+1}\,dx = \int \frac{u^3+1}{u+1}\,\frac{du}{u}$$

NOW BY LONG DIVISION, THE SUM OF CUBES IDENTITY FROM ABOVE, OR JUST MANIPULATIONS...

$$= \int \frac{u^2(u+1)-u(u+1)+(u+1)}{u+1}\,\frac{du}{u} = \int (u^2-u+1)\,\frac{du}{u}$$

$$= \int u - 1 + \frac{1}{u}\,du = \frac{1}{2}u^2 - u + \ln|u| + C$$

$$= \frac{1}{2}e^{2x} - e^{x} + \ln(e^{x}) + C = \frac{1}{2}e^{2x} - e^{x} + x + C$$

327. $$\int \frac{\sqrt{x}}{x-1}\,dx = 2\sqrt{x} + \ln\left|\frac{\sqrt{x}-1}{\sqrt{x}+1}\right| + C$$

$$\int \frac{\sqrt{x}}{x-1}\,dx = \dots \text{ BY SUBSTITUTION} \qquad \boxed{u=\sqrt{x} \quad u^2 = x \quad 2u\,du = dx}$$

$$= \int \frac{u}{u^2-1}(2u\,du) = \int \frac{2u^2}{u^2-1}\,du$$

IMPROPER FRACTION, NEED TO BE DIVIDED OUT

$$= \int \frac{2(u^2-1)+2}{u^2-1}\,du = \int 2 + \frac{2}{u^2-1}\,du = \int 2 + \frac{2}{(u-1)(u+1)}\,du$$

PARTIAL FRACTIONS BY INSPECTION (COVER UP)

$$= \int 2 + \frac{1}{u-1} - \frac{1}{u+1}\,du = 2u + \ln|u-1| - \ln|u+1| + C$$

$$= 2u + \ln\left|\frac{u-1}{u+1}\right| + C = 2\sqrt{x} + \ln\left|\frac{\sqrt{x}-1}{\sqrt{x}+1}\right| + C$$

328. $$\int 2^x\,3^x\,dx = \frac{6^x}{\ln 6} + C$$

$$\int 2^x 3^x\,dx = \int (2\times 3)^x\,dx = \int 6^x\,dx$$

$$= \int \frac{1}{\ln 6}(6^x \ln 6)\,dx = \frac{6^x}{\ln 6} + C$$

329. $$\int 3^{2x+1}\,dx = \begin{bmatrix} \dfrac{3^{2x+1}}{2\ln 3} + C \\ \dfrac{3\times 9^{x}}{\ln 9} + C \end{bmatrix}$$

$$\int 3^{2x+1}\,dx = \frac{1}{2\ln 3}\int 3^{2x+1}\times \ln 3 \times 2\,dx$$

BY REVERSE CHAIN RULE (RECOGNITION), SINCE

$$\frac{d}{dx}\left(a^{x}\right) = a^{x}\ln a$$

$$= \frac{3^{2x+1}}{2\ln 3} + C$$

VARIATION

$$\int 3^{2x+1}\,dx = \int 3^{2x}\times 3^{1}\,dx = 3\int \left(3^{2}\right)^{x} dx = 3\int 9^{x}\,dx$$

$$= \frac{3}{\ln 9}\int 9^{x}\ln 9\,dx = \frac{3\times 9^{x}}{\ln 9} + C$$

330. $$\int \frac{2x^{\frac{3}{2}}+1}{x^{\frac{5}{2}}+2x}\,dx = \ln\left(x^{2}+2\sqrt{x}\right) + C$$

$$\int \frac{2x^{\frac{3}{2}}+1}{x^{\frac{5}{2}}+2x}\,dx = \dots \text{ BY SUBSTITUTION}$$

$$u = x^{\frac{1}{2}}, \quad u^{2} = x, \quad 2u\,du = dx$$

$$= \int \frac{2u^{3}+1}{u^{5}+2u^{2}}\,(2u\,du) = \int \frac{4u^{3}+2}{u^{4}+2u}\,du$$

BY INSPECTION AS THIS IS OF THE FORM $\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + C$

$$= \ln\left|u^{4}+2u\right| + C = \ln\left(x^{2}+2\sqrt{x}\right) + C$$

331. $$\int \frac{1}{x\left(1+\sqrt{x}\right)}\,dx = 2\ln\left(\frac{\sqrt{x}}{1+\sqrt{x}}\right) + C$$

$$\int \frac{1}{x\left(1+\sqrt{x}\right)}\,dx = \dots \text{ BY SUBSTITUTION}$$

$$u = \sqrt{x}, \quad u^{2} = x, \quad 2u\,du = dx$$

$$= \int \frac{1}{u^{2}(1+u)}\,(2u\,du) = \int \frac{2}{u(u+1)}\,du$$

PARTIAL FRACTIONS BY INSPECTION (COVER UP)

$$= \int \frac{2}{u} - \frac{2}{u+1}\,du = 2\ln|u| - 2\ln|u+1| + C = 2\ln\left|\frac{u}{u+1}\right| + C$$

$$= 2\ln\left(\frac{\sqrt{x}}{1+\sqrt{x}}\right) + C$$

[THE SUBSTITUTION $u = 1+\sqrt{x}$ ALSO WORKS]

332. $\displaystyle\int \frac{\sqrt{x}}{\sqrt{x}-1}\,dx = x + 2\sqrt{x} + 2\ln\left|\sqrt{x}-1\right| + C$

$\displaystyle\int \frac{\sqrt{x}}{\sqrt{x}-1}\,dx = \ldots$ BY SUBSTITUTION

$u = \sqrt{x}$
$u^2 = x$
$2u\,du = dx$

$\displaystyle = \int \frac{u}{u-1}(2u\,du) = \int \frac{2u^2}{u-1}\,du$

BY A SECOND SUBSTITUTION OR MANIPULATION (DIVISION)

$\displaystyle = \int \frac{2u(u-1)+2(u-1)+2}{u-1}\,du = \int 2u + 2 + \frac{2}{u-1}\,du$

$= u^2 + 2u + 2\ln|u-1| + C = x + 2\sqrt{x} + 2\ln|\sqrt{x}-1| + C$

[THE SUBSTITUTION $u = \sqrt{x}-1$ ALSO WORKS]

333. $\displaystyle\int \frac{1}{x\left(1+x^2\right)}\,dx = \ln\left|\frac{x}{\sqrt{x^2+1}}\right| + C$

$\displaystyle\int \frac{1}{x(1+x^2)}\,dx = \ldots$ BY PARTIAL FRACTIONS

$\displaystyle\frac{1}{x(1+x^2)} \equiv \frac{A}{x} + \frac{Bx+C}{1+x^2}$
$1 \equiv A(1+x^2) + x(Bx+C)$
$1 \equiv A + Ax^2 + Bx^2 + Cx$
$1 \equiv (A+B)x^2 + Cx + A$
$\therefore A=1,\ C=0,\ B=-1$

$\displaystyle \ldots = \int \frac{1}{x} - \frac{x}{1+x^2}\,dx = \ln|x| - \tfrac{1}{2}\ln(1+x^2) + C$

$\displaystyle = \ln|x| - \ln(1+x^2)^{\frac{1}{2}} + C = \ln\left|\frac{x}{\sqrt{1+x^2}}\right| + C$

ALTERNATIVE BY TRIGONOMETRIC SUBSTITUTION

$\displaystyle\int \frac{1}{x(1+x^2)}\,dx = \int \frac{1}{\tan\theta(1+\tan^2\theta)}(\sec^2\theta\,d\theta)$

$\displaystyle = \int \frac{\cancel{\sec^2\theta}}{\tan\theta\,\cancel{\sec^2\theta}}\,d\theta = \int \frac{1}{\tan\theta}\,d\theta$

$\displaystyle = \int \cot\theta\,d\theta = \ln|\sin\theta| + C$

$\displaystyle = \ln\left|\frac{x}{\sqrt{x^2+1}}\right| + C$

$x = \tan\theta$
$dx = \sec^2\theta\,d\theta$

$\tan\theta = \frac{x}{1}$
$\sin\theta = \frac{x}{\sqrt{x^2+1}}$

334. $\displaystyle\int \sin^4 x \sin 2x\,dx = \tfrac{1}{3}\sin^6 x + C$

$\displaystyle\int \sin^4 x \sin 2x\,dx = \int \sin^4 x(2\sin x\cos x)\,dx = \int 2\sin^5 x\cos x\,dx$

BY REVERSE CHAIN RULE

$= \frac{1}{3}\sin^6 x + C$

[THE SUBSTITUTION $u = \sin x$, ALSO WORKS]

335. $\displaystyle\int (\tan x+\cot x)^2 \ dx = \tan x - \cot x + C$

$$\int (\tan x+\cot x)^2 dx = \int \tan^2 x + 2\tan x\cot x + \cot^2 x \ dx$$
$$= \int (\sec^2 x - 1) + 2\cancel{\tan x}\left(\frac{1}{\cancel{\tan x}}\right) + (\text{cosec}^2 x - 1) \ dx$$
$$= \int \sec^2 x + \text{cosec}^2 x \ dx = \tan x - \cot x + C$$

336. $\displaystyle\int \frac{x^2+2x-2}{x^2-2x+2} \ dx = x + \ln\left(x^2-2x+2\right) + C$

$$\int \frac{x^2+2x-2}{x^2-2x+2} dx = \dots$$ BY DIVISION OR MANIPULATION
$$= \int \frac{(x^2-2x+2)+4x-4}{x^2-2x+2} dx = \int 1 + \frac{4x-4}{x^2-2x+2} dx$$
$$= \int 1 + 2\left(\frac{2x-2}{x^2-2x+2}\right) dx = x + 2\ln|x^2-2x+2| + C$$

OF THE FORM $\int \frac{f'(x)}{f(x)} dx = \ln|f(x)| + C$

337. $\displaystyle\int x\sin x\cos x \ dx = -\frac{1}{4}x\cos 2x + \frac{1}{8}\sin 2x + C$

$$\int x\sin x\cos x \, dx = \int x \times \tfrac{1}{2}(2\sin x\cos x) \, dx = \int \tfrac{1}{2}x\sin 2x \, dx$$

INTEGRATION BY PARTS

$\frac{1}{2}x$	$\frac{1}{2}$
$-\frac{1}{2}\cos 2x$	$\sin 2x$

$$= -\tfrac{1}{4}x\cos 2x - \int -\tfrac{1}{4}\cos 2x \, dx$$
$$= -\tfrac{1}{4}x\cos 2x + \int \tfrac{1}{4}\cos 2x \, dx$$
$$= -\tfrac{1}{4}x\cos 2x + \tfrac{1}{8}\sin 2x + C$$

338. $\displaystyle\int \sec x\tan^3 x \ dx = -\frac{1}{3}\sec^3 x - \sec x + C$

$$\int \sec x\tan^3 x \, dx = \int \sec x\tan x(\tan^2 x) \, dx = \int \sec x\tan x(\sec^2 x - 1) \, dx$$
$$= \int \sec^3 x\tan x - \sec x\tan x \, dx =$$ BY REVERSE CHAIN RULE (INSPECTION) AS $\frac{d}{dx}(\sec x) = \sec x\tan x$
$$= \tfrac{1}{3}\sec^3 x - \sec x + C$$

[THE SUBSTITUTION $u=\sec x$ WORKS WELL TOO]

339. $$\int \frac{1}{\operatorname{cosec} x - \cot x}\,dx = \begin{bmatrix} \ln|\sin x| - \ln|\operatorname{cosec} x + \cot x| + C \\ \ln|1-\cos x| + C \end{bmatrix}$$

$$\int \frac{1}{\operatorname{cosec}x - \cot x}\,dx = \int \frac{\operatorname{cosec}x + \cot x}{(\operatorname{cosec}x - \cot x)(\operatorname{cosec}x + \cot x)}\,dx$$

$$= \int \frac{\operatorname{cosec}x + \cot x}{\operatorname{cosec}^2 x - \cot^2 x}\,dx = \int \frac{\operatorname{cosec}x + \cot x}{(1+\cot^2 x) - \cot^2 x}\,dx = \int \operatorname{cosec}x + \cot x\,dx$$

$$= -\ln|\operatorname{cosec}x + \cot x| + \ln|\sin x| + C$$

$$\ldots = \ln\left|\frac{\sin x}{\operatorname{cosec}x + \cot x}\right| + C = \ln\left|\frac{\sin x}{\frac{1}{\sin x} + \frac{\cos x}{\sin x}}\right| + C$$

$$= \ln\left|\frac{\sin^2 x}{1+\cos x}\right| + C = \ln\left|\frac{1-\cos^2 x}{1+\cos x}\right| + C$$

$$= \ln\left|\frac{(1-\cos x)(1+\cos x)}{1+\cos x}\right| + C = \ln|1-\cos x| + C$$

ALTERNATIVE

$$\int \frac{1}{\operatorname{cosec}x - \cot x}\,dx = \int \frac{1\sin x}{\operatorname{cosec}x\sin x - \cot x\sin x}\,dx$$

$$= \int \frac{\sin x}{1-\cos x}\,dx \quad \text{WHICH IS OF THE FORM } \int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + C$$

$$= \ln|1-\cos x| + C$$

340. $$\int \frac{2x^{\frac{1}{4}}+1}{4x^{\frac{5}{4}}+4x}\,dx = \ln\left(x^{\frac{1}{2}} + x^{\frac{1}{4}}\right) + C$$

$$\int \frac{2x^{\frac{1}{4}}+1}{4x^{\frac{5}{4}}+4x}\,dx \;\ldots\; \text{BY SUBSTITUTION} \ldots \qquad u = x^{\frac{1}{4}},\quad u^4 = x,\quad 4u^3\,du = dx$$

$$= \int \frac{2u+1}{4u^5+4u^4}(4u^3\,du) = \int \frac{2u+1}{u^2+u}\,du$$

WHICH IS OF THE FORM

$$\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + C$$

$$= \ln|u^2+u| + C = \ln\left(x^{\frac{1}{2}} + x^{\frac{1}{4}}\right) + C$$

341. $$\int \frac{\ln(x-1)}{\sqrt{x}}\,dx = 2\sqrt{x}\ln|x-1| - 4\sqrt{x} + 2\ln\left|\frac{\sqrt{x}+1}{\sqrt{x}-1}\right| + C$$

$$\int \frac{\ln(x-1)}{\sqrt{x}}\,dx = \int x^{-\frac{1}{2}}\ln(x-1)\,dx \ldots \text{BY PARTS}$$

$\ln(x-1)$	$\frac{1}{x-1}$
$2x^{\frac{1}{2}}$	$x^{-\frac{1}{2}}$

$$= 2x^{\frac{1}{2}}\ln|x-1| - \int \frac{2x^{\frac{1}{2}}}{x-1}\,dx$$

BY SUBSTITUTION

$u = x^{\frac{1}{2}}$, $u^2 = x$, $2u\,du = dx$

$$= 2x^{\frac{1}{2}}\ln|x-1| - \int \frac{2u}{u^2-1}(2u\,du)$$

$$= 2x^{\frac{1}{2}}\ln|x-1| - \int \frac{4u^2}{u^2-1}\,du \quad \text{IMPROPER}$$

$$= 2x^{\frac{1}{2}}\ln|x-1| - \int \frac{4(u^2-1)+4}{u^2-1}\,du$$

$$= 2x^{\frac{1}{2}}\ln|x-1| - \int 4 + \frac{4}{u^2-1}\,du$$

$$= 2x^{\frac{1}{2}}\ln|x-1| - \int 4 + \frac{4}{(u-1)(u+1)}\,du$$

PARTIAL FRACTIONS BY INSPECTION

$$= 2x^{\frac{1}{2}}\ln|x-1| - \int 4 + \frac{2}{u-1} - \frac{2}{u+1}\,du$$

$$= 2x^{\frac{1}{2}}\ln|x-1| - 4u - 2\ln|u-1| + 2\ln|u+1| + C$$

$$= 2x^{\frac{1}{2}}\ln|x-1| - 4\sqrt{x} + 2\ln\left|\frac{u+1}{u-1}\right| + C$$

$$= 2\sqrt{x}\ln|x-1| - 4\sqrt{x} + 2\ln\left|\frac{\sqrt{x}+1}{\sqrt{x}-1}\right| + C$$

SUBSTITUTION $u=\sqrt{x}$ FIRST, FOLLOWED BY INTEGRATION BY PARTS ALSO WORKS

342. $$\int e^{x+e^{x}}\ dx = e^{e^{x}} + C$$

$$\int e^{e^x+x}\,dx = \int e^{e^x} \times e^x\,dx = \ldots$$ BY REVERSE CHAIN RULE (INSPECTION)

$$= e^{e^x} + C$$

[THE SUBSTITUTION $u = e^x$ ALSO WORKS]

343. $$\int \frac{1}{x\left(1+\sqrt{x}\right)^2}\ dx = 2\ln\left[\frac{\sqrt{x}}{1+\sqrt{x}}\right] + \frac{2}{1+\sqrt{x}} + C$$

$$\int \frac{1}{x(1+\sqrt{x})^2}\,dx = \ldots$$ BY SUBSTITUTION ...

$u = \sqrt{x}$

$u^2 = x$

$2u\,du = dx$

$$= \int \frac{1}{u^2(1+u)^2}(2u\,du) = \int \frac{2}{u(1+u)^2}\,du$$

BY PARTIAL FRACTIONS

$$\frac{2}{u(1+u)^2} \equiv \frac{A}{u} + \frac{B}{(1+u)^2} + \frac{C}{1+u}$$

$$\boxed{2 \equiv A(1+u)^2 + Cu(1+u) + Bu}$$

- IF $u=-1$: $2 = -B$, $B = -2$
- IF $u=0$: $2 = A$
- IF $u=1$: $2 = 4A + 2C + B$, $2 = 8 + 2C - 2$, $-4 = 2C$, $C = -2$

$$\ldots = \int \frac{2}{u} - \frac{2}{1+u} - \frac{2}{(1+u)^2}\,du = 2\ln|u| - 2\ln|1+u| + \frac{2}{1+u} + C$$

$$= 2\ln\left|\frac{u}{1+u}\right| + \frac{2}{u+1} + C = 2\ln\left(\frac{\sqrt{x}}{1+\sqrt{x}}\right) + \frac{2}{1+\sqrt{x}} + C$$

344. $$\int x^2 e^{-x}\ dx = -e^{-x}\left(x^2+2x+2\right) + C$$

$$\int x^2 e^{-x}\,dx = \ldots$$ INTEGRATION BY PARTS

x^2	$2x$
$-e^{-x}$	e^{-x}

$$= -x^2e^{-x} - \int -2xe^{-x}\,dx = -x^2e^{-x} + \int 2xe^{-x}\,dx$$

INTEGRATION BY PARTS AGAIN

$2x$	2
$-e^{-x}$	e^{-x}

$$= -x^2e^{-x} + \left[2xe^{-x} - \int -2e^{-x}\,dx\right]$$

$$= -x^2e^{-x} + 2xe^{-x} + \int 2e^{-x}\,dx$$

$$= -x^2e^{-x} + 2xe^{-x} - 2e^{-x} + C = -e^{-x}(x^2+2x+2) + C$$

345. $$\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\ dx = 2e^{\sqrt{x}} + C$$

$$\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx = \int x^{-\frac{1}{2}} e^{x^{\frac{1}{2}}}\,dx = \dots$$ BY REVERSE CHAIN RULE (INSPECTION)

$$= 2e^{x^{\frac{1}{2}}} + C = 2e^{\sqrt{x}} + C$$

[THE SUBSTITUTION $u = \sqrt{x}$ ALSO WORKS WELL]

346. $$\int \sqrt{x}\,e^{\sqrt{x}}\ dx = 2e^{\sqrt{x}}\left(x + 2\sqrt{x} + 2\right) + C$$

$$\int \sqrt{x}\,e^{\sqrt{x}}\,dx = \dots$$ BY SUBSTITUTION ...

$u = \sqrt{x}$

$u^2 = x$

$2u\,du = dx$

$$= \int u e^{u}(2u\,du) = \underbrace{\int 2u^2 e^{u}\,du}_{\text{INTEGRATION BY PARTS}}$$

$$\begin{array}{c|c} 2u^2 & 4u \\ \hline e^{u} & e^{u} \end{array}$$

$$= 2u^2e^{u} - \underbrace{\int 4ue^{u}\,du}_{\text{INTEGRATION BY PARTS AGAIN}}$$

$$\begin{array}{c|c} 4u & 4 \\ \hline e^{u} & e^{u} \end{array}$$

$$= 2u^2e^{u} - \left[4ue^{u} - \int 4e^{u}\,du\right]$$

$$= 2u^2e^{u} - 4ue^{u} + \int 4e^{u}\,du$$

$$= 2u^2e^{u} - 4ue^{u} + 4e^{u} + C$$

$$= 2e^{u}(u^2 - 2u + 2) + C = 2e^{\sqrt{x}}(x - 2\sqrt{x} + 2) + C$$

347. $$\int \frac{4x^2 - x + 1}{(x-1)(2x-1)}\ dx = 2x + 4\ln|x-1| - \frac{3}{2}\ln|2x-1| + C$$

$$\int \frac{4x^2 - x + 1}{(x-1)(2x-1)}\,dx = \int \frac{4x^2 - x + 1}{2x^2 - 3x + 1}\,dx$$ ← IMPROPER

$$= \int \frac{2(2x^2 - 3x + 1) + 5x - 1}{2x^2 - 3x + 1}\,dx = \int 2 + \frac{5x - 1}{2x^2 - 3x + 1}\,dx$$

$$= \int 2 + \frac{5x - 1}{(x-1)(2x-1)}\,dx = \dots$$ PARTIAL FRACTIONS BY INSPECTION

$$= \int 2 + \frac{4}{x-1} - \frac{3}{2x-1}\,dx = 2x + 4\ln|x-1| - \frac{3}{2}\ln|2x-1| + C$$

348. $$\int e^{x}\cos x \ dx = \frac{1}{2}e^{-x}\left(\sin x + \cos x\right) + C$$

$\int e^{x}\cos x \, dx$ ... BY PARTS TWICE

e^{x}	e^{x}
$\sin x$	$\cos x$

$\int e^{x}\cos x \, dx = e^{x}\sin x - \underbrace{\int e^{x}\sin x \, dx}_{\text{BY PARTS}}$

e^{x}	e^{x}
$-\cos x$	$\sin x$

$\int e^{x}\cos x \, dx = e^{x}\sin x - \left[-e^{x}\cos x - \int -e^{x}\cos x \, dx\right]$

$\int e^{x}\cos x \, dx = e^{x}\sin x + e^{x}\cos x - \int e^{x}\cos x \, dx$

$2\int e^{x}\cos x \, dx = e^{x}\sin x + e^{x}\cos x$

$\int e^{x}\cos x \, dx = \frac{1}{2}e^{x}(\sin x + \cos x) + C$

ALTERNATIVE BY DIFFERENTIATION

$\Rightarrow \frac{d}{dx}\left[e^{x}(A\cos x + B\sin x)\right] \equiv e^{x}\cos x$

$\Rightarrow e^{x}(A\cos x + B\sin x) + e^{x}(-A\sin x + B\cos x) \equiv e^{x}\cos x$

$\Rightarrow (A+B)\cos x + (B-A)\sin x \equiv \cos x$

$A+B = 1$
$B-A = 0$ $\quad \therefore \ A = B = \frac{1}{2}$

$\therefore \int e^{x}\cos x \, dx = e^{x}\left(\frac{1}{2}\cos x + \frac{1}{2}\sin x\right) + C$

[THE ABOVE INTEGRAL CAN ALSO BE DONE BY COMPLEX NUMBERS – NOT SHOWN HERE]

349. $$\int \frac{5^{2x}}{5^{2x}+3} \ dx = \frac{\ln\left(5^{2x}+3\right)}{\ln 25} + C$$

$\int \frac{5^{2x}}{5^{2x}+3} \, dx = \frac{1}{2\ln 5}\int \frac{5^{2x}\times 2\times \ln 5}{5^{2x}+3} \, dx$

OF THE FORM $\int \frac{f'(x)}{f(x)} \, dx = \ln|f(x)| + C$

SINCE $\frac{d}{dx}\left[a^{f(x)}\right] = a^{f(x)} \times f'(x) \times \ln a$

$= \frac{\ln(5^{2x}+3)}{2\ln 5} + C = \frac{\ln(5^{2x}+3)}{\ln 25} + C$

[THE SUBSTITUTION $u = 5^{2x}+3$ ALSO WORKS WELL]

350. $$\int \frac{3x^{5}}{x^{3}-1} \ dx = x^{3} + \ln\left|x^{3}-1\right| + C$$

$\int \frac{3x^{5}}{x^{3}-1} \, dx = \ldots$ BY DIVISION OR MANIPULATION

$= \int \frac{3x^{2}(x^{3}-1) + 3x^{2}}{x^{3}-1} \, dx = \int 3x^{2} + \frac{3x^{2}}{x^{3}-1} \, dx$

OF THE FORM $\int \frac{f'(x)}{f(x)} \, dx = \ln|f(x)| + C$

$= x^{3} + \ln|x^{3}-1| + C$

[THE SUBSTITUTION $u = x^{3}-1$ ALSO WORKS]

351. $$\int \frac{x-2}{x^2-4x-2}\,dx = \frac{1}{2}\ln\left|x^2-4x-2\right| + C$$

$$\int \frac{x-2}{x^2-4x-2}\,dx = \frac{1}{2}\int \frac{2(x-2)}{x^2-4x-2}\,dx$$

$$= \frac{1}{2}\int \frac{2x-4}{x^2-4x-2}\,dx$$

THIS OF THE FORM $\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)|+C$

$$= \frac{1}{2}\ln\left|x^2-4x-2\right| + C$$

[THE SUBSTITUTION $u = x^2-4x-2$ ALSO WORKS]

352. $$\int \frac{1}{(x-1)\sqrt{x^2-1}}\,dx = -\sqrt{\frac{x+1}{x-1}} + C$$

$$\int \frac{1}{(x-1)\sqrt{x^2-1}}\,dx = \dots$$ BY SUBSTITUTION

$x-1 = \frac{1}{u}$

$dx = -\frac{1}{u^2}\,du$

$x^2-1 = (x+1)(x-1)$

$x^2-1 = \left(\frac{1}{u}+2\right)\frac{1}{u}$

$x^2-1 = \frac{1}{u^2}+\frac{2}{u}$

$$= \int \frac{1}{\frac{1}{u}\sqrt{\frac{1}{u^2}+\frac{2}{u}}}\left(-\frac{1}{u^2}\,du\right)$$

$$= -\int \frac{1}{u\left(\frac{1+2u}{u^2}\right)^{\frac{1}{2}}}\,du = -\int \frac{1}{(1+2u)^{\frac{1}{2}}}\,du$$

$$= -\int (1+2u)^{-\frac{1}{2}}\,du = -(1+2u)^{\frac{1}{2}} + C$$

$$= -\left(1+2\left(\frac{1}{x-1}\right)\right)^{\frac{1}{2}} + C = -\sqrt{1+\frac{2}{x-1}} + C = -\sqrt{\frac{x-1+2}{x-1}} + C$$

$$= -\sqrt{\frac{x+1}{x-1}} + C$$

ALTERNATIVE BY TRIGONOMETRIC SUBSTITUTION

$\theta = \text{arcsec}\,x$

$\sec\theta = x$

$\sec\theta\tan\theta\,d\theta = dx$

$$\int \frac{1}{(x-1)\sqrt{x^2-1}}\,dx = \dots$$

$$= \int \frac{1}{(\sec\theta-1)\cancel{\sqrt{\sec^2\theta-1}}}\left(\sec\theta\,\cancel{\tan\theta}\,d\theta\right)$$

$$= \int \frac{\sec\theta}{\sec\theta-1}\,d\theta = \int \frac{\sec\theta(\sec\theta+1)}{(\sec\theta-1)(\sec\theta+1)}\,d\theta$$

$$= \int \frac{\sec^2\theta+\sec\theta}{\sec^2\theta-1}\,d\theta = \int \frac{\sec^2\theta+\sec\theta}{\tan^2\theta}\,d\theta = \int \frac{\sec^2\theta\cos^2\theta+\sec\theta\cos^2\theta}{\tan^2\theta\cos^2\theta}\,d\theta$$

$$= \int \frac{1+\cos\theta}{\sin^2\theta}\,d\theta = \int \frac{1}{\sin^2\theta}+\frac{\cos\theta}{\sin^2\theta}\,d\theta = \int \text{cosec}^2\theta + \cot\theta\,\text{cosec}\,\theta\,d\theta$$

$$= -\cot\theta - \text{cosec}\,\theta + C = -\left(\frac{\cos\theta}{\sin\theta}+\frac{1}{\sin\theta}\right) = -\frac{1+\cos\theta}{\sin\theta} + C$$

$$= -\frac{1+\cos\theta}{\sqrt{1-\cos^2\theta}} + C = -\frac{1+\cos\theta}{\sqrt{(1-\cos\theta)(1+\cos\theta)}} + C = -\sqrt{\frac{1+\cos\theta}{1-\cos\theta}} + C$$

$$= -\sqrt{\frac{1\sec\theta+\cos\theta\sec\theta}{1\sec\theta-\cos\theta\sec\theta}} + C = -\sqrt{\frac{\sec\theta+1}{\sec\theta-1}} + C = -\sqrt{\frac{x+1}{x-1}} + C$$

353. $$\int \frac{3x^3-x^2+10x-3}{x^2+3}\,dx = \frac{3}{2}x^2 - x + \frac{1}{2}\ln\left(x^2+3\right) + C$$

$$\int \frac{3x^3-x^2+10x-3}{x^2+3}\,dx = \dots$$ BY DIVISION OR MANIPULATION

$$= \int \frac{3x(x^2+3)-(x^2+3)+x}{x^2+3}\,dx = \int 3x-1+\frac{x}{x^2+3}\,dx$$

$$= \int 3x-1+\frac{1}{2}\left(\frac{2x}{x^2+3}\right)\,dx$$

OF THE FORM $\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)|+C$

$$= \frac{3}{2}x^2 - x + \frac{1}{2}\ln(x^2+3) + C$$

354. $$\int \frac{4}{\left(4+x^2\right)^{\frac{3}{2}}}\,dx = \frac{x}{\sqrt{4+x^2}} + C$$

$\int \frac{4}{(4+x^2)^{\frac{3}{2}}}\,dx = \dots$ BY TRIGONOMETRIC SUBSTITUTION

$= \int \frac{4}{(4+4\tan^2\theta)^{\frac{3}{2}}}\,(2\sec^2\theta\,d\theta)$

$= \int \frac{\cancel{8}\sec^2\theta}{\cancel{8}(1+\tan^2\theta)^{\frac{3}{2}}}\,d\theta = \int \frac{\sec^2\theta}{(\sec^2\theta)^{\frac{3}{2}}}\,d\theta$

$= \int \frac{\sec^2\theta}{\sec^3\theta}\,d\theta = \int \frac{1}{\sec\theta}\,d\theta = \int \cos\theta\,d\theta$

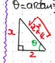
$= \sin\theta + C = \frac{x}{\sqrt{x^2+4}} + C$

$x = 2\tan\theta$

$dx = 2\sec^2\theta\,d\theta$

$\tan\theta = \frac{x}{2}$

$\theta = \arctan\frac{x}{2}$



$\sin\theta = \frac{x}{\sqrt{x^2+4}}$

355. $$\int \frac{(x+1)^2}{x^2+1}\,dx = x + \ln\left(x^2+1\right) + C$$

$\int \frac{(x+1)^2}{x^2+1}\,dx = \int \frac{x^2+2x+1}{x^2+1}\,dx = \int \frac{(x^2+1)+2x}{x^2+1}\,dx$

$= \int 1 + \frac{2x}{x^2+1}\,dx = x + \ln(x^2+1) + C$

OF THE FORM $\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + C$

356. $$\int \frac{\cos x + \tan x}{1+\tan^2 x}\,dx = \sin x - \frac{1}{3}\sin^3 x + \frac{1}{2}\sin^2 x + C$$

$\int \frac{\cos x + \tan x}{1+\tan^2 x}\,dx = \int \frac{\cos x + \tan x}{\sec^2 x}\,dx$

$= \int \frac{\cos x}{\sec^2 x} + \frac{\tan x}{\sec^2 x}\,dx = \int \cos^3 x + \tan x\cos^2 x\,dx$

$= \int \cos x\cos^2 x + \frac{\sin x}{\cos x}\cos^2 x\,dx = \int \cos x(1-\sin^2 x) + \sin x\cos x\,dx$

$= \int \cos x - \underbrace{\cos x\sin^2 x + \sin x\cos x}_{\text{BY REVERSE CHAIN RULE (RECOGNITION)}}\,dx$

$= \sin x - \frac{1}{3}\sin^3 x + \frac{1}{2}\sin^2 x + C$

ALTERNATIVE BY SUBSTITUTION

$\int \frac{\cos x + \tan x}{1+\tan^2 x}\,dx = \int \frac{\cos x + \tan x}{\sec^2 x}\,dx$

$= \int \frac{\cos x + \tan x}{\sec^2 x}\left(\frac{du}{\cos x}\right) = \int \frac{\cos x + \tan x}{\sec x}\,du$

$= \int \cos^2 x + \tan x\cos x\,du$

$= \int 1 - \sin^2 x + \frac{\sin x}{\cancel{\cos x}}\cancel{\cos x}\,du = \int 1 - \sin^2 x + \sin x\,du$

$= \int 1 - u^2 + u\,du = u - \frac{1}{3}u^3 + \frac{1}{2}u^2 + C$

$= \sin x - \frac{1}{3}\sin^3 x + \frac{1}{2}\sin^2 x + C$

$u = \sin x$

$\frac{du}{dx} = \cos x$

$dx = \frac{du}{\cos x}$

357. $$\int \frac{\sqrt{1+\sqrt{x}}}{\sqrt{x}}\,dx = \frac{4}{3}\left(1+\sqrt{x}\right)^{\frac{3}{2}} + C$$

$$\int \frac{\sqrt{\sqrt{x}+1}}{\sqrt{x}}\,dx = \int x^{-\frac{1}{2}}\left(x^{\frac{1}{2}}+1\right)^{\frac{1}{2}}\,dx$$

BY REVERSE CHAIN RULE (RECOGNITION)

$$= \frac{4}{3}\left(x^{\frac{1}{2}}+1\right)^{\frac{3}{2}} + C$$

358. $$\int \frac{4x^3\sqrt{x^4+1}}{1+\sqrt{x^4+1}}\,dx = x^4 - 2\sqrt{x^4+1} + 2\ln\left(1+\sqrt{x^4+1}\right) + C$$

$$\int \frac{4x^3\sqrt{x^4+1}}{1+\sqrt{x^4+1}}\,dx = \dots$$ BY SUBSTITUTION

$$u = \sqrt{x^4+1}$$
$$u^2 = x^4+1$$
$$2u\,du = 4x^3\,dx$$
$$dx = \frac{u}{2x^3}\,du$$

$$= \int \frac{\cancel{4x^3}\,u}{1+u}\left(\frac{u}{\cancel{2x^3}}\,du\right) = \int \frac{2u^2}{u+1}\,du$$

MANIPULATE OR LONG DIVIDE, SO ANOTHER SUBSTITUTION IS NOT USED

$$= \int \frac{2u(u+1)-2(u+1)+2}{u+1}\,du = \int 2u - 2 + \frac{2}{u+1}\,du$$

$$= u^2 - 2u + 2\ln|u+1| + C = (x^4+1) - 2\sqrt{x^4+1} + 2\ln\left(1+\sqrt{x^4+1}\right)$$

$$= x^4 - 2\sqrt{x^4+1} + 2\ln\left(1+\sqrt{x^4+1}\right) + C$$

[THE SUBSTITUTION $u = 1+\sqrt{x^4+1}$]

359. $$\int (\ln x)^2\,dx = 2x + x\left(\ln|x|\right)^2 - 2x\ln|x| + C$$

$$\int (\ln x)^2\,dx = \dots$$ INTEGRATION BY PARTS

$(\ln x)^2$	$\frac{2}{x}\ln x$
x	1

$$= x\left(\ln|x|\right)^2 - \underbrace{\int 2\ln x\,dx}_{\text{BY PARTS}}$$

$\ln x$	$\frac{1}{x}$
$2x$	2

$$= x\left(\ln|x|\right)^2 - \left\{2x\ln|x| - \int 2\,dx\right\}$$

$$= x\left(\ln|x|\right)^2 - 2x\ln|x| + \int 2\,dx$$

$$= x\left(\ln|x|\right)^2 - 2x\ln|x| + 2x + C$$

360. $$\int \sqrt{18\cos x \sin 2x}\ dx = \begin{cases} \dfrac{4}{\sqrt{\sin^3 x}} + C & \text{if } \cos x > 0 \\ -\dfrac{4}{\sqrt{\sin^3 x}} + C & \text{if } \cos x > 0 \end{cases}$$

$$\int \sqrt{18\cos x \sin 2x}\ dx = \int \sqrt{18\cos x(2\sin x\cos x)}\ dx = \int \sqrt{36\sin x\cos^2 x}\ dx$$

$$= \int 6|\cos x|(\sin x)^{\frac{1}{2}}\ dx = \dots \text{ BY REVERSE CHAIN RULE}$$

$$= \begin{cases} 4(\sin x)^{\frac{3}{2}} & \text{IF } \cos x > 0 \\ -4(\sin x)^{\frac{3}{2}} & \text{IF } \cos x < 0 \end{cases} \Rightarrow \begin{cases} \dfrac{4}{\sqrt{\sin^3 x}} + C & \text{IF } \cos x > 0 \\ -\dfrac{4}{\sqrt{\sin^3 x}} + C & \text{IF } \cos x < 0 \end{cases}$$

361. $$\int \frac{1}{\sqrt{x^5 + x^2}}\ dx = \begin{cases} \dfrac{1}{2}\ln\left|\dfrac{\sqrt{x^3+1}-1}{\sqrt{x^3+1}+1}\right| + C & \text{if } x > 0 \\ -\dfrac{1}{2}\ln\left|\dfrac{\sqrt{x^3+1}-1}{\sqrt{x^3+1}+1}\right| + C & \text{if } x < 0 \end{cases}$$

$$\int \frac{1}{\sqrt{x^5+x^2}}\ dx = \int \frac{1}{|x|\sqrt{x^3+1}}\ dx = \dots \text{ SUBSTITUTION}$$

$$u = \sqrt{x^3+1}, \quad u^2 = x^3+1, \quad 2u\,du = 3x^2\,dx, \quad dx = \frac{2u\,du}{3x^2}$$

$$\int \frac{1}{|x|\,u}\,\frac{2u}{3x^2}\,du = \frac{2}{3}\int \frac{1}{|x|x^2}\,du$$

- IF $x \geqslant 0$

$$\dots = \frac{2}{3}\int \frac{1}{x^3}\,du = \frac{2}{3}\int \frac{1}{u^2-1}\,du$$

$$= \frac{2}{3}\int \frac{1}{(u-1)(u+1)}\,du \dots \text{ PARTIAL FRACTIONS BY INSPECTION}$$

$$= \frac{2}{3}\int \frac{\frac{1}{2}}{u-1} - \frac{\frac{1}{2}}{u+1}\,du = \frac{1}{3}\int \frac{1}{u-1} - \frac{1}{u+1}\,du$$

$$= \frac{1}{3}\ln\left|\frac{u-1}{u+1}\right| + C = \frac{1}{2}\ln\left|\frac{\sqrt{x^3+1}-1}{\sqrt{x^3+1}+1}\right| + C, \text{ IF } x \geqslant 0$$

- IF $x \leqslant 0$

$$= -\frac{2}{3}\int \frac{1}{x^3}\,du = \dots -\frac{1}{2}\ln\left|\frac{\sqrt{x^3+1}-1}{\sqrt{x^3+1}+1}\right| + C, \text{ IF } x \leqslant 0$$

362. $$\int \sqrt{\sin^2 x + (\cos x - 1)^2}\ dx = \begin{cases} 4\cos\left(\frac{1}{2}x\right) + C & \text{if } \sin\left(\frac{1}{2}x\right) < 0 \\ -4\cos\left(\frac{1}{2}x\right) + C & \text{if } \sin\left(\frac{1}{2}x\right) > 0 \end{cases}$$

$$\int \sqrt{\sin^2 x + (\cos x - 1)^2}\ dx = \int \sqrt{\sin^2 x + \cos^2 x - 2\cos x + 1}\ dx$$

$$= \int \sqrt{2 - 2\cos x}\ dx = \int \sqrt{2 - 2\left(1 - 2\sin^2 \tfrac{x}{2}\right)}\ dx$$

$$= \int \sqrt{4\sin^2 \tfrac{x}{2}} = \int 2\left|\sin \tfrac{x}{2}\right|\ dx$$

$$\text{IF } \sin \tfrac{x}{2} \geqslant 0 \ \dots = \int 2\sin \tfrac{x}{2}\ dx = -4\cos \tfrac{x}{2} + C$$

$$\text{IF } \sin \tfrac{x}{2} \leqslant 0 \ \dots = \int -2\sin \tfrac{x}{2}\ dx = 4\cos \tfrac{x}{2} + C$$

363. $\displaystyle\int x(\sin x+\cos x)\ dx = \left[\begin{array}{l} x(\sin x-\cos x)+\sin x+\cos x + C \\ (1+x)\sin x+(1-x)\cos x + C \end{array}\right]$

$\int x(\sin x+\cos x)\,dx = \ldots$ INTEGRATION BY PARTS

x	1
$-\cos x+\sin x$	$\sin x+\cos x$

$= x(\cos x+\sin x) - \int -\cos x+\sin x\,dx$

$= x(\sin x-\cos x) + \int \cos x-\sin x\,dx$

$= x(\sin x-\cos x) + \sin x + \cos x + C$

$= (1+x)\sin x + (1-x)\cos x + C$

364. $\displaystyle\int \left(\frac{1}{x^2}+\frac{1}{x^3}\right)e^{\frac{1}{x}}\ dx = -\frac{1}{x}e^{\frac{1}{x}} + C$

$\int \left(\frac{1}{x^2}+\frac{1}{x^3}\right)e^{\frac{1}{x}}\,dx = \ldots$ BY SUBSTITUTION

$u = \frac{1}{x}$

$du = -\frac{1}{x^2}\,dx$

$dx = -x^2\,du$

$dx = -\frac{1}{u^2}\,du$

$= \int (u^2+u^3)e^u\left(-\frac{1}{u^2}\,du\right)$

$= \int -(1+u)e^u\,du$

BY PARTS

$-(1+u)$	-1
e^u	e^u

$= -(1+u)e^u - \int -e^u\,du$

$= -(1+u)e^u + \int e^u\,du$

$= \cancel{-e^u} - ue^u + \cancel{e^u} + C$

$= -\frac{1}{x}e^{\frac{1}{x}} + C$

365. $\displaystyle\int \frac{\sec^4 x}{\sqrt{\tan x}}\ dx = 2(\tan x)^{\frac{1}{2}} + \frac{2}{5}(\tan x)^{\frac{5}{2}} + C$

$\int \frac{\sec^4 x}{\sqrt{\tan x}}\,dx = \int \sec^2 x\sec^2 x(\tan x)^{-\frac{1}{2}}\,dx$

$= \int \sec^2 x(1+\tan^2 x)(\tan x)^{-\frac{1}{2}}\,dx = \int \sec^2 x(\tan x)^{-\frac{1}{2}} + \sec^2 x(\tan x)^{\frac{3}{2}}\,dx$

BY REVERSE CHAIN RULE (INSPECTION)

$= 2(\tan x)^{\frac{1}{2}} + \frac{2}{5}(\tan x)^{\frac{5}{2}} + C$

THE SUBSTITUTION $u=\tan x$ OR $u=\sqrt{\tan x}$ ALSO WORK WELL

366. $\displaystyle\int \frac{\cos x}{(\cos x+\sin x)^3}\,dx = -\frac{1}{2(1+\tan x)^2}+C$

$$\int \frac{\cos x}{(\cos x+\sin x)^3}\,dx = \int \frac{\cos x}{\cos^3 x\left(1+\frac{\sin x}{\cos x}\right)^3}\,dx$$

$$= \int \sec^2 x(1+\tan x)^{-3}\,dx = \dots \text{ BY REVERSE CHAIN RULE (INSPECTION)}$$

$$= -\frac{1}{2}(1+\tan x)^{-2}+C = -\frac{1}{2(1+\tan x)^2}+C$$

ALTERNATIVE BY SUBSTITUTION

$$u=\tan x \qquad \frac{du}{dx}=\sec^2 x \qquad dx=\frac{du}{\sec^2 x} \qquad dx=\cos^2 x\,du$$

$$\int \frac{\cos x}{(\cos x+\sin x)^3}\,dx = \dots$$

$$= \int \frac{\cos x}{(\cos x+\sin x)^3}(\cos^2 x\,du)$$

$$= \int \frac{\cos^3 x}{(\cos x+\sin x)^3}\,du$$

$$= \int \left(\frac{\cos x}{\cos x+\sin x}\right)^3 du$$

$$= \int \left(\frac{\frac{\cos x}{\cos x}}{\frac{\cos x}{\cos x}+\frac{\sin x}{\cos x}}\right)^3 du = \int \left(\frac{1}{1+\tan x}\right)^3 du$$

$$= \int \frac{1}{(1+u)^3}\,du = \int (1+u)^{-3}\,du = -\frac{1}{2}(1+u)^{-2}+C$$

$$= -\frac{1}{(1+\tan x)^2}+C$$

367. $\displaystyle\int \frac{20x}{4-x^2}\,dx = -10\ln\left|4-x^2\right|+C$

$$\int \frac{20x}{4-x^2}\,dx = -10\int \frac{-2x}{4-x^2}\,dx = -10\ln|4-x^2|+C$$

OF THE FORM $\displaystyle\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)|+C$

THE SUBSTITUTION $u=4-x^2$ ALSO WORKS WELL, AS WELL AS PARTIAL FRACTIONS

368. $\displaystyle\int \frac{\sqrt{x}}{1+\sqrt{x}}\,dx = x-2\sqrt{x}+2\ln\left(1+\sqrt{x}\right)+C$

$$\int \frac{\sqrt{x}}{1+\sqrt{x}}\,dx = \dots \text{ BY SUBSTITUTION}$$

$$u=1+\sqrt{x} \qquad \sqrt{x}=u-1 \qquad x=(u-1)^2 \qquad dx=2(u-1)\,du$$

$$= \int \frac{u-1}{u}(2(u-1)\,du) = \int \frac{2(u-1)^2}{u}\,du$$

$$= \int \frac{2u^2-4u+2}{u}\,du = \int 2u-4+\frac{2}{u}\,du$$

$$= u^2-4u+2\ln|u|+C$$

$$= (1+\sqrt{x})^2-4(1+\sqrt{x})+2\ln(1+\sqrt{x})+C$$

$$= 1+2\sqrt{x}+x-4-4\sqrt{x}+2\ln(1+\sqrt{x})+C$$

$$= x-2\sqrt{x}+2\ln(1+\sqrt{x})+C$$

369. $$\int x\left(\sec^2 x - \operatorname{cosec}^2 x\right)\,dx = x\left(\tan x + \cot x\right) - \ln\left|\tan x\right| + C$$

$$\int x(\sec^2 x - \operatorname{cosec}^2 x)\,dx = \ldots$$ INTEGRATION BY PARTS

$$= x(\tan x + \cot x) - \int \tan x + \cot x \,dx \qquad \begin{array}{c|c} x & 1 \\ \hline \tan x + \cot x & \sec^2 x - \operatorname{cosec}^2 x \end{array}$$

STANDARD RESULT

$$= x(\tan x + \cot x) - \left[\ln|\sec x| + \ln|\sin x|\right] + C$$

$$= x(\tan x + \cot x) - \ln\left|\frac{\sin x}{\cos x}\right| + C$$

$$= x(\tan x + \cot x) - \ln|\tan x| + C$$

370. $$\int \frac{4x^2+4x}{\sqrt{2x+1}}\,dx = \frac{1}{5}(2x+1)^{\frac{5}{2}} - (2x+1)^{\frac{1}{2}} + C$$

$$\int \frac{4x^2+4x}{\sqrt{2x+1}}\,dx = \ldots$$ BY MANIPULATION

$$= \int \frac{(4x^2+4x+1)-1}{(2x+1)^{\frac{1}{2}}}\,dx = \int \frac{(2x+1)^2-1}{(2x+1)^{\frac{1}{2}}}\,dx$$

$$= \int (2x+1)^{\frac{3}{2}} - (2x+1)^{-\frac{1}{2}}\,dx = \frac{1}{5}(2x+1)^{\frac{5}{2}} - (2x+1)^{\frac{1}{2}} + C$$

THE SUBSTITUTION $u = 2x+1$ OR $u = \sqrt{2x+1}$ ALSO WORK WELL

371. $$\int x\tan^2 x\,dx = -\frac{1}{2}x^2 + x\tan x + \ln\left|\cos x\right| + C$$

$$\int x\tan^2 x\,dx = \int x(\sec^2 x - 1)\,dx = \int x\sec^2 x - x\,dx$$

INTEGRATION BY PARTS FOR THE FIRST TERM

$$= x\tan x - \int \tan x\,dx - \frac{1}{2}x^2 \qquad \begin{array}{c|c} x & 1 \\ \hline \tan x & \sec^2 x \end{array}$$

$$= -\frac{1}{2}x^2 + x\tan x - \ln|\sec x| + C$$

$$= -\frac{1}{2}x^2 + x\tan x + \ln|\cos x| + C$$

372. $$\int x\cos^2 3x\,dx = \frac{1}{4}x^2 + \frac{1}{12}x\sin 6x + \frac{1}{72}\cos 6x + C$$

$$\int x\cos^2 3x\,dx = \int x\left(\tfrac{1}{2} + \tfrac{1}{2}\cos 6x\right)dx$$

$$= \int \tfrac{1}{2}x + \tfrac{1}{2}x\cos 6x\,dx = \tfrac{1}{4}x^2 + \int \tfrac{1}{2}x\cos 6x\,dx$$

INTEGRATION BY PARTS

$$= \tfrac{1}{4}x^2 + \tfrac{1}{12}x\sin 6x - \int \tfrac{1}{12}\sin 6x\,dx \qquad \begin{array}{c|c} \frac{1}{2}x & \frac{1}{2} \\ \hline \frac{1}{6}\sin 6x & \cos 6x \end{array}$$

$$= \tfrac{1}{4}x^2 + \tfrac{1}{12}x\sin 6x + \tfrac{1}{72}\cos 6x + C$$

373. $\displaystyle\int \cos x(6\sin x - 2\sin 3x)^{\frac{2}{3}}\ dx = \tfrac{4}{3}\sin^3 x + C$

$$\int \cos x\,(6\sin x - 2\sin 3x)^{\frac{2}{3}}\,dx \ \dots$$ BY TRIG IDENTITIES ...

$$\boxed{\begin{aligned}\sin 3x = \sin(2x+x) &= \sin 2x\cos x + \cos 2x\sin x\\ &= (2\sin x\cos x)\cos x + (1-2\sin^2 x)\sin x\\ &= 2\sin x\cos^2 x + \sin x - 2\sin^3 x\\ &= 2\sin x(1-\sin^2 x) + \sin x - 2\sin^3 x\\ &= 2\sin x - 2\sin^3 x + \sin x - 2\sin^3 x\\ &= 3\sin x - 4\sin^3 x\end{aligned}}$$

$$= \int \cos x\left[\cancel{6\sin x} - 2(\cancel{3\sin x} - 4\sin^3 x)\right]^{\frac{2}{3}} dx$$

$$= \int \cos x\left[8\sin^3 x\right]^{\frac{2}{3}} dx = \int \cos x\left[4\sin^2 x\right] dx$$

$$= \int 4\cos x\sin^2 x\,dx = \underline{\tfrac{4}{3}\sin^3 x + C}$$

374. $\displaystyle\int \sqrt{x^2 - x^4}\ dx = \begin{cases} -\frac{1}{3}\left(1-x^2\right)^{\frac{3}{2}} + C & x > 0 \\ \frac{1}{3}\left(1-x^2\right)^{\frac{3}{2}} + C & x < 0 \end{cases}$

$$\int \sqrt{x^2 - x^4}\,dx = \int \sqrt{x^2(1-x^2)}\,dx = \int \sqrt{x^2}\,(1-x^2)^{\frac{1}{2}}\,dx$$

$$= \int |x|(1-x^2)^{\frac{1}{2}} = \begin{cases} \underline{-\frac{1}{3}(1-x^2)^{\frac{3}{2}} + C \quad \text{IF } x > 0} \\ \underline{\frac{1}{3}(1-x^2)^{\frac{3}{2}} + C \quad \text{IF } x < 0} \end{cases}$$

THE SUBSTITUTION $u = 1 - x^2$ OR $u = \sqrt{1-x^2}$ MAY ALSO BE USED

375. $\displaystyle\int \frac{x+3}{\sqrt[3]{x^2+6x}}\ dx = \tfrac{3}{4}\left(x^2+6x\right)^{\frac{2}{3}} + C$

$$\int \frac{x+3}{\sqrt[3]{x^2+6x}}\,dx = \int (x+3)(x^2+6x)^{-\frac{1}{3}}\,dx = \frac{1}{2}\int (2x+6)(x^2+6x)^{-\frac{1}{3}}\,dx$$

$= \dots$ BY REVERSE CHAIN RULE (INSPECTION)

$$= \underline{\frac{1}{2}\times\frac{3}{2}(x^2+6x)^{\frac{2}{3}} + C = \frac{3}{4}(x^2+6x)^{\frac{2}{3}} + C}$$

THE SUBSTITUTION $u = (x^2+6x)^{\frac{1}{3}}$ ALSO WORKS

376. $$\int \frac{1-x}{1-\sqrt{x}}\,dx = \left[\begin{array}{c} -\frac{2}{3}\left(1-\sqrt{x}\right)^3+3\left(1-\sqrt{x}\right)^3-4\left(1-\sqrt{x}\right)+C \\ \frac{2}{3}x^{\frac{3}{2}}+x+C \end{array}\right]$$

$\int \frac{1-x}{1-\sqrt{x}}\,dx = \dots$ BY MANIPULATION (DIFFERENCE OF SQUARES)

$= \int \frac{\cancel{(1-\sqrt{x})}(1+\sqrt{x})}{\cancel{1-\sqrt{x}}}\,dx = \int 1+x^{\frac{1}{2}}\,dx = x+\frac{2}{3}x^{\frac{3}{2}}+C$

ALTERNATIVE BY SUBSTITUTION (LENGTHY)

$\int \frac{1-x}{1-\sqrt{x}}\,dx = \int \frac{1-(1-u)^2}{u}\left[-2(1-u)\,du\right]$

$u = 1-\sqrt{x}$
$\sqrt{x} = 1-u$
$x = (1-u)^2$
$dx = -2(1-u)\,du$

$= \int \frac{\cancel{1}-\cancel{(1}-2u+u^2)}{u}\left[2(u-1)\,du\right]$

$= \int \frac{2(u-1)(2u-u^2)}{u}\,du = \int 2(u-1)(2-u)\,du$

$= -\int 2(u-1)(u-2)\,du = \int -2u^2+6u-4\,du$

$= -\frac{2}{3}u^3+3u^2-4u+C = -\frac{2}{3}\left[1-\sqrt{x}\right]^3+3(1-\sqrt{x})^2-4(1-\sqrt{x})+C$

$= -\frac{2}{3}\left[\cancel{1}-3\sqrt{x}+3x-x^{\frac{3}{2}}\right]+3\left[\cancel{1}-2\sqrt{x}+x\right]\cancel{-4}+4\sqrt{x}+C$

$= \frac{2}{3}x^{\frac{3}{2}}-2x+\cancel{2\sqrt{x}}-\cancel{6\sqrt{x}}+3x+\cancel{4\sqrt{x}}+C$

$= \frac{2}{3}x^{\frac{3}{2}}+x+C$ (AS BEFORE)

377. $$\int \frac{1}{\operatorname{cosec}2x-\cot 2x}\,dx = \ln\left|\sin x\right|+C$$

$\int \frac{1}{\operatorname{cosec}2x-\cot 2x}\,dx = \int \frac{1\sin 2x}{\operatorname{cosec}2x\sin 2x-\cot 2x\sin 2x}\,dx$

$= \int \frac{\sin 2x}{1-\cos 2x}\,dx = \int \frac{2\sin x\cos x}{1-(1-2\sin^2 x)}\,dx = \int \frac{2\sin x\cos x}{2\sin^2 x}\,dx$

$= \int \frac{\cos x}{\sin x}\,dx = \dots$ of the type $\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)|+C$

$= \ln|\sin x|+C$

378. $$\int \sec^3 x\tan^5 x\;dx = \frac{1}{7}\sec^7 x-\frac{2}{5}\sec^5 x+\frac{1}{3}\sec^3 x+C$$

$\int \sec^3 x\tan^5 x\,dx = \dots$ BY MANIPULATION ...

$= \int \sec^3 x\tan^4 x\tan x\,dx = \int \sec^3 x(\sec^2 x-1)^2\tan x\,dx$

$= \int \sec^3 x(\sec^4 x-2\sec^2 x+1)\tan x\,dx = \int(\sec^7 x-2\sec^5 x+\sec^3 x)\tan x\,dx$

$= \int(\sec^6 x-2\sec^4 x+\sec^2 x)\sec x\tan x$

BY REVERSE CHAIN RULE AS $\frac{d}{dx}(\sec x) = \sec x\tan x$

$= \frac{1}{7}\sec^7 x-\frac{2}{5}\sec^5 x+\frac{1}{3}\sec^3 x+C$

ALTERNATIVE BY SUBSTITUTION

$\int \sec^3 x\tan^5 x\,dx = \int \sec^3 x\tan^5 x\frac{du}{\sec x\tan x}$

$u = \sec x$
$\frac{du}{dx} = \sec x\tan x$
$dx = \frac{du}{\sec x\tan x}$

$= \int \sec^2 x\tan^4 x\,du = \int \sec^2 x(\sec^2 x-1)^2\,du$

$= \int u^2(u^2-1)^2\,du = \int u^6-2u^4+u^2\,du$

$= \frac{1}{7}u^7-\frac{2}{5}u^5+\frac{1}{3}u^3+C = \frac{1}{7}\sec^7 x-\frac{2}{5}\sec^5 x+\frac{1}{3}\sec^3 x+C$

THE SUBSTITUTION $u = \cos x$ ALSO WORKS WELL

379. $$\int \frac{\cosh x}{(\cosh x+\sinh x)^3}\,dx = \left[\begin{array}{l} -\dfrac{1}{2(1+\tanh x)^2}+C \\ -\dfrac{1}{4}e^{-2x}-\dfrac{1}{8}e^{-4x}+C \end{array}\right]$$

$$\int \frac{\cosh x}{(\cosh x+\sinh x)^3}\,dx = \ldots \text{ BY MANIPULATION } = \int \frac{\cosh x}{\cosh^3 x\left(1+\frac{\sinh x}{\cosh x}\right)^3}\,dx$$

$$= \int \frac{1}{\cosh^2 x\,(1+\tanh x)^3}\,dx = \int \operatorname{sech}^2 x\,(1+\tanh x)^{-3}\,dx$$

BY REVERSE CHAIN RULE (INSPECTION)

$$= -\frac{1}{2}(1+\tanh x)^{-2}+C = -\frac{1}{2(1+\tanh x)^2}+C$$

ALTERNATIVE BY EXPONENTIALS

$$\int \frac{\cosh x}{(\cosh x+\sinh x)^3}\,dx = \int \frac{\frac{1}{2}e^{x}+\frac{1}{2}e^{-x}}{\left(\frac{1}{2}e^{x}+\cancel{\frac{1}{2}e^{-x}}+\frac{1}{2}e^{x}-\cancel{\frac{1}{2}e^{-x}}\right)^3}\,dx \quad \int \frac{\frac{1}{2}e^{x}+\frac{1}{2}e^{-x}}{(e^{x})^3}\,dx$$

$$= \int \frac{\frac{1}{2}e^{x}+\frac{1}{2}e^{-x}}{e^{3x}}\,dx = \int \frac{1}{2}e^{-2x}+\frac{1}{2}e^{-4x}\,dx = -\frac{1}{4}e^{-2x}-\frac{1}{8}e^{-4x}+C$$

380. $$\int \cos x\cos^2 3x\,dx = \frac{1}{2}\sin x+\frac{1}{28}\sin 7x+\frac{1}{20}\sin 5x+C$$

$$\int \cos x\cos^2 3x\,dx = \ldots \text{ COSINE DOUBLE ANGLE } \ldots = \int \cos x\left(\frac{1}{2}+\frac{1}{2}\cos 6x\right)dx$$

$$= \int \frac{1}{2}\cos x+\frac{1}{2}\cos x\cos 6x\,dx = \ldots \text{ PROCEED AS BELOW}$$

$$\boxed{\begin{array}{l} \cos(6x+x) = \cos 7x = \cos 6x\cos x-\sin 6x\sin x \\ \cos(6x-x) = \cos 5x = \cos 6x\cos x+\sin 6x\sin x \qquad \text{ADDING} \\ \Rightarrow \cos 7x+\cos 5x = 2\cos 6x\cos x \\ \Rightarrow \frac{1}{2}\cos 6x\cos x = \frac{1}{4}\cos 7x+\frac{1}{4}\cos 5x \end{array}}$$

$$= \ldots \int \frac{1}{2}\cos x+\frac{1}{4}\cos 7x+\frac{1}{4}\cos 5x\,dx$$

$$= \frac{1}{2}\sin x+\frac{1}{28}\sin 7x+\frac{1}{20}\sin 5x+C$$

381. $$\int \frac{1}{\sqrt{x^{\frac{3}{2}}+4x}}\,dx = 4\sqrt{4+\sqrt{x}}+C$$

$$\int \frac{1}{\sqrt{x^{\frac{3}{2}}+4x}}\,dx = \text{ BY MANIPULATION}$$

$$= \int \frac{1}{\sqrt{x\left(x^{\frac{1}{2}}+4\right)}}\,dx = \int \frac{1}{\sqrt{x}\,\sqrt{x^{\frac{1}{2}}+4}}\,dx = \int x^{-\frac{1}{2}}\left(x^{\frac{1}{2}}+4\right)^{-\frac{1}{2}}dx$$

BY REVERSE CHAIN RULE (INSPECTION)

$$\frac{d}{dx}\left[\left(x^{\frac{1}{2}}+4\right)^{\frac{1}{2}}\right] = \frac{1}{2}\left(x^{\frac{1}{2}}+4\right)^{-\frac{1}{2}}\times\frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{4}x^{-\frac{1}{2}}\left(x^{\frac{1}{2}}+4\right)^{-\frac{1}{2}}$$

$$= 4\left(x^{\frac{1}{2}}+4\right)^{\frac{1}{2}}+C = 4\sqrt{\sqrt{x}+4}+C$$

ALTERNATIVE BY SUBSTITUTION

$$\int \frac{1}{\sqrt{x^{\frac{3}{2}}+4x}}\,dx = \int \frac{1}{\sqrt{u^3+4u^2}}\,(2u\,du) \qquad u = x^{\frac{1}{2}}$$

$$u^2 = x$$

$$= \int \frac{2u}{\sqrt{u^2}\,\sqrt{u+4}}\,du = \int \frac{2u}{+u\,(u+4)^{\frac{1}{2}}}\,du \qquad 2u\,du = dx$$

$$\sqrt{u^2} = |u| = \left|x^{\frac{1}{2}}\right| = x^{\frac{1}{2}}$$

$$= \int 2(u+4)^{-\frac{1}{2}}\,du = 4(u+4)^{\frac{1}{2}}+C = 4\left(x^{\frac{1}{2}}+4\right)^{\frac{1}{2}}+C$$

382. $$\int \frac{2}{(\cos x + 2\sin x)^2}\,dx = -\frac{1}{1+2\tan x} + C$$

$$\int \frac{2}{(\cos x + 2\sin x)^2}\,dx \quad \text{... MANIPULATE} \quad = \int \frac{2}{\cos^2 x\left(1+\frac{2\sin x}{\cos x}\right)^2}\,dx$$

$$= \int \frac{2}{\cos^2 x(1+2\tan x)^2}\,dx = \int 2(1+2\tan x)^{-2}\sec^2 x\,dx$$

BY REVERSE CHAIN RULE (INSPECTION) OR SUBSTITUTION

$$= -(1+2\tan x)^{-1} + C \quad = -\frac{1}{1+2\tan x} + C$$

383. $$\int \ln\left(\tfrac{1}{6}x+2\right)\,dx = (x+12)\ln|x+12| - (1+\ln 6)x + C$$

$$\int \ln\left(\tfrac{1}{6}x+2\right) = \text{... BY PARTS, FOLLOWED BY MANIPULATION/SUBSTITUTION}$$

$\ln(\frac{1}{6}x+2)$	$\frac{1}{6}\times\frac{1}{\frac{1}{6}x+2}$ $\frac{1}{x+12}$
x	1

$$= x\ln\left|\tfrac{1}{6}x+2\right| - \int \frac{x}{x+12}\,dx$$

$$= x\ln\left|\tfrac{1}{6}(x+12)\right| - \int \frac{(x+12)-12}{(x+12)}\,dx$$

$$= x\ln\tfrac{1}{6} + x\ln|x+12| - \int 1 - \frac{12}{x+12}\,dx$$

$$= -x\ln 6 + x\ln|x+12| - \left[x - 12\ln|x+12|\right] + C$$

$$= -x\ln 6 - x + x\ln|x+12| + 12\ln|x+12| + C$$

$$= (x+12)\ln|x+12| - (1+\ln 6)x + C$$

ALTERNATIVE BY SUBSTITUTION FIRST, FOLLOWED BY PARTS

$$\int \ln\left(\tfrac{1}{6}x+2\right)dx = \int (\ln u)(6\,du)$$

$$= \int 6\ln u\,du$$

$u = \frac{1}{6}x+2$

$\frac{du}{dx} = \frac{1}{6}$

$dx = 6\,du$

$\ln u$	$\frac{1}{u}$
$6u$	6

$$\dots = 6u\ln|u| - \int 6\,du$$

$$= 6u\ln|u| - 6u + C \quad = 6\left(\tfrac{1}{6}x+2\right)\ln\left|\tfrac{1}{6}x+2\right| - 6\left(\tfrac{1}{6}x+2\right) + C$$

$$= (x+12)\ln\left|\tfrac{1}{6}(x+12)\right| - (x+12) + C$$

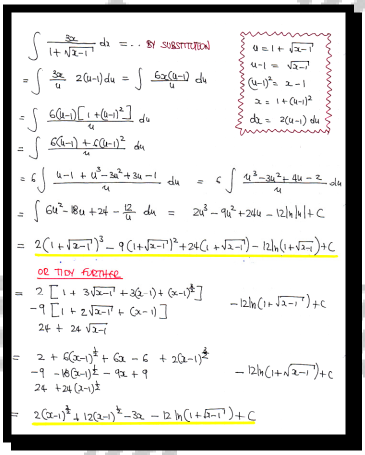
$$= (x+12)\left[\ln\tfrac{1}{6} + \ln|x+12|\right] - (x+12) + C$$

$$= (x+12)\ln\tfrac{1}{6} + (x+12)\ln|x+12| - (x+12) + C$$

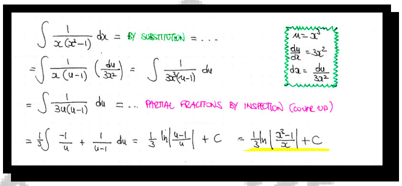
$$= -x\ln 6 - 12\ln 6 + (x+12)\ln|x+12| - x - 12 + C$$

$$= (x+12)\ln|x+12| - (1+\ln 6)x + C$$

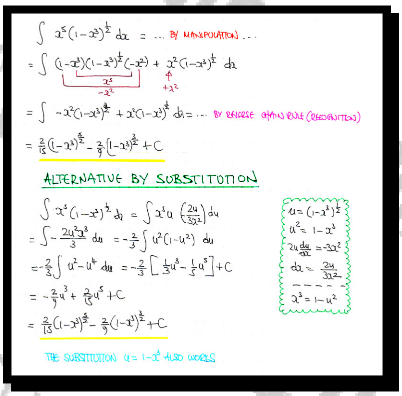
384. $$\int \frac{3x}{1+\sqrt{x-1}}\,dx = 2\left(x-1\right)^{\frac{3}{2}} + 12\left(x-1\right)^{\frac{1}{2}} - 3x - 12\ln\left(1+\sqrt{x-1}\right) + C$$



385. $$\int \frac{1}{x\left(x^3-1\right)}\,dx = \frac{1}{3}\ln\left|\frac{x^3-1}{x}\right| + C$$



386. $$\int x^5\left(1-x^3\right)^{\frac{1}{2}}\,dx = \frac{2}{15}\left(1-x^3\right)^{\frac{5}{2}} - \frac{2}{9}\left(1-x^3\right)^{\frac{3}{2}} + C$$



387. $$\int \frac{1}{x^{\frac{1}{2}} - x^{\frac{1}{4}}}\,dx = 2x^{\frac{1}{2}} + 4x^{\frac{1}{4}} + 4\ln\left|x^{\frac{1}{4}} - 1\right| + C$$

$$\int \frac{1}{x^{\frac{1}{2}} - x^{\frac{1}{4}}}\,dx = \dots$$ BY SUBSTITUTION

$u = x^{\frac{1}{4}}$
$u^4 = x$
$x = u^4$
$dx = 4u^3\,du$

$$= \int \frac{1}{u^2 - u}\left(4u^3\,du\right) = \int \frac{4u^3}{u(u-1)}\,du$$

$$= \int \frac{4u^2}{u-1}\,du$$ ANOTHER SUBSTITUTION, DIVISION OR MANIPULATION

$$= 4\int \frac{u(u-1) + (u-1) + 1}{(u-1)}\,du = 4\int u + 1 + \frac{1}{u-1}\,du$$

$$= 4\left[\tfrac{1}{2}u^2 + u + \ln|u-1|\right] + C = 2u^2 + 4u + 4\ln|u-1| + C$$

$$= 2x^{\frac{1}{2}} + 4x^{\frac{1}{4}} + 4\ln\left|x^{\frac{1}{4}} - 1\right| + C$$

388. $$\int \frac{1}{2x^2 - 2x + 1}\,dx = \arctan(2x-1) + C$$

$$\int \frac{1}{2x^2 - 2x + 1}\,dx = \int \frac{2}{4x^2 - 4x + 2}\,dx = \int \frac{2}{(2x-1)^2 + 1}\,dx$$

BY REVERSE CHAIN RULE OR SUBSTITUTION $u = 2x-1$

$$= \arctan(2x-1) + C$$

389. $$\int \frac{1}{\sqrt{-x^2 - 2x}}\,dx = \arcsin(x-1) + C$$

$$\int \frac{1}{\sqrt{-x^2 - 2x}}\,dx = \int \frac{1}{\sqrt{-(x^2+2x)}}\,dx = \int \frac{1}{\sqrt{1-(x^2+2x+1)}}$$

$$= \int \frac{1}{\sqrt{1-(x+1)^2}}\,dx$$

BY REVERSE CHAIN RULE OR SUBSTITUTION $u = x+1$

$$= \arcsin(x-1) + C$$

390. $$\int \frac{1}{\sqrt{9x^2 + 12x + 5}}\,dx = \frac{1}{3}\operatorname{arsinh}(3x+2) + C$$

$$\int \frac{1}{\sqrt{9x^2 + 12x + 5}}\,dx = \int \frac{1}{\sqrt{(3x+2)^2 + 1}}\,dx =$$

COMPLETE THE SQUARE BY INSPECTION

BY REVERSE CHAIN RULE (INSPECTION) OR SUBSTITUTION $u = 3x+2$

$$= \frac{1}{3}\operatorname{arsinh}(3x+2) + C$$

391. $\displaystyle\int \frac{1}{\sqrt{16x^2-8x}}\,dx = \frac{1}{4}\operatorname{arcosh}(4x-1) + C$

$\displaystyle\int \frac{1}{\sqrt{16x^2-8x}}\,dx = \text{... COMPLETING THE SQUARE ...}$

$\displaystyle= \int \frac{1}{\sqrt{(16x^2-8x+1)-1}}\,dx = \int \frac{1}{\sqrt{(4x-1)^2-1}}\,dx$

BY REVERSE CHAIN RULE (INSPECTION) OR SUBSTITUTION $u=4x-1$

$\displaystyle= \frac{1}{4}\operatorname{arcosh}(4x-1) + C$

392. $\displaystyle\int \sqrt{\frac{x}{1-x}}\,dx = \arcsin\sqrt{x} - \sqrt{x-x^2} + C$

$\displaystyle\int \sqrt{\frac{x}{1-x}}\,dx = \int \frac{\sqrt{x}}{\sqrt{1-x}}\,dx = \text{... SUBSTITUTION}$

$\displaystyle ... = \int \frac{\sin\theta}{\cancel{\sqrt{1-\sin^2\theta}}}(2\sin\theta\cancel{\cos\theta}\,d\theta)$

$\displaystyle= \int 2\sin^2\theta\,d\theta = \int 2\left(\tfrac{1}{2}-\tfrac{1}{2}\cos 2\theta\right)d\theta$

$\displaystyle= \int 1-\cos 2\theta\,d\theta = \theta - \tfrac{1}{2}\sin 2\theta + C$

$\displaystyle= \theta - \cancel{\tfrac{1}{2}}(\cancel{2}\sin\theta\cos\theta) + C$

$= \arcsin\sqrt{x} - \sqrt{x}\sqrt{1-x} + C$

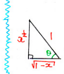
$= \arcsin\sqrt{x} - \sqrt{x-x^2} + C$

$x = \sin^2\theta$

$x^{\frac{1}{2}} = \sin\theta$

$\theta = \arcsin\sqrt{x}$

$dx = 2\sin\theta\cos\theta\,d\theta$



$\sin\theta = x^{\frac{1}{2}}$

$\cos\theta = \sqrt{1-x}$

393. $\displaystyle\int \frac{4x^2}{4x^2+1}\,dx = x - \frac{1}{2}\arctan 2x + C$

$\displaystyle\int \frac{4x^2}{4x^2+1}\,dx \text{ ... MANIPULATION ... } = \int \frac{(4x^2+1)-1}{(4x^2+1)}\,dx$

$\displaystyle= \int 1 - \frac{1}{4x^2+1}\,dx = \int 1 - \frac{1}{(2x)^2+1}\,dx$

BY REVERSE CHAIN RULE (INSPECTION)

$\displaystyle= x - \frac{1}{2}\arctan 2x + C$

394. $\displaystyle\int e^{\cos x}\sin x\cos x\,dx = (1-\cos x)e^{\cos x} + C$

$\displaystyle\int e^{\cos x}\sin x\cos x\,dx = \text{... INTEGRATION BY PARTS}$

$\displaystyle= (-\cos x)e^{\cos x} - \int(\sin x)e^{\cos x}\,dx$

$\displaystyle= -e^{\cos x}\cos x - \int e^{\cos x}\sin x\,dx$

REVERSE CHAIN RULE (INSPECTION)

$= -e^{\cos x}\cos x + e^{\cos x} + C$

$= e^{\cos x}(1-\cos x) + C$

$\cos x$	$-\sin x$
$-e^{\cos x}$	$e^{\cos x}\sin x$

395. $\displaystyle\int \left(9-4\sin^2 x\right)^{\frac{1}{2}} \sin x \, dx = -\frac{9}{4}\arcsin\left(\frac{2}{3}\cos x\right) - \frac{1}{2}\left(9-4\sin^2 x\right)^{\frac{1}{2}} \cos x + C$

$$\int (9-4\cos^2 x)^{\frac{1}{2}}\sin x\,dx = \int 9^{\frac{1}{2}}\left(1-\tfrac{4}{9}\cos^2 x\right)^{\frac{1}{2}}\sin x\,dx$$
$$= \int 3\left(1-\left(\tfrac{2}{3}\cos x\right)^2\right)^{\frac{1}{2}}\sin x\,dx = \dots$$

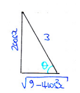
BY SUBSTITUTION

$$= \int 3\left[1-\sin^2\theta\right]^{\frac{1}{2}}\cancel{\sin x}\left[-\frac{3\cos\theta}{2\cancel{\sin x}}\,d\theta\right]$$
$$= \int -\tfrac{9}{2}\cos^2\theta\,d\theta$$
$$= \int -\tfrac{9}{2}\left(\tfrac{1}{2}+\tfrac{1}{2}\cos 2\theta\right)d\theta$$
$$= \int -\tfrac{9}{4}-\tfrac{9}{4}\cos 2\theta\,d\theta$$
$$= -\tfrac{9}{4}\theta-\tfrac{9}{8}\sin 2\theta + C$$
$$= -\tfrac{9}{4}\theta-\tfrac{9}{4}\sin\theta\cos\theta + C$$
$$= -\tfrac{9}{4}\arcsin\left(\tfrac{2}{3}\cos x\right)-\tfrac{9}{4}\left(\tfrac{2}{3}\cos x\right)\times\tfrac{1}{3}\sqrt{9-4\cos^2\theta}$$
$$= \underline{-\tfrac{9}{4}\arcsin\left(\tfrac{2}{3}\cos x\right)-\tfrac{1}{2}\cos x\sqrt{9-4\cos^2 x}+C}$$

$\sin\theta = \frac{2}{3}\cos x$

$\cos\theta\,d\theta = -\frac{2}{3}\sin x\,dx$

$dx = -\frac{3\cos\theta}{2\sin x}\,d\theta$



$\theta = \arcsin\left(\frac{2}{3}\cos x\right)$

$\cos\theta = \frac{1}{3}\sqrt{9-4\cos^2\theta}$

396. $\displaystyle\int \frac{2}{x+\sqrt{1-x^2}}\,dx = \arcsin x + \ln\left(x+\sqrt{1-x^2}\right) + C$

$$\int \frac{2}{x+\sqrt{1-x^2}}\,dx = \dots$$

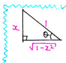
SUBSTITUTION ...

$x = \sin\theta$

$\theta = \arcsin x$

$dx = \cos\theta\,d\theta$

$$= \int \frac{2}{\sin\theta+\sqrt{1-\sin^2\theta}}\,(\cos\theta\,d\theta)$$
$$= \int \frac{2\cos\theta}{\sin\theta+\cos\theta}\,d\theta = \int \frac{(\sin\theta+\cos\theta)+(\cos\theta-\sin\theta)}{(\sin\theta+\cos\theta)}\,d\theta$$
$$= \int 1+\underbrace{\frac{\cos\theta-\sin\theta}{\sin\theta+\cos\theta}}_{\int\frac{f'(x)}{f(x)}dx = \ln|f(x)|+C}\,d\theta = \theta+\ln|\sin\theta+\cos\theta|+C$$
$$= \underline{\arcsin x+\ln\left|x+\sqrt{1-x^2}\right|+C}$$



$\sin\theta = \frac{x}{1} = x$

$\cos\theta = \frac{\sqrt{1-x^2}}{1}$

VARIATION...

$$\dots = \int \frac{2\cos\theta}{\sin\theta+\cos\theta}\,d\theta$$
$$= \int \frac{2\cos\theta(\cos\theta-\sin\theta)}{(\sin\theta+\cos\theta)(-\sin\theta+\cos\theta)}\,d\theta = \int \frac{2\cos^2\theta-2\cos\theta\sin\theta}{\cos^2\theta-\sin^2\theta}\,d\theta$$
$$= \int \frac{(1+\cos 2\theta)-\sin 2\theta}{\cos 2\theta}\,d\theta = \int \sec 2\theta+1-\tan 2\theta\,d\theta$$
$$= \theta+\tfrac{1}{2}\ln|\sec 2\theta+\tan 2\theta|-\tfrac{1}{2}\ln|\sec 2\theta|+C$$
$$= \theta+\tfrac{1}{2}\ln\left|\frac{\sec 2\theta+\tan 2\theta}{\sec 2\theta}\right|+C$$
$$= \theta+\tfrac{1}{2}\ln|1+\sin 2\theta|+C = \theta+\tfrac{1}{2}\ln|1+2\sin\theta\cos\theta|+C$$
$$= \theta+\tfrac{1}{2}\ln|\cos^2\theta+\sin^2\theta+2\sin\theta\cos\theta| = \theta+\tfrac{1}{2}\ln|(\cos\theta+\sin\theta)^2|+C$$
$$= \theta+\ln|\cos\theta+\sin\theta|+C \dots$$
$$= \underline{\arcsin x+\ln\left|x+\sqrt{1-x^2}\right|+C}$$

AS BEFORE

397. $\displaystyle\int \frac{1}{\sqrt{x+1}+\sqrt{x-1}}\,dx = \frac{1}{3}(x+1)^{\frac{3}{2}} - \frac{1}{3}(x-1)^{\frac{3}{2}} + C$

$$\int \frac{1}{\sqrt{x+1}+\sqrt{x-1}}\,dx = \dots$$

BY MANIPULATION

$$= \int \frac{1\left(\sqrt{x+1}-\sqrt{x-1}\right)}{\left(\sqrt{x+1}+\sqrt{x-1}\right)\left(\sqrt{x+1}-\sqrt{x-1}\right)}\,dx = \int \frac{\sqrt{x+1}-\sqrt{x-1}}{(\cancel{x}+1)-(\cancel{x}-1)}\,dx$$
$$= \int \tfrac{1}{2}(x+1)^{\frac{1}{2}}-\tfrac{1}{2}(x-1)^{\frac{1}{2}}\,dx = \underline{\tfrac{1}{3}(x+1)^{\frac{3}{2}}-\tfrac{1}{3}(x-1)^{\frac{3}{2}}+C}$$

398. $\displaystyle\int \cosh 2x \, dx = \frac{1}{2}\sin 2x + C$

$\int \cosh 2x \, dx = \dots$ BY INSPECTION $\dots \frac{1}{2}\sinh 2x + C$

399. $\displaystyle\int \mathrm{e}^{x}\left(\tan x + \sec^{2} x\right) dx = \mathrm{e}^{x}\tan x + C$

$\int e^{x}(\tan x + \sec^2 x) = \dots$ BY INSPECTION $\dots e^{x}\tan x + C$

$\frac{d}{dx}(e^{x}\tan x) = e^{x}\tan x + e^{x}(\sec^2 x)$

$= e^{x}(\tan x + \sec^2 x)$

400. $\displaystyle\int \frac{\sqrt{x+1}}{x+5}\, dx = 2\sqrt{x+1} - 4\arctan\left(\frac{\sqrt{x+1}}{2}\right) + C$

$\int \frac{\sqrt{x+1}}{x+5}\, dx = \dots$ BY SUBSTITUTION $\dots$

$u = \sqrt{x+1}$
$u^2 = x+1$
$x = u^2 - 1$
$dx = 2u\, du$
$x+5 = u^2+4$

$= \int \frac{u}{u^2+4}(2u\, du) = \int \frac{2u^2}{u^2+4}\, du$

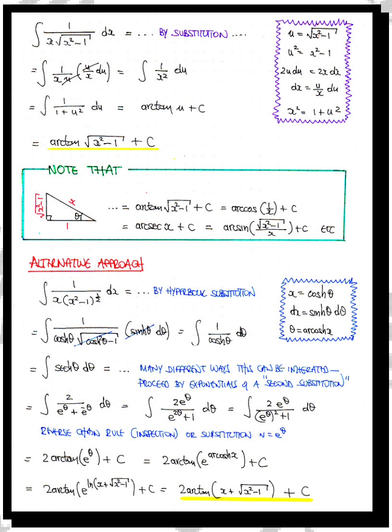
BY MANIPULATION OR SUBSTITUTION $u = 2\tan\theta$

$= \int \frac{2(u^2+4) - 8}{u^2+4}\, du = \int 2 - \frac{8}{u^2+4}\, du$

$= 2u - 8 \times \frac{1}{2}\arctan\left(\frac{u}{2}\right) + C$

$= 2\sqrt{x+1} - 4\arctan\left(\frac{\sqrt{x+1}}{2}\right) + C$

401. $$\int \frac{1}{x\sqrt{x^2-1}}\,dx = \begin{bmatrix} \arctan\sqrt{x^2-1} + C \\ \arcsin\left(\dfrac{\sqrt{x^2-1}}{x}\right) + C \\ \arccos\left(\dfrac{1}{x}\right) + C \\ \operatorname{arcsec} x + C \\ 2\arctan\left(x+\sqrt{x^2-1}\right) + C \end{bmatrix}$$



$\int \frac{1}{x\sqrt{x^2-1}}\,dx = \dots$ BY SUBSTITUTION ....

$u = \sqrt{x^2-1}$, $u^2 = x^2-1$, $2u\,du = 2x\,dx$, $dx = \frac{u}{x}\,du$, $x^2 = 1+u^2$

$= \int \frac{1}{x\cancel{u}}\left(\frac{\cancel{u}}{x}\,du\right) = \int \frac{1}{x^2}\,du$

$= \int \frac{1}{1+u^2}\,du = \arctan u + C$

$= \arctan\sqrt{x^2-1} + C$

NOTE THAT

$\dots = \arctan\sqrt{x^2-1} + C = \arccos\left(\frac{1}{x}\right) + C$

$= \operatorname{arcsec} x + C = \arcsin\left(\frac{\sqrt{x^2-1}}{x}\right) + C$ ETC

ALTERNATIVE APPROACH

$\int \frac{1}{x(x^2-1)^{\frac{1}{2}}}\,dx = \dots$ BY HYPERBOLIC SUBSTITUTION

$x = \cosh\theta$, $dx = \sinh\theta\,d\theta$, $\theta = \operatorname{arcosh} x$

$= \int \frac{1}{\cosh\theta\,\cancel{\sqrt{\cosh^2\theta-1}}}\left(\cancel{\sinh\theta}\,d\theta\right) = \int \frac{1}{\cosh\theta}\,d\theta$

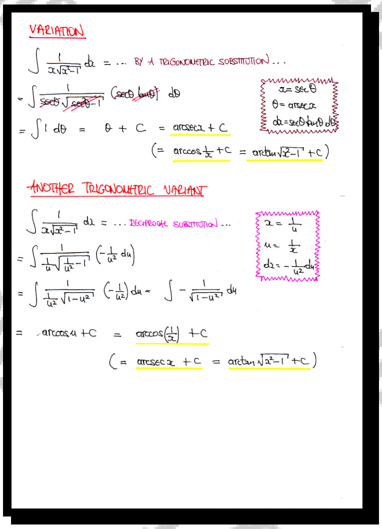
$= \int \operatorname{sech}\theta\,d\theta = \dots$ MANY DIFFERENT WAYS THIS CAN BE INTEGRATED — PROCEED BY EXPONENTIALS & A "SECOND SUBSTITUTION"

$= \int \frac{2}{e^{\theta}+e^{-\theta}}\,d\theta = \int \frac{2e^{\theta}}{e^{2\theta}+1}\,d\theta = \int \frac{2e^{\theta}}{(e^{\theta})^2+1}\,d\theta$

REVERSE CHAIN RULE (INSPECTION) OR SUBSTITUTION $u = e^{\theta}$

$= 2\arctan\left(e^{\theta}\right) + C = 2\arctan\left(e^{\operatorname{arcosh} x}\right) + C$

$= 2\arctan\left(e^{\ln(x+\sqrt{x^2-1})}\right) + C = 2\arctan\left(x+\sqrt{x^2-1}\right) + C$



VARIATION

$\int \frac{1}{x\sqrt{x^2-1}}\,dx = \dots$ BY A TRIGONOMETRIC SUBSTITUTION ...

$x = \sec\theta$, $\theta = \operatorname{arcsec} x$, $dx = \sec\theta\tan\theta\,d\theta$

$= \int \frac{1}{\cancel{\sec\theta}\,\cancel{\sqrt{\sec^2\theta-1}}}\left(\cancel{\sec\theta}\,\cancel{\tan\theta}\right)d\theta$

$= \int 1\,d\theta = \theta + C = \operatorname{arcsec} x + C$

$\left(= \arccos\frac{1}{x} + C = \arctan\sqrt{x^2-1} + C\right)$

ANOTHER TRIGONOMETRIC VARIANT

$\int \frac{1}{x\sqrt{x^2-1}}\,dx = \dots$ RECIPROCAL SUBSTITUTION ...

$x = \frac{1}{u}$, $u = \frac{1}{x}$, $dx = -\frac{1}{u^2}\,du$

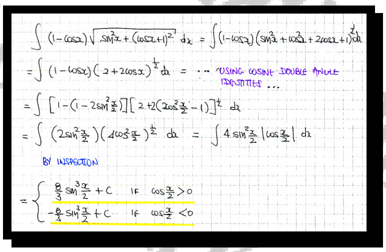
$= \int \frac{1}{\frac{1}{u}\sqrt{\frac{1}{u^2}-1}}\left(-\frac{1}{u^2}\,du\right)$

$= \int \frac{1}{\frac{1}{u^2}\sqrt{1-u^2}}\left(-\frac{1}{u^2}\right)du = \int -\frac{1}{\sqrt{1-u^2}}\,du$

$= \arccos u + C = \arccos\left(\frac{1}{x}\right) + C$

$\left(= \operatorname{arcsec} x + C = \arctan\sqrt{x^2-1} + C\right)$

402. $$\int (1-\cos x)\sqrt{(1+\cos x)^2+\sin^2 x}\,dx = \begin{cases} \frac{8}{3}\sin^2\left(\frac{1}{2}x\right) + C & \text{if } \cos\left(\frac{1}{2}x\right) > 0 \\ -\frac{8}{3}\sin^2\left(\frac{1}{2}x\right) + C & \text{if } \cos\left(\frac{1}{2}x\right) > 0 \end{cases}$$



$\int (1-\cos x)\sqrt{\sin^2 x + (\cos x+1)^2}\,dx = \int (1-\cos x)\left(\sin^2 x + \cos^2 x + 2\cos x + 1\right)^{\frac{1}{2}}dx$

$= \int (1-\cos x)(2+2\cos x)^{\frac{1}{2}}\,dx = \dots$ USING COSINE DOUBLE ANGLE IDENTITIES ...

$= \int \left[1-\left(1-2\sin^2\frac{x}{2}\right)\right]\left[2+2\left(2\cos^2\frac{x}{2}-1\right)\right]^{\frac{1}{2}}dx$

$= \int \left(2\sin^2\frac{x}{2}\right)\left(4\cos^2\frac{x}{2}\right)^{\frac{1}{2}}dx = \int 4\sin^2\frac{x}{2}\left|\cos\frac{x}{2}\right|dx$

BY INSPECTION

$= \begin{cases} \frac{8}{3}\sin^3\frac{x}{2} + C & \text{IF } \cos\frac{x}{2} > 0 \\ -\frac{8}{3}\sin^3\frac{x}{2} + C & \text{IF } \cos\frac{x}{2} < 0 \end{cases}$

403. $\displaystyle\int \frac{x+3}{x^2+1}\,dx = 3\arctan x + \tfrac{1}{2}\ln\left(x^2+1\right) + C$

$$\int \frac{x+3}{x^2+1}\,dx = \int \frac{x}{x^2+1} + \frac{3}{x^2+1}\,dx = \int \frac{1}{2}\left(\frac{2x}{x^2+1}\right) + 3\left(\frac{1}{x^2+1}\right)dx$$

BY INSPECTION FOR BOTH PARTS OF THE INTEGRAND

$$= \frac{1}{2}\ln\left(x^2+1\right) + 3\arctan x + C$$

404. $\displaystyle\int \frac{4x^3-3x^2+2x-1}{x+1}\,dx = \tfrac{4}{3}x^3 - \tfrac{7}{2}x^2 + 9x - 10\ln|x+1| + C$

$$\int \frac{4x^3-3x^2+2x-1}{x+1}\,dx =$$ BY SUBSTITUTION $u = x+1$, BY ALGEBRAIC DIVISION OR SIMILAR MANIPULATION

$$= \int \frac{4x^2(x+1) - 7x(x+1) + 9(x+1) - 10}{x+1}\,dx = \int 4x^2 - 7x + 9 - \frac{10}{x+1}\,dx$$

$$= \frac{4}{3}x^3 - \frac{7}{2}x^2 + 9x - 10\ln|x+1| + C$$

405. $\displaystyle\int \frac{1}{x^2+x+1}\,dx = \frac{2}{\sqrt{3}}\arctan\left(\frac{2x+1}{\sqrt{3}}\right) + C$

$$\int \frac{1}{x^2+x+1}\,dx =$$ COMPLETING THE SQUARE IN THE DENOMINATOR

$$= \int \frac{4}{4x^2+4x+4}\,dx = \int \frac{4}{(2x+1)^2+3}\,dx = \int \frac{4}{(2x+1)^2+\left(\sqrt{3}\right)^2}$$

BY INSPECTION OR USING THE SUBSTITUTION $u = 2x+1$

$$= \frac{4}{\sqrt{3}}\arctan\left(\frac{2x+1}{\sqrt{3}}\right) \times \frac{1}{2} + C = \frac{2}{\sqrt{3}}\arctan\left(\frac{2x+1}{\sqrt{3}}\right) + C$$

406. $\displaystyle\int \frac{9}{\left(9-x^2\right)^{\frac{3}{2}}}\,dx = \frac{x}{\sqrt{9-x^2}} + C$

$$\int \frac{9}{(9-x^2)^{\frac{3}{2}}}\,dx$$ ... BY A TRIGONOMETRIC SUBSTITUTION ...

$$= \int \frac{9}{(9-9\sin^2\theta)^{\frac{3}{2}}}(3\cos\theta\,d\theta)$$

$$= \int \frac{27\cos\theta}{[9(1-\sin^2\theta)]^{\frac{3}{2}}}\,d\theta = \int \frac{\cancel{27}\cos\theta}{\cancel{9^{\frac{3}{2}}}(\cos^2\theta)^{\frac{3}{2}}}\,d\theta$$

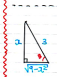
$$= \int \frac{\cos\theta}{\cos^3\theta}\,d\theta = \int \sec^2\theta\,d\theta$$

$$= \tan\theta + C = \frac{x}{(9-x^2)^{\frac{1}{2}}} + C$$

$x = 3\sin\theta$

$\theta = \arcsin\frac{x}{3}$

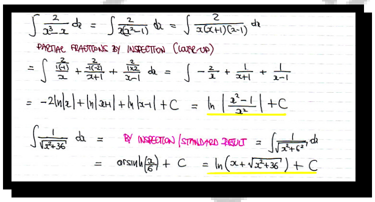
$dx = 3\cos\theta\,d\theta$



$\sin\theta = \frac{x}{3}$

$\tan\theta = \frac{x}{\sqrt{9-x^2}}$

407. $\displaystyle\int \frac{2}{x^3 - x}\,dx = \ln\left|\frac{x^2-1}{x^2}\right| + C$



$\int \frac{2}{x^3-x}\,dx = \int \frac{2}{x(x^2-1)}\,dx = \int \frac{2}{x(x+1)(x-1)}\,dx$

PARTIAL FRACTIONS BY INSPECTION (COVER-UP)

$= \int \frac{\frac{2}{1(-1)}}{x} + \frac{\frac{2}{-1(-2)}}{x+1} + \frac{\frac{2}{1\times 2}}{x-1}\,dx = \int -\frac{2}{x} + \frac{1}{x+1} + \frac{1}{x-1}$

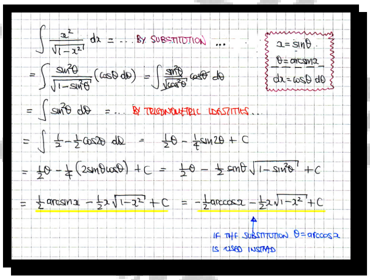
$= -2\ln|x| + \ln|x+1| + \ln|x-1| + C = \ln\left|\frac{x^2-1}{x^2}\right| + C$

$\int \frac{1}{\sqrt{x^2+36}}\,dx =$ BY INSPECTION / STANDARD RESULT $= \int \frac{1}{\sqrt{x^2+6^2}}\,dx$

$= \operatorname{arsinh}\left(\frac{x}{6}\right) + C = \ln\left(x + \sqrt{x^2+36}\right) + C$

408. $\displaystyle\int \frac{1}{\sqrt{x^2+36}}\,dx = \begin{bmatrix} \operatorname{arsinh}\left(\frac{1}{6}x\right) + C \\ \ln\left(x + \sqrt{x^2+36}\right) + C \end{bmatrix}$

409. $\displaystyle\int \frac{x^2}{\sqrt{1-x^2}}\,dx = \begin{bmatrix} \frac{1}{2}\arcsin x - \frac{1}{2}x\sqrt{1-x^2} + C \\ -\frac{1}{2}\arccos x - \frac{1}{2}x\sqrt{1-x^2} + C \end{bmatrix}$



$\int \frac{x^2}{\sqrt{1-x^2}}\,dx = \dots$ BY SUBSTITUTION ...

$x = \sin\theta$
$\theta = \arcsin x$
$dx = \cos\theta\,d\theta$

$= \int \frac{\sin^2\theta}{\sqrt{1-\sin^2\theta}}(\cos\theta\,d\theta) = \int \frac{\sin^2\theta}{\cancel{\sqrt{\cos^2\theta}}}\cancel{\cos\theta}\,d\theta$

$= \int \sin^2\theta\,d\theta = \dots$ BY TRIGONOMETRIC IDENTITIES ...

$= \int \frac{1}{2} - \frac{1}{2}\cos 2\theta\,d\theta = \frac{1}{2}\theta - \frac{1}{4}\sin 2\theta + C$

$= \frac{1}{2}\theta - \frac{1}{4}(2\sin\theta\cos\theta) + C = \frac{1}{2}\theta - \frac{1}{2}\sin\theta\sqrt{1-\sin^2\theta} + C$

$= \frac{1}{2}\arcsin x - \frac{1}{2}x\sqrt{1-x^2} + C = -\frac{1}{2}\arccos x - \frac{1}{2}x\sqrt{1-x^2} + C$

IF THE SUBSTITUTION $\theta = \arccos x$ IS USED INSTEAD

410. $\displaystyle\int \frac{\sqrt{x}}{x+1}\,dx = 2\sqrt{x} - 2\arctan\left(\sqrt{x}\right) + C$

$\displaystyle\int \frac{\sqrt{x}}{x+1}\,dx = \ldots$ BY SUBSTITUTION $\ldots$

$u = \sqrt{x}$
$u^2 = x$
$2u\,du = dx$

$\displaystyle = \int \frac{u}{u^2+1}(2u\,du) = \int \frac{2u^2}{u^2+1}\,du$

BY ALGEBRAIC DIVISION OR MANIPULATION

$\displaystyle = \int \frac{2(u^2+1)-2}{u^2+1}\,du = \int \frac{2(u^2+1)}{u^2+1} - \frac{2}{u^2+1}\,du$

$\displaystyle = \int 2 - \frac{2}{u^2+1}\,du = 2u - 2\arctan u + C$

$= 2\sqrt{x} - 2\arctan\sqrt{x} + C$

411. $\displaystyle\int \frac{1}{9x^2+64}\,dx = \frac{1}{24}\arctan\left(\frac{3}{8}x\right) + C$

$\displaystyle\int \frac{1}{9x^2+64}\,dx = \int \frac{1}{(3x)^2+8^2}\,dx = \ldots$ BY INSPECTION $\ldots$

$\displaystyle = \frac{1}{8}\arctan\left(\frac{3x}{8}\right) \times \frac{1}{3} = \frac{1}{24}\arctan\left(\frac{3x}{8}\right) + C$

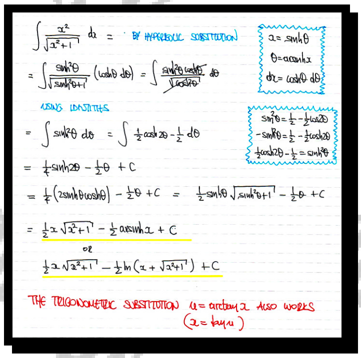
412. $\displaystyle\int \frac{x}{\sqrt{x^2+1}}\,dx = \sqrt{x^2+1} + C$

$\displaystyle\int \frac{x}{\sqrt{x^2+1}}\,dx = \int x(x^2+1)^{-\frac{1}{2}}\,dx$ BY REVERSE CHAIN RULE (INSPECTION)

$= (x^2+1)^{\frac{1}{2}} + C$

THE SUBSTITUTIONS $u = x^2+1$, $u = \sqrt{x^2+1}$, $x = \tan\theta$, $x = \sinh\theta$ ALSO WORK

413. $$\int \frac{x^2}{\sqrt{x^2+1}}\,dx = \left[\begin{array}{l} \frac{1}{2}x\sqrt{x^2+1}-\frac{1}{2}\operatorname{arsinh} x + C \\ \frac{1}{2}x\sqrt{x^2+1}-\frac{1}{2}\ln\left(x+\sqrt{x^2+1}\right)+C \end{array}\right]$$



$\int \frac{x^2}{\sqrt{x^2+1}}\,dx = \dots$ BY HYPERBOLIC SUBSTITUTION

$x = \sinh\theta$
$\theta = \operatorname{arsinh} x$
$dx = \cosh\theta\, d\theta$

$= \int \frac{\sinh^2\theta}{\sqrt{\sinh^2\theta+1}}(\cosh\theta\, d\theta) = \int \frac{\sinh^2\theta\cosh\theta}{\sqrt{\cosh^2\theta}}\,d\theta$

USING IDENTITIES

$\sinh^2\theta = \frac{1}{2}-\frac{1}{2}\cosh 2\theta$
$-\sinh^2\theta = \frac{1}{2}-\frac{1}{2}\cosh 2\theta$
$\frac{1}{2}\cosh 2\theta - \frac{1}{2} = \sinh^2\theta$

$= \int \sinh^2\theta\, d\theta = \int \frac{1}{2}\cosh 2\theta - \frac{1}{2}\,d\theta$

$= \frac{1}{4}\sinh 2\theta - \frac{1}{2}\theta + C$

$= \frac{1}{4}(2\sinh\theta\cosh\theta) - \frac{1}{2}\theta + C = \frac{1}{2}\sinh\theta\sqrt{\sinh^2\theta+1} - \frac{1}{2}\theta + C$

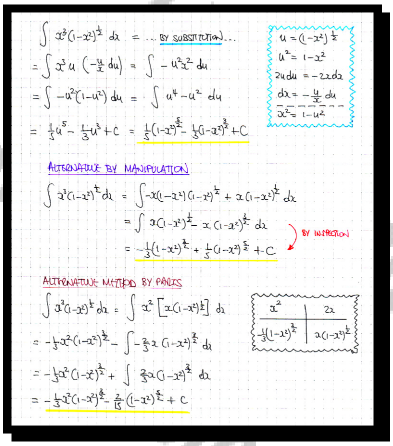
$= \frac{1}{2}x\sqrt{x^2+1} - \frac{1}{2}\operatorname{arsinh} x + C$

or

$\frac{1}{2}x\sqrt{x^2+1} - \frac{1}{2}\ln\left(x+\sqrt{x^2+1}\right) + C$

THE TRIGONOMETRIC SUBSTITUTION $u = \arctan x$ ALSO WORKS
$(x = \tan u)$

414. $$\int x^3\left(1-x^2\right)^{\frac{1}{2}}\,dx = \left[\begin{array}{l} \frac{1}{5}\left(1-x^2\right)^{\frac{5}{2}}-\frac{1}{3}\left(1-x^2\right)^{\frac{3}{2}}+C \\ -\frac{1}{3}x^2\left(1-x^2\right)^{\frac{3}{2}}-\frac{2}{15}\left(1-x^2\right)^{\frac{5}{2}}+C \end{array}\right]$$



$\int x^3(1-x^2)^{\frac{1}{2}}\,dx = \dots$ BY SUBSTITUTION...

$u = (1-x^2)^{\frac{1}{2}}$
$u^2 = 1-x^2$
$2u\,du = -2x\,dx$
$dx = -\frac{u}{x}\,du$
$x^2 = 1-u^2$

$= \int x^3 u\left(-\frac{u}{x}\,du\right) = \int -u^2x^2\,du$

$= \int -u^2(1-u^2)\,du = \int u^4 - u^2\,du$

$= \frac{1}{5}u^5 - \frac{1}{3}u^3 + C = \frac{1}{5}(1-x^2)^{\frac{5}{2}} - \frac{1}{3}(1-x^2)^{\frac{3}{2}} + C$

ALTERNATIVE BY MANIPULATION

$\int x^3(1-x^2)^{\frac{1}{2}}\,dx = \int -x(1-x^2)(1-x^2)^{\frac{1}{2}} + x(1-x^2)^{\frac{1}{2}}\,dx$

$= \int x(1-x^2)^{\frac{1}{2}} - x(1-x^2)^{\frac{3}{2}}\,dx$

BY INSPECTION

$= -\frac{1}{3}(1-x^2)^{\frac{3}{2}} + \frac{1}{5}(1-x^2)^{\frac{5}{2}} + C$

ALTERNATIVE METHOD BY PARTS

$\int x^3(1-x^2)^{\frac{1}{2}}\,dx = \int x^2\left[x(1-x^2)^{\frac{1}{2}}\right]dx$

x^2	$2x$
$-\frac{1}{3}(1-x^2)^{\frac{3}{2}}$	$x(1-x^2)^{\frac{1}{2}}$

$= -\frac{1}{3}x^2(1-x^2)^{\frac{3}{2}} - \int -\frac{2}{3}x(1-x^2)^{\frac{3}{2}}\,dx$

$= -\frac{1}{3}x^2(1-x^2)^{\frac{3}{2}} + \int \frac{2}{3}x(1-x^2)^{\frac{3}{2}}\,dx$

$= -\frac{1}{3}x^2(1-x^2)^{\frac{3}{2}} - \frac{2}{15}(1-x^2)^{\frac{5}{2}} + C$

415. $$\int \sinh x \cosh x \; dx = \begin{bmatrix} \frac{1}{2}\sinh^2 x + C \\ \frac{1}{2}\cosh^2 x + C \\ \frac{1}{4}\cosh 2x + C \end{bmatrix}$$

$$\int \sinh x \cosh x \, dx = \dots \text{BY REVERSE CHAIN RULE} \dots$$
$$= \tfrac{1}{2}\sinh^2 x + C \quad \text{OR} \quad \tfrac{1}{2}\cosh^2 x + C$$

ALTERNATIVE BY IDENTITIES

$$\int \sinh x \cosh x \, dx = \int \tfrac{1}{2}(2\sinh x \cosh x)\, dx = \int \tfrac{1}{2}\sinh 2x \, dx = \tfrac{1}{4}\cosh 2x + C$$

416. $$\int \frac{4^x + 2\times 16^x}{1+16^x} \; dx = \frac{1}{\ln 4}\Big[\arctan\left(4^x\right) \ln\left(1+4^{2x}\right)\Big] + C$$

$$\int \frac{4^x + 2\times 16^x}{1+16^x}\, dx = \int \frac{4^x + 2\times 4^{2x}}{1+4^{2x}} \qquad 16^x = (4^2)^x = 4^{2x}$$

NOW BY SUBSTITUTION

$$u = 4^x \qquad \frac{du}{dx} = 4^x(\ln 4) \qquad dx = \frac{du}{4^x \ln 4} \qquad dx = \frac{du}{u \ln 4}$$

$$= \int \frac{u + 2u^2}{1+u^2}\left(\frac{du}{u\ln 4}\right)$$
$$= \frac{1}{\ln 4}\int \frac{1+2u}{1+u^2}\, du = \frac{1}{\ln 4}\left[\int \frac{1}{1+u^2} + \frac{2u}{1+u^2}\, du\right.$$
$$= \frac{1}{\ln 4}\left[\arctan u + \ln(1+u^2)\right] + C$$
$$= \frac{1}{\ln 4}\left[\arctan(4^x) + \ln(1+4^{2x})\right] + C$$

417. $$\int \frac{\sin x}{\cos x + \cos^3 x} \; dx = \frac{1}{2}\ln\left(\frac{1+\cos^2 x}{\cos^2 x}\right) + C$$

$$\int \frac{\sin x}{\cos x + \cos^3 x}\, dx = \dots \text{BY SUBSTITUTION} \dots$$

$$u = \cos x \qquad \frac{du}{dx} = -\sin x \qquad dx = -\frac{du}{\sin x}$$

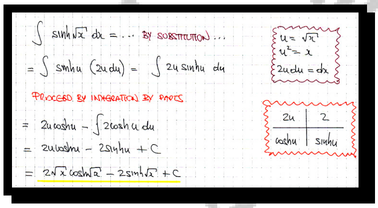
$$= \int \frac{\cancel{\sin x}}{u + u^3}\left(-\frac{du}{\cancel{\sin x}}\right) = \int -\frac{1}{u+u^3}\, du$$
$$= -\int \frac{1}{u(u^2+1)}\, du$$

BY PARTIAL FRACTIONS

$$\frac{1}{(u^2+1)u} \equiv \frac{A}{u} + \frac{Bu+C}{u^2+1}$$
$$1 \equiv A(u^2+1) + u(Bu+C)$$
$$1 \equiv (A+B)u^2 + Cu + A$$
$$\therefore C = 0, \quad A = 1, \quad B = -1$$

$$\dots = -\int \frac{1}{u} - \frac{u}{u^2+1}\, du = \int \frac{u}{u^2+1} - \frac{1}{u}\, du$$
$$= \tfrac{1}{2}\ln(u^2+1) - \ln u + C = \tfrac{1}{2}\left[\ln(u^2+1) - 2\ln u\right] + C$$
$$= \tfrac{1}{2}\left[\ln(u^2+1) - \ln u^2\right] = \tfrac{1}{2}\ln\left(\frac{u^2+1}{u^2}\right) + C = \tfrac{1}{2}\ln\left(\frac{1+\cos^2 x}{\cos^2 x}\right) + C$$

418. $$\int \sinh\sqrt{x}\ dx = 2\sqrt{x}\cosh\sqrt{x} - 2\sinh\sqrt{x} + C$$



$\int \sinh\sqrt{x}\,dx = \dots$ BY SUBSTITUTION ...

$u=\sqrt{x}$, $u^2 = x$, $2u\,du = dx$

$= \int \sinh u\,(2u\,du) = \int 2u\sinh u\,du$

PROCEED BY INTEGRATION BY PARTS

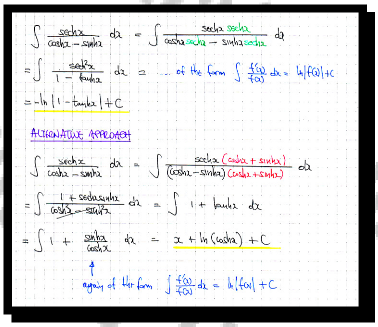
$2u$	2
$\cosh u$	$\sinh u$

$= 2u\cosh u - \int 2\cosh u\,du$

$= 2u\cosh u - 2\sinh u + C$

$= 2\sqrt{x}\cosh\sqrt{x} - 2\sinh\sqrt{x} + C$

419. $$\int \frac{\operatorname{sech} x}{\cosh x - \sinh x}\ dx = \begin{bmatrix} -\ln|1-\tanh x| + C \\ x + \ln(\cosh x) + C \end{bmatrix}$$



$\int \frac{\operatorname{sech}x}{\cosh x - \sinh x}\,dx = \int \frac{\operatorname{sech}x\operatorname{sech}x}{\cosh x\operatorname{sech}x - \sinh x\operatorname{sech}x}\,dx$

$= \int \frac{\operatorname{sech}^2 x}{1-\tanh x}\,dx = \dots$ of the form $\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + C$

$= -\ln|1-\tanh x| + C$

ALTERNATIVE APPROACH

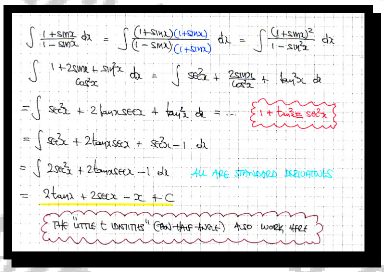
$\int \frac{\operatorname{sech}x}{\cosh x - \sinh x}\,dx = \int \frac{\operatorname{sech}x(\cosh x + \sinh x)}{(\cosh x - \sinh x)(\cosh x + \sinh x)}\,dx$

$= \int \frac{1+\operatorname{sech}x\sinh x}{\cosh^2 x - \sinh^2 x}\,dx = \int 1 + \tanh x\,dx$

$= \int 1 + \frac{\sinh x}{\cosh x}\,dx = x + \ln(\cosh x) + C$

again of the form $\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + C$

420. $$\int \frac{1+\sin x}{1-\sin x}\ dx = -x + 2\tan x + 2\sec x + C$$



$\int \frac{1+\sin x}{1-\sin x}\,dx = \int \frac{(1+\sin x)(1+\sin x)}{(1-\sin x)(1+\sin x)}\,dx = \int \frac{(1+\sin x)^2}{1-\sin^2 x}\,dx$

$\int \frac{1+2\sin x+\sin^2 x}{\cos^2 x}\,dx = \int \sec^2 x + \frac{2\sin x}{\cos^2 x} + \tan^2 x\,dx$

$= \int \sec^2 x + 2\tan x\sec x + \tan^2 x\,dx = \dots$ $\boxed{1+\tan^2 x \equiv \sec^2 x}$

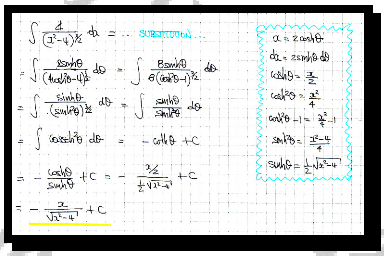
$= \int \sec^2 x + 2\tan x\sec x + \sec^2 x - 1\,dx$

$= \int 2\sec^2 x + 2\tan x\sec x - 1\,dx$ ALL ARE STANDARD DERIVATIVES

$= 2\tan x + 2\sec x - x + C$

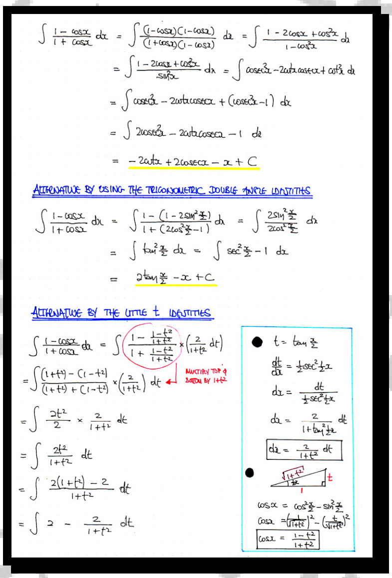
THE "LITTLE t IDENTITIES" (TAN-HALF-ANGLE) ALSO WORK HERE

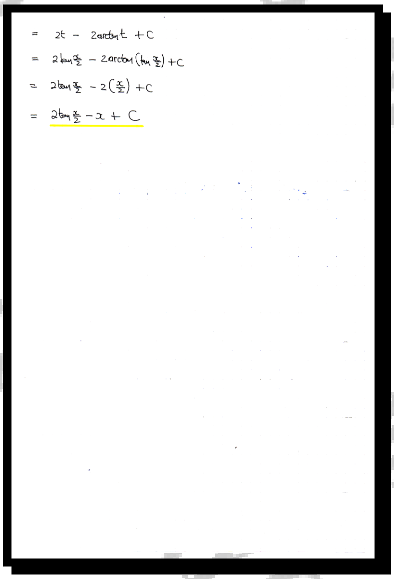
421. $$\int \frac{4}{\left(x^2-4\right)^{\frac{3}{2}}}\,dx = -\frac{x}{\sqrt{x^2-4}} + C$$



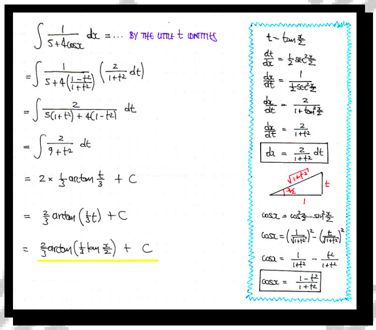
422. $$\int 4\sinh\left(\tfrac{1}{4}x\right)\,dx = 16\cosh\left(\tfrac{1}{4}x\right) + C$$

423. $$\int \frac{1-\cos x}{1+\cos x}\,dx = \begin{bmatrix} -x-2\cot x+2\operatorname{cosec} x + C \\ -x-2\tan\left(\tfrac{1}{2}x\right)+C \end{bmatrix}$$

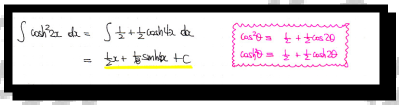




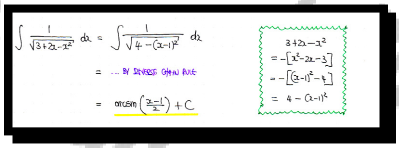
424. $\displaystyle\int \frac{1}{5+4\cos x}\,dx = \tfrac{2}{3}\arctan\left[\tfrac{1}{3}\tan\left(\tfrac{1}{2}x\right)\right] + C$



425. $\displaystyle\int \cosh^2 2x\,\,dx = \tfrac{1}{2}x + \tfrac{1}{2}\sinh 4x + C$



426. $\displaystyle\int \frac{1}{\sqrt{3+2x-x^2}}\,dx = \arcsin\left(\frac{x-1}{2}\right) + C$



427. $$\int \frac{1}{1+\sin 2x}\,dx = \left[\begin{array}{l} \frac{1}{2}\tan 2x - \frac{1}{2}\sec 2x + C \\ -\frac{1}{1+\tan x} + C \end{array}\right]$$

$$\int \frac{1}{1+\sin 2x}\,dx$$ ... BY MANIPULATION $\sin 2x \neq \pm 1$ $$= \int \frac{1(1-\sin 2x)}{(1+\sin 2x)(1-\sin 2x)}\,dx$$

$$= \int \frac{1-\sin 2x}{1-\sin^2 2x}\,dx = \int \frac{1-\sin 2x}{\cos^2 2x}\,dx = \int \frac{1}{\cos^2 2x} - \frac{\sin 2x}{\cos^2 2x}\,dx$$

$$= \int \sec^2 2x - \frac{\sin 2x}{\cos 2x}\frac{1}{\cos 2x}\,dx = \int \sec^2 2x - \sec 2x \tan 2x\,dx$$

$$= \frac{1}{2}\tan 2x - \frac{1}{2}\sec 2x + C$$

ALTERNATIVE BY SUBSTITUTION, SIMILAR TO THOSE IN "LITTLE t" IDENTITIES

$$\int \frac{1}{1+\sin 2x}\,dx = \int \frac{1}{1+2\sin x\cos x}\,dx$$

$$= \int \frac{\frac{1}{\cos^2 x}}{\frac{1}{\cos^2 x} + \frac{2\sin x\cos x}{\cos^2 x}}\,dx = \int \frac{\sec^2 x}{\sec^2 x + 2\tan x}\,dx$$

$$= \int \frac{1+\tan^2 x}{1+\tan^2 x + 2\tan x}\,dx$$

$$= \int \frac{\cancel{1+t^2}}{1+t^2+2t}\left(\frac{1}{\cancel{1+t^2}}\,dt\right)$$

$$= \int \frac{1}{t^2+2t+1}\,dt = \int \frac{1}{(t+1)^2}\,dt$$

$$= -\frac{1}{t+1} + C = -\frac{1}{1+\tan x} + C$$

$t = \tan x$

$\frac{dt}{dx} = \sec^2 x$

$\frac{dt}{dx} = 1 + \tan^2 x$

$\frac{dt}{dx} = 1 + t^2$

$dx = \frac{1}{1+t^2}\,dt$

428. $$\int \frac{588}{4x^2+49}\,dx = 42\arctan\left(\frac{2}{7}x\right) + C$$

$$\int \frac{588}{4x^2+49}\,dx = \int \frac{147}{x^2+\frac{49}{4}}\,dx = \int \frac{147}{x^2+\left(\frac{7}{2}\right)^2}\,dx$$

BY RECOGNITION (REVERSE CHAIN RULE)

$$= 147 \times \frac{2}{7} \times \arctan\left(\frac{x}{7/2}\right) + C = 42\arctan\left(\frac{2x}{7}\right) + C$$

429. $$\int \frac{x+1}{x^2+9}\,dx = \frac{1}{2}\ln\left(x^2+9\right) + \frac{1}{3}\arctan\left(\frac{1}{3}x\right) + C$$

$$\int \frac{x+1}{x^2+9}\,dx = \int \frac{x}{x^2+9} + \frac{1}{x^2+9}\,dx$$

"BOTTOM" DIFFERENTIATES TO "TOP"

STANDARD RESULT

$$= \frac{1}{2}\ln(x^2+9) + \frac{1}{3}\arctan\left(\frac{x}{3}\right) + C$$

430. $\displaystyle\int \sqrt{e^x - 1}\, dx = 2\sqrt{e^x - 1} - 2\arctan\left(\sqrt{e^x - 1}\right) + C$

$\displaystyle\int \sqrt{e^x - 1}\, dx = \ldots$ BY SUBSTITUTION ...

$u = \sqrt{e^x - 1}$
$u^2 = e^x - 1$
$2u\,du = e^x\,dx$
$dx = \frac{2u}{e^x}\,du$
$dx = \frac{2u}{u^2+1}\,du$

$\displaystyle = \int u\left(\frac{2u}{u^2+1}\,du\right) = \int \frac{2u^2}{u^2+1}\,du$

$\displaystyle = \int \frac{2(u^2+1) - 2}{u^2+1}\,du = \int 2 - \frac{2}{u^2+1}\,du$

$= 2u - 2\arctan u + C$

$= 2\sqrt{e^x - 1} - 2\arctan\left(\sqrt{e^x - 1}\right) + C$

431. $\displaystyle\int \frac{\sqrt{e^x}}{\sqrt{e^x + e^{-x}}}\, dx = \text{arsinh}\left(e^x\right) + C$

$\displaystyle\int \frac{\sqrt{e^x}}{\sqrt{e^x + e^{-x}}}\, dx = \ldots$ BY SUBSTITUTION ...

$u = e^x$
$\frac{du}{dx} = e^x$
$\frac{du}{dx} = u$
$dx = \frac{1}{u}\,du$

$\displaystyle = \int \frac{u^{\frac{1}{2}}}{\sqrt{u + u^{-1}}}\left(\frac{1}{u}\,du\right) = \int \frac{u^{-\frac{1}{2}}}{\left(u + \frac{1}{u}\right)^{\frac{1}{2}}}\,du$

$\displaystyle = \int \frac{1}{u^{\frac{1}{2}}} \times \frac{1}{\left(u + \frac{1}{u}\right)^{\frac{1}{2}}}\,du = \int \frac{1}{\sqrt{u\left(u + \frac{1}{u}\right)}}\,du$

$\displaystyle = \int \frac{1}{\sqrt{u^2 + 1}}\,du = \ldots$ STANDARD RESULT ...

$= \text{arsinh}\, u + C = \text{arsinh}\left(e^x\right) + C$

432. $\displaystyle\int \sqrt{1 - e^{2x}}\, dx = \sqrt{1 - e^{2x}} + \frac{1}{2}\ln\left|\frac{\sqrt{1 - e^{2x}} - 1}{\sqrt{1 - e^{2x}} + 1}\right| + C$

$\displaystyle\int \sqrt{1 - e^{2x}}\, dx = \ldots$ BY SUBSTITUTION ...

$u = \sqrt{1 - e^{2x}}$
$u^2 = 1 - e^{2x}$
$2u\,du = -2e^{2x}\,dx$
$dx = -\frac{u}{e^{2x}}\,du$
$dx = -\frac{u}{1-u^2}\,du$

$\displaystyle = \int u\left(\frac{-u}{1-u^2}\,du\right) = \int \frac{u^2}{u^2-1}\,du$

$\displaystyle = \int \frac{(u^2-1)+1}{u^2-1}\,du = \int 1 + \frac{1}{u^2-1}\,du$

... BY PARTIAL FRACTIONS ...

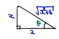
$\displaystyle = \int 1 + \frac{1}{(u-1)(u+1)}\,du = \int 1 + \frac{\frac{1}{2}}{u-1} - \frac{\frac{1}{2}}{u+1}\,du = u + \frac{1}{2}\ln|u-1| - \frac{1}{2}\ln|u+1| + C$

$\displaystyle = u + \frac{1}{2}\ln\left|\frac{u-1}{u+1}\right| + C = \sqrt{1 - e^{2x}} + \frac{1}{2}\ln\left|\frac{\sqrt{1 - e^{2x}} - 1}{\sqrt{1 - e^{2x}} + 1}\right| + C$

433. $$\int \frac{16}{\left(x^2+4\right)^{\frac{5}{2}}}\,dx = \frac{x}{\left(x^2+4\right)^{\frac{1}{2}}} - \frac{x^3}{3\left(x^2+4\right)^{\frac{3}{2}}} + C$$

$\int \frac{16}{(x^2+4)^{\frac{5}{2}}}\,dx = \dots$ BY TRIGONOMETRIC SUBSTITUTION ...

$x = 2\tan\theta$
$dx = 2\sec^2\theta\,d\theta$
$\tan\theta = \frac{x}{2}$



$\sin\theta = \frac{x}{(x^2+4)^{\frac{1}{2}}}$

$= \int \frac{16}{(4\tan^2\theta+4)^{\frac{5}{2}}}(2\sec^2\theta\,d\theta) = \int \frac{32\sec^2\theta}{32(1+\tan^2\theta)^{\frac{5}{2}}}\,d\theta$

$= \int \frac{\sec^2\theta}{(\sec^2\theta)^{\frac{5}{2}}}\,d\theta = \int \frac{\sec^2\theta}{\sec^5\theta}\,d\theta = \int \cos^3\theta\,d\theta$

$= \int \cos\theta\cos^2\theta\,d\theta = \int \cos\theta(1-\sin^2\theta)\,d\theta$

$= \int \cos\theta - \cos\theta\sin^2\theta\,d\theta = \sin\theta - \frac{1}{3}\sin^3\theta + C$

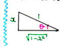
$= \frac{x}{(x^2+4)^{\frac{1}{2}}} - \frac{1}{3}\left(\frac{x}{(x^2+4)^{\frac{1}{2}}}\right)^3 + C = \frac{x}{(x^2+4)^{\frac{1}{2}}} - \frac{x^3}{3(x^2+4)^{3/2}} + C$

THE HYPERBOLIC SUBSTITUTION $x = 2\sinh\theta$ ALSO WORKS WELL

434. $$\int \sqrt{\left(1-x^2\right)^3}\,dx = \tfrac{3}{8}\arcsin x + \tfrac{1}{8}x\left(5-2x^2\right)\left(1-x^2\right)^{\frac{1}{2}} + C$$

$\int \sqrt{(1-x^2)^3}\,dx = \dots$ BY A TRIGONOMETRIC SUBSTITUTION

$x = \sin\theta$
$(\theta = \arcsin x)$
$dx = \cos\theta\,d\theta$



$\cos\theta = \sqrt{1-x^2}$

$= \int (1-\sin^2\theta)^{3/2}\cos\theta\,d\theta = \int (\cos^2\theta)^{\frac{3}{2}}\cos\theta\,d\theta$

$= \int \cos^4\theta\,d\theta = \int (\cos^2\theta)^2\,d\theta = \int \left(\frac{1}{2}+\frac{1}{2}\cos 2\theta\right)^2 d\theta$

$= \int \frac{1}{4} + \frac{1}{2}\cos 2\theta + \frac{1}{4}\cos^2 2\theta\,d\theta$

$= \int \frac{1}{4} + \frac{1}{2}\cos 2\theta + \frac{1}{4}\left(\frac{1}{2}+\frac{1}{2}\cos 4\theta\right)d\theta$

$= \int \frac{1}{4} + \frac{1}{2}\cos 2\theta + \frac{1}{8} + \frac{1}{8}\cos 4\theta\,d\theta = \int \frac{3}{8} + \frac{1}{2}\cos 2\theta + \frac{1}{8}\cos 4\theta\,d\theta$

$= \frac{3}{8}\theta + \frac{1}{4}\sin 2\theta + \frac{1}{32}\sin 4\theta + C = \frac{3}{8}\theta + \frac{1}{2}\sin\theta\cos\theta + \frac{1}{16}\sin 2\theta\cos 2\theta + C$

$= \frac{3}{8}\theta + \frac{1}{2}\sin\theta\cos\theta + \frac{1}{8}\sin\theta\cos\theta(1-2\sin^2\theta) + C$

$= \frac{3}{8}\arcsin x + \frac{1}{2}x\sqrt{1-x^2} + \frac{1}{8}x\sqrt{1-x^2}(1-2x^2) + C$

$= \frac{3}{8}\arcsin x + \frac{1}{8}x\sqrt{1-x^2}\left[4+(1-2x^2)\right] + C = \frac{3}{8}\arcsin x + \frac{1}{8}x(5-2x^2)(1-x^2)^{\frac{1}{2}} + C$

435. $$\int \cosh x \arctan\left(\sinh x\right)\,dx = \sin x \arctan\left(\sinh x\right) - \ln\left(\cosh x\right) + C$$

$\int \cosh x\arctan(\sinh x)\,dx = \dots$ INTEGRATION BY PARTS

$\arctan(\sinh x)$	$\frac{\cosh x}{1+\sinh^2 x}$
$\sinh x$	$\cosh x$

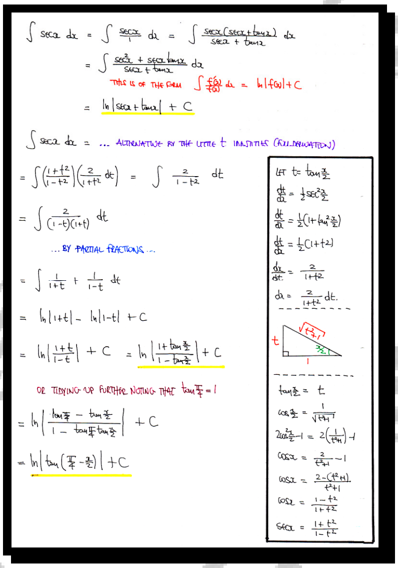
$= \sinh x\arctan(\sinh x) - \int \frac{\cosh x\sinh x}{1+\sinh^2 x}\,dx$

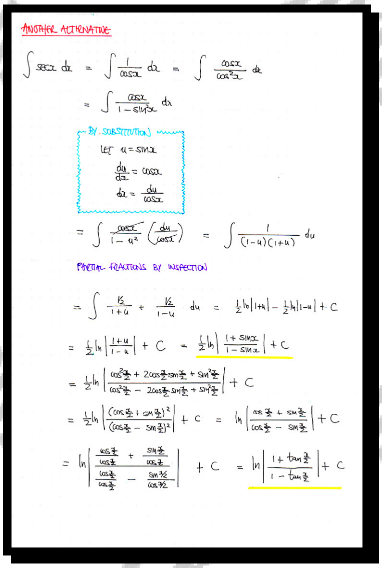
$= \sinh x\arctan(\sinh x) - \int \frac{\cosh x\sinh x}{\cosh^2 x}\,dx$

$= \sinh x\arctan(\sinh x) - \int \frac{\sinh x}{\cosh x}\,dx$

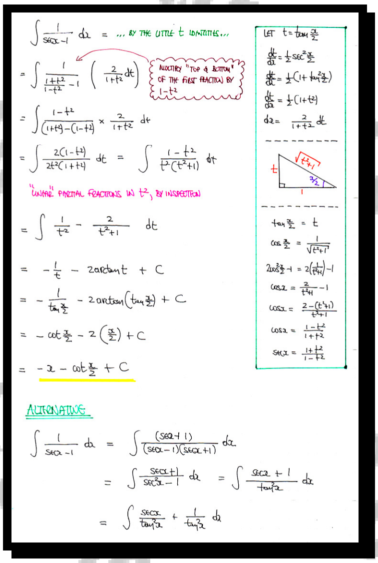
$= \sinh x\arctan(\sinh x) - \ln|\cosh x| + C$

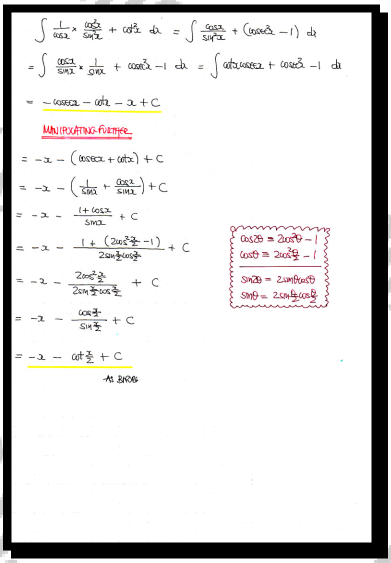
436. $$\int \sec x \; dx = \begin{bmatrix} \ln\left|\sec x + \tan x\right| + C \\ \ln\left|\tan\left(\frac{1}{4}\pi - \frac{1}{2}x\right)\right| + C \\ \frac{1}{2}\ln\left|\frac{1+\sin x}{1-\sin x}\right| + C \\ \ln\left|\frac{1+\tan\left(\frac{1}{2}x\right)}{1-\tan\left(\frac{1}{2}x\right)}\right| + C \end{bmatrix}$$



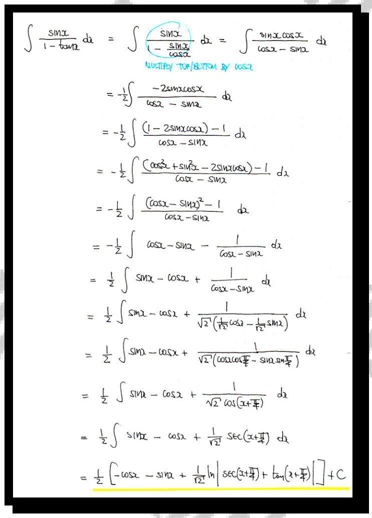


437. $$\int \frac{1}{\sec x - 1} \; dx = \begin{bmatrix} -x - \cot\left(\frac{1}{2}x\right) + C \\ -x - \cot x - \operatorname{cosec} x + C \end{bmatrix}$$

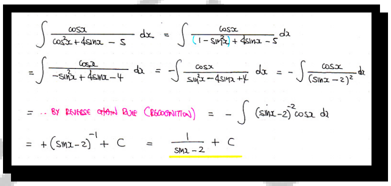




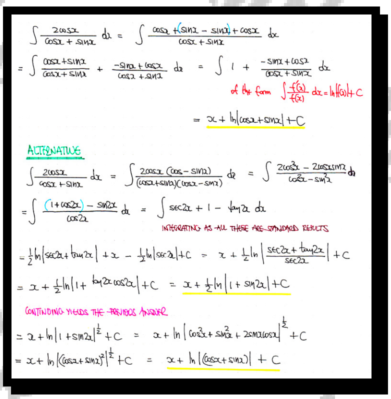
438. $\displaystyle\int \frac{\sin x}{1-\tan x}\,dx = \frac{1}{2}\left[-\cos x - \sin x + \frac{1}{\sqrt{2}}\ln\left|\sec\left(x+\frac{1}{4}\pi\right) + \tan\left(x+\frac{1}{4}\pi\right)\right|\right] + C$



439. $\displaystyle\int \frac{\cos x}{\cos^2 x + 4\sin x - 5}\,dx = \frac{1}{\sin x - 2} + C$



440. $\displaystyle\int \frac{2\cos x}{\cos x + \sin x}\,dx = \begin{bmatrix} x + \ln|\cos x + \sin x| + C \\ x + \frac{1}{2}\ln|1+\sin 2x| + C \end{bmatrix}$



441. $$\int \frac{18}{\left(\cos^2 x + 4\sin x - 5\right)\cos x}\,dx = \ldots$$

$$= \ldots \frac{6}{2-\sin x} - 8\ln\left(2-\sin x\right) + 9\ln\left|1-\sin x\right| - \ln\left|1-\sin x\right| + C$$

$$\int \frac{18}{(\cos^2 x + 4\sin x - 5)\cos x}\,dx = \int \frac{18}{(1-\sin^2 x + 4\sin x - 5)\cos x}\,dx$$

$$= \int \frac{18}{(-\sin^2 x + 4\sin x - 4)\cos x}\,dx = \int \frac{18}{-(\sin^2 x - 4\sin x + 4)\cos x}\,dx = \int \frac{18}{-(\sin x - 2)^2 \cos x}\,dx$$

PROCEED BY A SUBSTITUTION

$u = \sin x - 2$, $\sin x = u + 2$

$\frac{du}{dx} = \cos x$

$dx = \frac{du}{\cos x}$

$$\ldots = \int \frac{18}{-u^2 \cos x}\left(\frac{du}{\cos x}\right)$$

$$= \int \frac{-18}{u^2\cos^2 x}\,du = \int \frac{-18}{u^2(1-\sin^2 x)}\,du$$

$$= \int \frac{-18}{u^2\left[1-(u+2)^2\right]}\,du = \int \frac{-18}{u^2(-u^2-4u-3)}\,du = \int \frac{18}{u^2(u^2+4u+3)}\,du$$

$$= \int \frac{18}{u^2(u+1)(u+3)}\,du$$

BY PARTIAL FRACTIONS

$$\frac{18}{u^2(u+1)(u+3)} \equiv \frac{A}{u} + \frac{B}{u^2} + \frac{C}{u+1} + \frac{D}{u+3}$$

$$\boxed{18 \equiv Au(u+1)(u+3) + B(u+1)(u+3) + Cu^2(u+3) + Du^2(u+1)}$$

- IF $u=0$, $18 = 3B \Rightarrow \underline{B=6}$
- IF $u=-1$, $18 = 2C \Rightarrow \underline{C=9}$
- IF $u=-3$, $18 = -18D \Rightarrow \underline{D=-1}$
- IF $u=1$, $18 = 8A + 8B + 4C + 2D$
  $18 = 8A + 48 + 36 - 2$
  $\underline{A=-8}$

$$\ldots = \int \frac{6}{u^2} - \frac{8}{u} + \frac{9}{u+1} - \frac{1}{u+3}\,du = -\frac{6}{u} - 8\ln|u| + 9\ln|u+1| - \ln|u+3| + C$$

$$= -\frac{6}{\sin x - 2} - 8\ln|\sin x - 2| + 9\ln|\sin x - 1| - \ln|\sin x + 1| + C$$

$$= \underline{\frac{6}{2-\sin x} - 8\ln(2-\sin x) + 9\ln|1-\sin x| - \ln|1+\sin x| + C}$$

442. $$\int \frac{\tan x}{1+\sin^2 x}\,dx = \frac{1}{4}\ln\left|\frac{1+\sin^2 x}{\cos^2 x}\right| + C$$

$$\int \frac{\tan x}{1+\sin^2 x}\,dx = \ldots \text{ BY SUBSTITUTION}$$

$u = 1 + \sin^2 x$

$\frac{du}{dx} = 2\sin x\cos x$

$dx = \frac{1}{2\sin x\cos x}$

$\sin^2 x = u - 1$

$$= \int \frac{\tan x}{u} \times \frac{1}{2\sin x\cos x}\,du$$

$$= \int \frac{1}{u} \times \frac{\frac{\sin x}{\cos x}}{2\sin x\cos x}\,du$$

$$= \int \frac{1}{u} \times \frac{\sin x}{2\sin x\cos^2 x}\,du = \int \frac{1}{u}\,\frac{1}{2\cos^2 x}\,du = \int \frac{1}{2u(1-\sin^2 x)}\,du$$

$$= \int \frac{1}{2u(1-(u-1))}\,du = \int \frac{1}{2u(2-u)}\,du = \frac{1}{2}\int \frac{1}{u(2-u)}\,du$$

BY PARTIAL FRACTIONS

$$= \frac{1}{2}\int \frac{\frac{1}{2}}{u} + \frac{\frac{1}{2}}{2-u}\,du = \frac{1}{4}\int \frac{1}{u} + \frac{1}{2-u}\,du = \frac{1}{4}\left[\ln|u| - \ln|2-u|\right] + C$$

$$= \frac{1}{4}\ln\left|\frac{u}{2-u}\right| = \frac{1}{4}\ln\left|\frac{1+\sin^2 x}{2-(1+\sin^2 x)}\right| + C = \frac{1}{4}\ln\left|\frac{1+\sin^2 x}{1-\sin^2 x}\right| + C$$

$$= \underline{\frac{1}{4}\ln\left|\frac{1+\sin^2 x}{\cos^2 x}\right| + C}$$

443. $$\int (x+1)e^{x+1}\,dx = xe^{x+1} + C$$

$$\int (x+1)e^{x+1}\,dx = \ldots \text{ STANDARD INTEGRATION BY PARTS} \ldots$$

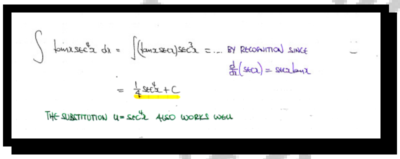
$$\begin{array}{c|c} x+1 & 1 \\ \hline e^{x+1} & e^{x+1} \end{array}$$

$$= (x+1)e^{x+1} - \int e^{x+1}\,dx$$

$$= (x+1)e^{x+1} - e^{x+1} + C$$

$$= \underline{xe^{x+1} + C}$$

444. $\displaystyle\int \tan x \sec^4 x \ dx = \frac{1}{4}\sec^4 x + C$

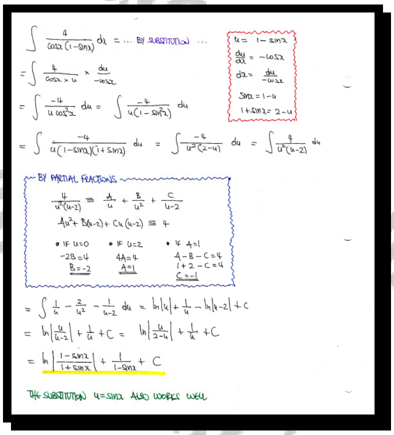


$\int \tan x \sec^4 x \, dx = \int (\tan x \sec x)\sec^3 x = \ldots$ BY RECOGNITION SINCE $\frac{d}{dx}(\sec x) = \sec x \tan x$

$= \frac{1}{4}\sec^4 x + C$

THE SUBSTITUTION $u = \sec^4 x$ ALSO WORKS WELL

445. $\displaystyle\int \frac{4}{(1-\sin x)\cos x} \ dx = \ln\left|\frac{1-\sin x}{1+\sin x}\right| + \frac{1}{1-\sin x} + C$



$\int \frac{4}{\cos x(1-\sin x)} dx = \ldots$ BY SUBSTITUTION $\ldots$

$u = 1-\sin x$

$\frac{du}{dx} = -\cos x$

$dx = \frac{du}{-\cos x}$

$\sin x = 1-u$

$1+\sin x = 2-u$

$= \int \frac{4}{\cos x \times u} \times \frac{du}{-\cos x}$

$= \int \frac{-4}{u\cos^2 x} du = \int \frac{-4}{u(1-\sin^2 x)} du$

$= \int \frac{-4}{u(1-\sin x)(1+\sin x)} du = \int \frac{-4}{u^2(2-u)} du = \int \frac{4}{u^2(u-2)} du$

BY PARTIAL FRACTIONS

$\frac{4}{u^2(u-2)} \equiv \frac{A}{u} + \frac{B}{u^2} + \frac{C}{u-2}$

$Au^2 + B(u-2) + Cu(u-2) \equiv 4$

- IF $u=0$: $-2B = 4$, $B=-2$
- IF $u=2$: $4A = 4$, $A=1$
- IF $A=1$: $A-B-C=4$, $1+2-C=4$, $C=-1$

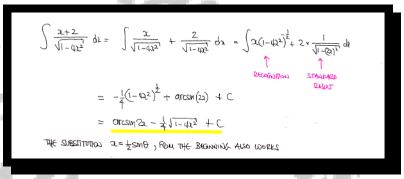
$= \int \frac{1}{u} - \frac{2}{u^2} - \frac{1}{u-2} du = \ln|u| + \frac{1}{u} - \ln|u-2| + C$

$= \ln\left|\frac{u}{u-2}\right| + \frac{1}{u} + C = \ln\left|\frac{u}{2-u}\right| + \frac{1}{u} + C$

$= \ln\left|\frac{1-\sin x}{1+\sin x}\right| + \frac{1}{1-\sin x} + C$

THE SUBSTITUTION $u = \sin x$ ALSO WORKS WELL

446. $\displaystyle\int \frac{x+2}{\sqrt{1-4x^2}} \ dx = \arcsin(2x) - \frac{1}{4}\sqrt{1-4x^2} + C$



$\int \frac{x+2}{\sqrt{1-4x^2}} dx = \int \frac{x}{\sqrt{1-4x^2}} + \frac{2}{\sqrt{1-4x^2}} dx = \int x(1-4x^2)^{-\frac{1}{2}} + 2 \times \frac{1}{\sqrt{1-(2x)^2}} dx$

RECOGNITION STANDARD RESULT

$= -\frac{1}{4}(1-4x^2)^{\frac{1}{2}} + \arcsin(2x) + C$

$= \arcsin 2x - \frac{1}{4}\sqrt{1-4x^2} + C$

THE SUBSTITUTION $x = \frac{1}{2}\sin\theta$, FROM THE BEGINNING ALSO WORKS

447. $$\int \frac{4x+1}{\sqrt{4x^2-9}}\ dx = \sqrt{4x^2-9} + \frac{1}{2}\operatorname{arcosh}\left(\frac{2x}{3}\right) + C$$

$$\int \frac{4x+1}{\sqrt{4x^2-9}}\,dx = \int \frac{4x}{\sqrt{4x^2-9}} + \frac{1}{\sqrt{4x^2-9}}\,dx = \int \underbrace{4x(4x^2-9)^{-\frac{1}{2}}}_{\text{RECOGNITION}} + \underbrace{\frac{1}{2\sqrt{x^2-\frac{9}{4}}}}_{\text{STANDARD RESULT}}\,dx$$

$$= (4x^2-9)^{\frac{1}{2}} + \frac{1}{2}\operatorname{arcosh}\left(\frac{x}{3/2}\right) + C = \sqrt{4x^2-9} + \frac{1}{2}\operatorname{arcosh}\left(\frac{2x}{3}\right) + C$$

THE SUBSTITUTION $x = \frac{3}{2}\cosh\theta$, FROM THE BEGINNING ALSO WORKS

448. $$\int 8x\ln(2x+1)\ dx = \left(4x^2-1\right)\ln(2x+1) + 4x - 2x^2 + C$$

$\ln(2x+1)$	$\frac{2}{2x+1}$
$4x^2$	$8x$

$$\int 8x\ln(2x+1)\,dx = \text{... INTEGRATION BY PARTS ...}$$

$$= 4x^2\ln(2x+1) - \int \frac{8x^2}{2x+1}\,dx = \text{... MANIPULATION OR LONG DIVISION}$$

$$= 4x^2\ln|2x+1| - \int \frac{4x(2x+1) - 2(2x+1) + 2}{2x+1}\,dx$$

$$= 4x^2\ln|2x+1| - \int 4x - 2 + \frac{2}{2x+1}\,dx$$

$$= 4x^2\ln|2x+1| - \left[2x^2 - 2x + \ln|2x+1|\right] + C$$

$$= 4x^2\ln|2x+1| - 2x^2 + 2x - \ln|2x+1| + C$$

$$= (4x^2-1)\ln|2x+1| + 2x - 2x^2 + C$$

449. $$\int \cos 3x \sin 6x\ dx = \begin{cases} -\frac{2}{9}\cos^2 3x + C \\ -\frac{1}{18}\left[\cos 9x + 3\cos 3x\right] + C \\ -\frac{1}{9}\left[2\cos 3x\cos 6x + \sin 3x \sin 6x\right] + C \end{cases}$$

$$\int \cos 3x\sin 6x\,dx = \int \cos 3x(2\sin 3x\cos 3x)\,dx = \int 2\sin 3x\cos^2 3x\,dx$$

BY RECOGNITION

$$= -\frac{2}{9}\cos^3 3x + C$$

ALTERNATIVE BY TRIG IDENTITIES

$$\left.\begin{aligned} \sin(6x+3x) &= \sin 6x\cos 3x + \cos 6x\sin 3x \\ \sin(6x-3x) &= \sin 6x\cos 3x - \cos 6x\sin 3x \end{aligned}\right\}\ \text{ADDING}$$

$$\sin 9x + \sin 3x = 2\sin 6x\cos 3x$$

$$\sin 6x\cos 3x = \frac{1}{2}\sin 9x + \frac{1}{2}\sin 3x$$

$$\dots = \int \frac{1}{2}\sin 9x + \frac{1}{2}\sin 3x\,dx = -\frac{1}{18}\cos 9x - \frac{1}{6}\cos 3x + C$$

ALTERNATIVE BY DOUBLE INTEGRATION BY PARTS

$\cos 3x$	$-3\sin 3x$
$-\frac{1}{6}\cos 6x$	$\sin 6x$

$$\int \cos 3x\sin 6x\,dx = -\frac{1}{6}\cos 3x\cos 6x - \int \frac{1}{2}\sin 3x\cos 6x\,dx$$

BY PARTS AGAIN

$\frac{1}{2}\sin 3x$	$\frac{3}{2}\cos 3x$
$\frac{1}{6}\sin 6x$	$\cos 6x$

$$\int \cos 3x\sin 6x\,dx = -\frac{1}{6}\cos 3x\cos 6x - \left[\frac{1}{12}\sin 3x\sin 6x - \frac{1}{4}\int \sin 6x\cos 3x\,dx\right]$$

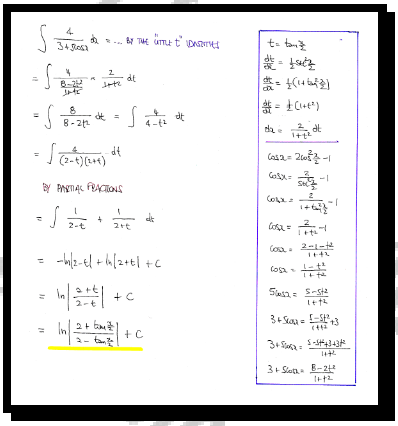
$$\int \cos 3x\sin 6x\,dx = -\frac{1}{6}\cos 3x\cos 6x - \frac{1}{12}\sin 3x\sin 6x + \frac{1}{4}\int \sin 6x\cos 3x\,dx$$

$$4\int \cos 3x\sin 6x\,dx = -\frac{2}{3}\cos 3x\cos 6x - \frac{1}{3}\sin 3x\sin 6x + \int \sin 6x\cos 3x\,dx$$

$$3\int \cos 3x\sin 6x\,dx = -\frac{1}{3}\left[2\cos 3x\cos 6x + \sin 3x\sin 6x\right] + C$$

$$\int \cos 3x\sin 6x\,dx = -\frac{1}{9}\left[2\cos 3x\cos 6x + \sin 3x\sin 6x\right] + C$$

450. $\displaystyle\int \frac{4}{3+5\cos x}\,dx = \ln\left|\frac{2+\tan\left(\frac{1}{2}x\right)}{2-\tan\left(\frac{1}{2}x\right)}\right| + C$



$\displaystyle\int \frac{4}{3+5\cos x}\,dx = \ldots$ BY THE "LITTLE t" IDENTITIES

$\displaystyle = \int \frac{4}{\frac{8-2t^2}{1+t^2}} \times \frac{2}{1+t^2}\,dt$

$\displaystyle = \int \frac{8}{8-2t^2}\,dt = \int \frac{4}{4-t^2}\,dt$

$\displaystyle = \int \frac{4}{(2-t)(2+t)}\,dt$

BY PARTIAL FRACTIONS

$\displaystyle = \int \frac{1}{2-t} + \frac{1}{2+t}\,dt$

$= -\ln|2-t| + \ln|2+t| + C$

$\displaystyle = \ln\left|\frac{2+t}{2-t}\right| + C$

$\displaystyle = \ln\left|\frac{2+\tan\frac{x}{2}}{2-\tan\frac{x}{2}}\right| + C$

$t = \tan\frac{x}{2}$

$\frac{dt}{dx} = \frac{1}{2}\sec^2\frac{x}{2}$

$\frac{dt}{dx} = \frac{1}{2}\left(1+\tan^2\frac{x}{2}\right)$

$\frac{dt}{dx} = \frac{1}{2}(1+t^2)$

$dx = \frac{2}{1+t^2}\,dt$

$\cos x = 2\cos^2\frac{x}{2} - 1$

$\cos x = \frac{2}{\sec^2\frac{x}{2}} - 1$

$\cos x = \frac{2}{1+\tan^2\frac{x}{2}} - 1$

$\cos x = \frac{2}{1+t^2} - 1$

$\cos x = \frac{2-1-t^2}{1+t^2}$

$\cos x = \frac{1-t^2}{1+t^2}$

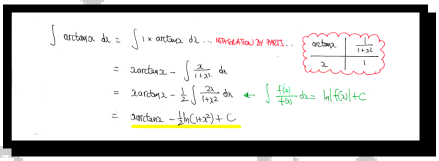
$5\cos x = \frac{5-5t^2}{1+t^2}$

$3+5\cos x = \frac{5-5t^2}{1+t^2} + 3$

$3+5\cos x = \frac{5-5t^2+3+3t^2}{1+t^2}$

$3+5\cos x = \frac{8-2t^2}{1+t^2}$

451. $\displaystyle\int \arctan x\,\,dx = x\arctan x - \frac{1}{2}\ln\left(1+x^2\right) + C$



$\displaystyle\int \arctan x\,dx = \int 1 \times \arctan x\,dx \ldots$ INTEGRATION BY PARTS . .

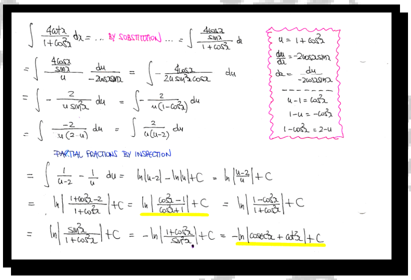
$\arctan x$	$\frac{1}{1+x^2}$
x	1

$\displaystyle = x\arctan x - \int \frac{x}{1+x^2}\,dx$

$\displaystyle = x\arctan x - \frac{1}{2}\int \frac{2x}{1+x^2}\,dx \leftarrow \int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + C$

$= x\arctan x - \frac{1}{2}\ln(1+x^2) + C$

452. $\displaystyle\int \frac{4\cot x}{1+\cos^2 x}\,dx = \begin{bmatrix} \ln\left|\dfrac{1-\cos^2 x}{1+\cos^2 x}\right| + C \\ -\ln\left|\operatorname{cosec}^2 x + \cot^2 x\right| + C \end{bmatrix}$



$\displaystyle\int \frac{4\cot x}{1+\cos^2 x}\,dx = \ldots$ BY SUBSTITUTION $\displaystyle\ldots = \int \frac{\frac{4\cos x}{\sin x}}{1+\cos^2 x}\,dx$

$\displaystyle = \int \frac{\frac{4\cos x}{\sin x}}{u}\,\frac{du}{-2\cos x\sin x} = \int -\frac{4\cos x}{2u\sin^2 x\cos x}\,du$

$\displaystyle = \int -\frac{2}{u\sin^2 x}\,du = \int -\frac{2}{u(1-\cos^2 x)}\,du$

$\displaystyle = \int \frac{-2}{u(2-u)}\,du = \int \frac{2}{u(u-2)}\,du$

PARTIAL FRACTIONS BY INSPECTION

$\displaystyle = \int \frac{1}{u-2} - \frac{1}{u}\,du = \ln|u-2| - \ln|u| + C = \ln\left|\frac{u-2}{u}\right| + C$

$\displaystyle = \ln\left|\frac{1+\cos^2 x-2}{1+\cos^2 x}\right| + C = \ln\left|\frac{\cos^2 x-1}{\cos^2 x+1}\right| + C = \ln\left|\frac{1-\cos^2 x}{1+\cos^2 x}\right| + C$

$\displaystyle = \ln\left|\frac{\sin^2 x}{1+\cos^2 x}\right| + C = -\ln\left|\frac{1+\cos^2 x}{\sin^2 x}\right| + C = -\ln\left|\operatorname{cosec}^2 x + \cot^2 x\right| + C$

$u = 1+\cos^2 x$

$\frac{du}{dx} = -2\cos x\sin x$

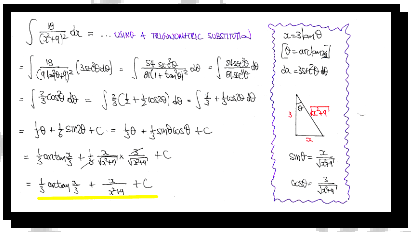
$dx = \frac{du}{-2\cos x\sin x}$

$u - 1 = \cos^2 x$

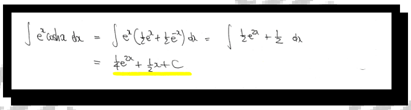
$1 - u = -\cos^2 x$

$1 - \cos^2 x = 2 - u$

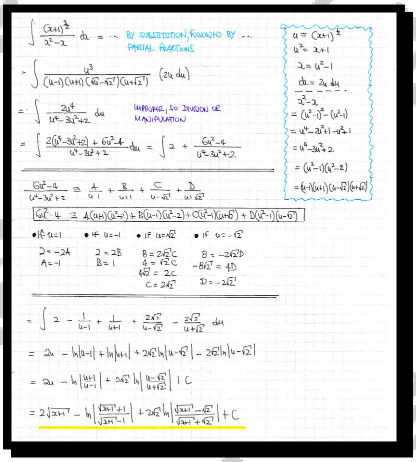
453. $$\int \frac{18}{\left(x^2+9\right)^2}\ dx = \frac{1}{3}\arctan\left(\frac{1}{3}x\right)+\frac{x}{x^2+9}+C$$



454. $$\int e^x \cosh x\ dx = \frac{1}{4}e^{2x}+\frac{1}{2}x+C$$



455. $$\int \frac{(x+1)^{\frac{3}{2}}}{x^2-x}\ dx = 2\sqrt{x+1}-\ln\left|\frac{\sqrt{x+1}+1}{\sqrt{x+1}-1}\right|+2\sqrt{2}\,\text{n}\left|\frac{\sqrt{x+1}-\sqrt{2}}{\sqrt{x+1}+\sqrt{2}}\right|+C$$



456. $$\int \frac{x^4}{\left(x^5-1\right)^{\frac{3}{2}}}\,dx = -\frac{2}{5\sqrt{x^5-1}} + C$$

$\displaystyle\int \frac{x^4}{(x^5-1)^{3/2}}\,dx = \int x^4(x^5-1)^{-\frac{3}{2}}\,dx = \ldots$ BY INSPECTION ...

$= -\frac{2}{5}(x^5-1)^{-\frac{1}{2}} + C = -\dfrac{2}{5\sqrt{x^5-1}} + C$

THE SUBSTITUTION $u = x^5-1$ OR $u=\sqrt{x^5-1}$ ALSO WORK WELL

457. $$\int \frac{9}{e^x\sqrt{e^{2x}-9}}\,dx = \frac{\sqrt{e^{2x}-9}}{e^x} + C$$

$\displaystyle\int \frac{9}{e^x\sqrt{e^{2x}-9}}\,dx = \ldots$ BY SUBSTITUTION ...

$u = \frac{1}{e^x}$, $e^x = \frac{1}{u}$, $e^x\,dx = -\frac{1}{u^2}\,du$, $\frac{1}{u}\,dx = -\frac{1}{u^2}\,du$, $dx = -\frac{1}{u}\,du$

$\displaystyle = \int \frac{9}{\frac{1}{u}\sqrt{\frac{1}{u^2}-9}}\left(-\frac{1}{u}\,du\right) = \int \frac{-9}{\sqrt{\frac{1}{u^2}-9}}\,du$

$\displaystyle = \int \frac{-9}{\sqrt{\frac{1-9u^2}{u^2}}}\,du = \int \frac{-9u}{\sqrt{1-9u^2}}\,du = \int -9u(1-9u^2)^{-\frac{1}{2}}\,du$

$= (1-9u^2)^{\frac{1}{2}} + C = \left(1-\frac{9}{e^{2x}}\right)^{\frac{1}{2}} + C$

$= \left(\frac{e^{2x}-9}{e^{2x}}\right)^{\frac{1}{2}} + C = \dfrac{\sqrt{e^{2x}-9}}{e^x} + C$

ALTERNATIVE APPROACHES USING DOUBLE SUBSTITUTIONS

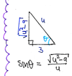
$\displaystyle\int \frac{9}{e^x\sqrt{e^{2x}-9}}\,dx = \ldots\ u = e^x \ldots = \int \frac{9}{u\sqrt{u^2-9}}\left(\frac{1}{u}\,du\right)$

$u = e^x$, $du = e^x\,dx$, $du = u\,dx$, $dx = \frac{1}{u}\,du$

$\displaystyle = \int \frac{9}{u^2\sqrt{u^2-9}}\,du$

NOW A TRIGONOMETRIC SUBSTITUTION TO FOLLOW

$u = 3\sec\theta$, $\theta = \operatorname{arcsec}\frac{u}{3}$, $\theta = \arccos\left(\frac{3}{u}\right)$, $du = 3\sec\theta\tan\theta$



$\sin\theta = \frac{\sqrt{u^2-9}}{u}$

$\displaystyle = \int \frac{\cancel{9}}{\cancel{9}\sec^2\theta\sqrt{9\sec^2\theta-9}}(3\sec\theta\tan\theta)\,d\theta$

$\displaystyle = \int \frac{3\sec\theta\tan\theta}{\sec^2\theta\sqrt{9(\sec^2\theta-1)}}\,d\theta$

$\displaystyle = \int \frac{\cancel{3}\cancel{\sec\theta}\cancel{\tan\theta}}{\sec^{\cancel{2}}\theta(\cancel{3}\cancel{\tan\theta})}\,d\theta$

$\displaystyle = \int \frac{1}{\sec\theta}\,d\theta = \int \cos\theta\,d\theta$

$= \sin\theta + C$

$= \frac{\sqrt{u^2-9}}{u} + C$

$= \dfrac{\sqrt{e^{2x}-9}}{e^x} + C$

ANOTHER APPROACH USING HYPERBOLIC SUBSTITUTIONS

$\displaystyle\int \frac{9}{e^x\sqrt{e^{2x}-9}}\,dx = \int \frac{9\tanh\theta}{3\cosh\theta\sqrt{9\cosh^2\theta-9}}\,d\theta$

$e^x = 3\cosh\theta$, $dx\,e^x = 3\sinh\theta\,d\theta$, $dx = \frac{3\sinh\theta\,d\theta}{e^x}$, $dx = \frac{\cancel{3}\sinh\theta}{\cancel{3}\cosh\theta}\,d\theta$, $dx = \tanh\theta\,d\theta$

$\theta = \operatorname{arcosh}\frac{e^x}{3}$ so $\sinh\theta = \sqrt{\cosh^2\theta-1}$

$\displaystyle = \int \frac{9\tanh\theta}{3\cosh\theta\sqrt{9(\cosh^2\theta-1)}}\,d\theta = \int \frac{9\tanh\theta}{(3\cosh\theta)(3\sinh\theta)}\,d\theta$

$\displaystyle = \int \frac{1}{\cancel{\sinh\theta}\cosh\theta} \times \frac{\cancel{\sinh\theta}}{\cosh\theta}\,d\theta = \int \frac{1}{\cosh^2\theta}\,d\theta$

$\displaystyle = \int \operatorname{sech}^2\theta\,d\theta = \tanh\theta + C = \frac{\sinh\theta}{\cosh\theta} + C$

$= \dfrac{\sqrt{\cosh^2\theta-1}}{\cosh\theta} + C = \dfrac{\sqrt{\left(\frac{1}{3}e^x\right)^2-1}}{\frac{e^x}{3}} + C$

$= \dfrac{\sqrt{\frac{1}{9}e^{2x}-1}}{\frac{e^x}{3}} + C = \dfrac{\cancel{\frac{1}{3}}\sqrt{e^{2x}-9}}{\cancel{\frac{1}{3}}\,e^x} + C$

$= \sqrt{\dfrac{e^{2x}-9}{e^{2x}}} + C$

458. $$\int \frac{1}{\sqrt{1-x-x^2}}\,dx = \arcsin\left(\frac{2x+1}{\sqrt{5}}\right) + C$$

$\displaystyle\int \frac{1}{\sqrt{1-x-x^2}}\,dx = \ldots$ COMPLETING THE SQUARE

$1-x-x^2 = -(x^2+x-1) = -\left[\left(x+\frac{1}{2}\right)^2-\frac{1}{4}-1\right] = \frac{5}{4}-\left(x+\frac{1}{2}\right)^2 = \left(\frac{\sqrt{5}}{2}\right)^2-\left(x+\frac{1}{2}\right)^2$

$\displaystyle = \int \frac{1}{\sqrt{\left(\frac{\sqrt{5}}{2}\right)^2-\left(x+\frac{1}{2}\right)^2}}\,dx$

$= \arcsin\left(\dfrac{x+\frac{1}{2}}{\sqrt{5}/2}\right) + C$

$= \arcsin\left(\dfrac{2x+1}{\sqrt{5}}\right) + C$

459. $$\int \frac{1-3x^3}{\sqrt{1-x^2}}\,dx = \arcsin x + \left(2+x^2\right)\sqrt{1-x^2} + C$$

$$\int \frac{1-3x^3}{\sqrt{1-x^2}}\,dx = \int \frac{1}{\sqrt{1-x^2}} - \frac{3x^3}{\sqrt{1-x^2}}\,dx$$
$$= \arcsin x - 3\int \frac{x^3}{\sqrt{1-x^2}}\,dx$$
$$= \arcsin x - 3\int \frac{x^3}{\cancel{u}}\left(\frac{-\cancel{u}}{x}\right)du$$
$$= \arcsin x - 3\int -x^2\,du$$
$$= \arcsin x + 3\int 1-u^2\,du$$
$$= \arcsin x + 3u - u^3 + C$$
$$= \arcsin x + 3(1-x^2)^{\frac{1}{2}} - (1-x^2)^{\frac{3}{2}} + C$$
$$= \arcsin x + (1-x^2)^{\frac{1}{2}}(3-(1+x^2)) + C$$
$$= \arcsin x + (2+x^2)(1-x^2)^{\frac{1}{2}} + C$$

$u = \sqrt{1-x^2}$
$u^2 = 1-x^2$
$\cancel{2}u\frac{du}{dx} = -\cancel{2}x$
$dx = -\frac{u}{x}\,du$

ALTERNATIVE BY TRIGONOMETRIC SUBSTITUTIONS AND MANIPULATIONS

$$\int \frac{1-3x^3}{\sqrt{1-x^2}}\,dx = \ldots = \int \frac{1-3\sin^3\theta}{\sqrt{1-\sin^2\theta}}(\cos\theta\,d\theta)$$
$$= \int \frac{1-3\sin^3\theta}{\cancel{\cos\theta}}(\cancel{\cos\theta}\,d\theta) = \int 1-3\sin^3\theta\,d\theta$$
$$= \int 1-3\sin\theta(1-\cos^2\theta)\,d\theta$$
$$= \int 1-3\sin\theta+3\sin\theta\cos^2\theta\,d\theta$$
$$= \theta + 3\cos\theta - \cos^3\theta + C$$
$$= \theta + \cos\theta(3-\cos^2\theta) + C$$
$$= \arcsin x + \sqrt{1-x^2}\left[3-(1-x^2)\right] + C$$
$$= \arcsin x + (1-x^2)^{\frac{1}{2}}(2+x^2) + C$$

$\theta = \arcsin x$
$x = \sin\theta$
$dx = \cos\theta\,d\theta$

$\sin\theta = x$
$\cos\theta = \sqrt{1-x^2}$

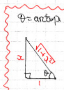
460. $$\int \frac{8}{\left(1+x^2\right)^3}\,dx = 3\arctan x + \frac{3x}{1+x^2} + \frac{2x}{\left(1+x^2\right)^2} + C$$

$$\int \frac{8}{(1+x^2)^3}\,dx = \ldots$$ PARTIAL FRACTIONS WOULD WORK BUT TOO LENGTHY WITH A TRIPLE REPEATED IRREDUCIBLE FACTOR — USE $\tan$ SUBSTITUTION)

$$= \int \frac{8}{(1+\tan^2\theta)^3}(\sec^2\theta\,d\theta) = \int \frac{8\sec^2\theta}{\sec^6\theta}\,d\theta$$
$$= \int \frac{8}{\sec^4\theta}\,d\theta = \int 8\cos^4\theta\,d\theta = \int 8\left(\tfrac{1}{2}+\tfrac{1}{2}\cos 2\theta\right)^2 d\theta$$
$$= \int 8\left(\tfrac{1}{4}+\tfrac{1}{2}\cos 2\theta+\tfrac{1}{4}\cos^2 2\theta\,d\theta\right) = \int 2+4\cos 2\theta+2\cos^2 2\theta\,d\theta$$
$$= \int 2+4\cos 2\theta+2\left(\tfrac{1}{2}+\tfrac{1}{2}\cos 4\theta\right)d\theta = \int 3+4\cos 2\theta+\cos 4\theta\,d\theta$$
$$= 3\theta + 2\sin 2\theta + \tfrac{1}{4}\sin 4\theta + C = 3\theta + 4\sin\theta\cos\theta + \tfrac{1}{2}\sin 2\theta\cos 2\theta + C$$
$$= 3\theta + 4\sin\theta\cos\theta + \sin\theta\cos\theta(2\cos^2\theta-1) + C$$
$$= 3\theta + 4\sin\theta\cos\theta + 2\sin\theta\cos^3\theta - \sin\theta\cos\theta + C$$
$$= 3\theta + 3\sin\theta\cos\theta + 2\sin\theta\cos^3\theta + C$$
$$= 3\arctan x + 3\left(\frac{x}{\sqrt{1+x^2}}\right)\left(\frac{1}{\sqrt{1+x^2}}\right) + 2\left(\frac{x}{\sqrt{1+x^2}}\right)\left(\frac{1}{\sqrt{1+x^2}}\right)^3 + C$$
$$= 3\arctan x + \frac{3x}{1+x^2} + \frac{2x}{(1+x^2)^2} + C$$

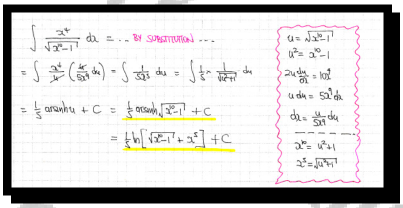
$\theta = \arctan x$
$x = \tan\theta$
$dx = \sec^2\theta\,d\theta$

$\theta = \arctan x$



$\sin\theta = \frac{x}{\sqrt{1+x^2}}$
$\cos\theta = \frac{1}{\sqrt{1+x}}$

461. $$\int \frac{x^4}{\sqrt{x^{10}-1}}\,dx = \begin{cases} \frac{1}{5}\operatorname{arsinh}\left[\sqrt{x^{10}-1}\right] + C \\ \frac{1}{5}\ln\left[x^5+\sqrt{x^{10}-1}\right] + C \end{cases}$$



$\int \frac{x^4}{\sqrt{x^{10}-1}}\,dx = \dots$ BY SUBSTITUTION ...

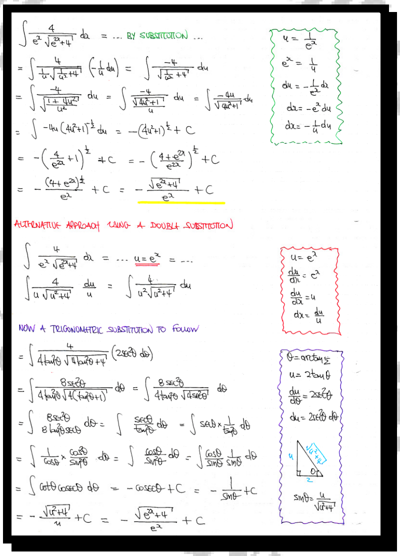
$= \int \frac{x^4}{u}\left(\frac{u}{5x^9}\,du\right) = \int \frac{1}{5x^5}\,du = \int \frac{1}{5}\times\frac{1}{\sqrt{u^2+1}}\,du$

$= \frac{1}{5}\operatorname{arsinh} u + C = \frac{1}{5}\operatorname{arsinh}\sqrt{x^{10}-1} + C$

$= \frac{1}{5}\ln\left[\sqrt{x^{10}-1}+x^5\right] + C$

$u = \sqrt{x^{10}-1}$
$u^2 = x^{10}-1$
$2u\frac{du}{dx} = 10x^9$
$u\,du = 5x^9\,dx$
$dx = \frac{u}{5x^9}\,du$
$x^{10} = u^2+1$
$x^5 = \sqrt{u^2+1}$

462. $$\int \frac{4}{e^x\sqrt{e^{2x}+4}}\,dx = -\frac{\sqrt{e^{2x}+4}}{e^x} + C$$



$\int \frac{4}{e^x\sqrt{e^{2x}+4}}\,dx = \dots$ BY SUBSTITUTION ...

$= \int \frac{4}{\frac{1}{u}\sqrt{\frac{1}{u^2}+4}}\left(-\frac{1}{u}\,du\right) = \int \frac{-4}{\sqrt{\frac{1}{u^2}+4}}\,du$

$= \int \frac{-4}{\sqrt{\frac{1+4u^2}{u^2}}}\,du = \int \frac{-4}{\frac{\sqrt{4u^2+1}}{u}}\,du = \int \frac{-4u}{\sqrt{4u^2+1}}\,du$

$= \int -4u(4u^2+1)^{-\frac{1}{2}}\,du = -(4u^2+1)^{\frac{1}{2}} + C$

$= -\left(\frac{4}{e^{2x}}+1\right)^{\frac{1}{2}} + C = -\left(\frac{4+e^{2x}}{e^{2x}}\right)^{\frac{1}{2}} + C$

$= -\frac{(4+e^{2x})^{\frac{1}{2}}}{e^x} + C = -\frac{\sqrt{e^{2x}+4}}{e^x} + C$

$u = \frac{1}{e^x}$
$e^x = \frac{1}{u}$
$du = -\frac{1}{e^x}\,dx$
$dx = -e^x\,du$
$dx = -\frac{1}{u}\,du$

ALTERNATIVE APPROACH USING A DOUBLE SUBSTITUTION

$\int \frac{4}{e^x\sqrt{e^{2x}+4}}\,dx = \dots \; u = e^x \; = \dots$

$\int \frac{4}{u\sqrt{u^2+4}}\,\frac{du}{u} = \int \frac{4}{u^2\sqrt{u^2+4}}\,du$

$u = e^x$
$\frac{du}{dx} = e^x$
$\frac{du}{dx} = u$
$dx = \frac{du}{u}$

NOW A TRIGONOMETRIC SUBSTITUTION TO FOLLOW

$= \int \frac{4}{4\tan^2\theta\sqrt{4\tan^2\theta+4}}(2\sec^2\theta\,d\theta)$

$= \int \frac{8\sec^2\theta}{4\tan^2\theta\sqrt{4(\tan^2\theta+1)}}\,d\theta = \int \frac{8\sec^2\theta}{4\tan^2\theta\sqrt{4\sec^2\theta}}\,d\theta$

$= \int \frac{8\sec^2\theta}{8\tan^2\theta\sec\theta}\,d\theta = \int \frac{\sec\theta}{\tan^2\theta}\,d\theta = \int \sec\theta\times\frac{1}{\tan^2\theta}\,d\theta$

$= \int \frac{1}{\cos\theta}\times\frac{\cos^2\theta}{\sin^2\theta}\,d\theta = \int \frac{\cos\theta}{\sin^2\theta}\,d\theta = \int \frac{\cos\theta}{\sin\theta}\frac{1}{\sin\theta}\,d\theta$

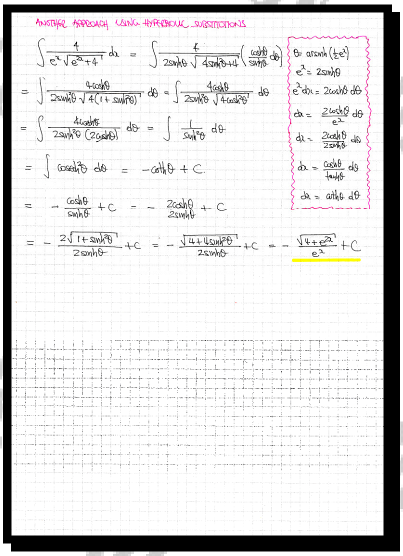
$= \int \cot\theta\operatorname{cosec}\theta\,d\theta = -\operatorname{cosec}\theta + C = -\frac{1}{\sin\theta} + C$

$= -\frac{\sqrt{u^2+4}}{u} + C = -\frac{\sqrt{e^{2x}+4}}{e^x} + C$

$\theta = \arctan\frac{u}{2}$
$u = 2\tan\theta$
$\frac{du}{d\theta} = 2\sec^2\theta$
$du = 2\sec^2\theta\,d\theta$

u, $\sqrt{u^2+4}$, θ, 2

$\sin\theta = \frac{u}{\sqrt{u^2+4}}$



ANOTHER APPROACH USING HYPERBOLIC SUBSTITUTIONS

$\int \frac{4}{e^x\sqrt{e^{2x}+4}}\,dx = \int \frac{4}{2\sinh\theta\sqrt{4\sinh^2\theta+4}}\left(\frac{\cosh\theta}{\sinh\theta}\,d\theta\right)$

$= \int \frac{4\cosh\theta}{2\sinh^2\theta\sqrt{4(1+\sinh^2\theta)}}\,d\theta = \int \frac{4\cosh\theta}{2\sinh^2\theta\sqrt{4\cosh^2\theta}}\,d\theta$

$= \int \frac{4\cosh\theta}{2\sinh^2\theta(2\cosh\theta)}\,d\theta = \int \frac{1}{\sinh^2\theta}\,d\theta$

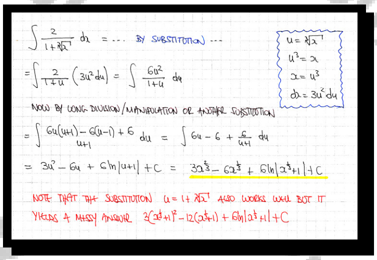
$= \int \operatorname{cosech}^2\theta\,d\theta = -\coth\theta + C$

$= -\frac{\cosh\theta}{\sinh\theta} + C = -\frac{2\cosh\theta}{2\sinh\theta} + C$

$= -\frac{2\sqrt{1+\sinh^2\theta}}{2\sinh\theta} + C = -\frac{\sqrt{4+4\sinh^2\theta}}{2\sinh\theta} + C = -\frac{\sqrt{4+e^{2x}}}{e^x} + C$

$\theta = \operatorname{arsinh}\left(\frac{1}{2}e^x\right)$
$e^x = 2\sinh\theta$
$e^x\,dx = 2\cosh\theta\,d\theta$
$dx = \frac{2\cosh\theta}{e^x}\,d\theta$
$dx = \frac{2\cosh\theta}{2\sinh\theta}\,d\theta$
$dx = \frac{\cosh\theta}{\tanh\theta}\,d\theta$
$dx = \coth\theta\,d\theta$

463. $$\int \frac{2}{1+\sqrt[3]{x}}\,dx = 3x^{\frac{2}{3}} - 6x^{\frac{1}{3}} + 6\ln\left|1+x^{\frac{1}{3}}\right| + C$$



$\int \frac{2}{1+\sqrt[3]{x}}\,dx = \dots$ BY SUBSTITUTION ...

$= \int \frac{2}{1+u}(3u^2\,du) = \int \frac{6u^2}{1+u}\,du$

NOW BY LONG DIVISION/MANIPULATION OR ANOTHER SUBSTITUTION

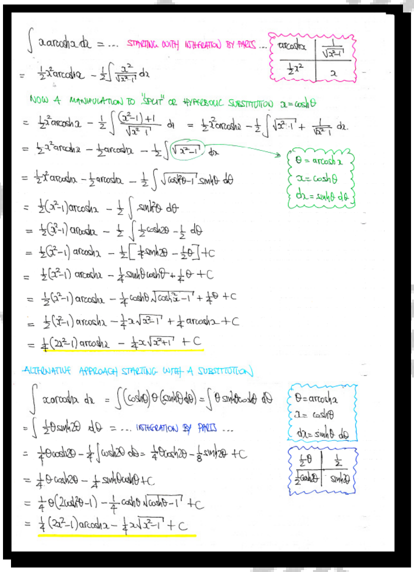
$= \int \frac{6u(u+1) - 6(u-1) + 6}{u+1}\,du = \int 6u - 6 + \frac{6}{u+1}\,du$

$= 3u^2 - 6u + 6\ln|u+1| + C = 3x^{\frac{2}{3}} - 6x^{\frac{1}{3}} + 6\ln|x^{\frac{1}{3}}+1| + C$

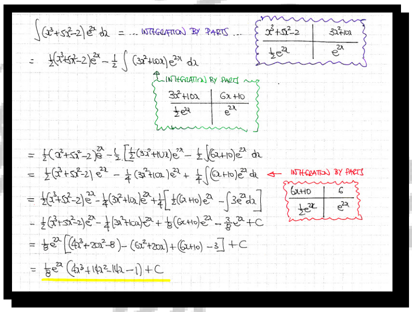
$u = \sqrt[3]{x}$
$u^3 = x$
$x = u^3$
$dx = 3u^2\,du$

NOTE THAT THE SUBSTITUTION $u = 1+\sqrt[3]{x}$ ALSO WORKS WELL BUT IT YIELDS A MESSY ANSWER $3(x^{\frac{1}{3}}+1)^2 - 12(x^{\frac{1}{3}}+1) + 6\ln|x^{\frac{1}{3}}+1| + C$

464. $\displaystyle\int x\,\text{arcosh}\,x\ dx = \frac{1}{2}\left(2x^2-1\right)\text{arcosh}\,x - \frac{1}{4}x\sqrt{x^2-1} + C$



465. $\displaystyle\int \left(x^3+5x^2-2\right)e^{2x}\ dx = \frac{1}{8}\left(4x^3+14x^2-14x-1\right)e^{2x} + C$



466. $$\int \frac{1}{x\sqrt{x^2-2}}\,dx = \begin{cases} \dfrac{1}{\sqrt{2}}\arccos\left[\dfrac{\sqrt{2}}{x}\right] + C \\ -\dfrac{1}{\sqrt{2}}\arcsin\left[\dfrac{\sqrt{2}}{x}\right] + C \\ \dfrac{1}{\sqrt{2}}\arctan\left[\dfrac{\sqrt{x^2-2}}{2}\right] + C \end{cases}$$

$\int \frac{1}{x\sqrt{x^2-2}}\,dx = \dots$ START WITH A SUBSTITUTION

$u = \frac{1}{x}$
$x = \frac{1}{u}$
$dx = -\frac{1}{u^2}\,du$

$= \int \frac{1}{\frac{1}{u}\sqrt{\frac{1}{u^2}-2}}\left(-\frac{1}{u^2}\,du\right) = \int \frac{-1}{\frac{1}{u}\sqrt{\frac{1-2u^2}{u^2}}\,u^2}\,du$

$= \int \frac{-1}{\frac{1}{u}\frac{\sqrt{1-2u^2}}{u}u^2}\,du = \int \frac{-1}{\sqrt{1-2u^2}}\,du$

$= \frac{-1}{\sqrt{2}}\int \frac{1}{\sqrt{\frac{1}{2}-u^2}}\,du = \frac{1}{\sqrt{2}}\int \frac{-1}{\sqrt{\left(\frac{1}{\sqrt{2}}\right)^2-u^2}}\,du = \frac{1}{\sqrt{2}}\arccos\left(\frac{u}{1/\sqrt{2}}\right)+C$

$= \frac{1}{\sqrt{2}}\arccos(\sqrt{2}u)+C = \frac{1}{\sqrt{2}}\arccos\left(\frac{\sqrt{2}}{x}\right)+C$

$= -\frac{1}{\sqrt{2}}\arcsin\left(\frac{\sqrt{2}}{x}\right)+C$

$= \frac{1}{\sqrt{2}}\arctan\left(\sqrt{\frac{x^2-2}{2}}\right)+C$

THE SUBSTITUTION $x = \sqrt{2}\sec\theta$ ALSO WORKS WELL

467. $$\int \sqrt{\frac{1+x}{1-x}}\,dx = \begin{cases} \arcsin x - \sqrt{1-x^2} + C \\ -\arccos x - \sqrt{1-x^2} + C \end{cases}$$

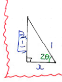
$\int \sqrt{\frac{1+x}{1-x}}\,dx = \int \frac{\sqrt{1+x}}{\sqrt{1-x}}\,dx = \int \frac{\sqrt{1+x}\sqrt{1+x}}{\sqrt{1-x}\sqrt{1+x}}\,dx = \int \frac{1+x}{\sqrt{1-x^2}}\,dx$

$= \int \frac{1}{\sqrt{1-x^2}} + \frac{x}{\sqrt{1-x^2}}\,dx = \int \frac{1}{\sqrt{1-x^2}} + x(1-x^2)^{-\frac{1}{2}}\,dx$

$= \arcsin x - (1-x^2)^{\frac{1}{2}} + C = -\arccos x - (1-x^2)^{\frac{1}{2}} + C$

ALTERNATIVE BY SUBSTITUTION

$\theta = \frac{1}{2}\arccos x$
$x = \cos 2\theta$
$dx = 2\sin 2\theta\,d\theta$



$\int \sqrt{\frac{1+x}{1-x}}\,dx = \int \sqrt{\frac{1+\cos 2\theta}{1-\cos 2\theta}}\,d\theta$

$= \int \sqrt{\frac{1+(2\cos^2\theta-1)}{1-(1-2\sin^2\theta)}}\,(-2\sin 2\theta\,d\theta)$

$= \int \sqrt{\frac{2\cos^2\theta}{2\sin^2\theta}}\,(-4\sin\theta\cos\theta\,d\theta) = \int -4\cos^2\theta\,d\theta$

$= \int -4\left(\frac{1}{2}+\frac{1}{2}\cos 2\theta\right)d\theta = \int -2-2\cos 2\theta\,d\theta$

$= -2\theta - \sin 2\theta + C = -\arccos x - \sqrt{1-x^2} + C$

468. $$\int \sqrt[3]{3\sin 2x - 2\sin 3x\cos x}\,dx = -\frac{4}{3}\sqrt[3]{\cos^4 x} + C$$

$\int \sqrt[3]{3\sin 2x - 2\sin 3x\cos x}\,dx \dots$ USING THE TRIPLE ANGLE IDENTITY

$\sin 3x \equiv 3\sin x - 4\sin^3 x$

$= \int \left[6\sin x\cos x - 2\cos x(3\sin x - 4\sin^3 x)\right]^{\frac{1}{3}}\,dx$

$= \int \left[\cancel{6\sin x\cos x} - \cancel{6\sin x\cos x} + 8\sin^3 x\cos x\right]^{\frac{1}{3}}\,dx = \int 2\sin x(\cos x)^{\frac{1}{3}}\,dx$

$= \frac{2}{-\frac{4}{3}}(\cos x)^{\frac{4}{3}} + C = -\frac{4}{3}\sqrt[3]{\cos^4 x} + C$

469. $\displaystyle\int \frac{25}{25+x^2}\,dx = 5\arctan\left(\tfrac{1}{5}x\right) + C$

$$\int \frac{25}{25+x^2}\,dx = \dots \text{ STANDARD ARCTAN INTEGRAL } = 25\int \frac{1}{x^2+5^2}\,dx$$
$$= \frac{25}{5}\arctan\frac{x}{5} + C = 5\arctan\frac{x}{5} + C$$

470. $\displaystyle\int \frac{4}{x^3+2x}\,dx = \ln\left[\frac{x^2}{x^2+2}\right] + C$

$$\int \frac{4}{x^3+2x}\,dx = \int \frac{4}{x(x^2+2)}\,dx = \dots \text{ BY PARTIAL FRACTIONS}$$
$$\frac{4}{x(x^2+2)} \equiv \frac{A}{x} + \frac{Bx+C}{x^2+2} \Longrightarrow 4 \equiv A(x^2+2) + Bx^2 + Cx$$
$$\Longrightarrow 4 \equiv (A+B)x^2 + Cx + 2A$$
$$\therefore C=0 \quad A=2 \quad B=-2$$
$$\int \frac{2}{x} - \frac{2x}{x^2+2}\,dx = 2\ln|x| - \ln|x^2+2| + C = \ln|x^2| - \ln|x^2+2| + C$$
$$= \ln\left(\frac{x^2}{x^2+2}\right) + C$$

471. $\displaystyle\int \frac{1}{x^2-4}\,dx = \begin{cases} \dfrac{1}{4}\ln\left|\dfrac{x-2}{x+2}\right| + C \\ -\dfrac{1}{2}\operatorname{artanh}\left[\dfrac{2+x}{2-x}\right] + C \end{cases}$

$$\int \frac{1}{x^2-4}\,dx = \int \frac{1}{(x-2)(x+2)}\,dx = \int \frac{\frac{1}{4}}{x-2} - \frac{\frac{1}{4}}{x+2}\,dx$$
$$= \tfrac{1}{4}\ln|x-2| - \tfrac{1}{4}\ln|x+2| + C = \tfrac{1}{4}\ln\left|\frac{x-2}{x+2}\right| + C$$

ALTERNATIVE BY MANIPULATING INTO STANDARD RESULTS FOR artanh OR arcoth

$$\int \frac{1}{x^2-4}\,dx = -\int \frac{1}{4-x^2}\,dx = -\frac{1}{2}\int \frac{2}{4-x^2}\,dx$$
$$= -\tfrac{1}{2}\operatorname{artanh}\left(\tfrac{1}{2}x\right) + C$$
$$= -\tfrac{1}{4}\ln\left(\frac{1+\frac{1}{2}x}{1-\frac{1}{2}x}\right) + C$$
$$= -\tfrac{1}{4}\ln\left(\frac{2+x}{2-x}\right) + C$$
$$= \tfrac{1}{4}\ln\left(\frac{2+x}{2-x}\right) + C$$

$$\frac{d}{dx}(\operatorname{artanh} x) = \frac{1}{1-x^2}$$
$$\frac{d}{dx}\left(\operatorname{artanh}\tfrac{1}{2}x\right) = \frac{\frac{1}{2}}{1-\frac{1}{4}x^2} = \frac{2}{4-x^2}$$

472. $\displaystyle\int 4\coth^2 2x\,dx = 4x - 2\coth 2x + C$

$$\int 4\coth^2 2x\,dx = \dots \left[\begin{array}{l} 1+\cot^2 x \equiv \operatorname{cosec}^2 x \\ 1-\coth^2 x \equiv -\operatorname{cosech}^2 x \end{array}\right] = \int 4(1+\operatorname{cosech}^2 2x)\,dx$$
$$= \int 4 + 4\operatorname{cosech}^2 2x\,dx = 4x - 2\coth 2x + C$$

473. $$\int \frac{x^2}{\sqrt{x^2-16}}\, dx = \begin{cases} 8\operatorname{arcosh}\left(\frac{1}{4}x\right) + \frac{1}{2}x\sqrt{x^2-16} + C \\ \ln\left(x+\sqrt{x^2-16}\right) + \frac{1}{2}x\sqrt{x^2-16} + C \end{cases}$$

$\int \frac{x^2}{\sqrt{x^2-16}}\,dx = \ldots$ MANIPULATE FIRST & SUBSTITUTE AFTER …

$= \int \frac{(x^2-16)+16}{\sqrt{x^2-16}}\,dx = \int (x^2-16)^{\frac{1}{2}} + \frac{16}{\sqrt{x^2-16}}\,dx = 16\operatorname{arcosh}\frac{x}{4} + \int \sqrt{x^2-16}\,dx$

$\theta = \operatorname{arcosh}\frac{x}{4}$
$x = 4\cosh\theta$
$dx = 4\sinh\theta\,d\theta$

$= 16\operatorname{arcosh}\frac{x}{4} + \int \sqrt{16\cosh^2\theta-16}\,(4\sinh\theta\,d\theta)$

$= 16\operatorname{arcosh}\frac{x}{4} + \int 4\sqrt{\cosh^2\theta-1}\,(4\sinh\theta\,d\theta)$

$= 16\operatorname{arcosh}\frac{x}{4} + \int 16\sinh^2\theta\,d\theta$

$\sin\theta \equiv \frac{1}{2} - \frac{1}{2}\cos 2\theta$
$-\sinh^2\theta = \frac{1}{2} - \frac{1}{2}\cosh 2\theta$

$= 16\operatorname{arcosh}\frac{x}{4} + \int 16\left(\frac{1}{2}\cosh 2\theta - \frac{1}{2}\right)d\theta$

$= 16\operatorname{arcosh}\frac{x}{4} + \int 8\cosh 2\theta - 8\,d\theta$

$= 16\operatorname{arcosh}\frac{x}{4} + 4\sinh 2\theta - 8\theta + C = 16\operatorname{arcosh}\frac{x}{4} + 8\sinh\theta\cosh\theta - 8\operatorname{arcosh}\frac{x}{4} + C$

$= 8\operatorname{arcosh}\frac{x}{4} + 8\cosh\theta\sqrt{\cosh^2\theta-1} + C = 8\operatorname{arcosh}\frac{x}{4} + 8\times\frac{1}{4}\times x\times\sqrt{\frac{x^2}{16}-1} + C$

$= 8\operatorname{arcosh}\frac{x}{4} + 2x\times\sqrt{\frac{x^2-16}{16}} + C = 8\operatorname{arcosh}\frac{x}{4} + \frac{1}{2}x\sqrt{x^2-16} + C$

$\left[\operatorname{arcosh}\frac{x}{4} = \ln\left[\frac{x}{4} + \sqrt{\frac{x^2}{16}-1}\right] = \ln\left[\frac{1}{4}\left(x+\sqrt{x^2-16}\right)\right] = \ln\frac{1}{4} + \ln\left(x+\sqrt{x^2-16}\right)\right]$

$= \ln\left(x+\sqrt{x^2-16}\right) + \frac{1}{2}x\sqrt{x^2-16} + C$

NOTE THAT USING $x = 4\cosh\theta$ STRAIGHT AWAY ALSO WORKS WELL

474. $$\int \frac{\arctan x}{1+x^2}\, dx = \frac{1}{2}(\arctan x)^2 + C$$

$\int \frac{\arctan x}{1+x^2}\,dx = \frac{1}{2}(\arctan x)^2 + C$ BY INSPECTION

THE SUBSTITUTION $u = \arctan x$ ALSO WORKS WELL

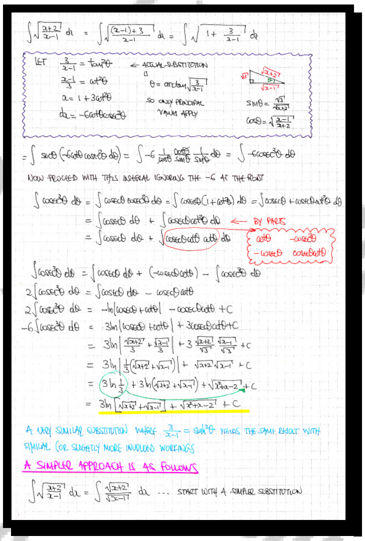
475. $$\int \frac{1}{\sqrt{25-9x^2}}\, dx = \frac{1}{3}\arcsin\left(\frac{3}{5}x\right) + C$$

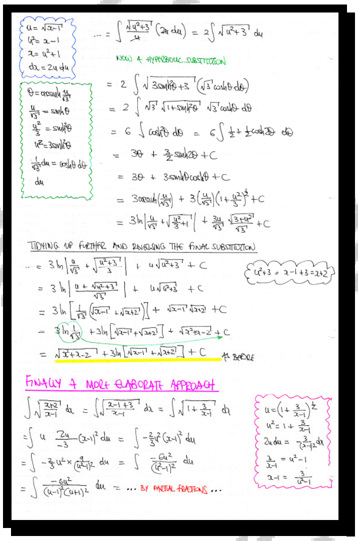
$\int \frac{1}{\sqrt{25-9x^2}}\,dx = \int \frac{1}{3\sqrt{\frac{25}{9}-x^2}}\,dx = \frac{1}{3}\int \frac{1}{\sqrt{\left(\frac{5}{3}\right)^2-x^2}}\,dx$

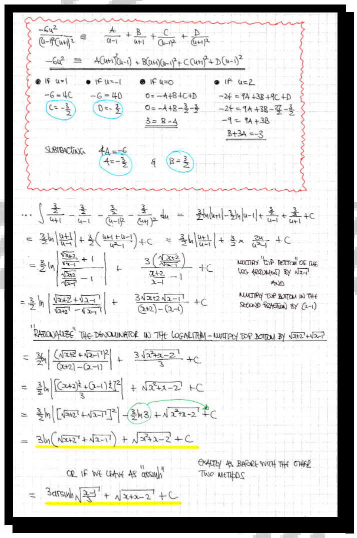
$= \ldots$ BY INSPECTION $\ldots = \frac{1}{3}\arcsin\left(\frac{x}{5/3}\right) + C$

$= \frac{1}{3}\arcsin\left(\frac{3x}{5}\right) + C$

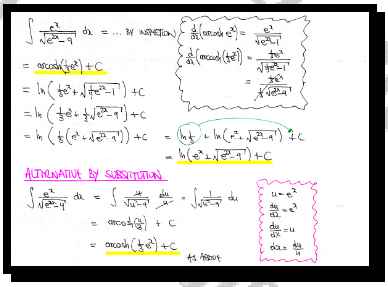
476. $$\int \frac{\sqrt{x+2}}{\sqrt{x-1}}\,dx = \begin{cases} 3\ln\left[\sqrt{x+2}+\sqrt{x-1}\right]+\sqrt{x^2+x-2}+C \\ 3\operatorname{arsinh}\sqrt{\dfrac{x-1}{3}}+\sqrt{x^2+x-2}+C \end{cases}$$



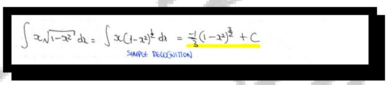




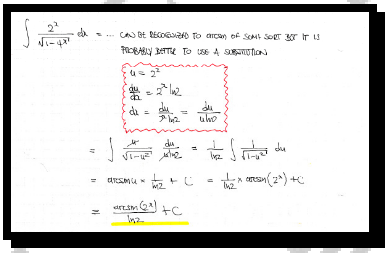
477. $$\int \frac{e^x}{\sqrt{e^{2x}-9}}\,dx = \begin{cases} \ln\left[e^x+\sqrt{e^{2x}-9}\right]+C \\ \operatorname{arcosh}\left[\frac{1}{3}e^x\right]+C \end{cases}$$



478. $$\int x\sqrt{1-x^2}\,dx = -\tfrac{1}{3}\left(1-x^2\right)^{\frac{3}{2}}+C$$



479. $$\int \frac{2^x}{\sqrt{1-4^x}}\,dx = \frac{\arcsin\left(2^x\right)}{\ln 2} + C$$



$\int \frac{2^x}{\sqrt{1-4^x}}\,dx = \dots$ CAN BE RECOGNIZED TO arcsin OF SOME SORT BUT IT IS PROBABLY BETTER TO USE A SUBSTITUTION

$u = 2^x$

$\frac{du}{dx} = 2^x \ln 2$

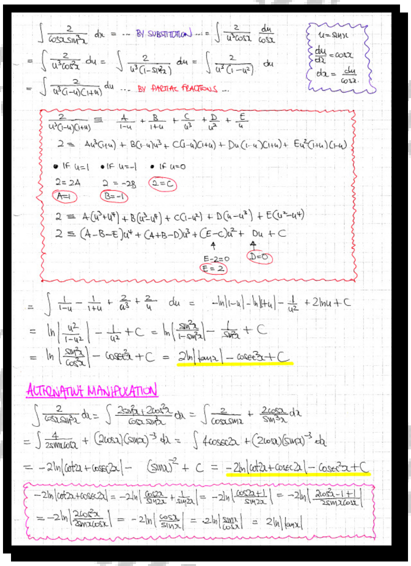
$dx = \frac{du}{2^x \ln 2} = \frac{du}{u \ln 2}$

$= \int \frac{\cancel{u}}{\sqrt{1-u^2}} \frac{du}{\cancel{u}\ln 2} = \frac{1}{\ln 2}\int \frac{1}{\sqrt{1-u^2}}\,du$

$= \arcsin u \times \frac{1}{\ln 2} + C = \frac{1}{\ln 2} \times \arcsin\left(2^x\right) + C$

$= \frac{\arcsin\left(2^x\right)}{\ln 2} + C$

480. $$\int \frac{2}{\cos x \sin^3 x}\,dx = \begin{cases} 2\ln\left|\tan x\right| - \operatorname{cosec}^2 x + C \\ -2\ln\left|\cot 2x + \operatorname{cosec} 2x\right| - \operatorname{cosec}^2 x + C \end{cases}$$



$\int \frac{2}{\cos x \sin^3 x}\,dx = \dots$ BY SUBSTITUTION $\dots = \int \frac{2}{u^3 \cos x}\frac{du}{\cos x}$

$u = \sin x$

$\frac{du}{dx} = \cos x$

$dx = \frac{du}{\cos x}$

$= \int \frac{2}{u^3 \cos^2 x}\,du = \int \frac{2}{u^3(1-\sin^2 x)}\,du = \int \frac{2}{u^3(1-u^2)}\,du$

$= \int \frac{2}{u^3(1-u)(1+u)}\,du \dots$ BY PARTIAL FRACTIONS …

$\frac{2}{u^3(1-u)(1+u)} \equiv \frac{A}{1-u} + \frac{B}{1+u} + \frac{C}{u^3} + \frac{D}{u^2} + \frac{E}{u}$

$2 \equiv Au^3(1+u) + B(1-u)u^3 + C(1-u)(1+u) + Du(1-u)(1+u) + Eu^2(1+u)(1-u)$

- IF $u=1$ • IF $u=-1$ • IF $u=0$

$2 = 2A$ $\quad 2 = -2B$ $\quad 2 = C$

$A = 1$ $\quad B = -1$

$2 \equiv A(u^3+u^4) + B(u^3-u^4) + C(1-u^2) + D(u-u^3) + E(u^2-u^4)$

$2 \equiv (A-B-E)u^4 + (A+B-D)u^3 + (E-C)u^2 + Du + C$

$E - 2 = 0$ $\quad D = 0$

$E = 2$

$= \int \frac{1}{1-u} - \frac{1}{1+u} + \frac{2}{u^3} + \frac{2}{u}\,du = -\ln|1-u| - \ln|1+u| - \frac{1}{u^2} + 2\ln u + C$

$= \ln\left|\frac{u^2}{1-u^2}\right| - \frac{1}{u^2} + C = \ln\left|\frac{\sin^2 x}{1-\sin^2 x}\right| - \frac{1}{\sin^2 x} + C$

$= \ln\left|\frac{\sin^2 x}{\cos^2 x}\right| - \operatorname{cosec}^2 x + C = 2\ln|\tan x| - \operatorname{cosec}^2 x + C$

ALTERNATIVE MANIPULATION

$\int \frac{2}{\cos x \sin^3 x}\,dx = \int \frac{2\sin^2 x + 2\cos^2 x}{\cos x \sin^3 x}\,dx = \int \frac{2}{\cos x \sin x} + \frac{2\cos x}{\sin^3 x}\,dx$

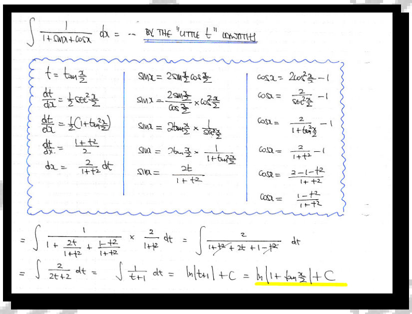
$= \int \frac{4}{2\sin x \cos x} + (2\cos x)(\sin x)^{-3}\,dx = \int 4\operatorname{cosec} 2x + (2\cos x)(\sin x)^{-3}\,dx$

$= -2\ln|\cot 2x + \operatorname{cosec} 2x| - (\sin x)^{-2} + C = -2\ln|\cot 2x + \operatorname{cosec} 2x| - \operatorname{cosec}^2 x + C$

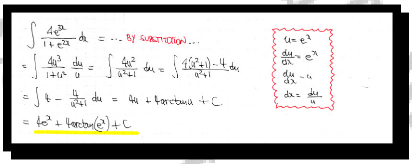
$-2\ln|\cot 2x + \operatorname{cosec} 2x| = -2\ln\left|\frac{\cos 2x}{\sin 2x} + \frac{1}{\sin 2x}\right| = -2\ln\left|\frac{\cos 2x + 1}{\sin 2x}\right| = -2\ln\left|\frac{2\cos^2 x - 1 + 1}{2\sin x \cos x}\right|$

$= -2\ln\left|\frac{2\cos^2 x}{2\sin x \cos x}\right| = -2\ln\left|\frac{\cos x}{\sin x}\right| = 2\ln\left|\frac{\sin x}{\cos x}\right| = 2\ln|\tan x|$

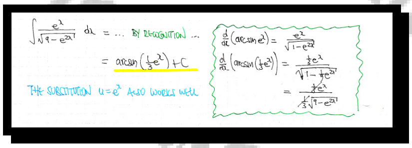
481. $$\int \frac{1}{1+\sin x+\cos x}\,dx = \ln\left|1+\tan\left(\tfrac{1}{2}x\right)\right| + C$$



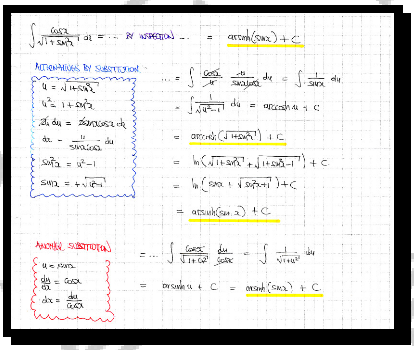
482. $$\int \frac{4\mathrm{e}^{3x}}{1+\mathrm{e}^{2x}}\,dx = 4\mathrm{e}^{x}+\arctan\left(\mathrm{e}^{x}\right) + C$$



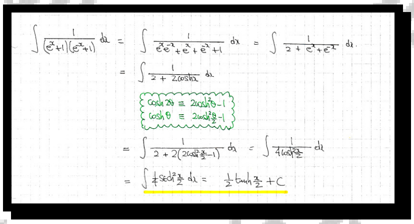
483. $$\int \frac{\mathrm{e}^{x}}{\sqrt{9-\mathrm{e}^{2x}}}\,dx = \arcsin\left(\tfrac{1}{3}\mathrm{e}^{x}\right) + C$$



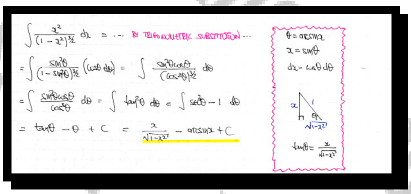
484. $$\int \frac{\cos x}{\sqrt{1+\sin^2 x}}\,dx = \begin{cases} \operatorname{arsinh}(\sin x) + C \\ \operatorname{arcosh}\left(\sqrt{1+\sin^2 x}\right) + C \end{cases}$$



485. $$\int \frac{1}{\left(e^x+1\right)\left(e^x-1\right)}\,dx = \frac{1}{2}\tanh\left(\frac{1}{2}x\right) + C$$



486. $$\int \frac{x^2}{\left(1-x^2\right)^{\frac{3}{2}}}\,dx = \frac{x}{\sqrt{1-x^2}} - \arcsin x + C$$



487. $$\int \frac{1}{5\cosh x + 4\sinh x}\,dx = \frac{2}{3}\arctan\left(3e^{x}\right) + C$$

$$\int \frac{1}{5\cosh x + 4\sinh x}\,dx = \dots \text{ SWITCH INTO EXPONENTIALS}$$

$$= \int \frac{1}{\frac{5}{2}(e^{x}+e^{-x}) + \frac{4}{2}(e^{x}-e^{-x})}\,dx = \int \frac{2}{5e^{x}+5e^{-x}+4e^{x}-4e^{-x}}\,dx = \int \frac{2}{9e^{x}+e^{-x}}\,dx$$

BY SUBSTITUTION

$u = e^{x}$

$\frac{du}{dx} = e^{x}$

$\frac{du}{dx} = u$

$dx = \frac{du}{u}$

$$\dots = \int \frac{2}{9u + u^{-1}}\left(\frac{du}{u}\right) = \int \frac{2}{9u^2+1}\,du$$

$$= \frac{2}{9}\int \frac{1}{u^2 + \frac{1}{9}}\,du = \frac{2}{9}\int \frac{1}{u^2 + \left(\frac{1}{3}\right)^2}\,du$$

$$= \frac{2}{9} \times \frac{1}{\frac{1}{3}} \times \arctan\left(\frac{u}{\frac{1}{3}}\right) + C = \frac{2}{3}\arctan(3u) + C$$

$$= \frac{2}{3}\arctan\left(3e^{x}\right) + C$$

488. $$\int \frac{x+3}{\sqrt{4-x^2}}\,dx = 3\arcsin\left(\tfrac{1}{2}x\right) - \sqrt{4-x^2} + C$$

$$\int \frac{x+3}{\sqrt{4-x^2}}\,dx = \dots \text{ SPLIT THE FRACTION } = \int \frac{x}{\sqrt{4-x^2}} + \frac{3}{\sqrt{4-x^2}}\,dx$$

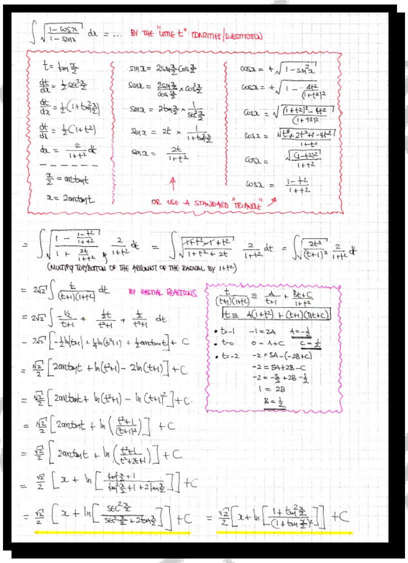
$$= \int x(4-x^2)^{-\frac{1}{2}} + \frac{3}{\sqrt{4-x^2}}\,dx$$

BOTH "STANDARD" OR RECOGNIZABLE

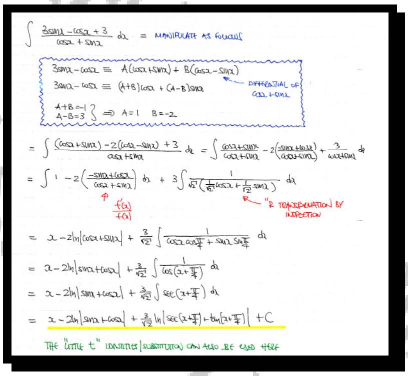
$$= -(4-x^2)^{\frac{1}{2}} + 3\arcsin\frac{x}{2} + C$$

$$= 3\arcsin\left(\frac{x}{2}\right) - \sqrt{4-x^2} + C$$

489. $$\int \sqrt{\frac{1-\cos x}{1-\sin x}}\,dx = \begin{cases} \dfrac{1}{\sqrt{2}}\left[x+\ln\left[\dfrac{\sec^2\left(\frac{1}{2}x\right)}{\sec^2\left(\frac{1}{2}x\right)+2\tan\left(\frac{1}{2}x\right)}\right]\right]+C \\[2ex] \dfrac{1}{\sqrt{2}}\left[x+\ln\left[\dfrac{1+\tan^2\left(\frac{1}{2}x\right)}{\left[1+\tan\left(\frac{1}{2}x\right)\right]^2}\right]\right]+C \end{cases}$$



490. $$\int \frac{3\sin x-\cos x+3}{\cos x+\sin x}\,dx = x-2\ln\left|\sin x+\cos x\right|+\frac{3}{\sqrt{2}}\ln\left|\tan\left(x+\tfrac{1}{4}\pi\right)+\sec\left(x+\tfrac{1}{4}\pi\right)\right|+C$$



491. $$\int \frac{4x-1}{\sqrt{x^2+16}}\,dx = 4\sqrt{x^2+16} - \text{arsinh}\left(\tfrac{1}{4}x\right) + C$$

$$\int \frac{4x-1}{\sqrt{x^2+16}}\,dx = \ldots$$ SPLIT AND RECOGNIZE $$= \int \frac{4x}{\sqrt{x^2+16}} - \frac{1}{\sqrt{x^2+16}}\,dx$$

$$= \int 4x(x^2+16)^{-\frac{1}{2}} - \frac{1}{\sqrt{x^2+16}}\,dx = 4(x^2+16)^{\frac{1}{2}} - \text{arsinh}\frac{x}{4} + C$$

$$= 4\sqrt{x^2+16} - \text{arsinh}\frac{x}{4} + C$$

492. $$\int \frac{1}{4x^2+4x+2}\,dx = \frac{1}{2}\arctan(2x+1) + C$$

$$\int \frac{1}{4x^2+4x+2}\,dx = \frac{1}{4}\int \frac{1}{x^2+x+\frac{1}{2}}\,dx = \frac{1}{4}\int \frac{1}{(x+\frac{1}{2})^2+\frac{1}{4}}\,dx$$

$$= \frac{1}{4}\int \frac{1}{(x+\frac{1}{2})^2+(\frac{1}{2})^2}\,dx = \frac{1}{4}\times\frac{1}{\frac{1}{2}}\arctan\left(\frac{x+\frac{1}{2}}{\frac{1}{2}}\right) + C$$

$$= \frac{1}{2}\arctan(2x+1) + C$$

493. $$\int \frac{1}{1+8\sin^2 x}\,dx = \frac{1}{3}\arctan(3\tan x) + C$$

$$\int \frac{1}{1+8\sin^2 x}\,dx = \ldots$$ MANIPULATE AS FOLLOWS $$= \int \frac{\frac{1}{\cos^2 x}}{\frac{1}{\cos^2 x} + \frac{8\sin^2 x}{\cos^2 x}}\,dx$$

$$= \int \frac{\sec^2 x}{\sec^2 x + 8\tan^2 x}\,dx = \int \frac{\sec^2 x}{(1+\tan^2 x)+8\tan^2 x}\,dx$$

$$= \int \frac{\sec^2 x}{9\tan^2 x+1}\,dx = \frac{1}{9}\int \frac{\sec^2 x}{\tan^2 x+\frac{1}{9}}\,dx$$

THIS CAN BE RECOGNISED AS THE DIFFERENTIAL OF $\arctan(3\tan x)$, OR USE THE SUBSTITUTION $u=\tan x$

$$= \frac{1}{9}\times\frac{1}{\frac{1}{3}}\arctan(3\tan x) + C = \frac{1}{3}\arctan(3\tan x) + C$$

NOTE THAT THE "LITTLE t" SUBSTITUTION $t=\tan x$ WORKS WELL TOO

494. $$\int \frac{1}{\sqrt{\sin x\cos^3 x}}\,dx = 2\sqrt{\tan x} + C$$

$$\int \frac{1}{\sqrt{\sin x\cos^3 x}}\,dx = \int \frac{1}{\sqrt{\frac{\sin x}{\cos x}\times\cos^4 x}}\,dx = \int \frac{1}{\sqrt{\tan x\cos^4 x}}\,dx$$

$$= \int \frac{1}{\sqrt{\tan x}\sqrt{\cos^4 x}}\,dx = \int (\tan x)^{-\frac{1}{2}}\sec^2 x\,dx$$

BY RECOGNITION OR SUBSTITUTION $u=\sqrt{\tan x}$

$$= 2(\tan x)^{\frac{1}{2}} + C = 2\sqrt{\tan x} + C$$

495. $$\int \frac{\sin x}{16+9\cos^2 x}\,dx = -\frac{1}{12}\arctan\left(\frac{3\cos x}{4}\right) + C$$

$$\int \frac{\sin x}{16+9\cos^2 x}\,dx = \quad \text{THIS IS AN OBVIOUS DIFFERENTIAL OF } \arctan(k\cos x)$$

$$= \frac{1}{9}\int \frac{\sin x}{\frac{16}{9}+\cos^2 x}\,dx = \frac{1}{9}\int \frac{\sin x}{\left(\frac{4}{3}\right)^2+\cos^2 x}\,dx$$

$$= \frac{1}{9}\times\frac{-1}{4/3}\arctan\left(\frac{\cos x}{4/3}\right)+C = -\frac{1}{12}\arctan\left(\frac{3\cos x}{4}\right)+C$$

THE SUBSTITUTION $u=\sin x$ OR $\cos x = \frac{4}{3}\tan\theta$ BOTH WORK WELL

496. $$\int \sinh^4 x \coth x\; dx = \frac{1}{4}\sinh^4 x + C$$

$$\int \sinh^4 x\coth x\,dx = \int \sinh^4 x\,\frac{\cosh x}{\sinh x}\,dx = \int \sinh^3 x\cosh x\,dx$$

$$= \frac{1}{4}\sinh^4 x + C \quad \text{(BY INSPECTION)}$$

497. $$\int \frac{5\cos x - \sin x}{3\cos x + 2\sin x}\,dx = x + \ln\left|3\cos x + 2\sin x\right| + C$$

$$\int \frac{5\cos x-\sin x}{3\cos x+2\sin x}\,dx = \ldots$$

WRITE THE NUMERATOR IN THE FORM

$$A(5\cos x-\sin x)+B\frac{d}{dx}(3\cos x+2\sin x)$$

IE $A(5\cos x-\sin x)+B(-3\sin x+2\cos x)\equiv \text{NUMERATOR}$

HERE THE NUMBERS ARE VERY EASY

$$= \int \frac{3\cos x+2\sin x-3\sin x+2\cos x}{3\cos x+2\sin x}\,dx$$

$$= \int \frac{3\cos x+2\sin x}{3\cos x+2\sin x}+\frac{-3\sin x+2\cos x}{3\cos x+2\sin x}\,dx$$

OF THE FORM $\int \frac{f'(x)}{f(x)}\,dx$

$$= \int 1+\frac{2\cos x-3\sin x}{2\sin x+3\cos x}\,dx = x+\ln|2\sin x+3\cos x|+C$$

THIS CAN ALSO BE DONE (VERY LENGTHY) BY THE SUBSTITUTION $t=\tan\frac{x}{2}$

498. $$\int \frac{3x-4}{x^2+2x+17}\,dx = \frac{3}{2}\ln\left(x^2+2x+17\right) - \frac{7}{2}\arctan\left(\frac{x+1}{2}\right) + C$$

$$\int \frac{3x-4}{x^2+2x+17}\,dx$$

AS THE DENOMINATOR IS IRREDUCIBLE, COMPLETE THE SQUARE

$$= \int \frac{3(x+1)-7}{(x+1)^2+16}\,dx = \int \frac{3(x+1)}{(x+1)^2+16} - \frac{7}{(x+1)^2+16}\,dx$$

ANOTHER SUBSTITUTION $u=x+1 \Longleftrightarrow x=u-1$ IS POSSIBLE

$$= \frac{3}{2}\int \frac{2(x+1)}{(x+1)^2+16}\,dx - 7\int \frac{1}{(x+1)^2+4^2}\,dx$$

OF THE FORM $\int \frac{f'(x)}{f(x)}\,dx$ STANDARD ARCTAN

$$= \frac{3}{2}\ln\left((x+1)^2+16\right) - 7\arctan\left(\frac{x+1}{2}\right)\times\frac{1}{2} = \frac{3}{2}\ln(x^2+2x+17)-\frac{7}{2}\arctan\left(\frac{x+1}{2}\right)+C$$

499. $$\int \frac{x+4}{\sqrt{x^2+8x+3}}\,dx = \sqrt{x^2+8x+3} + C$$

$\int \frac{x+4}{\sqrt{x^2+8x+3}}\,dx = \dots$ NOTE THAT $\frac{d}{dx}(x^2+8x+3) = 2x+8 = 2(x+4)$

$= \int (x+4)(x^2+8x+3)^{-\frac{1}{2}}\,dx$

BY RECOGNITION OR SUBSTITUTION $u = x^2+8x+3$

$= \frac{1}{2}\int (2x+8)(x^2+8x+3)\,dx$

$= (x^2+8x+3)^{\frac{1}{2}} + C = \sqrt{x^2+8x+3} + C$

500. $$\int \operatorname{arsinh}\left(\sqrt{x}\right)\,dx = \tfrac{1}{2}x(2x+1)\operatorname{arsinh}\left(\sqrt{x}\right) - \tfrac{1}{2}\sqrt{x^2+x} + C$$

$\int \operatorname{arsinh}(\sqrt{x})\,dx = \dots$ BY SUBSTITUTION …

$u = \operatorname{arsinh}(\sqrt{x})$

$\sinh u = \sqrt{x}$

$x = \sinh^2 u$

$dx = 2\sinh u\cosh u\,du$

$dx = \sinh 2u\,du$

$\cosh u = +\sqrt{1+\sinh^2 u}$

$\cosh u = \sqrt{1+x}$

$= \int u(\sinh 2u)\,du =$ INTEGRATION BY PARTS

u	1
$\frac{1}{2}\cosh 2u$	$\sinh 2u$

$= \frac{1}{2}u\cosh 2u - \int \frac{1}{2}\cosh 2u\,du$

$= \frac{1}{2}u\cosh 2u - \frac{1}{4}\sinh 2u + C$

$= \frac{1}{2}u(1+2\sinh^2 u) - \frac{1}{2}\sinh u\cosh u + C$

$\left\{\begin{matrix}\cos 2\theta \equiv 1-2\sin^2\theta \\ \cosh 2\theta \equiv 1+2\sinh^2\theta\end{matrix}\right\}$ $\left\{\begin{matrix}\sin 2\theta \equiv 2\sin\theta\cos\theta \\ \sinh 2\theta \equiv 2\sinh\theta\cosh\theta\end{matrix}\right\}$

$= \frac{1}{2}(\operatorname{arsinh}\sqrt{x})(1+2x) - \frac{1}{2}\sqrt{x}(\sqrt{1+x}) + C$

$= \frac{1}{2}(2x+1)\operatorname{arsinh}\sqrt{x} - \frac{1}{2}\sqrt{x^2+x} + C$

ALTERNATIVE SUBSTITUTION IS $u = \sqrt{x}$ … LEADS TO $\int 2u\operatorname{arsinh}u\,du$, FOLLOWED BY INTEGRATION BY PARTS IS AN ALTERNATIVE

501. $$\int \frac{1}{2x^2+7x+3}\,dx = \frac{1}{5}\ln\left|\frac{2x+1}{x+3}\right| + C$$

$\int \frac{1}{2x^2+7x+3}\,dx = \int \frac{1}{(2x+1)(x+3)}\,dx \dots$ PARTIAL FRACTIONS BY INSPECTION

$= \int \frac{2/5}{2x+1} + \frac{-1/5}{x+3}\,dx = \int \frac{2/5}{2x+1} - \frac{1/5}{x+3}\,dx$

$= \frac{1}{5}\ln|2x+1| - \frac{1}{5}\ln|x+3| + C = \frac{1}{5}\ln\left|\frac{2x+1}{x+3}\right| + C$

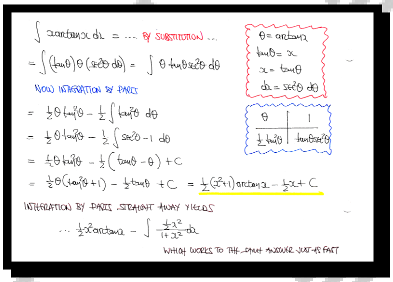
502. $$\int \frac{10\sinh 2x}{\cosh^2 2x}\,dx = -5\operatorname{sech} 2x + C$$

$\int \frac{10\sinh 2x}{\cosh^2 2x}\,dx = \int \frac{10\sinh 2x}{\cosh 2x} \times \frac{1}{\cosh 2x}\,dx = \int 10\tanh 2x\operatorname{sech} 2x$

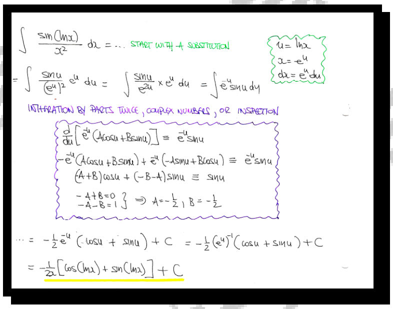
NOW $\frac{d}{dx}[\operatorname{sech} x] = -\operatorname{sech} x\tanh x$

$= -5\operatorname{sech} 2x + C$

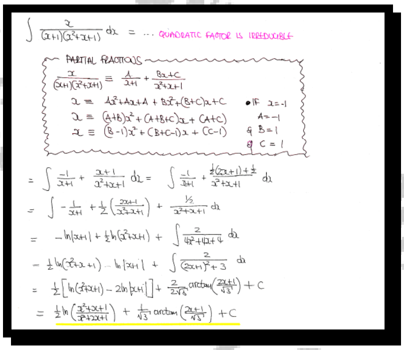
503. $$\int x\arctan x \ dx = \frac{1}{2}\left(x^2+1\right)\arctan x - \frac{1}{2}x + C$$



504. $$\int \frac{\sin\left(\ln x\right)}{x^2} \ dx = -\frac{1}{2x}\left[\cos\left(\ln x\right)+\sin\left(\ln x\right)\right] + C$$



505. $$\int \frac{x}{\left(x+1\right)\left(x^2+x+1\right)} \ dx = \frac{1}{2}\ln\left(\frac{x^2+x+1}{x^2+2x+1}\right) + \frac{1}{\sqrt{3}}\arctan\left(\frac{2x+1}{\sqrt{3}}\right) + C$$



506. $\displaystyle\int x\left(2^{x}\right)\,dx=\frac{2^{x}}{\ln 2}\left[x-\frac{1}{\ln 2}\right]+C$

$$\int x(2^x)\,dx = \text{INTEGRATION BY PARTS}$$

$$= \frac{x(2^x)}{\ln 2} - \int \frac{1}{\ln 2}2^x\,dx$$

$$= \frac{x(2^x)}{\ln 2} - \frac{1}{(\ln 2)^2}(2^x)+C$$

$$= \frac{2^x}{\ln 2}\left[x-\frac{1}{\ln 2}\right]+C$$

x	1
$\frac{1}{\ln 2}2^x$	2^x

507. $\displaystyle\int e^{2x}\sinh x\,dx=\frac{1}{6}e^{3x}-\frac{1}{2}e^{x}+C$

$$\int e^{2x}\sinh x\,dx = \int e^{2x}\left(\frac{e^x}{2}-\frac{e^{-x}}{2}\right)dx = \int \frac{1}{2}e^{3x}-\frac{1}{2}e^{x}+C$$

$$= \frac{1}{6}e^{3x}-\frac{1}{2}e^{x}+C$$

508. $\displaystyle\int \frac{x^2}{\left(x^2+8\right)^{\frac{3}{2}}}\,dx=\ln\left[x+\sqrt{x^2+8}\right]-\frac{x}{\sqrt{x^2+8}}+C$

$$\int \frac{x^2}{(x^2+8)^{\frac{3}{2}}}\,dx = \ldots \text{HYPERBOLIC SUBSTITUTION} \ldots$$

$$= \int \frac{8\sinh^2\theta}{(8\cosh^2\theta)^{3/2}}\sqrt{8}\cosh\theta\,d\theta = \int \frac{\cancel{8\sqrt{8}}\sinh^2\theta\cosh\theta}{\cancel{8^{3/2}}\cosh^3\theta}\,d\theta$$

$$= \int \frac{\sinh^2\theta}{\cosh^2\theta}\,d\theta = \int \tanh^2\theta\,d\theta = \int 1-\operatorname{sech}^2\theta\,d\theta$$

$$1+\tan^2\theta=\sec^2\theta \qquad 1-\tanh^2\theta=\operatorname{sech}^2\theta$$

$$= \theta-\tanh\theta+C = \theta-\frac{\sinh\theta}{\cosh\theta}+C = \theta-\frac{\sinh\theta}{\sqrt{1+\sinh^2\theta}}+C$$

$$= \operatorname{arsinh}\left(\frac{x}{\sqrt{8}}\right)-\frac{\frac{x}{\sqrt{8}}}{\sqrt{1+\frac{x^2}{8}}}+C = \ln\left[\frac{x}{\sqrt{8}}+\sqrt{\frac{x^2}{8}+1}\right]-\frac{x/\sqrt{8}}{\sqrt{\frac{8+x^2}{8}}}+C$$

$$= \ln\left[\frac{x}{\sqrt{8}}+\frac{1}{\sqrt{8}}\sqrt{x^2+8}\right]-\frac{x/\sqrt{8}}{\sqrt{x^2+8}/\sqrt{8}}+C = \ln\left[\frac{1}{\sqrt{8}}\left(x+\sqrt{x^2+8}\right)\right]-\frac{x}{\sqrt{x^2+8}}+C$$

$$= \ln\left(\frac{1}{\sqrt{8}}\right)+\ln\left(x+\sqrt{x^2+8}\right)-\frac{x}{\sqrt{x^2+8}}+C = \ln\left[x+\sqrt{x^2+8}\right]-\frac{x}{\sqrt{x^2+8}}+C$$

$x=\sqrt{8}\sinh\theta$

$\theta=\operatorname{arsinh}\frac{x}{\sqrt{8}}$

$x^2+8=8\sinh^2\theta+8$

$=8(1+\sinh^2\theta)$

$=8\cosh^2\theta$

$dx=\sqrt{8}\cosh\theta$

509. $\displaystyle\int \ln\left(1+x^2\right)\,dx=x\ln\left(1+x^2\right)-2x+2\arctan x+C$

$$\int \ln(1+x^2)\,dx = \ldots \text{INTEGRATION BY PARTS}$$

$$= x\ln(1+x^2)-\int\frac{2x^2}{1+x^2}\,dx$$

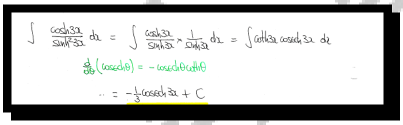
$$= x\ln(1+x^2)-2\int\frac{(x^2+1)-1}{x^2+1}\,dx = x\ln(1+x^2)-2\int 1-\frac{1}{1+x^2}\,dx$$

$$= x\ln(1+x^2)-2\left[x-\arctan x\right]+C$$

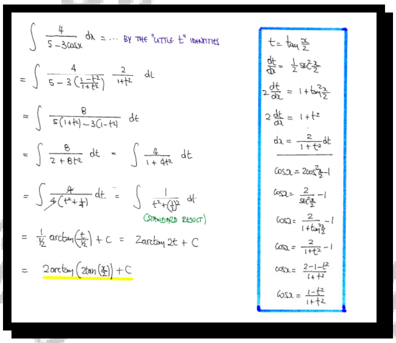
$$= x\ln(1+x^2)-2x+2\arctan x+C$$

$\ln(1+x^2)$	$\frac{2x}{1+x^2}$
x	1

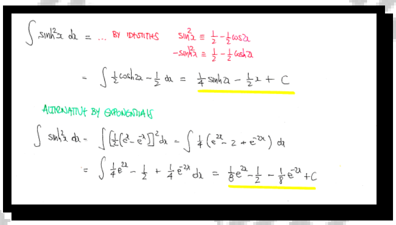
510. $\displaystyle\int \frac{\cosh 3x}{\sinh^2 3x}\, dx = -\frac{1}{3}\operatorname{cosech} 3x + C$



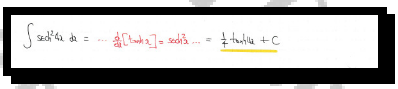
511. $\displaystyle\int \frac{4}{5-3\cos x}\, dx = 2\arctan\left[2\tan\left(\frac{1}{2}x\right)\right] + C$



512. $\displaystyle\int \sinh^2 x\, dx = \begin{bmatrix} \frac{1}{2}\sinh 2x - \frac{1}{2}x + C \\ \frac{1}{8}e^{2x} - \frac{1}{2} - \frac{1}{8}e^{-2x} + C \end{bmatrix}$



513. $\displaystyle\int \operatorname{sech}^2 4x\, dx = \frac{1}{4}\tanh 4x + C$



514. $$\int \frac{1}{\sqrt{64+9x^2}}\,dx = \frac{1}{3}\operatorname{arsinh}\left(\frac{3}{8}x\right) + C$$

$$\int \frac{1}{\sqrt{64+9x^2}}\,dx = \int \frac{1}{3\sqrt{x^2+\frac{64}{9}}}\,dx = \int \frac{1}{3\sqrt{x^2+\left(\frac{8}{3}\right)^2}}\,dx$$

(STANDARD RESULT)

$$= \frac{1}{3}\times \operatorname{arsinh}\left(\frac{x}{8/3}\right)+C = \frac{1}{3}\operatorname{arsinh}\left(\frac{3x}{8}\right)+C$$

515. $$\int \frac{8}{\sqrt{16x^2-1}}\,dx = 2\operatorname{arcosh}4x + C$$

$$\int \frac{8}{\sqrt{16x^2-1}}\,dx = \int \frac{8}{4\sqrt{x^2-\frac{1}{16}}}\,dx = \int \frac{2}{\sqrt{x^2-\left(\frac{1}{4}\right)^2}}\,dx$$

(STANDARD RESULT)

$$= 2\operatorname{arcosh}\left(\frac{x}{1/4}\right)+C = 2\operatorname{arcosh}4x + C$$

516. $$\int \frac{1}{\sqrt{x}\cos^2\sqrt{x}}\,dx = 2\tan\sqrt{x} + C$$

$\int \frac{1}{\sqrt{x}\cos^2\sqrt{x}}\,dx = \ldots$ BY RECOGNITION AS $\ldots = \int x^{-\frac{1}{2}}\sec^2\left(x^{\frac{1}{2}}\right)\,dx$

$$= 2\tan x^{\frac{1}{2}} + C = 2\tan\sqrt{x} + C$$

THE SUBSTITUTION $u=\sqrt{x}$ ALSO WORKS WELL HERE

517. $$\int \frac{6}{(1+\cos x)^2}\,dx = 3\tan\left(\frac{1}{2}x\right) + \tan^3\left(\frac{1}{2}x\right) + C$$

$$\int \frac{6}{(1+\cos x)^2}\,dx = \int \frac{6}{\left[1+\left(2\cos^2\frac{x}{2}-1\right)\right]^2}\,dx = \int \frac{6}{4\cos^4\frac{x}{2}}\,dx$$

$$= \int \frac{3}{2}\sec^4\frac{x}{2}\,dx = \frac{3}{2}\int \sec^2\frac{x}{2}\sec^2\frac{x}{2}\,dx$$

$$= \frac{3}{2}\int \sec^2\frac{x}{2}\left(1+\tan^2\frac{x}{2}\right)dx = \frac{3}{2}\int \sec^2\frac{x}{2} + \sec^2\frac{x}{2}\tan^2\frac{x}{2}\,dx$$

BY RECOGNITION SINCE $\frac{d}{dx}(\tan x) = \sec^2 x$

$$= \frac{3}{2}\left[2\tan\frac{x}{2} + \frac{2}{3}\tan^3\frac{x}{2}\right]+C$$

$$= 3\tan\frac{x}{2} + \tan^3\frac{x}{2} + C$$

THE SUBSTITUTION $t=\tan\frac{x}{2}$ ALSO WORKS VERY WELL

518. $$\int \sin^3 x\sqrt{\left(\cos^2 x\sin x\right)^2+\left(\sin^2 x\cos x\right)^2}\,dx=\tfrac{1}{5}\sin^5 x+C$$

$$\int \sin^3 x\sqrt{(\cos^2 x\sin x)^2+(\sin^2 x\cos x)^2}\,dx = \int \sin^3 x\sqrt{\cos^4 x\sin^2 x+\sin^4 x\cos^2 x}\,dx$$

$$=\int \sin^3 x\sqrt{\sin^2 x\cos^2 x\,(\cos^2 x+\sin^2 x)}\,dx = \int \sin^3 x\sqrt{\sin^2 x\cos^2 x}\,dx$$

$$=\int \sin^4 x\cos x\,dx = \tfrac{1}{5}\sin^5 x + C$$

519. $$\int \frac{3x^2-2x+5}{(x+3)^2}\,dx=3\ln|x+3|+\frac{20}{x+3}-\frac{19}{(x+3)^2}+C$$

$$\int \frac{3x^2-2x+5}{(x+3)^3}\,dx = \ldots \text{ BY SUBSTITUTION} \ldots$$

$u=x+3$
$x=u-3$
$dx=du$

$$=\int \frac{3(u-3)^2-2(u-3)+5}{u^3}\,du = \int \frac{3u^2-18u+27-2u+6+5}{u}\,du$$

$$=\int \frac{3u^2-20u+38}{u^3}\,du = \int \frac{3}{u}-\frac{20}{u^2}+\frac{38}{u^3}\,du$$

$$=3\ln|u|+\frac{20}{u}-\frac{19}{u^2}+C = 3\ln|x+3|+\frac{20}{x+3}-\frac{19}{(x+3)^2}+C$$

PARTIAL FRACTIONS OF THE FORM $\frac{A}{x+3}+\frac{B}{(x+3)^2}+\frac{C}{(x+3)^2}$ ALSO WORK

520. $$\int \frac{1}{\sqrt{x^2+x+1}}\,dx=\operatorname{arsinh}\left(\frac{2x+1}{\sqrt{3}}\right)+C$$

$$\int \frac{1}{\sqrt{x^2+x+1}}\,dx = \int \frac{1}{\sqrt{(x+\frac{1}{2})^2-\frac{1}{4}+1}}\,dx = \int \frac{1}{\sqrt{(x+\frac{1}{2})^2+\frac{3}{4}}}\,dx$$

$= \ldots$ USE THE SUBSTITUTION $u=x+\frac{1}{2}$
$du=dx$

$$=\int \frac{1}{\sqrt{u^2+\left(\frac{\sqrt{3}}{2}\right)^2}}\,dx$$

STANDARD RESULT

$$=\operatorname{arsinh}\left(\frac{u}{\sqrt{3}/2}\right)+C = \operatorname{arsinh}\left(\frac{2u}{\sqrt{3}}\right)+C$$

$$=\operatorname{arsinh}\left(\frac{2x+1}{\sqrt{3}}\right)+C$$

521. $$\int \frac{\sqrt{25-x^2}}{x}\,dx = \sqrt{25-x^2} + \frac{5}{2}\ln\left|\frac{\sqrt{25-x^2}-5}{\sqrt{25-x^2}+5}\right| + C$$

$\int \frac{\sqrt{25-x^2}}{x}\,dx = \dots$ BY SUBSTITUTION ...

$u = \sqrt{25-x^2}$
$u^2 = 25-x^2$
$2u\frac{du}{dx} = -2x$
$dx = -\frac{u}{x}\,du$
$x^2 = 25-u^2$

$= \int \frac{u}{x}\left(-\frac{u}{x}\,du\right) = \int -\frac{u^2}{x^2}\,du = \int -\frac{u^2}{25-u^2}\,du$

$= \int \frac{u^2}{u^2-25}\,du$ MANIPULATE THE IMPROPER FRACTION

$= \int \frac{u^2-25+25}{u^2-25}\,du = \int 1+\frac{25}{u^2-25}\,du$

$= \int 1+\frac{25}{(u-5)(u+5)}\,du = \int 1+\frac{\frac{5}{2}}{u-5}-\frac{\frac{5}{2}}{u+5}\,du$

(PARTIAL FRACTIONS BY COVER UP)

$= u+\frac{5}{2}\ln|u-5|-\frac{5}{2}\ln|u+5|+C = u+\frac{5}{2}\ln\left|\frac{u-5}{u+5}\right|+C$

$= \sqrt{25-x^2}+\frac{5}{2}\ln\left|\frac{\sqrt{25-x^2}-5}{\sqrt{25-x^2}+5}\right|+C$

522. $$\int \frac{1}{\sqrt{5-4x-x^2}}\,dx = \arcsin\left(\frac{x+2}{3}\right) + C$$

$\int \frac{1}{\sqrt{5-4x-x^2}}\,dx = \int \frac{1}{\sqrt{-(x^2+4x-5)}}\,dx = \int \frac{1}{\sqrt{-[(x+2)^2-9]}}\,dx$

$= \int \frac{1}{\sqrt{9-(x+2)^2}}\,dx$

BY THE SUBSTITUTION $u = x+2$, $dx = du$

$= \int \frac{1}{\sqrt{9-u^2}}\,du$

$= \arcsin\left(\frac{u}{3}\right)+C$

$= \arcsin\left(\frac{x+2}{3}\right)+C$

523. $$\int e^x \sin x\,dx = \frac{1}{2}e^x(\sin x - \cos x) + C$$

$\int e^x \sin x\,dx = \dots$ DOUBLE INTEGRATION BY PARTS

$\sin x$	$\cos x$
e^x	e^x

$= e^x\sin x - \int e^x\cos x\,dx$

BY PARTS AGAIN

$\cos x$	$-\sin x$
e^x	e^x

$\int e^x\sin x\,dx = e^x\sin x - \left[e^x\cos x + \int e^x\sin x\,dx\right]$

$\int e^x\sin x\,dx = e^x\sin x - e^x\cos x - \int e^x\sin x\,dx$

$2\int e^x\sin x\,dx = e^x(\sin x-\cos x)$

$\int e^x\sin x\,dx = \frac{1}{2}e^x(\sin x-\cos x)+C$

ALTERNATIVE BY COMPLEX NUMBERS

$\int e^x\sin x\,dx = \text{Im}\left[\int e^x e^{ix}\,dx\right] = \text{Im}\left[\int e^{(1+i)x}\,dx\right]$

$= \text{Im}\left[\frac{1}{1+i}e^{(1+i)x}\right] = \text{Im}\left[\frac{1-i}{(1+i)(1-i)}e^x e^{ix}\right]$

$= e^x\,\text{Im}\left[\frac{1-i}{2}(\cos x+i\sin x)\right] = \frac{1}{2}e^x\,\text{Im}\left[(1-i)(\cos x+i\sin x)\right]$

$= \frac{1}{2}e^x(\sin x-\cos x)+C$

ALTERNATIVE BY DIFFERENTIATION

$\frac{d}{dx}\left[e^x(A\cos x+B\sin x)\right] = e^x(-A\sin x+B\cos x)+e^x(A\cos x+B\sin x)$
$= e^x\left[(A+B)\cos x+(B-A)\sin x\right]$

NOW $\left.\begin{array}{l}A+B=0\\B-A=1\end{array}\right\} \Rightarrow 2B=1$, $B=\frac{1}{2}$ & $A=-\frac{1}{2}$

$\therefore\ \frac{d}{dx}\left[e^x\left(\frac{1}{2}\sin x-\frac{1}{2}\cos x\right)\right] = e^x\sin x$

$\therefore\ \int e^x\sin x\,dx = \frac{1}{2}e^x(\sin x-\cos x)+C$

524. $$\int \frac{2x^2+5x-1}{x^3+x^2-2x}\,dx = 2\ln|x-1| + \frac{1}{2}\ln\left|\frac{x}{x+2}\right| + C$$

$$\int \frac{2x^2+5x-1}{x^3+x^2-2x}\,dx = \int \frac{2x^2-5x-1}{x(x^2+x-2)}\,dx = \int \frac{2x^2-5x-1}{x(x-1)(x+2)}\,dx$$

BY PARTIAL FRACTIONS

$$\frac{2x^2+5x-1}{x(x-1)(x+2)} \equiv \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+2}$$

$$2x^2-5x-1 \equiv A(x-1)(x+2) + Bx(x+2) + C(x-1)x$$

- IF $x=0$: $-1 = -2A$, $A = \frac{1}{2}$
- IF $x=1$: $2+5-1 = 3B$, $3B = 6$, $B = 2$
- IF $x=-2$: $8-10-1 = 6C$, $6C = -3$, $C = -\frac{1}{2}$

$$\dots = \int \frac{\frac{1}{2}}{x} + \frac{2}{x-1} - \frac{\frac{1}{2}}{x+2}\,dx = \frac{1}{2}\ln|x| + 2\ln|x-1| - \frac{1}{2}\ln|x+2| + C$$

$$= 2\ln|x-1| + \frac{1}{2}\ln\left|\frac{x}{x+2}\right| + C$$

525. $$\int x^3 e^{x^2} \sin x\,dx = \frac{1}{2}e^{x^2}\left(x^2-1\right) + C$$

$$\int x^3 e^{x^2}\,dx = \dots \text{ REWRITE AS } = \int x^2\left(xe^{x^2}\right)dx$$

BY PARTS

x^2	$2x$
$\frac{1}{2}e^{x^2}$	xe^{x^2}

$$= \frac{1}{2}x^2e^{x^2} - \int xe^{x^2}\,dx$$

$$= \frac{1}{2}x^2e^{x^2} - \frac{1}{2}e^{x^2} + C$$

$$= \frac{1}{2}e^{x^2}(x^2-1) + C$$

THE SUBSTITUTION $u=x^2$, FOLLOWED BY INTEGRATION BY PARTS IS ALSO GOOD

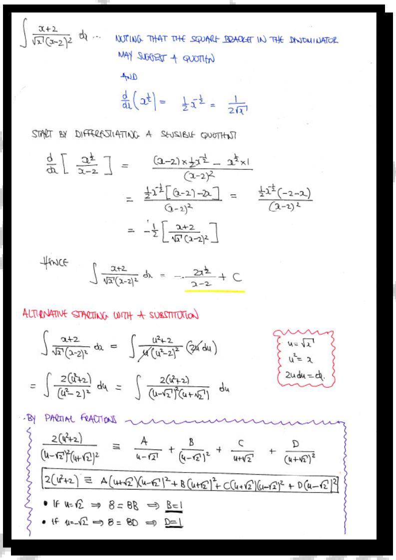
526. $$\int \frac{\tan\sqrt{x}}{\sqrt{x}}\,dx = 2\ln\left|\sec\sqrt{x}\right| + C$$

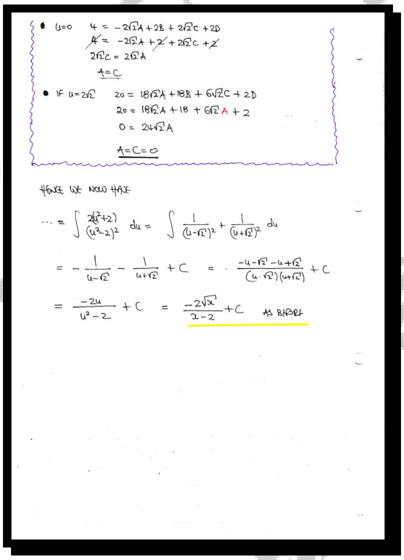
$$\int \frac{\tan\sqrt{x}}{\sqrt{x}}\,dx = \int x^{-\frac{1}{2}}\tan\left(x^{\frac{1}{2}}\right)dx$$ BY RECOGNITION & STANDARD RESULTS

$$= 2\ln\left|\sec x^{\frac{1}{2}}\right| + C = 2\ln\left|\sec\sqrt{x}\right| + C$$

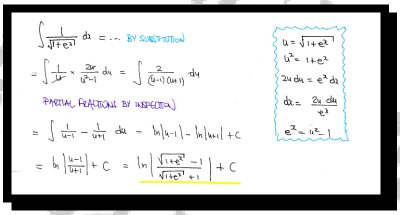
THE SUBSTITUTION $u=\sqrt{x}$ ALSO WORKS WELL

527. $$\int \frac{x+2}{(x-2)^2\sqrt{x}}\,dx = -\frac{2\sqrt{x}}{x-2} + C$$





528. $$\int \frac{1}{\sqrt{1+e^x}}\,dx = \ln\left|\frac{\sqrt{1+e^x}\; - 1}{\sqrt{1+e^x}\; + 1}\right| + C$$



529. $$\int \operatorname{cosec}^3 x \, dx = \begin{bmatrix} \frac{1}{2}\ln\left|\tan\left(\frac{1}{2}x\right)\right| - \frac{1}{2}\operatorname{cosec} x \cot x + C \\ -\ln\left|\cot x + \operatorname{cosec} x\right| - \frac{1}{2}\operatorname{cosec} x \cot x + C \end{bmatrix}$$

$$\int \operatorname{cosec}^3 x \, dx = \int \operatorname{cosec} x \operatorname{cosec}^2 x \, dx = \int \operatorname{cosec} x (1 + \cot^2 x) \, dx$$
$$= \int \operatorname{cosec} x \, dx + \int \operatorname{cosec} x + \operatorname{cosec} x \cot^2 x \, dx$$
$$= \int \operatorname{cosec} x \, dx + \int \underbrace{(\operatorname{cosec} x \cot x) \cot x}_{\text{BY PARTS}} \, dx$$

$\cot x$	$-\operatorname{cosec}^2 x$
$-\operatorname{cosec} x$	$\operatorname{cosec} x \cot x$

$$\int \operatorname{cosec}^3 x \, dx = \int \operatorname{cosec} x \, dx + \left[-\cot x \operatorname{cosec} x - \int \operatorname{cosec}^3 x \, dx \right]$$

STANDARD RESULT (OR $-\ln|\cot x + \operatorname{cosec} x|$)

$$\int \operatorname{cosec}^3 x \, dx = \ln\left|\tan \frac{x}{2}\right| - \cot x \operatorname{cosec} x - \int \operatorname{cosec}^3 x \, dx$$
$$2\int \operatorname{cosec}^3 x \, dx = \ln\left|\tan \frac{x}{2}\right| - \cot x \operatorname{cosec} x$$
$$\int \operatorname{cosec}^3 x \, dx = \frac{1}{2}\ln\left|\tan \frac{x}{2}\right| - \frac{1}{2}\operatorname{cosec} x \cot x + C$$

530. $$\int \frac{1}{e^{2x} - 3e^x} \, dx = \frac{1}{9}\ln\left|1 - 3e^{-x}\right| - \frac{1}{3}e^{-x} + C$$

$$\int \frac{1}{e^{2x} - 3e^x} \, dx = \int \frac{1}{e^x(e^x - 3)} \, dx$$
$$= \int \frac{1}{u(u-3)} \frac{du}{u} = \int \frac{1}{u^2(u-3)} \, du$$

BY SUBSTITUTION !

$u = e^x$

$\frac{du}{dx} = e^x$

$du = u \, dx$

$dx = \frac{du}{u}$

BY PARTIAL FRACTIONS

$$\frac{1}{u^2(u-3)} \equiv \frac{A}{u} + \frac{B}{u^2} + \frac{C}{u-3}$$

$$\boxed{1 \equiv Au(u-3) + B(u-3) + Cu^2}$$

- IF $u = 3$: $1 = 9C$, $C = \frac{1}{9}$
- IF $u = 0$: $1 = -3B$, $B = -\frac{1}{3}$
- IF $u = 1$: $1 = -2A - 2B + C$, $1 = -2A + \frac{2}{3} + \frac{1}{9}$, $9 = -18A + 6 + 1$, $18A = -2$, $A = -\frac{1}{9}$

$$= \int \frac{-\frac{1}{9}}{u} - \frac{\frac{1}{3}}{u^2} + \frac{\frac{1}{9}}{u-3} \, du = -\frac{1}{9}\ln|u| + \frac{1}{3u} + \frac{1}{9}\ln|u-3| + C$$
$$= \frac{1}{9}\ln\left|\frac{u-3}{u}\right| + \frac{1}{3u} + C = \frac{1}{9}\ln\left|1 - \frac{3}{u}\right| + \frac{1}{3u} + C$$
$$= \frac{1}{9}\ln\left|1 - \frac{3}{e^x}\right| + \frac{1}{3e^x} + C = \frac{1}{9}\ln\left|1 - 3e^{-x}\right| + \frac{1}{3}e^{-x} + C$$

531. $$\int \frac{\cosh x}{1 + \sinh x} \, dx = \ln\left|1 + \sinh x\right| + C$$

$$\int \frac{\cosh x}{1 + \sinh x} \, dx = \text{OF THE FORM } \int \frac{f'(x)}{f(x)} \, dx = \ln|f(x)| + C$$
$$= \ln|1 + \sinh x| + C$$

OR USE THE SUBSTITUTION $u = 1 + \sinh x$

532. $$\int \frac{1}{6\sinh 2x + 9\cosh 2x}\,dx = \begin{cases} \dfrac{\sqrt{5}}{15}\arctan\left(\sqrt{5}\,\mathrm{e}^{2x}\right) + C \\[2mm] \dfrac{\sqrt{5}}{15}\arctan\left(\dfrac{2+3\tanh x}{\sqrt{5}}\right) + C \\[2mm] \dfrac{\sqrt{5}}{30}\arctan\left[\sinh\left(2x+\tfrac{1}{2}\ln 5\right)\right] + C \end{cases}$$

$\int \frac{1}{6\sinh 2x + 9\cosh 2x}\,dx = \dots$ SWITCHING TO EXPONENTIALS ...

$= \int \frac{1}{6\left(\frac{e^{2x}}{2}-\frac{e^{-2x}}{2}\right)+9\left(\frac{e^{2x}}{2}+\frac{e^{-2x}}{2}\right)}\,dx = \int \frac{1}{\frac{15}{2}e^{2x}+\frac{3}{2}e^{-2x}}\,dx = \frac{2}{3}\int \frac{1}{5e^{2x}+e^{-2x}}\,dx$

CONTINUE BY SUBSTITUTION

$u = e^{2x}$ & $5e^{2x}+e^{-2x}$

$\frac{du}{dx} = 2e^{2x}$ $\quad 5u+\frac{1}{u}$

$\frac{du}{dx} = 2u$

$dx = \frac{du}{2u}$

$= \frac{2}{3}\int \frac{1}{5u+\frac{1}{u}}\,\frac{du}{2u}$

$= \frac{2}{3}\int \frac{1}{10u^2+2}\,du$

$= \frac{1}{15}\int \frac{1}{u^2+\frac{1}{5}}\,du$

$= \frac{1}{15}\int \frac{1}{u^2+\left(\frac{1}{\sqrt{5}}\right)^2}\,du$

$= \frac{1}{15}\times\frac{1}{\frac{1}{\sqrt{5}}}\times\arctan\left(\frac{u}{\frac{1}{\sqrt{5}}}\right)+C$

$= \frac{\sqrt{5}}{15}\arctan\left(\sqrt{5}e^{2x}\right)+C$

ALTERNATIVE BY "LITTLE t" IDENTITIES APPROACH

LET $t = \tanh x$

$\frac{dt}{dx} = \operatorname{sech}^2 x$

$\frac{dt}{dx} = 1-\tanh^2 x$

$\frac{dt}{dx} = 1-t^2$

$dx = \frac{dt}{1-t^2}$

$\sin 2t = \frac{2\tan t}{1+\tan^2 t}$ & $\cos 2t = \frac{1-\tan^2 t}{1+\tan^2 t}$

OBSERVE RULE

$\sinh 2t = \frac{2\tanh t}{1-\tanh^2 t}$ $\quad \cosh 2t = \frac{1+\tanh^2 t}{1-\tanh^2 t}$

$\sinh 2t = \frac{2t}{1-t^2}$ $\quad \cosh 2t = \frac{1+t^2}{1-t^2}$

$\int \frac{1}{6\sinh 2x+9\cosh 2x}\,dx = \int \frac{1}{6\left(\frac{2t}{1-t^2}\right)+9\left(\frac{1+t^2}{1-t^2}\right)}\,\frac{dt}{1-t^2}$

$= \int \frac{1}{12t+9+9t^2}\,dt = \frac{1}{9}\int \frac{1}{t^2+\frac{4t}{3}+1}\,dt$

$= \frac{1}{9}\int \frac{1}{\left(t-\frac{2}{3}\right)^2-\frac{4}{9}+1}\,dt = \frac{1}{9}\int \frac{1}{\left(t-\frac{2}{3}\right)^2+\frac{5}{9}}\,dt$

ANOTHER SUBSTITUTION

$u = t+\frac{2}{3}$

$du = dt$

$= \frac{1}{9}\int \frac{1}{u^2+\left(\frac{\sqrt{5}}{3}\right)^2}\,du = \frac{1}{9}\times\frac{1}{\frac{\sqrt{5}}{3}}\times\arctan\left(\frac{u}{\frac{\sqrt{5}}{3}}\right)+C$

$= \frac{1}{3\sqrt{5}}\arctan\left(\frac{3u}{\sqrt{5}}\right)+C = \frac{\sqrt{5}}{15}\arctan\left(\frac{3t+2}{\sqrt{5}}\right)+C$

$= \frac{\sqrt{5}}{15}\arctan\left(\frac{2+3\tanh x}{\sqrt{5}}\right)+C$

VARIANT USING "R TRANSFORMATIONS"

$6\sinh 2x + 9\cosh 2x \equiv R\cosh(2x+\theta)$

$\equiv R\cosh 2x\cosh\theta + R\sinh 2x\sinh 2\theta$

$\left.\begin{array}{l} R\cosh\theta = 9 \\ R\sinh\theta = 6 \end{array}\right\}$ SQUARE & ADD $R^2 = 81-36$ & $\tanh\theta = \frac{2}{3}$

$R = \sqrt{45}$

$\theta = \operatorname{artanh}\frac{2}{3}$

$\theta = \frac{1}{2}\ln\left|\frac{1+\frac{2}{3}}{1-\frac{2}{3}}\right|$

$\theta = \frac{1}{2}\ln 5$

$\int \frac{1}{6\sinh 2x+9\cosh 2x}\,dx = \int \frac{1}{\sqrt{45}\cosh\left(2x+\frac{1}{2}\ln 5\right)}\,dx$

$= \frac{1}{3\sqrt{5}}\int \operatorname{sech}\left(2x+\frac{1}{2}\ln 5\right)\,dx$

NOW $\int \operatorname{sech} x\,dx = \arctan(\sinh x)$

$= \frac{\sqrt{5}}{15}\arctan\left(\sinh\left(2x+\frac{1}{2}\ln 5\right)\right)\times\frac{1}{2}+C$

$= \frac{\sqrt{5}}{30}\arctan\left[\sinh\left(2x+\frac{1}{2}\ln 5\right)\right]+C$

533. $$\int \sqrt{1+4\sinh^2 x\cosh^2 x}\;dx = \frac{1}{2}\sinh 2x + C$$

$\int \sqrt{1+4\sinh^2 x\cosh^2 x}\,dx = \int \sqrt{1+(2\sinh x\cosh x)^2}\,dx$

$= \int \sqrt{1+\sinh^2 2x}\,dx = \int \sqrt{\cosh^2 2x}\,dx = \int \cosh 2x\,dx$

$= \frac{1}{2}\sinh 2x + C$

534. $$\int \frac{\operatorname{arsinh} x}{\sqrt{x^2+1}}\;dx = \frac{1}{2}(\operatorname{arsinh})^2 + C$$

$\int \frac{\operatorname{arsinh} x}{\sqrt{x^2+1}}\,dx = \dots$ BY RECOGNITION $\dots = \frac{1}{2}(\operatorname{arsinh} x)^2 + C$

THE SUBSTITUTION $u = \operatorname{arsinh} x$ CAN ALSO BE USED

535. $$\int \frac{x}{\sqrt{x^2+1}}\,dx = \sqrt{x^2+1} + C$$

$\int \frac{x}{\sqrt{x^2+1}}\,dx = \ldots$ BY RECOGNITION $\ldots = \int x(x^2+1)^{-\frac{1}{2}}\,dx$

$= (x^2+1)^{\frac{1}{2}} + C$

$= \sqrt{x^2+1} + C$

THE SUBSTITUTION $u = x^2+1$ OR $u = \sqrt{x^2+1}$ ALSO WORK WELL

536. $$\int \sqrt{x^2+4}\,dx = 2\ln\left(x+\sqrt{x^2+4}\right) + \frac{1}{2}x\sqrt{x^2+4} + C$$

$x = 2\sinh t$
$dx = 2\cosh t\,dt$
$t = \text{arsinh}\frac{x}{2}$

$\int \sqrt{x^2+4}\,dx = \ldots$ BY SUBSTITUTION $\ldots$

$= \int \sqrt{4\sinh^2 t+4}\,(2\cosh t\,dt) = \int 4\cosh t\sqrt{1+\sinh^2 t}\,dt$

$= \int 4\cosh^2 t\,dt = \int 4\left(\frac{1}{2}+\frac{1}{2}\cosh 2t\right)dt$

$= \int 2+2\cosh 2t\,dt = 2t+\sinh 2t + C = 2t + 2\sinh t\cosh t + C$

$= 2t+2\sinh t(1+\sinh^2 t)^{\frac{1}{2}} + C = 2\text{arsinh}\frac{x}{2} + x\left(1+\frac{x^2}{4}\right)^{\frac{1}{2}} + C$

$= 2\ln\left[\frac{x}{2}+\sqrt{\frac{x^2}{4}+1}\right] + x\sqrt{\frac{x^2+4}{4}} + C$

$= 2\ln\left[\frac{x}{2}+\sqrt{\frac{x^2+4}{4}}\right] + \frac{x}{2}\sqrt{x^2+4} + C$

$= 2\ln\left[\frac{x}{2}+\frac{1}{2}\sqrt{x^2+4}\right] + \frac{1}{2}x\sqrt{x^2+4} + C$

$= 2\ln\left[\frac{1}{2}\left(x+\sqrt{x^2+4}\right)\right] + \frac{1}{2}x\sqrt{x^2+4} + C$

$= 2\ln\frac{1}{2} + 2\ln\left(x+\sqrt{x^2+4}\right) + \frac{1}{2}x\sqrt{x^2+4} + C$

$= 2\ln\left(x+\sqrt{x^2+4}\right) + \frac{1}{2}x\sqrt{x^2+4} + C$

537. $$\int \frac{1}{4x^{\frac{1}{2}}\left(1+x^{\frac{1}{4}}\right)}\,dx = x^{\frac{1}{4}} - \ln\left(1+x^{\frac{1}{4}}\right) + C$$

$u = x^{\frac{1}{4}}$
$u^2 = x^{\frac{1}{2}}$
$x = u^4$
$dx = 4u^3\,du$

$\int \frac{1}{4x^{\frac{1}{2}}(1+x^{\frac{1}{4}})}\,dx = \ldots$ BY SUBSTITUTION $\ldots$

$= \int \frac{1}{4u^2(1+u)}(4u^3\,du) = \int \frac{4u^3}{4u^2(1+u)}\,du$

$= \int \frac{u}{u+1}\,du = \ldots$ MANIPULATING $\ldots = \int \frac{u+1-1}{u+1}\,du$

$= \int 1-\frac{1}{u+1}\,du = u - \ln|u+1| + C = x^{\frac{1}{4}} - \ln\left(1+x^{\frac{1}{4}}\right) + C$

538. $$\int \frac{6x^2}{\sqrt{1-4x^6}}\,dx = \left[\begin{array}{l} \arcsin\left(2x^3\right) + C \\ \arccos\left(\sqrt{1-4x^6}\right) + C \end{array}\right.$$

$\int \frac{6x^2}{\sqrt{1-4x^6}} = \ldots$ BY RECOGNITION .. $\arcsin(2x^3) + C$

OR BY SUBSTITUTION

$u = \sqrt{1-4x^6}$
$u^2 = 1-4x^6$
$2u\frac{du}{dx} = -24x^5$
$dx = \frac{u\,du}{-12x^5}$

$= \int \frac{6x^2}{u}\left(\frac{u}{-12x^5}\right)du$

$= \int -\frac{1}{2x^3}\,du$

BUT $4x^6 = 1-u^2$
$2x^3 = \sqrt{1-u^2}$

$= \int -\frac{1}{\sqrt{1-u^2}}\,du$

$= \arccos u + C$

$= \arccos\left(\sqrt{1-4x^6}\right) + C$

LOOKING AT THE TRIANGLE BELOW

$\theta = \arccos\sqrt{1-4x^6}$
$\cos\theta = \sqrt{1-4x^6}$

I.E. $\arccos\sqrt{1-4x^6} = \arcsin(2x^3)$

539. $$\int \frac{x^2}{\sqrt{1-x^2}}\,dx = \frac{1}{2}\arcsin x + \frac{1}{2}x\sqrt{1-x^2} + C$$

$\int \frac{x^2}{\sqrt{1-x^2}}\,dx = \ldots$ BY SUBSTITUTION ..

$\theta = \arcsin x$
$x = \sin\theta$
$dx = \cos\theta\,d\theta$

$= \int \frac{\sin^2\theta}{\cancel{\sqrt{1-\sin^2\theta}}}\left(\cancel{\cos\theta}\,d\theta\right) = \int \frac{1}{2} - \frac{1}{2}\cos 2\theta\,d\theta$

$= \frac{1}{2}\theta - \frac{1}{4}\sin 2\theta + C = \frac{1}{2}\theta + \frac{1}{2}\sin\theta\cos\theta + C$

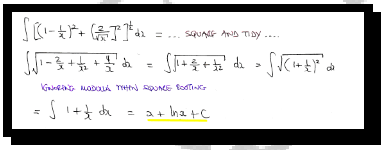
$= \frac{1}{2}\theta + \frac{1}{2}\sin\theta\sqrt{1-\sin^2\theta} + C = \frac{1}{2}\arcsin x + \frac{1}{2}x\sqrt{1-x^2} + C$

THE SUBSTITUTION $\theta = \arccos x$, $x = \cos\theta$ ALSO WORKS WELL

540. $$\int \frac{1}{\sqrt{4-x^2}}\,dx = \left[\begin{array}{r} \arcsin\left(\frac{1}{2}x\right) + C \\ -\arccos\left(\frac{1}{2}x\right) + C \end{array}\right.$$

$\int \frac{1}{\sqrt{4-x^2}}\,dx = \ldots$ STANDARD RESULT ... $= \arcsin\left(\frac{x}{2}\right) + C$
$= -\arccos\left(\frac{x}{2}\right) + C$

541. $\displaystyle\int \sqrt{\left(1-\frac{1}{x}\right)^2+\left(\frac{2}{\sqrt{x}}\right)^2}\;dx = x+\ln x + C$



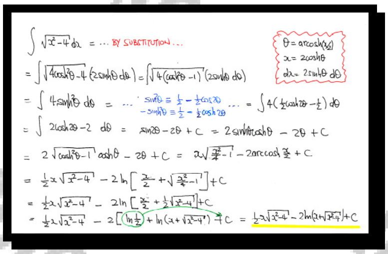
$\displaystyle\int \left[\left(1-\frac{1}{x}\right)^2+\left[\frac{2}{\sqrt{x}}\right]^2\right]^{\frac{1}{2}} dx = \dots$ SQUARE AND TIDY ....

$\displaystyle\int \sqrt{1-\frac{2}{x}+\frac{1}{x^2}+\frac{4}{x}}\;dx = \int \sqrt{1+\frac{2}{x}+\frac{1}{x^2}}\;dx = \int \sqrt{\left(1+\frac{1}{x}\right)^2}\;dx$

IGNORING MODULI WHEN SQUARE ROOTING

$\displaystyle = \int 1+\frac{1}{x}\;dx = x+\ln x + C$

542. $\displaystyle\int \sqrt{x^2-4}\;dx = \frac{1}{2}x\sqrt{x^2-4} - 2\ln\left(x+\sqrt{x^2-4}\right) + C$



$\displaystyle\int \sqrt{x^2-4}\;dx = \dots$ BY SUBSTITUTION ...

$\theta = \operatorname{arcosh}(x/2)$
$x = 2\cosh\theta$
$dx = 2\sinh\theta\,d\theta$

$\displaystyle = \int \sqrt{4\cosh^2\theta-4}\,(2\sinh\theta\,d\theta) = \int \sqrt{4(\cosh^2\theta-1)}\,(2\sinh\theta\,d\theta)$

$\displaystyle = \int 4\sinh^2\theta\,d\theta = \dots$ $\sin^2\theta \equiv \frac{1}{2}-\frac{1}{2}\cos 2\theta$, $-\sinh^2\theta \equiv \frac{1}{2}-\frac{1}{2}\cosh 2\theta$ $\dots = \int 4\left(\frac{1}{2}\cosh 2\theta-\frac{1}{2}\right)d\theta$

$\displaystyle = \int 2\cosh 2\theta - 2\;d\theta = \sinh 2\theta - 2\theta + C = 2\sinh\theta\cosh\theta - 2\theta + C$

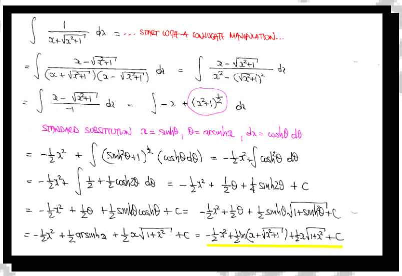
$\displaystyle = 2\sqrt{\cosh^2\theta-1}\,\cosh\theta - 2\theta + C = x\sqrt{\frac{x^2}{4}-1} - 2\operatorname{arcosh}\frac{x}{2} + C$

$\displaystyle = \frac{1}{2}x\sqrt{x^2-4} - 2\ln\left[\frac{x}{2}+\sqrt{\frac{x^2}{4}-1}\right]+C$

$\displaystyle = \frac{1}{2}x\sqrt{x^2-4} - 2\ln\left[\frac{x}{2}+\frac{1}{2}\sqrt{x^2-4}\right]+C$

$\displaystyle = \frac{1}{2}x\sqrt{x^2-4} - 2\left[\ln\frac{1}{2}+\ln\left(x+\sqrt{x^2-4}\right)\right]+C = \frac{1}{2}x\sqrt{x^2-4} - 2\ln\left(x+\sqrt{x^2-4}\right)+C$

543. $\displaystyle\int \frac{1}{x+\sqrt{x^2+1}}\;dx = -\frac{1}{2}x^2+\frac{1}{2}\ln\left(x+\sqrt{x^2+1}\right)+\frac{1}{2}x\sqrt{x^2+1}+C$



$\displaystyle\int \frac{1}{x+\sqrt{x^2+1}}\;dx = \dots$ START WITH A CONJUGATE MANIPULATION...

$\displaystyle = \int \frac{x-\sqrt{x^2+1}}{\left(x+\sqrt{x^2+1}\right)\left(x-\sqrt{x^2+1}\right)}\;dx = \int \frac{x-\sqrt{x^2+1}}{x^2-\left(\sqrt{x^2+1}\right)^2}\;dx$

$\displaystyle = \int \frac{x-\sqrt{x^2+1}}{-1}\;dx = \int -x+(x^2+1)^{\frac{1}{2}}\;dx$

STANDARD SUBSTITUTION $x=\sinh\theta$, $\theta=\operatorname{arsinh}x$, $dx=\cosh\theta\,d\theta$

$\displaystyle = -\frac{1}{2}x^2+\int(\sinh^2\theta+1)^{\frac{1}{2}}(\cosh\theta\,d\theta) = -\frac{1}{2}x^2+\int\cosh^2\theta\,d\theta$

$\displaystyle = -\frac{1}{2}x^2+\int\frac{1}{2}+\frac{1}{2}\cosh 2\theta\,d\theta = -\frac{1}{2}x^2+\frac{1}{2}\theta+\frac{1}{4}\sinh 2\theta + C$

$\displaystyle = -\frac{1}{2}x^2+\frac{1}{2}\theta+\frac{1}{2}\sinh\theta\cosh\theta + C = -\frac{1}{2}x^2+\frac{1}{2}\theta+\frac{1}{2}\sinh\theta\sqrt{1+\sinh^2\theta}+C$

$\displaystyle = -\frac{1}{2}x^2+\frac{1}{2}\operatorname{arsinh}x+\frac{1}{2}x\sqrt{1+x^2}+C = -\frac{1}{2}x^2+\frac{1}{2}\ln\left(x+\sqrt{x^2+1}\right)+\frac{1}{2}x\sqrt{1+x^2}+C$

544. $$\int \frac{1}{\sqrt{x^2+9}}\,dx = \begin{cases} \text{arsinh}\left(\frac{1}{3}x\right) + C \\ \ln\left(x+\sqrt{x^2+9}\right) + C \end{cases}$$

$\int \frac{1}{\sqrt{x^2+9}}\,dx = \ldots$ STANDARD RESULT $\ldots = \text{arsinh}\left(\frac{1}{3}x\right)+C$

$= \ln\left[\frac{x}{3}+\sqrt{\frac{1}{9}x^2+1}\right]+C = \ln\left[\frac{x}{3}+\frac{\sqrt{x^2+9}}{3}\right]+C$

$= \ln\left[\frac{1}{3}\left(x+\sqrt{x^2+9}\right)\right]+C$

$= \ln\frac{1}{3} + \ln\left(x+\sqrt{x^2+9}\right)+C = \ln\left(x+\sqrt{x^2+9}\right)+C$

545. $$\int \frac{1}{\sqrt{x}\sqrt{1+\sqrt{x}}}\,dx = 4\sqrt{1+\sqrt{x}} + C$$

$\int \frac{1}{\sqrt{x}\sqrt{\sqrt{x}+1}}\,dx = \ldots$ THIS DOES NOT WORK TOO WELL $\ldots \int \frac{1}{\sqrt{x+\sqrt{x}}}\,dx$

REWRITING AS INDICES

$\ldots = \int x^{-\frac{1}{2}}\left(x^{\frac{1}{2}}+1\right)^{-\frac{1}{2}} = \ldots$ BY RECOGNITION $\ldots = 4\left(x^{\frac{1}{2}}+1\right)^{\frac{1}{2}}+C$

$= 4\sqrt{\sqrt{x}+1}+C$

THE SUBSTITUTION $u=\sqrt{x}$ ALSO WORKS WELL

546. $$\int \frac{1}{x\left(1+x^2\right)^3}\,dx = \frac{3+2x^2}{4\left(1+x^2\right)^2}\,\frac{1}{2}\ln\left(\frac{x^2}{1+x^2}\right) + C$$

$\int \frac{1}{x(x^2+1)^3}\,dx = \ldots$ START BY A SUBSTITUTION $\ldots$

$u = x^2+1$, $\frac{du}{dx} = 2x$, $dx = \frac{du}{2x}$, $x^2 = u-1$

$= \int \frac{1}{x u^3}\left(\frac{du}{2x}\right) = \int \frac{1}{2x^2u^3}\,du = \int \frac{1}{2(u-1)u^3}\,du$

BY PARTIAL FRACTIONS

$\frac{1}{(u-1)u^3} \equiv \frac{A}{u-1}+\frac{B}{u^3}+\frac{C}{u^2}+\frac{D}{u}$

$\boxed{1 \equiv Au^3 + B(u-1) + Cu(u-1) + Du^2(u-1)}$

- IF $u=1$: $1 = A$
- IF $u=0$: $1=-B$, $B=-1$

$1 \equiv u^3+1-u+Cu^2-Cu+Du^3-Du^2$

$1 \equiv (D+1)u^3+(C-D)u^2+(-1-C)u+1$

$\therefore D=-1$ & $C=-1$

$\ldots = \frac{1}{2}\int \frac{1}{u-1}-\frac{1}{u^3}-\frac{1}{u^2}-\frac{1}{u}\,du = \frac{1}{2}\left[\ln|u-1|+\frac{1}{2u^2}+\frac{1}{u}-\ln|u|\right]+C$

$= \frac{1}{2}\left[\ln\left|\frac{u-1}{u}\right| + \frac{1+2u}{2u^2}\right]+C = \frac{1}{2}\ln\left|\frac{x^2}{x^2+1}\right| + \frac{1+2(x^2+1)}{4(1+x^2)^2}+C$

$= \frac{1}{2}\ln\left(\frac{x^2}{x^2+1}\right) + \frac{2x^2+3}{4(1+x^2)^2}+C$

ALTERNATIVE SUBSTITUTION WITHOUT PARTIAL FRACTIONS

$\int \frac{1}{x(x^2+1)^3}\,dx = \ldots$ BY SUBSTITUTION $\ldots$

$x^2+1 = \frac{1}{u}$, $2x\,dx = -\frac{1}{u^2}\,du$, $dx = -\frac{1}{2xu^2}\,du$

$= \int \frac{1}{x\left(\frac{1}{u}\right)^3}\left(-\frac{du}{2xu^2}\right) = \int \frac{-1}{\frac{x}{u^3}\times 2xu^2}\,du$

$= \int \frac{-1}{\frac{2x^2}{u}}\,du = \int -\frac{u}{2x^2}\,du = -\frac{1}{2}\int \frac{u}{x^2}\,du$

$= -\frac{1}{2}\int \frac{u}{\frac{1}{u}-1}\,du = \frac{1}{2}\int \frac{u}{1-\frac{1}{u}}\,du = \frac{1}{2}\int \frac{u^2}{u-1}\,du$

DIVIDE OR MANIPULATE

$= \frac{1}{2}\int \frac{u(u-1)+(u-1)+1}{u-1}\,du = \frac{1}{2}\int u+1+\frac{1}{u-1}\,du$

$= \frac{1}{2}\left[\frac{1}{2}u^2+u+\ln|u-1|\right]+C$

$= \frac{1}{2}\left[\frac{1}{2}\frac{1}{(1+x^2)^2}+\frac{1}{1+x^2}+\ln\left|\frac{1}{x^2+1}-1\right|\right]+C$

$= \frac{1}{4}\frac{1}{(x^2+1)^2}+\frac{1}{2}\frac{1}{(x^2+1)}+\ln\left|\frac{1-x^2-1}{x^2+1}\right|+C$

$= \frac{1+2(x^2+1)}{4(x^2+1)^2}+\ln\left|\frac{-x^2}{x^2+1}\right|+C = \frac{2x^2+3}{4(x^2+1)^2}+\ln\left(\frac{x^2}{x^2+1}\right)+C$

ANOTHER APPROACH VIA HYPERBOLIC SUBSTITUTION CAN BE EXPLORED

$\int \frac{1}{x(x^2+1)^3}\,dx = \ldots$ BY SUBSTITUTION $\ldots$

$x = \sinh\theta$, $dx = \cosh\theta\,d\theta$

$= \int \frac{1}{\sinh\theta(\sinh^2\theta+1)^3}(\cosh\theta\,d\theta)$

$= \int \frac{\cosh\theta}{\sinh\theta\cosh^6\theta}\,d\theta = \int \frac{1}{\sinh\theta\cosh^5\theta}\,d\theta = \int \frac{\cosh^2\theta-\sinh^2\theta}{\sinh\theta\cosh^5\theta}\,d\theta$

$= \int \frac{1}{\sinh\theta\cosh^3\theta} - \frac{\sinh\theta}{\cosh^5\theta}\,d\theta = \int \frac{\cosh^2\theta-\sinh^2\theta}{\sinh\theta\cosh^3\theta} - \frac{\sinh\theta}{\cosh^5\theta}\,d\theta$

$= \int \frac{1}{\sinh\theta\cosh\theta} - \frac{\sinh\theta}{\cosh^3\theta} - \frac{\sinh\theta}{\cosh^5\theta}\,d\theta$

$= \int \frac{2}{2\sinh\theta\cosh\theta} - \tanh\theta\,\text{sech}^2\theta - \tanh\theta\,\text{sech}^4\theta\,d\theta$

$= \int \frac{2}{\sinh 2\theta} - (\tanh\theta\,\text{sech}\theta)\text{sech}\theta - (\tanh\theta\,\text{sech}\theta)\text{sech}^3\theta\,d\theta$

$= \left[\int \frac{2\sinh 2\theta}{\sinh^2 2\theta}\,d\theta\right] + \frac{1}{2}\text{sech}^2\theta + \frac{1}{4}\text{sech}^4\theta$

$= \int \frac{2\sinh 2\theta}{\cosh^2 2\theta - 1}\,d\theta + \frac{1}{2}\text{sech}^2\theta + \frac{1}{4}\text{sech}^4\theta$

$= -\,\text{artanh}(\cosh 2\theta) + \frac{1}{2}\text{sech}^2\theta + \frac{1}{4}\text{sech}^4\theta + C$

WHICH WILL EVENTUALLY GIVE THE SAME ANSWER AFTER USING THE LOGARITHMIC FORM FOR THE artanh

547. $\displaystyle\int \frac{\sin x \cos x}{\cos^2 x + 3\cos x + 2}\,dx = \ln(1+\cos x) - 2\ln(2+\cos x) + C$

$\displaystyle\int \frac{\sin x\cos x}{\cos^2 x + 3\cos x + 2}\,dx = \dots$ BY SUBSTITUTION ...

$u = \cos x$
$\frac{du}{dx} = -\sin x$
$dx = \frac{du}{-\sin x}$

$\displaystyle = \int \frac{\cancel{\sin x} \times u}{u^2+3u+2}\left(\frac{du}{\cancel{-\sin x}}\right) = \int \frac{-u}{(u+2)(u+1)}\,du$

BY PARTIAL FRACTIONS (COVER UP)

$\displaystyle = \int \frac{1}{u+1} - \frac{2}{u+2}\,du = \ln|u+1| - 2\ln|u+2| + C$

$= \ln(1+\cos x) - 2\ln(2+\cos x) + C$

548. $\displaystyle\int \sin(\ln x)\,dx = \frac{1}{2}x\left[\sin(\ln x) - \cos(\ln x)\right] + C$

$\displaystyle\int \sin(\ln x)\,dx = \dots$ BY SUBSTITUTION ...

$u = \ln x$
$\frac{du}{dx} = \frac{1}{x}$
$dx = x\,du$
$x = e^u$
$dx = e^u\,du$

$\displaystyle = \int \sin u\,(e^u\,du) = \int e^u \sin u\,du$

NOW BY DOUBLE PARTS, COMPLEX NUMBERS OR INSPECTION VIA DIFFERENTIATION

$\frac{d}{du}\left(e^u(A\sin u + B\cos u)\right) \equiv e^u \sin u$

$e^u(A\sin u + B\cos u) + e^u(A\cos u - B\sin u) \equiv e^u\sin u$

$(A+B)e^u\cos u + (A-B)e^u\sin u \equiv e^u \sin u$

$A+B=0$
$A-B=1$ $\quad \therefore \ A=\frac{1}{2} \quad B=-\frac{1}{2}$

$\displaystyle \Rightarrow \int e^u \sin u\,du = \frac{1}{2}e^u(\sin u - \cos u) + C$

$= \frac{1}{2}x\left(\sin(\ln x) - \cos(\ln x) + C\right.$

549. $\displaystyle\int -\sinh\left(\frac{1}{2}x\right)dx = -2\cosh\left(\frac{1}{2}x\right) + C$

$\displaystyle\int -\sinh\left(\tfrac{1}{2}x\right)dx = \dots$ STANDARD RESULT ...

$= -2\cosh\left(\tfrac{1}{2}x\right) + C$

550. $\displaystyle\int (\cos x)\left[\ln(\sin x)\right]dx = \left[1 - \ln|\sin x|\right]\sin x + C$

$\displaystyle\int \cos x \ln(\sin x)\,dx = \dots$ INTEGRATION BY PARTS

$\ln(\sin x)$	$\cot x$
$-\sin x$	$\cos x$

$\displaystyle = -\sin x \ln|\sin x| + \int \sin x \cot x\,dx$

$\displaystyle = -\sin x \ln|\sin x| + \int \cos x\,dx$

$= -\sin x \ln|\sin x| + \sin x + C = (1 - \ln|\sin x|)\sin x + C$

551. $\displaystyle\int \sqrt{1-x^2}\ dx = \frac{1}{2}\arcsin x + \frac{1}{2}x\sqrt{1-x^2} + C$

$\displaystyle\int \sqrt{1-x^2}\ dx = \ldots$ BY SUBSTITUTION $\ldots$

$\theta = \text{arc}$
$x = \sin\theta$
$dx = \cos\theta\, d\theta$

$\displaystyle = \int \sqrt{1-\sin^2\theta}\,(\cos\theta\, d\theta) = \int \cos^2\theta\, d\theta$

$\displaystyle = \int \tfrac{1}{2} + \tfrac{1}{2}\cos 2\theta\, d\theta = \tfrac{1}{2}\theta + \tfrac{1}{4}\sin 2\theta + C = \tfrac{1}{2}\theta + \tfrac{1}{2}\sin\theta\cos\theta + C$

$= \tfrac{1}{2}\theta + \tfrac{1}{2}\sin\theta\sqrt{1-\sin^2\theta} + C = \tfrac{1}{2}\arcsin x + \tfrac{1}{2}x\sqrt{1-x^2} + C$

552. $\displaystyle\int \frac{3}{x^2-4x+13}\ dx = \arctan\left(\frac{x-2}{3}\right) + C$

$\displaystyle\int \frac{3}{x^2-4x+13}\,dx = \int \frac{3}{(x-2)^2+9}\,dx =$ STANDARD ARCTAN IN $(x-2)$

$= \tfrac{1}{3}\times 3 \times \arctan\left(\tfrac{x-2}{3}\right) + C = \arctan\left(\tfrac{x-2}{3}\right) + C$

553. $\displaystyle\int \frac{\sin x}{\cos^5 x}\ dx = \frac{1}{4}\sec^4 x + C$

$\displaystyle\int \frac{\sin x}{\cos^5 x}\,dx = \int \frac{\sin x}{\cos x}\times\frac{1}{\cos^4 x}\,dx = \int \tan x\sec^4 x\,dx$

$\displaystyle = \int (\tan x\sec x)\sec^3 x\,dx \ldots$ $\tfrac{d}{dx}(\sec x) = \sec x\tan x$

$= \tfrac{1}{4}\sec^4 x + C$

554. $\displaystyle\int 3\tanh^2 x\ dx = 3x - 3\tanh x + C$

$\displaystyle\int 3\tanh^2 x\,dx = \ldots$ $1+\tan^2 x \equiv \sec^2 x$, $1-\tanh^2 x \equiv \text{sech}^2 x$ $\displaystyle \ldots = \int 3(1-\text{sech}^2 x)\,dx$

$\displaystyle = \int 3 - 3\,\text{sech}^2 x\,dx = 3x - 3\tanh x + C$

$\tfrac{d}{dx}(\tanh x) = \text{sech}^2 x$

555. $$\int \frac{1}{\sqrt{x^2-4x+13}}\,dx = \begin{bmatrix} \operatorname{arsinh}\left(\dfrac{x-2}{3}\right)+C \\ \ln\left[x-2+\sqrt{x^2-4x+13}+C\right] \end{bmatrix}$$

$$\int \frac{1}{\sqrt{x^2-4x+13}}\,dx = \int \frac{1}{\sqrt{(x-2)^2+9}}\,dx = \text{STANDARD ARSINH IN } x-2$$

$$= \operatorname{arsinh}\left(\frac{x-2}{3}\right)+C$$

OR IN LOGARITHMS

$$= \ln\left[\frac{x-2}{3}+\sqrt{\frac{(x-2)^2}{9}+1}\right]+C = \ln\left[\frac{x-2}{3}+\frac{1}{3}\sqrt{x^2-4x+13}\right]+C$$

$$= \ln\left[\frac{1}{3}\left[x-2+\sqrt{x^2-4x+13}\right]\right]+C = \ln\frac{1}{3}+\ln\left[x-2+\sqrt{x^2-4x+13}\right]+C$$

$$= \ln\left[x-2+\sqrt{x^2-4x+13}\right]+C$$

556. $$\int \frac{3}{\sqrt{4-x^4}}\,dx = \frac{1}{2}\arcsin\left(\frac{1}{2}x^2\right)+C$$

$$\int \frac{x}{\sqrt{4-x^4}}\,dx = \ldots \text{ BY INSPECTION/RECOGNITION OF ARCSINE/ARCCOS}$$

$$= \frac{1}{2}\arcsin\frac{x^2}{2}+C$$

$$\frac{d}{dx}\left[\arcsin\frac{x^2}{2}\right] = 2x\,\frac{1}{\sqrt{1-\frac{x^4}{4}}} = \frac{2x}{\sqrt{4-x^4}}$$

THE SUBSTITUTION $x^2 = 2\sin\theta$, IE $\sin\theta = \frac{x^2}{2}$ ALSO WORKS WELL

557. $$\int \frac{1}{\sqrt{16x^2-9}}\,dx = \begin{bmatrix} \dfrac{1}{4}\operatorname{arcosh}\left(\dfrac{4}{3}x\right)+C \\ \dfrac{1}{4}\ln\left[4x+\sqrt{16x^2-9}+C\right] \end{bmatrix}$$

$$\int \frac{1}{\sqrt{16x^2-9}}\,dx = \frac{1}{4}\int \frac{1}{\sqrt{x^2-\frac{9}{16}}}\,dx = \frac{1}{4}\int \frac{1}{\sqrt{x^2-\left(\frac{3}{4}\right)^2}}\,dx$$

STANDARD ARCCOSH INTEGRAL

$$= \frac{1}{4}\operatorname{arcosh}\left(\frac{x}{3/4}\right)+C = \frac{1}{4}\operatorname{arcosh}\left(\frac{4x}{3}\right)+C$$

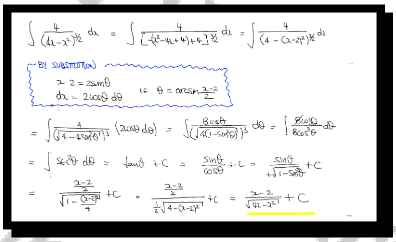
OR IN LOGARITHMIC FORM

$$= \frac{1}{4}\ln\left[\frac{4x}{3}+\sqrt{\frac{16x^2}{9}-1}\right]+C = \frac{1}{4}\ln\left[\frac{4x}{3}+\frac{1}{3}\sqrt{16x^2-9}\right]+C$$

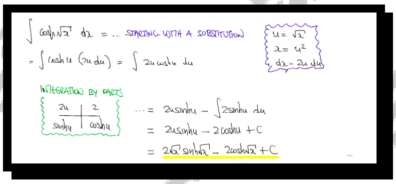
$$= \frac{1}{4}\ln\left[\frac{1}{3}\left(4x+\sqrt{16x^2-9}\right)\right]+C = \frac{1}{4}\ln\frac{1}{3}+\frac{1}{4}\ln\left(4x+\sqrt{16x^2-9}\right)+C$$

$$= \frac{1}{4}\ln\left[4x+\sqrt{16x^2-9}\right]+C$$

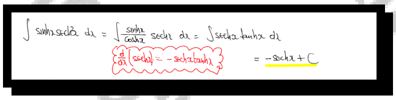
558. $$\int \frac{4}{\left(4x-x^2\right)^{\frac{3}{2}}}\,dx = \frac{x-2}{\sqrt{4x-x^2}} + C$$



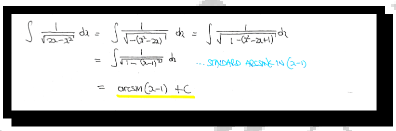
559. $$\int \cosh\sqrt{x}\,dx = 2\sqrt{x}\,\sinh\sqrt{x} - 2\cosh\sqrt{x} + C$$



560. $$\int \sinh x\,\operatorname{sech}^2 x\,dx = -\operatorname{sech} x + C$$



561. $$\int \frac{1}{\sqrt{2x-x^2}}\,dx = \arcsin\left(x-1\right) + C$$



562. $$\int \frac{8}{x^4-1}\,dx = 2\ln\left|\frac{x-1}{x+1}\right| - 4\arctan x + C$$

$$\int \frac{8}{x^4-1}\,dx = \int \frac{8}{(x^2-1)(x^2+1)}\,dx = \int \frac{8}{(x-1)(x+1)(x^2+1)}\,dx$$

BY PARTIAL FRACTIONS

$$\frac{8}{(x-1)(x+1)(x^2+1)} \equiv \frac{A}{x-1} + \frac{B}{x+1} + \frac{Cx+D}{x^2+1}$$

$$\boxed{8 \equiv A(x+1)(x^2+1) + B(x-1)(x^2+1) + (Cx+D)(x+1)(x-1)}$$

- IF $x=1$: $8 = 4A$, $A=2$
- IF $x=-1$: $8 = -4B$, $B=-2$
- IF $x=0$: $8 = A-B-D$, $8 = 2+2-D$, $D=-4$
- IF $x=2$: $8 = 15A+5B+3(2C+D)$, $8 = 30-10+3(2C-4)$, $8 = 20+6C-12$, $C=0$

$$= \int \frac{2}{x-1} - \frac{2}{x+1} - \frac{4}{x^2+1}\,dx = 2\ln|x-1| - 2\ln|x+1| - 4\arctan x + C$$

$$= 2\ln\left|\frac{x-1}{x+1}\right| - 4\arctan x + C$$

563. $$\int \frac{1}{x\left(x^2+1\right)}\,dx = \begin{cases} \ln|x| - \frac{1}{2}\ln\left(x^2+1\right) + C \\ \frac{1}{2}\ln\left(\frac{x^2}{x^2+1}\right) + C \end{cases}$$

$$\int \frac{1}{x(x^2+1)}\,dx = \dots$$

PARTIAL FRACTIONS

$$\frac{1}{x(x^2+1)} \equiv \frac{A}{x} + \frac{Bx+C}{1+x^2}$$

$$1 \equiv A(1+x^2) + (Bx+C)x$$

$$1 \equiv (A+B)x^2 + Cx + A$$

$$A=1,\ C=0,\ B=-1$$

$$= \int \frac{1}{x} - \frac{x}{x^2+1}\,dx = \ln x - \frac{1}{2}\ln(x^2+1) + C$$

$$= \frac{1}{2}\left[2\ln x - \ln(x^2+1)\right] + C = \frac{1}{2}\left[\ln x^2 - \ln(x^2+1)\right] + C$$

$$= \frac{1}{2}\ln\left(\frac{x^2}{x^2+1}\right) + C$$

564. $$\int \frac{1}{x\sqrt{1-(\ln x)^2}}\,dx = \arcsin(\ln x) + C$$

$$\int \frac{1}{x\sqrt{1-(\ln x)^2}}\,dx = \int \frac{1}{x} \times \frac{1}{\sqrt{1-(\ln x)^2}}\,dx = \dots \text{ BY INSPECTION}$$

$$= \arcsin(\ln x) + C$$

THE SUBSTITUTION $u = \ln x$ ALSO WORKS WELL

565. $$\int \frac{6}{(1+\cos x)^2}\,dx = 3\tan\left(\frac{1}{2}x\right) + \tan^3\left(\frac{1}{2}x\right) + C$$

$$\int \frac{6}{(1+\cos x)^2}\,dx = \int \frac{6}{\left(1+2\cos^2\frac{x}{2}-1\right)^2}\,dx = \int \frac{6}{4\cos^4\frac{x}{2}}\,dx$$
$$= \int \frac{3}{2}\sec^4\frac{x}{2}\,dx = \int \frac{3}{2}\sec^2\frac{x}{2}\left(1+\tan^2\frac{x}{2}\right)dx.$$
$$= \int \frac{3}{2}\sec^2\frac{x}{2} + \frac{3}{2}\sec^2\frac{x}{2}\tan^2\frac{x}{2}\,dx$$

(BY INSPECTION) $= 3\tan\frac{x}{2} + \tan^3\frac{x}{2} + C$

566. $$\int \frac{x}{1+\sqrt{x}}\,dx = \frac{2}{3}x^{\frac{3}{2}} - x + 2x^{\frac{1}{2}} - 2\ln\left(1+\sqrt{x}\right) + C$$

$\int \frac{x}{1+\sqrt{x}}\,dx = \dots$ BY SUBSTITUTION...

$$\boxed{\begin{aligned} u &= 1+\sqrt{x} \\ u-1 &= \sqrt{x} \\ x &= (u-1)^2 \\ \frac{dx}{du} &= 2(u-1) \\ dx &= 2(u-1)\,du \end{aligned}}$$

$$= \int \frac{(u-1)^2}{u}\left(2(u-1)\,du\right) = \int \frac{2(u-1)^3}{u}\,du$$
$$= 2\int \frac{u^3-3u^2+3u-1}{u}\,du = 2\int u^2-3u+3-\frac{1}{u}\,du$$
$$= 2\left[\frac{1}{3}u^3 - \frac{3}{2}u^2 + 3u - \ln|u|\right] + C$$
$$= \frac{2}{3}(1+\sqrt{x})^3 - 3(1+\sqrt{x})^2 + 6(1+\sqrt{x}) - 2\ln(1+\sqrt{x}) + C$$

THE SUBSTITUTION $u=\sqrt{x}$ PRODUCES A NEATER ANSWER

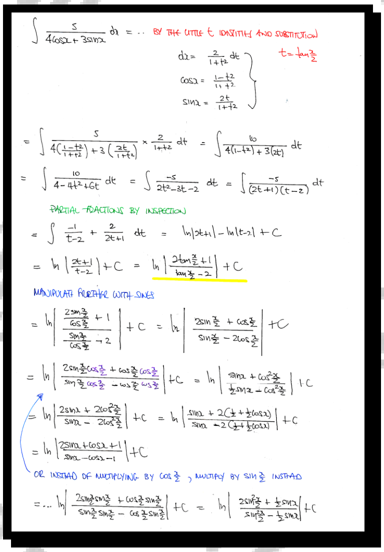
$$\boxed{\begin{aligned} u &= \sqrt{x} \\ u^2 &= x \\ 2u\,du &= dx \end{aligned}}$$

$$\int \frac{u^2}{1+u}(2u\,du) = \int \frac{2u^3}{1+u}\,du = 2\int \frac{u^3}{u+1}\,du$$

BY DIVISION OR MANIPULATION

$$= 2\int \frac{u^2(u+1) - u(u+1) + (u+1) - 1}{u+1}\,du$$
$$= 2\int u^2 - u + 1 - \frac{1}{u+1}\,du = 2\left[\frac{1}{3}u^3 - \frac{1}{2}u^2 + u - \ln|u+1|\right] + C$$
$$= \frac{2}{3}u^3 - u^2 + 2u - 2\ln|u+1| + C = \frac{2}{3}x^{\frac{3}{2}} - x + 2x^{\frac{1}{2}} - 2\ln(1+\sqrt{x}) + C$$

567. $$\int \frac{15}{4\cos x + 3\sin x}\,dx = \begin{cases} \ln\left|\dfrac{2\tan\left(\frac{1}{2}x\right)+1}{\tan\left(\frac{1}{2}x\right)-2}\right| + C \\ \ln\left|\dfrac{2\sin x + \cos x + 1}{\sin x - \cos x - 1}\right| + C \\ \ln\left|\dfrac{\sin x - 2\cos x + 2}{1 - \cos x - \sin x}\right| + C \\ \ln\left|\dfrac{5 + 4\sin x - 3\cos x}{4\cos x + 3\sin x}\right| + C \end{cases}$$


$\int \frac{5}{4\cos x + 3\sin x}\,dx = \dots$ BY THE LITTLE t IDENTITIES AND SUBSTITUTION $t = \tan\frac{x}{2}$

$dx = \frac{2}{1+t^2}\,dt$, $\cos x = \frac{1-t^2}{1+t^2}$, $\sin x = \frac{2t}{1+t^2}$

$= \int \frac{5}{4\left(\frac{1-t^2}{1+t^2}\right) + 3\left(\frac{2t}{1+t^2}\right)} \times \frac{2}{1+t^2}\,dt = \int \frac{10}{4(1-t^2)+3(2t)}\,dt$

$= \int \frac{10}{4-4t^2+6t}\,dt = \int \frac{-5}{2t^2-3t-2}\,dt = \int \frac{-5}{(2t+1)(t-2)}\,dt$

PARTIAL FRACTIONS BY INSPECTION

$= \int \frac{-1}{t-2} + \frac{2}{2t+1}\,dt = \ln|2t+1| - \ln|t-2| + C$

$= \ln\left|\frac{2t+1}{t-2}\right| + C = \ln\left|\frac{2\tan\frac{x}{2}+1}{\tan\frac{x}{2}-2}\right| + C$

MANIPULATE FURTHER WITH SINES

$= \ln\left|\frac{\frac{2\sin\frac{x}{2}}{\cos\frac{x}{2}}+1}{\frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}-2}\right| + C = \ln\left|\frac{2\sin\frac{x}{2}+\cos\frac{x}{2}}{\sin\frac{x}{2}-2\cos\frac{x}{2}}\right| + C$

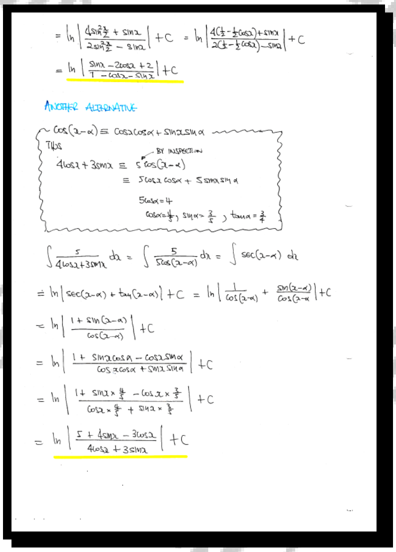
$= \ln\left|\frac{2\sin\frac{x}{2}\cos\frac{x}{2}+\cos\frac{x}{2}\cos\frac{x}{2}}{\sin\frac{x}{2}\cos\frac{x}{2}-2\cos\frac{x}{2}\cos\frac{x}{2}}\right| + C = \ln\left|\frac{\sin x + \cos^2\frac{x}{2}}{\frac{1}{2}\sin x - 2\cos^2\frac{x}{2}}\right| + C$

$= \ln\left|\frac{2\sin x + 2\cos^2\frac{x}{2}}{\sin x - 4\cos^2\frac{x}{2}}\right| + C = \ln\left|\frac{\sin x + 2\left(\frac{1}{2}+\frac{1}{2}\cos x\right)}{\sin x - 2\left(\frac{1}{2}+\frac{1}{2}\cos x\right)}\right| + C$

$= \ln\left|\frac{2\sin x + \cos x + 1}{\sin x - \cos x - 1}\right| + C$

OR INSTEAD OF MULTIPLYING BY $\cos\frac{x}{2}$, MULTIPLY BY $\sin\frac{x}{2}$ INSTEAD

$= \dots \ln\left|\frac{2\sin\frac{x}{2}\sin\frac{x}{2}+\cos\frac{x}{2}\sin\frac{x}{2}}{\sin\frac{x}{2}\sin\frac{x}{2}-2\cos\frac{x}{2}\sin\frac{x}{2}}\right| + C = \ln\left|\frac{2\sin^2\frac{x}{2}+\frac{1}{2}\sin x}{\sin^2\frac{x}{2}-\sin x}\right| + C$


$= \ln\left|\frac{4\sin^2\frac{x}{2}+\sin x}{2\sin^2\frac{x}{2}-2\sin x}\right| + C = \ln\left|\frac{4\left(\frac{1}{2}-\frac{1}{2}\cos x\right)+\sin x}{2\left(\frac{1}{2}-\frac{1}{2}\cos x\right)-\sin x}\right| + C$

$= \ln\left|\frac{\sin x - 2\cos x + 2}{1-\cos x-\sin x}\right| + C$

ANOTHER ALTERNATIVE

$\cos(x-\alpha) \equiv \cos x\cos\alpha + \sin x\sin\alpha$

THUS

$4\cos x + 3\sin x \equiv \overset{\text{BY INSPECTION}}{5}\cos(x-\alpha)$

$\equiv 5\cos x\cos\alpha + 5\sin x\sin\alpha$

$5\cos\alpha = 4$

$\cos\alpha = \frac{4}{5}$, $\sin\alpha = \frac{3}{5}$, $\tan\alpha = \frac{3}{4}$

$\int \frac{5}{4\cos x + 3\sin x}\,dx = \int \frac{5}{5\cos(x-\alpha)}\,dx = \int \sec(x-\alpha)\,dx$

$= \ln|\sec(x-\alpha)+\tan(x-\alpha)| + C = \ln\left|\frac{1}{\cos(x-\alpha)} + \frac{\sin(x-\alpha)}{\cos(x-\alpha)}\right| + C$

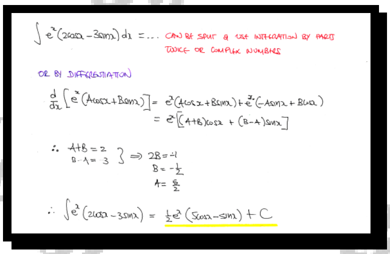
$= \ln\left|\frac{1+\sin(x-\alpha)}{\cos(x-\alpha)}\right| + C$

$= \ln\left|\frac{1+\sin x\cos\alpha - \cos x\sin\alpha}{\cos x\cos\alpha + \sin x\sin\alpha}\right| + C$

$= \ln\left|\frac{1+\sin x \times \frac{4}{5} - \cos x \times \frac{3}{5}}{\cos x \times \frac{4}{5} + \sin x \times \frac{3}{5}}\right| + C$

$= \ln\left|\frac{5+4\sin x - 3\cos x}{4\cos x + 3\sin x}\right| + C$

568. $$\int e^{x}\left(2\cos x - 3\sin x\right)\,dx = \tfrac{1}{2}e^{x}\left(5\cos x - \sin x\right) + C$$


$\int e^x(2\cos x - 3\sin x)\,dx = \dots$ CAN BE SPLIT & USE INTEGRATION BY PARTS TWICE OR COMPLEX NUMBERS

OR BY DIFFERENTIATION

$\frac{d}{dx}\left[e^x(A\cos x + B\sin x)\right] = e^x(A\cos x + B\sin x) + e^x(-A\sin x + B\cos x)$

$= e^x\left[(A+B)\cos x + (B-A)\sin x\right]$

$\therefore \left.\begin{array}{l} A+B=2 \\ B-A=-3 \end{array}\right\} \Rightarrow 2B=-1$

$B = -\frac{1}{2}$

$A = \frac{5}{2}$

$\therefore \int e^x(2\cos x - 3\sin x) = \frac{1}{2}e^x(5\cos x - \sin x) + C$

569. $\displaystyle\int \log_2 x \, dx = \frac{1}{\ln 2}\left(x \ln x - x\right) + C$

$$\int \log_2 x \, dx = \int \frac{\log_e x}{\log_e 2} dx = \int \frac{\ln x}{\ln 2} dx = \frac{1}{\ln 2}\int \ln x \, dx$$

STANDARD RESULT OR USE INTEGRATION BY PARTS BY WRITING $1 \times \ln x$

$$= \frac{1}{\ln 2}\left[x\ln x - x\right] + C$$

570. $\displaystyle\int \sqrt{1+\cos 3x} \, dx = \frac{2\sqrt{2}}{3}\sin\left(\frac{3}{2}x\right) + C$

$$\int \sqrt{1+\cos 3x}\, dx = \int \sqrt{1+\left(2\cos^2 \tfrac{3}{2}x - 1\right)}\, dx = \int \sqrt{2\cos^2 \tfrac{3}{2}x}\, dx$$

IGNORING "MODULUS" APPROACH

$$= \int \sqrt{2}\cos \tfrac{3}{2}x \, dx = \frac{2\sqrt{2}}{3}\sin \tfrac{3}{2}x + C$$

571. $\displaystyle\int \frac{x^3}{1+x^8} \, dx = \frac{1}{4}\arctan\left(x^4\right) + C$

$$\int \frac{x^3}{1+x^8} dx = \dots$$ BY INSPECTION SINCE $\frac{d}{dx}\left(\arctan x^4\right) = \frac{1}{1+\left(x^4\right)^2} \times 4x^3$

$$= \frac{1}{4}\arctan x^4 + C$$

572. $\displaystyle\int \ln\left(x+\sqrt{x}\right) dx = x\ln\left(x+\sqrt{x}\right) - \ln\left(1+\sqrt{x}\right) - x + \sqrt{x} + C$

$$\int \ln\left(x+\sqrt{x}\right) dx = \dots$$ START BY A LOGICAL SUBSTITUTION

$u = \sqrt{x}$

$u^2 = x$

$dx = 2u\,du$

$$= \int \ln\left(u^2+u\right)\left(2u\,du\right) = \int 2u\ln\left(u^2+u\right) du$$

NOW INTEGRATION BY PARTS

$\ln(u^2+u)$	$\frac{2u+1}{u^2+u}$
u^2	$2u$

$$= u^2\ln\left(u^2+u\right) - \int \frac{u^2(2u+1)}{u(u+1)} du$$

$$= u^2\ln\left(u^2+u\right) - \int \frac{2u^2+u}{u+1} du$$

MANIPULATION OR LONG DIVISION

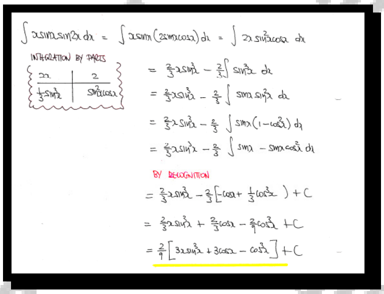
$$= u^2\ln\left(u^2+u\right) - \int \frac{2u(u+1)-(u+1)+1}{(u+1)} du$$

$$= u^2\ln\left(u^2+u\right) - \int 2u - 1 + \frac{1}{u+1} \, du$$

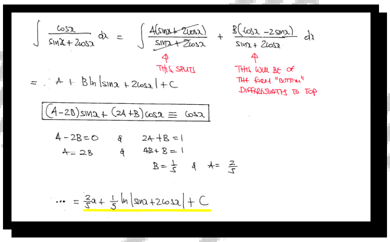
$$= u^2\ln\left(u^2+u\right) - u^2 + u - \ln(u+1) + C$$

$$= x\ln\left(x+\sqrt{x}\right) - x + \sqrt{x} - \ln\left(1+\sqrt{x}\right) + C$$

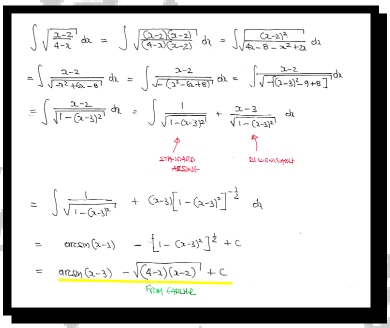
573. $\displaystyle\int x\sin x\sin 2x\,dx = \tfrac{2}{9}\left(3x\sin^3 x + 3\cos x - \cos^3 x\right) + C$



574. $\displaystyle\int \frac{\cos x}{\sin x + 2\cos x}\,dx = \tfrac{2}{5}x + \tfrac{1}{5}\ln\left|\sin x + 2\cos x\right| + C$



575. $\displaystyle\int \sqrt{\frac{x-2}{4-x}}\,dx = \arcsin(x-3) - \sqrt{(4-x)(x-2)} + C$



576. $\displaystyle\int \frac{\sinh^3 x}{\cosh^2 x}\,dx = \cosh x + \operatorname{sech} x + C$

$$\int \frac{\sinh^3 x}{\cosh^2 x}\,dx = \int \frac{\sinh x \sinh^2 x}{\cosh^2 x}\,dx = \int \frac{\sinh x(\cosh^2 x - 1)}{\cosh^2 x}\,dx$$

$$= \int \frac{\sinh x \cosh^2 x}{\cosh^2 x} - \frac{\sinh x}{\cosh^2 x}\,dx = \int \sinh x - \frac{\sinh x}{\cosh x}\times\frac{1}{\cosh x}\,dx$$

$$= \int \sinh x - \tanh x \operatorname{sech} x\,dx = \cosh x + \operatorname{sech} x + C$$

ALTERNATIVE BY SUBSTITUTION

$$\int \frac{\sinh^3 x}{\cosh^2 x}\,dx = \int \frac{\sinh^3 x}{u^2}\,\frac{du}{\sinh x} = \int \frac{\sinh^2 x}{u^2}\,du$$

$$= \int \frac{\cosh^2 x - 1}{u^2}\,du = \int \frac{u^2-1}{u^2}\,du = \int 1 - \frac{1}{u^2}\,du$$

$$= u + \frac{1}{u} + C = \cosh x + \frac{1}{\cosh x} + C = \cosh x + \operatorname{sech} x + C$$

$u = \cosh x$, $\frac{du}{dx} = \sinh x$, $dx = \frac{du}{\sinh x}$

577. $\displaystyle\int \frac{\sec^2 x}{4+\tan^2 x}\,dx = \frac{1}{2}\arctan\left(\frac{1}{2}\tan x\right) + C$

$$\int \frac{\sec^2 x}{4+\tan^2 x}\,dx = \dots$$

NOTING THAT $\frac{d}{dx}\left(\arctan(A\tan x)\right) = \frac{A\sec^2 x}{1+A^2\tan^2 x}$

IF $A=\frac{1}{2}$ $\quad \frac{d}{dx}\left(\arctan\left(\frac{1}{2}\tan x\right)\right) = \frac{\frac{1}{2}\sec^2 x}{1+\frac{1}{4}\tan^2 x} = \frac{2\sec^2 x}{4+\tan^2 x}$

$$\therefore \int \frac{\sec^2 x}{4+\tan^2 x}\,dx = \frac{1}{2}\arctan\left(\frac{1}{2}\tan x\right) + C$$

ALTERNATIVE BY SUBSTITUTION

$$\int \frac{\sec^2 x}{4+\tan^2 x}\,dx = \int \frac{\cancel{\sec^2 x}}{4+u^2}\,\frac{du}{\cancel{\sec^2 x}} = \int \frac{1}{4+u^2}\,du$$

$$= \int \frac{1}{u^2+2^2}\,du = \frac{1}{2}\arctan\left(\frac{u}{2}\right) + C$$

$$= \frac{1}{2}\arctan\left(\frac{1}{2}\tan x\right) + C$$

$u = \tan x$, $\frac{du}{dx} = \sec^2 x$, $dx = \frac{du}{\sec^2 x}$

578. $\displaystyle\int \frac{4x+5}{x^2+2x+2}\,dx = 2\ln\left(x^2+2x+2\right) + \arctan(x+1) + C$

$$\int \frac{4x+5}{x^2+2x+2}\,dx = \dots$$ COMPLETE THE SQUARE INTO ARCTANS & SPLITTING

$$= \int \frac{(4x+4)+1}{(x+1)^2+1}\,dx = \int \frac{4(x+1)}{(x+1)^2+1} + \frac{1}{(x+1)^2+1}\,dx$$

("BOTTOM" DIFFERENTIATES TO "TOP") (STANDARD ARCTAN)

$$= 2\ln\left[(x+1)^2+1\right] + \arctan(x+1) + C$$

$$= 2\ln\left(x^2+2x+2\right) + \arctan(x+1) + C$$

579. $$\int \frac{1}{(x-2)^{\frac{1}{2}}(4-x)^{\frac{1}{2}}}\,dx = \arcsin(x-3) + C$$

$$\int \frac{1}{(x-2)^{\frac{1}{2}}(4-x)^{\frac{1}{2}}}\,dx = \int \frac{1}{\sqrt{-x^2+6x-8}}\,dx = \int \frac{1}{\sqrt{-(x^2-6x+8)}}\,dx$$

$$= \int \frac{1}{\sqrt{-[(x-3)^2-9+8]}}\,dx = \int \frac{1}{\sqrt{1-(x-3)^2}}\,dx = \arcsin(x-3) + C$$

580. $$\int \frac{x^3}{x-2}\,dx = \frac{1}{2}x^3 + x^2 + 4x + 8\ln|x-2| + C$$

BY LONG DIVISION OR MANIPULATION

$$\int \frac{x^3}{x-2}\,dx = \int \frac{x^2(x-2)+2x(x-2)+4(x-2)+8}{(x-2)}\,dx = \int x^2+2x+4+\frac{8}{x-2}\,dx$$

$$= \frac{1}{3}x^3 + x^2 + 4x + 8\ln|x-2| + C$$

THE SUBSTITUTION $u=x-2$ ALSO WORK WELL

581. $$\int x^5 e^{x^3}\,dx = \frac{1}{3}e^{x^3}\left(x^3-1\right) + C$$

$$\int x^5 e^{x^3}\,dx = \int x^3 x^2 e^{x^3}\,dx = \dots \text{ INTEGRATION BY PARTS}$$

x^3	$3x^2$
$\frac{1}{3}e^{x^3}$	$x^2e^{x^3}$

$\frac{d}{dx}\left(e^{x^3}\right) = 3x^2e^{x^3}$

$$= \frac{1}{3}x^3e^{x^3} - \int x^2e^{x^3}\,dx$$

$$= \frac{1}{3}x^3e^{x^3} - \frac{1}{3}e^{x^3} + C$$

$$= \frac{1}{3}e^{x^3}(x^3-1) + C$$

THE SUBSTITUTION $u=x^3$ FIRST, SIMPLIFIES THE INTEGRAL ENOUGH SO THE INTEGRATION BY PARTS WHICH FOLLOWS IS A FORMALITY

582. $$\int \frac{x^2-12x-6}{(x-3)\left(x^2+2\right)}\,dx = 2\ln\left(x^2+2\right) - 3\ln|x-3| + C$$

$$\int \frac{x^2-12x-6}{(x-3)(x^2+2)}\,dx = \dots \text{ BY PARTIAL FRACTIONS} \dots$$

$$\frac{x^2-12x-6}{(x-3)(x^2+2)} \equiv \frac{A}{x-3} + \frac{Bx+C}{x^2+2}$$

$$x^2-12x-6 \equiv A(x^2+2) + (x-3)(Bx+C)$$

$$x^2-12x-6 \equiv (A+B)x^2 + (C-3B)x + (2A-3C)$$

NOW WE HAVE:
- IF $x=3 \Rightarrow 9-36-6=11A$, $11A=-33$, $\boxed{A=-3}$
- $A+B=1$, $-3+B=1$, $\boxed{B=4}$
- $C-3B=-12$, $C-12=-12$, $\boxed{C=0}$

$$\dots = \int -\frac{3}{x-3} + \frac{4x}{x^2+2}\,dx = \int 2\left(\frac{2x}{x^2+2}\right) - \frac{3}{x-3}\,dx = 2\ln(x^2+2) - 3\ln|x-3| + C$$

583. $\displaystyle\int \sin x \sin^3\left(\tfrac{1}{2}x\right)\, dx = \tfrac{4}{5}\sin^5\left(\tfrac{1}{2}x\right) + C$

$\displaystyle\int \sin x \sin^3\left(\frac{x}{2}\right) dx = \int \left(2\sin\frac{x}{2}\cos\frac{x}{2}\right)\sin^3\frac{x}{2}\, dx = \int 2\sin^4\frac{x}{2}\cos\frac{x}{2}\, dx$

SIN2A ≡ 2SINACOSA

(RECOGNITION)

$\displaystyle = \frac{2}{5}\times 2\sin^5\frac{x}{2} + C = \frac{4}{5}\sin^5\frac{x}{2} + C$

584. $\displaystyle\int \frac{\cot x}{\sqrt{1-\cos 2x}}\, dx = -\frac{1}{\sqrt{2}}\operatorname{cosec} x + C$

$\displaystyle\int \frac{\cot x}{\sqrt{1-\cos 2x}}\, dx = \int \frac{\cot x}{\sqrt{1-(1-2\sin^2 x)}}\, dx = \int \frac{\cot x}{\sqrt{2\sin^2 x}}\, dx$

IGNORING MODULI

$\displaystyle = \int \frac{1}{\sqrt{2}}\frac{\cot x}{\sin x}\, dx = \frac{1}{\sqrt{2}}\int \cot x \operatorname{cosec} x\, dx = -\frac{1}{\sqrt{2}}\operatorname{cosec} x + C$

585. $\displaystyle\int \tan^5 x \sec x\, dx = \tfrac{1}{5}\sec^5 x - \tfrac{2}{3}\sec^3 x + \sec x + C$

$\displaystyle\int \tan^5 x \sec x\, dx = \int \tan^2 x \tan^2 x (\tan x \sec x)\, dx = \int (\sec^2 x - 1)^2 \sec x \tan x\, dx$

DIFFERENTIAL OF SECx

$\displaystyle = \int (\sec^4 x - 2\sec^2 x + 1)(\sec x \tan x)\, dx$

$\displaystyle = \int \sec^4 x(\sec x \tan x) - 2\sec^2 x(\sec x \tan x) + (\sec x \tan x)\, dx$

$\displaystyle = \frac{1}{5}\sec^5 x - \frac{2}{3}\sec^3 x + \sec x + C$

586. $\displaystyle\int \frac{2\cosh x \sinh x}{1+\cosh^4 x}\, dx = \arctan\left(\cosh^2 x\right) + C$

$\displaystyle\int \frac{2\cosh x \sinh x}{1+\cosh^4 x}\, dx = \ldots$ BY RECOGNITION OF AN ARCTAN

$\displaystyle = \arctan(\cosh^2 x) + C$

ALTERNATIVE BY SUBSTITUTION

$u = \cosh^2 x$

$\dfrac{du}{dx} = 2\sinh x\cosh x$

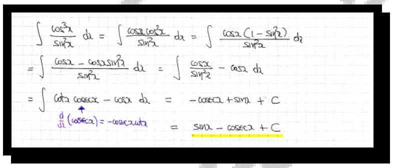
$dx = \dfrac{du}{2\sinh x\cosh x}$

$\displaystyle \ldots = \int \frac{2\cosh x\sinh x}{1+u^2}\times\frac{du}{2\sinh x\cosh x}$

$\displaystyle = \int \frac{1}{1+u^2}\, du = \arctan u + C$

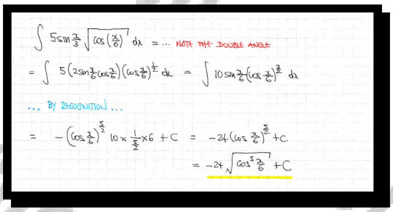
$\displaystyle = \arctan(\cosh^2 x) + C$

587. $\displaystyle\int \frac{\cos^3 x}{\sin^2 x}\,dx = \sin x - \operatorname{cosec} x + C$



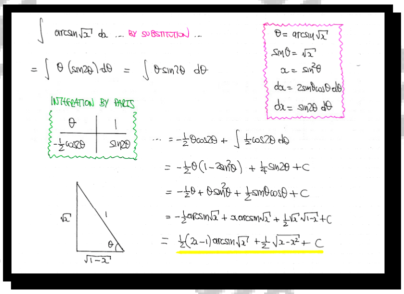
$$\int \frac{\cos^3 x}{\sin^2 x}\,dx = \int \frac{\cos x\cos^2 x}{\sin^2 x}\,dx = \int \frac{\cos x(1-\sin^2 x)}{\sin^2 x}\,dx$$
$$= \int \frac{\cos x - \cos x\sin^2 x}{\sin^2 x}\,dx = \int \frac{\cos x}{\sin^2 x} - \cos x\,dx$$
$$= \int \cot x\operatorname{cosec} x - \cos x\,dx = -\operatorname{cosec} x + \sin x + C$$
$\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x\cot x$
$$= \sin x - \operatorname{cosec} x + C$$

588. $\displaystyle\int 5\sin\left(\tfrac{1}{3}x\right)\sqrt{\cos\left(\tfrac{1}{6}x\right)}\,dx = -24\sqrt{\cos^5\left(\tfrac{1}{6}x\right)} + C$



$$\int 5\sin\frac{x}{3}\sqrt{\cos\left(\frac{x}{6}\right)}\,dx = \dots$$ NOTE THE DOUBLE ANGLE
$$= \int 5\left(2\sin\frac{x}{6}\cos\frac{x}{6}\right)\left(\cos\frac{x}{6}\right)^{\frac{1}{2}}dx = \int 10\sin\frac{x}{6}\left(\cos\frac{x}{6}\right)^{\frac{3}{2}}dx$$
... BY RECOGNITION ...
$$= -\left(\cos\frac{x}{6}\right)^{\frac{5}{2}} 10\times\frac{1}{\frac{5}{2}}\times 6 + C = -24\left(\cos\frac{x}{6}\right)^{\frac{5}{2}} + C$$
$$= -24\sqrt{\cos^5\frac{x}{6}} + C$$

589. $\displaystyle\int \arcsin\sqrt{x}\,dx = \tfrac{1}{2}(2x-1)\arcsin\sqrt{x} + \tfrac{1}{2}\sqrt{x-x^2} + C$



$\int \arcsin\sqrt{x}\,dx$ ... BY SUBSTITUTION ...

$\theta = \arcsin\sqrt{x}$
$\sin\theta = \sqrt{x}$
$x = \sin^2\theta$
$dx = 2\sin\theta\cos\theta\,d\theta$
$dx = \sin 2\theta\,d\theta$

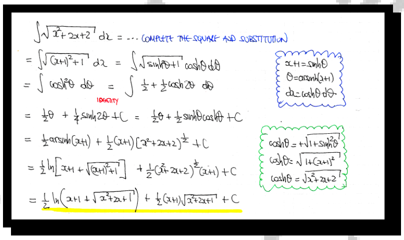
$$= \int \theta(\sin 2\theta)\,d\theta = \int \theta\sin 2\theta\,d\theta$$

INTEGRATION BY PARTS

θ	1
$-\frac{1}{2}\cos 2\theta$	$\sin 2\theta$

$$\dots = -\tfrac{1}{2}\theta\cos 2\theta + \int \tfrac{1}{2}\cos 2\theta\,d\theta$$
$$= -\tfrac{1}{2}\theta(1-2\sin^2\theta) + \tfrac{1}{4}\sin 2\theta + C$$
$$= -\tfrac{1}{2}\theta + \theta\sin^2\theta + \tfrac{1}{2}\sin\theta\cos\theta + C$$
$$= -\tfrac{1}{2}\arcsin\sqrt{x} + x\arcsin\sqrt{x} + \tfrac{1}{2}\sqrt{x}\sqrt{1-x} + C$$
$$= \tfrac{1}{2}(2x-1)\arcsin\sqrt{x} + \tfrac{1}{2}\sqrt{x-x^2} + C$$

590. $\displaystyle\int \sqrt{x^2+2x+2}\, dx = \tfrac{1}{2}\ln\left[x+1+\sqrt{x^2+2x+2}\right]+\tfrac{1}{2}(x+1)\sqrt{x^2+2x+2}+C$



$\int \sqrt{x^2+2x+2}\, dx = \dots$ COMPLETE THE SQUARE AND SUBSTITUTION

$= \int \sqrt{(x+1)^2+1}\, dx = \int \sqrt{\sinh^2\theta+1}\cosh\theta\, d\theta$

$= \int \cosh^2\theta\, d\theta = \int \frac{1}{2}+\frac{1}{2}\cosh 2\theta\, d\theta$

IDENTITY

$= \frac{1}{2}\theta+\frac{1}{4}\sinh 2\theta + C = \frac{1}{2}\theta+\frac{1}{2}\sinh\theta\cosh\theta + C$

$= \frac{1}{2}\text{arsinh}(x+1)+\frac{1}{2}(x+1)\left(x^2+2x+2\right)^{\frac{1}{2}}+C$

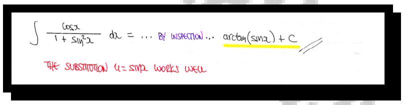
$= \frac{1}{2}\ln\left[x+1+\sqrt{(x+1)^2+1}\right]+\frac{1}{2}\left(x^2+2x+2\right)^{\frac{1}{2}}(x+1)+C$

$= \frac{1}{2}\ln\left(x+1+\sqrt{x^2+2x+1}\right)+\frac{1}{2}(x+1)\sqrt{x^2+2x+1}+C$

$x+1=\sinh\theta$
$\theta=\text{arsinh}(x+1)$
$dx=\cosh\theta\, d\theta$

$\cosh\theta=+\sqrt{1+\sinh^2\theta}$
$\cosh\theta=\sqrt{1+(x+1)^2}$
$\cosh\theta=\sqrt{x^2+2x+2}$

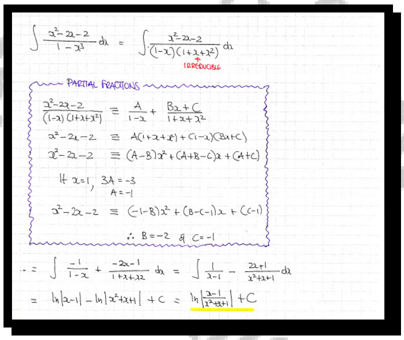
591. $\displaystyle\int \frac{\cos x}{1+\sin^2 x}\, dx = \arctan(\sin x)+C$



$\int \frac{\cos x}{1+\sin^2 x}\, dx = \dots$ BY INSPECTION $\dots$ $\arctan(\sin x)+C$

THE SUBSTITUTION $u=\sin x$ WORKS WELL

592. $\displaystyle\int \frac{x^2-2x-2}{1-x^3}\, dx = \ln\left|\frac{x-1}{x^2+x+1}\right|+C$



$\int \frac{x^2-2x-2}{1-x^3}\, dx = \int \frac{x^2-2x-2}{(1-x)(1+x+x^2)}\, dx$

IRREDUCIBLE

PARTIAL FRACTIONS

$\frac{x^2-2x-2}{(1-x)(1+x+x^2)} \equiv \frac{A}{1-x}+\frac{Bx+C}{1+x+x^2}$

$x^2-2x-2 \equiv A(1+x+x^2)+(1-x)(Bx+C)$

$x^2-2x-2 \equiv (A-B)x^2+(A+B-C)x+(A+C)$

If $x=1$, $3A=-3$
$A=-1$

$x^2-2x-2 \equiv (-1-B)x^2+(B-C-1)x+(C-1)$

$\therefore B=-2$ & $C=-1$

$\dots = \int \frac{-1}{1-x}+\frac{-2x-1}{1+x+x^2}\, dx = \int \frac{1}{x-1}-\frac{2x+1}{x^2+x+1}\, dx$

$= \ln|x-1|-\ln|x^2+x+1|+C = \ln\left|\frac{x-1}{x^2+x+1}\right|+C$

593. $$\int \frac{3x^3-4x^2+5x-2}{(x-1)(x^2+1)}\,dx = 3x+\ln\left|\frac{x-1}{x^2+1}\right| + C$$

$$\int \frac{3x^3-4x^2+5x-2}{(x-1)(x^2+1)} = \dots$$ IMPROPER PARTIAL FRACTIONS WITH IRREDUCIBLE FACTOR

$$\frac{3x^3-4x^2+5x-2}{(x-1)(x^2+1)} \equiv 3 + \frac{A}{x-1} + \frac{Bx+C}{x^2+1}$$

$$3x^3-4x^2+5x-2 \equiv 3(x-1)(x^2+1)+A(x^2+1)+(x-1)(Bx+C)$$

- IF $x=1$: $2A=2$, $A=1$
- IF $x=0$: $-2=-3+A-C$, $C=0$
- IF $x=2$: $24-16+10-2=15+5+2B$, $-4=2B$, $B=-2$

$$= \int 3+\frac{1}{x-1}-\frac{2x}{x^2+1}\,dx = 3x+\ln|x-1|-\ln(x^2+1)+C$$

"BOTTOM" DIFFERENTIATES TO "TOP"

$$= 3x+\ln\left|\frac{x-1}{x^2+1}\right|+C$$

594. $$\int \frac{8x^2-29x-10}{\left(x^2-x-2\right)^2}\,dx = \frac{x+10}{x^2-x-2}+3\ln\left|\frac{x-2}{x+1}\right| + C$$

$$\int \frac{8x^2-29x-10}{(x^2-x-2)^2}\,dx = \int \frac{8x^2-29x-10}{(x+1)^2(x-2)^2}\,dx$$

PARTIAL FRACTIONS WITH TWO DISTINCT REPEATED FACTORS

$$\frac{8x^2-29x-10}{(x+1)^2(x-2)^2} \equiv \frac{A}{(x+1)^2}+\frac{B}{(x-2)^2}+\frac{C}{x+1}+\frac{D}{x-2}$$

$$8x^2-29x-10 \equiv A(x-2)^2+B(x+1)^2+C(x-2)^2(x+1)+D(x+1)^2(x-2)$$

- If $x=2$: $32-58-10=9B$, $-36=9B$, $B=-4$
- IF $x=-1$: $8+29-10=9A$, $27=9A$, $A=3$
- IF $x=3$: $72-87-10=A+16B+4C+16D$, $-25=3-64+4C+16D$, $4C+16D=36$, $C+4D=9$
- IF $x=-2$: $32+58-10=16A+B-16C-4D$, $80=48-4-16C-4D$, $36=-16C-4D$, $4D+16C=-36$, $D+4C=-9$

SOLVING SIMULTANEOUSLY

$$\left.\begin{array}{l} C+4D=9 \\ 4C+D=-9 \end{array}\right\} \Rightarrow \left.\begin{array}{l} C+4D=9 \\ -16C-4D=36 \end{array}\right\} \Rightarrow -15C=45$$

$C=-3$ & $D=3$

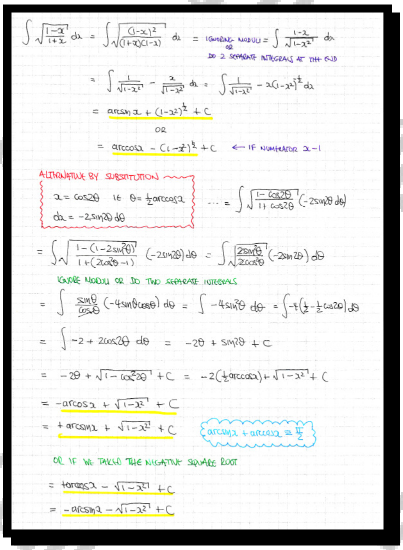
$$\dots = \int \frac{3}{(x+1)^2}-\frac{4}{(x-2)^2}-\frac{3}{x+1}+\frac{3}{x-2}\,dx$$

$$= -\frac{3}{x+1}+\frac{4}{x-2}+3\ln|x-2|-3\ln|x+1|+C$$

$$= \frac{4x+4-3x+6}{x^2-x-2}+3\ln\left|\frac{x-2}{x+1}\right|+C$$

$$= \frac{x+10}{x^2-x-2}+3\ln\left|\frac{x-2}{x+1}\right|+C$$

595. $$\int \sqrt{\frac{1-x}{1+x}}\, dx = \begin{cases} \arcsin x + \sqrt{1-x^2} + C \\ \arccos x - \sqrt{1-x^2} + C \\ -\arcsin x - \sqrt{1-x^2} + C \\ -\arccos x + \sqrt{1-x^2} + C \end{cases}$$



$\int \sqrt{\frac{1-x}{1+x}}\, dx = \int \sqrt{\frac{(1-x)^2}{(1+x)(1-x)}}\, dx$ = IGNORING MODULI = $\int \frac{1-x}{\sqrt{1-x^2}}\, dx$
OR
DO 2 SEPARATE INTEGRALS AT THE END

$= \int \frac{1}{\sqrt{1-x^2}} - \frac{x}{\sqrt{1-x^2}}\, dx = \int \frac{1}{\sqrt{1-x^2}} - x(1-x^2)^{-\frac{1}{2}}\, dx$

$= \arcsin x + (1-x^2)^{\frac{1}{2}} + C$

OR

$= \arccos x - (1-x^2)^{\frac{1}{2}} + C$ ← IF NUMERATOR $x-1$

ALTERNATIVE BY SUBSTITUTION

$x = \cos 2\theta$ ie $\theta = \frac{1}{2}\arccos x$
$dx = -2\sin 2\theta\, d\theta$

$\dots = \int \sqrt{\frac{1-\cos 2\theta}{1+\cos 2\theta}}\,(-2\sin 2\theta\, d\theta)$

$= \int \sqrt{\frac{1-(1-2\sin^2\theta)}{1+(2\cos^2\theta-1)}}\,(-2\sin 2\theta)\, d\theta = \int \sqrt{\frac{2\sin^2\theta}{2\cos^2\theta}}\,(-2\sin 2\theta)\, d\theta$

IGNORE MODULI OR DO TWO SEPARATE INTEGRALS

$= \int \frac{\sin\theta}{\cancel{\cos\theta}}(-4\sin\theta\cos\theta)\, d\theta = \int -4\sin^2\theta\, d\theta = \int -4\left(\frac{1}{2}-\frac{1}{2}\cos 2\theta\right) d\theta$

$= \int -2 + 2\cos 2\theta\, d\theta = -2\theta + \sin 2\theta + C$

$= -2\theta + \sqrt{1-\cos^2 2\theta} + C = -2\left(\frac{1}{2}\arccos x\right) + \sqrt{1-x^2} + C$

$= -\arccos x + \sqrt{1-x^2} + C$

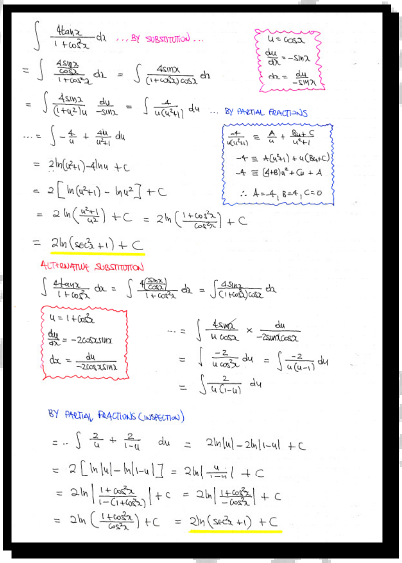
$= +\arcsin x + \sqrt{1-x^2} + C$ $\quad \arcsin x + \arccos x \equiv \frac{\pi}{2}$

OR IF WE TAKE THE NEGATIVE SQUARE ROOT

$= +\arccos x - \sqrt{1-x^2} + C$

$= -\arcsin x - \sqrt{1-x^2} + C$

596. $$\int \frac{4\tan x}{1+\cos^2 x}\, dx = 2\ln\left(1+\sec^2 x\right) + C$$



$\int \frac{4\tan x}{1+\cos^2 x}\, dx$ ... BY SUBSTITUTION ...

$u = \cos x$
$\frac{du}{dx} = -\sin x$
$dx = \frac{du}{-\sin x}$

$= \int \frac{\frac{4\sin x}{\cos x}}{1+\cos^2 x}\, dx = \int \frac{4\sin x}{(1+\cos^2 x)\cos x}\, dx$

$= \int \frac{4\sin x}{(1+u^2)u}\, \frac{du}{-\sin x} = \int \frac{-4}{u(u^2+1)}\, du$ ... BY PARTIAL FRACTIONS

$\frac{-4}{u(u^2+1)} \equiv \frac{A}{u} + \frac{Bu+C}{u^2+1}$
$-4 \equiv A(u^2+1) + u(Bu+C)$
$-4 \equiv (A+B)u^2 + Cu + A$
$\therefore A=-4,\ B=4,\ C=0$

$\dots = \int -\frac{4}{u} + \frac{4u}{u^2+1}\, du$

$= 2\ln(u^2+1) - 4\ln u + C$

$= 2\left[\ln(u^2+1) - \ln u^2\right] + C$

$= 2\ln\left(\frac{u^2+1}{u^2}\right) + C = 2\ln\left(\frac{1+\cos^2 x}{\cos^2 x}\right) + C$

$= 2\ln(\sec^2 x + 1) + C$

ALTERNATIVE SUBSTITUTION

$\int \frac{4\tan x}{1+\cos^2 x}\, dx = \int \frac{4\left(\frac{\sin x}{\cos x}\right)}{1+\cos^2 x}\, dx = \int \frac{4\sin x}{(1+\cos^2 x)\cos x}\, dx$

$u = 1+\cos^2 x$
$\frac{du}{dx} = -2\cos x\sin x$
$dx = \frac{du}{-2\cos x\sin x}$

$\dots = \int \frac{4\cancel{\sin x}}{u\cos x} \times \frac{du}{-2\cancel{\sin x}\cos x}$

$= \int \frac{-2}{u\cos^2 x}\, du = \int \frac{-2}{u(u-1)}\, du$

$= \int \frac{2}{u(1-u)}\, du$

BY PARTIAL FRACTIONS (INSPECTION)

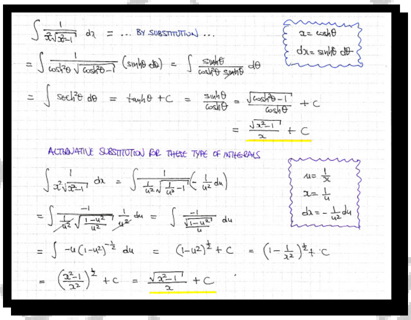
$= \dots \int \frac{2}{u} + \frac{2}{1-u}\, du = 2\ln|u| - 2\ln|1-u| + C$

$= 2\left[\ln|u| - \ln|1-u|\right] = 2\ln\left|\frac{u}{1-u}\right| + C$

$= 2\ln\left|\frac{1+\cos^2 x}{1-(1+\cos^2 x)}\right| + C = 2\ln\left|\frac{1+\cos^2 x}{-\cos^2 x}\right| + C$

$= 2\ln\left(\frac{1+\cos^2 x}{\cos^2 x}\right) + C = 2\ln(\sec^2 x + 1) + C$

597. $$\int \frac{1}{x^2\sqrt{x^2-1}}\,dx = \frac{\sqrt{x^2-1}}{x} + C$$



$\int \frac{1}{x^2\sqrt{x^2-1}}\,dx = \dots$ BY SUBSTITUTION ...

$x = \cosh\theta$
$dx = \sinh\theta\,d\theta$

$= \int \frac{1}{\cosh^2\theta\sqrt{\cosh^2\theta-1}}(\sinh\theta\,d\theta) = \int \frac{\sinh\theta}{\cosh^2\theta\,\sinh\theta}\,d\theta$

$= \int \operatorname{sech}^2\theta\,d\theta = \tanh\theta + C = \frac{\sinh\theta}{\cosh\theta} = \frac{\sqrt{\cosh^2\theta-1}}{\cosh\theta} + C$

$= \frac{\sqrt{x^2-1}}{x} + C$

ALTERNATIVE SUBSTITUTION FOR THESE TYPE OF INTEGRALS

$u = \frac{1}{x}$
$x = \frac{1}{u}$
$dx = -\frac{1}{u^2}\,du$

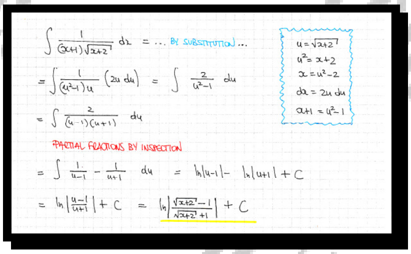
$\int \frac{1}{x^2\sqrt{x^2-1}}\,dx = \int \frac{1}{\frac{1}{u^2}\sqrt{\frac{1}{u^2}-1}}\left(-\frac{1}{u^2}\,du\right)$

$= \int \frac{-1}{\frac{1}{u^2}\sqrt{\frac{1-u^2}{u^2}}}\frac{1}{u^2}\,du = \int \frac{-1}{\frac{\sqrt{1-u^2}}{u}}\,du$

$= \int -u(1-u^2)^{-\frac{1}{2}}\,du = (1-u^2)^{\frac{1}{2}} + C = \left(1-\frac{1}{x^2}\right)^{\frac{1}{2}} + C$

$= \left(\frac{x^2-1}{x^2}\right)^{\frac{1}{2}} + C = \frac{\sqrt{x^2-1}}{x} + C$

598. $$\int \frac{1}{(x+1)\sqrt{x+2}}\,dx = \ln\left|\frac{\sqrt{x+2}-1}{\sqrt{x+2}+1}\right| + C$$



$\int \frac{1}{(x+1)\sqrt{x+2}}\,dx = \dots$ BY SUBSTITUTION ...

$u = \sqrt{x+2}$
$u^2 = x+2$
$x = u^2-2$
$dx = 2u\,du$
$x+1 = u^2-1$

$= \int \frac{1}{(u^2-1)u}(2u\,du) = \int \frac{2}{u^2-1}\,du$

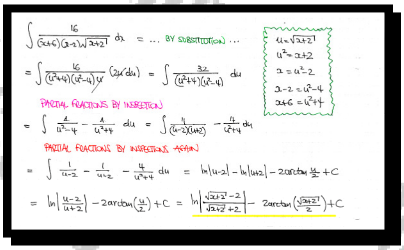
$= \int \frac{2}{(u-1)(u+1)}\,du$

PARTIAL FRACTIONS BY INSPECTION

$= \int \frac{1}{u-1} - \frac{1}{u+1}\,du = \ln|u-1| - \ln|u+1| + C$

$= \ln\left|\frac{u-1}{u+1}\right| + C = \ln\left|\frac{\sqrt{x+2}-1}{\sqrt{x+2}+1}\right| + C$

599. $$\int \frac{1}{(x+6)(x-2)\sqrt{x+2}}\,dx = \ln\left|\frac{\sqrt{x+2}-2}{\sqrt{x+2}+2}\right| - 2\arctan\left(\frac{\sqrt{x+2}}{2}\right) + C$$



$\int \frac{16}{(x+6)(x-2)\sqrt{x+2}}\,dx = \dots$ BY SUBSTITUTION ...

$u = \sqrt{x+2}$
$u^2 = x+2$
$x = u^2-2$
$x-2 = u^2-4$
$x+6 = u^2+4$

$= \int \frac{16}{(u^2+4)(u^2-4)\cancel{u}}(2\cancel{u}\,du) = \int \frac{32}{(u^2+4)(u^2-4)}\,du$

PARTIAL FRACTIONS BY INSPECTION

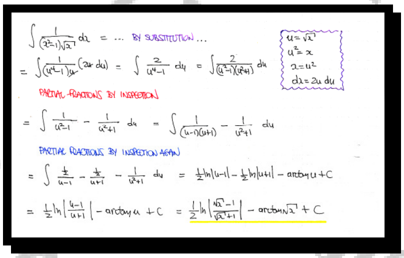
$= \int \frac{4}{u^2-4} - \frac{4}{u^2+4}\,du = \int \frac{4}{(u-2)(u+2)} - \frac{4}{u^2+4}\,du$

PARTIAL FRACTIONS BY INSPECTIONS AGAIN

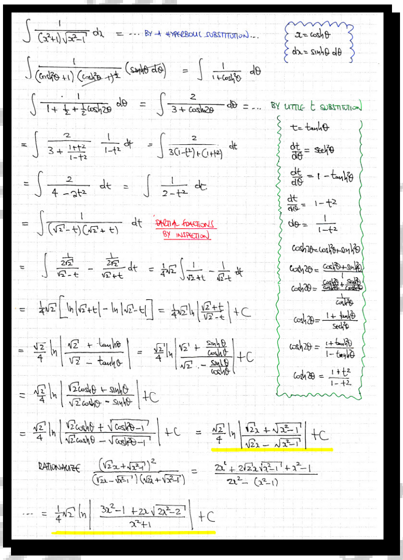
$= \int \frac{1}{u-2} - \frac{1}{u+2} - \frac{4}{u^2+4}\,du = \ln|u-2| - \ln|u+2| - 2\arctan\frac{u}{2} + C$

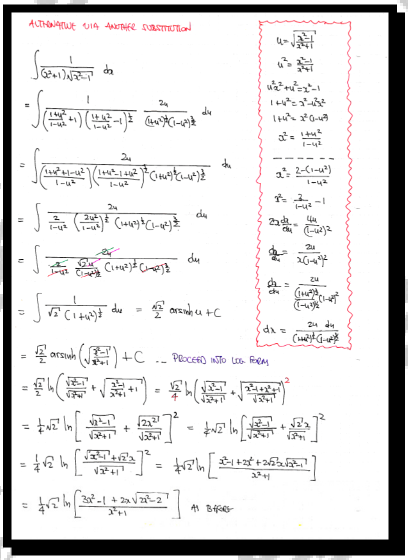
$= \ln\left|\frac{u-2}{u+2}\right| - 2\arctan\left(\frac{u}{2}\right) + C = \ln\left|\frac{\sqrt{x+2}-2}{\sqrt{x+2}+2}\right| - 2\arctan\left(\frac{\sqrt{x+2}}{2}\right) + C$

600. $$\int \frac{1}{\left(x^2-1\right)\sqrt{x}}\,dx = \ln\left|\frac{\sqrt{x}-1}{\sqrt{x}+1}\right| - \arctan\sqrt{x} + C$$



601. $$\int \frac{1}{\left(x^2+1\right)\sqrt{x^2-1}}\,dx = \begin{cases} \dfrac{\sqrt{2}}{4}\ln\left|\dfrac{\sqrt{2}x+\sqrt{x^2-1}}{\sqrt{2}x-\sqrt{x^2-1}}\right| + C \\[2ex] \dfrac{\sqrt{2}}{4}\ln\left|\dfrac{3x^2-1+2x\sqrt{2x^2-2}}{x^2+1}\right| + C \end{cases}$$





602. $$\int \frac{1}{(x+1)\sqrt{x^2+4x+2}}\,dx = -\arcsin\left[\frac{x}{\left(x+1\sqrt{2}\right)}\right]+C$$

$$\int \frac{1}{(x+1)\sqrt{x^2+4x+2}}\,dx = \dots \text{ SUBSTITUTION } \dots$$

$u=\dfrac{1}{x+1}$

$x+1=\dfrac{1}{u}$

$dx=-\dfrac{1}{u^2}\,du$

x^2+4x+2

$=(x+1)^2+2x+1$

$=(x+1)^2+2(x+1)-1$

$=\dfrac{1}{u^2}+\dfrac{2}{u}-1$

$=\dfrac{1+2u-u^2}{u^2}$

$$=\int u\times\frac{1}{\sqrt{\frac{1+2u-u^2}{u^2}}}\times\left(-\frac{1}{u^2}\,du\right)$$

$$=\int -\frac{u}{\frac{\sqrt{1+2u-u^2}}{u}}\times\frac{1}{u^2}\,du$$

$$=\int -\frac{1}{\sqrt{1+2u-u^2}}\,du$$

COMPLETING THE SQUARE

$1+2u-u^2=-\left[u^2-2u-1\right]=-\left[(u-1)^2-2\right]$

$=2-(u-1)^2$

$$=\int -\frac{1}{\sqrt{2-(u-1)^2}}\,du=\int -\frac{1}{\sqrt{\sqrt{2}^2-(u-1)^2}}\,du=-\arcsin\left(\frac{u-1}{\sqrt{2}}\right)+C$$

$$=-\arcsin\left(\frac{\frac{1}{x+1}-1}{\sqrt{2}}\right)+C=-\arcsin\left(\frac{x+1-1}{\sqrt{2}(x+1)}\right)+C=-\arcsin\left(\frac{x}{\sqrt{2}(x+1)}\right)+C$$

603. $$\int \sqrt{36-x^2}\,dx = 18\arcsin\left(\tfrac{1}{6}x\right)+\tfrac{1}{2}x\sqrt{36-x^2}+C$$

$$\int\sqrt{36-x^2}\,dx = \dots \text{ BY SUBSTITUTION } \dots$$

$\theta=\arcsin\left(\frac{x}{6}\right)$

$x=6\sin\theta$

$dx=6\cos\theta\,d\theta$

$$=\int\sqrt{36-36\sin^2\theta}\,(6\cos\theta\,d\theta)=\int\sqrt{36(1-\sin^2\theta)}\,6\cos\theta\,d\theta$$

$$=\int(6\cos\theta)(6\cos\theta)\,d\theta=\int 36\cos^2\theta\,d\theta=\int 36\left(\tfrac{1}{2}+\tfrac{1}{2}\cos2\theta\right)d\theta=\int 18+18\cos2\theta\,d\theta$$

$$=18\theta+9\sin2\theta+C=18\theta+18\sin\theta\cos\theta+C=18\theta+(6\sin\theta)(3\cos\theta)+C$$

$\cos\theta=+\sqrt{1-\sin^2\theta}$

$\cos\theta=\sqrt{1-\frac{x^2}{36}}=\frac{1}{6}\sqrt{36-x^2}$

$$=18\arcsin\left(\frac{x}{6}\right)+x\left(\frac{1}{2}\sqrt{36-x^2}\right)+C=18\arcsin\frac{x}{6}+\frac{1}{2}x\sqrt{36-x^2}+C$$

604. $$\int \frac{x+2}{\left(x^2+4x+1\right)^{\frac{3}{2}}}\,dx = -\sqrt{x^2+4x+1}+C$$

$$\int\frac{x+2}{(x^2+4x+1)^{3/2}}\,dx=\dots \text{ BY INSPECTION } \dots \int(x+2)(x^2+4x+1)^{-\frac{3}{2}}\,dx$$

$$=-(x^2+4x+1)^{-\frac{1}{2}}+C=-\sqrt{x^2+4x+1}+C$$

ALTERNATIVE BY SUBSTITUTION AFTER COMPLETING THE SQUARE ON THE DENOMINATOR

$x^2+4x+1=(x+2)^2-3$

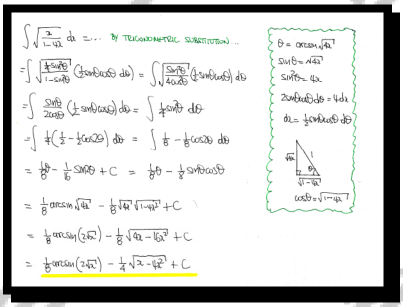
$u=x+2$

$du=dx$

$$\therefore=\int\frac{x+2}{\left[(x+2)^2-3\right]^{3/2}}\,dx=\int\frac{u}{(u^2-3)^{\frac{3}{2}}}\,du$$

$$=\int u(u^2-3)^{-\frac{3}{2}}\,du=-(u^2-3)^{-\frac{1}{2}}+C=-\sqrt{x^2+4x+1}+C$$

605. $\displaystyle\int \sqrt{\frac{x}{1-4x}}\,dx = \frac{1}{8}\arcsin\left(2\sqrt{x}\right) - \frac{1}{4}\sqrt{x-4x^2} + C$



$\displaystyle\int \sqrt{\frac{x}{1-4x}}\,dx = \dots$ BY TRIGONOMETRIC SUBSTITUTION ...

$\displaystyle= \int \sqrt{\frac{\frac{1}{4}\sin^2\theta}{1-\sin^2\theta}}\left(\tfrac{1}{2}\sin\theta\cos\theta\,d\theta\right) = \int \sqrt{\frac{\sin^2\theta}{4\cos^2\theta}}\left(\tfrac{1}{2}\sin\theta\cos\theta\right)d\theta$

$\displaystyle= \int \frac{\sin\theta}{2\cos\theta}\left(\tfrac{1}{2}\sin\theta\cos\theta\right)d\theta = \int \tfrac{1}{4}\sin^2\theta\,d\theta$

$\displaystyle= \int \tfrac{1}{4}\left(\tfrac{1}{2}-\tfrac{1}{2}\cos 2\theta\right)d\theta = \int \tfrac{1}{8}-\tfrac{1}{8}\cos 2\theta\,d\theta$

$= \frac{1}{8}\theta - \frac{1}{16}\sin 2\theta + C = \frac{1}{8}\theta - \frac{1}{8}\sin\theta\cos\theta$

$= \frac{1}{8}\arcsin\sqrt{4x} - \frac{1}{8}\sqrt{4x}\sqrt{1-4x^2} + C$

$= \frac{1}{8}\arcsin\left(2\sqrt{x}\right) - \frac{1}{8}\sqrt{4x-16x^2} + C$

$= \frac{1}{8}\arcsin\left(2\sqrt{x}\right) - \frac{1}{4}\sqrt{x-4x^2} + C$

$\theta = \arcsin\sqrt{4x}$

$\sin\theta = \sqrt{4x}$

$\sin^2\theta = 4x$

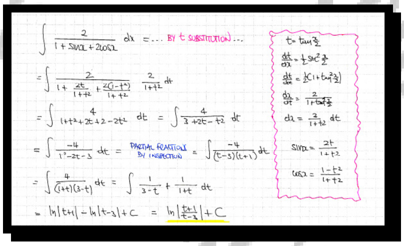
$2\sin\theta\cos\theta\,d\theta = 4\,dx$

$dx = \frac{1}{2}\sin\theta\cos\theta\,d\theta$

$\sqrt{4x}$, 1, θ, $\sqrt{1-4x}$

$\cos\theta = \sqrt{1-4x}$

606. $\displaystyle\int \frac{2}{1+\sin x+2\cos x}\,dx = \ln\left[\frac{\tan\left(\frac{1}{2}x\right)+1}{\tan\left(\frac{1}{2}x\right)-3}\right] + C$



$\displaystyle\int \frac{2}{1+\sin x+2\cos x}\,dx = \dots$ BY t SUBSTITUTION ...

$\displaystyle= \int \frac{2}{1+\frac{2t}{1+t^2}+\frac{2(1-t^2)}{1+t^2}}\;\frac{2}{1+t^2}\,dt$

$\displaystyle= \int \frac{4}{1+t^2+2t+2-2t^2}\,dt = \int \frac{4}{3+2t-t^2}\,dt$

$\displaystyle= \int \frac{-4}{t^2-2t-3}\,dt =$ PARTIAL FRACTIONS BY INSPECTION $\displaystyle= \int \frac{-4}{(t-3)(t+1)}\,dt$

$\displaystyle= \int \frac{4}{(1+t)(3-t)}\,dt = \int \frac{1}{3-t}+\frac{1}{1+t}\,dt$

$= \ln|t+1| - \ln|t-3| + C = \ln\left|\frac{t+1}{t-3}\right| + C$

$t = \tan\frac{x}{2}$

$\frac{dt}{dx} = \frac{1}{2}\sec^2\frac{x}{2}$

$\frac{dt}{dx} = \frac{1}{2}\left(1+\tan^2\frac{x}{2}\right)$

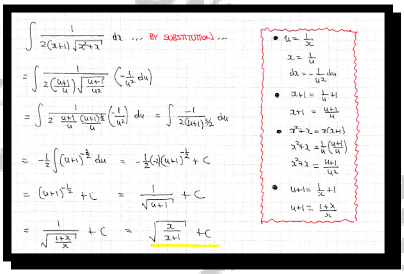
$\frac{dx}{dt} = \frac{2}{1+\tan^2\frac{x}{2}}$

$dx = \frac{2}{1+t^2}\,dt$

$\sin x = \frac{2t}{1+t^2}$

$\cos x = \frac{1-t^2}{1+t^2}$

607. $\displaystyle\int \frac{1}{2(x+1)\sqrt{x^2+x}}\,dx = \sqrt{\frac{x}{x+1}} + C$



$\displaystyle\int \frac{1}{2(x+1)\sqrt{x^2+x}}\,dx \dots$ BY SUBSTITUTION ...

$\displaystyle= \int \frac{1}{2\left(\frac{u+1}{u}\right)\sqrt{\frac{u+1}{u^2}}}\left(-\frac{1}{u^2}\,du\right)$

$\displaystyle= \int \frac{1}{2\,\frac{u+1}{u}\,\frac{(u+1)^{\frac{1}{2}}}{u}}\left(-\frac{1}{u^2}\right)du = \int \frac{-1}{2(u+1)^{3/2}}\,du$

$\displaystyle= -\frac{1}{2}\int (u+1)^{-\frac{3}{2}}\,du = -\frac{1}{2}(-2)(u+1)^{-\frac{1}{2}} + C$

$\displaystyle= (u+1)^{-\frac{1}{2}} + C = \frac{1}{\sqrt{u+1}} + C$

$\displaystyle= \frac{1}{\sqrt{\frac{1+x}{x}}} + C = \sqrt{\frac{x}{x+1}} + C$

- $u = \frac{1}{x}$
  $x = \frac{1}{u}$
  $dx = -\frac{1}{u^2}\,du$
- $x+1 = \frac{1}{u}+1$
  $x+1 = \frac{u+1}{u}$
- $x^2+x = x(x+1)$
  $x^2+x = \frac{1}{u}\left(\frac{u+1}{u}\right)$
  $x^2+x = \frac{u+1}{u^2}$
- $u+1 = \frac{1}{x}+1$
  $u+1 = \frac{1+x}{x}$

608. $$\int \frac{\ln(2x-1)}{2x-1}\,dx = \frac{1}{2}\big[\ln(2x-1)\big]^2 + C$$

$\int \frac{\ln(2x-1)}{2x-1}\,dx = \dots$ BY INSPECTION $\dots = \frac{1}{2}[\ln(2x-1)]^2 + C$

THE SUBSTITUTION $u = 2x-1$ ALSO WORKS WELL

609. $$\int \cos(\ln x)\,dx = \tfrac{1}{2}x\big[\sin(\ln x) + \cos(\ln x)\big] + C$$

$\int \cos(\ln x)\,dx \dots$ BY SUBSTITUTION $\dots$

$u = \ln x$

$\frac{du}{dx} = \frac{1}{x}$

$dx = x\,du$

$x = e^u$

$\int \cos u\,(x\,du) = \int e^u \cos u\,du$

INTEGRATION BY PARTS, COMPLEX NUMBERS, OR INSPECTION

$\frac{d}{du}\left(e^u(A\sin u + B\cos u)\right) \equiv e^u \cos u$

$\cancel{e^u}(A\sin u + B\cos u) + \cancel{e^u}(A\cos u - B\sin u) \equiv \cancel{e^u}\cos u$

$(A-B)\sin u + (A+B)\cos u \equiv \cos u$

$A+B=1$
$A-B=0$ $\quad A=\frac{1}{2} \quad B=\frac{1}{2}$

$= e^u\left(\frac{1}{2}\sin u + \frac{1}{2}\cos u\right) + C = \frac{1}{2}e^u(\sin u + \cos u) + C$

$= \frac{1}{2}x[\sin(\ln x) + \cos(\ln x)] + C$

610. $$\int \arcsin x\,dx = x\arcsin x + \sqrt{1-x^2} + C$$

$\int \arcsin x\,dx = \int 1 \times \arcsin x\,dx = \dots$ BY PARTS$\dots$

$\arcsin x$	$\frac{1}{\sqrt{1-x^2}}$
x	1

$= x\arcsin x - \int \frac{x}{\sqrt{1-x^2}}\,dx$

$= x\arcsin x - \int x(1-x^2)^{-\frac{1}{2}}\,dx$

$= x\arcsin x + (1-x^2)^{\frac{1}{2}} + C$

611. $$\int \frac{1}{\sqrt{x^2-16}}\,dx = \begin{cases} \operatorname{arcosh}\left(\frac{1}{4}x\right) + C \\ \ln\left(x + \sqrt{x^2-16}\right) + C \end{cases}$$

$\int \frac{1}{\sqrt{x^2-16}}\,dx = \int \frac{1}{\sqrt{x^2-4^2}}\,dx =$ STANDARD RESULT

$= \operatorname{arcosh}\left(\frac{x}{4}\right) + C$

$= \ln\left(\frac{x}{4} + \sqrt{\frac{x^2}{16}-1}\right) + C = \ln\left(\frac{x}{4} + \sqrt{\frac{x^2-16}{16}}\right) + C$

$= \ln\left(\frac{x}{4} + \frac{\sqrt{x^2-16}}{4}\right) + C = \ln\left[\frac{1}{4}\left(x + \sqrt{x^2-16}\right)\right] + C$

$= \ln\frac{1}{4} + \ln\left(x + \sqrt{x^2-16}\right) + C = \ln\left(x + \sqrt{x^2-16}\right) + C$

612. $$\int \frac{3x^4+x^3+2x^2+x-2}{(x+1)x^5}\,dx = \frac{1-2x+x^2-4x^3}{2x^4}+\ln\left|\frac{x}{x+1}\right|+C$$

$$\int \frac{3x^4+x^3+2x^2+x-2}{(x+1)x^5}\,dx = \int \frac{3x^4+x^3+2x^2+x-2}{x+1}\times\frac{1}{x^5}\,dx$$

DIVIDE "BACKWARDS"

$$\begin{array}{r} -2+3x-x^2+2x^3+x^4 \\ 1+x\,\big)\overline{-2+x+2x^2+x^3+3x^4} \\ \underline{2+2x} \\ 3x+2x^2+x^3+3x^4 \\ \underline{-3x-3x^2} \\ -x^2+x^3+3x^4 \\ \underline{x^2+x^3} \\ 2x^3+3x^4 \\ \underline{-2x^3-2x^4} \\ x^4 \\ \underline{-x^4-x^5} \\ -x^5 \end{array}$$

$$= \int\left(-2+3x-x^2+2x^3+x^4-\frac{x^5}{1+x}\right)\frac{1}{x^5}\,dx$$

$$= \int -\frac{2}{x^5}+\frac{3}{x^4}-\frac{1}{x^3}+\frac{2}{x^2}+\frac{1}{x}-\frac{1}{1+x}\,dx$$

$$= \frac{1}{2x^4}-\frac{1}{x^3}+\frac{1}{2x^2}-\frac{2}{x}+\ln|x|-\ln|x+1|+C$$

$$= \frac{1-2x+x^2-4x^3}{2x^4}+\ln\left|\frac{x}{x+1}\right|+C$$

613. $$\int \frac{e^{\arcsin x}}{\sqrt{1-x^2}}\,dx = e^{\arcsin x}+C$$

$$\int \frac{e^{\arcsin x}}{\sqrt{1-x^2}}\,dx = \dots \text{ BY INSPECTION } \dots = e^{\arcsin x}+C$$

THE SUBSTITUTION $u=\arcsin x$ CAN ALSO BE USED

614. $$\int \sqrt{\frac{3+2\sqrt{x}}{4x}}\,dx = \frac{1}{3}\left(3+2\sqrt{x}\right)^{\frac{3}{2}}+C$$

$$\int\sqrt{\frac{2\sqrt{x}+3}{4x}}\,dx = \int\left(\frac{2x^{\frac{1}{2}}+3}{4x}\right)^{\frac{1}{2}}dx = \int \frac{1}{2}x^{-\frac{1}{2}}\left(2x^{\frac{1}{2}}+3\right)^{\frac{1}{2}}dx$$

$$= \dots \text{ BY INSPECTION } \dots \ \frac{1}{3}\left(2x^{\frac{1}{2}}+3\right)^{\frac{3}{2}}+C$$

THE SUBSTITUTION $u=\sqrt{x}$ ALSO WORKS WELL

615. $$\int \frac{2x+1}{\sqrt{9-x^2}}\,dx = \arcsin\left(\tfrac{1}{3}x\right)-2\left(9-x^2\right)^{\frac{1}{2}}+C$$

$$\int \frac{2x+1}{\sqrt{9-x^2}}\,dx = \int \frac{2x}{\sqrt{9-x^2}}+\frac{1}{\sqrt{9-x^2}}\,dx = \int 2x\left(9-x^2\right)^{-\frac{1}{2}}+\frac{1}{\sqrt{9-x^2}}\,dx$$

$$= -2\left(9-x^2\right)^{\frac{1}{2}}+\arcsin\frac{x}{3}+C$$

616. $\displaystyle\int \frac{\sinh x}{\cosh x - 1}\, dx = \ln(\cosh x - 1) + C$

$\displaystyle\int \frac{\sinh x}{\cosh x - 1}\, dx = \ldots$ OF THE FORM $\displaystyle\int \frac{f'(x)}{f(x)}\, dx = \ln|f(x)| + C$

$= \ln|\cosh x - 1| + C$

617. $\displaystyle\int \frac{1}{x\sqrt{1-x^2}}\, dx = \begin{cases} \dfrac{1}{2}\ln\left|\dfrac{\sqrt{1-x^2}-1}{\sqrt{1-x^2}+1}\right| + C \\[2ex] \ln\left|\dfrac{1-\sqrt{1-x^2}}{x}\right| + C \end{cases}$

$\displaystyle\int \frac{1}{x\sqrt{1-x^2}}\, dx = \ldots$ BY SUBSTITUTION ...

$u = \sqrt{1-x^2}$

$u^2 = 1 - x^2$

$2u\,du = -2x\,dx$

$dx = -\dfrac{u\,du}{x}$

$x^2 = 1 - u^2$

$\displaystyle = \int \frac{1}{x\cancel{u}}\left(\frac{-\cancel{u}}{x}\right) du = \int \frac{-1}{x^2}\, du$

$\displaystyle = \int \frac{-1}{1-u^2}\, du = \int \frac{1}{u^2-1}\, du = \int \frac{1}{(u-1)(u+1)}\, du$

PARTIAL FRACTIONS BY INSPECTION

$\displaystyle = \int \frac{\frac{1}{2}}{u-1} + \frac{\frac{1}{2}}{u+1}\, du = \frac{1}{2}\ln|u-1| - \frac{1}{2}\ln|u+1| + C = \frac{1}{2}\ln\left|\frac{u-1}{u+1}\right| + C$

$\displaystyle = \frac{1}{2}\ln\left|\frac{\sqrt{1-x^2}-1}{\sqrt{1-x^2}+1}\right| + C$

ALTERNATIVE BY TRIGONOMETRIC SUBSTITUTIONS

$\theta = \arcsin x$

$x = \sin\theta$

$dx = \cos\theta\, d\theta$

$\displaystyle\int \frac{1}{x\sqrt{1-x^2}}\, dx = \int \frac{1}{\sin\theta\sqrt{1-\sin^2\theta}}\,(\cos\theta\, d\theta)$

$\displaystyle = \int \operatorname{cosec}\theta\, d\theta = -\ln|\operatorname{cosec}\theta + \cot\theta| + C$

$\displaystyle = -\ln\left|\frac{1}{\sin\theta} + \frac{\cos\theta}{\sin\theta}\right| = -\ln\left|\frac{1+\cos\theta}{\sin\theta}\right| + C = -\ln\left|\frac{(1+\cos\theta)(1-\cos\theta)}{\sin\theta(1-\cos\theta)}\right| + C$

$\displaystyle = -\ln\left|\frac{1-\cos^2\theta}{\sin\theta(1-\cos\theta)}\right| = -\ln\left|\frac{\sin^2\theta}{\sin\theta(1-\cos\theta)}\right| = -\ln\left|\frac{\sin\theta}{1-\cos\theta}\right| + C$

$\displaystyle = \ln\left|\frac{1-\cos\theta}{\sin\theta}\right| = \ln\left|\frac{1-(1-\sin^2\theta)^{\frac{1}{2}}}{\sin\theta}\right| + C = \ln\left|\frac{1-\sqrt{1-x^2}}{x}\right| + C$

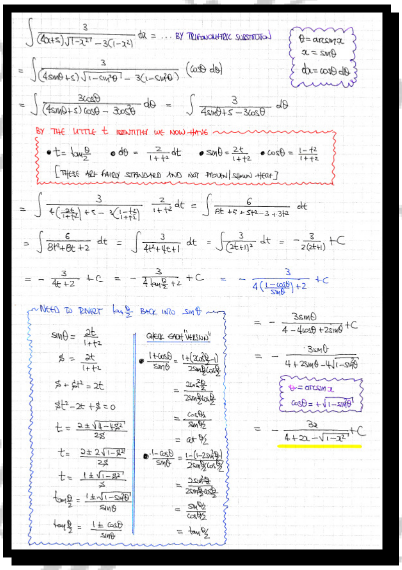
618. $\displaystyle\int \frac{1}{\sqrt{x^2-2x}}\, dx = \operatorname{arcosh}(x-1) + C$

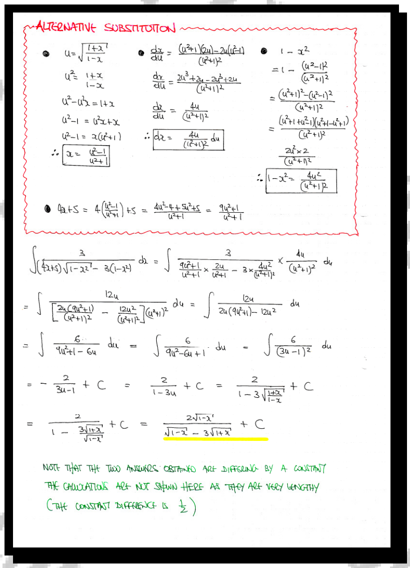
$\displaystyle\int \frac{1}{\sqrt{x^2-2x}}\, dx = \int \frac{1}{\sqrt{(x^2-2x+1)-1}}\, dx = \int \frac{1}{\sqrt{(x-1)^2-1}}\, dx$

$= \ldots$ STANDARD ARCOSH INTEGRAL ...

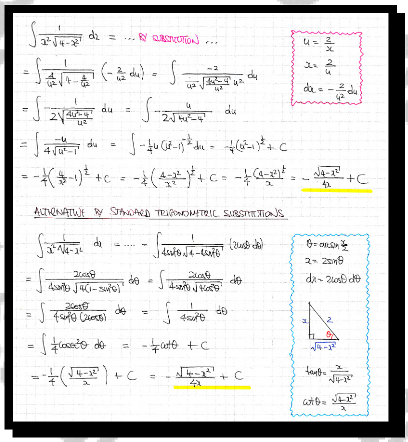
$\displaystyle = \operatorname{arcosh}\left(\frac{x-1}{1}\right) + C = \operatorname{arccosh}(x-1) + C$

619. $$\int \frac{3}{(4x+5)\sqrt{1-x^2}-3\left(1-x^2\right)}\,dx = \begin{cases} \dfrac{-3x}{4+2x-\sqrt{1-x^2}} + C \\[2ex] \dfrac{2\sqrt{1-x}}{\sqrt{1-x}-3\sqrt{1+x}} + C \end{cases}$$





620. $$\int \frac{1}{x^2\sqrt{4-x^2}}\,dx = -\frac{\sqrt{4-x^2}}{4x} + C$$



621. $$\int \frac{\arcsin x}{\sqrt{1-x^2}}\,dx = \frac{1}{2}(\arcsin x)^2 + C$$

$$\int \frac{\arcsin x}{\sqrt{1-x^2}}\,dx = \dots \text{ BY INSPECTION } \dots = \frac{1}{2}(\arcsin x)^2 + C$$

THE SUBSTITUTION $u = \arcsin x$ ALSO WORKS WELL

622. $$\int \frac{3}{x\sqrt{3x^2+2x-1}}\,dx = \begin{cases} \arccos\left(\dfrac{1+x}{2x}\right) + C \\ -\arcsin\left(\dfrac{1+x}{2x}\right) + C \end{cases}$$

$$\int \frac{1}{x\sqrt{3x^2+2x-1}} = \dots \text{ START WITH A SUBSTITUTION}$$

$u = \frac{1}{x}$

$x = \frac{1}{u}$

$dx = -\frac{1}{u^2}\,du$

$$= \int \frac{1}{\frac{1}{u}\sqrt{\frac{3}{u^2}+\frac{2}{u}-1}}\left(-\frac{1}{u^2}\,du\right) = \int \frac{-1}{u\sqrt{\frac{3-2u-u^2}{u^2}}}\,du$$

$$= \int \frac{-1}{u\frac{\sqrt{-u^2-2u+3}}{u}}\,du = \int -\frac{1}{\sqrt{-[u^2+2u-3]}}\,du$$

$$= \int \frac{-1}{\sqrt{-[(u+1)^2-4]}}\,du = \int \frac{-1}{\sqrt{4-(u+1)^2}}\,du = -\arcsin\left(\frac{u+1}{2}\right)+C$$

OR

$$\arccos\left(\frac{u+1}{2}\right)+C$$

$$= \arccos\left(\frac{\frac{1}{x}+1}{2}\right)+C = \arccos\left(\frac{1+x}{2x}\right)+C \quad \text{OR} \quad -\arcsin\left(\frac{x+1}{2x}\right)+C$$

623. $$\int \frac{3}{x\sqrt{x^2+2}}\,dx = \begin{cases} \dfrac{\sqrt{2}}{4}\ln\left|\dfrac{\sqrt{x^2+2}-\sqrt{2}}{\sqrt{x^2+2}+\sqrt{2}}\right| + C \\ \dfrac{\sqrt{2}}{2}\ln\left|\dfrac{\sqrt{x^2+2}-\sqrt{2}}{x}\right| + C \end{cases}$$

$$\int \frac{1}{x(\sqrt{x^2+2})}\,dx = \text{ BY SUBSTITUTION } \dots$$

$u = \sqrt{x^2+2}$

$u^2 = x^2+2$

$2u\,du = 2x\,dx$

$dx = \frac{u}{x}\,du$

$x^2 = u^2-2$

$$= \int \frac{1}{xu}\left(\frac{u}{x}\,du\right) = \int \frac{1}{x^2}\,du = \int \frac{1}{u^2-2}\,du$$

$$= \int \frac{1}{(u-\sqrt{2})(u+\sqrt{2})}\,du \quad \text{PARTIAL FRACTIONS BY INSPECTION}$$

$$= \int \frac{\frac{1}{2\sqrt{2}}}{u-\sqrt{2}} - \frac{\frac{1}{2\sqrt{2}}}{u+\sqrt{2}}\,du = \frac{1}{2\sqrt{2}}\Big[\ln|u-\sqrt{2}| - \ln|u+\sqrt{2}|\Big]+C$$

$$= \frac{\sqrt{2}}{4}\ln\left|\frac{u-\sqrt{2}}{u+\sqrt{2}}\right|+C = \frac{\sqrt{2}}{4}\ln\left|\frac{\sqrt{x^2+2}-\sqrt{2}}{\sqrt{x^2+2}+\sqrt{2}}\right|+C$$

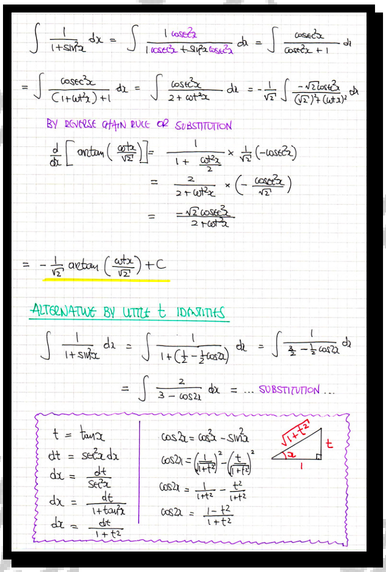
CAN BE MANIPULATED FURTHER.

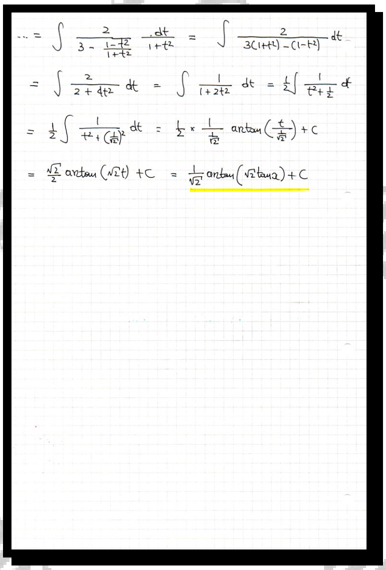
$$= \frac{\sqrt{2}}{4}\ln\left|\frac{(\sqrt{x^2+2}-\sqrt{2})(\sqrt{x^2+2}-\sqrt{2})}{(\sqrt{x^2+2}+\sqrt{2})(\sqrt{x^2+2}-\sqrt{2})}\right|+C = \frac{\sqrt{2}}{4}\ln\left|\frac{(\sqrt{x^2+2}-\sqrt{2})^2}{(x^2+\cancel{2})-\cancel{2}}\right|$$

$$= \frac{\sqrt{2}}{4}\ln\left[\frac{\sqrt{x^2+2}-\sqrt{2}}{x}\right]^2 + C = \frac{\sqrt{2}}{2}\ln\left|\frac{\sqrt{x^2+2}-\sqrt{2}}{x}\right|+C$$

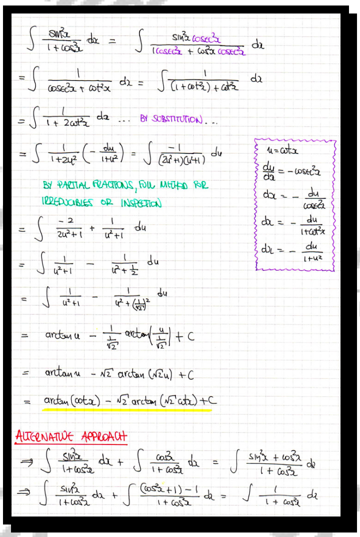
NOTE THAT THE FORM $\int \frac{1}{u^2-2}\,du$ CAN BE INTEGRATED VIA artanh AND THEN TIDY TO THE SAME ANSWER VIA ITS LOGARITHMIC FORM

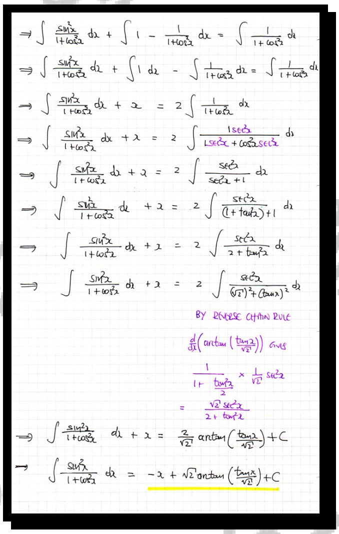
624. $$\int \frac{1}{1+\sin^2 x}\,dx = \begin{bmatrix} -\dfrac{1}{\sqrt{2}}\arctan\left(\dfrac{\cot x}{\sqrt{2}}\right) + C \\ \dfrac{1}{\sqrt{2}}\arctan\left(\sqrt{2}\,\tan x\right) + C \end{bmatrix}$$

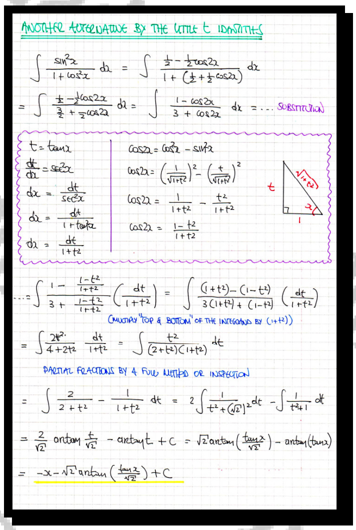




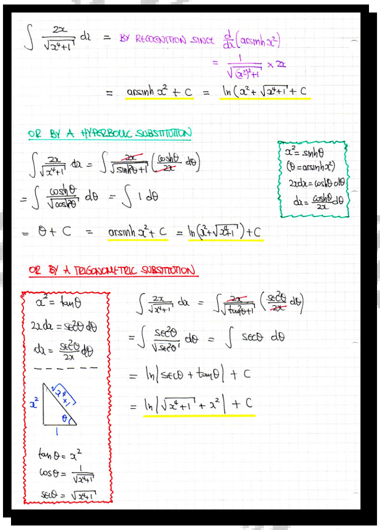
625. $$\int \frac{\sin^2 x}{1+\cos^2 x}\,dx = \begin{bmatrix} -x-\sqrt{2}\,\arctan\left(\dfrac{\tan x}{\sqrt{2}}\right) + C \\ \arctan(\cot x)-\sqrt{2}\,\arctan\left(\sqrt{2}\,\cot x\right) + C \end{bmatrix}$$







626. $$\int \frac{2x}{\sqrt{x^4+1}}\,dx = \begin{bmatrix} \ln\left[x^2+\sqrt{x^4+1}\right] + C \\ \operatorname{arsinh}\left(x^2\right) + C \end{bmatrix}$$



$\int \frac{2x}{\sqrt{x^4+1}}\,dx =$ BY RECOGNITION SINCE $\frac{d}{dx}\left(\operatorname{arsinh} x^2\right) = \frac{1}{\sqrt{(x^2)^2+1}} \times 2x$

$= \operatorname{arsinh} x^2 + C = \ln\left(x^2+\sqrt{x^4+1}\right) + C$

OR BY A HYPERBOLIC SUBSTITUTION

$x^2 = \sinh\theta$
$(\theta = \operatorname{arsinh} x^2)$
$2x\,dx = \cosh\theta\,d\theta$
$dx = \frac{\cosh\theta}{2x}\,d\theta$

$\int \frac{2x}{\sqrt{x^4+1}}\,dx = \int \frac{\cancel{2x}}{\sqrt{\sinh^2\theta+1}}\left(\frac{\cosh\theta}{\cancel{2x}}\,d\theta\right)$

$= \int \frac{\cosh\theta}{\sqrt{\cosh^2\theta}}\,d\theta = \int 1\,d\theta$

$= \theta + C = \operatorname{arsinh} x^2 + C = \ln\left(x^2+\sqrt{x^4+1}\right) + C$

OR BY A TRIGONOMETRIC SUBSTITUTION

$x^2 = \tan\theta$
$2x\,dx = \sec^2\theta\,d\theta$
$dx = \frac{\sec^2\theta}{2x}\,d\theta$

$\int \frac{2x}{\sqrt{x^4+1}}\,dx = \int \frac{\cancel{2x}}{\sqrt{\tan^2\theta+1}}\left(\frac{\sec^2\theta}{\cancel{2x}}\,d\theta\right)$

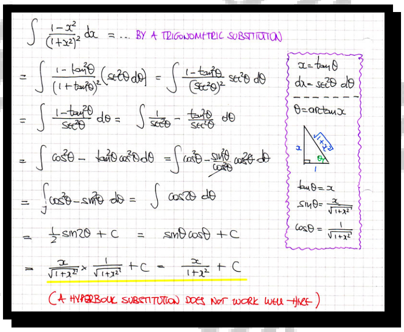
$= \int \frac{\sec^2\theta}{\sqrt{\sec^2\theta}}\,d\theta = \int \sec\theta\,d\theta$

$= \ln\left|\sec\theta + \tan\theta\right| + C$

$= \ln\left|\sqrt{x^4+1} + x^2\right| + C$

$\tan\theta = x^2$
$\cos\theta = \frac{1}{\sqrt{x^4+1}}$
$\sec\theta = \sqrt{x^4+1}$

627. $$\int \frac{1-x^2}{\left(1+x^2\right)^2}\,dx = \frac{x}{1+x^2} + C$$



$\int \frac{1-x^2}{(1+x^2)^2}\,dx = \ldots$ BY A TRIGONOMETRIC SUBSTITUTION

$= \int \frac{1-\tan^2\theta}{(1+\tan^2\theta)^2}\left(\sec^2\theta\,d\theta\right) = \int \frac{1-\tan^2\theta}{(\sec^2\theta)^2}\sec^2\theta\,d\theta$

$= \int \frac{1-\tan^2\theta}{\sec^2\theta}\,d\theta = \int \frac{1}{\sec^2\theta} - \frac{\tan^2\theta}{\sec^2\theta}\,d\theta$

$= \int \cos^2\theta - \tan^2\theta\cos^2\theta\,d\theta = \int \cos^2\theta - \frac{\sin^2\theta}{\cancel{\cos^2\theta}}\cancel{\cos^2\theta}\,d\theta$

$= \int \cos^2\theta - \sin^2\theta\,d\theta = \int \cos 2\theta\,d\theta$

$= \frac{1}{2}\sin 2\theta + C = \sin\theta\cos\theta + C$

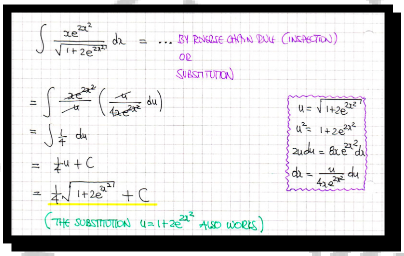
$= \frac{x}{\sqrt{1+x^2}} \times \frac{1}{\sqrt{1+x^2}} + C = \frac{x}{1+x^2} + C$

(A HYPERBOLIC SUBSTITUTION DOES NOT WORK WELL HERE)

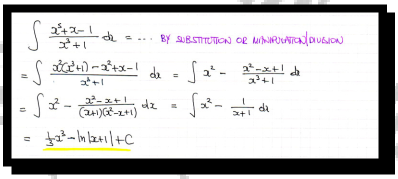
$x = \tan\theta$
$dx = \sec^2\theta\,d\theta$
$\theta = \arctan x$

$\tan\theta = x$
$\sin\theta = \frac{x}{\sqrt{1+x^2}}$
$\cos\theta = \frac{1}{\sqrt{1+x^2}}$

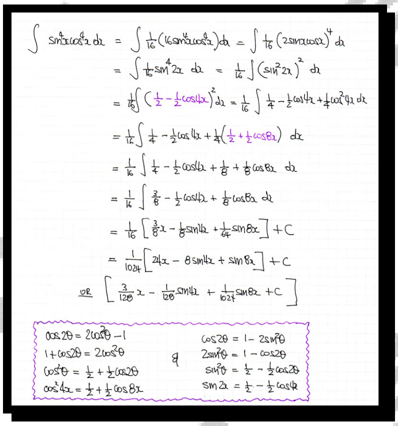
628. $\displaystyle\int \frac{x\,\mathrm{e}^{2x^2}}{\sqrt{1+2\mathrm{e}^{2x^2}}}\,dx = \frac{1}{4}\sqrt{1+2\mathrm{e}^{2x^2}} + C$



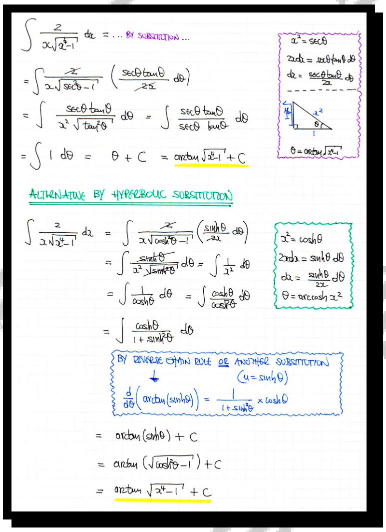
629. $\displaystyle\int \frac{x^5+x-1}{x^3+1}\,dx = \frac{1}{3}x^3 - \ln|x+1| + C$



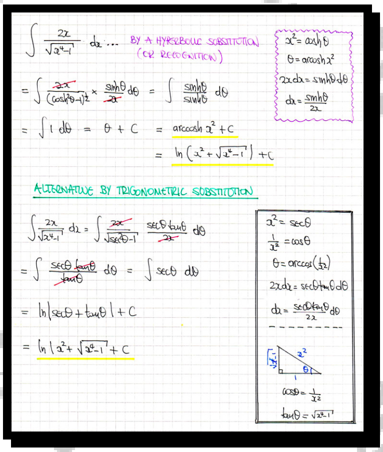
630. $\displaystyle\int \sin^4 x\cos^4 x\ dx = \frac{3}{128}x - \frac{1}{128}\sin 4x + \frac{1}{1024}\sin 8x + C$



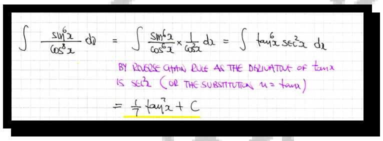
631. $$\int \frac{2}{x\sqrt{x^4-1}}\,dx = \arctan\left(\sqrt{x^4-1}\right) + C$$



632. $$\int \frac{2x}{\sqrt{x^4-1}}\,dx = \left[\begin{array}{l} \ln\left[x^2+\sqrt{x^4-1}\right] + C \\ \operatorname{arcosh}\left(x^2\right) + C \end{array}\right]$$



633. $\displaystyle\int \frac{\sin^6 x}{\cos^8 x}\,dx = \frac{1}{7}\tan^7 x + C$

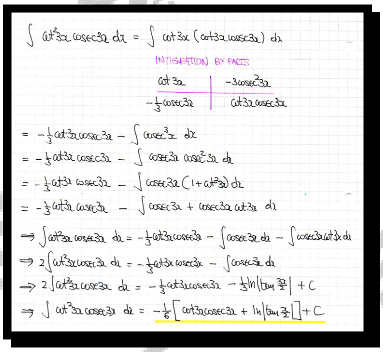


$\displaystyle\int \frac{\sin^6 x}{\cos^8 x}\,dx = \int \frac{\sin^6 x}{\cos^6 x}\times\frac{1}{\cos^2 x}\,dx = \int \tan^6 x\sec^2 x\,dx$

BY REVERSE CHAIN RULE AS THE DERIVATIVE OF $\tan x$ IS $\sec^2 x$ (OR THE SUBSTITUTION $u=\tan x$)

$= \frac{1}{7}\tan^7 x + C$

634. $\displaystyle\int \cot^2 3x\,\mathrm{cosec}\,3x\,dx = -\frac{1}{6}\left[\cot 3x\,\mathrm{cosec}\,3x + \ln\left|\tan\left(\frac{3}{2}x\right)\right|\right] + C$



$\displaystyle\int \cot^2 3x\,\mathrm{cosec}\,3x\,dx = \int \cot 3x\,(\cot 3x\,\mathrm{cosec}\,3x)\,dx$

INTEGRATION BY PARTS

$\cot 3x$	$-3\,\mathrm{cosec}^2 3x$
$-\frac{1}{3}\mathrm{cosec}\,3x$	$\cot 3x\,\mathrm{cosec}\,3x$

$= -\frac{1}{3}\cot 3x\,\mathrm{cosec}\,3x - \int \mathrm{cosec}^3 x\,dx$

$= -\frac{1}{3}\cot 3x\,\mathrm{cosec}\,3x - \int \mathrm{cosec}\,3x\,\mathrm{cosec}^2 3x\,dx$

$= -\frac{1}{3}\cot 3x\,\mathrm{cosec}\,3x - \int \mathrm{cosec}\,3x\,(1+\cot^2 3x)\,dx$

$= -\frac{1}{3}\cot 3x\,\mathrm{cosec}\,3x - \int \mathrm{cosec}\,3x + \mathrm{cosec}\,3x\cot 3x\,dx$

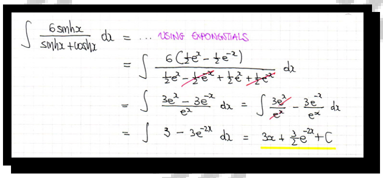
$\Rightarrow \int \cot^2 3x\,\mathrm{cosec}\,3x\,dx = -\frac{1}{3}\cot 3x\,\mathrm{cosec}\,3x - \int \mathrm{cosec}\,3x\,dx - \int \mathrm{cosec}\,3x\cot 3x\,dx$

$\Rightarrow 2\int \cot^2 3x\,\mathrm{cosec}\,3x\,dx = -\frac{1}{3}\cot 3x\,\mathrm{cosec}\,3x - \int \mathrm{cosec}\,3x\,dx$

$\Rightarrow 2\int \cot^2 3x\,\mathrm{cosec}\,3x\,dx = -\frac{1}{3}\cot 3x\,\mathrm{cosec}\,3x - \frac{1}{3}\ln\left|\tan\frac{3x}{2}\right| + C$

$\Rightarrow \int \cot^2 3x\,\mathrm{cosec}\,3x\,dx = -\frac{1}{6}\left[\cot 3x\,\mathrm{cosec}\,3x + \ln\left|\tan\frac{x}{2}\right|\right] + C$

635. $\displaystyle\int \frac{6\sinh x}{\sinh x + \cosh x}\,dx = 3x + \frac{3}{2}e^{-2x} + C$



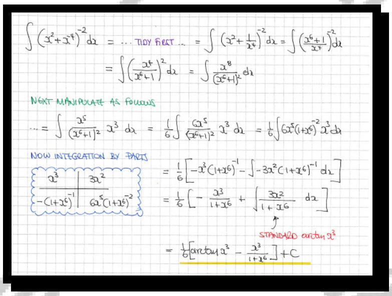
$\displaystyle\int \frac{6\sinh x}{\sinh x+\cosh x}\,dx = \dots$ USING EXPONENTIALS

$\displaystyle = \int \frac{6\left(\frac{1}{2}e^x - \frac{1}{2}e^{-x}\right)}{\frac{1}{2}e^x - \cancel{\frac{1}{2}e^{-x}} + \frac{1}{2}e^x + \cancel{\frac{1}{2}e^{-x}}}\,dx$

$\displaystyle = \int \frac{3e^x - 3e^{-x}}{e^x}\,dx = \int \frac{3\cancel{e^x}}{\cancel{e^x}} - \frac{3e^{-x}}{e^x}\,dx$

$\displaystyle = \int 3 - 3e^{-2x}\,dx = 3x + \frac{3}{2}e^{-2x} + C$

636. $\displaystyle\int \left(x^2+x^{-4}\right)^{-2}\,dx = \frac{1}{6}\left[\arctan\left(x^3\right) - \frac{x^3}{1+x^6}\right] + C$



$\int (x^2+x^{-4})^{-2}\,dx = \ldots$ TIDY FIRST $\ldots = \int \left(x^2+\frac{1}{x^4}\right)^{-2} dx = \int \left(\frac{x^6+1}{x^4}\right)^{-2} dx$

$= \int \left(\frac{x^4}{x^6+1}\right)^2 dx = \int \frac{x^8}{(x^6+1)^2}\,dx$

NEXT MANIPULATE AS FOLLOWS

$\ldots = \int \frac{x^5}{(x^6+1)^2}\,x^3\,dx = \frac{1}{6}\int \frac{6x^5}{(x^6+1)^2}\,x^3\,dx = \frac{1}{6}\int 6x^5(1+x^6)^{-2}x^3\,dx$

NOW INTEGRATION BY PARTS

x^3	$3x^2$
$-(1+x^6)^{-1}$	$6x^5(1+x^6)^{-2}$

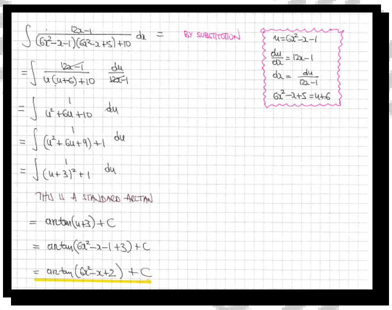
$= \frac{1}{6}\left[-x^3(1+x^6)^{-1} - \int -3x^2(1+x^6)^{-1}\,dx\right]$

$= \frac{1}{6}\left[-\frac{x^3}{1+x^6} + \int \frac{3x^2}{1+x^6}\,dx\right]$

STANDARD $\arctan x^3$

$= \frac{1}{6}\left[\arctan x^3 - \frac{x^3}{1+x^6}\right] + C$

637. $\displaystyle\int \frac{12x-1}{\left(6x^2-x-1\right)\left(6x^2-x+5\right)+10}\,dx = \arctan\left(6x^2-x+2\right) + C$



$\int \frac{12x-1}{(6x^2-x-1)(6x^2-x+5)+10}\,dx =$ BY SUBSTITUTION

$u = 6x^2-x-1$

$\frac{du}{dx} = 12x-1$

$dx = \frac{du}{12x-1}$

$6x^2-x+5 = u+6$

$= \int \frac{\cancel{12x-1}}{u(u+6)+10}\,\frac{du}{\cancel{12x-1}}$

$= \int \frac{1}{u^2+6u+10}\,du$

$= \int \frac{1}{(u^2+6u+9)+1}\,du$

$= \int \frac{1}{(u+3)^2+1}\,du$

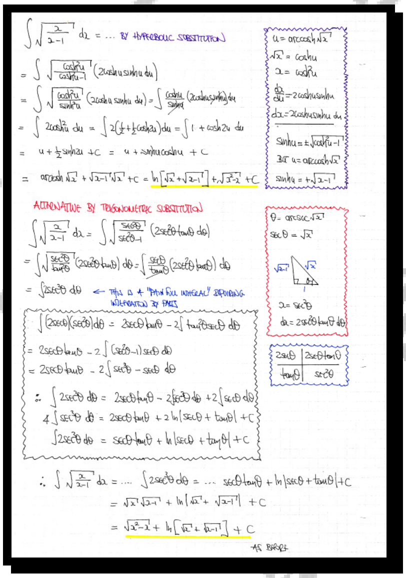
THIS IS A STANDARD ARCTAN

$= \arctan(u+3) + C$

$= \arctan(6x^2-x-1+3) + C$

$= \arctan(6x^2-x+2) + C$

638. $\displaystyle\int \sqrt{\frac{x}{x-1}}\,dx = \frac{1}{6}\left[\arctan\left(x^3\right) - \frac{x^3}{1+x^6}\right] + C$



$\displaystyle\int \sqrt{\frac{x}{x-1}}\,dx = \ldots$ BY HYPERBOLIC SUBSTITUTION

$= \displaystyle\int \sqrt{\frac{\cosh^2 u}{\cosh^2 u-1}}\,(2\cosh u\sinh u\,du)$

$= \displaystyle\int \sqrt{\frac{\cosh^2 u}{\sinh^2 u}}\,(2\cosh u\sinh u\,du) = \int \frac{\cosh u}{\sinh u}(2\cosh u\sinh u)\,du$

$= \displaystyle\int 2\cosh^2 u\,du = \int 2\left(\tfrac{1}{2}+\tfrac{1}{2}\cosh 2u\right)du = \int 1+\cosh 2u\,du$

$= u + \tfrac{1}{2}\sinh 2u + C = u + \sinh u\cosh u + C$

$= \text{arcosh}\sqrt{x} + \sqrt{x-1}\sqrt{x} + C = \ln\left[\sqrt{x}+\sqrt{x-1}\right] + \sqrt{x^2-x} + C$

$u = \text{arcosh}\sqrt{x}$, $\sqrt{x} = \cosh u$, $x = \cosh^2 u$, $\frac{dx}{du} = 2\cosh u\sinh u$, $dx = 2\cosh u\sinh u\,du$

$\sinh u = \pm\sqrt{\cosh^2 u-1}$, BUT $u = \text{arcosh}\sqrt{x}$, $\sinh u = +\sqrt{x-1}$

ALTERNATIVE BY TRIGONOMETRIC SUBSTITUTION

$\displaystyle\int \sqrt{\frac{x}{x-1}}\,dx = \int \sqrt{\frac{\sec^2\theta}{\sec^2\theta-1}}\,(2\sec^2\theta\tan\theta\,d\theta)$

$= \displaystyle\int \sqrt{\frac{\sec^2\theta}{\tan^2\theta}}\,(2\sec^2\theta\tan\theta)\,d\theta = \int \frac{\sec\theta}{\tan\theta}(2\sec^2\theta\tan\theta)\,d\theta$

$= \displaystyle\int 2\sec^3\theta\,d\theta$ ← THIS IS A "PAINFUL INTEGRAL" REQUIRING INTEGRATION BY PARTS

$\theta = \text{arcsec}\sqrt{x}$, $\sec\theta = \sqrt{x}$, $\sqrt{x-1}$, $\sqrt{x}$, 1, $x = \sec^2\theta$, $dx = 2\sec^2\theta\tan\theta\,d\theta$

$\displaystyle\int (2\sec\theta)(\sec^2\theta)\,d\theta = 2\sec\theta\tan\theta - 2\int \tan^2\theta\sec\theta\,d\theta$

$= 2\sec\theta\tan\theta - 2\displaystyle\int (\sec^2\theta-1)\sec\theta\,d\theta$

$= 2\sec\theta\tan\theta - 2\displaystyle\int \sec^3\theta - \sec\theta\,d\theta$

$\therefore \displaystyle\int 2\sec^3\theta\,d\theta = 2\sec\theta\tan\theta - 2\int\sec^3\theta\,d\theta + 2\int\sec\theta\,d\theta$

$4\displaystyle\int \sec^3\theta\,d\theta = 2\sec\theta\tan\theta + 2\ln|\sec\theta+\tan\theta| + C$

$\displaystyle\int 2\sec^3\theta\,d\theta = \sec\theta\tan\theta + \ln|\sec\theta+\tan\theta| + C$

$2\sec\theta$	$2\sec\theta\tan\theta$
$\tan\theta$	$\sec^2\theta$

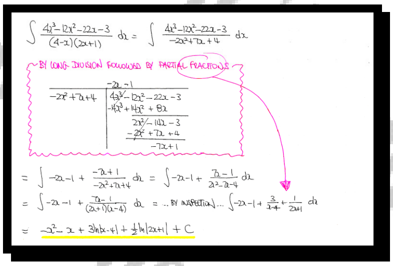
$\therefore \displaystyle\int \sqrt{\frac{x}{x-1}}\,dx = \ldots \int 2\sec^3\theta\,d\theta = \ldots \sec\theta\tan\theta + \ln|\sec\theta+\tan\theta| + C$

$= \sqrt{x}\sqrt{x-1} + \ln\left|\sqrt{x}+\sqrt{x-1}\right| + C$

$= \sqrt{x^2-x} + \ln\left[\sqrt{x}+\sqrt{x-1}\right] + C$

AS BEFORE

639. $\displaystyle\int \frac{4x^3-12x^2-22x-3}{(4-x)(x+1)}\,dx = -x^2 - x + 3\ln|x-4| + \frac{1}{2}\ln|2x+1| + C$



$\displaystyle\int \frac{4x^3-12x^2-22x-3}{(4-x)(2x+1)}\,dx = \int \frac{4x^3-12x^2-22x-3}{-2x^2+7x+4}\,dx$

BY LONG DIVISION FOLLOWED BY PARTIAL FRACTIONS

Quotient: $-2x-1$; divisor $-2x^2+7x+4$ into $4x^3-12x^2-22x-3$

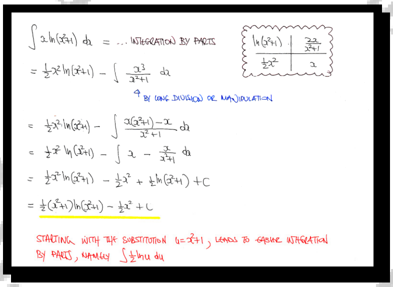
$-4x^3+14x^2+8x$ … $2x^2-14x-3$; $-2x^2+7x+4$ … $-7x+1$

$= \displaystyle\int -2x-1 + \frac{-7x+1}{-2x^2+7x+4}\,dx = \int -2x-1+\frac{7x-1}{2x^2-7x-4}\,dx$

$= \displaystyle\int -2x-1+\frac{7x-1}{(2x+1)(x-4)}\,dx = \ldots$ BY INSPECTION $\ldots \displaystyle\int -2x-1+\frac{3}{x-4}+\frac{1}{2x+1}\,dx$

$= -x^2 - x + 3\ln|x-4| + \tfrac{1}{2}\ln|2x+1| + C$

640. $\displaystyle\int x\ln\left(x^2+1\right)dx = \frac{1}{2}\left(x^2+1\right)\ln\left(x^2+1\right) - \frac{1}{2}x^2 + C$



$\displaystyle\int x\ln(x^2+1)\,dx = \ldots$ INTEGRATION BY PARTS

$\ln(x^2+1)$	$\frac{2x}{x^2+1}$
$\frac{1}{2}x^2$	x

$= \tfrac{1}{2}x^2\ln(x^2+1) - \displaystyle\int \frac{x^3}{x^2+1}\,dx$

BY LONG DIVISION OR MANIPULATION

$= \tfrac{1}{2}x^2\ln(x^2+1) - \displaystyle\int \frac{x(x^2+1)-x}{x^2+1}\,dx$

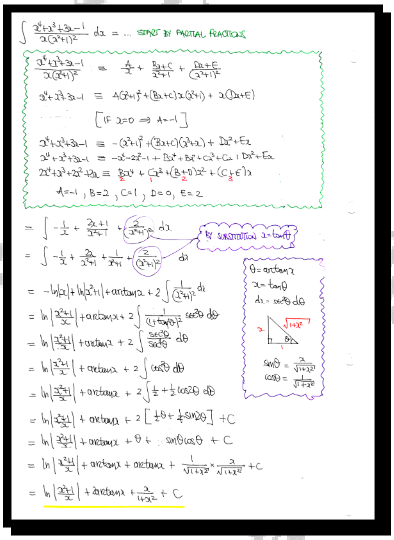
$= \tfrac{1}{2}x^2\ln(x^2+1) - \displaystyle\int x - \frac{x}{x^2+1}\,dx$

$= \tfrac{1}{2}x^2\ln(x^2+1) - \tfrac{1}{2}x^2 + \tfrac{1}{2}\ln(x^2+1) + C$

$= \tfrac{1}{2}(x^2+1)\ln(x^2+1) - \tfrac{1}{2}x^2 + C$

STARTING WITH THE SUBSTITUTION $u = x^2+1$, LEADS TO EASIER INTEGRATION BY PARTS, NAMELY $\int \frac{1}{2}\ln u\,du$

641. $$\int \frac{x^4+x^3+3x-1}{x\left(x^2+1\right)^2}\,dx = \ln\left|\frac{x^2+1}{x}\right| + 2\arctan x + \frac{x}{x^2+1} + C$$



$\int \frac{x^4+x^3+3x-1}{x(x^2+1)^2}\,dx = \dots$ START BY PARTIAL FRACTIONS

$\frac{x^4+x^3+3x-1}{x(x^2+1)^2} \equiv \frac{A}{x} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2}$

$x^4+x^3+3x-1 \equiv A(x^2+1)^2 + (Bx+C)x(x^2+1) + x(Dx+E)$

$[\text{IF } x=0 \Rightarrow A=-1]$

$x^4+x^3+3x-1 \equiv -(x^2+1)^2 + (Bx+C)(x^3+x) + Dx^2+Ex$

$x^4+x^3+3x-1 \equiv -x^4-2x^2-1 + Bx^4+Bx^2+Cx^3+Cx+Dx^2+Ex$

$2x^4+x^3+2x^2+3x \equiv Bx^4 + Cx^3 + (B+D)x^2 + (C+E)x$

$A=-1,\ B=2,\ C=1,\ D=0,\ E=2$

$= \int -\frac{1}{x} + \frac{2x+1}{x^2+1} + \frac{2}{(x^2+1)^2}\,dx$ BY SUBSTITUTION $x=\tan\theta$

$= \int -\frac{1}{x} + \frac{2x}{x^2+1} + \frac{1}{x^2+1} + \frac{2}{(x^2+1)^2}\,dx$

$= -\ln|x| + \ln|x^2+1| + \arctan x + 2\int \frac{1}{(x^2+1)^2}\,dx$

$= \ln\left|\frac{x^2+1}{x}\right| + \arctan x + 2\int \frac{1}{(1+\tan^2\theta)^2}\sec^2\theta\,d\theta$

$= \ln\left|\frac{x^2+1}{x}\right| + \arctan x + 2\int \frac{\sec^2\theta}{\sec^4\theta}\,d\theta$

$= \ln\left|\frac{x^2+1}{x}\right| + \arctan x + 2\int \cos^2\theta\,d\theta$

$= \ln\left|\frac{x^2+1}{x}\right| + \arctan x + 2\int \frac{1}{2}+\frac{1}{2}\cos 2\theta\,d\theta$

$= \ln\left|\frac{x^2+1}{x}\right| + \arctan x + 2\left[\frac{1}{2}\theta + \frac{1}{4}\sin 2\theta\right] + C$

$= \ln\left|\frac{x^2+1}{x}\right| + \arctan x + \theta + \sin\theta\cos\theta + C$

$= \ln\left|\frac{x^2+1}{x}\right| + \arctan x + \arctan x + \frac{1}{\sqrt{1+x^2}} \times \frac{x}{\sqrt{1+x^2}} + C$

$= \ln\left|\frac{x^2+1}{x}\right| + 2\arctan x + \frac{x}{1+x^2} + C$

$\theta = \arctan x$

$x = \tan\theta$

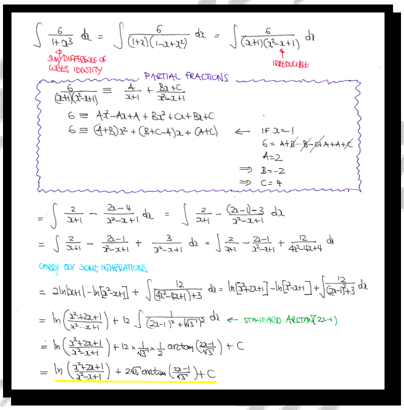
$dx = \sec^2\theta\,d\theta$

x, $\sqrt{1+x^2}$, 1, θ

$\sin\theta = \frac{x}{\sqrt{1+x^2}}$

$\cos\theta = \frac{1}{\sqrt{1+x^2}}$

642. $$\int \frac{6}{1+x^3}\,dx = \ln\left[\frac{x^2+2x+1}{x^2-x+1}\right] + 2\sqrt{3}\arctan\left(\frac{2x-1}{\sqrt{3}}\right) + C$$



$\int \frac{6}{1+x^3}\,dx = \int \frac{6}{(1+x)(1-x+x^2)}\,dx = \int \frac{6}{(x+1)(x^2-x+1)}\,dx$

SUM/DIFFERENCE OF CUBES IDENTITY

IRREDUCIBLE

PARTIAL FRACTIONS

$\frac{6}{(x+1)(x^2-x+1)} \equiv \frac{A}{x+1} + \frac{Bx+C}{x^2-x+1}$

$6 \equiv Ax^2-Ax+A+Bx^2+Cx+Bx+C$

$6 \equiv (A+B)x^2 + (B+C-A)x + (A+C)$ $\leftarrow$ IF $x=-1$

$6 = A+\cancel{B}-\cancel{B}-\cancel{C}+A+A+\cancel{C}$

$A=2$

$\Rightarrow B=-2$

$\Rightarrow C=4$

$= \int \frac{2}{x+1} - \frac{2x-4}{x^2-x+1}\,dx = \int \frac{2}{x+1} - \frac{(2x-1)-3}{x^2-x+1}\,dx$

$= \int \frac{2}{x+1} - \frac{2x-1}{x^2-x+1} + \frac{3}{x^2-x+1}\,dx = \int \frac{2}{x+1} - \frac{2x-1}{x^2-x+1} + \frac{12}{4x^2-4x+4}\,dx$

CARRY OUT SOME INTEGRATIONS

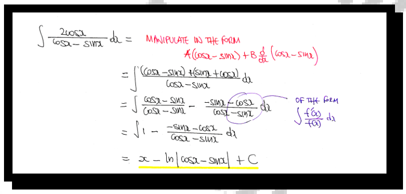
$= 2\ln|x+1| - \ln[x^2-x+1] + \int \frac{12}{(4x^2-4x+1)+3}\,dx = \ln[x^2+2x+1] - \ln[x^2-x+1] + \int \frac{12}{(2x-1)^2+3}\,dx$

$= \ln\left(\frac{x^2+2x+1}{x^2-x+1}\right) + 12\int \frac{1}{(2x-1)^2+(\sqrt{3})^2}\,dx$ $\leftarrow$ STANDARD ARCTAN$(2x-1)$

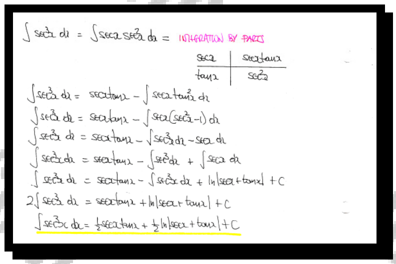
$= \ln\left(\frac{x^2+2x+1}{x^2-x+1}\right) + 12 \times \frac{1}{\sqrt{3}} \times \frac{1}{2}\arctan\left(\frac{2x-1}{\sqrt{3}}\right) + C$

$= \ln\left(\frac{x^2+2x+1}{x^2-x+1}\right) + 2\sqrt{3}\arctan\left(\frac{2x-1}{\sqrt{3}}\right) + C$

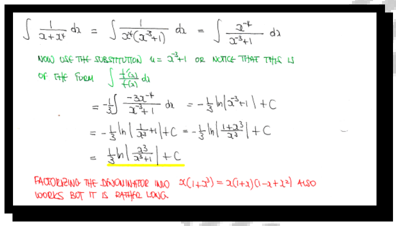
643. $$\int \frac{2\cos x}{\cos x - \sin x}\,dx = x - \ln\left|\cos x - \sin x\right| + C$$



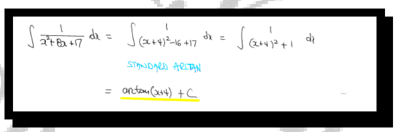
644. $$\int \sec^3 x\,dx = \frac{1}{2}\sec x\tan x + \frac{1}{2}\ln\left|\sec x + \tan x\right| + C$$



645. $$\int \frac{1}{x+x^4}\,dx = \frac{1}{3}\ln\left|\frac{x^3}{x^3+1}\right| + C$$

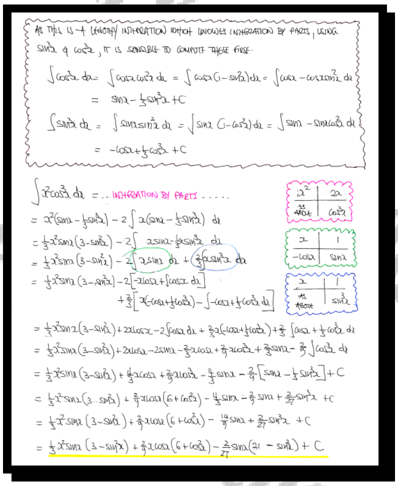


646. $$\int \frac{1}{x^2+8x+17}\,dx = \arctan(x+4) + C$$

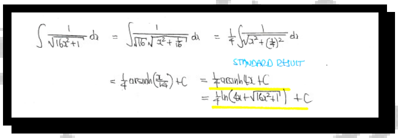


647. $$\int x^2 \cos^3 x \, dx = \ldots$$

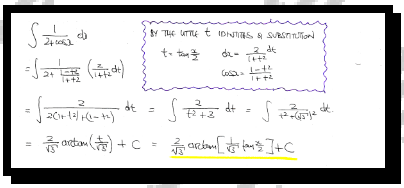
$$\ldots = \frac{1}{3}x^2\left(3-\sin^2 x\right)\sin x + \frac{2}{9}x\left(6+\cos^2 x\right)\cos x - \frac{2}{27}\left(21-\sin^2 x\right)\sin x \; + \; C$$



648. $$\int \frac{1}{\sqrt{16x^2+1}}\, dx = \begin{bmatrix} \frac{1}{4}\ln\left[4x+\sqrt{16x^2+1}\right] \; + \; C \\ \frac{1}{4}\operatorname{arsinh}(4x) \; + \; C \end{bmatrix}$$



649. $$\int \frac{1}{2+\cos x}\, dx = \frac{2}{\sqrt{3}}\arctan\left[\frac{1}{\sqrt{3}}\tan\left(\frac{1}{2}x\right)\right] \; + \; C$$



650. $$\int \frac{\sin x}{\sin\left(x+\frac{1}{4}\pi\right)}\,dx = \frac{\sqrt{2}}{2}\left[x-\ln\left|\sin\left(x+\frac{1}{4}\pi\right)\right|\right] + C$$

$\displaystyle\int \frac{\sin x}{\sin(x+\frac{\pi}{4})}\,dx$ = FIRST A SIMPLE SUBSTITUTION TO FACILITATE SPLITTING OF THE FRACTION LATER

$u = x+\frac{\pi}{4}$

$dx = du$

$$= \int \frac{\sin(u-\pi/4)}{\sin u}\,du = \int \frac{\sin u\cos\pi/4 - \cos u\sin\pi/4}{\sin u}\,du$$

$$= \int \cos\tfrac{\pi}{4} - \cot u\sin\tfrac{\pi}{4}\,du = \frac{\sqrt{2}}{2}\int 1-\cot u\,du$$

$$= \frac{\sqrt{2}}{2}\left[u-\ln|\sin u|\right]+C = \frac{\sqrt{2}}{2}\left[x+\frac{\pi}{4}-\ln\left|\sin\left(x+\frac{\pi}{4}\right)\right|\right]+C$$

$$= \frac{\sqrt{2}}{2}\left[x-\ln\left|\sin(x+\pi/4)\right|\right]+\left(\frac{\sqrt{2}\pi}{8}+C\right) = \frac{\sqrt{2}}{2}\left[x-\ln\left|\sin\left(x+\frac{\pi}{4}\right)\right|\right]+C$$

651. $$\int \frac{1}{\tanh x+\tan x}\,dx = \ln\left|\sinh x\cos x+\sin x\cosh x\right| + C$$

$$\int \frac{1}{\tanh x+\tan x}\,dx = \int \frac{1}{\frac{\sinh x}{\cosh x}+\frac{\sin x}{\cos x}}\,dx$$

$$= \int \frac{1}{\frac{\sinh x\cos x+\sin x\cosh x}{\cosh x\cos x}}\,dx = \int \frac{\cosh x\cos x}{\sinh x\cos x+\sin x\cosh x}\,dx$$

NOW OBSERVE THE FOLLOWING

$$\frac{d}{dx}\left[\sinh x\cos x+\sin x\cosh x\right] = \cosh x\cos x - \cancel{\sinh x\sin x} + \cos x\cosh x + \cancel{\sin x\sinh x}$$
$$= 2\cosh x\cos x$$

WHICH MEANS THAT THE INTEGRAND IS OF THE FORM $\frac{f'(x)}{f(x)}$

$$\therefore \int \frac{1}{\tanh x+\tan x}\,dx = \frac{1}{2}\ln\left|\sinh x\cos x+\sin x\cosh x\right|+C$$

652. $$\int \frac{(2x+1)^2}{x(x+1)^4}\,dx = \frac{1}{3(x+1)^3}-\frac{3}{2(x+1)^2}+\frac{1}{x+1}+\ln\left|\frac{x}{x+1}\right| + C$$

$\displaystyle\int \frac{(2x+1)^2}{x(x+1)^4}\,dx$ WE CANNOT AVOID THE PARTIAL FRACTION, BUT IT IS BETTER TO START WITH A SUBSTITUTION

$u = x+1$

$x = u-1$

$du = dx$

$$= \int \frac{[2(u-1)+1]^2}{(u-1)\,u^4}\,du = \int \frac{(2u-1)^2}{u-1}\times\frac{1}{u^4}\,du = \int \frac{4u^2-4u+1}{u-1}\,\frac{1}{u^4}\,du$$

"LONG DIVISION BACKWARDS"

$$\begin{array}{r|l}
 & -1+3u-u^2-u^3 \\
-1+u & 1-4u+4u^2 \\
 & \cancel{1}+u \\
 & -3u+4u^2 \\
 & \cancel{+3u}-3u^2 \\
 & u^2 \\
 & \cancel{-u^2}+u^3 \\
 & u^3 \\
 & \cancel{-u^3}+u^4 \\
 & u^4
\end{array}$$

STOP HERE

$$= \int\left[-1+3u-u^2-u^3+\frac{u^4}{u-1}\right]\times\frac{1}{u^4}\,du = \int -\frac{1}{u^4}+\frac{3}{u^3}-\frac{1}{u^2}-\frac{1}{u}+\frac{1}{u-1}\,du$$

$$= \frac{1}{3u^3}-\frac{3}{2u^2}+\frac{1}{u}-\ln|u|+\ln|u-1|+C = \frac{1}{3(x+1)^3}-\frac{3}{2(x+1)^2}+\frac{1}{x+1}+\ln\left|\frac{x}{x+1}\right|+C$$

653. $$\int \frac{1}{x^2\left(1+x^4\right)^{\frac{3}{4}}}\,dx = -\frac{1}{x}\left(1+x^4\right)^{\frac{1}{4}} + C$$

$$\int \frac{1}{x^2(1+x^4)^{3/4}}\,dx = \int \frac{1}{x^2\left[x^4\right]^{3/4}(x^{-4}+1)^{3/4}}\,dx = \int \frac{1}{x^2x^3(x^{-4}+1)^{3/4}}\,dx$$

$$= \int x^{-5}(x^{-4}+1)^{-\frac{3}{4}}\,dx$$

THE SUBSTITUTION $u=1+x^{-4}$ WORKS WELL OR RECOGNITION

$$= -(x^{-4}+1)^{\frac{1}{4}}+C = -\left(\frac{1}{x^4}+1\right)^{\frac{1}{4}}+C$$

$$= -\left(\frac{1+x^4}{x^4}\right)^{\frac{1}{4}}+C = -\frac{1}{x}(1+x^4)^{\frac{1}{4}}+C$$

654. $$\int \frac{1}{x\left(x-\sqrt{x^2-1}\right)}\,dx = x+\sqrt{x^2-1}+\arcsin\left(\frac{1}{x}\right) + C$$

$\displaystyle\int \frac{1}{x(x-\sqrt{x^2-1})}\,dx = \ldots$ START WITH A SIMPLE STANDARD SUBSTITUTION

$$= \int \frac{1}{\frac{1}{u}\left(\frac{1}{u}-\sqrt{\frac{1}{u^2}-1}\right)}\left(-\frac{1}{u^2}\,du\right) = \int \frac{-1}{\frac{1}{u^2}-\frac{1}{u}\sqrt{\frac{1-u^2}{u^2}}}\left(\frac{du}{u^2}\right)$$

$$= \int \frac{-1}{1-u\frac{\sqrt{1-u^2}}{u}}\,du = \int \frac{-1}{1-\sqrt{1-u^2}}\,du$$

$x=\frac{1}{u}$

$u=\frac{1}{x}$

$dx=-\frac{1}{u^2}\,du$

ANOTHER SUBSTITUTION

$$= \int \frac{-1}{1-\sqrt{1-\sin^2\theta}}(\cos\theta\,d\theta) = \int \frac{-\cos\theta}{1-\cos\theta}\,d\theta$$

$$= \int \frac{-\cos\theta(1+\cos\theta)}{(1-\cos\theta)(1+\cos\theta)}\,d\theta = \int \frac{-\cos\theta-\cos^2\theta}{1-\cos^2\theta}\,d\theta$$

$$= \int \frac{-\cos\theta-\cos^2\theta}{\sin^2\theta}\,d\theta = \int -\frac{\cos\theta}{\sin^2\theta}-\cot^2\theta\,d\theta$$

$$= \int -\frac{\cos\theta}{\sin\theta}\times\frac{1}{\sin\theta}-(\operatorname{cosec}^2\theta-1)\,d\theta$$

$$= \int -\operatorname{cosec}\theta\cot\theta-\operatorname{cosec}^2\theta+1\,d\theta = \operatorname{cosec}\theta+\cot\theta+\theta+C$$

$$= \frac{1}{u}+\frac{\sqrt{1-u^2}}{u}+\arcsin u+C = x+\sqrt{x^2-1}+\arcsin\left(\frac{1}{x}\right)+C$$

$$\left[\frac{\sqrt{1-u^2}}{u} = \sqrt{\frac{1-u^2}{u^2}} = \sqrt{\frac{1}{u^2}-1} = \sqrt{x^2-1}\right]$$

$\theta=\arcsin u$

$u=\sin\theta$

$du=\cos\theta\,d\theta$

$\operatorname{cosec}\theta=\frac{1}{u}$

$\cot\theta=\frac{\sqrt{1-u^2}}{u}$

655. $$\int \frac{\sqrt{x^2-1}}{x}\,dx = \sqrt{x^2-1}-\arctan\left(\sqrt{x^2-1}\right) + C$$

$\displaystyle\int \frac{\sqrt{x^2-1}}{x}\,dx = \ldots$ BY SUBSTITUTION…

$$= \int \frac{u}{x}\left(\frac{u}{x}\,du\right) = \int \frac{u^2}{x^2}\,du = \int \frac{u^2}{u^2+1}\,du$$

$$= \int \frac{(u^2+1)-1}{u^2+1}\,du = \int 1-\frac{1}{u^2+1}\,du$$

$$= u-\arctan u+C = \sqrt{x^2-1}-\arctan\left(\sqrt{x^2-1}\right)+C$$

$u=\sqrt{x^2-1}$

$u^2=x^2-1$

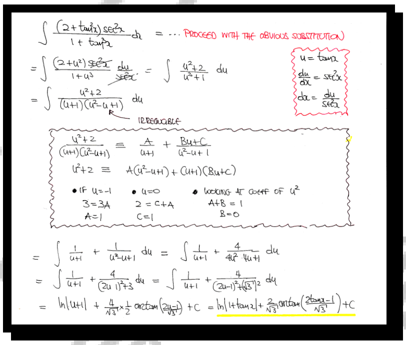
$2u\,du=2x\,dx$

$dx=\frac{u}{x}\,du$

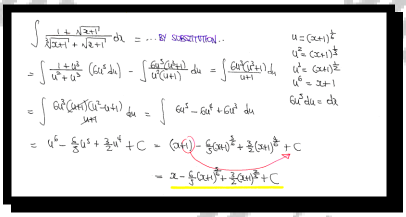
$x^2=u^2+1$

NOTE THAT THE SUBSTITUTION $x=\cosh\theta$ ALSO WORKS BUT IT IS LENGTHY

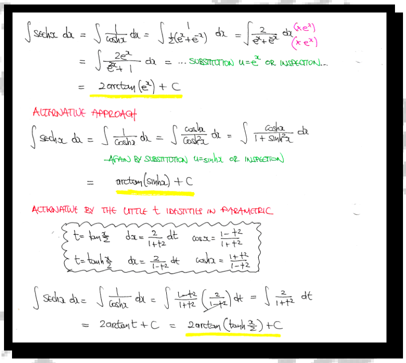
656. $$\int \frac{\left(2+\tan^2 x\right)\sec^2 x}{1+\tan^3 x}\,dx = \ln\left|1+\tan x\right| + \frac{2}{\sqrt{3}}\arctan\left(\frac{2\tan x - 1}{\sqrt{3}}\right) + C$$



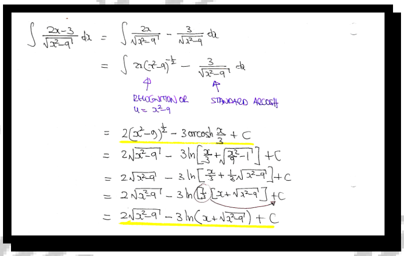
657. $$\int \frac{1+\sqrt{x+1}}{\sqrt[3]{x+1}+\sqrt{x+1}}\,dx = x - \tfrac{6}{5}(x+1)^{\frac{5}{6}} + \tfrac{3}{2}(x+1)^{\frac{2}{3}} + C$$



658. $$\int \operatorname{sech} x \, dx = \begin{cases} 2\arctan\left(e^x\right) + C \\ \arctan\left(\sinh x\right) + C \\ 2\arctan\left(\tanh \tfrac{1}{2}x\right) + C \end{cases}$$



659. $$\int \frac{2x-3}{\sqrt{x^2-9}}\,dx = \begin{cases} 2\sqrt{x^2-9} - 3\,\text{arcosh}\left(\frac{1}{3}x\right) + C \\ 2\sqrt{x^2-9} - 3\ln\left(x+\sqrt{x^2-9}\right) + C \end{cases}$$

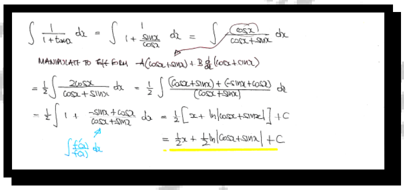


$$\int \frac{2x-3}{\sqrt{x^2-9}}\,dx = \int \frac{2x}{\sqrt{x^2-9}} - \frac{3}{\sqrt{x^2-9}}\,dx$$
$$= \int 2x(x^2-9)^{-\frac{1}{2}} - \frac{3}{\sqrt{x^2-9}}\,dx$$

RECOGNITION OR $u = x^2-9$ — STANDARD ARCOSH

$$= 2(x^2-9)^{\frac{1}{2}} - 3\,\text{arcosh}\,\frac{x}{3} + C$$
$$= 2\sqrt{x^2-9} - 3\ln\left[\frac{x}{3} + \sqrt{\frac{x^2}{9}-1}\right] + C$$
$$= 2\sqrt{x^2-9} - 3\ln\left[\frac{x}{3} + \frac{1}{3}\sqrt{x^2-9}\right] + C$$
$$= 2\sqrt{x^2-9} - 3\ln\left(\frac{1}{3}\left[x+\sqrt{x^2-9}\right]\right) + C$$
$$= 2\sqrt{x^2-9} - 3\ln\left(x+\sqrt{x^2-9}\right) + C$$

660. $$\int \frac{1}{1+\tan x}\,dx = \frac{1}{2}x + \frac{1}{2}\ln|\cos x + \sin x| + C$$



$$\int \frac{1}{1+\tan x}\,dx = \int \frac{1}{1+\frac{\sin x}{\cos x}}\,dx = \int \frac{\cos x}{\cos x + \sin x}\,dx$$

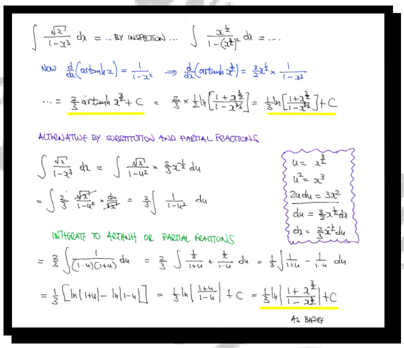
MANIPULATE TO THE FORM $A(\cos x + \sin x) + B\frac{d}{dx}(\cos x + \sin x)$

$$= \frac{1}{2}\int \frac{2\cos x}{\cos x + \sin x}\,dx = \frac{1}{2}\int \frac{(\cos x + \sin x) + (-\sin x + \cos x)}{(\cos x + \sin x)}\,dx$$
$$= \frac{1}{2}\int 1 + \frac{-\sin x + \cos x}{\cos x + \sin x}\,dx = \frac{1}{2}\left[x + \ln|\cos x + \sin x|\right] + C$$

$\int \frac{f'(x)}{f(x)}\,dx$

$$= \frac{1}{2}x + \frac{1}{2}\ln|\cos x + \sin x| + C$$

661. $$\int \frac{\sqrt{x}}{1-x^3}\,dx = \begin{cases} \frac{2}{3}\,\text{artanh}\left(x^{\frac{3}{2}}\right) + C \\ \frac{1}{3}\ln\left|\frac{1+x^{\frac{3}{2}}}{1-x^{\frac{3}{2}}}\right| + C \end{cases}$$



$$\int \frac{\sqrt{x}}{1-x^3}\,dx = \ldots \text{BY INSPECTION} \ldots \int \frac{x^{\frac{1}{2}}}{1-\left(x^{\frac{3}{2}}\right)^2}\,dx = \ldots$$

NOW $\frac{d}{dx}(\text{artanh}\,x) = \frac{1}{1-x^2} \Rightarrow \frac{d}{dx}\left(\text{artanh}\,x^{\frac{3}{2}}\right) = \frac{3}{2}x^{\frac{1}{2}} \times \frac{1}{1-x^3}$

$$\ldots = \frac{2}{3}\,\text{artanh}\,x^{\frac{3}{2}} + C = \frac{2}{3} \times \frac{1}{2}\ln\left[\frac{1+x^{\frac{3}{2}}}{1-x^{\frac{3}{2}}}\right] = \frac{1}{3}\ln\left[\frac{1+x^{\frac{3}{2}}}{1-x^{\frac{3}{2}}}\right] + C$$

ALTERNATIVE BY SUBSTITUTION AND PARTIAL FRACTIONS

$$\int \frac{\sqrt{x}}{1-x^3}\,dx = \int \frac{\sqrt{x}}{1-u^2} \times \frac{2}{3}x^{-\frac{1}{2}}\,du$$
$$= \int \frac{2}{3}\,\frac{\cancel{\sqrt{x}}}{1-u^2} \times \frac{du}{\cancel{\sqrt{x}}} = \frac{2}{3}\int \frac{1}{1-u^2}\,du$$

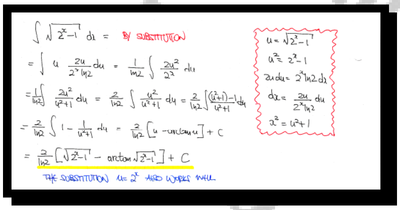
$u = x^{\frac{3}{2}}$
$u^2 = x^3$
$2u\,du = 3x^2$
$du = \frac{3}{2}x^{\frac{1}{2}}\,dx$
$dx = \frac{2}{3}x^{-\frac{1}{2}}\,du$

INTEGRATE TO ARTANH OR PARTIAL FRACTIONS

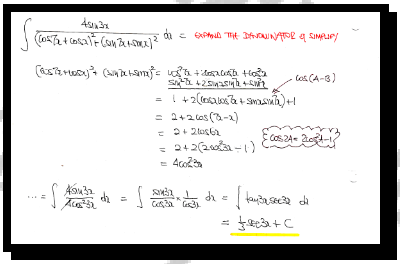
$$= \frac{2}{3}\int \frac{1}{(1-u)(1+u)}\,du = \frac{2}{3}\int \frac{\frac{1}{2}}{1+u} + \frac{\frac{1}{2}}{1-u}\,du = \frac{1}{3}\int \frac{1}{1+u} - \frac{1}{1-u}\,du$$
$$= \frac{1}{3}\left[\ln|1+u| - \ln|1-u|\right] = \frac{1}{3}\ln\left|\frac{1+u}{1-u}\right| + C = \frac{1}{3}\ln\left|\frac{1+x^{\frac{3}{2}}}{1-x^{\frac{3}{2}}}\right| + C$$

AS BEFORE

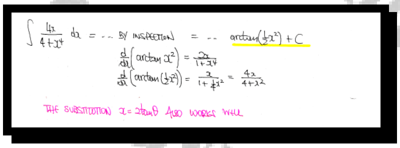
662. $$\int \sqrt{2^x - 1}\, dx = \frac{2}{\ln 2}\left[\sqrt{2^x - 1} - \arctan\left(\sqrt{2^x - 1}\right)\right] + C$$



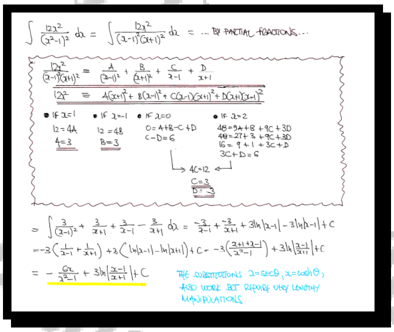
663. $$\int \frac{4\sin 3x}{(\cos 7x + \cos x)^2 + (\sin 7x + \sin x)^2}\, dx = \frac{1}{3}\sec 3x + C$$



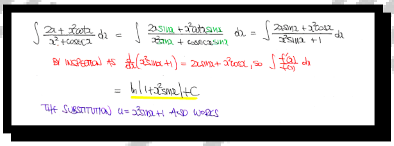
664. $$\int \frac{4x}{4 + x^4}\, dx = \arctan\left(\frac{1}{2}x^2\right) + C$$



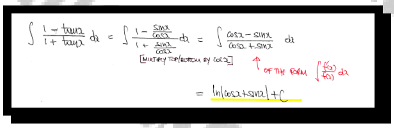
665. $$\int \frac{12x^2}{\left(x^2-1\right)^2}\,dx = -\frac{6x}{x^2-1} + 3\ln\left|\frac{x-1}{x+1}\right| + C$$



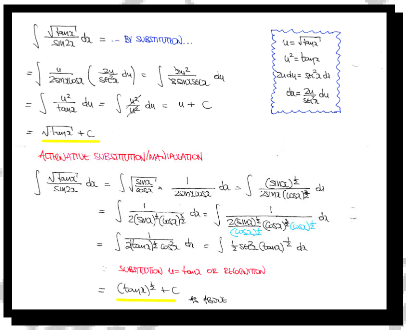
666. $$\int \frac{2x + x^2 \cot x}{x^2 + \operatorname{cosec} x}\,dx = \ln\left|1 + x^2 \sin x\right| + C$$



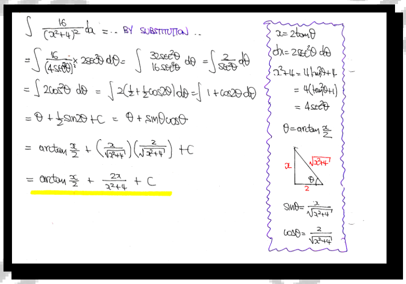
667. $$\int \frac{1-\tan x}{1+\tan x}\,dx = \ln\left|\cos x + \sin x\right| + C$$



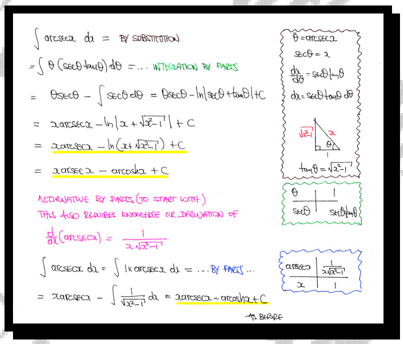
668. $\displaystyle\int \frac{\sqrt{\tan x}}{\sin 2x}\,dx = \sqrt{\tan x} \;+\; C$



669. $\displaystyle\int \frac{16}{\left(x^2+4\right)^2}\,dx = \frac{2x}{x^2+4} + \arctan\left(\frac{1}{2}x\right) \;+\; C$



670. $\displaystyle\int \operatorname{arcsec} x\,dx = \begin{cases} x\operatorname{arcsec} x - \operatorname{arcosh} x \;+\; C \\ x\operatorname{arcsec} x - \ln\left(x+\sqrt{x^2-1}\right) \;+\; C \end{cases}$



671. $$\int \frac{x^2}{1-2\sqrt{x}}\,dx = -\frac{1}{5}x^{\frac{5}{2}} - \frac{1}{8}x^2 - \frac{1}{12}x^{\frac{3}{2}} - \frac{1}{16}x - \frac{1}{32}\ln\left|1-2\sqrt{x}\right| + C$$

$$= \frac{1}{32}\left[\frac{1}{5}u^5 - \frac{5}{4}u^4 + \frac{10}{3}u^3 - 5u^2 + 5u - \ln|u|\right] + C, \qquad u = 1-2\sqrt{x}$$

$\int \frac{x^2}{1-2\sqrt{x}}\,dx = \ldots$ BY SUBSTITUTION ...

$u = 1-2\sqrt{x}$
$2\sqrt{x} = 1-u$
$4x = (1-u)^2$
$4\,dx = -2(1-u)\,du$
$dx = \frac{1}{2}(u-1)\,du$
$16x^2 = (1-u)^4$
$16x^2 = (u-1)^4$

$\frac{1}{16}\int \frac{16x^2}{1-2\sqrt{x}}\,dx = \frac{1}{16}\int \frac{(u-1)^4}{u}\,\frac{1}{2}(u-1)\,du = \frac{1}{32}\int \frac{(u-1)^5}{u}\,du$

$= \frac{1}{32}\int \frac{u^5 - 5u^4 + 10u^3 - 10u^2 + 5u - 1}{u}\,du$

$= \frac{1}{32}\int u^4 - 5u^3 + 10u^2 - 10u + 5 - \frac{1}{u}\,du$

$= \frac{1}{32}\left[\frac{1}{5}u^5 - \frac{5}{4}u^4 + \frac{10}{3}u^3 - 5u^2 + 5u - \ln|u|\right] + C$

$= \frac{1}{32}\left[\frac{1}{5}(1-2\sqrt{x})^5 - \frac{5}{4}(1-2\sqrt{x})^4 + \frac{10}{3}(1-2\sqrt{x})^3 - 5(1-2\sqrt{x})^2 + 5(1-2\sqrt{x}) - \ln|1-2\sqrt{x}|\right] + C$

TIDY FURTHER TO

$= -\frac{1}{5}x^{\frac{5}{2}} - \frac{1}{8}x^2 - \frac{1}{12}x^{\frac{3}{2}} - \frac{1}{16}x - \frac{1}{32}\ln|1-2\sqrt{x}| + C$

672. $$\int \frac{1-x}{(1+x)^2\sqrt{x}}\,dx = \frac{2\sqrt{x}}{x+1} + C$$

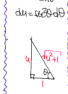
$\int \frac{1-x}{\sqrt{x}(1+x)^2}\,dx = \ldots$ BY SUBSTITUTION ...

$u = \sqrt{x}$
$u^2 = x$
$dx = 2u\,du$

$\int \frac{1-u^2}{u(1+u^2)^2}(2u\,du) = \int \frac{2(1-u^2)}{(1+u^2)^2}\,du$

A TRIGONOMETRIC SUBSTITUTION TO FOLLOW

$\theta = \arctan u$
$u = \tan\theta$
$du = \sec^2\theta\,d\theta$



$\sin\theta = \frac{u}{\sqrt{u^2+1}}$
$\cos\theta = \frac{1}{\sqrt{u^2+1}}$

$= \int \frac{2(1-\tan^2\theta)}{(1+\tan^2\theta)^2}(\sec^2\theta\,d\theta) = \int \frac{2\sec^2\theta(1-\tan^2\theta)}{\sec^4\theta}\,d\theta$

$= \int \frac{2(1-\tan^2\theta)}{\sec^2\theta}\,d\theta = 2\int \frac{1}{\sec^2\theta} - \frac{\tan^2\theta}{\sec^2\theta}\,d\theta$

$= 2\int \cos^2\theta - \frac{\sin^2\theta}{\cancel{\cos^2\theta}} \times \cancel{\cos^2\theta}\,d\theta = 2\int \cos^2\theta - \sin^2\theta\,d\theta$

$= \int 2\cos 2\theta\,d\theta = \sin 2\theta + C$

$= 2\sin\theta\cos\theta + C = 2\left(\frac{u}{\sqrt{u^2+1}}\right)\left(\frac{1}{\sqrt{u^2+1}}\right) + C$

$= \frac{2u}{u^2+1} + C = \frac{2\sqrt{x}}{x+1} + C$

ALTERNATIVE APPROACH AFTER THE FIRST SUBSTITUTION

$\int \frac{1-x}{\sqrt{x}(1+x)^2}\,dx = \ldots = \int \frac{2(1-u^2)}{(1+u^2)^2}\,du$

PROCEED BY PARTIAL FRACTIONS

$\frac{2(1-u^2)}{(1+u^2)^2} \equiv \frac{Ax+B}{1+u^2} + \frac{Cx+D}{(1+u^2)^2}$

$2(1-u^2) \equiv (Ax+B)(1+u^2) + Cx + D$

- IF $u = i$: $4 = Ci + D$; $D = 4$ & $C = 0$
- IF $u = 0$: $2 = B + D$; $B = -2$
- IF $u = 1$: $0 = 2(A+B) + C + D$; $0 = 2(A-2) + 4$; $0 = 2A$; $A = 0$

$= \int \frac{4}{(1+u^2)^2} - \frac{2}{1+u^2}\,du = \ldots$ YET AGAIN A TRIGONOMETRIC SUBSTITUTION

$= \int \frac{4}{(1+u^2)^2}\,du - 2\arctan u$ ← MORE ARCTANS SHOULD COME OUT OF THIS, TO CANCEL IT

$\theta = \arctan u$
$\tan\theta = u$
$du = \sec^2\theta\,d\theta$

$\sin\theta = \frac{u}{\sqrt{u^2+1}}$
$\cos\theta = \frac{1}{\sqrt{u^2+1}}$

$= \int \frac{4}{(1+\tan^2\theta)^2}(\sec^2\theta\,d\theta) - 2\arctan u$

$= \int \frac{4\sec^2\theta}{\sec^4\theta}\,d\theta - 2\arctan u$

$= \int 4\cos^2\theta\,d\theta - 2\arctan u$

$= \int 4\left(\frac{1}{2} + \frac{1}{2}\cos 2\theta\right)d\theta - 2\arctan u$

$= \int 2 + 2\cos 2\theta\,d\theta - 2\arctan u$

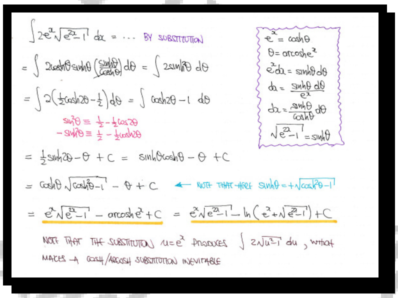
$= 2\theta + \sin 2\theta - 2\arctan u + C$

$= \cancel{2\arctan u} + 2\sin\theta\cos\theta - \cancel{2\arctan u} + C$

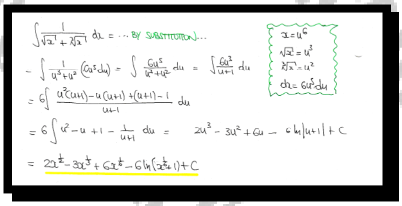
$= 2\left(\frac{u}{\sqrt{u^2+1}}\right)\left(\frac{1}{\sqrt{u^2+1}}\right) + C = \frac{2u}{u^2+1} + C = \frac{2\sqrt{x}}{x+1} + C$

AS BEFORE

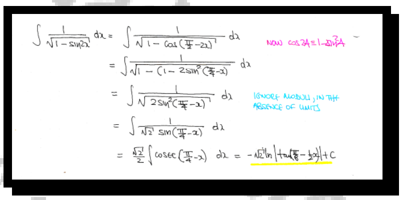
673. $$\int 2e^{x}\sqrt{e^{2x}-1}\,dx = \begin{cases} e^{x}\sqrt{e^{2x}-1}-\operatorname{arcosh} e^{x} + C \\ e^{x}\sqrt{e^{2x}-1}-\ln\left(e^{x}+\sqrt{e^{2x}-1}\right) + C \end{cases}$$



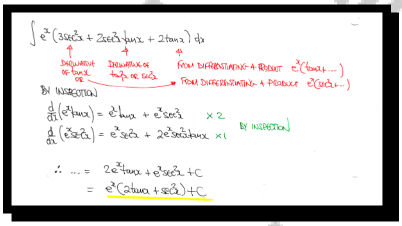
674. $$\int \frac{1}{\sqrt{x}+\sqrt[3]{x}}\,dx = 2x^{\frac{1}{2}}-3x^{\frac{1}{3}}-6\ln\left|1+x^{\frac{1}{6}}\right| + C$$



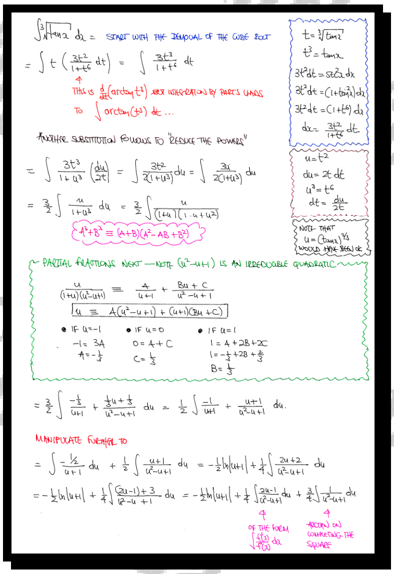
675. $$\int \frac{1}{\sqrt{1-\sin 2x}}\,dx = -\sqrt{2}\ln\left|\tan\left(\frac{1}{8}\pi-\frac{1}{2}x\right)\right| + C$$

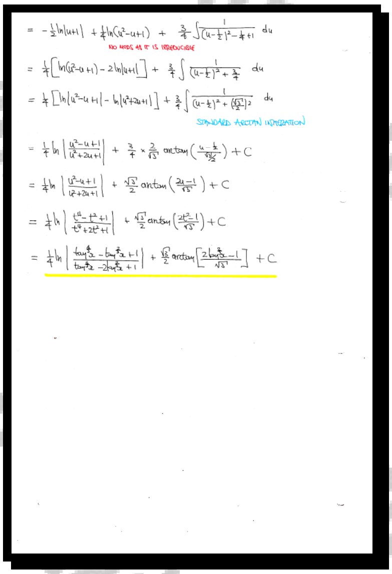


676. $$\int e^{x}\left(3\sec^{2}x+2\sec^{2}x\tan x+2\tan x\right)\,dx=e^{x}\left(\sec^{2}x+2\tan x\right)\ +\ C$$

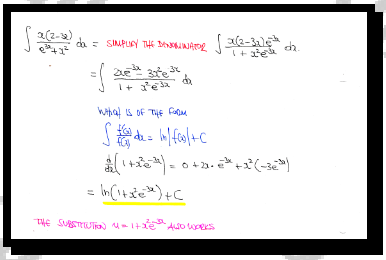


677. $$\int \sqrt[3]{\tan x}\ dx=\frac{1}{4}\ln\left|\frac{\tan^{\frac{4}{3}}x-\tan^{\frac{2}{3}}x+1}{\tan^{\frac{4}{3}}x-2\tan^{\frac{2}{3}}x+1}\right|+\frac{\sqrt{3}}{2}\arctan\left[\frac{2\tan^{\frac{2}{3}}x-1}{\sqrt{3}}\right]\ +\ C$$

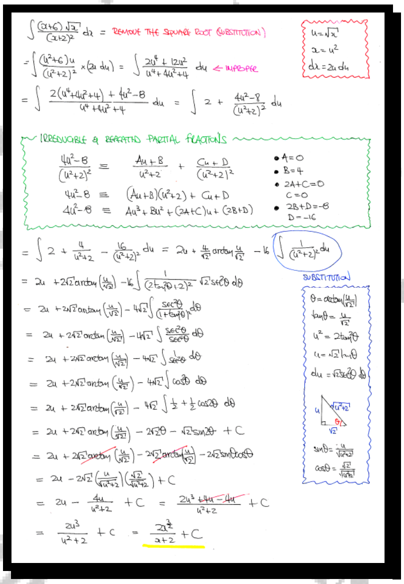


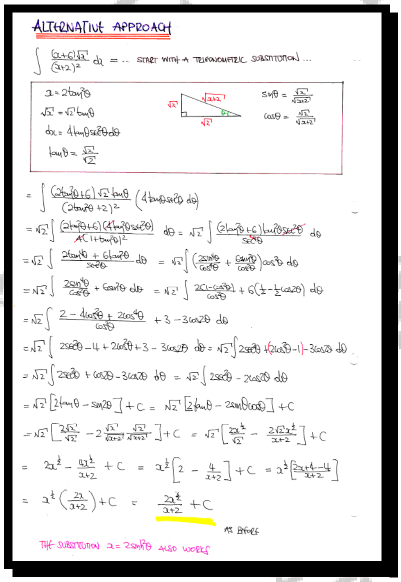


678. $$\int \frac{x(2-3x)}{x^{2}+e^{3x}}\ dx=\ln\left(1+x^{2}e^{-3x}\right)\ +\ C$$

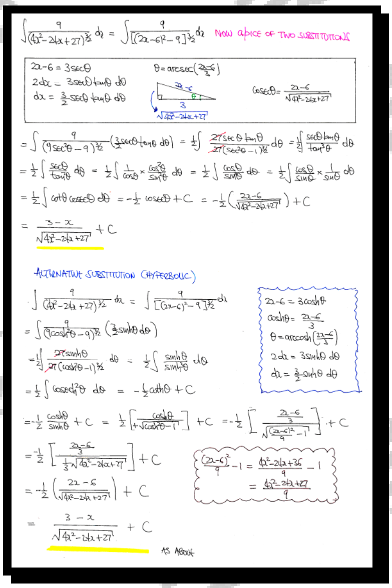


679. $$\int \frac{(x+6)}{(x+2)^2}\,dx = \frac{2x^{\frac{3}{2}}}{x+2} + C$$





680. $$\int \frac{9}{\left(4x^2-24x+27\right)^{\frac{3}{2}}}\,dx = \frac{3-x}{\sqrt{4x^2-24x+27}} + C$$



681. $$\int \frac{\sqrt{x-x^2}}{x^4}\,dx = -\frac{2}{3}\left(\frac{1}{x}-1\right)^{\frac{3}{2}} - \frac{2}{5}\left(\frac{1}{x}-1\right)^{\frac{5}{2}} + C$$

$$\int \frac{\sqrt{x-x^2}}{x^4}\,dx = \dots \text{ BY SUBSTITUTION}$$

$$x=\frac{1}{u}, \quad u=\frac{1}{x}, \quad dx=-\frac{1}{u^2}\,du$$

$$= \int \frac{\sqrt{\frac{1}{u}-\frac{1}{u^2}}}{\frac{1}{u^4}}\left(-\frac{1}{u^2}\,du\right) = \int \frac{\sqrt{\frac{u-1}{u^2}}}{\frac{1}{u^4}}\left(-\frac{1}{u^2}\right)du$$

$$= \int -u^2\,\frac{\sqrt{u-1}}{u}\,du = \int -u(u-1)^{\frac{1}{2}}\,du \quad \dots \text{ ANOTHER SUBSTITUTION OR MANIPULATION}$$

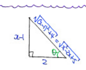
$$= \int [-1-(u-1)](u-1)^{\frac{1}{2}}\,du = \int -(u-1)^{\frac{1}{2}} - (u-1)^{\frac{3}{2}}\,du = -\frac{2}{3}(u-1)^{\frac{3}{2}} - \frac{2}{5}(u-1)^{\frac{5}{2}} + C$$

$$= -\frac{2}{3}\left(\frac{1}{x}-1\right)^{\frac{3}{2}} - \frac{2}{5}\left(\frac{1}{x}-1\right)^{\frac{5}{2}} + C$$

682. $$\int \frac{16}{\left(x^2-2x+5\right)^2}\,dx = \arctan\left[\tfrac{1}{2}(x-1)\right] + \frac{2(x-1)}{x^2-2x+5} + C$$

$$\int \frac{16}{(x^2-2x+5)^2} = \int \frac{16}{\left[(x-1)^2+4\right]^2}\,dx \quad \dots \text{ BY SUBSTITUTION}$$

$$x-1 = 2\tan\theta, \quad dx = 2\sec^2\theta\,d\theta, \quad \theta = \arctan\left(\frac{x-1}{2}\right), \quad \tan\theta = \frac{x-1}{2}$$



$$\sin\theta = \frac{x-1}{\sqrt{x^2-2x+5}}, \quad \cos\theta = \frac{2}{\sqrt{x^2-2x+5}}$$

$$= \int \frac{16}{(4\tan^2\theta+4)^2}\left(2\sec^2\theta\,d\theta\right) = \int \frac{32\sec^2\theta}{16(\tan^2\theta+1)^2}\,d\theta = \int \frac{2\sec^2\theta}{\sec^4\theta}\,d\theta$$

$$= \int \frac{2}{\sec^2\theta}\,d\theta = \int 2\cos^2\theta\,d\theta = \int 1+\cos 2\theta\,d\theta = \theta + \frac{1}{2}\sin 2\theta + C$$

$$= \theta + \sin\theta\cos\theta + C = \arctan\left(\frac{x-1}{2}\right) + \frac{x-1}{\sqrt{x^2-2x+5}}\,\frac{2}{\sqrt{x^2-2x+5}} + C$$

$$= \arctan\left(\frac{x-1}{2}\right) + \frac{2(x-1)}{x^2-2x+5} + C$$

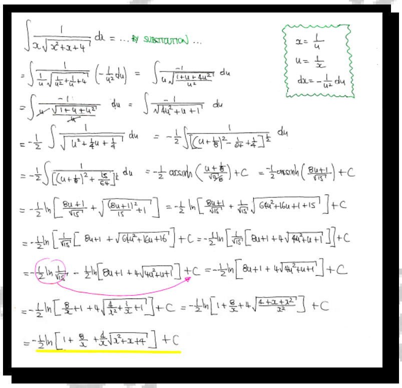
683. $$\int \frac{\tanh x}{\operatorname{sech}^2 x - \tanh^2 x}\,dx = \frac{1}{2}\ln\left|1-\sinh^2 x\right| + C$$

$$\int \frac{\tanh x}{\operatorname{sech}^2 x - \tanh^2 x}\,dx = \int \frac{\frac{\sinh x}{\cosh x}}{\frac{1}{\cosh^2 x} - \frac{\sinh^2 x}{\cosh^2 x}}\,dx = \int \frac{\cosh x \sinh x}{1-\sinh^2 x}\,dx$$

$$= \frac{1}{2}\int \frac{2\cosh x \sinh x}{1-\sinh^2 x}\,dx \quad \text{OF THE FORM } \int \frac{f'(x)}{f(x)}\,dx$$

$$= \frac{1}{2}\ln\left|1-\sinh^2 x\right| + C$$

684. $$\int \frac{1}{x\sqrt{x^2+x+4}}\,dx = -\frac{1}{2}\ln\left|1+\frac{8}{x}+\frac{4}{x}\sqrt{x^2+x+4}\right| + C$$



$\int \frac{1}{x\sqrt{x^2+x+4}}\,dx = \dots$ BY SUBSTITUTION …

$x = \frac{1}{u}$, $u = \frac{1}{x}$, $dx = -\frac{1}{u^2}\,du$

$= \int \frac{1}{\frac{1}{u}\sqrt{\frac{1}{u^2}+\frac{1}{u}+4}}\left(-\frac{1}{u^2}\,du\right) = \int \frac{-1}{u\sqrt{\frac{1+u+4u^2}{u^2}}}\,du$

$= \int \frac{-1}{\cancel{u}\frac{\sqrt{1+u+4u^2}}{\cancel{u}}}\,du = \int \frac{-1}{\sqrt{4u^2+u+1}}\,du$

$= -\frac{1}{2}\int \frac{1}{\sqrt{u^2+\frac{1}{4}u+\frac{1}{4}}}\,du = -\frac{1}{2}\int \frac{1}{\left[\left(u+\frac{1}{8}\right)^2-\frac{1}{64}+\frac{1}{4}\right]^{\frac{1}{2}}}\,du$

$= -\frac{1}{2}\int \frac{1}{\left[\left(u+\frac{1}{8}\right)^2+\frac{15}{64}\right]^{\frac{1}{2}}}\,du = -\frac{1}{2}\,\text{arsinh}\left(\frac{u+\frac{1}{8}}{\sqrt{15}/8}\right)+C = -\frac{1}{2}\,\text{arsinh}\left(\frac{8u+1}{\sqrt{15}}\right)+C$

$= -\frac{1}{2}\ln\left[\frac{8u+1}{\sqrt{15}}+\sqrt{\frac{(8u+1)^2}{15}+1}\right] = -\frac{1}{2}\ln\left[\frac{8u+1}{\sqrt{15}}+\frac{1}{\sqrt{15}}\sqrt{64u^2+16u+1+15}\right]+C$

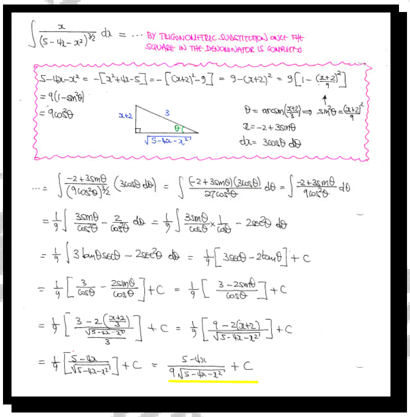
$= -\frac{1}{2}\ln\left[\frac{1}{\sqrt{15}}\left[8u+1+\sqrt{64u^2+16u+16}\right]\right]+C = -\frac{1}{2}\ln\left[\frac{1}{\sqrt{15}}\left[8u+1+4\sqrt{4u^2+u+1}\right]\right]+C$

$= -\frac{1}{2}\ln\frac{1}{\sqrt{15}} - \frac{1}{2}\ln\left[8u+1+4\sqrt{4u^2+u+1}\right]+C = -\frac{1}{2}\ln\left[8u+1+4\sqrt{4u^2+u+1}\right]+C$

$= -\frac{1}{2}\ln\left[\frac{8}{x}+1+4\sqrt{\frac{4}{x^2}+\frac{1}{x}+1}\right]+C = -\frac{1}{2}\ln\left[1+\frac{8}{x}+4\sqrt{\frac{4+x+x^2}{x^2}}\right]+C$

$= -\frac{1}{2}\ln\left[1+\frac{8}{x}+\frac{4}{x}\sqrt{x^2+x+4}\right]+C$

685. $$\int \frac{x}{\left(5-4x-x^2\right)^{\frac{3}{2}}}\,dx = \frac{5-4x}{9\sqrt{5-4x-x^2}} + C$$



$\int \frac{x}{(5-4x-x^2)^{3/2}}\,dx = \dots$ BY TRIGONOMETRIC SUBSTITUTION ONCE THE SQUARE IN THE DENOMINATOR IS COMPLETED

$5-4x-x^2 = -\left[x^2+4x-5\right] = -\left[(x+2)^2-9\right] = 9-(x+2)^2 = 9\left[1-\frac{(x+2)^2}{9}\right]$

$= 9(1-\sin^2\theta)$

$= 9\cos^2\theta$

$x+2$, 3, θ, $\sqrt{5-4x-x^2}$

$\theta = \arcsin\left(\frac{x+2}{3}\right) \Rightarrow \sin^2\theta = \frac{(x+2)^2}{9}$

$x = -2+3\sin\theta$

$dx = 3\cos\theta\,d\theta$

$\dots = \int \frac{-2+3\sin\theta}{(9\cos^2\theta)^{3/2}}(3\cos\theta\,d\theta) = \int \frac{(-2+3\sin\theta)(3\cos\theta)}{27\cos^3\theta}\,d\theta = \int \frac{-2+3\sin\theta}{9\cos^2\theta}\,d\theta$

$= \frac{1}{9}\int \frac{3\sin\theta}{\cos^2\theta}-\frac{2}{\cos^2\theta}\,d\theta = \frac{1}{9}\int \frac{3\sin\theta}{\cos\theta}\times\frac{1}{\cos\theta}-2\sec^2\theta\,d\theta$

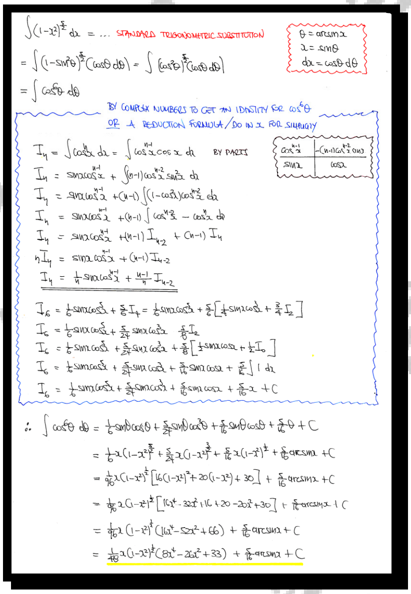
$= \frac{1}{9}\int 3\tan\theta\sec\theta-2\sec^2\theta\,d\theta = \frac{1}{9}\left[3\sec\theta-2\tan\theta\right]+C$

$= \frac{1}{9}\left[\frac{3}{\cos\theta}-\frac{2\sin\theta}{\cos\theta}\right]+C = \frac{1}{9}\left[\frac{3-2\sin\theta}{\cos\theta}\right]+C$

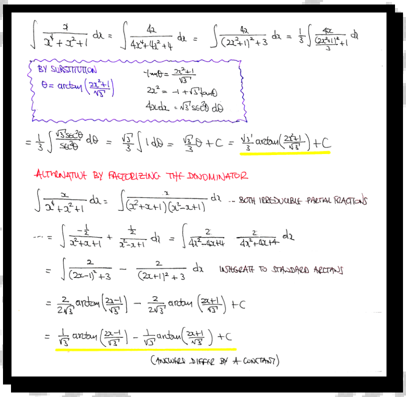
$= \frac{1}{9}\left[\frac{3-2\left(\frac{x+2}{3}\right)}{\frac{\sqrt{5-4x-x^2}}{3}}\right]+C = \frac{1}{9}\left[\frac{9-2(x+2)}{\sqrt{5-4x-x^2}}\right]+C$

$= \frac{1}{9}\left[\frac{5-4x}{\sqrt{5-4x-x^2}}\right]+C = \frac{5-4x}{9\sqrt{5-4x-x^2}}+C$

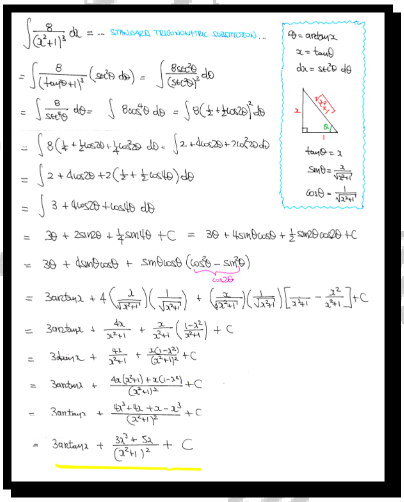
686. $$\int \left(1-x^{2}\right)^{\frac{5}{2}}\,dx=\frac{1}{48}x\left(8x^{4}-26x^{2}+33\right)\left(1-x^{2}\right)^{\frac{1}{2}}+\frac{5}{16}\arcsin x + C$$



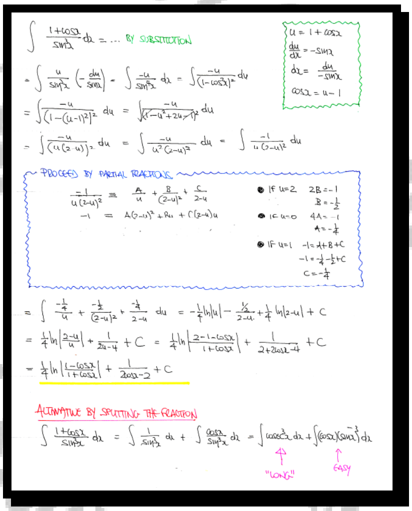
687. $$\int \frac{x}{x^{4}+x^{2}+1}\,dx=\begin{cases}\dfrac{1}{\sqrt{3}}\arctan\left(\dfrac{2x^{2}+1}{\sqrt{3}}\right)+C\\[2ex] \dfrac{1}{\sqrt{3}}\arctan\left(\dfrac{2x-1}{\sqrt{3}}\right)-\dfrac{1}{\sqrt{3}}\arctan\left(\dfrac{2x+1}{\sqrt{3}}\right)+C\end{cases}$$



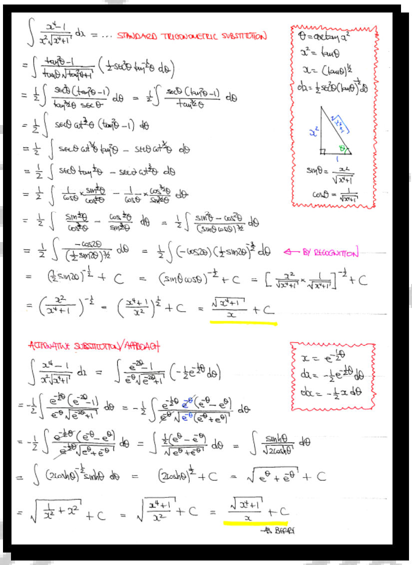
688. $$\int \frac{8}{\left(x^2+1\right)^3}\,dx = 3\arctan x + \frac{3x^3+5x}{\left(x^2+1\right)^2} + C$$

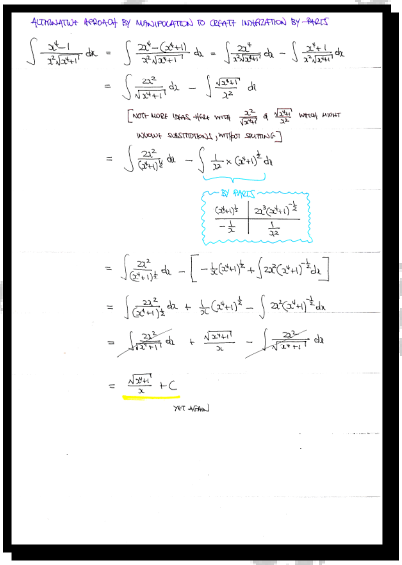


689. $$\int \frac{1+\cos x}{\sin^2 x}\,dx = \frac{1}{4}\ln\left|\frac{1-\cos x}{1+\cos x}\right| + \frac{1}{2\cos x - 2} + C$$

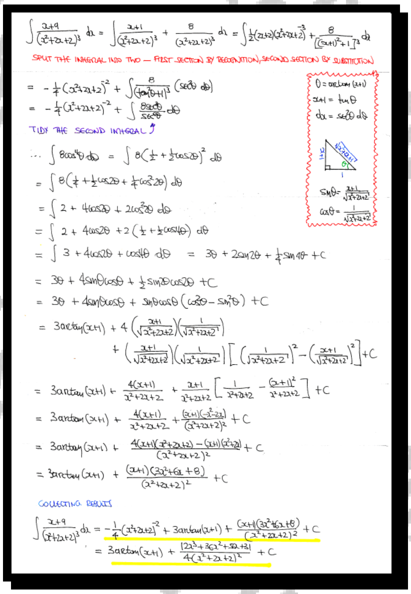


690. $$\int \frac{x^4-1}{x^2\sqrt{x^4+1}}\,dx = \frac{\sqrt{x^4+1}}{x} + C$$





691. $$\int \frac{x+9}{\left(x^2+2x+2\right)^3}\,dx = 3\arctan(x+1) + \frac{12x^3+36x^2+56x+31}{\left(x^2+2x+2\right)^2} + C$$



692. $$\int \frac{\sec^2 x - 1}{\sec^3 x + \tan^3 x}\,dx = \frac{1}{3}\ln\left(1+\sin^3 x\right) + C$$

$\displaystyle\int \frac{\sec^2 x - 1}{\sec^3 x + \tan^3 x}\,dx =$ … SWITCH INTO SINES & COSINES $= \displaystyle\int \frac{\frac{\sin^2 x}{\cos^2 x}}{\frac{1}{\cos^3 x}+\frac{\sin^3 x}{\cos^3 x}}\,dx$

$= \left[\displaystyle\int \frac{\tan^2 x}{\sec^3 x + \tan^3 x}\,dx\right] \leftarrow$ BETTER

$=$ … MULTIPLY TOP & BOTTOM OF THE FRACTION BY $\cos^3 x$ … $= \displaystyle\int \frac{\sin^2 x\cos x}{1+\sin^3 x}\,dx$

$=$ … OF THE FORM $\displaystyle\int \frac{f'(x)}{f(x)}\,dx$ … $= \frac{1}{3}\displaystyle\int \frac{3\sin^2 x\cos x}{1+\sin^3 x}\,dx = \frac{1}{3}\ln\left[1+\sin^3 x\right]+C$

693. $$\int \tan x \sec 2x\,dx = -\frac{1}{2}\ln\left|1-\tan^2 x\right| + C$$

$\displaystyle\int \tan x\sec 2x\,dx = \int \frac{\tan x}{\cos 2x}\,dx = \int \frac{\tan x}{\cos^2 x - \sin^2 x}\,dx = \int \frac{\frac{\tan x}{\cos^2 x}}{\frac{\cos^2 x}{\cos^2 x}-\frac{\sin^2 x}{\cos^2 x}}\,dx$

$= \displaystyle\int \frac{\tan x\sec^2 x}{1-\tan^2 x}\,dx = -\frac{1}{2}\int \frac{-2\tan x\sec^2 x}{1-\tan^2 x}\,dx$

$=$ … OF THE FORM $\displaystyle\int \frac{f'(x)}{f(x)}\,dx$ … $= -\frac{1}{2}\ln\left|1-\tan^2 x\right| + C$

694. $$\int \frac{5\cos x}{(2+\sin x)(3+4\sin x)}\,dx = \ln\left|\frac{3+4\sin x}{2+\sin x}\right| + C$$

$\displaystyle\int \frac{5\cos x}{4\sin^2 x + 11\sin x + 6}\,dx =$ … EVIDENT SUBSTITUTION FOLLOWED BY PARTIAL FRACTIONS

$u = \sin x$

$\frac{du}{dx} = \cos x$

$dx = \frac{du}{\cos x}$

$= \displaystyle\int \frac{5\cancel{\cos x}}{4u^2+11u+6}\left(\frac{du}{\cancel{\cos x}}\right) = \int \frac{5}{(4u+3)(u+2)}\,du$

BY INSPECTION

$= \displaystyle\int \frac{\frac{5}{5/4}}{4u+3} + \frac{\frac{5}{-5}}{u+2}\,du = \int \frac{4}{4u+3} - \frac{1}{u+2}\,du$

$= \ln|4u+3| - \ln|u+2| + C = \ln\left|\frac{4u+3}{u+2}\right| + C = \ln\left|\frac{3+4u}{2+u}\right|$

$= \ln\left|\frac{3+4\sin x}{2+\sin x}\right| + C$

695. $$\int \frac{x^2+3x+3}{(x+1)^3}\left(e^{-x}\sin x\right)dx = \frac{-e^{-x}\left[(x+2)\sin x+(x+1)\cos x\right]}{2(x+1)^2} + C$$

$$\int \frac{x^2+3x+3}{(x+1)^3}e^{-x}\sin x\,dx = \dots$$ START WITH PARTIAL FRACTIONS

$$\frac{x^2+3x+3}{(x+1)^3} \equiv \frac{A}{x+1}+\frac{B}{(x+1)^2}+\frac{C}{(x+1)^3}$$
$$x^2+3x+3 \equiv A(x+1)^2+B(x+1)+C$$
$$x^2+3x+3 \equiv Ax^2+(2A+B)x+(A+B+C) \qquad A=1,\ B=1,\ C=1$$

$$= \int\left[\frac{1}{x+1}+\frac{1}{(x+1)^2}+\frac{1}{(x+1)^3}\right]e^{-x}\sin x\,dx = \int\frac{1}{x+1}e^{-x}\sin x\,dx+\int\frac{1}{(x+1)^2}e^{-x}\sin x\,dx+\int\frac{1}{(x+1)^3}e^{-x}\sin x\,dx$$

NEXT FIND THE INTEGRAL OF $e^{-x}\sin x$ [DOUBLE PARTS | COMPLEX NUMBERS OR INSPECTION]

$$\frac{d}{dx}\left[e^{-x}(A\sin x+B\cos x)\right] = -e^{-x}(A\sin x+B\cos x)+e^{-x}(A\cos x-B\sin x)$$
$$= e^{-x}\big[\underbrace{(-A-B)}_{1}\sin x+\underbrace{(-B+A)}_{0}\cos x\big]$$
$$A=B=-\tfrac{1}{2}$$

NEXT INTEGRATION BY PARTS FOR THE FOLLOWING SECTIONS

- $$\int\frac{1}{x+1}e^{-x}\sin x\,dx = -\frac{1}{2(x+1)}e^{-x}(\sin x+\cos x)-\frac{1}{2}\int\underbrace{\frac{1}{(x+1)^2}e^{-x}(\sin x+\cos x)}_{\text{NOTICE THIS PART}}dx$$

$\frac{1}{x+1}$	$-\frac{1}{(x+1)^2}$
$-\frac{1}{2}e^{-x}(\sin x+\cos x)$	e^{-x}

- $$\int\frac{1}{(x+1)^3}e^{-x}\sin x\,dx = -\frac{1}{2(x+1)^2}e^{-x}\sin x+\frac{1}{2}\int\underbrace{\frac{e^{-x}}{(x+1)^2}(-\sin x+\cos x)}_{\text{NOTICE THIS PART AGAIN}}dx$$

$e^{-x}\sin x$	$e^{-x}(\cos x-\sin x)$
$-\frac{1}{2}\frac{1}{(x+1)^2}$	$\frac{1}{(x+1)^3}$

$$\frac{d}{dx}\left[e^{-x}\sin x\right] = -e^{-x}\sin x+e^{-x}\cos x = e^{-x}(\cos x-\sin x)$$

COLLECTING THE RESULTS SO FAR

$$= \int\frac{1}{(x+1)}e^{-x}\sin x\,dx+\int\frac{1}{(x+1)^2}e^{-x}\sin x\,dx+\int\frac{1}{(x+1)^3}e^{-x}\sin x\,dx$$

$$= -\frac{1}{2(x+1)}e^{-x}(\cos x+\sin x)-\frac{1}{2}\int\frac{1}{(x+1)^2}e^{-x}(\cos x+\sin x)\,dx$$
$$+\int\frac{1}{(x+1)^2}e^{-x}\sin x$$
$$-\frac{1}{2(x+1)^2}e^{-x}\sin x \qquad +\frac{1}{2}\int\frac{1}{(x+1)^2}e^{-x}(\cos x-\sin x)\,dx$$

THESE SHOULD CANCEL

$$= -\frac{1}{2(x+1)}e^{-x}(\cos x+\sin x)-\frac{1}{2(x+1)^2}e^{-x}\sin x$$
$$+\frac{1}{2}\int e^{-x}\left(\frac{1}{(x+1)^2}\right)\left[\cancel{-\cos x-\sin x+2\sin x+\cos x-\sin x}\right]dx.$$

$$= -\frac{1}{2}\left(\frac{e^{-x}}{(x+1)^2}\right)\left[(x+1)(\cos x+\sin x)+\sin x\right]+C$$

$$= -\frac{e^{-x}}{2(x+1)^2}\left[(x+2)\sin x+(x+1)\cos x\right]+C$$

$$= \frac{-e^{-x}\left[(x+2)\sin x+(x+1)\cos x\right]}{2(x+1)^2}+C$$

696. $$\int \frac{\sin^6 x+\cos^6 x}{\sin^2 x\cos^2 x}\,dx = \left[\begin{array}{l} -3x+\tan x-\cot x + C \\ -3x-2\cot 2x + C \end{array}\right.$$

$$\int\frac{\sin^6x+\cos^6x}{\sin^2x\cos^2x}dx = \int\frac{\sin^6x}{\sin^2x\cos^2x}+\frac{\cos^6x}{\sin^2x\cos^2x}dx$$
$$= \int\frac{\sin^4x}{\cos^2x}+\frac{\cos^4x}{\sin^2x}dx = \int\frac{(1-\cos^2x)^2}{\cos^2x}+\frac{(1-\sin^2x)^2}{\sin^2x}dx$$
$$= \int\frac{1-2\cos^2x+\cos^4x}{\cos^2x}+\frac{1-2\sin^2x+\sin^4x}{\sin^2x}dx$$
$$= \int\sec^2x-2+\cos^2x+\operatorname{cosec}^2x-2+\sin^2x\,dx$$
$$= \int\sec^2x+\operatorname{cosec}^2x-4+1\,dx = \int\sec^2x+\operatorname{cosec}^2x-3\,dx$$
$$= \tan x-\cot x-3x+C$$

ALTERNATIVE USING SUM OF CUBES $A^3+B^3 \equiv (A+B)(A^2-AB+B^2)$

$$\int\frac{\sin^6x+\cos^6x}{\sin^2x\cos^2x}dx = \int\frac{(\sin^2x)^3+(\cos^2x)^3}{\sin^2x\cos^2x}dx$$
$$= \int\frac{\cancel{(\sin^2x+\cos^2x)}(\sin^4x-\sin^2x\cos^2x+\cos^4x)}{\sin^2x\cos^2x}dx$$
$$= \int\frac{\sin^4x}{\sin^2x\cos^2x}-\frac{\sin^2x\cos^2x}{\sin^2x\cos^2x}+\frac{\cos^4x}{\sin^2x\cos^2x}dx = \int\frac{\sin^2x}{\cos^2x}-1+\frac{\cos^2x}{\sin^2x}dx$$
$$= \int\tan^2x-1+\cot^2x\,dx = \int(\sec^2x-1)-1+(\operatorname{cosec}^2x-1)\,dx$$
$$= \int\sec^2x+\operatorname{cosec}^2x-3\,dx = \tan x-\cot x-3x+C$$

697. $$\int \frac{3\sin^2 x\cos^2 x}{\left(\cos^3 x-\sin^3 x\right)^2}\,dx=\frac{1}{1-\tan^3 x}+C$$

$$\int \frac{3\sin^2 x\cos^2 x}{(\cos^3 x-\sin^3 x)^3}\,dx=\int \frac{3\sin^2 x\cos^2 x}{\cos^6 x\left(1-\frac{\sin^3 x}{\cos^3 x}\right)^2}\,dx$$

$$=\int \frac{3\sin^2 x\cos^2 x}{\cos^6 x}\left(1-\tan^3 x\right)^{-2}\,dx=\int 3\tan^2 x\sec^2 x\left(1-\tan^3 x\right)^{-2}\,dx$$

BY RECOGNITION OR SUBSTITUTION $u=\tan x$

$$=\left(1-\tan^3 x\right)^{-1}+C=\frac{1}{1-\tan^3 x}+C$$

698. $$\int \frac{\sin^8 x-\cos^8 x}{1-\frac{1}{2}\sin^2 2x}\,dx=\frac{1}{2}\sin 2x+C$$

$$\int \frac{\sin^8 x-\cos^8 x}{1-\frac{1}{2}\sin^2 2x}\,dx=\int \frac{(\sin^4 x)^2-(\cos^4 x)^2}{1-\frac{1}{2}(\sin 2x)^2}\,dx$$

$$=\int \frac{(\sin^4 x-\cos^4 x)(\sin^4 x+\cos^4 x)}{1-\frac{1}{2}(2\sin x\cos x)^2}\,dx=\int \frac{(\sin^4 x-\cos^4 x)(\sin^4 x+\cos^4 x)}{1-2\sin^2 x\cos^2 x}\,dx$$

$$=\int \frac{(\sin^4 x-\cos^4 x)(\sin^4 x+\cos^4 x)}{(1)^2-2\sin^2 x\cos^2 x}\,dx=\int \frac{(\sin^4 x-\cos^4 x)(\sin^4 x+\cos^4 x)}{(\cos^2 x+\sin^2 x)^2-2\sin^2 x\cos^2 x}\,dx$$

$$=\int \frac{(\sin^4 x-\cos^4 x)(\sin^4 x+\cos^4 x)}{\cos^4 x+\cancel{2\cos^2 x\sin^2 x}+\sin^4 x-\cancel{2\sin^2 x\cos^2 x}}\,dx$$

$$=\int \frac{(\sin^4 x-\cos^4 x)\cancel{(\cos^4 x+\sin^4 x)}}{\cancel{\cos^4 x+\sin^4 x}}\,dx=\int \sin^4 x-\cos^4 x\,dx$$

$$=\int (\sin^2 x-\cos^2 x)\cancel{(\sin^2 x+\cos^2 x)}\,dx=\int \cos 2x\,dx$$

$$=\frac{1}{2}\sin 2x+C$$

699. $$\int 12x^2\arctan x\,dx=4x^3\arctan x-2x^2+2\ln\left(x^2+1\right)+\text{C}$$

ATTEMPT THIS WITHOUT SUBSTITUTIONS/PARTS, JUST A BIT OF COMMON SENSE

$$\frac{d}{dx}\left[4x^3\arctan x\right]=12x^2\arctan x+4x^3\times\frac{1}{1+x^2}$$

INTEGRATE WITH RESPECT TO x

$$\int \frac{d}{dx}\left[4x^3\arctan x\right]dx=\int 12x^2\arctan x\,dx+\int \frac{4x^3}{1+x^2}\,dx$$

$$4x^3\arctan x=\int 12x^2\arctan x\,dx+\int \frac{4x(x^2+1)+4x}{x^2+1}\,dx$$

$$4x^3\arctan x=\int 12x^2\arctan x\,dx+\int 4x+\frac{4x}{x^2+1}\,dx$$

$$4x^3\arctan x=\int 12x^2\arctan x\,dx+\int 4x\,dx+2\int \frac{2x}{x^2+1}\,dx$$

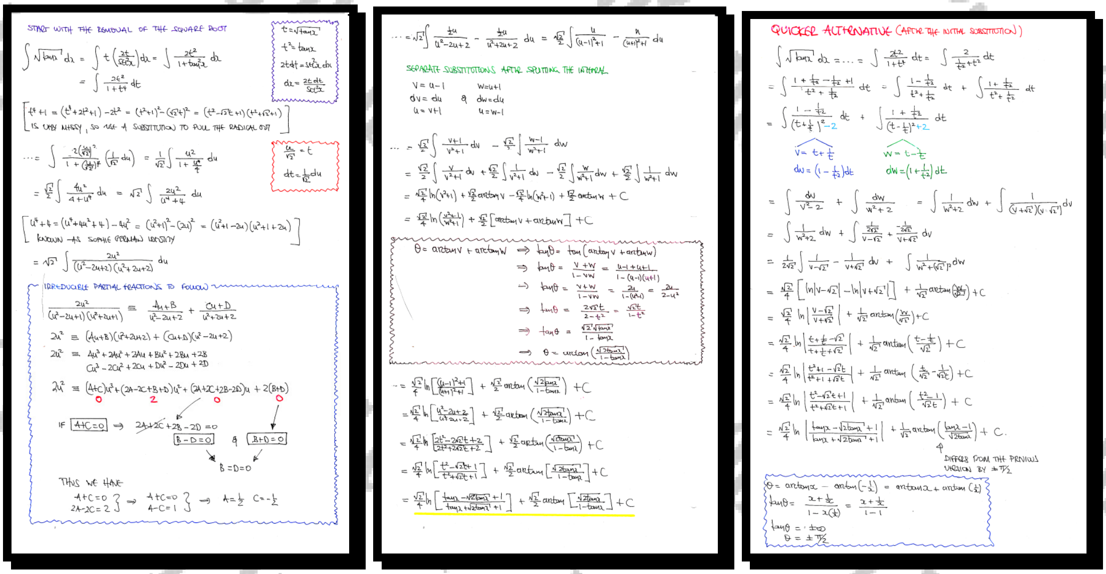
$$4x^3\arctan x=\int 12x^2\arctan x\,dx+2x^2+2\ln(x^2+1)+C$$

REARRANGE TO GET RESULT

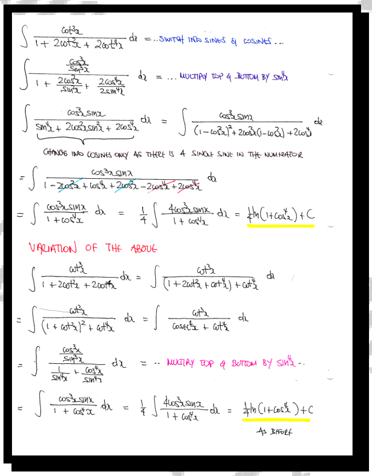
$$\int 12x^2\arctan x\,dx=4x^3\arctan x-2x^2-2\ln(x^2+1)$$

700. $\displaystyle\int \sqrt{\tan x}\, dx = \ldots$

$$\ldots = \begin{cases} \dfrac{\sqrt{2}}{4}\ln\left|\dfrac{\tan x - \sqrt{2\tan x}+1}{\tan x+\sqrt{2\tan x}+1}\right| + \dfrac{\sqrt{2}}{2}\arctan\left[\dfrac{\sqrt{2\tan x}}{1-\tan x}\right] + C \\ \dfrac{\sqrt{2}}{4}\ln\left|\dfrac{\tan x - \sqrt{2\tan x}+1}{\tan x+\sqrt{2\tan x}+1}\right| + \dfrac{\sqrt{2}}{2}\arctan\left[\dfrac{-1+\tan x}{\sqrt{2\tan x}}\right] + C \end{cases}$$



701. $\displaystyle\int \frac{\cot^3 x}{1+2\cot^2 x+2\cot^4 x}\, dx = -\frac{1}{4}\ln\left(1+\cos^4 x\right) + C$



702. $\displaystyle\int \left(2x^2+1\right)e^{x^2}\ dx = x e^{x^2} + C$

LOOKING AT THE EXPONENTIAL FUNCTION

$\frac{d}{dx}\left[e^{x+\frac{1}{x}}\right] = e^{x+\frac{1}{x}}\left(1-\frac{1}{x^2}\right)$ REARRANGE TO OBTAIN THIS EXPRESSION IN THE INTEGRAND

$\int \left(1+x-\frac{1}{x}\right)e^{x+\frac{1}{x}}\,dx = \int \left[1+x\left(1-\frac{1}{x^2}\right)\right]e^{x+\frac{1}{x}}\,dx$

$= \int e^{x+\frac{1}{x}} + x\left(1-\frac{1}{x^2}\right)e^{x+\frac{1}{x}}\,dx$

$= \int e^{x+\frac{1}{x}}\,dx + \int \underbrace{x\left(1-\frac{1}{x^2}\right)e^{x+\frac{1}{x}}}\,dx$

INTEGRATION BY PARTS

x	1
$e^{x+\frac{1}{x}}$	$\left(1-\frac{1}{x^2}\right)e^{x+\frac{1}{x}}$

$= \cancel{\int e^{x+\frac{1}{x}}\,dx} + xe^{x+\frac{1}{x}} - \cancel{\int e^{x+\frac{1}{x}}\,dx}$

$= xe^{x+\frac{1}{x}} + C$

703. $\displaystyle\int \left(1+x-\frac{1}{x}\right)e^{x+\frac{1}{x}}\ dx = x e^{x+\frac{1}{x}} + C$

LOOKING AT THE EXPONENTIAL FUNCTION

$\frac{d}{dx}\left[e^{x+\frac{1}{x}}\right] = e^{x+\frac{1}{x}}\left(1-\frac{1}{x^2}\right)$ REARRANGE TO OBTAIN THIS EXPRESSION IN THE INTEGRAND

$\int \left(1+x-\frac{1}{x}\right)e^{x+\frac{1}{x}}\,dx = \int \left[1+x\left(1-\frac{1}{x^2}\right)\right]e^{x+\frac{1}{x}}\,dx$

$= \int e^{x+\frac{1}{x}} + x\left(1-\frac{1}{x^2}\right)e^{x+\frac{1}{x}}\,dx$

$= \int e^{x+\frac{1}{x}}\,dx + \int \underbrace{x\left(1-\frac{1}{x^2}\right)e^{x+\frac{1}{x}}}\,dx$

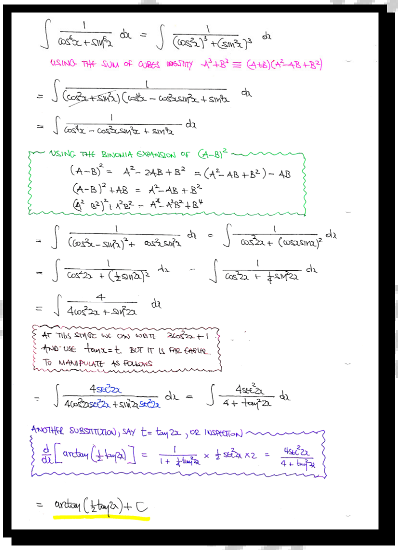
INTEGRATION BY PARTS

x	1
$e^{x+\frac{1}{x}}$	$\left(1-\frac{1}{x^2}\right)e^{x+\frac{1}{x}}$

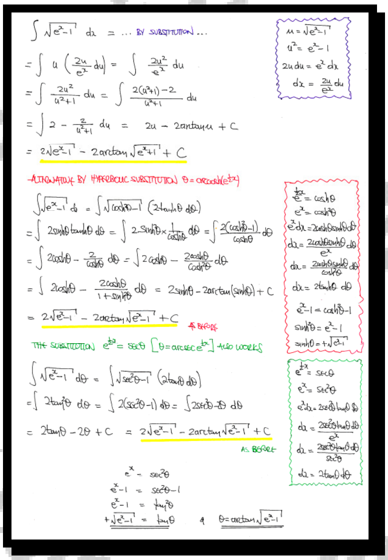
$= \cancel{\int e^{x+\frac{1}{x}}\,dx} + xe^{x+\frac{1}{x}} - \cancel{\int e^{x+\frac{1}{x}}\,dx}$

$= xe^{x+\frac{1}{x}} + C$

704. $$\int \frac{1}{\cos^6 x+\sin^6 x}\,dx=\arctan\left(\frac{1}{2}\tan 2x\right)+C$$



705. $$\int \sqrt{e^x-1}\,dx=2\sqrt{e^x-1}-2\arctan\left(\sqrt{e^x-1}\right)+C$$



706. $$\int \frac{1}{\sqrt{e^{x}-1}}\,dx = 2\arctan\left(\sqrt{e^{x}-1}\right) + C$$

$$\int \frac{1}{\sqrt{e^{x}-1}}\,dx = \text{BY SUBSTITUTION}$$

$$= \int \frac{1}{u}\left(\frac{2u}{u^2+1}\,du\right) = \int \frac{2}{u^2+1}\,du$$

$$= 2\arctan u + C = 2\arctan\sqrt{e^{x}-1} + C$$

$u = \sqrt{e^{x}-1}$

$u^2 = e^{x}-1$

$2u\,du = e^{x}\,dx$

$dx = \frac{2u}{e^{x}}\,du$

$dx = \frac{2u}{u^2+1}\,du$

THE SUBSTITUTIONS $\theta = \text{arcosh}\left(e^{\frac{1}{2}x}\right)$ & $\theta = \text{arcsec}\left(e^{\frac{1}{2}x}\right)$ ALSO WORK, BUT ARE "LENGTHY"...

707. $$\int \frac{1}{x^4+3x^2+2}\,dx = \ln\left(x^2+2\right) - \frac{1}{2}\ln\left(x^2+1\right) + C$$

$$\int \frac{x^3}{x^4+3x^2+2}\,dx = \int \frac{x^3}{(x^2+1)(x^2+2)}\,dx = \dots \text{ IRREDUCIBLE PARTIAL FRACTIONS}$$

$$\frac{x^3}{(x^2+1)(x^2+2)} \equiv \frac{Ax+B}{x^2+1} + \frac{Cx+D}{x^2+2}$$

$$x^3 \equiv (Ax+B)(x^2+2) + (Cx+D)(x^2+1)$$

$$x^3 \equiv \begin{bmatrix} Ax^3 + Bx^2 + 2Ax + 2B \\ Cx^3 + Dx^2 + Cx + D \end{bmatrix}$$

$A+C=1 \quad B+D=0 \quad 2A+C=0 \quad 2B+D=0$

$$\left.\begin{matrix} A+C=1 \\ 2A+C=0 \end{matrix}\right\} \Rightarrow A=-1,\ C=2$$

$$\left.\begin{matrix} B+D=0 \\ 2B+D=0 \end{matrix}\right\} \Rightarrow B=D=0$$

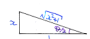
$$= \int \frac{-x}{x^2+1} + \frac{2x}{x^2+2}\,dx = -\frac{1}{2}\ln(x^2+1) + \ln(x^2+2) + C$$

$$= \ln(x^2+2) - \frac{1}{2}\ln(x^2+1)$$

708. $$\int x\arccos\left[\frac{1-x^2}{1+x^2}\right]dx = -x + \left(1+x^2\right)\arctan x + \text{C}$$

THE ARGUMENT OF THE ARCCOS IS VERY SPECIAL

- LET $x = \tan\frac{\theta}{2}$ OR ACTUAL SUBSTITUTION $\theta = 2\arctan x$
- $\frac{1-x^2}{1+x^2} = \frac{1-\tan^2\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}} = \frac{1-\tan^2\theta/2}{\sec^2\theta/2} = \cos^2\frac{\theta}{2} - \tan^2\frac{\theta}{2}\cos^2\frac{\theta}{2} = \cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2} = \cos\theta$
- $dx = \sec^2\frac{\theta}{2} \times \frac{1}{2}\,d\theta$



$\Rightarrow \sec^2\frac{\theta}{2} = x^2+1$

$$\int x\arccos\left(\frac{1-x^2}{1+x^2}\right)dx = \int \tan\frac{\theta}{2}\,\underbrace{\arccos(\cos\theta)}_{\theta}\left(\frac{1}{2}\sec^2\frac{\theta}{2}\,d\theta\right)$$

$$= \int \frac{1}{2}\theta\left[\tan\frac{\theta}{2}\sec^2\frac{\theta}{2}\right]d\theta = \dots \text{ INTEGRATION BY PARTS } \dots$$

$\frac{1}{2}\theta$	$\frac{1}{2}$
$\tan^2\frac{\theta}{2}$	$\tan\frac{\theta}{2}\sec^2\frac{\theta}{2}$

$$= \frac{1}{2}\theta\tan^2\frac{\theta}{2} - \int \frac{1}{2}\tan^2\frac{\theta}{2}\,d\theta = \frac{1}{2}\theta\tan^2\frac{\theta}{2} - \frac{1}{2}\int \sec^2\frac{\theta}{2} - 1\,d\theta$$

$$= \frac{1}{2}\theta\tan^2\frac{\theta}{2} - \frac{1}{2}\left[2\tan\frac{\theta}{2}\right] + \frac{1}{2}\theta + C$$

$$= \frac{1}{2}\theta\left(\tan^2\frac{\theta}{2}+1\right) - \tan\frac{\theta}{2} + C = \frac{1}{2}\theta\sec^2\frac{\theta}{2} - \frac{1}{2}\tan\frac{\theta}{2} + C$$

$$= \frac{1}{2}(2\arctan x)(x^2+1) - x + C = -x + (1+x^2)\arctan x + C$$

709. $$\int 2x^2\sec^2 x\tan x \ dx = x^2\sec^2 x - 2x\sec x + 2\ln|\sec x| \ + \ C$$

$\int 2x^2\sec^2 x\tan x \ dx = \dots$ INTEGRATION BY PARTS

x^2	$2x$
$\tan^2 x$	$2\sec^2 x\tan x$

$= x^2\tan^2 x - \int 2x\tan^2 x \ dx$

$= x^2\tan^2 x - \int 2x(\sec^2 x - 1)\ dx = x^2\tan^2 x - \int 2x\sec^2 x - 2x \ dx.$

$= x^2\tan^2 x + \int 2x \ dx - \int 2x\sec^2 x \ dx$

BY PARTS

$2x$	2
$\tan x$	$\sec^2 x$

$= x^2\tan^2 x + x^2 - \left[2x\tan x - \int 2\tan x \ dx\right]$

$= x^2\tan^2 x + x^2 - 2x\tan x + 2\int \tan x \ dx$

$= x^2(\tan^2 x + 1) - 2x\tan x + 2\ln|\sec x| + C$

$= x^2\sec^2 x - 2x\tan x + 2\ln|\sec x| + C$

ALTERNATIVE (MORE EFFECTIVE)

$\int 2x^2\sec^2 x\tan x \ dx = \dots$ BY PARTS AGAIN

x^2	$2x$
$\sec^2 x$	$2\sec^2 x\tan x$

$= x^2\sec^2 x - \underbrace{\int 2x\sec^2 x \ dx}$

BY PARTS AGAIN, IDENTICAL TO ABOVE

$= x^2\sec^2 x - \left[2x\tan x - \int 2\tan x \ dx\right]$

$= x^2\sec^2 x - 2x\tan x + \int 2\tan x \ dx$

$= x^2\sec^2 x - 2x\tan x + 2\ln|\sec x| + C$

AS BEFORE

710. $$\int \ln\left(\sqrt{x} + \sqrt{x+1}\right) dx = \tfrac{1}{2}(2x+1)\ln\left[\sqrt{x} + \sqrt{x+1}\right] - \tfrac{1}{2}\sqrt{x^2+x} \ + \ C$$

$\int \ln\left(\sqrt{x} + \sqrt{x+1}\right) dx = \dots$ THIS IS SIMPLY THE $\text{arsinh}\sqrt{x}$!! ... SUBSTITUTION

$\theta = \text{arsinh}\sqrt{x}$
$\sinh\theta = \sqrt{x}$
$x = \sinh^2\theta$
$dx = 2\sinh\theta\cosh\theta \ d\theta$
$x+1 = \cosh^2\theta$

$= \int \theta(2\sinh\theta\cosh\theta \ d\theta) = \int \theta\sinh 2\theta \ d\theta$

INTEGRATION BY PARTS OR "INSPECTION"

$\frac{d}{d\theta}(\theta\cosh 2\theta) = 1\times\cosh 2\theta + \theta\times 2\sinh 2\theta$

$\int \frac{d}{d\theta}(\theta\cosh 2\theta)\, d\theta = \int \cosh 2\theta \ d\theta + 2\int \theta\sinh 2\theta \ d\theta$

$\theta\cosh 2\theta = \frac{1}{2}\sinh 2\theta + 2\int \theta\sinh 2\theta \ d\theta$

$\int \theta\sinh 2\theta \ d\theta = \frac{1}{2}\theta\cosh 2\theta - \frac{1}{4}\sinh 2\theta + C$

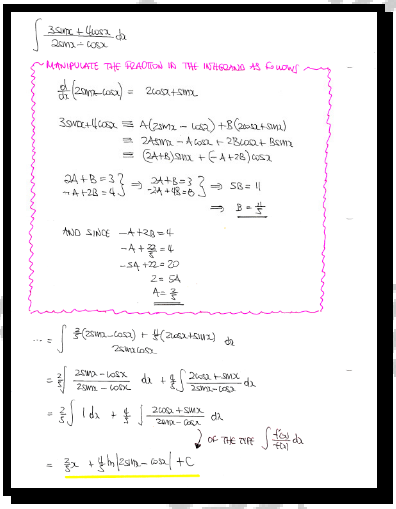
$\rightarrow \int \ln\left(\sqrt{x} + \sqrt{x+1}\right) dx = \frac{1}{2}\theta(\cosh^2\theta + \sinh^2\theta) - \frac{1}{2}\sinh\theta\cosh\theta + C$

$= \frac{1}{2}\text{arsinh}\sqrt{x}\left[x+1+x\right] - \frac{1}{2}\sqrt{x}\sqrt{x+1} + C$

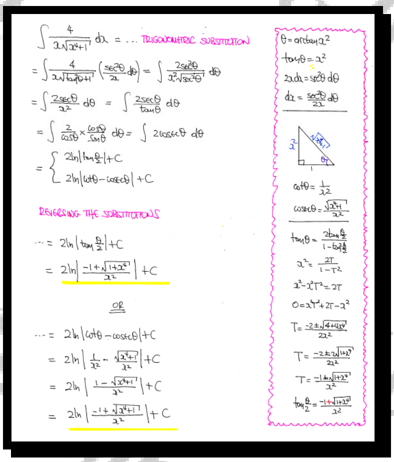
$= \frac{1}{2}(2x+1)\ln\left[\sqrt{x} + \sqrt{x+1}\right] - \frac{1}{2}\sqrt{x^2+x} + C$

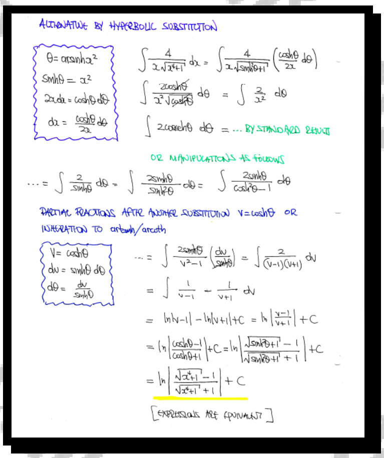
THE SUBSTITUTION $\theta = \arctan\sqrt{x}$ ALSO WORKS BUT IT IS EXTREMELY LENGTHY

711. $$\int \frac{3\sin x + 4\cos x}{2\sin x - \cos x}\,dx = \frac{2}{5}x + \frac{11}{5}\ln\left|2\sin x - 4\cos x\right| + C$$

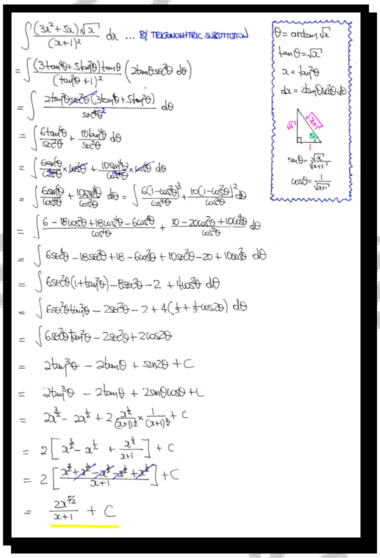


712. $$\int \frac{4}{x\sqrt{x^4+1}}\,dx = \begin{cases} \ln\left[\dfrac{\sqrt{x^4+1}-1}{\sqrt{x^4+1}+1}\right] + C \\[2ex] 2\ln\left[\dfrac{1-\sqrt{x^4+1}}{x^2}\right] + C \end{cases}$$

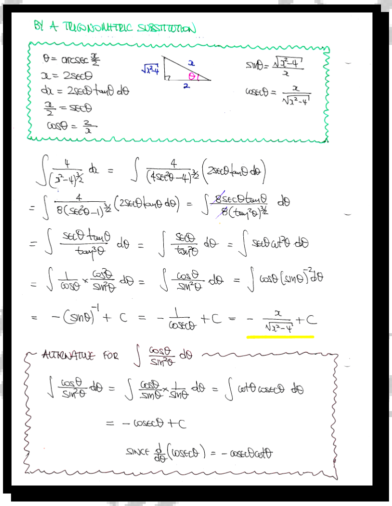


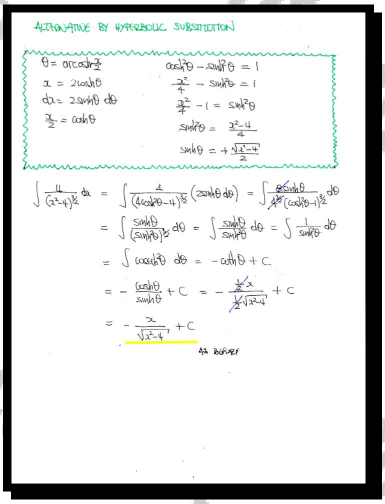


713. $$\int \frac{\left(3x^2+5x\right)\sqrt{x}}{(x+1)^2}\,dx = \frac{2x^{\frac{5}{2}}}{x+1} + C$$



714. $$\int \frac{4}{\left(x^2-4\right)^{\frac{3}{2}}}\,dx = -\frac{x}{\sqrt{x^2-4}} + C$$





715. $$\int \frac{1}{(1-x)\sqrt{1-2x}}\,dx = -2\arctan\left(\sqrt{1-2x}\right) + C$$

BY TRIGONOMETRIC SUBSTITUTIONS

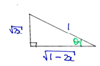
$\theta = \arcsin\sqrt{2x}$

$\sin\theta = \sqrt{2x}$

$2x = \sin^2\theta$

$2\,dx = 2\sin\theta\cos\theta\,d\theta$

$dx = \sin\theta\cos\theta\,d\theta$



$\sec\theta = \frac{1}{\sqrt{1-2x}}$

$$\int \frac{1}{(1-x)\sqrt{1-2x}}\,dx = \int \frac{1}{\left(1-\frac{1}{2}\sin^2\theta\right)\left(\frac{1}{\sec\theta}\right)}(\sin\theta\cos\theta\,d\theta)$$

$$= \int \frac{\sin\theta\cancel{\cos\theta}}{\frac{1}{2}(2-\sin^2\theta)\cancel{\cos\theta}}\,d\theta = \int \frac{2\sin\theta}{2-\sin^2\theta}\,d\theta$$

$$= \int \frac{2\sin\theta\sec^2\theta}{2\sec^2\theta - \sin^2\theta\sec^2\theta}\,d\theta = \int \frac{2\tan\theta\sec\theta}{2\sec^2\theta - \tan^2\theta}\,d\theta$$

LOOKING AT THE NUMERATOR, $\frac{d}{d\theta}(\sec\theta) = \sec\theta\tan\theta$

$$= \int \frac{2\tan\theta\sec\theta}{2\sec^2\theta - (\sec^2\theta - 1)}\,d\theta = \int \frac{2\tan\theta\sec\theta}{\sec^2\theta + 1}$$

ANOTHER SUBSTITUTION OR RECOGNITION

$$= 2\arctan(\sec\theta) + C = 2\arctan\left(\frac{1}{\sqrt{1-2x}}\right) + C$$

$$= 2\operatorname{arccot}\left(\sqrt{1-2x}\right) + C = 2\left[\frac{\pi}{2} - \arctan\sqrt{1-2x}\right] + C$$

$$= -2\arctan\sqrt{1-2x} + (\pi + C) = -2\arctan\sqrt{1-2x} + C$$

"AT THE END"

- $\arctan\frac{1}{x} = \operatorname{arccot}x \iff \arctan x = \operatorname{arccot}\frac{1}{x}$
- $\arctan x + \operatorname{arccot} x = \frac{\pi}{2}$

716. $$\int \sqrt{\tan x} - \sqrt{\cot x}\,dx = -\sqrt{2}\ln\left[\cos x + \sin x + \sqrt{\sin 2x}\right] + C$$

$$\int \sqrt{\tan x} - \sqrt{\cot x}\,dx = \int \frac{\sqrt{\sin x}}{\sqrt{\cos x}} - \frac{\sqrt{\cos x}}{\sqrt{\sin x}}\,dx = \int \frac{\sin x - \cos x}{\sqrt{\sin x\cos x}}\,dx$$

$$= \int \frac{\sin x - \cos x}{\sqrt{\frac{1}{2}\sin 2x}}\,dx = \frac{1}{\frac{1}{\sqrt{2}}}\int \frac{\sin x - \cos x}{\sqrt{\sin 2x}}\,dx$$

MANIPULATE THE DENOMINATOR AS FOLLOWS – $\sin 2x$ WAS NOT NECESSARY

$$\sqrt{\sin 2x} = \sqrt{\sin 2x + 1 - 1} = \sqrt{2\sin x\cos x + \cos^2 x + \sin^2 x - 1}$$

$$= \sqrt{(\cos x + \sin x)^2 - 1}$$

NEXT NOTE THAT $\frac{d}{dx}(\cos x + \sin x) = \cos x - \sin x =$ "NUMERATOR"

$u = \cos x + \sin x$

$\frac{du}{dx} = \cos x - \sin x$

$dx = \frac{du}{\cos x - \sin x}$

$$\ldots = \sqrt{2}\int \frac{\sin x - \cos x}{\sqrt{(\cos x + \sin x)^2 - 1}}\,dx = \sqrt{2}\int \frac{\overset{-1}{\cancel{\sin x - \cos x}}}{\sqrt{u^2 - 1}}\,\frac{du}{\cancel{\cos x - \sin x}}$$

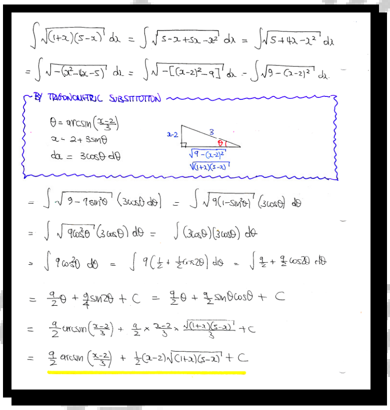
$$\ldots = -\sqrt{2}\int \frac{1}{\sqrt{u^2 - 1}}\,du = -\sqrt{2}\,\operatorname{arcosh} u + C = -\sqrt{2}\ln\left[u + \sqrt{u^2 - 1}\right] + C$$

$$= -\sqrt{2}\ln\left[\cos x + \sin x + \sqrt{(\cos x + \sin x)^2 - 1}\right] + C$$

$$= -\sqrt{2}\ln\left[\cos x + \sin x + \sqrt{\cancel{\cos^2 x + \sin^2 x} + 2\cos x\sin x \cancel{- 1}}\right] + C$$

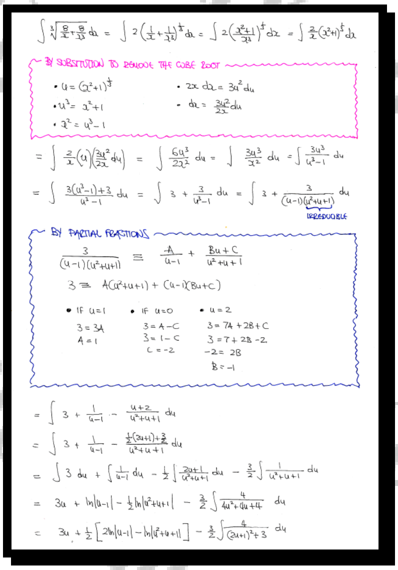
$$= -\sqrt{2}\ln\left[\cos x + \sin x + \sqrt{\sin 2x}\right] + C$$

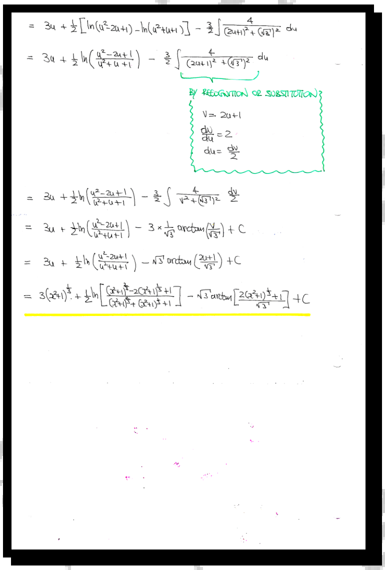
717. $$\int \sqrt{(1+x)(5-x)}\,dx = \frac{9}{2}\arcsin\left[\frac{1}{3}(x-2)\right] + \frac{1}{2}(x-2)\sqrt{(1+x)(5-x)} + C$$



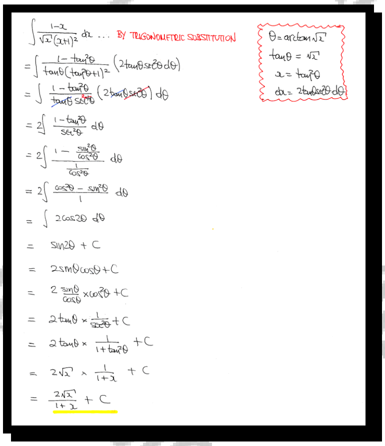
718. $$\int \sqrt[3]{\frac{8}{x}+\frac{8}{x^3}}\,dx = \ldots$$

$$3\left(x^2+1\right)^{\frac{1}{3}} + \frac{1}{2}\ln\left[\frac{\left(x^2+1\right)^{\frac{2}{3}} - 2\left(x^2+1\right)^{\frac{1}{3}} + 1}{\left(x^2+1\right)^{\frac{2}{3}} + \left(x^2+1\right)^{\frac{1}{3}} + 1}\right] - \sqrt{3}\arctan\left[\frac{2\left(x^2+1\right)^{\frac{1}{3}} + 1}{\sqrt{3}}\right] + C$$





719. $$\int \frac{1-x}{\sqrt{x}\,(x+1)^2}\,dx = \frac{2\sqrt{x}}{x+1} + C$$



$\int \frac{1-x}{\sqrt{x}(x+1)^2}\,dx \ldots$ BY TRIGONOMETRIC SUBSTITUTION

$\theta = \arctan\sqrt{x}$

$\tan\theta = \sqrt{x}$

$x = \tan^2\theta$

$dx = 2\tan\theta\sec^2\theta\,d\theta$

$= \int \frac{1-\tan^2\theta}{\tan\theta(\tan^2\theta+1)^2}(2\tan\theta\sec^2\theta\,d\theta)$

$= \int \frac{1-\tan^2\theta}{\cancel{\tan\theta}\sec^{\cancel{4}\,2}\theta}(2\cancel{\tan\theta}\cancel{\sec^2\theta})\,d\theta$

$= 2\int \frac{1-\tan^2\theta}{\sec^2\theta}\,d\theta$

$= 2\int \frac{1-\frac{\sin^2\theta}{\cos^2\theta}}{\frac{1}{\cos^2\theta}}\,d\theta$

$= 2\int \frac{\cos^2\theta-\sin^2\theta}{1}\,d\theta$

$= \int 2\cos 2\theta\,d\theta$

$= \sin 2\theta + C$

$= 2\sin\theta\cos\theta + C$

$= 2\frac{\sin\theta}{\cos\theta}\times\cos^2\theta + C$

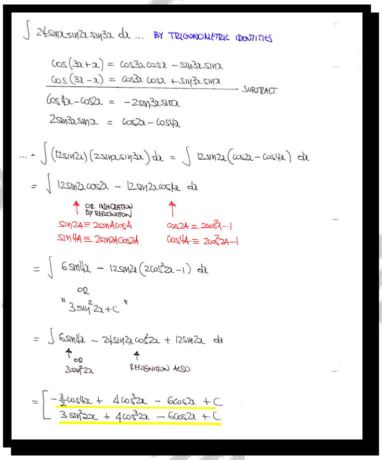
$= 2\tan\theta\times\frac{1}{\sec^2\theta} + C$

$= 2\tan\theta\times\frac{1}{1+\tan^2\theta} + C$

$= 2\sqrt{x}\times\frac{1}{1+x} + C$

$= \frac{2\sqrt{x}}{1+x} + C$

720. $$\int 24\sin x\sin 2x\sin 3x\,dx = \begin{cases} 3\sin^2 2x + 4\cos^3 2x - 6\cos 2x + C \\ -\frac{3}{2}\cos 4xx + 4\cos^3 2x - 6\cos 2x + C \end{cases}$$



$\int 24\sin x\sin 2x\sin 3x\,dx \ldots$ BY TRIGONOMETRIC IDENTITIES

$\cos(3x+x) = \cos 3x\cos x - \sin 3x\sin x$

$\cos(3x-x) = \cos 3x\cos x + \sin 3x\sin x$ SUBTRACT

$\cos 4x - \cos 2x = -2\sin 3x\sin x$

$2\sin 3x\sin x = \cos 2x - \cos 4x$

$\ldots = \int (12\sin 2x)(2\sin x\sin 3x)\,dx = \int 12\sin 2x(\cos 2x - \cos 4x)\,dx$

$= \int 12\sin 2x\cos 2x - 12\sin 2x\cos 4x\,dx$

OR INTEGRATION BY RECOGNITION

$\sin 2A \equiv 2\sin A\cos A$

$\sin 4A \equiv 2\sin 2A\cos 2A$

$\cos 2A \equiv 2\cos^2 A - 1$

$\cos 4A \equiv 2\cos^2 2A - 1$

$= \int 6\sin 4x - 12\sin 2x(2\cos^2 2x - 1)\,dx$

OR

"$3\sin^2 2x + C$"

$= \int 6\sin 4x - 24\sin 2x\cos^2 2x + 12\sin 2x\,dx$

OR $3\sin^2 2x$

RECOGNITION ALSO

$= \begin{cases} -\frac{3}{2}\cos 4x + 4\cos^3 2x - 6\cos 2x + C \\ 3\sin^2 2x + 4\cos^3 2x - 6\cos 2x + C \end{cases}$

721. $$\int \frac{\sec^2 x}{\sqrt{\sec x+\tan x}}\,dx=(\sec x+\tan x)^{\frac{1}{2}}-\frac{1}{3}(\sec x+\tan x)^{-\frac{3}{2}}+C$$

$$\int \frac{\sec^2 x}{\sqrt{\sec x+\tan x}}\,dx \ \ldots$$ LOOKING AT THE DERIVATIVE "IN THE DENOMINATOR"

$$\frac{d}{dx}(\sec x+\tan x)=\sec^2 x+\sec x\tan x$$

$$=\frac{1}{2}\int \frac{\sec^2 x+\sec x\tan x-\sec x\tan x+\sec^2 x}{\sqrt{\sec x+\tan x}}\,dx$$

$$=\frac{1}{2}\int \frac{\sec^2 x+\sec x\tan x}{\sqrt{\sec x+\tan x}}\,dx+\frac{1}{2}\int \frac{\sec^2 x-\sec x\tan x}{\sqrt{\sec x+\tan x}}\,dx$$

$$=\frac{1}{2}\int \frac{\sec^2 x+\sec x\tan x}{\sqrt{\sec x+\tan x}}\,dx+\frac{1}{2}\int \frac{\sec x(\sec x-\tan x)(\sec x+\tan x)}{\sqrt{\sec x+\tan x}\,(\sec x+\tan x)}\,dx$$

$$=\frac{1}{2}\int \frac{\sec^2 x+\sec x\tan x}{\sqrt{\sec x+\tan x}}\,dx+\frac{1}{2}\int \frac{\sec x(\cancel{\sec^2 x-\tan^2 x})}{(\sec x+\tan x)^{3/2}}\,dx$$

$$=\frac{1}{2}\int \frac{\sec^2 x+\sec x\tan x}{\sqrt{\sec x+\tan x}}\,dx+\frac{1}{2}\int \frac{\sec x}{(\sec x+\tan x)^{3/2}}\,dx$$

$$=\frac{1}{2}\int \frac{\sec^2 x+\sec x\tan x}{\sqrt{\sec x+\tan x}}\,dx+\frac{1}{2}\int \frac{\sec x(\sec x+\tan x)}{(\sec x+\tan x)^{3/2}(\sec x+\tan x)}\,dx$$

$$=\frac{1}{2}\int \frac{\sec^2 x+\sec x\tan x}{(\sec x+\tan x)^{1/2}}\,dx+\frac{1}{2}\int \frac{\sec^2 x+\sec x\tan x}{(\sec x+\tan x)^{5/2}}\,dx$$

THESE ARE NOW "EXACT DIFFERENTIALS" OF THE ARGUMENT OF THE RADICAL

$$=\frac{1}{2}\int (\sec^2 x+\sec x\tan x)(\sec x+\tan x)^{-\frac{1}{2}}\,dx+\frac{1}{2}\int(\sec^2 x+\sec x\tan x)(\sec x+\tan x)^{-\frac{5}{2}}\,dx$$

$$=\frac{1}{2}\times\frac{1}{\frac{1}{2}}(\sec x+\tan x)^{\frac{1}{2}}+\frac{1}{2}\times\frac{1}{-\frac{3}{2}}(\sec x+\tan x)^{-\frac{3}{2}}+C$$

$$=(\sec x+\tan x)^{\frac{1}{2}}-\frac{1}{3}(\sec x+\tan x)^{-\frac{3}{2}}+C$$

722. $$\int \ln(\ln x)+\frac{1}{(\ln x)^2}\,dx=x\ln(\ln x)-\frac{x}{\ln x}+C$$

$$\int \ln(\ln x)+\frac{1}{(\ln x)^2}\,dx=\int \ln(\ln x)\,dx+\int\frac{1}{(\ln x)^2}\,dx$$

PROCEED BY INTEGRATION BY PARTS IN EACH INTEGRAL SEPARATELY, WITH THE HOPE OF CANCELLING THINGS OUT

- $$\int 1\times\ln(\ln x)\,dx=x\ln(\ln x)-\underbrace{\int\frac{1}{\ln x}\,dx}$$

$\ln(\ln x)$	$\frac{1}{x\ln x}$
x	1

NOT INTEGRABLE SO SOMEHOW WE NEED TO CANCEL IT

- $$\int\frac{1}{(\ln x)^2}\,dx=\cancel{\frac{x}{(\ln x)^2}+2\int\frac{1}{(\ln x)^3}\,dx}$$

$(\ln x)^{-2}$	$-2(\ln x)^{-3}\frac{1}{x}$
x	1

FAILURE BUT THIS LINE GIVES AN INSIGHT FOR THE POWERS OF $\ln x$

ONE MORE INTEGRATION BY PARTS MIGHT "MAP $\int\frac{1}{\ln x}\,dx$ TO $\int\frac{1}{(\ln x)^2}$"

- $$\int\frac{1}{\ln x}\,dx=\frac{x}{\ln x}+\int\frac{1}{(\ln x)^2}\,dx$$

$$\int\frac{1}{(\ln x)^2}\,dx=-\frac{x}{\ln x}+\int\frac{1}{\ln x}\,dx$$

$(\ln x)^{-1}$	$-\frac{1}{x}(\ln x)^{-2}$
x	1

FINALLY WE HAVE BY ADDING

$$\int\ln(\ln x)\,dx=x\ln(\ln x)-\int\frac{1}{\ln x}\,dx \longleftarrow$$ 1ST INTEGRATION BY PARTS

$$\int\frac{1}{(\ln x)^2}\,dx=-\frac{x}{\ln x}+\int\frac{1}{\ln x}\,dx \longleftarrow$$ 3RD INTEGRATION BY PARTS

$$\int\ln(\ln x)+\frac{1}{(\ln x)^2}\,dx=x\ln(\ln x)-\frac{x}{\ln x}+C$$

723. $\displaystyle\int \sqrt{(2x+5)(2x-3)}\,dx = \ldots$

$$\frac{1}{4}(2x+1)\sqrt{(2x+5)(2x-3)} - 4\ln\left[2x+1+\sqrt{(2x+5)(2x-3)}\right] + C$$

$$\int \sqrt{(2x+5)(2x-3)}\,dx = \int \sqrt{4x^2+4x-15}\,dx = \int \sqrt{(2x+1)^2-16}\,dx$$

BY A HYPERBOLIC SUBSTITUTION

- $\theta = \text{arccosh}\left(\frac{2x+1}{4}\right)$
- $2x+1 = 4\cosh\theta$
- $2\,dx = 4\sinh\theta\,d\theta$
- $dx = 2\sinh\theta\,d\theta$

$(4\cosh\theta)^2 - (4\sinh\theta)^2 = 16$

$16\cosh^2\theta - 16\sinh^2\theta = 16$

$16\cosh^2\theta - 16 = 16\sinh^2\theta$

$4\sinh\theta = +\sqrt{16\cosh^2\theta - 16}$

$= \int \sqrt{(4\cosh\theta)^2-16}\,(2\sinh\theta\,d\theta) = \int \sqrt{16(\cosh^2\theta-1)}\,(2\sinh\theta\,d\theta)$

$= \int \sqrt{16\sinh^2\theta}\,(2\sinh\theta)\,d\theta = \int 8\sinh^2\theta\,d\theta$

$\sin^2\theta \equiv \frac{1}{2} - \frac{1}{2}\cos 2\theta$

$-\sinh^2\theta = \frac{1}{2} - \frac{1}{2}\cosh 2\theta$

$= \int 8\left(\frac{1}{2}\cosh 2\theta - \frac{1}{2}\right)d\theta = \int 4\cosh 2\theta - 4\,d\theta$

$= 2\sinh 2\theta - 4\theta + C = 4\sinh\theta\cosh\theta - 4\theta + C$

$= \frac{1}{4}(4\sinh\theta)(4\cosh\theta) - 4\theta + C$

$= \frac{1}{4}\left[4\cosh\theta\right]\left[\sqrt{16\cosh^2\theta - 16}\right] - 4\theta + C$

$= \frac{1}{4}(2x+1) \times \sqrt{(2x+1)^2-16} - 4\,\text{arccosh}\left(\frac{2x+1}{4}\right) + C$

$= \frac{1}{4}(2x+1)\sqrt{4x^2+4x-15} - 4\ln\left[\frac{2x+1}{4} + \sqrt{\left(\frac{2x+1}{4}\right)^2 - 1}\right] + C$

$= \frac{1}{4}(2x+1)\sqrt{(2x+5)(2x-3)} - 4\ln\left[\frac{2x+1}{4} + \sqrt{\frac{(2x+1)^2-16}{16}}\right] + C$

$= \frac{1}{4}(2x+1)\sqrt{(2x+5)(2x-3)} - 4\ln\left[\frac{1}{4}\left[(2x+1) + \sqrt{(2x+1)^2-16}\right]\right] + C$

$= \frac{1}{4}(2x+1)\sqrt{(2x+5)(2x-3)} - 4\ln\left[\frac{1}{4}\left[2x+1 + \sqrt{(2x+5)(2x-3)}\right]\right] + C$

$= \frac{1}{4}(2x+1)\sqrt{(2x+5)(2x-3)} - 4\left[\ln\frac{1}{4} + \ln\left[2x+1+\sqrt{(2x+5)(2x-3)}\right]\right] + C$

$= \frac{1}{4}(2x+1)\sqrt{(2x+5)(2x-3)} - 4\ln\left[2x+1+\sqrt{(2x+5)(2x-3)}\right] + C$

724. $\displaystyle\int \frac{3x}{x-\sqrt{x^2-1}}\,dx = x^3 + \left(x^2-1\right)^{\frac{3}{2}} + C$

$$\int \frac{3x}{x-\sqrt{x^2-1}}\,dx = \int \frac{3x\left[x+\sqrt{x^2-1}\right]}{\left[x-\sqrt{x^2-1}\right]\left[x+\sqrt{x^2-1}\right]}\,dx$$

$= \int \frac{3x^2 + 3x\sqrt{x^2-1}}{x^2-(x^2-1)}\,dx = \int \frac{3x^2 + 3x(x^2-1)^{\frac{1}{2}}}{1}\,dx = \int 3x^2 + 3x(x^2-1)^{\frac{1}{2}}\,dx$

$= x^3 + (x^2-1)^{\frac{3}{2}} + C$

ALTERNATIVE BY HYPERBOLIC SUBSTITUTION

$\theta = \text{arccosh}\,x$

$\cosh\theta = x$

$dx = \sinh\theta\,d\theta$

$\int \frac{3x}{x-\sqrt{x^2-1}}\,dx = \int \frac{3\cosh\theta}{\cosh\theta - \sqrt{\cosh^2\theta - 1}}(\sinh\theta\,d\theta)$

$= \int \frac{3\cosh\theta\sinh\theta}{\cosh\theta - \sinh\theta}\,d\theta = \int \frac{3\cosh\theta\sinh\theta(\cosh\theta+\sinh\theta)}{(\cosh\theta-\sinh\theta)(\cosh\theta+\sinh\theta)}\,d\theta$

$= \int \frac{3\cosh^2\theta\sinh\theta + 3\cosh\theta\sinh^2\theta}{\cancel{\cosh^2\theta - \sinh^2\theta}}\,d\theta = \int 3\cosh^2\theta\sinh\theta + 3\cosh\theta\sinh^2\theta\,d\theta$

$= \cosh^3\theta + \sinh^3\theta + C = \cosh^3\theta + \left(+\sqrt{\cosh^2\theta - 1}\right)^3 + C$

$= x^3 + (x^2-1)^{\frac{3}{2}} + C$ AS BEFORE

ALTERNATIVE BY TRIGONOMETRIC SUBSTITUTION

$\theta = \text{arcsec}\,x$

$\sec\theta = x$

$dx = \sec\theta\tan\theta\,d\theta$

$\int \frac{3x}{x-\sqrt{x^2-1}}\,dx = \int \frac{3\sec\theta}{\sec\theta - \sqrt{\sec^2\theta - 1}}(\sec\theta\tan\theta\,d\theta)$

$= \int \frac{3\sec^2\theta\tan\theta}{\sec\theta - \tan\theta}\,d\theta = \int \frac{3\sec^2\theta\tan\theta(\sec\theta+\tan\theta)}{(\sec\theta-\tan\theta)(\sec\theta+\tan\theta)}\,d\theta$

$= \int \frac{3\sec^3\theta\tan\theta + 3\sec^2\theta\tan^2\theta}{\cancel{\sec^2\theta - \tan^2\theta}}\,d\theta = \int 3\sec^3\theta\tan\theta + 3\sec^2\theta\tan^2\theta\,d\theta$

NOTE THAT

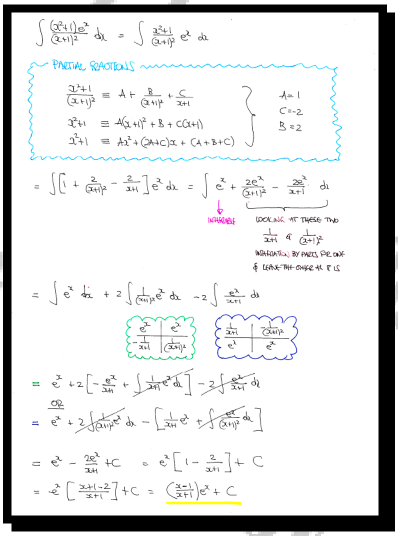
$\frac{d}{d\theta}(\sec^3\theta) = 3\sec^2\theta(\sec\theta\tan\theta) = 3\sec^3\theta\tan\theta$

$\frac{d}{d\theta}(\tan^3\theta) = 3\tan^2\theta \times \sec^2\theta = 3\tan^2\theta\sec^2\theta$

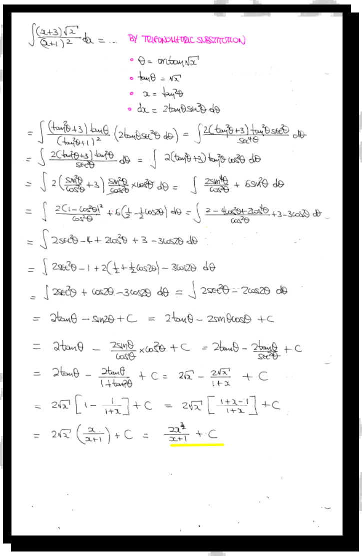
$= \sec^3\theta + \tan^3\theta + C = \sec^3\theta + \left(+\sqrt{\sec^2\theta - 1}\right)^3 + C$

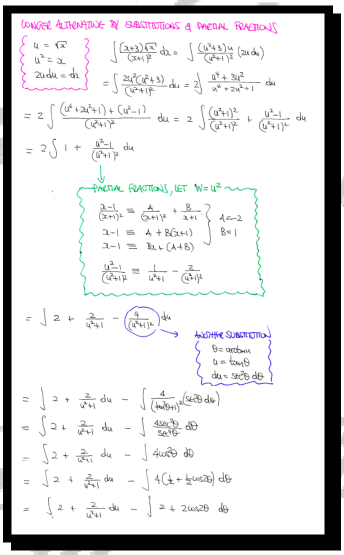
$= x^3 + (x^2-1)^{\frac{3}{2}} + C$

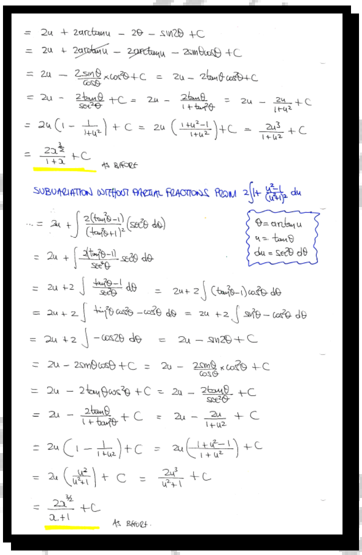
725. $$\int \frac{(x^2+1)e^x}{(x+1)^2}\,dx = \left[\frac{x-1}{x+1}\right]e^x + C$$



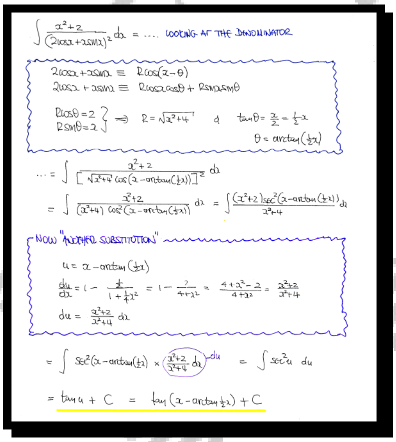
726. $$\int \frac{(x+3)\sqrt{x}}{(x+1)^2}\,dx = \frac{2x^{\frac{3}{2}}}{x+1} + C$$







727. $$\int \frac{x^2+2}{(2\cos x + x\sin x)^2}\,dx = \tan\left[x - \arctan\left(\frac{1}{2}x\right)\right] + C$$



$$\int \frac{x^2+2}{(2\cos x + x\sin x)^2}\,dx = \dots$$ LOOKING AT THE DINOMINATOR

$$2\cos x + x\sin x \equiv R\cos(x-\theta)$$
$$2\cos x + x\sin x \equiv R\cos x\cos\theta + R\sin x\sin\theta$$
$$\left.\begin{array}{l} R\cos\theta = 2 \\ R\sin\theta = x \end{array}\right\} \Rightarrow R = \sqrt{x^2+4} \quad \& \quad \tan\theta = \frac{x}{2} = \frac{1}{2}x$$
$$\theta = \arctan\left(\tfrac{1}{2}x\right)$$

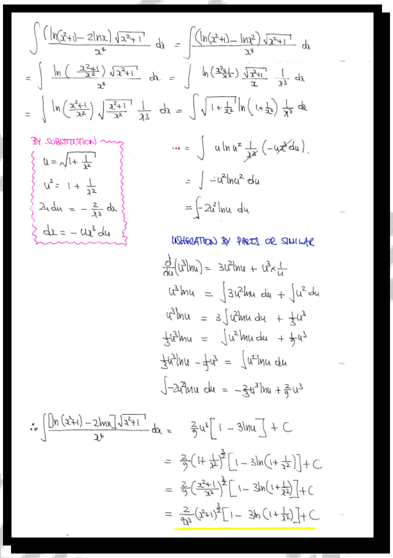
$$\dots = \int \frac{x^2+2}{\left[\sqrt{x^2+4}\cos\left(x-\arctan\left(\frac{1}{2}x\right)\right)\right]^2}\,dx$$
$$= \int \frac{x^2+2}{(x^2+4)\cos^2\left(x-\arctan\left(\frac{1}{2}x\right)\right)}\,dx = \int \frac{(x^2+2)\sec^2\left(x-\arctan\left(\frac{1}{2}x\right)\right)}{x^2+4}\,dx$$

NOW "ANOTHER SUBSTITUTION"

$$u = x - \arctan\left(\tfrac{1}{2}x\right)$$
$$\frac{du}{dx} = 1 - \frac{\frac{1}{2}}{1+\frac{1}{4}x^2} = 1 - \frac{2}{4+x^2} = \frac{4+x^2-2}{4+x^2} = \frac{x^2+2}{x^2+4}$$
$$du = \frac{x^2+2}{x^2+4}\,dx$$

$$= \int \sec^2\left(x-\arctan\left(\tfrac{1}{2}x\right)\right) \times \underbrace{\frac{x^2+2}{x^2+4}\,dx}_{du} = \int \sec^2 u\,du$$
$$= \tan u + C = \tan\left(x - \arctan\tfrac{1}{2}x\right) + C$$

728. $$\int \frac{\left[\ln\left(x^2+1\right) - 2\ln x\right]\sqrt{x^2+1}}{x^4}\,dx = \frac{2}{9x^3}\left(x^2+1\right)^{\frac{3}{2}}\left[1 - 3\ln\left(\frac{x^2+1}{x^2}\right)\right] + C$$



$$\int \frac{\left(\ln(x^2+1) - 2\ln x\right)\sqrt{x^2+1}}{x^4}\,dx = \int \frac{\left(\ln(x^2+1) - \ln x^2\right)\sqrt{x^2+1}}{x^4}\,dx$$
$$= \int \frac{\ln\left(\frac{x^2+1}{x^2}\right)\sqrt{x^2+1}}{x^4}\,dx = \int \ln\left(\frac{x^2+1}{x^2}\right)\frac{\sqrt{x^2+1}}{x}\frac{1}{x^3}\,dx$$
$$= \int \ln\left(\frac{x^2+1}{x^2}\right)\sqrt{\frac{x^2+1}{x^2}}\,\frac{1}{x^3}\,dx = \int \sqrt{1+\frac{1}{x^2}}\ln\left(1+\frac{1}{x^2}\right)\frac{1}{x^3}\,dx$$

BY SUBSTITUTION

$$u = \sqrt{1+\frac{1}{x^2}}$$
$$u^2 = 1 + \frac{1}{x^2}$$
$$2u\,du = -\frac{2}{x^3}\,dx$$
$$dx = -ux^3\,du$$

$$\dots = \int u\ln u^2\,\frac{1}{\cancel{x^3}}\left(-u\cancel{x^3}\,du\right)$$
$$= \int -u^2\ln u^2\,du$$
$$= \int -2u^2\ln u\,du$$

INTEGRATION BY PARTS OR SIMILAR

$$\frac{d}{du}\left(u^3\ln u\right) = 3u^2\ln u + u^3 \times \frac{1}{u}$$
$$u^3\ln u = \int 3u^2\ln u\,du + \int u^2\,du$$
$$u^3\ln u = 3\int u^2\ln u\,du + \frac{1}{3}u^3$$
$$\frac{1}{3}u^3\ln u = \int u^2\ln u\,du + \frac{1}{9}u^3$$
$$\frac{1}{3}u^3\ln u - \frac{1}{9}u^3 = \int u^2\ln u\,du$$
$$\int -2u^2\ln u\,du = -\frac{2}{3}u^3\ln u + \frac{2}{9}u^3$$

$$\therefore \int \frac{\left[\ln(x^2+1) - 2\ln x\right]\sqrt{x^2+1}}{x^4}\,dx = \frac{2}{9}u^3\left[1 - 3\ln u\right] + C$$
$$= \frac{2}{9}\left(1+\frac{1}{x^2}\right)^{\frac{3}{2}}\left[1 - 3\ln\left(1+\frac{1}{x^2}\right)\right] + C$$
$$= \frac{2}{9}\left(\frac{x^2+1}{x^2}\right)^{\frac{3}{2}}\left[1 - 3\ln\left(1+\frac{1}{x^2}\right)\right] + C$$
$$= \frac{2}{9x^3}\left(x^2+1\right)^{\frac{3}{2}}\left[1 - 3\ln\left(1+\frac{1}{x^2}\right)\right] + C$$

729. $$\int \frac{\arcsin\sqrt{x}-\arccos\sqrt{x}}{\arcsin\sqrt{x}+\arccos\sqrt{x}}\,dx = \frac{2}{\pi}\left[\sqrt{x-x^2}+(2x-1)\arcsin\sqrt{x}\right] - x + C$$

$\displaystyle\int \frac{\arcsin\sqrt{x}-\arccos\sqrt{x}}{\arcsin\sqrt{x}+\arccos\sqrt{x}}\,dx$ ... START BY SUBSTITUTION

$y=\sqrt{x}$
$y^2=x$
$dx=2y\,dy$

$= \displaystyle\int \frac{\arcsin y-\arccos y}{\arcsin y+\arccos y}(2y\,dy)$

$\arcsin A+\arccos A \equiv \frac{\pi}{2}$

$= \displaystyle\int \frac{\arcsin y-\left(\frac{\pi}{2}-\arcsin y\right)}{\pi/2}(2y\,dy)$

$= \displaystyle\int \frac{4y}{\pi}\left(2\arcsin y-\frac{\pi}{2}\right)dy = \int \frac{8}{\pi}y\arcsin y - 2y\,dy$

$= -y^2+\frac{8}{\pi}\displaystyle\int y\arcsin y\,dy$

NOW INTEGRATION BY PARTS OR SUBSTITUTION FOLLOWED BY INTEGRATION BY PARTS AFTER

$\theta=\arcsin y$
$y=\sin\theta$
$dy=\cos\theta\,d\theta$
$\cos\theta=+\sqrt{1-y^2}$

$= -y^2+\frac{8}{\pi}\displaystyle\int(\sin\theta)\theta\cos\theta\,d\theta$

$= -y^2+\frac{8}{\pi}\displaystyle\int\theta\sin\theta\cos\theta\,d\theta$

$= -y^2+\frac{8}{\pi}\displaystyle\int\theta\left(\tfrac{1}{2}\sin2\theta\right)d\theta$

$= -y^2+\frac{1}{\pi}\displaystyle\int 4\theta\sin2\theta\,d\theta$

NOW INTEGRATION BY PARTS OR SIMILAR MANIPULATION

$\Rightarrow \frac{d}{d\theta}[\theta\cos2\theta] = 1\times\cos2\theta-2\theta\sin2\theta$

$\Rightarrow \frac{d}{d\theta}[2\theta\cos2\theta] = 2\cos2\theta-4\theta\sin2\theta$

$\Rightarrow \displaystyle\int\frac{d}{d\theta}[2\theta\cos2\theta]\,d\theta = \int 2\cos2\theta\,d\theta-\int4\theta\sin2\theta\,d\theta$

$\Rightarrow 2\theta\cos2\theta = \sin2\theta-\displaystyle\int4\theta\sin2\theta$

$\Rightarrow \displaystyle\int4\theta\sin2\theta = \sin2\theta-2\theta\cos2\theta+C$

RETURNING TO THE INTEGRAL – MAIN LINE

$\displaystyle\int \frac{\arcsin\sqrt{x}-\arccos\sqrt{x}}{\arcsin\sqrt{x}+\arccos\sqrt{x}}\,dx = -y^2+\frac{1}{\pi}\int4\theta\sin2\theta\,d\theta$

$= -y^2+\frac{1}{\pi}[\sin2\theta-2\theta\cos2\theta]+C$

$= -y^2+\frac{1}{\pi}[2\sin\theta\cos\theta-2\theta(1-2\sin^2\theta)]+C$

$= -y^2+\frac{2}{\pi}[\sin\theta\cos\theta-\theta(1-2\sin^2\theta)]+C$

$= -y^2+\frac{2}{\pi}\left[y\sqrt{1-y^2}-(\arcsin y)(1-2y^2)\right]+C$

$= -y^2+\frac{2}{\pi}\left[y\sqrt{1-y^2}+(2y^2-1)\arcsin y\right]+C$

$= -x+\frac{2}{\pi}\left[\sqrt{x}\sqrt{1-x}+(2x-1)\arcsin\sqrt{x}\right]+C$

$= \frac{2}{\pi}\left[\sqrt{x-x^2}+(2x-1)\arcsin\sqrt{x}\right]-x+C$

730. $$\int \frac{e^{3x}(6x-5)}{(2x-1)^4}\,dx = \frac{e^{3x}}{2x-1} + C$$

$\displaystyle\int\frac{e^{3x}(6x-5)}{(2x-1)^2}\,dx$ ... INTEGRATION BY PARTS...

$e^{3x}(6x-5)$	$3e^{3x}(6x-5)+6e^{3x}$ $e^{3x}(18x-9)$ $9e^{3x}(2x-1)$
$-\frac{1}{2(2x-1)}$	$\frac{1}{(2x-1)^2}$

$= -\dfrac{e^{3x}(6x-5)}{2(2x-1)}+\displaystyle\int\frac{9}{2}e^{3x}\,dx$

$= -\frac{1}{2}e^{3x}\left(\frac{6x-5}{2x-1}\right)+\frac{3}{2}e^{3x}+C$

$= \frac{1}{2}e^{3x}\left[3-\frac{6x-5}{2x-1}\right]+C$

$= \frac{1}{2}e^{3x}\left[\frac{\cancel{6x}-3-\cancel{6x}+5}{2x-1}\right]+C = \frac{1}{2}e^{3x}\left(\frac{2}{2x-1}\right)+C = \frac{e^{3x}}{2x-1}+C$

ALTERNATIVE STARTING WITH PARTIAL FRACTIONS IN THE INTEGRAND

- $\dfrac{6x-5}{(2x-1)^2} \equiv \dfrac{A}{2x-1}+\dfrac{B}{(2x-1)^2}$
- $6x-5 \equiv A(2x-1)+B$
- $6x-5 \equiv 2Ax+(B-A)$

$A=3$
$B=-2$

RETURNING TO THE INTEGRAL

$\displaystyle\int\frac{e^{3x}(6x-5)}{(2x-1)^2}\,dx = \int\frac{3e^{3x}}{2x-1}-\frac{2e^{3x}}{(2x-1)^2}\,dx$

$= \displaystyle\int\frac{3e^{3x}}{2x-1}\,dx-\int\frac{2e^{3x}}{(2x-1)^2}\,dx$

NOW INTEGRATION BY PARTS ON THE FIRST INTEGRAL ONLY (OR THE SECOND INTEGRAL ONLY)

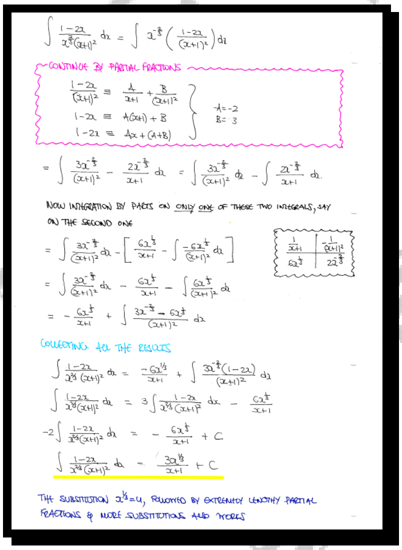
$\frac{1}{2x-1}$	$\frac{-2}{(2x-1)^2}$
e^{3x}	$3e^{3x}$

$= \left[\dfrac{e^{3x}}{2x-1}-\displaystyle\int\frac{-2e^{3x}}{(2x-1)^2}\,dx\right]-\int\frac{2e^{3x}}{(2x-1)^2}\,dx$

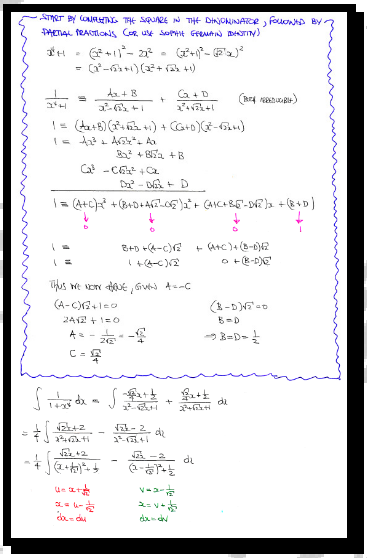
$= \dfrac{e^{3x}}{2x-1}+\cancel{\displaystyle\int\frac{2e^{3x}}{(2x-1)^2}\,dx}-\cancel{\int\frac{2e^{3x}}{(2x-1)^2}\,dx}$

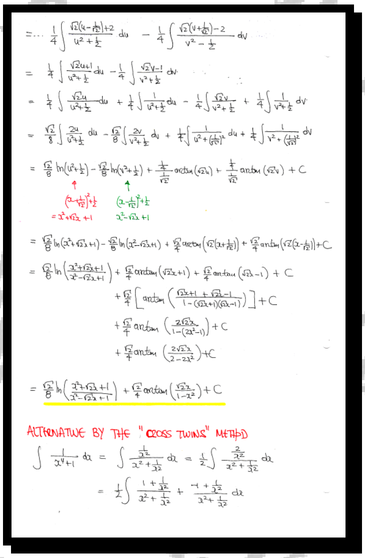
$= \dfrac{e^{3x}}{2x-1}+C$ AS BEFORE

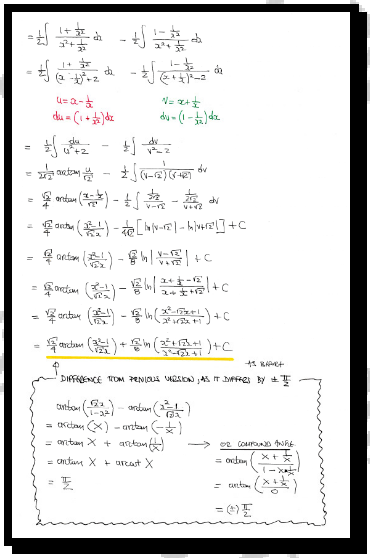
731. $$\int \frac{1-2x}{x^{\frac{2}{3}}(x+1)^4}\,dx = \frac{3x^{\frac{1}{3}}}{x+1} + C$$



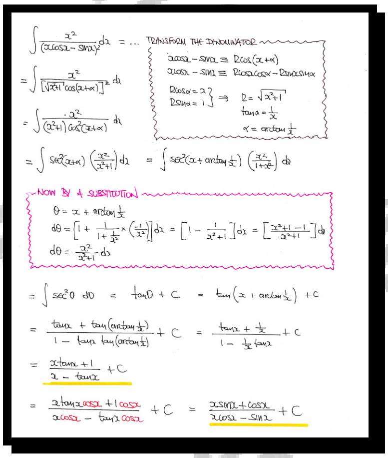
732. $$\int \frac{1}{x^4+1}\,dx = \left[\frac{\sqrt{2}}{8}\ln\left[\frac{x^2+\sqrt{2}\,x+1}{x^2-\sqrt{2}\,x+1}\right] + \frac{\sqrt{2}}{4}\arctan\left[\frac{\sqrt{2}\,x}{1-x^2}\right] + C \atop \frac{\sqrt{2}}{8}\ln\left[\frac{x^2+\sqrt{2}\,x+1}{x^2-\sqrt{2}\,x+1}\right] + \frac{\sqrt{2}}{4}\arctan\left[\frac{x^2-1}{\sqrt{2}\,x}\right] + C\right.$$



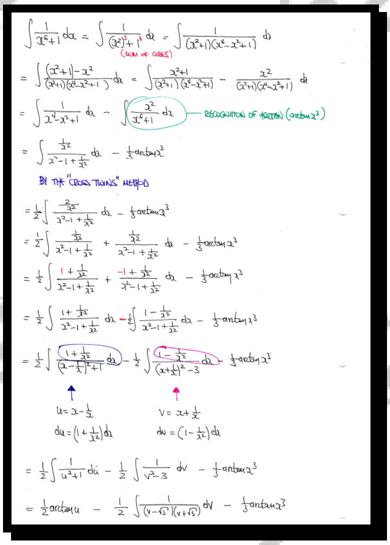


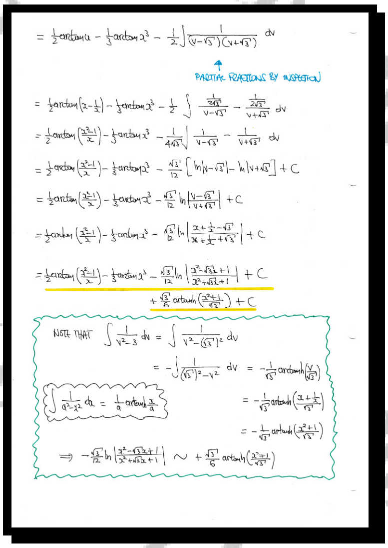


733. $$\int \frac{x^2}{(x\cos x - \sin x)^2}\,dx = \begin{cases} \dfrac{x\tan x + 1}{x - \tan x} + C \\ \dfrac{x\sin x + \cos x}{x\cos x - \sin x} + C \end{cases}$$



734. $$\int \frac{1}{x^6+1}\,dx = \begin{cases} \dfrac{1}{2}\arctan\left[\dfrac{x^2-1}{x}\right] - \dfrac{1}{3}\arctan x^3 - \dfrac{\sqrt{3}}{12}\ln\left[\dfrac{x^2-\sqrt{3}\,x+1}{x^2+\sqrt{3}\,x+1}\right] + C \\ \dfrac{1}{2}\arctan\left[\dfrac{x^2-1}{x}\right] - \dfrac{1}{3}\arctan x^3 + \dfrac{\sqrt{3}}{6}\operatorname{artanh}\left[\dfrac{x^2+1}{\sqrt{3}}\right] + C \end{cases}$$





735. $$\int \frac{x^2+1}{x^4-x^2+1}\,dx = \arctan\left(\frac{x^2-1}{x}\right) + C$$

$\int \frac{x^2+1}{x^4-x^2+1}\,dx = \int \frac{1+\frac{1}{x^2}}{x^2-1+\frac{1}{x^2}}$ $\quad \frac{d}{dx}\left(x-\frac{1}{x}\right) = 1+\frac{1}{x^2}$

$= \int \frac{x+\frac{1}{x^2}}{\left(x-\frac{1}{x}\right)^2+1}\,dx =$ BY ANOTHER SUBSTITUTION OR RECOGNITION

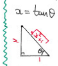
$= \arctan\left(x-\frac{1}{x}\right)+C = \arctan\left(\frac{x^2-1}{x}\right)+C$

736. $$\int \frac{x^2\arctan x}{1+x^2}\,dx = x\arctan x - \frac{1}{2}(\arctan x)^2 - \frac{1}{2}\ln\left(x^2+1\right) + C$$

$\int \frac{x^2\arctan x}{1+x^2}\,dx = \ldots$ START WITH A SUBSTITUTION

$\theta = \arctan x$

$x = \tan\theta$



$dx = \sec^2\theta\,d\theta$

$= \int \frac{(\tan^2\theta)\theta}{\cancel{1+\tan^2\theta}}\left(\cancel{\sec^2\theta}\,d\theta\right) = \int \theta\tan^2\theta\,d\theta$

$= \int \theta\left(\sec^2\theta-1\right)d\theta = \int \theta\sec^2\theta - \theta\,d\theta$

$= \int -\theta\,d\theta + \int \theta\sec^2\theta\,d\theta \leftarrow$ BY PARTS

θ	1
$\tan\theta$	$\sec^2\theta$

$= -\frac{1}{2}\theta^2 + \theta\tan\theta - \int \tan\theta\,d\theta$

$= \theta\tan\theta - \frac{1}{2}\theta^2 - \ln|\sec\theta| + C$

$= x\arctan x - \frac{1}{2}(\arctan x)^2 - \ln\sqrt{x^2+1} + C$

$= x\arctan x - \frac{1}{2}(\arctan x)^2 - \frac{1}{2}\ln(x^2+1) + C$

737. $$\int \frac{x}{1+x^4}\,dx = \frac{1}{2}\arctan x^2 + C$$

$\int \frac{x}{1+x^4}\,dx =$ BY RECOGNITION OR LET $u=x^2$

$\frac{d}{dx}\left(\arctan x^2\right) = \frac{2x}{1+x^4}$

$\ldots = \frac{1}{2}\arctan x^2 + C$

738. $$\int \frac{1}{1-\cot x}\,dx = \frac{1}{2}x + \frac{1}{2}\ln|\sin x - \cos x| + C$$

$\int \frac{1}{1-\cot x}\,dx = \int \frac{1\sin x}{1\sin x - \cot x\sin x}\,dx = \frac{1}{2}\int \frac{2\sin x}{\sin x-\cos x}\,dx$

$= \frac{1}{2}\int \frac{(\sin x-\cos x)+(\sin x+\cos x)}{\sin x-\cos x}\,dx = \frac{1}{2}\int 1 + \underbrace{\frac{\sin x+\cos x}{\sin x-\cos x}}_{\int \frac{f'(x)}{f(x)}dx = \ln|f(x)|+C}\,dx$

$= \frac{1}{2}x + \frac{1}{2}\ln|\sin x-\cos x| + C$

739. $$\int \frac{\sin\left(x^{-2}\right)}{x^5}\,dx = \frac{1}{2}\left[\frac{1}{x^2}\cos\left(\frac{1}{x^2}\right) - \sin\left(\frac{1}{x^2}\right)\right] + C$$

$\int \frac{\sin(x^{-2})}{x^5}\,dx = \dots$ SUBSTITUTION ...

$u = \frac{1}{x^2} = x^{-2}$

$\frac{du}{dx} = -\frac{2}{x^3}$

$dx = -\frac{1}{2}x^3\,du$

$= \int \frac{\sin u}{x^5}\left(-\frac{1}{2}x^3\right)du = \int -\frac{\sin u}{2x^2}\,du$

$= \int -\frac{1}{2}u\sin u\,du = \dots$ INTEGRATION BY PARTS

$-\frac{1}{2}u$	$-\frac{1}{2}$
$-\cos u$	$\sin u$

$= \frac{1}{2}u\cos u - \int \frac{1}{2}\cos u\,du = \frac{1}{2}u\cos u - \frac{1}{2}\sin u + C$

$= \frac{1}{2}\left[\frac{1}{x^2}\cos\left(\frac{1}{x^2}\right) - \sin\left(\frac{1}{x^2}\right)\right] + C$

740. $$\int 3^{\ln x}\,dx = \frac{x\left(3^{\ln x}\right)}{1+\ln 3} + C$$

$\int 3^{\ln x}\,dx = \dots$ SUBSTITUTION ...

$u = \ln x$

$e^u = x$

$dx = e^u\,du$

$= \int 3^u e^u\,du = \int (3e)^u\,du$

USING $\frac{d}{dx}(a^x) = a^x \ln a$

$\int a^x\,dx = \frac{1}{\ln a}a^x + C$

$= \frac{1}{\ln(3e)}(3e)^u + C = \frac{3^u \times e^u}{\ln 3 + \ln e} + C = \frac{3^{\ln x} \times x}{\ln 3 + 1} + C$

$= \frac{x\left(3^{\ln x}\right)}{1+\ln 3} + C$

741. $$\int \tan^5 x\,dx = \frac{1}{4}\sec^4 x - \sec^2 x + \ln|\sec x| + C$$

$\int \tan^5 x\,dx = \int \tan x \tan^4 x\,dx = \int \tan x(\sec^2 x - 1)^2\,dx$

$= \int \tan x(\sec^4 x - 2\sec^2 x + 1)\,dx = \int \tan x\sec^4 x - 2\sec^2 x\tan x + \tan x\;dx$

$= \int \tan x\sec x(\sec^3 x - 2\sec x) + \tan x\;dx$ ... BY RECOGNITION SINCE $\frac{d}{dx}(\sec x) = \sec x\tan x$

$= \frac{1}{4}\sec^4 x - \sec^2 x + \ln|\sec x| + C$

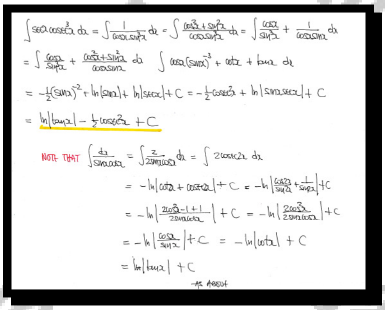
742. $$\int \tan^3 x\sec^3 x\,dx = \frac{1}{5}\sec^5 x - \frac{1}{3}\sec^3 x + C$$

$\int \tan^3 x\sec^3 x\,dx = \int \tan x\tan^2 x\sec^3 x\,dx = \int \tan x(\sec^2 x - 1)\sec^3 x\,dx$

$= \int (\tan x\sec x)(\sec^4 x - \sec^2 x)\dots$ NOW $\frac{d}{dx}(\sec x) = \sec x\tan x$

$= \frac{1}{5}\sec^5 x - \frac{1}{3}\sec^3 x + C$

743. $$\int \sec x \operatorname{cosec}^3 x \, dx = \ln|\tan x| - \tfrac{1}{2}\operatorname{cosec}^2 x + C$$



$\int \sec x \operatorname{cosec}^3 x \, dx = \int \frac{1}{\cos x \sin^3 x} \, dx = \int \frac{\cos^2 x + \sin^2 x}{\cos x \sin^3 x} \, dx = \int \frac{\cos x}{\sin^3 x} + \frac{1}{\cos x \sin x} \, dx$

$= \int \frac{\cos x}{\sin^3 x} + \frac{\cos^2 x + \sin^2 x}{\cos x \sin x} \, dx \quad \int \cos x (\sin x)^{-3} + \cot x + \tan x \, dx$

$= -\frac{1}{2}(\sin x)^{-2} + \ln|\sin x| + \ln|\sec x| + C = -\frac{1}{2}\operatorname{cosec}^2 x + \ln|\sin x \sec x| + C$

$= \ln|\tan x| - \frac{1}{2}\operatorname{cosec}^2 x + C$

NOTE THAT $\int \frac{dx}{\sin x \cos x} = \int \frac{2}{2\sin x \cos x} \, dx = \int 2\operatorname{cosec} 2x \, dx$

$= -\ln|\cot 2x + \operatorname{cosec} 2x| + C = -\ln\left|\frac{\cos 2x}{\sin 2x} + \frac{1}{\sin 2x}\right| + C$

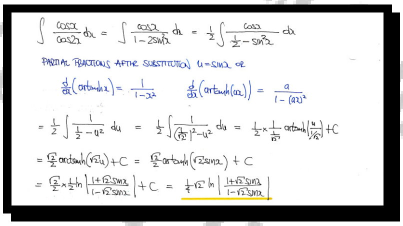
$= -\ln\left|\frac{2\cos^2 x - 1 + 1}{2\sin x \cos x}\right| + C = -\ln\left|\frac{2\cos^2 x}{2\sin x \cos x}\right| + C$

$= -\ln\left|\frac{\cos x}{\sin x}\right| + C = -\ln|\cot x| + C$

$= \ln|\tan x| + C$

–AS ABOVE

744. $$\int \frac{\cos x}{\cos 2x} \, dx = \frac{1}{4}\sqrt{2} \ln\left|\frac{1+\sqrt{2}\sin x}{1-\sqrt{2}\sin x}\right| + C$$



$\int \frac{\cos x}{\cos 2x} \, dx = \int \frac{\cos x}{1 - 2\sin^2 x} \, dx = \frac{1}{2}\int \frac{\cos x}{\frac{1}{2} - \sin^2 x} \, dx$

PARTIAL FRACTIONS AFTER SUBSTITUTION $u = \sin x$ OR

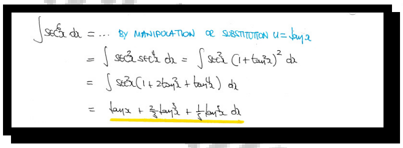
$\frac{d}{dx}(\operatorname{artanh} x) = \frac{1}{1-x^2} \qquad \frac{d}{dx}(\operatorname{artanh}(ax)) = \frac{a}{1-(ax)^2}$

$= \frac{1}{2}\int \frac{1}{\frac{1}{2} - u^2} \, du = \frac{1}{2}\int \frac{1}{\left(\frac{1}{\sqrt{2}}\right)^2 - u^2} \, du = \frac{1}{2} \times \frac{1}{\frac{1}{\sqrt{2}}} \operatorname{artanh}\left(\frac{u}{1/\sqrt{2}}\right) + C$

$= \frac{\sqrt{2}}{2}\operatorname{artanh}(\sqrt{2}u) + C = \frac{\sqrt{2}}{2}\operatorname{artanh}(\sqrt{2}\sin x) + C$

$= \frac{\sqrt{2}}{2} \times \frac{1}{2}\ln\left|\frac{1+\sqrt{2}\sin x}{1-\sqrt{2}\sin x}\right| + C = \frac{1}{4}\sqrt{2}\ln\left|\frac{1+\sqrt{2}\sin x}{1-\sqrt{2}\sin x}\right|$

745. $$\int \sec^6 x \, dx = \tan x + \tfrac{2}{3}\tan^3 x + \tfrac{1}{5}\tan^5 x + C$$



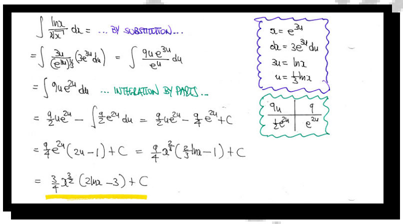
$\int \sec^6 x \, dx = \ldots$ BY MANIPULATION OR SUBSTITUTION $u = \tan x$

$= \int \sec^2 x \sec^4 x \, dx = \int \sec^2 x (1 + \tan^2 x)^2 \, dx$

$= \int \sec^2 x (1 + 2\tan^2 x + \tan^4 x) \, dx$

$= \tan x + \frac{2}{3}\tan^3 x + \frac{1}{5}\tan^5 x \, dx$

746. $$\int \frac{\ln x}{\sqrt[3]{x}}\,dx = \frac{3}{4}x^{\frac{3}{2}}\left(2\ln x - 3\right) + C$$



$\int \frac{\ln x}{\sqrt[3]{x}}\,dx = \ldots$ BY SUBSTITUTION...

$= \int \frac{3u}{(e^{3u})^{\frac{1}{3}}}\left(3e^{3u}\,du\right) = \int \frac{9ue^{3u}}{e^{u}}\,du$

$= \int 9ue^{2u}\,du$ ... INTEGRATION BY PARTS...

$= \frac{9}{2}ue^{2u} - \int \frac{9}{2}e^{2u}\,du = \frac{9}{2}ue^{2u} - \frac{9}{4}e^{2u} + C$

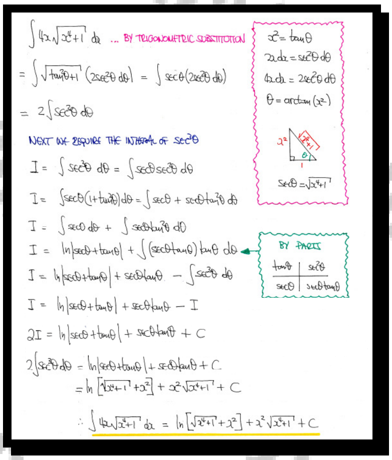
$= \frac{9}{4}e^{2u}(2u-1) + C = \frac{9}{4}x^{\frac{2}{3}}\left(\frac{2}{3}\ln x - 1\right) + C$

$= \frac{3}{4}x^{\frac{3}{2}}(2\ln x - 3) + C$

$x = e^{3u}$
$dx = 3e^{3u}\,du$
$3u = \ln x$
$u = \frac{1}{3}\ln x$

$9u$	9
$\frac{1}{2}e^{2u}$	e^{2u}

747. $$\int 4x\sqrt{x^4+1}\,dx = \ln\left[x^2+\sqrt{x^4+1}\right] + x^2\sqrt{x^4+1} + C$$



$\int 4x\sqrt{x^4+1}\,dx$ ... BY TRIGONOMETRIC SUBSTITUTION

$= \int \sqrt{\tan^2\theta+1}\,(2\sec^2\theta\,d\theta) = \int \sec\theta(2\sec^2\theta\,d\theta)$

$= 2\int \sec^3\theta\,d\theta$

NEXT WE REQUIRE THE INTEGRAL OF $\sec^3\theta$

$I = \int \sec^3\theta\,d\theta = \int \sec\theta\sec^2\theta\,d\theta$

$I = \int \sec\theta(1+\tan^2\theta)\,d\theta = \int \sec\theta + \sec\theta\tan^2\theta\,d\theta$

$I = \int \sec\theta\,d\theta + \int \sec\theta\tan^2\theta\,d\theta$

$I = \ln|\sec\theta+\tan\theta| + \int(\sec\theta\tan\theta)\tan\theta\,d\theta$

$I = \ln|\sec\theta+\tan\theta| + \sec\theta\tan\theta - \int \sec^3\theta\,d\theta$

$I = \ln|\sec\theta+\tan\theta| + \sec\theta\tan\theta - I$

$2I = \ln|\sec\theta+\tan\theta| + \sec\theta\tan\theta + C$

$2\int \sec^3\theta\,d\theta = \ln|\sec\theta+\tan\theta| + \sec\theta\tan\theta + C$

$= \ln\left[\sqrt{x^4+1}+x^2\right] + x^2\sqrt{x^4+1} + C$

$\therefore \int 4x\sqrt{x^4+1}\,dx = \ln\left[\sqrt{x^4+1}+x^2\right] + x^2\sqrt{x^4+1} + C$

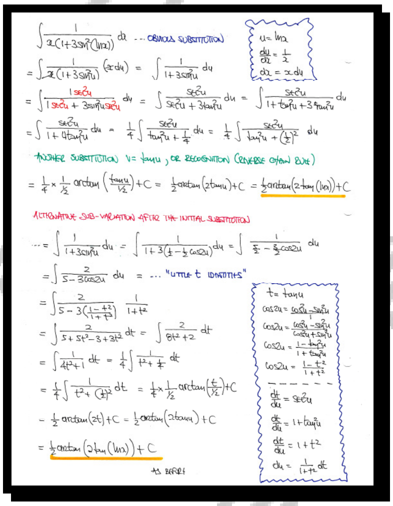
$x^2 = \tan\theta$
$2x\,dx = \sec^2\theta\,d\theta$
$4x\,dx = 2\sec^2\theta\,d\theta$
$\theta = \arctan(x^2)$

$\sec\theta = \sqrt{x^4+1}$

BY PARTS

$\tan\theta$	$\sec^2\theta$
$\sec\theta$	$\sec\theta\tan\theta$

748. $$\int \frac{1}{x\left[1+3\sin^2(\ln x)\right]}\,dx = \frac{1}{2}\arctan\left[2\tan(\ln x)\right] + C$$



$\int \frac{1}{x(1+3\sin^2(\ln x))}\,dx$ ... OBVIOUS SUBSTITUTION

$u = \ln x$, $\frac{du}{dx} = \frac{1}{x}$, $dx = x\,du$

$= \int \frac{1}{x(1+3\sin^2 u)}(x\,du) = \int \frac{1}{1+3\sin^2 u}\,du$

$= \int \frac{1\sec^2 u}{1\sec^2 u + 3\sin^2 u\sec^2 u}\,du = \int \frac{\sec^2 u}{\sec^2 u + 3\tan^2 u}\,du = \int \frac{\sec^2 u}{1+\tan^2 u + 3\tan^2 u}\,du$

$= \int \frac{\sec^2 u}{1+4\tan^2 u}\,du = \frac{1}{4}\int \frac{\sec^2 u}{\tan^2 u + \frac{1}{4}}\,du = \frac{1}{4}\int \frac{\sec^2 u}{\tan^2 u + \left(\frac{1}{2}\right)^2}\,du$

ANOTHER SUBSTITUTION $v = \tan u$, OR RECOGNITION (REVERSE CHAIN RULE)

$= \frac{1}{4}\times\frac{1}{\frac{1}{2}}\arctan\left(\frac{\tan u}{\frac{1}{2}}\right) + C = \frac{1}{2}\arctan(2\tan u) + C = \frac{1}{2}\arctan(2\tan(\ln x)) + C$

ALTERNATIVE SUB-VARIATION AFTER THE INITIAL SUBSTITUTION

$\dots = \int \frac{1}{1+3\sin^2 u}\,du = \int \frac{1}{1+3\left(\frac{1}{2}-\frac{1}{2}\cos 2u\right)}\,du = \int \frac{1}{\frac{5}{2}-\frac{3}{2}\cos 2u}\,du$

$= \int \frac{2}{5-3\cos 2u}\,du = \dots$ "LITTLE t IDENTITIES"

$t = \tan u$
$\cos 2u = \frac{\cos^2 u - \sin^2 u}{1}$
$\cos 2u = \frac{\cos^2 u - \sin^2 u}{\cos^2 u + \sin^2 u}$
$\cos 2u = \frac{1-\tan^2 u}{1+\tan^2 u}$
$\cos 2u = \frac{1-t^2}{1+t^2}$
$\frac{dt}{du} = \sec^2 u$
$\frac{dt}{du} = 1+\tan^2 u$
$\frac{dt}{du} = 1+t^2$
$du = \frac{1}{1+t^2}\,dt$

$= \int \frac{2}{5-3\left(\frac{1-t^2}{1+t^2}\right)}\,\frac{1}{1+t^2}$

$= \int \frac{2}{5+5t^2-3+3t^2}\,dt = \int \frac{2}{8t^2+2}\,dt$

$= \int \frac{1}{4t^2+1}\,dt = \frac{1}{4}\int \frac{1}{t^2+\frac{1}{4}}\,dt$

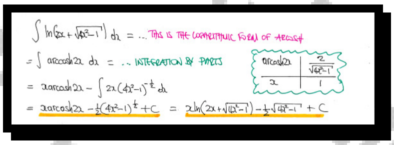
$= \frac{1}{4}\int \frac{1}{t^2+\left(\frac{1}{2}\right)^2}\,dt = \frac{1}{4}\times\frac{1}{\frac{1}{2}}\arctan\left(\frac{t}{\frac{1}{2}}\right) + C$

$= \frac{1}{2}\arctan(2t) + C = \frac{1}{2}\arctan(2\tan u) + C$

$= \frac{1}{2}\arctan(2\tan(\ln x)) + C$

AS BEFORE

749. $$\int \ln\left[2x+\sqrt{4x^2-1}\right]dx = \begin{cases} x\,\text{arcosh}\,2x - \frac{1}{2}\sqrt{4x^2-1} + C \\ x\ln\left[2x+\sqrt{4x^2-1}\right] - \frac{1}{2}\sqrt{4x^2-1} + C \end{cases}$$



$\int \ln(2x+\sqrt{4x^2-1})\,dx = \dots$ THIS IS THE LOGARITHMIC FORM OF ARCOSH

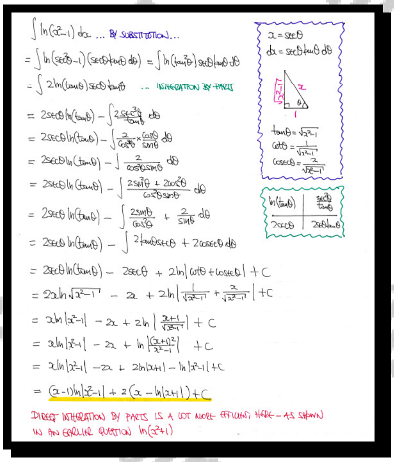
$= \int \text{arcosh}\,2x\,dx = \dots$ INTEGRATION BY PARTS

$\text{arcosh}\,2x$	$\frac{2}{\sqrt{4x^2-1}}$
x	1

$= x\,\text{arcosh}\,2x - \int 2x(4x^2-1)^{-\frac{1}{2}}\,dx$

$= x\,\text{arcosh}\,2x - \frac{1}{2}(4x^2-1)^{\frac{1}{2}} + C = x\ln(2x+\sqrt{4x^2-1}) - \frac{1}{2}\sqrt{4x^2-1} + C$

750. $$\int \ln\left(x^2-1\right)\,dx = (x-1)\ln\left|x^2-1\right| + 2\left[x-\ln|x+1|\right] + C$$


$\int \ln(x^2-1)\,dx$ ... BY SUBSTITUTION...

$= \int \ln(\sec^2\theta-1)(\sec\theta\tan\theta\,d\theta) = \int \ln(\tan^2\theta)\sec\theta\tan\theta\,d\theta$

$= \int 2\ln(\tan\theta)\sec\theta\tan\theta$ ... INTEGRATION BY PARTS

$= 2\sec\theta\ln(\tan\theta) - \int \frac{2\sec^3\theta}{\tan\theta}\,d\theta$

$= 2\sec\theta\ln(\tan\theta) - \int \frac{2}{\cos^3\theta}\times\frac{\cos\theta}{\sin\theta}\,d\theta$

$= 2\sec\theta\ln(\tan\theta) - \int \frac{2}{\cos^2\theta\sin\theta}\,d\theta$

$= 2\sec\theta\ln(\tan\theta) - \int \frac{2\sin^2\theta+2\cos^2\theta}{\cos^2\theta\sin\theta}\,d\theta$

$= 2\sec\theta\ln(\tan\theta) - \int \frac{2\sin\theta}{\cos^2\theta} + \frac{2}{\sin\theta}\,d\theta$

$= 2\sec\theta\ln(\tan\theta) - \int 2\tan\theta\sec\theta + 2\operatorname{cosec}\theta\,d\theta$

$= 2\sec\theta\ln(\tan\theta) - 2\sec\theta + 2\ln|\cot\theta+\operatorname{cosec}\theta| + C$

$= 2x\ln\sqrt{x^2-1} - 2x + 2\ln\left|\frac{1}{\sqrt{x^2-1}}+\frac{x}{\sqrt{x^2-1}}\right| + C$

$= x\ln|x^2-1| - 2x + 2\ln\left|\frac{x+1}{\sqrt{x^2-1}}\right| + C$

$= x\ln|x^2-1| - 2x + \ln\left|\frac{(x+1)^2}{x^2-1}\right| + C$

$= x\ln|x^2-1| - 2x + 2\ln|x+1| - \ln|x^2-1| + C$

$= (x-1)\ln|x^2-1| + 2(x-\ln|x+1|) + C$

$x=\sec\theta$

$dx = \sec\theta\tan\theta\,d\theta$

$\tan\theta=\sqrt{x^2-1}$

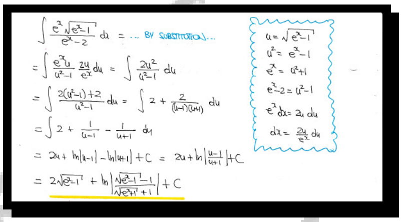
$\cot\theta=\frac{1}{\sqrt{x^2-1}}$

$\operatorname{cosec}\theta=\frac{x}{\sqrt{x^2-1}}$

$\ln(\tan\theta)$	$\frac{\sec^2\theta}{\tan\theta}$
$2\sec\theta$	$2\sec\theta\tan\theta$

DIRECT INTEGRATION BY PARTS IS A LOT MORE EFFICIENT HERE – AS SHOWN IN AN EARLIER QUESTION $\ln(x^2+1)$

751. $$\int \frac{e^x\sqrt{e^x-1}}{e^x-2}\,dx = 2\sqrt{e^x-1} + \ln\left|\frac{\sqrt{e^x-1}-1}{\sqrt{e^x-1}+1}\right| + C$$


$\int \frac{e^x\sqrt{e^x-1}}{e^x-2}\,dx = $ ... BY SUBSTITUTION...

$= \int \frac{e^x u}{u^2-1}\,\frac{2u}{e^x}\,du = \int \frac{2u^2}{u^2-1}\,du$

$= \int \frac{2(u^2-1)+2}{u^2-1}\,du = \int 2 + \frac{2}{(u-1)(u+1)}\,du$

$= \int 2 + \frac{1}{u-1} - \frac{1}{u+1}\,du$

$= 2u + \ln|u-1| - \ln|u+1| + C = 2u + \ln\left|\frac{u-1}{u+1}\right| + C$

$= 2\sqrt{e^x-1} + \ln\left|\frac{\sqrt{e^x-1}-1}{\sqrt{e^x+1}+1}\right| + C$

$u=\sqrt{e^x-1}$

$u^2=e^x-1$

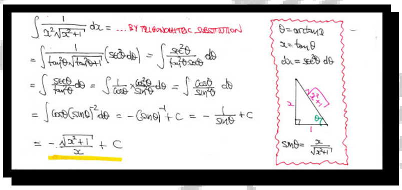
$e^x=u^2+1$

$e^x-2=u^2-1$

$e^x\,dx=2u\,du$

$dx=\frac{2u}{e^x}\,du$

752. $$\int \frac{1}{x^2\sqrt{x^2+1}}\,dx = -\frac{\sqrt{x^2+1}}{x} + C$$


$\int \frac{1}{x^2\sqrt{x^2+1}}\,dx = $ ... BY TRIGONOMETRIC SUBSTITUTION

$= \int \frac{1}{\tan^2\theta\sqrt{\tan^2\theta+1}}(\sec^2\theta\,d\theta) = \int \frac{\sec^2\theta}{\tan^2\theta\sec\theta}\,d\theta$

$= \int \frac{\sec\theta}{\tan^2\theta}\,d\theta = \int \frac{1}{\cos\theta}\times\frac{\cos^2\theta}{\sin^2\theta}\,d\theta = \int \frac{\cos\theta}{\sin^2\theta}\,d\theta$

$= \int \cos\theta(\sin\theta)^{-2}\,d\theta = -(\sin\theta)^{-1} + C = -\frac{1}{\sin\theta} + C$

$= -\frac{\sqrt{x^2+1}}{x} + C$

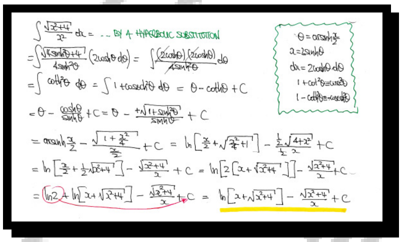
$\theta=\arctan x$

$x=\tan\theta$

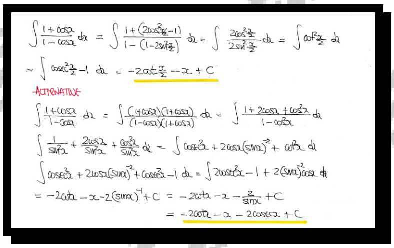
$dx=\sec^2\theta\,d\theta$

$\sin\theta=\frac{x}{\sqrt{x^2+1}}$

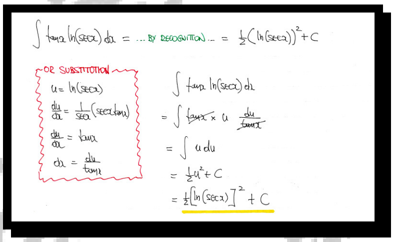
753. $$\int \frac{\sqrt{x^2+4}}{x^2}\,dx = \ln\left[x+\sqrt{x^2+4}\right] - \frac{\sqrt{x^2+4}}{x} + C$$



754. $$\int \frac{1+\cos x}{1-\cos x}\,dx = \begin{cases} -x-2\cot\left(\frac{1}{2}x\right) + C \\ -x-2\cot x - \operatorname{cosec} x + C \end{cases}$$



755. $$\int \tan x \ln(\sec x)\,dx = \frac{1}{2}\left[\ln(\sec x)\right]^2 + C$$



756. $$\int \frac{\arctan x}{x^2}\,dx = -\frac{\arctan x}{x} + \ln|x| - \frac{1}{2}\ln\left(x^2+1\right) + C$$

$$\int \frac{\arctan x}{x^2}\,dx = \ldots \text{ INTEGRATION BY PARTS OR MANIPULATIONS}$$

$$\frac{d}{dx}\left(\frac{1}{x}\arctan x\right) = -\frac{1}{x^2}\arctan x + \frac{1}{x}\,\frac{1}{x^2+1}$$

$$\frac{1}{x}\arctan x = \int -\frac{1}{x^2}\arctan x\,dx + \int \frac{1}{x(x^2+1)}\,dx$$

$$\int \frac{\arctan x}{x^2}\,dx = -\frac{1}{x}\arctan x + \int \frac{1}{x} + \frac{-x}{x^2+1}\,dx$$

(BY INSPECTION)

$$\int \frac{\arctan x}{x^2}\,dx = -\frac{1}{x}\arctan x + \ln|x| - \frac{1}{2}\ln(x^2+1) + C$$

757. $$\int \frac{\sin x\cos x + x\ln x}{x\cos^2 x}\,dx = (\ln x)(\tan x) + C$$

$$\int \frac{\sin x\cos x + x\ln x}{x\cos^2 x}\,dx = \int \frac{\sin x}{x\cos x} + \frac{\ln x}{\cos^2 x}\,dx$$
$$= \int \frac{1}{x}\tan x + (\ln x)\sec^2 x\,dx$$
$$= \int \frac{1}{x}\tan x\,dx + \int (\ln x)\sec^2 x\,dx$$

INTEGRATE BY PARTS ONE OF THE TWO INTEGRALS – LEAVE THE OTHER

$\ln x$	$\frac{1}{x}$
$\tan x$	$\sec^2 x$

$$= \cancel{\int \frac{1}{x}\tan x\,dx} + (\ln x)\tan x - \cancel{\int \frac{1}{x}\tan x\,dx}$$
$$= \ln x\tan x + C$$

758. $$\int \frac{x^2\left(x^4+1\right)}{\sqrt[4]{x^4+2}}\,dx = \frac{1}{6}\left(x^8+2x^4\right)^{\frac{3}{4}} + C$$

$$\int \frac{x^2(x^4+1)}{\sqrt[4]{x^4+2}}\,dx \ \ldots \text{ MANIPULATE AS FOLLOWS} \ldots \int x^2(x^4+1)(x^4+2)^{-\frac{1}{4}}\,dx$$

$$= \int x^3(x^4+1)\,x^{-1}(x^4+2)^{-\frac{1}{4}}\,dx = \int (x^7+x^3)(x^4)^{-\frac{1}{4}}(x^4+2)^{-\frac{1}{4}}\,dx$$

$$= \int (x^7+x^3)\left[x^4(x^4+2)\right]^{-\frac{1}{4}}\,dx = \int (x^7+x^3)(x^8+2x^4)^{-\frac{1}{4}}\,dx$$

NOW NOTE THAT $\frac{d}{dx}(x^8+2x^4) = 8x^7+8x^3 = 8(x^7+x^3)$

$$= \frac{1}{8}\int 8(x^7+x^3)(x^8+2x^4)^{-\frac{1}{4}}\,dx$$

BY RECOGNITION (REVERSE CHAIN RULE) OR THE SUBSTITUTION $u = x^8+2x^4$

$$= \frac{1}{8}\times\frac{4}{3}(x^8+2x^4)^{\frac{3}{4}} + C = \frac{1}{6}(x^8+2x^4)^{\frac{3}{4}} + C$$

759. $\displaystyle\int \sqrt{1+\tan x}\, dx =$

$$\left[\begin{array}{l} \dfrac{1}{\sqrt{2\sqrt{2}-2}}\arctan\left[\dfrac{1-\sqrt{2}+\tan x}{\sqrt{\left(2\sqrt{2}-2\right)\left(1+\tan x\right)}}\right]-\dfrac{1}{\sqrt{2\sqrt{2}-2}}\operatorname{artanh}\left[\dfrac{1+\sqrt{2}+\tan x}{\sqrt{\left(2\sqrt{2}+2\right)\left(1+\tan x\right)}}\right]+C \\ \dfrac{1}{\sqrt{2\sqrt{2}-2}}\arctan\left[\dfrac{1-\sqrt{2}+\tan x}{\sqrt{\left(2\sqrt{2}-2\right)\left(1+\tan x\right)}}\right]-\dfrac{1}{\sqrt{2\sqrt{2}-2}}\ln\left[\dfrac{1+\sqrt{2}+\tan x-\sqrt{\left(2\sqrt{2}+2\right)\left(1+\tan x\right)}}{1+\sqrt{2}+\tan x+\sqrt{\left(2\sqrt{2}+2\right)\left(1+\tan x\right)}}\right]+C \end{array}\right]$$

$\displaystyle\int \sqrt{1+\tan x}\, dx$ .... BY SUBSTITUTION

$$\boxed{\begin{aligned} t&=\sqrt{1+\tan x} \\ t^2&=1+\tan x \\ t^2-1&=\tan x \\ 2t\,dt&=\sec^2 x\,dx \\ 2t\,dt&=(1+\tan^2 x)\,dx \\ 2t\,dt&=\left[1+(t^2-1)^2\right]dx \\ 2t\,dt&=(t^4-2t^2+2)\,dx \\ dx&=\frac{2t}{t^4-2t^2+2}\,dt \end{aligned}}$$

$$= \int t\left(\frac{2t}{t^4-2t^2+2}\right)dt = \int \frac{2t^2}{t^4-2t^2+2}\,dt$$

$$= \int \frac{2}{t^2-2+\frac{2}{t^2}}\,dt$$

$$= \int \frac{1}{\left(t^2+\frac{2}{t^2}\right)-2}\,dt + \int \frac{1}{\left(t^2+\frac{2}{t^2}\right)-2}\,dt$$

PREPARE THESE INTEGRALS FOR THE "CROSS TWIN" METHOD

$$\left(t-\frac{\sqrt{2}}{t}\right)^2 = t^2-2\sqrt{2}+\frac{2}{t^2} \qquad \left(t+\frac{\sqrt{2}}{t}\right)^2 = t^2+2\sqrt{2}+\frac{2}{t^2}$$

$$\left(t-\frac{\sqrt{2}}{t}\right)^2+2\sqrt{2} = t^2+\frac{2}{t^2} \qquad \left(t+\frac{\sqrt{2}}{t}\right)^2-2\sqrt{2} = t^2+\frac{2}{t^2}$$

$$t^2+\frac{2}{t^2}-2 = \left(t-\frac{\sqrt{2}}{t}\right)^2+2\sqrt{2}-2 \qquad \left(t-\frac{\sqrt{2}}{t}\right)^2-2\sqrt{2}-2 = t^2+\frac{2}{t^2}-2$$

$$t^2+\frac{2}{t^2}-2 = \left(t-\frac{\sqrt{2}}{t}\right)^2+a^2 \qquad t^2+\frac{2}{t^2}-2 = \left(t+\frac{\sqrt{2}}{t}\right)^2-b^2$$

WHERE $a^2=2\sqrt{2}-2$ $\qquad$ WHERE $b^2=2\sqrt{2}+2$

RETURNING TO THE TWO INTEGRALS NOTING THAT THEIR DENOMINATORS ARE IDENTICAL

$$= \int \frac{1+\frac{\sqrt{2}}{t^2}}{t^2+\frac{2}{t^2}-2}\,dt + \int \frac{1-\frac{\sqrt{2}}{t^2}}{t^2+\frac{2}{t^2}-2}\,dt$$

$$= \int \frac{1+\frac{\sqrt{2}}{t^2}}{\left(t-\frac{\sqrt{2}}{t}\right)^2+a^2}\,dt + \int \frac{1-\frac{\sqrt{2}}{t^2}}{\left(t+\frac{\sqrt{2}}{t}\right)^2-b^2}\,dt$$

NOW THE CROSS TWINS STANDARD SUBSTITUTIONS

$$\boxed{\begin{aligned} u&=t-\frac{\sqrt{2}}{t} \\ du&=\left(1+\frac{\sqrt{2}}{t^2}\right)dt \end{aligned}} \qquad \boxed{\begin{aligned} v&=t+\frac{\sqrt{2}}{t} \\ dv&=\left(1-\frac{\sqrt{2}}{t}\right)dt \end{aligned}}$$

$$= \int \frac{du}{u^2+a^2} + \int \frac{dv}{v^2-b^2}$$

NOW PROCEED ON THE SECOND INTEGRAL BY PARTIAL FRACTIONS OR INTEGRATION TO artanh / arcoth

$$= \int \frac{du}{u^2+a^2} - \int \frac{dv}{b^2-v^2}$$

$$\boxed{\begin{aligned} \int \frac{1}{b^2-x^2}\,dx &= \frac{1}{b}\operatorname{artanh}\left(\frac{x}{b}\right)+C \\ &= \frac{1}{b}\operatorname{arcoth}\left(\frac{x}{b}\right)+C \end{aligned}}$$

$$= \frac{1}{a}\arctan\left(\frac{u}{a}\right)-\frac{1}{b}\operatorname{artanh}\left(\frac{v}{b}\right)+C$$

$$= \frac{1}{a}\arctan\left(\frac{t-\frac{\sqrt{2}}{t}}{a}\right)-\frac{1}{b}\operatorname{artanh}\left(\frac{t+\frac{\sqrt{2}}{t}}{b}\right)+C$$

$$= \frac{1}{a}\arctan\left(\frac{t^2-\sqrt{2}}{at}\right)-\frac{1}{b}\operatorname{artanh}\left(\frac{t^2+\sqrt{2}}{bt}\right)+C$$

$$= \frac{1}{\sqrt{2\sqrt{2}-2}}\arctan\left(\frac{1-\sqrt{2}+\tan x}{\sqrt{2\sqrt{2}-2}\sqrt{1+\tan x}}\right)-\frac{1}{\sqrt{2\sqrt{2}+2}}\operatorname{artanh}\left(\frac{1+\sqrt{2}+\tan x}{\sqrt{2\sqrt{2}+2}\sqrt{1+\tan x}}\right)+C$$

$$= \frac{1}{\sqrt{2\sqrt{2}-2}}\arctan\left(\frac{1-\sqrt{2}+\tan x}{\sqrt{(2\sqrt{2}-2)(1+\tan x)}}\right)-\frac{1}{\sqrt{2\sqrt{2}+2}}\operatorname{artanh}\left(\frac{1+\sqrt{2}+\tan x}{\sqrt{(2\sqrt{2}+2)(1+\tan x)}}\right)+C$$

ALTERNATIVE BY PARTIAL FRACTIONS FOR THE "SECOND" INTEGRAL

$$\int \frac{1}{v^2-b^2}\,dv = \int \frac{1}{(v-b)(v+b)}\,dv = \int \frac{\frac{1}{2b}}{v-b}-\frac{\frac{1}{2b}}{v+b}\,dv$$

$$= \frac{1}{2b}\int \frac{1}{v-b}-\frac{1}{v+b}\,dv = \frac{1}{2b}\ln\left|\frac{v-b}{v+b}\right|+C = \frac{1}{2b}\ln\left|\frac{t+\frac{\sqrt{2}}{t}-b}{t+\frac{\sqrt{2}}{t}+b}\right|+C$$

$$= \frac{1}{2b}\ln\left|\frac{t^2-bt+\sqrt{2}}{t^2+bt+\sqrt{2}}\right| = \frac{1}{2\sqrt{2\sqrt{2}+2}}\ln\left|\frac{1+\tan x-\sqrt{(2\sqrt{2}+2)(1+\tan x)}+\sqrt{2}}{1+\tan x+\sqrt{(2\sqrt{2}+2)(1+\tan x)}+\sqrt{2}}\right|+C$$

$$= \frac{1}{2\sqrt{2\sqrt{2}+2}}\ln\left|\frac{1+\sqrt{2}+\tan x-\sqrt{(2\sqrt{2}+2)(1+\tan x)}}{1+\sqrt{2}+\tan x+\sqrt{(2\sqrt{2}+2)(1+\tan x)}}\right|+C$$

$$\therefore = \frac{1}{\sqrt{2\sqrt{2}-2}}\arctan\left[\frac{1-\sqrt{2}+\tan x}{\sqrt{(2\sqrt{2}-2)(1+\tan x)}}\right]-\frac{1}{2\sqrt{2\sqrt{2}+2}}\ln\left|\frac{1+\sqrt{2}+\tan x-\sqrt{(2\sqrt{2}+2)(1+\tan x)}}{1+\sqrt{2}+\tan x+\sqrt{(2\sqrt{2}+2)(1+\tan x)}}\right|+C$$