INTEGRATION STRUCTURED EXAM QUESTIONS PART II

Question 1 (**)

$$\frac{4}{(x-1)\left(x^2+1\right)} \equiv \frac{A}{x-1} + \frac{Bx+C}{x^2+1}.$$

a) Find the values of A, B and C in the above identity.

b) Hence find

$$\int \frac{4}{(x-1)\left(x^2+1\right)}\,dx.$$

$\boxed{A=2}$, $\boxed{B=-2}$, $\boxed{C=-2}$, $\boxed{2\ln|x-1| - \ln\left(x^2+1\right) - 2\arctan x + C}$

(a) $\frac{4}{(x-1)(x^2+1)} \equiv \frac{A}{x-1} + \frac{Bx+C}{x^2+1}$

$4 \equiv A(x^2+1) + (x-1)(Bx+C)$

If $x=1$, $4 = 2A \Rightarrow A = 2$

If $x=0$, $4 = A - C \Rightarrow C = -2$

If $x=2$, $4 = 5A + 2B + C$

$4 = 10 + 2B - 2$

$-4 = 2B$

$B = -2$

(b) $\int \frac{4}{(x-1)(x^2+1)}\,dx = \int \frac{2}{x-1} - \frac{2x+2}{x^2+1}\,dx$

$= \int \frac{2}{x-1} - \frac{2x}{x^2+1} - \frac{2}{x^2+1}\,dx$

$= 2\ln|x-1| - \ln|x^2+1| - 2\arctan x + C$

Question 2 (**)

Find an exact value for

$$\int_0^{\sqrt{3}} \frac{3}{\sqrt{4-x^2}}\,dx.$$

$\boxed{\pi}$

$\int_0^{\sqrt{3}} \frac{3}{\sqrt{4-x^2}}\,dx = \int_0^{\sqrt{3}} \frac{3}{\sqrt{2^2-x^2}}\,dx = \left[3\arcsin\frac{x}{2}\right]_0^{\sqrt{3}}$

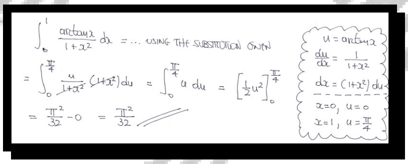
$= 3\arcsin\frac{\sqrt{3}}{2} - 3\arcsin 0 = 3 \times \frac{\pi}{3} = \pi$

Question 3 **(**)**

By using the substitution $u = \arctan x$, or otherwise, find an exact value for

$$\int_0^1 \frac{\arctan x}{1+x^2}\ dx .$$

$$\boxed{\frac{\pi^2}{32}}$$



$\int_0^1 \frac{\arctan x}{1+x^2}\,dx = \dots$ USING THE SUBSTITUTION GIVEN

$= \int_0^{\frac{\pi}{4}} \frac{u}{1+x^2}(1+x^2)\,du = \int_0^{\frac{\pi}{4}} u\,du = \left[\frac{1}{2}u^2\right]_0^{\frac{\pi}{4}}$

$= \frac{\pi^2}{32} - 0 = \frac{\pi^2}{32}$

$u = \arctan x$

$\frac{du}{dx} = \frac{1}{1+x^2}$

$dx = (1+x^2)\,du$

$x=0,\ u=0$

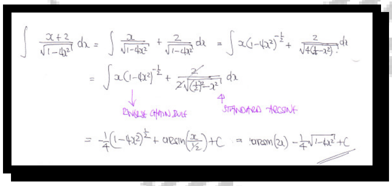
$x=1,\ u=\frac{\pi}{4}$

Question 4 **(**)**

Find an expression for

$$\int \frac{x+2}{\sqrt{1-4x^2}}\ dx .$$

$$\boxed{\arcsin 2x - \frac{1}{4}\sqrt{1-4x^2} + C}$$



$\int \frac{x+2}{\sqrt{1-4x^2}}\,dx = \int \frac{x}{\sqrt{1-4x^2}} + \frac{2}{\sqrt{1-4x^2}}\,dx = \int x(1-4x^2)^{-\frac{1}{2}} + \frac{2}{\sqrt{4(\frac{1}{4}-x^2)}}\,dx$

$= \int x(1-4x^2)^{-\frac{1}{2}} + \frac{2}{2\sqrt{(\frac{1}{2})^2-x^2}}\,dx$

REVERSE CHAIN RULE — STANDARD ARCSINE

$= -\frac{1}{4}(1-4x^2)^{\frac{1}{2}} + \arcsin\left(\frac{x}{\frac{1}{2}}\right) + C = \arcsin(2x) - \frac{1}{4}\sqrt{1-4x^2} + C$

Question 5 **(**)**

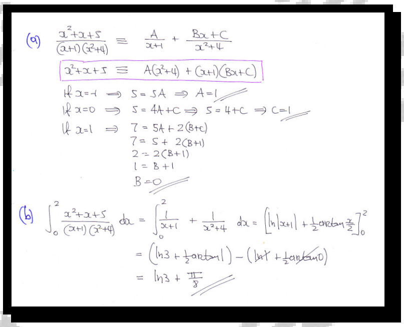
$$\frac{x^2+x+5}{(x+1)\left(x^2+4\right)} \equiv \frac{A}{x+1} + \frac{Bx+C}{x^2+4}.$$

a) Find the values of A, B and C in the above identity.

b) Hence find the exact value of

$$\int_0^2 \frac{x^2+x+5}{(x+1)\left(x^2+4\right)}\, dx.$$

$\boxed{A=1}$, $\boxed{B=0}$, $\boxed{C=1}$, $\boxed{\frac{\pi}{8}+\ln 3}$



(a) $\frac{x^2+x+5}{(x+1)(x^2+4)} \equiv \frac{A}{x+1} + \frac{Bx+C}{x^2+4}$

$x^2+x+5 \equiv A(x^2+4) + (x+1)(Bx+C)$

If $x=-1 \Rightarrow 5 = 5A \Rightarrow A=1$

If $x=0 \Rightarrow 5 = 4A+C \Rightarrow 5 = 4+C \Rightarrow C=1$

If $x=1 \Rightarrow 7 = 5A+2(B+C)$

$7 = 5+2(B+1)$

$2 = 2(B+1)$

$1 = B+1$

$B=0$

(b) $\int_0^2 \frac{x^2+x+5}{(x+1)(x^2+4)}\, dx = \int_0^2 \frac{1}{x+1} + \frac{1}{x^2+4}\, dx = \left[\ln|x+1| + \frac{1}{2}\arctan\frac{x}{2}\right]_0^2$

$= \left(\ln 3 + \frac{1}{2}\arctan 1\right) - \left(\ln 1 + \frac{1}{2}\arctan 0\right)$

$= \ln 3 + \frac{\pi}{8}$

Question 6 (**)

Find an exact value for

$$\int_0^{\frac{1}{2}} \frac{6x+1}{\sqrt{1-x^2}} \, dx \, .$$

$$\boxed{\frac{\pi}{6} - 3\sqrt{3} + 6}$$

$$\int_0^{\frac{1}{2}} \frac{6x+1}{\sqrt{1-x^2}} \, dx = \int_0^{\frac{1}{2}} \frac{6x}{\sqrt{1-x^2}} + \frac{1}{\sqrt{1-x^2}} \, dx = \int_0^{\frac{1}{2}} 6x(1-x^2)^{-\frac{1}{2}} + \frac{1}{(1-x^2)^{\frac{1}{2}}} \, dx$$

REVERSE CHAIN RULE; STANDARD ARCSINE

$$= \left[-6(1-x^2)^{\frac{1}{2}} + \arcsin x\right]_0^{\frac{1}{2}}$$

$$= \left[\arcsin x - 6\sqrt{1-x^2}\right]_0^{\frac{1}{2}} = \left(\arcsin\tfrac{1}{2} - 6\sqrt{\tfrac{3}{4}}\right) - \left(\arcsin 0 - 6\right)$$

$$= \frac{\pi}{6} - 3\sqrt{3} + 6$$

Question 7 (**)

Use the substitution $u = \sin x$ to find an exact value in terms of natural logarithms for

$$\int_0^{\frac{\pi}{2}} \frac{\cos x}{\sqrt{1+\sin^2 x}} \, dx \, .$$

$$\boxed{\ln\left(1+\sqrt{2}\right)}$$

$$\int_0^{\frac{\pi}{2}} \frac{\cos x}{\sqrt{1+\sin^2 x}} \, dx \quad \text{USING THE SUBSTITUTION GIVEN}$$

$$= \int_0^1 \frac{\cos x}{\sqrt{1+u^2}} \, \frac{du}{\cos x} = \int_0^1 \frac{1}{\sqrt{1+u^2}} \, du$$

$$= \left[\text{arsinh}\, u\right]_0^1 = \text{arsinh}\, 1 - \text{arsinh}\, 0$$

$$= \ln\left(1+\sqrt{2}\right)$$

$u = \sin x$

$\frac{du}{dx} = \cos x$

$dx = \frac{du}{\cos x}$

$x = 0, \; u = 0$

$x = \frac{\pi}{2}, \; u = 1$

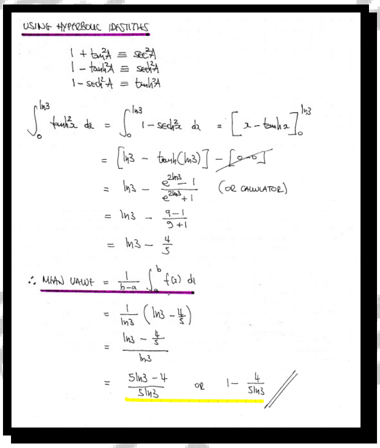
Question 8 (**)

The function f is defined as

$$f(x) \equiv \tanh^2 x\,,\ x \in \mathbb{R}\,,\ 0 \le x \le \ln 3\,.$$

Determine the mean value of f, in its entire domain.

$\boxed{}$, $\boxed{1 - \dfrac{4}{5\ln 3} \approx 0.272}$



USING HYPERBOLIC IDENTITIES

$1 + \tan^2 A \equiv \sec^2 A$

$1 - \tanh^2 A \equiv \text{sech}^2 A$

$1 - \text{sech}^2 A \equiv \tanh^2 A$

$$\int_0^{\ln 3} \tanh^2 x \, dx = \int_0^{\ln 3} 1 - \text{sech}^2 x \, dx = \Big[x - \tanh x\Big]_0^{\ln 3}$$

$$= \big[\ln 3 - \tanh(\ln 3)\big] - [0 - 0]$$

$$= \ln 3 - \frac{e^{2\ln 3} - 1}{e^{2\ln 3} + 1} \quad \text{(OR CALCULATOR)}$$

$$= \ln 3 - \frac{9-1}{9+1}$$

$$= \ln 3 - \frac{4}{5}$$

$\therefore$ MEAN VALUE $= \dfrac{1}{b-a}\displaystyle\int_a^b f(x)\, dx$

$$= \frac{1}{\ln 3}\left(\ln 3 - \frac{4}{5}\right)$$

$$= \frac{\ln 3 - \frac{4}{5}}{\ln 3}$$

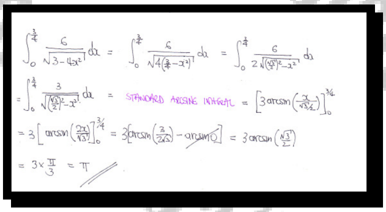
$$= \frac{5\ln 3 - 4}{5\ln 3} \quad \text{or} \quad 1 - \frac{4}{5\ln 3}$$

Question 9 **(**+)**

Find an exact value for

$$\int_0^{\frac{3}{4}} \frac{6}{\sqrt{3-4x^2}}\ dx .$$

$\boxed{\pi}$

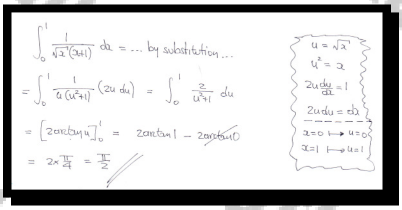


Question 10 **(**+)**

By using a suitable substitution, find in terms of π, the value of

$$\int_0^{1} \frac{1}{(x+1)\sqrt{x}}\ dx .$$

$\boxed{\frac{\pi}{2}}$



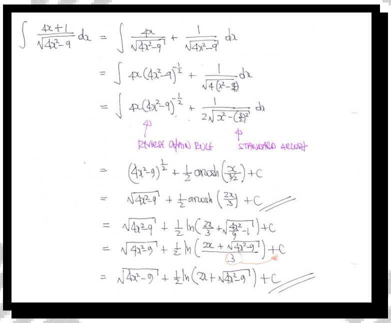
Question 11 **(**+)**

Show clearly that

$$\int \frac{4x+1}{\sqrt{4x^2-9}}\,dx = f(x)+\tfrac{1}{2}\ln\left[2x+f(x)\right]+C,$$

where $f(x)$ is a function to be found.

$$\boxed{f(x)=\sqrt{4x^2-9}}$$



$$\int \frac{4x+1}{\sqrt{4x^2-9}}\,dx = \int \frac{4x}{\sqrt{4x^2-9}} + \frac{1}{\sqrt{4x^2-9}}\,dx$$
$$= \int 4x(4x^2-9)^{-\frac{1}{2}} + \frac{1}{\sqrt{4(x^2-\frac{9}{4})}}\,dx$$
$$= \int 4x(4x^2-9)^{-\frac{1}{2}} + \frac{1}{2\sqrt{x^2-(\frac{3}{2})^2}}\,dx$$

REVERSE CHAIN RULE — STANDARD ARCOSH

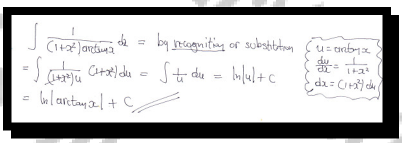
$$= (4x^2-9)^{\frac{1}{2}} + \tfrac{1}{2}\operatorname{arcosh}\left(\frac{x}{3/2}\right)+C$$
$$= \sqrt{4x^2-9} + \tfrac{1}{2}\operatorname{arcosh}\left(\frac{2x}{3}\right)+C$$
$$= \sqrt{4x^2-9} + \tfrac{1}{2}\ln\left(\frac{2x}{3}+\sqrt{\frac{4x^2}{9}-1}\right)+C$$
$$= \sqrt{4x^2-9} + \tfrac{1}{2}\ln\left(\frac{2x+\sqrt{4x^2-9}}{3}\right)+C$$
$$= \sqrt{4x^2-9} + \tfrac{1}{2}\ln\left(2x+\sqrt{4x^2-9}\right)+C$$

Question 12 **(**+)**

By using a suitable substitution, or otherwise, find

$$\int \frac{1}{(1+x^2)\arctan x}\,dx.$$

$$\boxed{\ln|\arctan x|+C}$$



$$\int \frac{1}{(1+x^2)\arctan x}\,dx = \text{by recognition or substitution}$$
$$= \int \frac{1}{(1+x^2)u}(1+x^2)\,du = \int \frac{1}{u}\,du = \ln|u|+C$$
$$= \ln|\arctan x|+C$$

$u=\arctan x$, $\frac{du}{dx}=\frac{1}{1+x^2}$, $dx=(1+x^2)\,du$

Question 13 (**+)

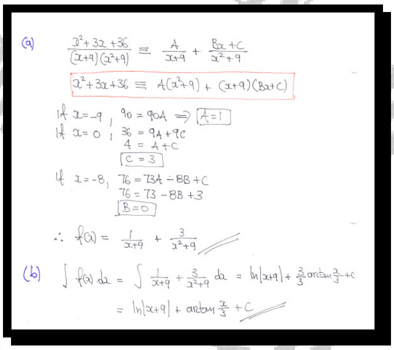
$$f(x) \equiv \frac{x^2+3x+36}{(x+9)\left(x^2+9\right)}.$$

a) Express $f(x)$ into partial fractions.

b) Hence find

$$\int f(x)\,dx.$$

$$\boxed{\frac{1}{x+9}+\frac{3}{x^2+9}}, \quad \boxed{\ln|x+9|+\arctan\frac{x}{3}+C}$$



(a) $\dfrac{x^2+3x+36}{(x+9)(x^2+9)} \equiv \dfrac{A}{x+9}+\dfrac{Bx+C}{x^2+9}$

$x^2+3x+36 \equiv A(x^2+9)+(x+9)(Bx+C)$

If $x=-9$, $90=90A \Rightarrow \boxed{A=1}$

If $x=0$, $36=9A+9C$

$4=A+C$

$\boxed{C=3}$

If $x=-8$, $76=73A-8B+C$

$76=73-8B+3$

$\boxed{B=0}$

$\therefore f(x)=\dfrac{1}{x+9}+\dfrac{3}{x^2+9}$

(b) $\int f(x)\,dx=\int \dfrac{1}{x+9}+\dfrac{3}{x^2+9}\,dx=\ln|x+9|+\frac{3}{3}\arctan\frac{x}{3}+C$

$=\ln|x+9|+\arctan\frac{x}{3}+C$

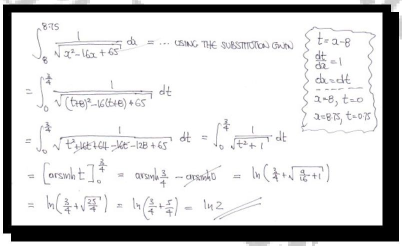
Question 14 **(***)**

Use the substitution $t = x - 8$ to find the exact value of

$$\int_{8}^{8.75} \frac{1}{\sqrt{x^2 - 16x + 65}} \, dx ,$$

giving the answer as a single natural logarithm.

$\boxed{\ln 2}$



Question 15 **(***)**

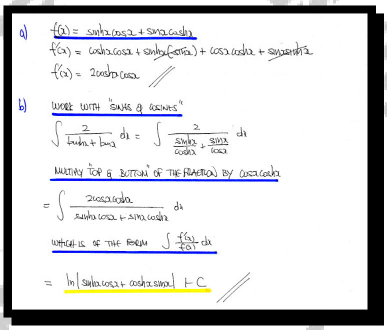
$$f(x) = \sinh x \cos x + \sin x \cosh x , \ x \in \mathbb{R} .$$

a) Find a simplified expression for $f'(x)$.

b) Use the answer to part (a) to find

$$\int \frac{2}{\tanh x + \tan x} \, dx .$$

FP2-K , $\boxed{f'(x) = 2\cosh x \cos x}$, $\boxed{\ln\left|\sinh x \cos x + \sin x \cosh x\right| + C}$



Question 16 (***)

Find the exact value of

$$\int_{2.5}^{7.5} \frac{15\sqrt{3}}{4x^2+75} \, dx .$$

$$\boxed{\frac{\pi}{4}}$$

$\int_{2.5}^{7.5} \frac{15\sqrt{3}}{4x^2+75}\,dx =$ DIVIDE TOP & BOTTOM OF THE INTEGRAND BY 4 ....

$= \int_{2.5}^{7.5} \frac{\frac{15}{4}\sqrt{3}}{x^2+\frac{75}{4}}\,dx = \frac{15}{4}\sqrt{3}\int_{2.5}^{7.5} \frac{1}{x^2+\left(\frac{\sqrt{75}}{2}\right)^2}\,dx = \dots$ standard arctan ...

$= \frac{15}{4}\sqrt{3} \times \frac{1}{\frac{\sqrt{75}}{2}} \times \left[\arctan\left(\frac{x}{\frac{\sqrt{75}}{2}}\right)\right]_{2.5}^{7.5} = \frac{15\sqrt{3}}{4} \times \frac{2}{\sqrt{75}} \times \left[\arctan\left(\frac{2x}{\sqrt{75}}\right)\right]_{2.5}^{7.5}$

$= \frac{15\sqrt{3}}{4} \times \frac{2}{5\sqrt{3}}\left[\arctan\left(\frac{15}{5\sqrt{3}}\right) - \arctan\left(\frac{5}{5\sqrt{3}}\right)\right] = \frac{3}{2}\left[\arctan\sqrt{3} - \arctan\frac{\sqrt{3}}{3}\right]$

$= \frac{3}{2}\left[\frac{\pi}{3} - \frac{\pi}{6}\right] = \frac{3}{2} \times \frac{\pi}{6} = \frac{\pi}{4}$

Question 17 (***)

Find the exact value of

$$\int_{0}^{1} \frac{2\sqrt{3}}{\sqrt{4\pi^2 - 3\pi^2 x^2}} \, dx .$$

$$\boxed{\frac{2}{3}}$$

$\int_0^1 \frac{2\sqrt{3}}{\sqrt{4\pi^2-3\pi^2x^2}}\,dx = \int_0^1 \frac{2\sqrt{3}}{\sqrt{3\pi^2\left(\frac{4}{3}-x^2\right)}} = \int_0^1 \frac{2\sqrt{3}}{\sqrt{3}\,\pi\sqrt{\frac{4}{3}-x^2}}\,dx$

$= \frac{2}{\pi}\int \frac{1}{\sqrt{\left(\frac{2}{\sqrt{3}}\right)^2 - x^2}}\,dx = \left[\frac{2}{\pi}\arcsin\left(\frac{x}{2/\sqrt{3}}\right)\right]_0^1$

$= \frac{2}{\pi}\left[\arcsin\left(\frac{\sqrt{3}x}{2}\right)\right]_0^1$

$= \frac{2}{\pi}\left[\arcsin\left(\frac{\sqrt{3}}{2}\right) - \arcsin 0\right]$

$= \frac{2}{\pi} \times \frac{\pi}{3} = \frac{2}{3}$

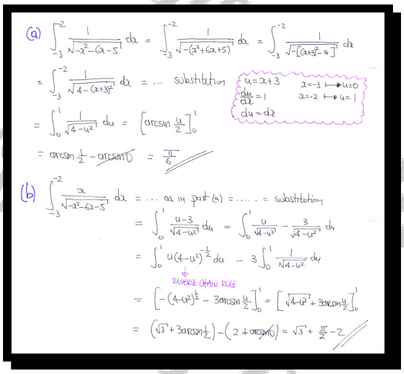
Question 18 (***)

Find the exact value of each of the following integrals.

a) $$\int_{-3}^{-2} \frac{1}{\sqrt{-x^2-6x-5}}\,dx.$$

b) $$\int_{-3}^{-2} \frac{x}{\sqrt{-x^2-6x-5}}\,dx.$$

$\boxed{\frac{\pi}{6}}$, $\boxed{\sqrt{3}+\frac{\pi}{2}-2}$

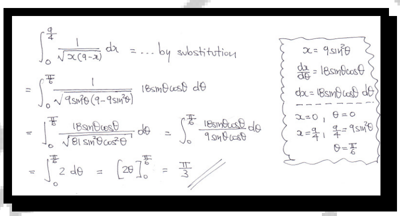


Question 19 (***)

By using the substitution $x = 9\sin^2\theta$, or otherwise, find the exact value of

$$\int_{0}^{\frac{9}{4}} \frac{1}{\sqrt{x(9-x)}}\,dx.$$

$\boxed{\frac{\pi}{3}}$

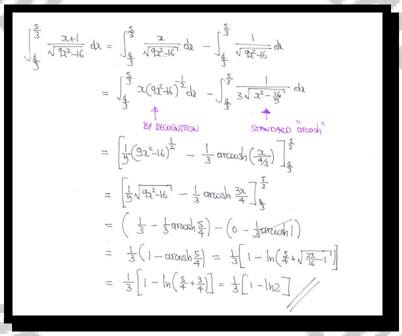


Question 20 (***)

Find the exact value of

$$\int_{\frac{4}{3}}^{\frac{5}{3}} \frac{x+1}{\sqrt{9x^2-16}}\, dx .$$

$$\boxed{\frac{1}{3}(1-\ln 2)}$$



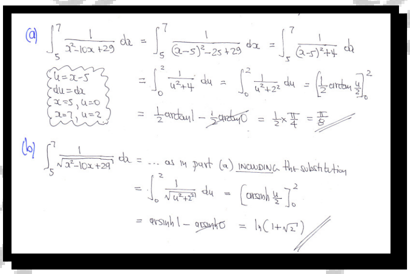
Question 21 (***)

Find the exact value of each of the following integrals.

a) $$\int_{5}^{7} \frac{1}{x^2-10x+29}\, dx .$$

b) $$\int_{5}^{7} \frac{x}{\sqrt{x^2-10x+29}}\, dx .$$

$$\boxed{\frac{\pi}{8}}, \ \boxed{\ln\left(1+\sqrt{2}\right)}$$

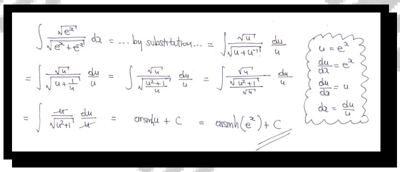


Question 22 (***)

Use the substitution $u = e^x$ to find

$$\int \frac{\sqrt{e^x}}{\sqrt{e^x + e^{-x}}}\, dx .$$

$$\boxed{\operatorname{arsinh}\left(e^x\right) + C}$$



$$\int \frac{\sqrt{e^x}}{\sqrt{e^x+e^{-x}}}\,dx = \dots \text{by substitution} \dots = \int \frac{\sqrt{u}}{\sqrt{u+u^{-1}}}\,\frac{du}{u}$$

$$= \int \frac{\sqrt{u}}{\sqrt{u+\frac{1}{u}}}\,\frac{du}{u} = \int \frac{\sqrt{u}}{\sqrt{\frac{u^2+1}{u}}}\,\frac{du}{u} = \int \frac{\sqrt{u}}{\frac{\sqrt{u^2+1}}{\sqrt{u}}}\,\frac{du}{u}$$

$$= \int \frac{u}{\sqrt{u^2+1}}\,\frac{du}{u} = \operatorname{arsinh} u + C = \operatorname{arsinh}\left(e^x\right) + C$$

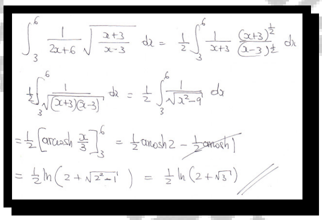
$u = e^x$, $\frac{du}{dx} = e^x$, $\frac{du}{dx} = u$, $dx = \frac{du}{u}$

Question 23 (***)

Find in exact simplified form in terms of natural logarithms

$$\int_3^6 \frac{1}{2x+6}\sqrt{\frac{x+3}{x-2}}\, dx .$$

$$\boxed{\frac{1}{2}\ln\left(2+\sqrt{3}\right)}$$



$$\int_3^6 \frac{1}{2x+6}\sqrt{\frac{x+3}{x-3}}\,dx = \frac{1}{2}\int_3^6 \frac{1}{x+3}\,\frac{(x+3)^{\frac{1}{2}}}{(x-3)^{\frac{1}{2}}}\,dx$$

$$\frac{1}{2}\int_3^6 \frac{1}{\sqrt{(x+3)(x-3)}}\,dx = \frac{1}{2}\int_3^6 \frac{1}{\sqrt{x^2-9}}\,dx$$

$$= \frac{1}{2}\left[\operatorname{arcosh}\frac{x}{3}\right]_3^6 = \frac{1}{2}\operatorname{arcosh}2 - \frac{1}{2}\operatorname{arcosh}1$$

$$= \frac{1}{2}\ln\left(2+\sqrt{2^2-1}\right) = \frac{1}{2}\ln\left(2+\sqrt{3}\right)$$

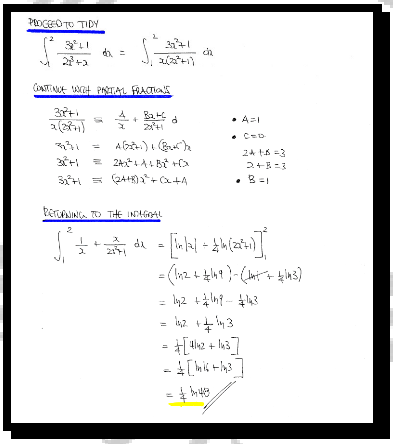
Question 24 (***)

Show that the exact value of the following integral

$$\int_1^2 \frac{3x^2+1}{2x^3+x}\,dx$$

is $\frac{1}{4}\ln 48$.

[ ], proof



PROCEED TO TIDY

$$\int_1^2 \frac{3x^2+1}{2x^3+x}\,dx = \int_1^2 \frac{3x^2+1}{x(2x^2+1)}\,dx$$

CONTINUE WITH PARTIAL FRACTIONS

$$\frac{3x^2+1}{x(2x^2+1)} \equiv \frac{A}{x} + \frac{Bx+C}{2x^2+1}$$

$$3x^2+1 \equiv A(2x^2+1) + (Bx+C)x$$

$$3x^2+1 \equiv 2Ax^2+A+Bx^2+Cx$$

$$3x^2+1 \equiv (2A+B)x^2+Cx+A$$

- $A=1$
- $C=0$

$2A+B=3$

$2+B=3$

- $B=1$

RETURNING TO THE INTEGRAL

$$\int_1^2 \frac{1}{x} + \frac{x}{2x^2+1}\,dx = \left[\ln|x| + \frac{1}{4}\ln(2x^2+1)\right]_1^2$$

$$= \left(\ln 2 + \frac{1}{4}\ln 9\right) - \left(\ln 1 + \frac{1}{4}\ln 3\right)$$

$$= \ln 2 + \frac{1}{4}\ln 9 - \frac{1}{4}\ln 3$$

$$= \ln 2 + \frac{1}{4}\ln 3$$

$$= \frac{1}{4}\left[4\ln 2 + \ln 3\right]$$

$$= \frac{1}{4}\left[\ln 16 + \ln 3\right]$$

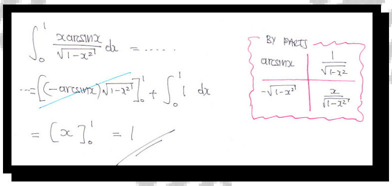
$$= \frac{1}{4}\ln 48$$

Question 25 (***)

$$\int_0^1 \frac{x \arcsin x}{\sqrt{1-x^2}} \, dx .$$

Show that value of the above definite integral is 1.

proof



Question 26 (***+)

$$f(x) \equiv x \arctan x , \ x \in \mathbb{R} .$$

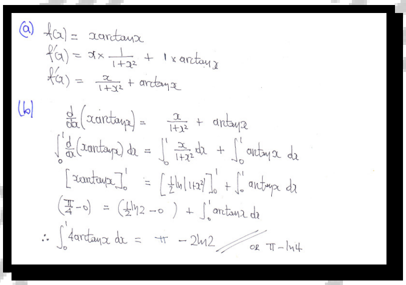
a) Find an expression for $f'(x)$.

b) Use the answer to part (a) to find the exact value of

$$\int_0^1 4 \arctan x \ dx .$$

You may **not** use standard integration by parts to obtain the answer to part (b).

$f'(x) = \frac{x}{1+x^2} + \arctan x$, $\pi - \ln 4$

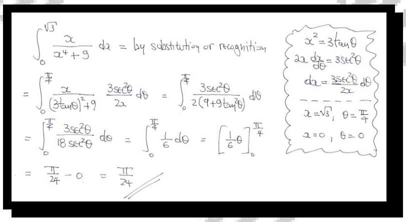


Question 27 **(***+)**

By using the substitution $x^2 = 3\tan\theta$, or otherwise, find the exact value of

$$\int_0^{\sqrt{3}} \frac{x}{x^4+9}\, dx .$$

$\boxed{\dfrac{\pi}{24}}$



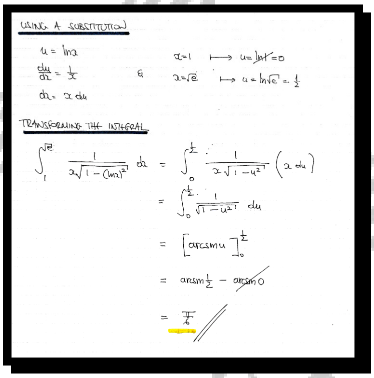
Question 28 **(***+)**

Use an appropriate substitution to find an exact value for the following integral.

$$\int_1^{\sqrt{e}} \frac{1}{x\sqrt{1-(\ln x)^2}}\, dx .$$

You may assume that the integral converges.

$\boxed{}$, $\boxed{\frac{1}{6}\pi}$



Question 29 (***+)

The curves C_1 and C_2 have respective equations

$$y = 18\cosh x, \ x \in \mathbb{R} \qquad \text{and} \qquad y = 12 + 14\sinh x, \ x \in \mathbb{R}.$$

a) Find the exact coordinates of the points of intersection between C_1 and C_2.

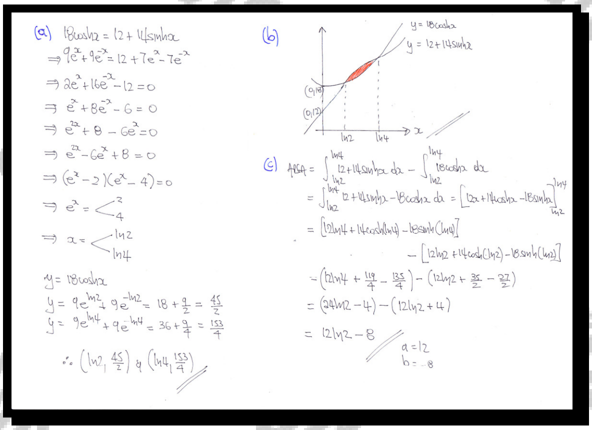
b) Sketch in the same diagram the graph of C_1 and the graph of C_2.

c) Show that the finite region bounded by the graphs of C_1 and C_2 has an area of

$$a\ln 2 + b,$$

where a and b are integers to be found.

$\boxed{\left(\ln 2, \frac{45}{2}\right) \ \& \ \left(\ln 4, \frac{153}{4}\right)}$, $\boxed{12\ln 2 - 8}$



Question 30 (***+)

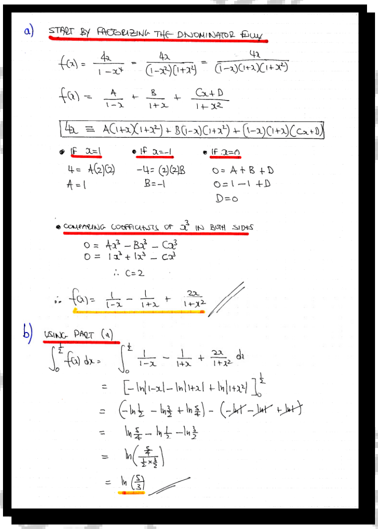
$$f(x) \equiv \frac{4x}{1-x^4}.$$

a) Express $f(x)$ into partial fractions.

b) Hence find, as a single natural logarithm, the value of

$$\int_0^{\frac{1}{2}} f(x)\ dx.$$

$\boxed{}$, $\boxed{f(x) = \frac{1}{1-x} - \frac{1}{1+x} + \frac{2x}{1+x^2}}$, $\boxed{\ln\frac{5}{3}}$



a) START BY FACTORIZING THE DENOMINATOR FULLY

$$f(x) = \frac{4x}{1-x^4} = \frac{4x}{(1-x^2)(1+x^2)} = \frac{4x}{(1-x)(1+x)(1+x^2)}$$

$$f(x) = \frac{A}{1-x} + \frac{B}{1+x} + \frac{Cx+D}{1+x^2}$$

$$4x \equiv A(1+x)(1+x^2) + B(1-x)(1+x^2) + (1-x)(1+x)(Cx+D)$$

- IF $x=1$: $4 = A(2)(2)$, $A = 1$
- IF $x=-1$: $-4 = (2)(2)B$, $B = -1$
- IF $x=0$: $0 = A + B + D$, $0 = 1 - 1 + D$, $D = 0$

- COMPARING COEFFICIENTS OF x^3 IN BOTH SIDES

$$0 = Ax^3 - Bx^3 - Cx^3$$
$$0 = 1x^3 + 1x^3 - Cx^3$$
$$\therefore C = 2$$

$$\therefore f(x) = \frac{1}{1-x} - \frac{1}{1+x} + \frac{2x}{1+x^2}$$

b) USING PART (a)

$$\int_0^{\frac{1}{2}} f(x)\,dx = \int_0^{\frac{1}{2}} \frac{1}{1-x} - \frac{1}{1+x} + \frac{2x}{1+x^2}\,dx$$
$$= \Big[-\ln|1-x| - \ln|1+x| + \ln|1+x^2|\Big]_0^{\frac{1}{2}}$$
$$= \left(-\ln\tfrac{1}{2} - \ln\tfrac{3}{2} + \ln\tfrac{5}{4}\right) - \left(-\ln 1 - \ln 1 + \ln 1\right)$$
$$= \ln\tfrac{5}{4} - \ln\tfrac{1}{2} - \ln\tfrac{3}{2}$$
$$= \ln\left(\frac{\frac{5}{4}}{\frac{1}{2}\times\frac{3}{2}}\right)$$
$$= \ln\left(\frac{5}{3}\right)$$

Question 31 (***+)

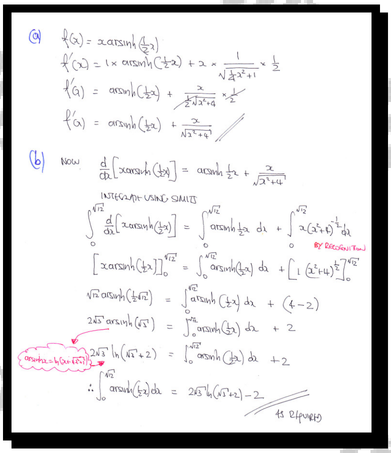
$$f(x) = x \operatorname{arsinh}\left(\tfrac{1}{2}x\right), \quad x \in \mathbb{R}.$$

a) Find a simplified expression for $f'(x)$.

b) Use the answer to part **(a)** to show that

$$\int_0^{\sqrt{12}} \operatorname{arsinh}\left(\tfrac{1}{2}x\right)\, dx = 2\sqrt{3}\ln\left(2+\sqrt{3}\right) - 2.$$

$$\boxed{f'(x) = \operatorname{arsinh}\left(\tfrac{1}{2}x\right) + \frac{x}{\sqrt{x^2+4}}}$$


(a) $f(x) = x\operatorname{arsinh}\left(\tfrac{1}{2}x\right)$

$f'(x) = 1 \times \operatorname{arsinh}\left(\tfrac{1}{2}x\right) + x \times \frac{1}{\sqrt{\tfrac{1}{4}x^2+1}} \times \tfrac{1}{2}$

$f'(x) = \operatorname{arsinh}\left(\tfrac{1}{2}x\right) + \frac{x}{\tfrac{1}{2}\sqrt{x^2+4}} \times \tfrac{1}{2}$

$f'(x) = \operatorname{arsinh}\left(\tfrac{1}{2}x\right) + \frac{x}{\sqrt{x^2+4}}$

(b) NOW $\frac{d}{dx}\left[x\operatorname{arsinh}\left(\tfrac{1}{2}x\right)\right] = \operatorname{arsinh}\tfrac{1}{2}x + \frac{x}{\sqrt{x^2+4}}$

INTEGRATE USING LIMITS

$\int_0^{\sqrt{12}} \frac{d}{dx}\left[x\operatorname{arsinh}\left(\tfrac{1}{2}x\right)\right] = \int_0^{\sqrt{12}} \operatorname{arsinh}\tfrac{1}{2}x \, dx + \int_0^{\sqrt{12}} x(x^2+4)^{-\frac{1}{2}}\, dx$ BY RECOGNITION

$\left[x\operatorname{arsinh}\left(\tfrac{1}{2}x\right)\right]_0^{\sqrt{12}} = \int_0^{\sqrt{12}} \operatorname{arsinh}\left(\tfrac{1}{2}x\right) dx + \left[1\,(x^2+4)^{\frac{1}{2}}\right]_0^{\sqrt{12}}$

$\sqrt{12}\operatorname{arsinh}\left(\tfrac{1}{2}\sqrt{12}\right) = \int_0^{\sqrt{12}} \operatorname{arsinh}\left(\tfrac{1}{2}x\right) dx + (4-2)$

$2\sqrt{3}\operatorname{arsinh}\left(\sqrt{3}\right) = \int_0^{\sqrt{12}} \operatorname{arsinh}\left(\tfrac{1}{2}x\right) dx + 2$

$\operatorname{arsinh}x = \ln\left(x+\sqrt{x^2+1}\right)$

$2\sqrt{3}\ln\left(\sqrt{3}+2\right) = \int_0^{\sqrt{12}} \operatorname{arsinh}\left(\tfrac{1}{2}x\right) dx + 2$

$\therefore \int_0^{\sqrt{12}} \operatorname{arsinh}\left(\tfrac{1}{2}x\right) dx = 2\sqrt{3}\ln\left(\sqrt{3}+2\right) - 2$

AS REQUIRED

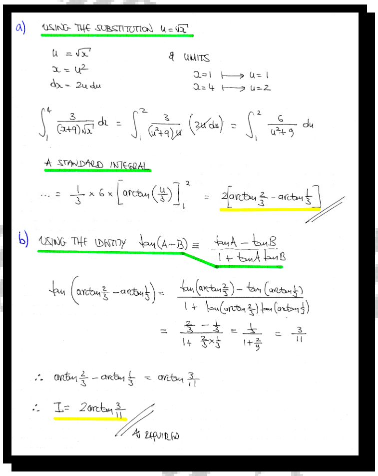
Question 32 **(***+)**

$$I = \int_{1}^{4} \frac{3}{(x+9)\sqrt{x}}\ dx .$$

a) By using a suitable substitution find an exact value for I .

b) Show clearly that the answer to part **(a)** can be written as $2\arctan\frac{3}{11}$.

$$I = 2\left(\arctan\frac{2}{3} - \arctan\frac{1}{3}\right)$$



a) USING THE SUBSTITUTION $u=\sqrt{x}$

$u=\sqrt{x}$, $x=u^2$, $dx=2u\,du$

& LIMITS: $x=1 \mapsto u=1$, $x=4 \mapsto u=2$

$$\int_1^4 \frac{3}{(x+9)\sqrt{x}}\,dx = \int_1^2 \frac{3}{(u^2+9)u}(2u\,du) = \int_1^2 \frac{6}{u^2+9}\,du$$

A STANDARD INTEGRAL

$$\ldots = \frac{1}{3}\times 6\times\left[\arctan\left(\frac{u}{3}\right)\right]_1^2 = 2\left[\arctan\frac{2}{3}-\arctan\frac{1}{3}\right]$$

b) USING THE IDENTITY $\tan(A-B) \equiv \frac{\tan A - \tan B}{1+\tan A\tan B}$

$$\tan\left(\arctan\frac{2}{3}-\arctan\frac{1}{3}\right) = \frac{\tan(\arctan\frac{2}{3}) - \tan(\arctan\frac{1}{3})}{1+\tan(\arctan\frac{2}{3})\tan(\arctan\frac{1}{3})} = \frac{\frac{2}{3}-\frac{1}{3}}{1+\frac{2}{3}\times\frac{1}{3}} = \frac{\frac{1}{3}}{1+\frac{2}{9}} = \frac{3}{11}$$

$\therefore \arctan\frac{2}{3} - \arctan\frac{1}{3} = \arctan\frac{3}{11}$

$\therefore I = 2\arctan\frac{3}{11}$ AS REQUIRED

Question 33 (***+)

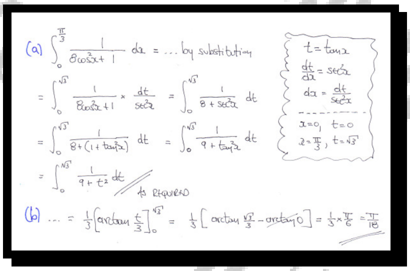
$$I = \int_{0}^{\frac{\pi}{3}} \frac{1}{1+8\cos^2 x}\, dx .$$

a) By using the substitution $t = \tan x$, or otherwise, show clearly that

$$I = \int_{0}^{\sqrt{3}} \frac{1}{9+t^2}\, dt .$$

b) Hence find the exact value of I.

$$\boxed{\frac{\pi}{18}}$$


(a) $\int_0^{\frac{\pi}{3}} \frac{1}{8\cos^2 x+1}\,dx = \ldots$ by substitution

$= \int_0^{\sqrt{3}} \frac{1}{8\cos^2 x+1} \times \frac{dt}{\sec^2 x} = \int_0^{\sqrt{3}} \frac{1}{8+\sec^2 x}\,dt$

$= \int_0^{\sqrt{3}} \frac{1}{8+(1+\tan^2 x)}\,dt = \int_0^{\sqrt{3}} \frac{1}{9+\tan^2 x}\,dt$

$= \int_0^{\sqrt{3}} \frac{1}{9+t^2}\,dt$ as REQUIRED

$t = \tan x$, $\frac{dt}{dx} = \sec^2 x$, $dx = \frac{dt}{\sec^2 x}$

$x = 0,\ t = 0$; $x = \frac{\pi}{3},\ t = \sqrt{3}$

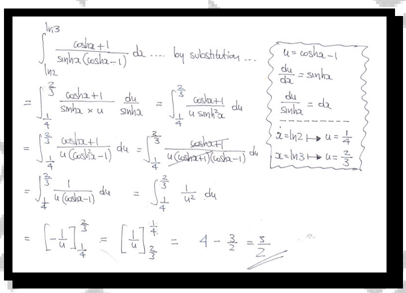
(b) $\ldots = \frac{1}{3}\left[\arctan \frac{t}{3}\right]_0^{\sqrt{3}} = \frac{1}{3}\left[\arctan \frac{\sqrt{3}}{3} - \arctan 0\right] = \frac{1}{3} \times \frac{\pi}{6} = \frac{\pi}{18}$

Question 34 (***+)

By using the substitution $u = \cosh x - 1$, or otherwise, find the value of

$$\int_{\ln 2}^{\ln 3} \frac{\cosh x + 1}{\sinh x(\cosh x - 1)}\, dx .$$

$$\boxed{\frac{5}{2}}$$


$\int_{\ln 2}^{\ln 3} \frac{\cosh x+1}{\sinh x(\cosh x-1)}\,dx \ldots$ by substitution ...

$= \int_{\frac{1}{4}}^{\frac{2}{3}} \frac{\cosh x+1}{\sinh x \times u}\,\frac{du}{\sinh x} = \int_{\frac{1}{4}}^{\frac{2}{3}} \frac{\cosh x+1}{u\sinh^2 x}\,du$

$= \int_{\frac{1}{4}}^{\frac{2}{3}} \frac{\cosh x+1}{u(\cosh^2 x-1)}\,du = \int_{\frac{1}{4}}^{\frac{2}{3}} \frac{\cosh x+1}{u(\cosh x+1)(\cosh x-1)}\,du$

$= \int_{\frac{1}{4}}^{\frac{2}{3}} \frac{1}{u(\cosh x-1)}\,du = \int_{\frac{1}{4}}^{\frac{2}{3}} \frac{1}{u^2}\,du$

$= \left[-\frac{1}{u}\right]_{\frac{1}{4}}^{\frac{2}{3}} = \left[\frac{1}{u}\right]_{\frac{2}{3}}^{\frac{1}{4}} = 4 - \frac{3}{2} = \frac{5}{2}$

$u = \cosh x - 1$, $\frac{du}{dx} = \sinh x$, $\frac{du}{\sinh x} = dx$

$x = \ln 2 \mapsto u = \frac{1}{4}$; $x = \ln 3 \mapsto u = \frac{2}{3}$

Question 35 (***+)

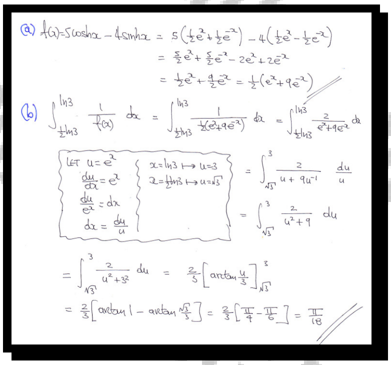
$$f(x) = 5\cosh x - 4\sinh x,\ x \in \mathbb{R}.$$

a) Find a simplified expression for $f(x)$ in terms of e^x.

b) Hence by using the substitution $u = e^x$, or otherwise, show that

$$\int_{\frac{1}{2}\ln 3}^{\ln 3} \frac{1}{f(x)}\ dx = \frac{\pi}{18}.$$

$$\boxed{f(x) = \tfrac{1}{2}e^x + \tfrac{9}{2}e^{-x}}$$


(a) $f(x) = 5\cosh x - 4\sinh x = 5\left(\frac{1}{2}e^x + \frac{1}{2}e^{-x}\right) - 4\left(\frac{1}{2}e^x - \frac{1}{2}e^{-x}\right)$
$= \frac{5}{2}e^x + \frac{5}{2}e^{-x} - 2e^x + 2e^{-x}$
$= \frac{1}{2}e^x + \frac{9}{2}e^{-x} = \frac{1}{2}(e^x + 9e^{-x})$

(b) $\int_{\frac{1}{2}\ln 3}^{\ln 3} \frac{1}{f(x)}\,dx = \int_{\frac{1}{2}\ln 3}^{\ln 3} \frac{1}{\frac{1}{2}(e^x+9e^{-x})}\,dx = \int_{\frac{1}{2}\ln 3}^{\ln 3} \frac{2}{e^x+9e^{-x}}\,dx$

LET $u = e^x$, $\frac{du}{dx} = e^x$, $\frac{du}{e^x} = dx$, $dx = \frac{du}{u}$; $x = \ln 3 \mapsto u = 3$, $x = \frac{1}{2}\ln 3 \mapsto u = \sqrt{3}$

$= \int_{\sqrt{3}}^{3} \frac{2}{u + 9u^{-1}}\,\frac{du}{u} = \int_{\sqrt{3}}^{3} \frac{2}{u^2+9}\,du$

$= \int_{\sqrt{3}}^{3} \frac{2}{u^2+3^2}\,du = \frac{2}{3}\left[\arctan\frac{u}{3}\right]_{\sqrt{3}}^{3}$

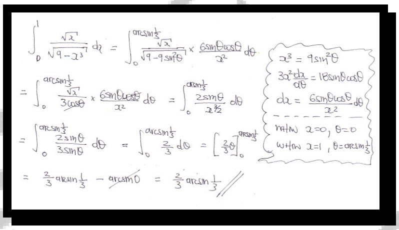
$= \frac{2}{3}\left[\arctan 1 - \arctan\frac{\sqrt{3}}{3}\right] = \frac{2}{3}\left[\frac{\pi}{4} - \frac{\pi}{6}\right] = \frac{\pi}{18}$

Question 36 (***+)

By using the substitution $x^3 = 9\sin^2\theta$, or otherwise, find the exact value of

$$\int_0^1 \frac{\sqrt{x}}{\sqrt{9-x^3}}\ dx.$$

$$\boxed{\tfrac{2}{3}\arcsin\left(\tfrac{1}{3}\right)}$$


$\int_0^1 \frac{\sqrt{x}}{\sqrt{9-x^3}}\,dx = \int_0^{\arcsin\frac{1}{3}} \frac{\sqrt{x}}{\sqrt{9-9\sin^2\theta}} \times \frac{6\sin\theta\cos\theta}{x^2}\,d\theta$

$= \int_0^{\arcsin\frac{1}{3}} \frac{\sqrt{x}}{3\cos\theta} \times \frac{6\sin\theta\cos\theta}{x^2}\,d\theta = \int_0^{\arcsin\frac{1}{3}} \frac{2\sin\theta}{x^{3/2}}\,d\theta$

$= \int_0^{\arcsin\frac{1}{3}} \frac{2\sin\theta}{3\sin\theta}\,d\theta = \int_0^{\arcsin\frac{1}{3}} \frac{2}{3}\,d\theta = \left[\frac{2}{3}\theta\right]_0^{\arcsin\frac{1}{3}}$

$= \frac{2}{3}\arcsin\frac{1}{3} - \arcsin 0 = \frac{2}{3}\arcsin\frac{1}{3}$

$x^3 = 9\sin^2\theta$, $3x^2\frac{dx}{d\theta} = 18\sin\theta\cos\theta$, $dx = \frac{6\sin\theta\cos\theta}{x^2}\,d\theta$; when $x = 0$, $\theta = 0$; when $x = 1$, $\theta = \arcsin\frac{1}{3}$

Question 37 **(***+)**

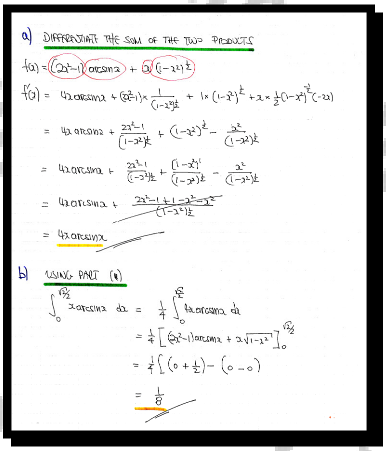
$$f(x) \equiv \left(2x^2 - 1\right)\arcsin x + x\sqrt{1-x^2}, \quad -1 \le x \le 1.$$

a) Find a simplified expression for $f'(x)$.

b) Hence find

$$\int_0^{\frac{\sqrt{2}}{2}} x\arcsin x \, dx.$$

$\boxed{}$, $\boxed{f'(x) = 4x\arcsin x}$, $\boxed{\frac{1}{8}}$



a) DIFFERENTIATE THE SUM OF THE TWO PRODUCTS

$f(x) = (2x^2-1)\arcsin x + x(1-x^2)^{\frac{1}{2}}$

$f'(x) = 4x\arcsin x + (2x^2-1)\times\frac{1}{(1-x^2)^{\frac{1}{2}}} + 1\times(1-x^2)^{\frac{1}{2}} + x\times\frac{1}{2}(1-x^2)^{-\frac{1}{2}}(-2x)$

$= 4x\arcsin x + \frac{2x^2-1}{(1-x^2)^{\frac{1}{2}}} + (1-x^2)^{\frac{1}{2}} - \frac{x^2}{(1-x^2)^{\frac{1}{2}}}$

$= 4x\arcsin x + \frac{2x^2-1}{(1-x^2)^{\frac{1}{2}}} + \frac{(1-x^2)^1}{(1-x^2)^{\frac{1}{2}}} - \frac{x^2}{(1-x^2)^{\frac{1}{2}}}$

$= 4x\arcsin x + \frac{2x^2-1+1-x^2-x^2}{(1-x^2)^{\frac{1}{2}}}$

$= 4x\arcsin x$

b) USING PART (a)

$\int_0^{\sqrt{2}/2} x\arcsin x \, dx = \frac{1}{4}\int_0^{\frac{\sqrt{2}}{2}} 4x\arcsin x \, dx$

$= \frac{1}{4}\left[(2x^2-1)\arcsin x + x\sqrt{1-x^2}\right]_0^{\sqrt{2}/2}$

$= \frac{1}{4}\left[\left(0+\frac{1}{2}\right) - (0-0)\right]$

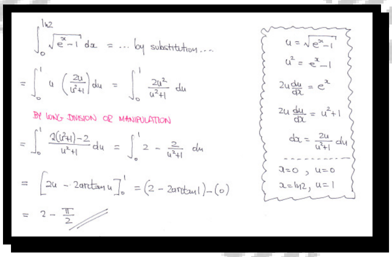
$= \frac{1}{8}$

Question 38 **(***+)**

By using the substitution $u=\sqrt{e^{x}-1}$, or otherwise, find the exact value of

$$\int_{0}^{\ln 2} \sqrt{e^{x}-1}\ dx .$$

$$\boxed{2-\frac{\pi}{2}}$$

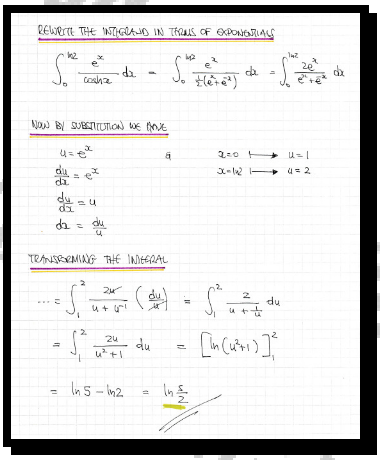


Question 39 **(***+)**

Find in exact simplified form the value of

$$\int_{0}^{\ln 2} \frac{e^{x}}{\cosh x}\ dx .$$

$\boxed{}$, $\boxed{\ln \frac{5}{2}}$



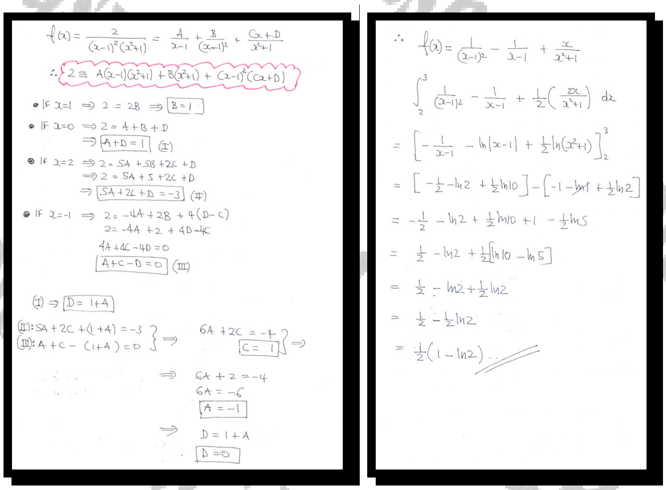
Question 40 (***+)

$$f(x)=\frac{2}{(x-1)^2\left(x^2+1\right)},\ x\neq 0.$$

Use partial fractions to show that

$$\int_2^3 f(x)\ dx=\frac{1}{2}(1-\ln 2).$$

proof



$$f(x)=\frac{2}{(x-1)^2(x^2+1)}=\frac{A}{x-1}+\frac{B}{(x-1)^2}+\frac{Cx+D}{x^2+1}$$

$$\therefore\ 2\equiv A(x-1)(x^2+1)+B(x^2+1)+(x-1)^2(Cx+D)$$

- If $x=1 \Rightarrow 2=2B \Rightarrow \boxed{B=1}$
- If $x=0 \Rightarrow 2=A+B+D \Rightarrow \boxed{A+D=1}$ (I)
- If $x=2 \Rightarrow 2=5A+5B+2C+D \Rightarrow 2=5A+5+2C+D \Rightarrow \boxed{5A+2C+D=-3}$ (II)
- If $x=-1 \Rightarrow 2=-4A+2B+4(D-C)$, $2=-4A+2+4D-4C$, $4A+4C-4D=0$, $\boxed{A+C-D=0}$ (III)

(I) $\Rightarrow \boxed{D=1+A}$

(II): $5A+2C+(1+A)=-3$, (III): $A+C-(1+A)=0$ $\Rightarrow$ $6A+2C=-4$, $\boxed{C=1}$ $\Rightarrow$ $6A+2=-4$, $6A=-6$, $\boxed{A=-1}$ $\Rightarrow$ $D=1+A$, $\boxed{D=0}$

$$\therefore\ f(x)=\frac{1}{(x-1)^2}-\frac{1}{x-1}+\frac{x}{x^2+1}$$

$$\int_2^3 \frac{1}{(x-1)^2}-\frac{1}{x-1}+\frac{1}{2}\left(\frac{2x}{x^2+1}\right)\ dx$$

$$=\left[-\frac{1}{x-1}-\ln|x-1|+\frac{1}{2}\ln(x^2+1)\right]_2^3$$

$$=\left[-\frac{1}{2}-\ln 2+\frac{1}{2}\ln 10\right]-\left[-1-\ln 1+\frac{1}{2}\ln 5\right]$$

$$=-\frac{1}{2}-\ln 2+\frac{1}{2}\ln 10+1-\frac{1}{2}\ln 5$$

$$=\frac{1}{2}-\ln 2+\frac{1}{2}[\ln 10-\ln 5]$$

$$=\frac{1}{2}-\ln 2+\frac{1}{2}\ln 2$$

$$=\frac{1}{2}-\frac{1}{2}\ln 2$$

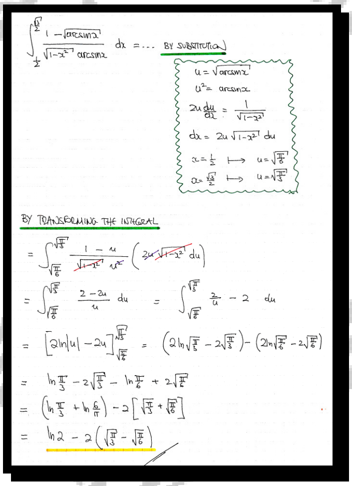
$$=\frac{1}{2}(1-\ln 2)$$

Question 41 (***+)

Use an appropriate substitution to find an exact value for the following integral.

$$\int_{\frac{1}{2}}^{\frac{1}{2}\sqrt{3}} \frac{1-\sqrt{\arcsin x}}{\sqrt{1-x^2}\ \arcsin x}\ dx .$$

$\boxed{}$, $\boxed{\ln 2 \ - \ 2\left(\sqrt{\frac{1}{3}\pi}-\sqrt{\frac{1}{6}\pi}\right)}$



$$\int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} \frac{1-\sqrt{\arcsin x}}{\sqrt{1-x^2}\ \arcsin x}\ dx = \dots$$ BY SUBSTITUTION

$u = \sqrt{\arcsin x}$

$u^2 = \arcsin x$

$2u\frac{du}{dx} = \frac{1}{\sqrt{1-x^2}}$

$dx = 2u\sqrt{1-x^2}\ du$

$x = \frac{1}{2} \mapsto u = \sqrt{\frac{\pi}{6}}$

$x = \frac{\sqrt{3}}{2} \mapsto u = \sqrt{\frac{\pi}{3}}$

BY TRANSFORMING THE INTEGRAL

$$= \int_{\sqrt{\frac{\pi}{6}}}^{\sqrt{\frac{\pi}{3}}} \frac{1-u}{\sqrt{1-x^2}\ u^2}\left(2u\sqrt{1-x^2}\ du\right)$$

$$= \int_{\sqrt{\frac{\pi}{6}}}^{\sqrt{\frac{\pi}{3}}} \frac{2-2u}{u}\ du = \int_{\sqrt{\frac{\pi}{6}}}^{\sqrt{\frac{\pi}{3}}} \frac{2}{u} - 2\ du$$

$$= \Big[2\ln|u| - 2u\Big]_{\sqrt{\frac{\pi}{6}}}^{\sqrt{\frac{\pi}{3}}} = \left(2\ln\sqrt{\tfrac{\pi}{3}} - 2\sqrt{\tfrac{\pi}{3}}\right) - \left(2\ln\sqrt{\tfrac{\pi}{6}} - 2\sqrt{\tfrac{\pi}{6}}\right)$$

$$= \ln\frac{\pi}{3} - 2\sqrt{\frac{\pi}{3}} - \ln\frac{\pi}{6} + 2\sqrt{\frac{\pi}{6}}$$

$$= \left(\ln\frac{\pi}{3} + \ln\frac{6}{\pi}\right) - 2\left[\sqrt{\frac{\pi}{3}} + \sqrt{\frac{\pi}{6}}\right]$$

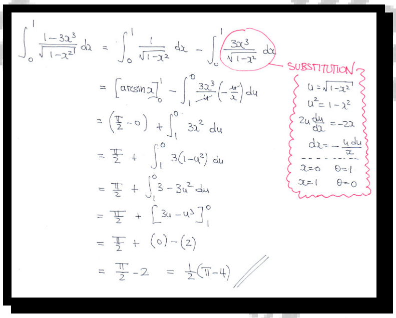
$$= \ln 2 - 2\left(\sqrt{\frac{\pi}{3}} - \sqrt{\frac{\pi}{6}}\right)$$

Question 42 (****)

Find the exact value of

$$\int_0^1 \frac{1-3x^3}{\sqrt{1-x^2}}\, dx.$$

$$\boxed{\frac{1}{2}(\pi-4)}$$

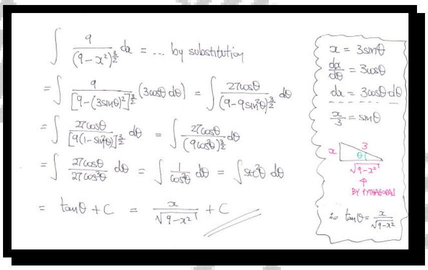


Question 43 (****)

Use a suitable trigonometric substitution to find an integrated expression for

$$\int \frac{9}{\left(9-x^2\right)^{\frac{3}{2}}}\, dx.$$

$$\boxed{\frac{x}{\sqrt{9-x^2}}+C}$$

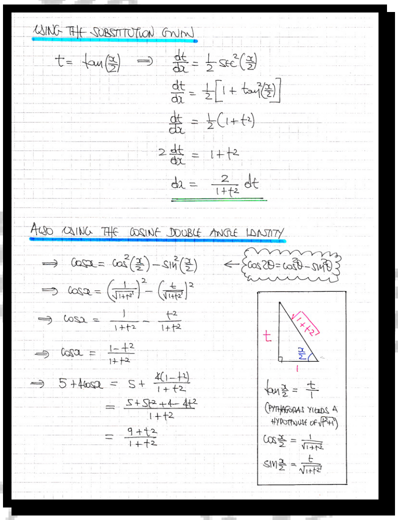


Question 44 (****)

Use the substitution $t = \tan\left(\frac{x}{2}\right)$ to find the value of

$$\int_0^{\frac{2\pi}{3}} \frac{1}{5+4\cos x}\, dx .$$

$\boxed{}$, $\boxed{\frac{\pi}{9}}$



USING THE SUBSTITUTION GIVEN

$$t = \tan\left(\frac{x}{2}\right) \implies \frac{dt}{dx} = \frac{1}{2}\sec^2\left(\frac{x}{2}\right)$$

$$\frac{dt}{dx} = \frac{1}{2}\left[1+\tan^2\left(\frac{x}{2}\right)\right]$$

$$\frac{dt}{dx} = \frac{1}{2}(1+t^2)$$

$$2\frac{dt}{dx} = 1+t^2$$

$$dx = \frac{2}{1+t^2}\,dt$$

ALSO USING THE COSINE DOUBLE ANGLE IDENTITY

$$\implies \cos x = \cos^2\left(\frac{x}{2}\right) - \sin^2\left(\frac{x}{2}\right) \quad \leftarrow \{\cos 2\theta = \cos^2\theta - \sin^2\theta\}$$

$$\implies \cos x = \left(\frac{1}{\sqrt{1+t^2}}\right)^2 - \left(\frac{t}{\sqrt{1+t^2}}\right)^2$$

$$\implies \cos x = \frac{1}{1+t^2} - \frac{t^2}{1+t^2}$$

$$\implies \cos x = \frac{1-t^2}{1+t^2}$$

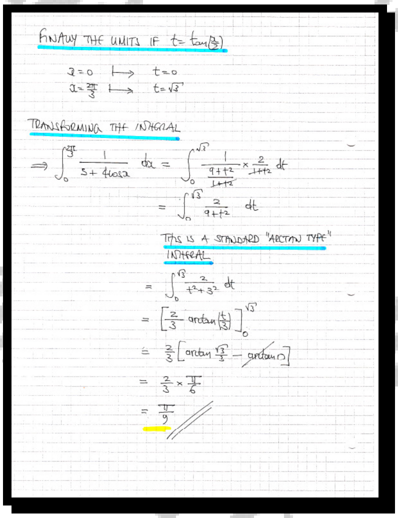
$$\implies 5+4\cos x = 5 + \frac{4(1-t^2)}{1+t^2} = \frac{5+5t^2+4-4t^2}{1+t^2} = \frac{9+t^2}{1+t^2}$$

$\tan\frac{x}{2} = \frac{t}{1}$

(PYTHAGORAS YIELDS A HYPOTENUSE OF $\sqrt{t^2+1}$)

$\cos\frac{x}{2} = \frac{1}{\sqrt{1+t^2}}$

$\sin\frac{x}{2} = \frac{t}{\sqrt{1+t^2}}$



FINALLY THE LIMITS IF $t = \tan\left(\frac{x}{2}\right)$

$x = 0 \longmapsto t = 0$

$x = \frac{2\pi}{3} \longmapsto t = \sqrt{3}$

TRANSFORMING THE INTEGRAL

$$\implies \int_0^{\frac{2\pi}{3}} \frac{1}{5+4\cos x}\,dx = \int_0^{\sqrt{3}} \frac{1}{\frac{9+t^2}{1+t^2}} \times \frac{2}{1+t^2}\,dt = \int_0^{\sqrt{3}} \frac{2}{9+t^2}\,dt$$

THIS IS A STANDARD "ARCTAN TYPE" INTEGRAL

$$= \int_0^{\sqrt{3}} \frac{2}{t^2+3^2}\,dt = \left[\frac{2}{3}\arctan\left(\frac{t}{3}\right)\right]_0^{\sqrt{3}} = \frac{2}{3}\left[\arctan\frac{\sqrt{3}}{3} - \arctan 0\right] = \frac{2}{3}\times\frac{\pi}{6} = \frac{\pi}{9}$$

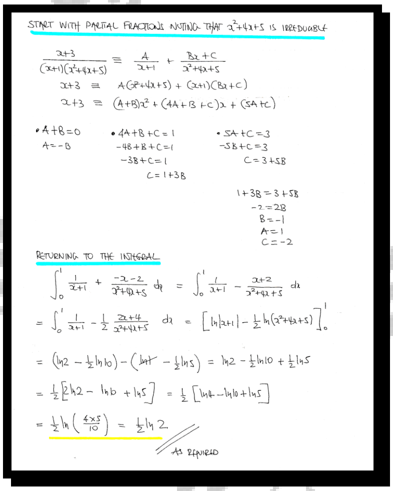
Question 45 **(****)**

Show that the exact value of the following integral

$$\int_0^1 \frac{x+3}{(x+1)\left(x^2+4x+5\right)}\, dx$$

is $\frac{1}{2}\ln 2$.

[ ], [proof]


START WITH PARTIAL FRACTIONS NOTING THAT x^2+4x+5 IS IRREDUCIBLE

$$\frac{x+3}{(x+1)(x^2+4x+5)} \equiv \frac{A}{x+1} + \frac{Bx+C}{x^2+4x+5}$$

$$x+3 \equiv A(x^2+4x+5) + (x+1)(Bx+C)$$

$$x+3 \equiv (A+B)x^2 + (4A+B+C)x + (5A+C)$$

- $A+B=0$, $A=-B$
- $4A+B+C=1$, $-4B+B+C=1$, $-3B+C=1$, $C=1+3B$
- $5A+C=3$, $-5B+C=3$, $C=3+5B$

$1+3B=3+5B$, $-2=2B$, $B=-1$, $A=1$, $C=-2$

RETURNING TO THE INTEGRAL

$$\int_0^1 \frac{1}{x+1} + \frac{-x-2}{x^2+4x+5}\, dx = \int_0^1 \frac{1}{x+1} - \frac{x+2}{x^2+4x+5}\, dx$$

$$= \int_0^1 \frac{1}{x+1} - \frac{1}{2}\frac{2x+4}{x^2+4x+5}\, dx = \left[\ln|x+1| - \frac{1}{2}\ln(x^2+4x+5)\right]_0^1$$

$$= \left(\ln 2 - \frac{1}{2}\ln 10\right) - \left(\ln 1 - \frac{1}{2}\ln 5\right) = \ln 2 - \frac{1}{2}\ln 10 + \frac{1}{2}\ln 5$$

$$= \frac{1}{2}\left[2\ln 2 - \ln 10 + \ln 5\right] = \frac{1}{2}\left[\ln 4 - \ln 10 + \ln 5\right]$$

$$= \frac{1}{2}\ln\left(\frac{4\times 5}{10}\right) = \frac{1}{2}\ln 2$$

AS REQUIRED

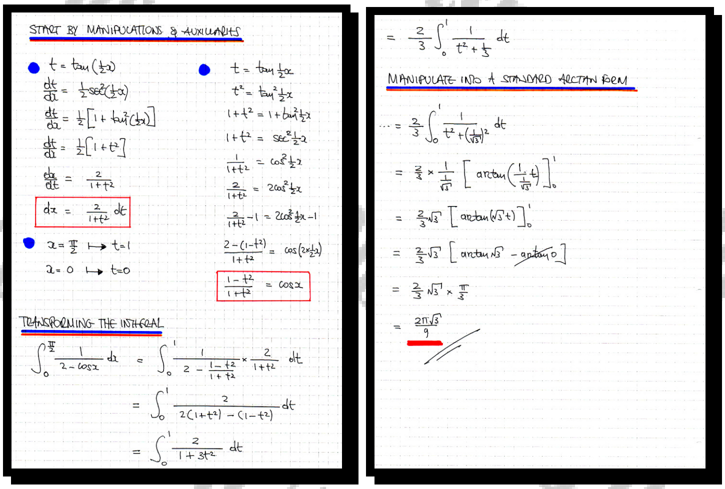
Question 46 (****)

Use the substitution $t = \tan\left(\frac{1}{2}x\right)$ to find an exact simplified value for

$$\int_0^{\frac{\pi}{2}} \frac{1}{2-\cos x}\, dx \, .$$

Any trigonometric identities to convert $\cos x$ in terms of t must be derived.

$\boxed{}$, $\boxed{\dfrac{2\pi\sqrt{3}}{9}}$



Question 47 (****)

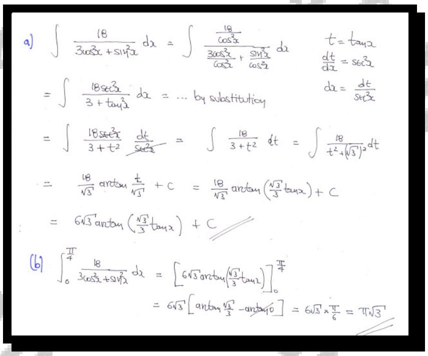
$$I = \int \frac{18}{3\cos^2 x + \sin^2 x}\, dx .$$

a) By using the substitution $t = \tan x$, or otherwise, show clearly that

$$I = 6\sqrt{3}\arctan\left(\frac{\sqrt{3}}{3}\tan x\right) + \text{constant} .$$

b) Hence find the exact value of $\displaystyle\int_0^{\frac{\pi}{4}} \frac{18}{3\cos^2 x + \sin^2 x}\, dx$.

$\boxed{\pi\sqrt{3}}$



a) $\displaystyle\int \frac{18}{3\cos^2 x + \sin^2 x}\,dx = \int \frac{\frac{18}{\cos^2 x}}{\frac{3\cos^2 x}{\cos^2 x} + \frac{\sin^2 x}{\cos^2 x}}\,dx$ $\quad t = \tan x$, $\frac{dt}{dx} = \sec^2 x$, $dx = \frac{dt}{\sec^2 x}$

$\displaystyle = \int \frac{18\sec^2 x}{3 + \tan^2 x}\,dx = \dots$ by substitution

$\displaystyle = \int \frac{18\sec^2 x}{3 + t^2}\,\frac{dt}{\sec^2 x} = \int \frac{18}{3 + t^2}\,dt = \int \frac{18}{t^2 + (\sqrt{3})^2}\,dt$

$\displaystyle = \frac{18}{\sqrt{3}}\arctan\frac{t}{\sqrt{3}} + C = \frac{18}{\sqrt{3}}\arctan\left(\frac{\sqrt{3}}{3}\tan x\right) + C$

$\displaystyle = 6\sqrt{3}\arctan\left(\frac{\sqrt{3}}{3}\tan x\right) + C$

(b) $\displaystyle\int_0^{\frac{\pi}{4}} \frac{18}{3\cos^2 x + \sin^2 x}\,dx = \left[6\sqrt{3}\arctan\left(\frac{\sqrt{3}}{3}\tan x\right)\right]_0^{\frac{\pi}{4}}$

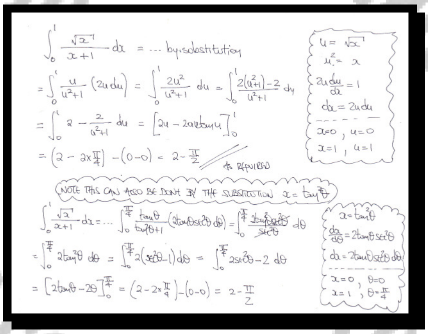
$\displaystyle = 6\sqrt{3}\left[\arctan\frac{\sqrt{3}}{3} - \arctan 0\right] = 6\sqrt{3}\times\frac{\pi}{6} = \pi\sqrt{3}$

Question 48 **(****)**

By using the substitution $u = \sqrt{x}$, or otherwise, find an exact value for

$$\int_0^1 \frac{\sqrt{x}}{x+1}\, dx .$$

$$\boxed{2 - \frac{\pi}{2}}$$

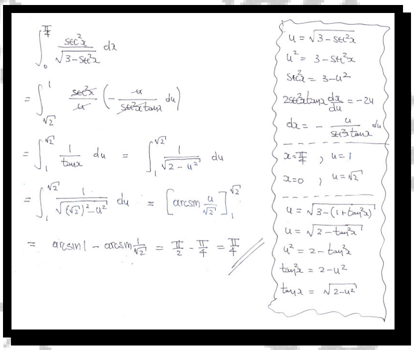


Question 49 **(****)**

By using the substitution $u = \sqrt{3 - \sec^2 x}$, or otherwise, find the exact value of

$$\int_0^{\frac{\pi}{4}} \frac{\sec^2 x}{\sqrt{3 - \sec^2 x}}\, dx .$$

$$\boxed{\frac{\pi}{4}}$$

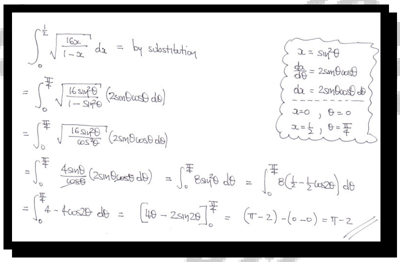


Question 50 (****)

By using a suitable trigonometric substitution, show clearly that

$$\int_0^{\frac{1}{2}} \sqrt{\frac{16x}{1-x}}\ dx = \pi - 2 .$$

proof

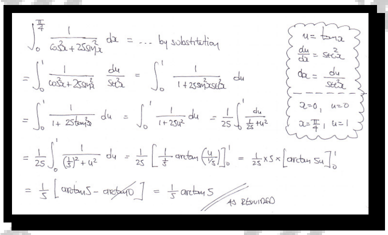


Question 51 (****)

By using the substitution $u = \tan x$, or otherwise, show clearly that

$$\int_0^{\frac{\pi}{4}} \frac{1}{\cos^2 x + 25\sin^2 x}\ dx = \frac{1}{5}\arctan 5 .$$

proof

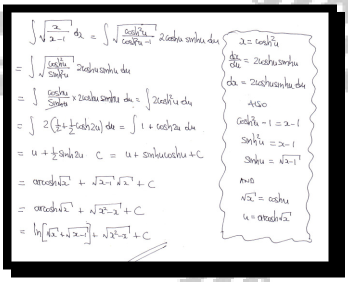


Question 52 **(****)**

By using the substitution $x = \cosh^2 u$, or otherwise, show that

$$\int \sqrt{\frac{x}{x-1}}\, dx = \ln\left(\sqrt{x} + \sqrt{x-1}\right) + \sqrt{x^2 - x} + \text{constant}.$$

proof



$$\int \sqrt{\frac{x}{x-1}}\, dx = \int \sqrt{\frac{\cosh^2 u}{\cosh^2 u - 1}}\, 2\cosh u \sinh u \, du$$

$$= \int \sqrt{\frac{\cosh^2 u}{\sinh^2 u}}\, 2\cosh u \sinh u \, du$$

$$= \int \frac{\cosh u}{\sinh u} \times 2\cosh u \sinh u \, du = \int 2\cosh^2 u \, du$$

$$= \int 2\left(\frac{1}{2} + \frac{1}{2}\cosh 2u\right) du = \int 1 + \cosh 2u \, du$$

$$= u + \frac{1}{2}\sinh 2u + C = u + \sinh u \cosh u + C$$

$$= \text{arcosh}\sqrt{x} + \sqrt{x-1}\sqrt{x} + C$$

$$= \text{arcosh}\sqrt{x} + \sqrt{x^2 - x} + C$$

$$= \ln\left[\sqrt{x} + \sqrt{x-1}\right] + \sqrt{x^2 - x} + C$$

$x = \cosh^2 u$

$\frac{dx}{du} = 2\cosh u \sinh u$

$dx = 2\cosh u \sinh u \, du$

ALSO

$\cosh^2 u - 1 = x - 1$

$\sinh^2 u = x - 1$

$\sinh u = \sqrt{x-1}$

AND

$\sqrt{x} = \cosh u$

$u = \text{arcosh}\sqrt{x}$

Question 53 (****)

$$\sin 2x \equiv \frac{2\tan x}{1+\tan^2 x}.$$

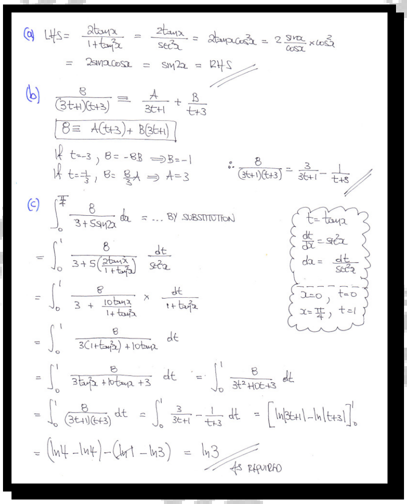
a) Prove the validity of the above trigonometric identity.

b) Express $\dfrac{8}{(3t+1)(t+3)}$ into partial fractions.

c) Hence use the substitution $t = \tan x$ to show that

$$\int_0^{\frac{\pi}{4}} \frac{8}{3+5\sin 2x}\,dx = \ln 3.$$

$$\boxed{\frac{8}{(3t+1)(t+3)} = \frac{3}{3t+1} - \frac{1}{t+3}}$$



(a) $\text{LHS} = \dfrac{2\tan x}{1+\tan^2 x} = \dfrac{2\tan x}{\sec^2 x} = 2\tan x\cos^2 x = 2\dfrac{\sin x}{\cos x}\times\cos^2 x$

$= 2\sin x\cos x = \sin 2x = \text{RHS}$

(b) $\dfrac{8}{(3t+1)(t+3)} \equiv \dfrac{A}{3t+1} + \dfrac{B}{t+3}$

$8 \equiv A(t+3) + B(3t+1)$

If $t=-3$, $8 = -8B \Rightarrow B = -1$

If $t = -\frac{1}{3}$, $8 = \frac{8}{3}A \Rightarrow A = 3$

$\therefore \dfrac{8}{(3t+1)(t+3)} = \dfrac{3}{3t+1} - \dfrac{1}{t+3}$

(c) $\displaystyle\int_0^{\frac{\pi}{4}} \frac{8}{3+5\sin 2x}\,dx = \dots$ BY SUBSTITUTION

$t = \tan x$, $\dfrac{dt}{dx} = \sec^2 x$, $dx = \dfrac{dt}{\sec^2 x}$

$x = 0,\ t = 0$; $x = \frac{\pi}{4},\ t = 1$

$= \displaystyle\int_0^1 \frac{8}{3+5\left(\frac{2\tan x}{1+\tan^2 x}\right)}\,\frac{dt}{\sec^2 x}$

$= \displaystyle\int_0^1 \frac{8}{3+\frac{10\tan x}{1+\tan^2 x}} \times \frac{dt}{1+\tan^2 x}$

$= \displaystyle\int_0^1 \frac{8}{3(1+\tan^2 x)+10\tan x}\,dt$

$= \displaystyle\int_0^1 \frac{8}{3\tan^2 x+10\tan x+3}\,dt = \int_0^1 \frac{8}{3t^2+10t+3}\,dt$

$= \displaystyle\int_0^1 \frac{8}{(3t+1)(t+3)}\,dt = \int_0^1 \frac{3}{3t+1} - \frac{1}{t+3}\,dt = \Big[\ln|3t+1| - \ln|t+3|\Big]_0^1$

$= (\ln 4 - \ln 4) - (\ln 1 - \ln 3) = \ln 3$

AS REQUIRED

Question 54 (****)

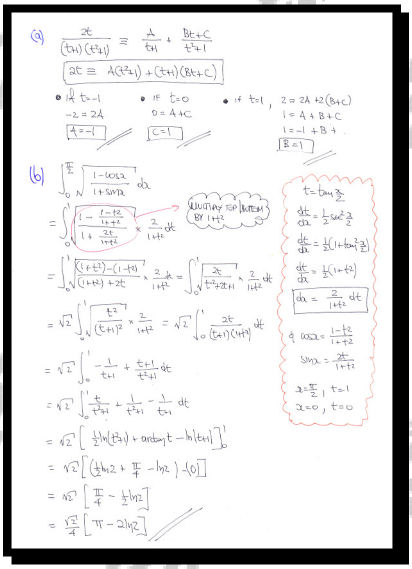
$$\frac{2t}{(t+1)\left(t^2+1\right)} \equiv \frac{A}{t+1} + \frac{Bt+C}{t^2+1}.$$

a) Determine the values of A, B and C in the above identity.

b) Hence find an value for

$$\int_0^{\frac{\pi}{2}} \sqrt{\frac{1-\cos x}{1+\sin x}}\, dx.$$

$\boxed{A=-1,\ B=1,\ C=1}$, $\boxed{\frac{\sqrt{2}}{2}(\pi - 2\ln 2)}$

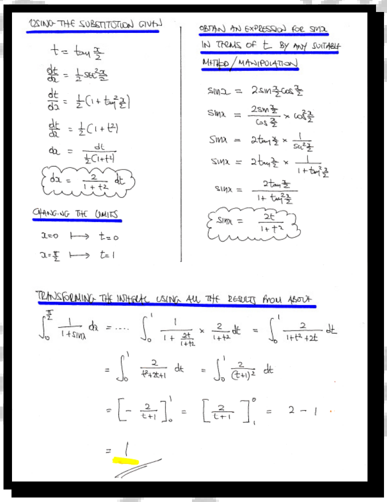


Question 55 (****)

Use the substitution $t = \tan\left(\frac{x}{2}\right)$ to find the value of

$$\int_0^{\frac{\pi}{2}} \frac{1}{1+\sin x} \, dx .$$

1

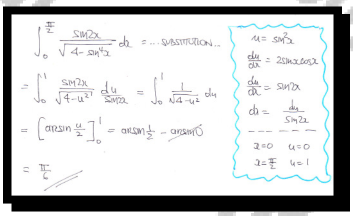


Question 56 (****)

Use suitable substitution to find the exact value of

$$\int_0^{\frac{\pi}{2}} \frac{\sin 2x}{\sqrt{4-\sin^4 x}} \, dx .$$

$\frac{\pi}{6}$



Question 57 **(****)**

$$I = \int \sqrt{\frac{x}{1-x}}\, dx$$

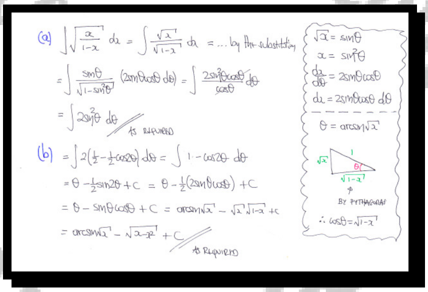
a) Use the substitution $\sqrt{x} = \sin\theta$ to show that

$$I = \int 2\sin^2\theta \, d\theta .$$

b) Hence show further that

$$I = \arcsin\sqrt{x} - \sqrt{x - x^2} + \text{constant}$$

proof



Question 58 (****)

The curve with the following equation is defined in the largest real domain.

$$y = (4x-3)\sqrt{-8\left(2x^2-3x+1\right)} + \arcsin(4x-3).$$

a) Show that

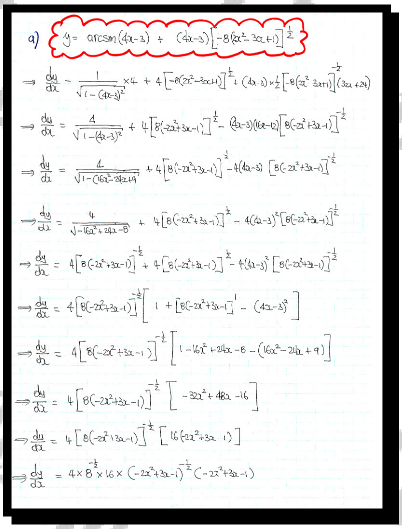
$$\frac{dy}{dx} = k\sqrt{-2x^2+3x-1},$$

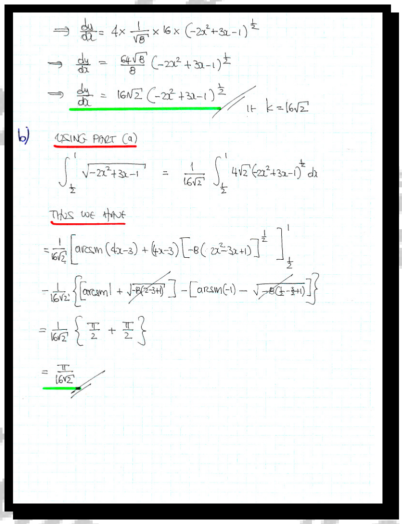
where k is an exact constant to be found.

b) Hence find the exact value of the following integral.

$$\int_{\frac{1}{2}}^{1} \sqrt{-2x^2+3x-1}\ dx.$$

$k = 16\sqrt{2}$, $\dfrac{\pi}{16\sqrt{2}}$



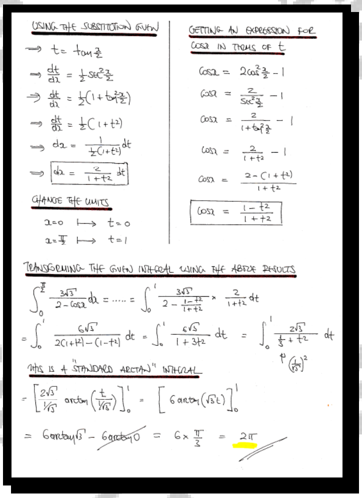


Question 59 (****)

Use the substitution $t = \tan\left(\frac{x}{2}\right)$ to find the exact value of

$$\int_0^{\frac{\pi}{2}} \frac{3\sqrt{3}}{2-\cos x}\, dx .$$

$\boxed{}$, $\boxed{2\pi}$



USING THE SUBSTITUTION GIVEN

$\Rightarrow t = \tan\frac{x}{2}$

$\Rightarrow \frac{dt}{dx} = \frac{1}{2}\sec^2\frac{x}{2}$

$\Rightarrow \frac{dt}{dx} = \frac{1}{2}\left(1+\tan^2\frac{x}{2}\right)$

$\Rightarrow \frac{dt}{dx} = \frac{1}{2}(1+t^2)$

$\Rightarrow dx = \frac{1}{\frac{1}{2}(1+t^2)}\,dt$

$\Rightarrow \boxed{dx = \frac{2}{1+t^2}\,dt}$

CHANGE THE LIMITS

$x=0 \longmapsto t=0$

$x=\frac{\pi}{2} \longmapsto t=1$

GETTING AN EXPRESSION FOR $\cos x$ IN TERMS OF t

$\cos x = 2\cos^2\frac{x}{2} - 1$

$\cos x = \frac{2}{\sec^2\frac{x}{2}} - 1$

$\cos x = \frac{2}{1+\tan^2\frac{x}{2}} - 1$

$\cos x = \frac{2}{1+t^2} - 1$

$\cos x = \frac{2-(1+t^2)}{1+t^2}$

$\boxed{\cos x = \frac{1-t^2}{1+t^2}}$

TRANSFORMING THE GIVEN INTEGRAL USING THE ABOVE RESULTS

$\int_0^{\frac{\pi}{2}} \frac{3\sqrt{3}}{2-\cos x}\,dx = \ldots\ldots = \int_0^1 \frac{3\sqrt{3}}{2-\frac{1-t^2}{1+t^2}} \times \frac{2}{1+t^2}\,dt$

$= \int_0^1 \frac{6\sqrt{3}}{2(1+t^2)-(1-t^2)}\,dt = \int_0^1 \frac{6\sqrt{3}}{1+3t^2}\,dt = \int_0^1 \frac{2\sqrt{3}}{\frac{1}{3}+t^2}\,dt$

THIS IS A "STANDARD ARCTAN" INTEGRAL

$= \left[\frac{2\sqrt{3}}{1/\sqrt{3}}\arctan\left(\frac{t}{1/\sqrt{3}}\right)\right]_0^1 = \left[6\arctan(\sqrt{3}t)\right]_0^1$

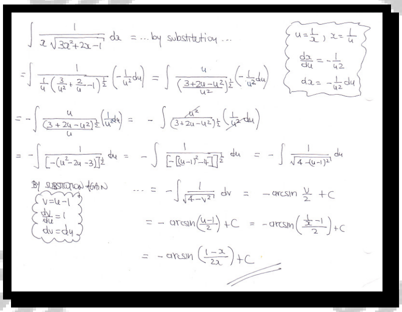
$= 6\arctan\sqrt{3} - 6\arctan 0 = 6 \times \frac{\pi}{3} = 2\pi$

Question 60 (****)

Use the substitution $u = \dfrac{1}{x}$ to find

$$\int \frac{1}{x\sqrt{3x^2+2x-1}}\,dx .$$

$$\boxed{-\arcsin\left(\frac{1-x}{2x}\right)+C}$$



$\int \frac{1}{x\sqrt{3x^2+2x-1}}\,dx = \ldots$ by substitution $\ldots$

$u = \frac{1}{x}$, $x = \frac{1}{u}$

$\frac{dx}{du} = -\frac{1}{u^2}$

$dx = -\frac{1}{u^2}\,du$

$= \int \frac{1}{\frac{1}{u}\left(\frac{3}{u^2}+\frac{2}{u}-1\right)^{\frac{1}{2}}}\left(-\frac{1}{u^2}du\right) = \int \frac{u}{\left(\frac{3+2u-u^2}{u^2}\right)^{\frac{1}{2}}}\left(-\frac{1}{u^2}du\right)$

$= -\int \frac{u}{\frac{(3+2u-u^2)^{\frac{1}{2}}}{u}}\left(\frac{1}{u^2}du\right) = -\int \frac{u^2}{(3+2u-u^2)^{\frac{1}{2}}}\left(\frac{1}{u^2}du\right)$

$= -\int \frac{1}{\left[-(u^2-2u-3)\right]^{\frac{1}{2}}}\,du = -\int \frac{1}{\left[-\left[(u-1)^2-4\right]\right]^{\frac{1}{2}}}\,du = -\int \frac{1}{\sqrt{4-(u-1)^2}}\,du$

BY SUBSTITUTION AGAIN

$v = u-1$

$\frac{dv}{du} = 1$

$dv = du$

$\ldots = -\int \frac{1}{\sqrt{4-v^2}}\,dv = -\arcsin\frac{v}{2}+C$

$= -\arcsin\left(\frac{u-1}{2}\right)+C = -\arcsin\left(\frac{\frac{1}{x}-1}{2}\right)+C$

$= -\arcsin\left(\frac{1-x}{2x}\right)+C$

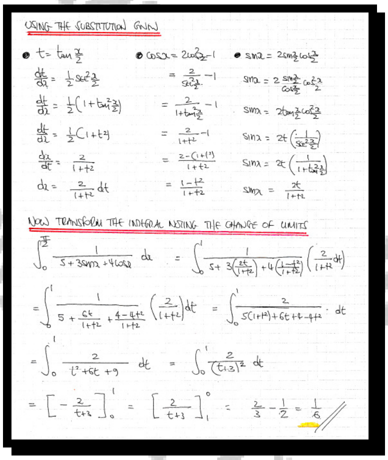
Question 61 (****+)

Use the substitution $t = \tan\left(\frac{x}{2}\right)$ to find the value of

$$\int_0^{\frac{\pi}{2}} \frac{1}{5++3\sin x + 4\cos x}\, dx .$$

All relevant results used in this evaluation must be carefully derived.

$\boxed{}$, $\boxed{\frac{1}{6}}$


USING THE SUBSTITUTION GIVEN

- $t = \tan\frac{x}{2}$, $\frac{dt}{dx} = \frac{1}{2}\sec^2\frac{x}{2}$, $\frac{dt}{dx} = \frac{1}{2}\left(1+\tan^2\frac{x}{2}\right)$, $\frac{dt}{dx} = \frac{1}{2}(1+t^2)$, $\frac{dx}{dt} = \frac{2}{1+t^2}$, $dx = \frac{2}{1+t^2}\,dt$
- $\cos x = 2\cos^2\frac{x}{2} - 1 = \frac{2}{\sec^2\frac{x}{2}} - 1 = \frac{2}{1+\tan^2\frac{x}{2}} - 1 = \frac{2}{1+t^2} - 1 = \frac{2-(1+t^2)}{1+t^2} = \frac{1-t^2}{1+t^2}$
- $\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2}$, $\sin x = 2\frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}\cos^2\frac{x}{2}$, $\sin x = 2\tan\frac{x}{2}\cos^2\frac{x}{2}$, $\sin x = 2t\left(\frac{1}{\sec^2\frac{x}{2}}\right)$, $\sin x = 2t\left(\frac{1}{1+\tan^2\frac{x}{2}}\right)$, $\sin x = \frac{2t}{1+t^2}$

NOW TRANSFORM THE INTEGRAL NOTING THE CHANGE OF LIMITS

$$\int_0^{\frac{\pi}{2}} \frac{1}{5+3\sin x+4\cos x}\,dx = \int_0^1 \frac{1}{5+3\left(\frac{2t}{1+t^2}\right)+4\left(\frac{1-t^2}{1+t^2}\right)}\left(\frac{2}{1+t^2}dt\right)$$

$$= \int_0^1 \frac{1}{5+\frac{6t}{1+t^2}+\frac{4-4t^2}{1+t^2}}\left(\frac{2}{1+t^2}\right)dt = \int_0^1 \frac{2}{5(1+t^2)+6t+4-4t^2}\,dt$$

$$= \int_0^1 \frac{2}{t^2+6t+9}\,dt = \int_0^1 \frac{2}{(t+3)^2}\,dt$$

$$= \left[-\frac{2}{t+3}\right]_0^1 = \left[\frac{2}{t+3}\right]_1^0 = \frac{2}{3} - \frac{1}{2} = \frac{1}{6}$$

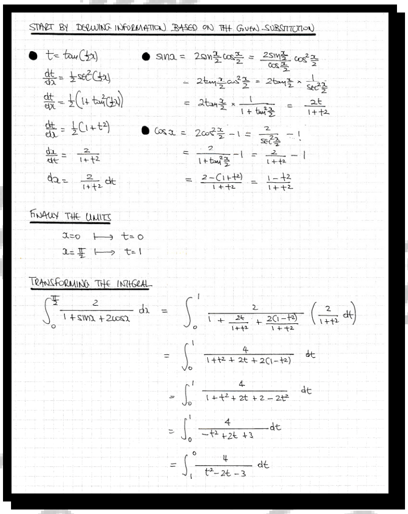
Question 62 (****+)

Use the substitution $t = \tan\left(\frac{1}{2}x\right)$ to find the exact value for the integral

$$\int_0^{\frac{1}{2}\pi} \frac{2}{1+\sin x + 2\cos x}\, dx$$

All relevant results used in this evaluation must be carefully derived.

$\boxed{}$, $\boxed{\ln 3}$



START BY DERIVING INFORMATION BASED ON THE GIVEN SUBSTITUTION

- $t = \tan(\frac{1}{2}x)$

$\frac{dt}{dx} = \frac{1}{2}\sec^2(\frac{1}{2}x)$

$\frac{dt}{dx} = \frac{1}{2}\left(1+\tan^2(\frac{1}{2}x)\right)$

$\frac{dt}{dx} = \frac{1}{2}(1+t^2)$

$\frac{dx}{dt} = \frac{2}{1+t^2}$

$dx = \frac{2}{1+t^2}\,dt$

- $\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2} = \frac{2\sin\frac{x}{2}}{\cos\frac{x}{2}}\cos^2\frac{x}{2}$

$= 2\tan\frac{x}{2}\cos^2\frac{x}{2} = 2\tan\frac{x}{2} \times \frac{1}{\sec^2\frac{x}{2}}$

$= 2\tan\frac{x}{2} \times \frac{1}{1+\tan^2\frac{x}{2}} = \frac{2t}{1+t^2}$

- $\cos x = 2\cos^2\frac{x}{2} - 1 = \frac{2}{\sec^2\frac{x}{2}} - 1$

$= \frac{2}{1+\tan^2\frac{x}{2}} - 1 = \frac{2}{1+t^2} - 1$

$= \frac{2-(1+t^2)}{1+t^2} = \frac{1-t^2}{1+t^2}$

FINALLY THE LIMITS

$x = 0 \longmapsto t = 0$

$x = \frac{\pi}{2} \longmapsto t = 1$

TRANSFORMING THE INTEGRAL

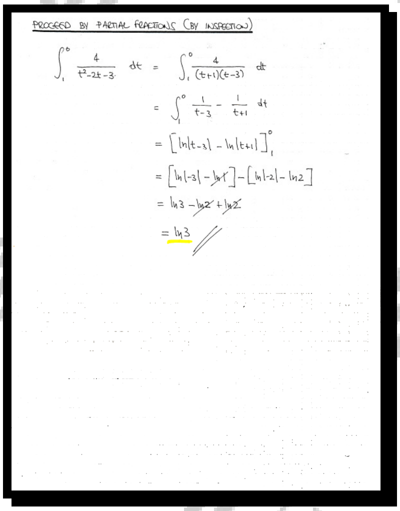
$$\int_0^{\frac{\pi}{2}} \frac{2}{1+\sin x+2\cos x}\,dx = \int_0^1 \frac{2}{1+\frac{2t}{1+t^2}+\frac{2(1-t^2)}{1+t^2}}\left(\frac{2}{1+t^2}\,dt\right)$$

$$= \int_0^1 \frac{4}{1+t^2+2t+2(1-t^2)}\,dt$$

$$= \int_0^1 \frac{4}{1+t^2+2t+2-2t^2}\,dt$$

$$= \int_0^1 \frac{4}{-t^2+2t+3}\,dt$$

$$= \int_1^0 \frac{4}{t^2-2t-3}\,dt$$



PROCEED BY PARTIAL FRACTIONS (BY INSPECTION)

$$\int_1^0 \frac{4}{t^2-2t-3}\,dt = \int_1^0 \frac{4}{(t+1)(t-3)}\,dt$$

$$= \int_1^0 \frac{1}{t-3} - \frac{1}{t+1}\,dt$$

$$= \Big[\ln|t-3| - \ln|t+1|\Big]_1^0$$

$$= \Big[\ln|-3| - \ln 1\Big] - \Big[\ln|-2| - \ln 2\Big]$$

$$= \ln 3 - \ln 2 + \ln 2$$

$$= \ln 3$$

Question 63 (****+)

$$y = \arcsin x\,,\ -1 \le x \le 1\,.$$

a) Show clearly that

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}\,.$$

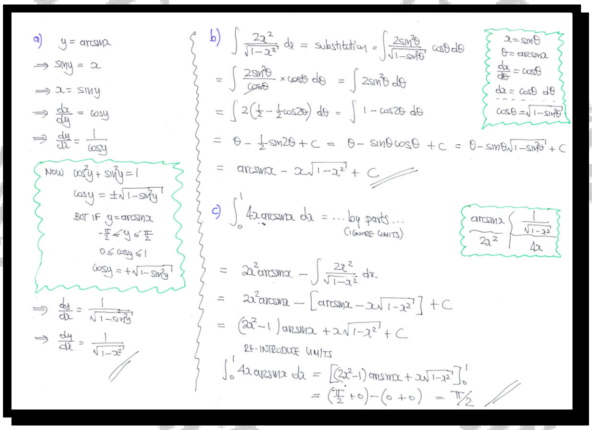
b) Use the substitution $x = \sin\theta$ to find

$$\int \frac{2x^2}{\sqrt{1-x^2}}\,dx\,.$$

c) Hence find an exact value for

$$\int_0^1 4x \arcsin x \; dx$$

$\boxed{\arcsin x - x\sqrt{1-x^2} + C}$, $\boxed{\frac{\pi}{2}}$


a) $y = \arcsin x$

$\Rightarrow \sin y = x$

$\Rightarrow x = \sin y$

$\Rightarrow \frac{dx}{dy} = \cos y$

$\Rightarrow \frac{dy}{dx} = \frac{1}{\cos y}$

NOW $\cos^2 y + \sin^2 y = 1$

$\cos y = \pm\sqrt{1-\sin^2 y}$

BUT IF $y = \arcsin x$

$-\frac{\pi}{2} \le y \le \frac{\pi}{2}$

$0 \le \cos y \le 1$

$\cos y = +\sqrt{1-\sin^2 y}$

$\Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{1-\sin^2 y}}$

$\Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$

b) $\int \frac{2x^2}{\sqrt{1-x^2}}\,dx$ = substitution $= \int \frac{2\sin^2\theta}{\sqrt{1-\sin^2\theta}}\cos\theta\,d\theta$

$x = \sin\theta$

$\theta = \arcsin x$

$\frac{dx}{d\theta} = \cos\theta$

$dx = \cos\theta\,d\theta$

$\cos\theta = \sqrt{1-\sin^2\theta}$

$= \int \frac{2\sin^2\theta}{\cos\theta} \times \cos\theta\,d\theta = \int 2\sin^2\theta\,d\theta$

$= \int 2\left(\frac{1}{2} - \frac{1}{2}\cos 2\theta\right) d\theta = \int 1 - \cos 2\theta\,d\theta$

$= \theta - \frac{1}{2}\sin 2\theta + C = \theta - \sin\theta\cos\theta + C = \theta - \sin\theta\sqrt{1-\sin^2\theta} + C$

$= \arcsin x - x\sqrt{1-x^2} + C$

c) $\int_0^1 4x \arcsin x\,dx = \ldots$ by parts ... (IGNORE LIMITS)

$\arcsin x$ | $\frac{1}{\sqrt{1-x^2}}$

$2x^2$ | $4x$

$= 2x^2 \arcsin x - \int \frac{2x^2}{\sqrt{1-x^2}}\,dx$

$= 2x^2 \arcsin x - \left[\arcsin x - x\sqrt{1-x^2}\right] + C$

$= (2x^2 - 1)\arcsin x + x\sqrt{1-x^2} + C$

RE-INTRODUCE LIMITS

$\int_0^1 4x \arcsin x\,dx = \left[(2x^2-1)\arcsin x + x\sqrt{1-x^2}\right]_0^1$

$= \left(\frac{\pi}{2} + 0\right) - (0 + 0) = \pi/2$

Question 64 (****+)

It is given that

$$x = -2 + \sqrt{3}\cosh\theta, \ \theta \geq 0 .$$

a) Show clearly that …

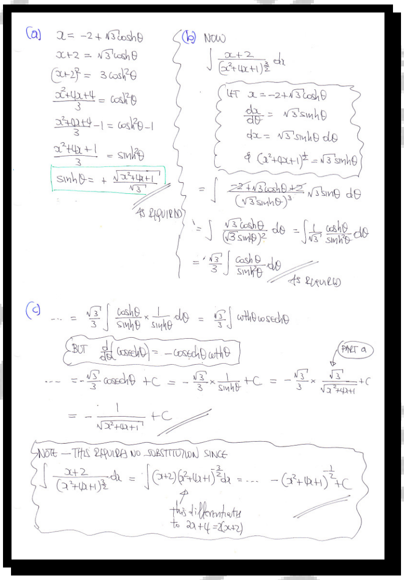
i. … $\sinh\theta = \dfrac{\sqrt{x^2+4x+1}}{\sqrt{3}}$.

ii. … $\displaystyle\int \frac{x+2}{\left(x^2+4x+1\right)^{\frac{3}{2}}}\,dx = \frac{\sqrt{3}}{3}\int \frac{\cosh\theta}{\sinh^2\theta}\,d\theta$.

b) By considering the derivative of $\operatorname{cosech}\theta$ find

$$\int \frac{x+2}{\left(x^2+4x+1\right)^{\frac{3}{2}}}\,dx$$

$$\boxed{-\left(x^2+4x+1\right)^{-\frac{1}{2}} + C}$$



(a) $x = -2+\sqrt{3}\cosh\theta$

$x+2 = \sqrt{3}\cosh\theta$

$(x+2)^2 = 3\cosh^2\theta$

$\frac{x^2+4x+4}{3} = \cosh^2\theta$

$\frac{x^2+4x+4}{3} - 1 = \cosh^2\theta - 1$

$\frac{x^2+4x+1}{3} = \sinh^2\theta$

$\sinh\theta = +\frac{\sqrt{x^2+4x+1}}{\sqrt{3}}$ AS REQUIRED

(b) NOW

$\int \frac{x+2}{(x^2+4x+1)^{\frac{3}{2}}}\,dx$

LET $x = -2+\sqrt{3}\cosh\theta$

$\frac{dx}{d\theta} = \sqrt{3}\sinh\theta$

$dx = \sqrt{3}\sinh\theta\,d\theta$

& $(x^2+4x+1)^{\frac{1}{2}} = \sqrt{3}\sinh\theta$

$= \int \frac{-2+\sqrt{3}\cosh\theta+2}{(\sqrt{3}\sinh\theta)^3}\sqrt{3}\sinh\theta\,d\theta$

$= \int \frac{\sqrt{3}\cosh\theta}{(\sqrt{3}\sinh\theta)^2}\,d\theta = \int \frac{1}{\sqrt{3}}\frac{\cosh\theta}{\sinh^2\theta}\,d\theta$

$= \frac{\sqrt{3}}{3}\int \frac{\cosh\theta}{\sinh^2\theta}\,d\theta$ AS REQUIRED

(c) $\dots = \frac{\sqrt{3}}{3}\int \frac{\cosh\theta}{\sinh\theta}\times\frac{1}{\sinh\theta}\,d\theta = \frac{\sqrt{3}}{3}\int \coth\theta\operatorname{cosech}\theta$

BUT $\frac{d}{d\theta}(\operatorname{cosech}\theta) = -\operatorname{cosech}\theta\coth\theta$

$\dots = -\frac{\sqrt{3}}{3}\operatorname{cosech}\theta + C = -\frac{\sqrt{3}}{3}\times\frac{1}{\sinh\theta} + C = -\frac{\sqrt{3}}{3}\times\frac{\sqrt{3}}{\sqrt{x^2+4x+1}} + C$ (PART a)

$= -\frac{1}{\sqrt{x^2+4x+1}} + C$

NOTE — THIS REQUIRES NO SUBSTITUTION SINCE

$\int \frac{x+2}{(x^2+4x+1)^{\frac{3}{2}}}\,dx = \int (x+2)(x^2+4x+1)^{-\frac{3}{2}}\,dx = \dots -(x^2+4x+1)^{-\frac{1}{2}} + C$

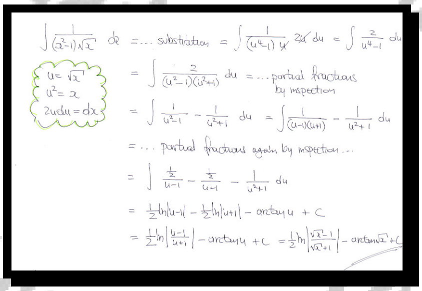
this differentiates to $2x+4 = 2(x+2)$

Question 65 (****+)

Use the substitution $u = \sqrt{x}$ to find

$$\int \frac{1}{\left(x^2 - 1\right)\sqrt{x}} \, dx \, .$$

$$\boxed{\frac{1}{2}\ln\left|\frac{\sqrt{x}-1}{\sqrt{x}+1}\right| - \arctan\sqrt{x} + C}$$


$$\int \frac{1}{(x^2-1)\sqrt{x}}\,dx = \dots \text{substitution} = \int \frac{1}{(u^4-1)u}\,2u\,du = \int \frac{2}{u^4-1}\,du$$

$u = \sqrt{x}$, $u^2 = x$, $2u\,du = dx$

$$= \int \frac{2}{(u^2-1)(u^2+1)}\,du = \dots \text{partial fractions by inspection}$$

$$= \int \frac{1}{u^2-1} - \frac{1}{u^2+1}\,du = \int \frac{1}{(u-1)(u+1)} - \frac{1}{u^2+1}\,du$$

$= \dots$ partial fractions again by inspection $\dots$

$$= \int \frac{\frac{1}{2}}{u-1} - \frac{\frac{1}{2}}{u+1} - \frac{1}{u^2+1}\,du$$

$$= \frac{1}{2}\ln|u-1| - \frac{1}{2}\ln|u+1| - \arctan u + C$$

$$= \frac{1}{2}\ln\left|\frac{u-1}{u+1}\right| - \arctan u + C = \frac{1}{2}\ln\left|\frac{\sqrt{x}-1}{\sqrt{x}+1}\right| - \arctan\sqrt{x} + C$$

Question 66 (****+)

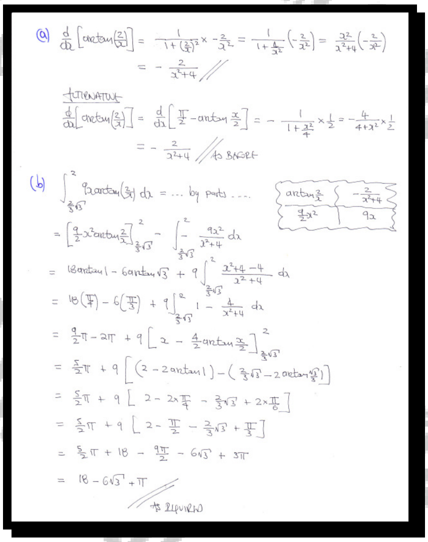
a) Find a simplified expression for

$$\frac{d}{dx}\left[\arctan\frac{2}{x}\right].$$

b) Hence show that

$$\int_{\frac{2}{3}\sqrt{3}}^{2} 9x\arctan\left(\frac{2}{x}\right)\, dx = \pi + 18 - 6\sqrt{3}$$

$$\boxed{\frac{d}{dx}\left[\arctan\frac{2}{x}\right] = -\frac{2}{x^2+4}}$$



(a) $\frac{d}{dx}\left[\arctan\left(\frac{2}{x}\right)\right] = \frac{1}{1+\left(\frac{2}{x}\right)^2} \times -\frac{2}{x^2} = \frac{1}{1+\frac{4}{x^2}}\left(-\frac{2}{x^2}\right) = \frac{x^2}{x^2+4}\left(-\frac{2}{x^2}\right)$
$= -\frac{2}{x^2+4}$

ALTERNATIVE
$\frac{d}{dx}\left[\arctan\left(\frac{2}{x}\right)\right] = \frac{d}{dx}\left[\frac{\pi}{2} - \arctan\frac{x}{2}\right] = -\frac{1}{1+\frac{x^2}{4}} \times \frac{1}{2} = -\frac{4}{4+x^2} \times \frac{1}{2}$
$= -\frac{2}{x^2+4}$ AS BEFORE

(b) $\int_{\frac{2}{3}\sqrt{3}}^{2} 9x\arctan\left(\frac{2}{x}\right) dx = \dots$ by parts ....

$\arctan\frac{2}{x}$	$-\frac{2}{x^2+4}$
$\frac{9}{2}x^2$	$9x$

$= \left[\frac{9}{2}x^2\arctan\frac{2}{x}\right]_{\frac{2}{3}\sqrt{3}}^{2} - \int_{\frac{2}{3}\sqrt{3}}^{2} -\frac{9x^2}{x^2+4}\, dx$
$= 18\arctan 1 - 6\arctan\sqrt{3} + 9\int_{\frac{2}{3}\sqrt{3}}^{2} \frac{x^2+4-4}{x^2+4}\, dx$
$= 18\left(\frac{\pi}{4}\right) - 6\left(\frac{\pi}{3}\right) + 9\int_{\frac{2}{3}\sqrt{3}}^{2} 1 - \frac{4}{x^2+4}\, dx$
$= \frac{9}{2}\pi - 2\pi + 9\left[x - \frac{4}{2}\arctan\frac{x}{2}\right]_{\frac{2}{3}\sqrt{3}}^{2}$
$= \frac{5}{2}\pi + 9\left[\left(2 - 2\arctan 1\right) - \left(\frac{2}{3}\sqrt{3} - 2\arctan\frac{\sqrt{3}}{3}\right)\right]$
$= \frac{5}{2}\pi + 9\left[2 - 2\times\frac{\pi}{4} - \frac{2}{3}\sqrt{3} + 2\times\frac{\pi}{6}\right]$
$= \frac{5}{2}\pi + 9\left[2 - \frac{\pi}{2} - \frac{2}{3}\sqrt{3} + \frac{\pi}{3}\right]$
$= \frac{5}{2}\pi + 18 - \frac{9\pi}{2} - 6\sqrt{3} + 3\pi$
$= 18 - 6\sqrt{3} + \pi$ AS REQUIRED

Question 67 (****+)

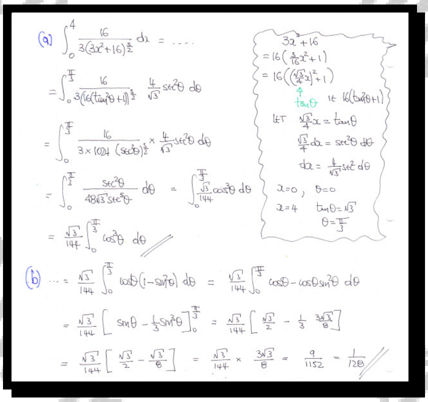
$$\int_0^4 \frac{16}{3\left(3x^2+16\right)^{\frac{5}{2}}}\ dx .$$

a) By using a suitable trigonometric substitution in terms of θ, show that the above integral can be transformed to

$$\frac{\sqrt{3}}{144}\int_0^{\frac{\pi}{3}} \cos^3\theta \ d\theta .$$

b) Hence evaluate the original integral.

$\boxed{\frac{1}{128}}$



Question 68 (****+)

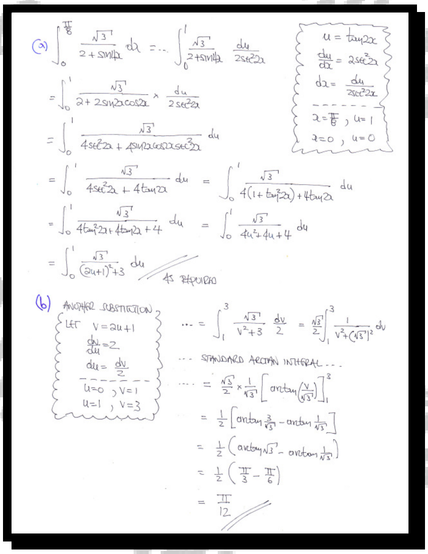
$$I = \int_0^{\frac{\pi}{8}} \frac{\sqrt{3}}{2+\sin 4x}\, dx .$$

a) Show that the substitution $u = \tan 2x$ transforms I into

$$J = \int_0^1 \frac{\sqrt{3}}{(2u+1)^2 + 3}\, du .$$

b) Hence find the exact value of I, giving the answer in terms of π.

$$\boxed{I = \frac{\pi}{12}}$$


(a) $\int_0^{\frac{\pi}{8}} \frac{\sqrt{3}}{2+\sin 4x}\,dx = \dots \int_0^1 \frac{\sqrt{3}}{2+\sin 4x}\,\frac{du}{2\sec^2 2x}$

$u = \tan 2x$, $\frac{du}{dx} = 2\sec^2 2x$, $dx = \frac{du}{2\sec^2 2x}$; $x = \frac{\pi}{8}, u = 1$; $x = 0, u = 0$

$= \int_0^1 \frac{\sqrt{3}}{2+2\sin 2x\cos 2x} \times \frac{du}{2\sec^2 2x}$

$= \int_0^1 \frac{\sqrt{3}}{4\sec^2 2x + 4\sin 2x\cos 2x\sec^2 2x}\,du$

$= \int_0^1 \frac{\sqrt{3}}{4\sec^2 2x + 4\tan 2x}\,du = \int_0^1 \frac{\sqrt{3}}{4(1+\tan^2 2x)+4\tan 2x}\,du$

$= \int_0^1 \frac{\sqrt{3}}{4\tan^2 2x + 4\tan 2x + 4}\,du = \int_0^1 \frac{\sqrt{3}}{4u^2+4u+4}\,du$

$= \int_0^1 \frac{\sqrt{3}}{(2u+1)^2+3}\,du$ // AS REQUIRED

(b) ANOTHER SUBSTITUTION

LET $v = 2u+1$, $\frac{dv}{du} = 2$, $du = \frac{dv}{2}$; $u=0, v=1$; $u=1, v=3$

$\dots = \int_1^3 \frac{\sqrt{3}}{v^2+3}\,\frac{dv}{2} = \frac{\sqrt{3}}{2}\int_1^3 \frac{1}{v^2+(\sqrt{3})^2}\,dv$

... STANDARD ARCTAN INTEGRAL ...

$\dots = \frac{\sqrt{3}}{2} \times \frac{1}{\sqrt{3}}\left[\arctan\left(\frac{v}{\sqrt{3}}\right)\right]_1^3$

$= \frac{1}{2}\left[\arctan\frac{3}{\sqrt{3}} - \arctan\frac{1}{\sqrt{3}}\right]$

$= \frac{1}{2}\left(\arctan\sqrt{3} - \arctan\frac{1}{\sqrt{3}}\right)$

$= \frac{1}{2}\left(\frac{\pi}{3} - \frac{\pi}{6}\right)$

$= \frac{\pi}{12}$

Question 69 (****+)

$$\sec x \equiv \frac{1+\tan^2\left(\frac{x}{2}\right)}{1-\tan^2\left(\frac{x}{2}\right)}.$$

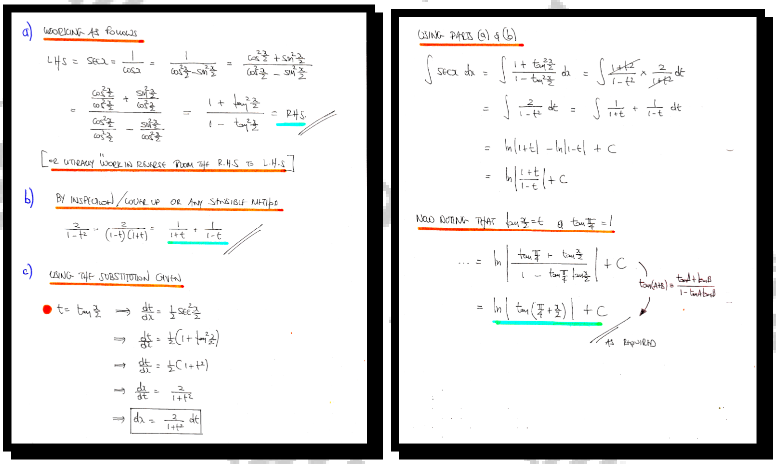
a) Prove the validity of the above trigonometric identity.

b) Express $\dfrac{2}{1-t^2}$ into partial fractions.

c) Hence use the substitution $t=\tan\left(\frac{x}{2}\right)$ to show that

$$\int \sec x \, dx = \ln\left|\tan\left(\frac{x}{2}+\frac{\pi}{4}\right)\right| + C.$$

$\boxed{}$, $\boxed{\dfrac{2}{1-t^2} = \dfrac{1}{1+t} + \dfrac{1}{1-t}}$

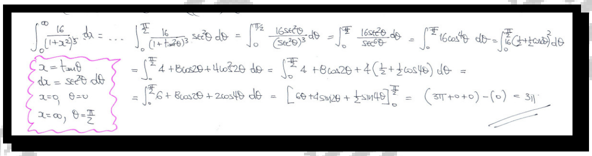


Question 70 (**+)**

Find an exact value for

$$\int_0^{\infty} \frac{16}{\left(1+x^2\right)^3}\, dx .$$

$\boxed{3\pi}$



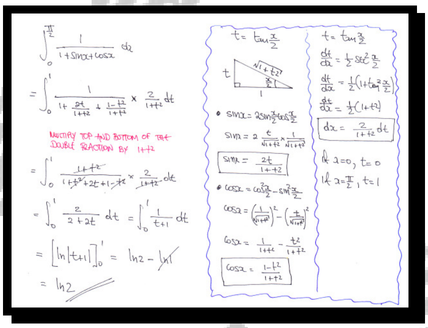
Question 71 (**+)**

$$I = \int_0^{\frac{\pi}{2}} \frac{1}{1+\sin x+\cos x}\, dx .$$

By using the substitution $t=\tan\left(\frac{x}{2}\right)$, or otherwise, show that

$$I = \ln 2 .$$

$\boxed{\text{proof}}$

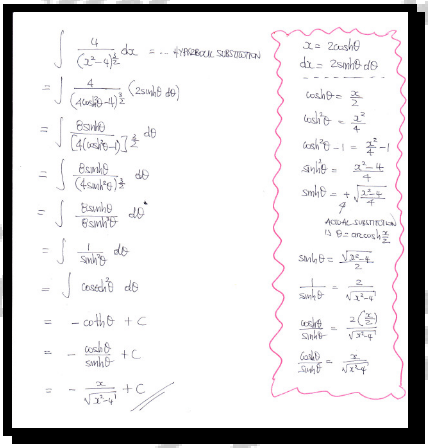


Question 72 (****+)

Use the substitution $x = 2\cosh\theta$, to find a simplified expression for

$$\int \frac{6}{\left(x^2-4\right)^{\frac{3}{2}}}\,dx\,.$$

$$\boxed{-\frac{x}{\sqrt{x^2-4}}+C}$$



$$\int \frac{4}{(x^2-4)^{\frac{3}{2}}}\,dx = \dots \text{ HYPERBOLIC SUBSTITUTION}$$

$$= \int \frac{4}{(4\cosh^2\theta-4)^{\frac{3}{2}}}(2\sinh\theta\,d\theta)$$

$$= \int \frac{8\sinh\theta}{[4(\cosh^2\theta-1)]^{\frac{3}{2}}}\,d\theta$$

$$= \int \frac{8\sinh\theta}{(4\sinh^2\theta)^{\frac{3}{2}}}\,d\theta$$

$$= \int \frac{8\sinh\theta}{8\sinh^3\theta}\,d\theta$$

$$= \int \frac{1}{\sinh^2\theta}\,d\theta$$

$$= \int \operatorname{cosech}^2\theta\,d\theta$$

$$= -\coth\theta + C$$

$$= -\frac{\cosh\theta}{\sinh\theta} + C$$

$$= -\frac{x}{\sqrt{x^2-4}} + C$$

$x = 2\cosh\theta$

$dx = 2\sinh\theta\,d\theta$

$\cosh\theta = \frac{x}{2}$

$\cosh^2\theta = \frac{x^2}{4}$

$\cosh^2\theta - 1 = \frac{x^2}{4} - 1$

$\sinh^2\theta = \frac{x^2-4}{4}$

$\sinh\theta = +\sqrt{\frac{x^2-4}{4}}$

ACTUAL SUBSTITUTION IS $\theta = \operatorname{arcosh}\frac{x}{2}$

$\sinh\theta = \frac{\sqrt{x^2-4}}{2}$

$\frac{1}{\sinh\theta} = \frac{2}{\sqrt{x^2-4}}$

$\frac{\cosh\theta}{\sinh\theta} = \frac{2\left(\frac{x}{2}\right)}{\sqrt{x^2-4}}$

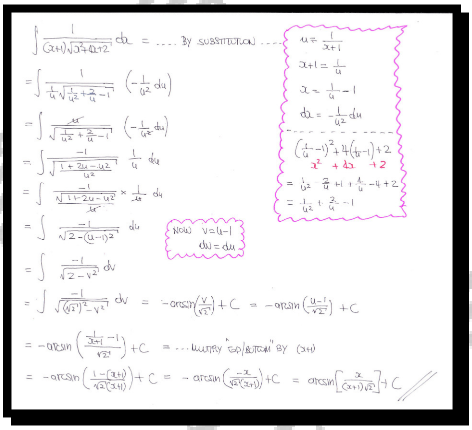
$\frac{\cosh\theta}{\sinh\theta} = \frac{x}{\sqrt{x^2-4}}$

Question 73 (****+)

Use the substitution $u = \dfrac{1}{x+1}$ to find

$$\int \frac{1}{(x+1)\sqrt{x^2+4x+2}}\,dx\,.$$

$$\boxed{\arcsin\left[\frac{x}{(x+1)\sqrt{2}}\right]+C}$$



$\int \frac{1}{(x+1)\sqrt{x^2+4x+2}}\,dx = \ldots$ BY SUBSTITUTION $\ldots$

$u = \frac{1}{x+1}$

$x+1 = \frac{1}{u}$

$x = \frac{1}{u} - 1$

$dx = -\frac{1}{u^2}\,du$

$\left(\frac{1}{u}-1\right)^2 + 4\left(\frac{1}{u}-1\right) + 2$

$x^2 \quad + 4x \quad + 2$

$= \frac{1}{u^2} - \frac{2}{u} + 1 + \frac{4}{u} - 4 + 2$

$= \frac{1}{u^2} + \frac{2}{u} - 1$

$= \int \frac{1}{\frac{1}{u}\sqrt{\frac{1}{u^2}+\frac{2}{u}-1}}\left(-\frac{1}{u^2}\,du\right)$

$= \int \frac{u}{\sqrt{\frac{1}{u^2}+\frac{2}{u}-1}}\left(-\frac{1}{u^2}\,du\right)$

$= \int \frac{-1}{\sqrt{\frac{1+2u-u^2}{u^2}}}\,\frac{1}{u}\,du$

$= \int \frac{-1}{\frac{\sqrt{1+2u-u^2}}{u}} \times \frac{1}{u}\,du$

$= \int \frac{-1}{\sqrt{2-(u-1)^2}}\,du$

NOW $v = u-1$, $dv = du$

$= \int \frac{-1}{\sqrt{2-v^2}}\,dv$

$= \int \frac{-1}{\sqrt{(\sqrt{2})^2-v^2}}\,dv = -\arcsin\left(\frac{v}{\sqrt{2}}\right) + C = -\arcsin\left(\frac{u-1}{\sqrt{2}}\right) + C$

$= -\arcsin\left(\frac{\frac{1}{x+1}-1}{\sqrt{2}}\right) + C = \ldots$ MULTIPLY "TOP/BOTTOM" BY $(x+1)$

$= -\arcsin\left(\frac{1-(x+1)}{\sqrt{2}(x+1)}\right) + C = -\arcsin\left(\frac{-x}{\sqrt{2}(x+1)}\right) + C = \arcsin\left[\frac{x}{(x+1)\sqrt{2}}\right] + C$

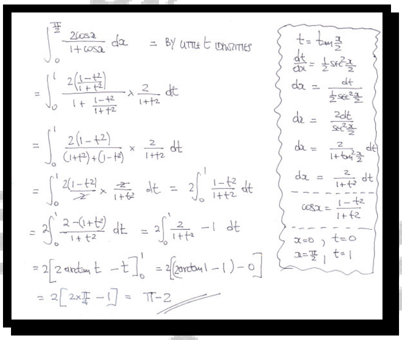
Question 74 (****+)

$$I = \int_0^{\frac{\pi}{2}} \frac{2\cos x}{1+\cos x} \, dx .$$

By using the substitution $t = \tan\left(\frac{x}{2}\right)$, or otherwise, show that

$$I = \pi - 2 .$$

proof

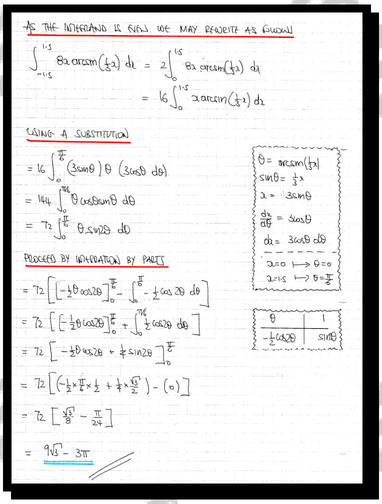


Question 75 **(****+)**

Find an exact value for

$$\int_{-1.5}^{1.5} 8x \arcsin\left(\frac{1}{3}x\right)\,dx.$$

$\boxed{}$, $\boxed{9\sqrt{3}-3\pi}$



AS THE INTEGRAND IS EVEN WE MAY REWRITE AS FOLLOWS

$$\int_{-1.5}^{1.5} 8x\arcsin\left(\tfrac{1}{3}x\right)dx = 2\int_0^{1.5} 8x\arcsin\left(\tfrac{1}{3}x\right)dx = 16\int_0^{1.5} x\arcsin\left(\tfrac{1}{3}x\right)dx$$

USING A SUBSTITUTION

$$= 16\int_0^{\frac{\pi}{6}} (3\sin\theta)\,\theta\,(3\cos\theta\,d\theta)$$

$$= 144\int_0^{\pi/6} \theta\cos\theta\sin\theta\,d\theta$$

$$= 72\int_0^{\frac{\pi}{6}} \theta\sin 2\theta\,d\theta$$

$\theta = \arcsin\left(\frac{1}{3}x\right)$, $\sin\theta = \frac{1}{3}x$, $x = 3\sin\theta$, $\frac{dx}{d\theta} = 3\cos\theta$, $dx = 3\cos\theta\,d\theta$

$x=0 \mapsto \theta=0$, $x=1.5 \mapsto \theta=\frac{\pi}{6}$

PROCEED BY INTEGRATION BY PARTS

θ	1
$-\frac{1}{2}\cos 2\theta$	$\sin 2\theta$

$$= 72\left[\left[-\tfrac{1}{2}\theta\cos 2\theta\right]_0^{\frac{\pi}{6}} - \int_0^{\frac{\pi}{6}} -\tfrac{1}{2}\cos 2\theta\,d\theta\right]$$

$$= 72\left[\left[-\tfrac{1}{2}\theta\cos 2\theta\right]_0^{\frac{\pi}{6}} + \int_0^{\pi/6} \tfrac{1}{2}\cos 2\theta\,d\theta\right]$$

$$= 72\left[-\tfrac{1}{2}\theta\cos 2\theta + \tfrac{1}{4}\sin 2\theta\right]_0^{\frac{\pi}{6}}$$

$$= 72\left[\left(-\tfrac{1}{2}\times\tfrac{\pi}{6}\times\tfrac{1}{2} + \tfrac{1}{4}\times\tfrac{\sqrt{3}}{2}\right) - (0)\right]$$

$$= 72\left[\frac{\sqrt{3}}{8} - \frac{\pi}{24}\right]$$

$$= 9\sqrt{3} - 3\pi$$

Question 76 (****+)

$$I = \int \frac{\operatorname{sech} x}{\cosh x - \sinh x} \, dx \, .$$

a) By multiplying the numerator and denominator of the integrand by $\operatorname{sech} x$, show that

$$I = -\ln\left(1 - \tanh x\right) + C \, ,$$

where C in an arbitrary constant.

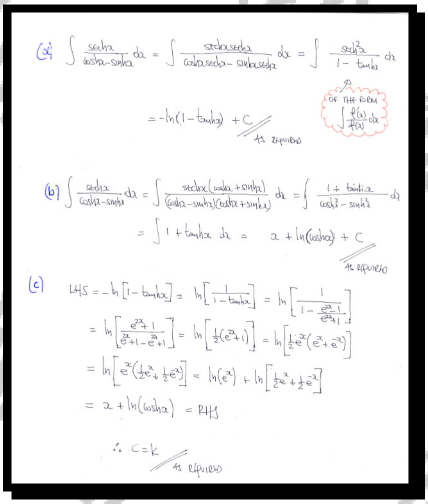
b) By multiplying the numerator and denominator of the integrand by $\left(\cosh x - \sinh x\right)$, show that

$$I = x + \ln\left(\cosh x\right) + K \, ,$$

where K in an arbitrary constant.

c) Show clearly that $C = K$.

proof



Question 77 (****+)

$$\sec x \equiv \frac{\cos x}{1-\sin^2 x}.$$

a) Prove the validity of the above trigonometric identity.

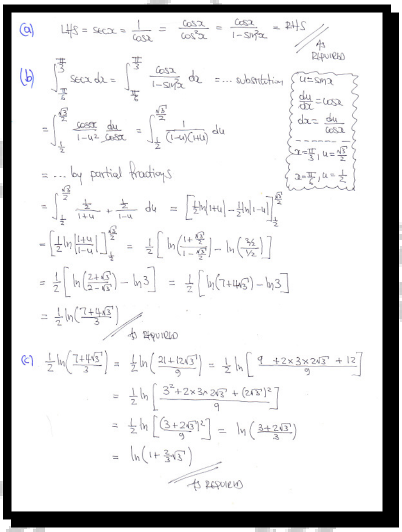
b) Use the substitution $u = \sin x$ to show that

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sec x \ dx = \frac{1}{2}\ln\left(\frac{7+4\sqrt{3}}{3}\right).$$

c) Show clearly that

$$\frac{1}{2}\ln\left(\frac{7+4\sqrt{3}}{3}\right) = \ln\left(1+\frac{2}{3}\sqrt{3}\right)$$

proof


(a) $\text{LHS} = \sec x = \frac{1}{\cos x} = \frac{\cos x}{\cos^2 x} = \frac{\cos x}{1-\sin^2 x} = \text{RHS}$ // AS REQUIRED

(b) $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sec x \, dx = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\cos x}{1-\sin^2 x} \, dx = \dots$ substitution

$u = \sin x$, $\frac{du}{dx} = \cos x$, $dx = \frac{du}{\cos x}$; $x = \frac{\pi}{3}, u = \frac{\sqrt{3}}{2}$; $x = \frac{\pi}{6}, u = \frac{1}{2}$

$= \int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} \frac{\cos x}{1-u^2} \frac{du}{\cos x} = \int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} \frac{1}{(1-u)(1+u)} \, du$

$= \dots$ by partial fractions

$= \int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} \frac{\frac{1}{2}}{1+u} + \frac{\frac{1}{2}}{1-u} \, du = \left[\frac{1}{2}\ln|1+u| - \frac{1}{2}\ln|1-u|\right]_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}}$

$= \left[\frac{1}{2}\ln\left|\frac{1+u}{1-u}\right|\right]_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} = \frac{1}{2}\left[\ln\left(\frac{1+\frac{\sqrt{3}}{2}}{1-\frac{\sqrt{3}}{2}}\right) - \ln\left(\frac{3/2}{1/2}\right)\right]$

$= \frac{1}{2}\left[\ln\left(\frac{2+\sqrt{3}}{2-\sqrt{3}}\right) - \ln 3\right] = \frac{1}{2}\left[\ln(7+4\sqrt{3}) - \ln 3\right]$

$= \frac{1}{2}\ln\left(\frac{7+4\sqrt{3}}{3}\right)$ // AS REQUIRED

(c) $\frac{1}{2}\ln\left(\frac{7+4\sqrt{3}}{3}\right) = \frac{1}{2}\ln\left(\frac{21+12\sqrt{3}}{9}\right) = \frac{1}{2}\ln\left[\frac{9+2\times3\times2\sqrt{3}+12}{9}\right]$

$= \frac{1}{2}\ln\left[\frac{3^2+2\times3\times2\sqrt{3}+(2\sqrt{3})^2}{9}\right]$

$= \frac{1}{2}\ln\left[\frac{(3+2\sqrt{3})^2}{9}\right] = \ln\left(\frac{3+2\sqrt{3}}{3}\right)$

$= \ln\left(1+\frac{2}{3}\sqrt{3}\right)$ // AS REQUIRED

Question 78 (****+)

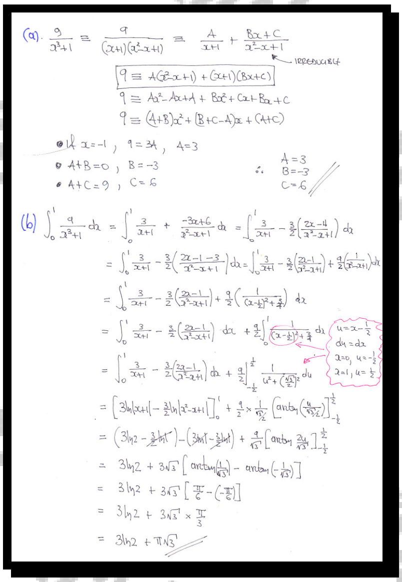
$$\frac{9}{x^3+1} \equiv \frac{A}{x+1} + \frac{Bx+C}{x^2-x+1}.$$

a) Find the value of each of the constants A, B and C in the above identity.

b) Hence find the exact value of

$$\int_0^1 \frac{9}{x^3-1}\, dx .$$

$\boxed{A=3}$, $\boxed{B=-3}$, $\boxed{C=6}$, $\boxed{3\ln 2+\pi\sqrt{3}}$


(a) $\frac{9}{x^3+1} \equiv \frac{9}{(x+1)(x^2-x+1)} \equiv \frac{A}{x+1} + \frac{Bx+C}{x^2-x+1}$ ← IRREDUCIBLE

$9 \equiv A(x^2-x+1)+(x+1)(Bx+C)$

$9 \equiv Ax^2-Ax+A+Bx^2+Cx+Bx+C$

$9 \equiv (A+B)x^2+(B+C-A)x+(A+C)$

- If $x=-1$, $9=3A$, $A=3$
- $A+B=0$, $B=-3$
- $A+C=9$, $C=6$

$\therefore$ $A=3$, $B=-3$, $C=6$

(b) $\int_0^1 \frac{9}{x^3+1}\,dx = \int_0^1 \frac{3}{x+1}+\frac{-3x+6}{x^2-x+1}\,dx = \int_0^1 \frac{3}{x+1}-\frac{3}{2}\left(\frac{2x-4}{x^2-x+1}\right)dx$

$= \int_0^1 \frac{3}{x+1}-\frac{3}{2}\left(\frac{2x-1-3}{x^2-x+1}\right)dx = \int_0^1 \frac{3}{x+1}-\frac{3}{2}\left(\frac{2x-1}{x^2-x+1}\right)+\frac{9}{2}\left(\frac{1}{x^2-x+1}\right)dx$

$= \int_0^1 \frac{3}{x+1}-\frac{3}{2}\left(\frac{2x-1}{x^2-x+1}\right)+\frac{9}{2}\left(\frac{1}{(x-\frac{1}{2})^2+\frac{3}{4}}\right)dx$

$= \int_0^1 \frac{3}{x+1}-\frac{3}{2}\left(\frac{2x-1}{x^2-x+1}\right)dx+\frac{9}{2}\int_0^1 \frac{1}{(x-\frac{1}{2})^2+\frac{3}{4}}\,dx$

$u=x-\frac{1}{2}$, $du=dx$, $x=0, u=-\frac{1}{2}$, $x=1, u=\frac{1}{2}$

$= \int_0^1 \frac{3}{x+1}-\frac{3}{2}\left(\frac{2x-1}{x^2-x+1}\right)dx+\frac{9}{2}\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{u^2+(\frac{\sqrt{3}}{2})^2}\,du$

$= \left[3\ln|x+1|-\frac{3}{2}\ln|x^2-x+1|\right]_0^1+\frac{9}{2}\times\frac{1}{\sqrt{3}/2}\left[\arctan\left(\frac{u}{\sqrt{3}/2}\right)\right]_{-\frac{1}{2}}^{\frac{1}{2}}$

$= \left(3\ln 2-\frac{3}{2}\ln 1\right)-\left(3\ln 1-\frac{3}{2}\ln 1\right)+\frac{9}{\sqrt{3}}\left[\arctan\frac{2u}{\sqrt{3}}\right]_{-\frac{1}{2}}^{\frac{1}{2}}$

$= 3\ln 2+3\sqrt{3}\left[\arctan\left(\frac{1}{\sqrt{3}}\right)-\arctan\left(-\frac{1}{\sqrt{3}}\right)\right]$

$= 3\ln 2+3\sqrt{3}\left[\frac{\pi}{6}-\left(-\frac{\pi}{6}\right)\right]$

$= 3\ln 2+3\sqrt{3}\times\frac{\pi}{3}$

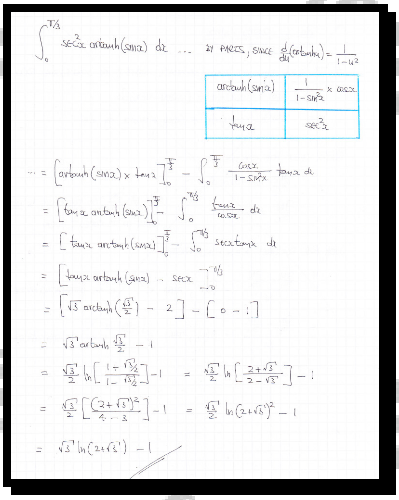
$= 3\ln 2+\pi\sqrt{3}$

Question 79 (****+)

Show that

$$\int_0^{\frac{1}{3}\pi} \sec^2 x \ \operatorname{artanh}(\sin x)\ dx \ = \ -1+\sqrt{3}\ln\left(2+\sqrt{3}\right).$$

proof



$$\int_0^{\pi/3} \sec^2 x \operatorname{artanh}(\sin x)\, dx \ \ldots \quad \text{BY PARTS, SINCE } \frac{d}{du}(\operatorname{artanh} u) = \frac{1}{1-u^2}$$

$\operatorname{artanh}(\sin x)$	$\frac{1}{1-\sin^2 x} \times \cos x$
$\tan x$	$\sec^2 x$

$$\ldots = \left[\operatorname{artanh}(\sin x) \times \tan x\right]_0^{\frac{\pi}{3}} - \int_0^{\frac{\pi}{3}} \frac{\cos x}{1-\sin^2 x} \tan x \, dx$$

$$= \left[\tan x \operatorname{artanh}(\sin x)\right]_0^{\frac{\pi}{3}} - \int_0^{\pi/3} \frac{\tan x}{\cos x}\, dx$$

$$= \left[\tan x \operatorname{artanh}(\sin x)\right]_0^{\frac{\pi}{3}} - \int_0^{\pi/3} \sec x \tan x \, dx$$

$$= \left[\tan x \operatorname{artanh}(\sin x) - \sec x\right]_0^{\pi/3}$$

$$= \left[\sqrt{3} \operatorname{artanh}\left(\frac{\sqrt{3}}{2}\right) - 2\right] - \left[0 - 1\right]$$

$$= \sqrt{3} \operatorname{artanh} \frac{\sqrt{3}}{2} - 1$$

$$= \frac{\sqrt{3}}{2} \ln\left[\frac{1+\sqrt{3}/2}{1-\sqrt{3}/2}\right] - 1 = \frac{\sqrt{3}}{2} \ln\left[\frac{2+\sqrt{3}}{2-\sqrt{3}}\right] - 1$$

$$= \frac{\sqrt{3}}{2}\left[\frac{(2+\sqrt{3})^2}{4-3}\right] - 1 = \frac{\sqrt{3}}{2} \ln(2+\sqrt{3})^2 - 1$$

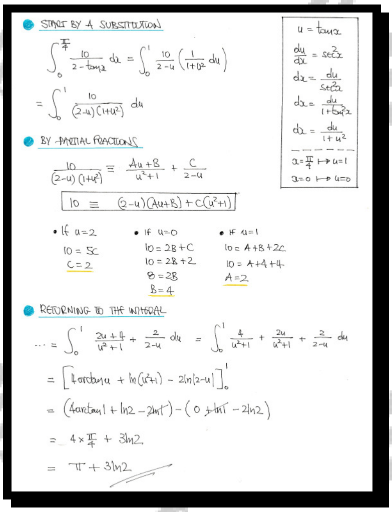
$$= \sqrt{3} \ln(2+\sqrt{3}) - 1$$

Question 80 (****+)

Use appropriate integration techniques to find an exact simplified value for

$$\int_{0}^{\frac{1}{4}\pi} \frac{10}{2-\tan x}\, dx \, .$$

$\boxed{}$, $\boxed{\pi + 3\ln 2}$



START BY A SUBSTITUTION

$$\int_0^{\frac{\pi}{4}} \frac{10}{2-\tan x}\,dx = \int_0^1 \frac{10}{2-u}\left(\frac{1}{1+u^2}\,du\right)$$

$$= \int_0^1 \frac{10}{(2-u)(1+u^2)}\,du$$

$u = \tan x$

$\frac{du}{dx} = \sec^2 x$

$dx = \frac{du}{\sec^2 x}$

$dx = \frac{du}{1+\tan^2 x}$

$dx = \frac{du}{1+u^2}$

$x = \frac{\pi}{4} \mapsto u = 1$

$x = 0 \mapsto u = 0$

BY PARTIAL FRACTIONS

$$\frac{10}{(2-u)(1+u^2)} \equiv \frac{Au+B}{u^2+1} + \frac{C}{2-u}$$

$$\boxed{10 \equiv (2-u)(Au+B) + C(u^2+1)}$$

- If $u=2$: $10 = 5C$, $C = 2$
- If $u=0$: $10 = 2B + C$, $10 = 2B + 2$, $8 = 2B$, $B = 4$
- If $u=1$: $10 = A + B + 2C$, $10 = A + 4 + 4$, $A = 2$

RETURNING TO THE INTEGRAL

$$\dots = \int_0^1 \frac{2u+4}{u^2+1} + \frac{2}{2-u}\,du = \int_0^1 \frac{4}{u^2+1} + \frac{2u}{u^2+1} + \frac{2}{2-u}\,du$$

$$= \left[4\arctan u + \ln(u^2+1) - 2\ln|2-u|\right]_0^1$$

$$= (4\arctan 1 + \ln 2 - 2\ln 1) - (0 + \ln 1 - 2\ln 2)$$

$$= 4 \times \frac{\pi}{4} + 3\ln 2$$

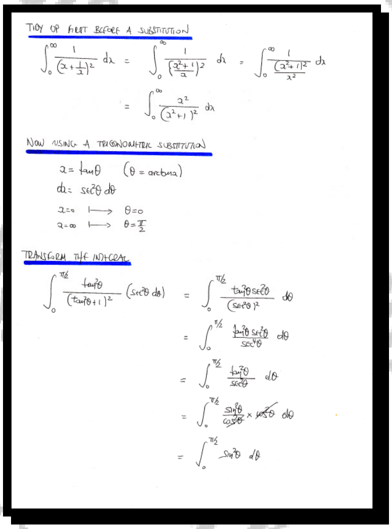
$$= \pi + 3\ln 2$$

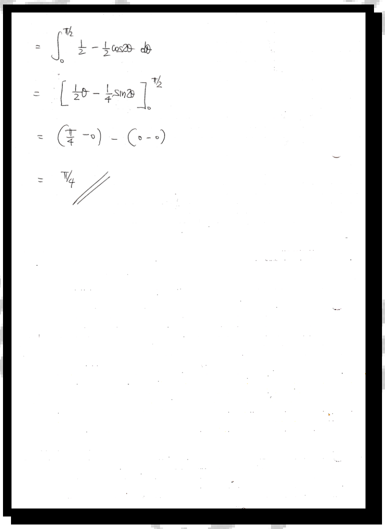
Question 81 (*****)

Use appropriate integration techniques to find an exact simplified value for

$$\int_0^{\infty} \frac{1}{\left(x+x^{-1}\right)^2}\, dx .$$

$\boxed{}, \boxed{\frac{\pi}{4}}$



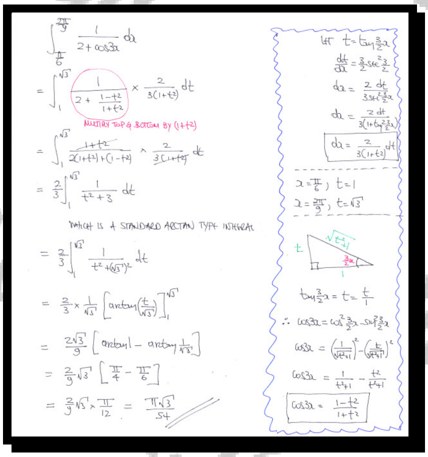


Question 82 (*****)

Use the substitution $t = \tan\left(\frac{3}{2}x\right)$ to find, in terms of π, the exact value of

$$\int_{\frac{\pi}{6}}^{\frac{2\pi}{9}} \frac{1}{2+\cos 3x}\, dx .$$

$$\boxed{\frac{\pi\sqrt{3}}{54}}$$

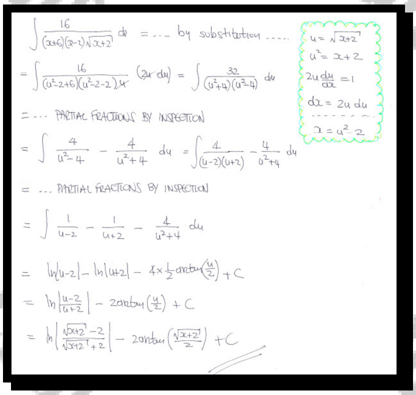


Question 83 (*****)

Use the substitution $u=\sqrt{x+2}$ to find

$$\int \frac{16}{(x+6)(x-2)\sqrt{x+2}}\,dx\,.$$

$$\boxed{\ln\left|\frac{\sqrt{x+2}-2}{\sqrt{x+2}+2}\right|-2\arctan\left(\frac{\sqrt{x+2}}{2}\right)+C}$$



$\int \frac{16}{(x+6)(x-2)\sqrt{x+2}}\,dx = \dots$ by substitution $\dots$

$u=\sqrt{x+2}$

$u^2=x+2$

$2u\frac{du}{dx}=1$

$dx=2u\,du$

$x=u^2-2$

$= \int \frac{16}{(u^2-2+6)(u^2-2-2)u}(2u\,du) = \int \frac{32}{(u^2+4)(u^2-4)}\,du$

$= \dots$ PARTIAL FRACTIONS BY INSPECTION

$= \int \frac{4}{u^2-4}-\frac{4}{u^2+4}\,du = \int \frac{4}{(u-2)(u+2)}-\frac{4}{u^2+4}\,du$

$= \dots$ PARTIAL FRACTIONS BY INSPECTION

$= \int \frac{1}{u-2}-\frac{1}{u+2}-\frac{4}{u^2+4}\,du$

$= \ln|u-2|-\ln|u+2|-4\times\frac{1}{2}\arctan\left(\frac{u}{2}\right)+C$

$= \ln\left|\frac{u-2}{u+2}\right|-2\arctan\left(\frac{u}{2}\right)+C$

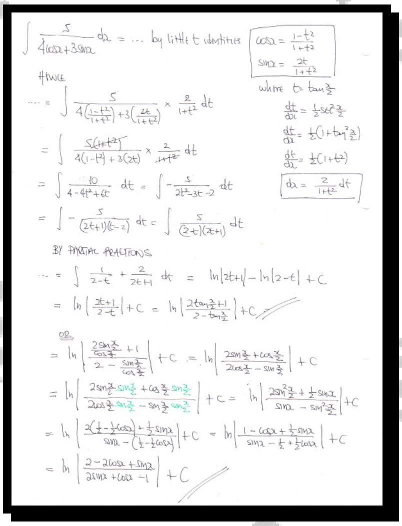
$= \ln\left|\frac{\sqrt{x+2}-2}{\sqrt{x+2}+2}\right|-2\arctan\left(\frac{\sqrt{x+2}}{2}\right)+C$

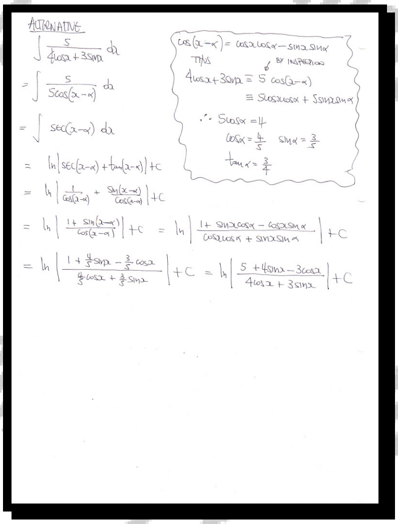
Question 84 (*****)

By using the substitution $t = \tan\left(\frac{x}{2}\right)$, or otherwise, show that

$$\int \frac{5}{4\cos x + 3\sin x}\, dx = \ln\left|\frac{2+\sin x - 2\cos x}{2\sin x + \cos x - 1}\right| + C .$$

proof



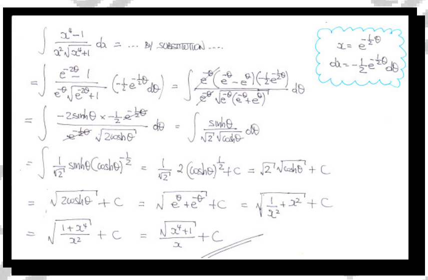


Question 85 (*****)

By using the substitution $x = e^{-\frac{1}{2}u}$, or otherwise, find a simplified expression for

$$\int \frac{x^4-1}{x^2\sqrt{x^4+1}}\, dx .$$

$$\frac{\sqrt{x^4+1}}{x} + C$$

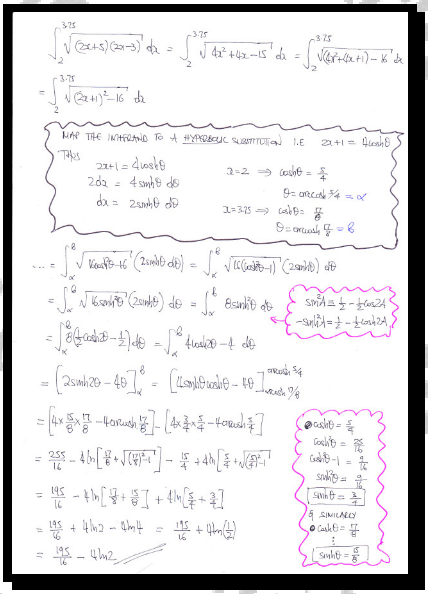


Question 86 (*****)

Use a suitable hyperbolic substitution to find the exact value of

$$\int_{2}^{3.75} \sqrt{(2x+5)(2x-3)}\ dx .$$

$$\boxed{\frac{195}{16} - 4\ln 2}$$



$$\int_{2}^{3.75} \sqrt{(2x+5)(2x-3)}\,dx = \int_{2}^{3.75} \sqrt{4x^2+4x-15}\,dx = \int_{2}^{3.75} \sqrt{(4x^2+4x+1)-16}\,dx$$

$$= \int_{2}^{3.75} \sqrt{(2x+1)^2-16}\,dx$$

MAP THE INTEGRAND TO A HYPERBOLIC SUBSTITUTION I.E $2x+1 = 4\cosh\theta$

THUS

$2x+1 = 4\cosh\theta$

$2dx = 4\sinh\theta\,d\theta$

$dx = 2\sinh\theta\,d\theta$

$x = 2 \Rightarrow \cosh\theta = \frac{5}{4}$

$\theta = \text{arcosh}\,5/4 = \alpha$

$x = 3.75 \Rightarrow \cosh\theta = \frac{17}{8}$

$\theta = \text{arcosh}\,\frac{17}{8} = \beta$

$$\dots = \int_{\alpha}^{\beta} \sqrt{16\cosh^2\theta - 16}\,(2\sinh\theta\,d\theta) = \int_{\alpha}^{\beta} \sqrt{16(\cosh^2\theta - 1)}\,(2\sinh\theta)\,d\theta$$

$$= \int_{\alpha}^{\beta} \sqrt{16\sinh^2\theta}\,(2\sinh\theta)\,d\theta = \int_{\alpha}^{\beta} 8\sinh^2\theta\,d\theta$$

$\sin^2 A \equiv \frac{1}{2} - \frac{1}{2}\cos 2A$

$-\sinh^2 A = \frac{1}{2} - \frac{1}{2}\cosh 2A$

$$= \int_{\alpha}^{\beta} 8\left(\tfrac{1}{2}\cosh 2\theta - \tfrac{1}{2}\right)d\theta = \int_{\alpha}^{\beta} 4\cosh 2\theta - 4\ d\theta$$

$$= \Big[2\sinh 2\theta - 4\theta\Big]_{\alpha}^{\beta} = \Big[4\sinh\theta\cosh\theta - 4\theta\Big]_{\text{arcosh}\,17/8}^{\text{arcosh}\,5/4}$$

$$= \left[4\times\frac{15}{8}\times\frac{17}{8} - 4\,\text{arcosh}\,\frac{17}{8}\right] - \left[4\times\frac{3}{4}\times\frac{5}{4} - 4\,\text{arcosh}\,\frac{5}{4}\right]$$

$$= \frac{255}{16} - 4\ln\left[\frac{17}{8} + \sqrt{\left(\frac{17}{8}\right)^2 - 1}\right] - \frac{15}{4} + 4\ln\left[\frac{5}{4} + \sqrt{\left(\frac{5}{4}\right)^2 - 1}\right]$$

$$= \frac{195}{16} - 4\ln\left[\frac{17}{8} + \frac{15}{8}\right] + 4\ln\left[\frac{5}{4} + \frac{3}{4}\right]$$

$$= \frac{195}{16} + 4\ln 2 - 4\ln 4 = \frac{195}{16} + 4\ln\left(\frac{1}{2}\right)$$

$$= \frac{195}{16} - 4\ln 2$$

- $\cosh\theta = \frac{5}{4}$

  $\cosh^2\theta = \frac{25}{16}$

  $\cosh^2\theta - 1 = \frac{9}{16}$

  $\sinh^2\theta = \frac{9}{16}$

  $\boxed{\sinh\theta = \frac{3}{4}}$

& SIMILARLY

- $\cosh\theta = \frac{17}{8}$

  $\vdots$

  $\boxed{\sinh\theta = \frac{15}{8}}$

Question 87 (*****)

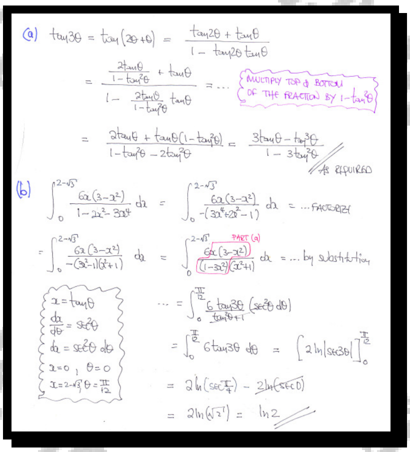
$$\tan 3\theta \equiv \frac{3\tan\theta - \tan^3\theta}{1-3\tan^2\theta}.$$

a) Prove the validity of the above trigonometric identity by writing $\tan 3\theta$ as $\tan(2\theta+\theta)$.

b) Hence, show clearly that

$$\int_0^{2-\sqrt{3}} \frac{6x(3-x^2)}{1-2x^2-3x^4}\, dx = \ln 2.$$

proof



(a) $\tan 3\theta = \tan(2\theta+\theta) = \frac{\tan 2\theta + \tan\theta}{1-\tan 2\theta\tan\theta}$

$= \frac{\frac{2\tan\theta}{1-\tan^2\theta} + \tan\theta}{1-\frac{2\tan\theta}{1-\tan^2\theta}\tan\theta} = \dots$ MULTIPLY TOP & BOTTOM OF THE FRACTION BY $1-\tan^2\theta$

$= \frac{2\tan\theta + \tan\theta(1-\tan^2\theta)}{1-\tan^2\theta-2\tan^2\theta} = \frac{3\tan\theta-\tan^3\theta}{1-3\tan^2\theta}$ // AS REQUIRED

(b) $\int_0^{2-\sqrt{3}} \frac{6x(3-x^2)}{1-2x^2-3x^4}\,dx = \int_0^{2-\sqrt{3}} \frac{6x(3-x^2)}{-(3x^4+2x^2-1)}\,dx = \dots$ FACTORIZE

$= \int_0^{2-\sqrt{3}} \frac{6x(3-x^2)}{-(3x^2-1)(x^2+1)}\,dx = \int_0^{2-\sqrt{3}} \frac{6x(3-x^2)}{(1-3x^2)(x^2+1)}\,dx = \dots$ by substitution (PART (a))

$x=\tan\theta$, $\frac{dx}{d\theta}=\sec^2\theta$, $dx=\sec^2\theta\,d\theta$, $x=0,\ \theta=0$, $x=2-\sqrt{3},\ \theta=\frac{\pi}{12}$

$\dots = \int_0^{\frac{\pi}{12}} \frac{6\tan 3\theta}{\tan^2\theta+1}(\sec^2\theta\,d\theta)$

$= \int_0^{\frac{\pi}{12}} 6\tan 3\theta\,d\theta = \left[2\ln|\sec 3\theta|\right]_0^{\frac{\pi}{12}}$

$= 2\ln\left(\sec\frac{\pi}{4}\right) - 2\ln(\sec 0)$

$= 2\ln(\sqrt{2}) = \ln 2$ //

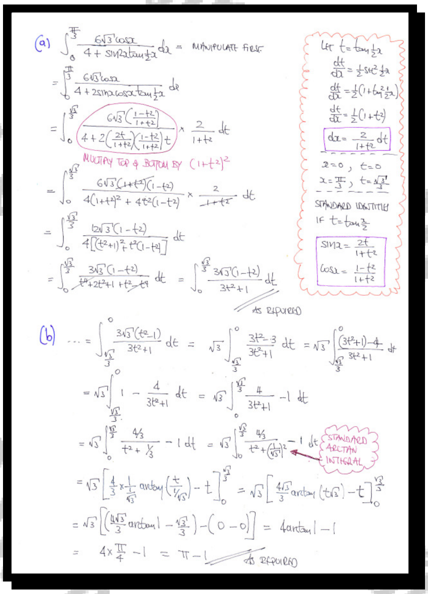
Question 88 (*****)

$$J = \int_{0}^{\frac{\pi}{3}} \frac{6\sqrt{3}\cos x}{4 + \sin 2x \tan\left(\frac{1}{2}x\right)} \, dx .$$

Use the substitution $t = \tan\left(\frac{1}{2}x\right)$ to show that

$$J = \pi - 1 .$$

[ ] , [proof]



(a) $\int_0^{\frac{\pi}{3}} \frac{6\sqrt{3}\cos x}{4+\sin 2x \tan\frac{1}{2}x}\,dx$ = MANIPULATE FIRST

$= \int_0^{\frac{\pi}{3}} \frac{6\sqrt{3}\cos x}{4+2\sin x\cos x\tan\frac{1}{2}x}\,dx$

$= \int_0^{\frac{\sqrt{3}}{3}} \frac{6\sqrt{3}\left(\frac{1-t^2}{1+t^2}\right)}{4+2\left(\frac{2t}{1+t^2}\right)\left(\frac{1-t^2}{1+t^2}\right)t} \times \frac{2}{1+t^2}\,dt$

MULTIPLY TOP & BOTTOM BY $(1+t^2)^2$

$= \int_0^{\frac{\sqrt{3}}{3}} \frac{6\sqrt{3}(1+t^2)(1-t^2)}{4(1+t^2)^2+4t^2(1-t^2)} \times \frac{2}{1+t^2}\,dt$

$= \int_0^{\frac{\sqrt{3}}{3}} \frac{12\sqrt{3}(1-t^2)}{4\left[(t^2+1)^2+t^2(1-t^2)\right]}\,dt$

$= \int_0^{\frac{\sqrt{3}}{3}} \frac{3\sqrt{3}(1-t^2)}{t^4+2t^2+1+t^2-t^4}\,dt = \int_0^{\frac{\sqrt{3}}{3}} \frac{3\sqrt{3}(1-t^2)}{3t^2+1}\,dt$ AS REQUIRED

LET $t = \tan\frac{1}{2}x$

$\frac{dt}{dx} = \frac{1}{2}\sec^2\frac{1}{2}x$

$\frac{dt}{dx} = \frac{1}{2}\left(1+\tan^2\frac{1}{2}x\right)$

$\frac{dt}{dx} = \frac{1}{2}(1+t^2)$

$dx = \frac{2}{1+t^2}\,dt$

$x=0,\ t=0$

$x=\frac{\pi}{3},\ t=\frac{\sqrt{3}}{3}$

STANDARD IDENTITIES IF $t = \tan\frac{x}{2}$

$\sin x = \frac{2t}{1+t^2}$

$\cos x = \frac{1-t^2}{1+t^2}$

(b) $\dots = \int_{\frac{\sqrt{3}}{3}}^{0} \frac{3\sqrt{3}(t^2-1)}{3t^2+1}\,dt = \sqrt{3}\int_{\frac{\sqrt{3}}{3}}^{0} \frac{3t^2-3}{3t^2+1}\,dt = \sqrt{3}\int_{\frac{\sqrt{3}}{3}}^{0} \frac{(3t^2+1)-4}{3t^2+1}\,dt$

$= \sqrt{3}\int_{\frac{\sqrt{3}}{3}}^{0} 1 - \frac{4}{3t^2+1}\,dt = \sqrt{3}\int_0^{\frac{\sqrt{3}}{3}} \frac{4}{3t^2+1} - 1\,dt$

$= \sqrt{3}\int_0^{\frac{\sqrt{3}}{3}} \frac{4/3}{t^2+1/3} - 1\,dt = \sqrt{3}\int_0^{\frac{\sqrt{3}}{3}} \frac{4/3}{t^2+\left(\frac{1}{\sqrt{3}}\right)^2} - 1\,dt$ STANDARD ARCTAN INTEGRAL

$= \sqrt{3}\left[\frac{4}{3}\times\frac{1}{\frac{1}{\sqrt{3}}}\arctan\left(\frac{t}{1/\sqrt{3}}\right) - t\right]_0^{\frac{\sqrt{3}}{3}} = \sqrt{3}\left[\frac{4\sqrt{3}}{3}\arctan\left(t\sqrt{3}\right) - t\right]_0^{\frac{\sqrt{3}}{3}}$

$= \sqrt{3}\left[\left(\frac{4\sqrt{3}}{3}\arctan 1 - \frac{\sqrt{3}}{3}\right) - (0-0)\right] = 4\arctan 1 - 1$

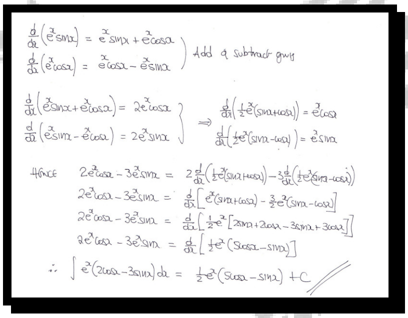
$= 4\times\frac{\pi}{4} - 1 = \pi - 1$ AS REQUIRED

Question 89 (*****)

By considering the derivatives of $e^x \sin x$ and $e^x \cos x$, find

$$\int e^x\left(2\cos x - 3\sin x\right)\,dx\,.$$

$$\boxed{\frac{1}{2}e^x\left(5\cos x - \sin x\right) + C}$$



$\frac{d}{dx}\left(e^x \sin x\right) = e^x \sin x + e^x \cos x$
$\frac{d}{dx}\left(e^x \cos x\right) = e^x \cos x - e^x \sin x$ } Add & subtract gives

$\frac{d}{dx}\left(e^x \sin x + e^x \cos x\right) = 2e^x \cos x$
$\frac{d}{dx}\left(e^x \sin x - e^x \cos x\right) = 2e^x \sin x$ } $\Rightarrow$ $\frac{d}{dx}\left(\frac{1}{2}e^x(\sin x + \cos x)\right) = e^x \cos x$, $\frac{d}{dx}\left(\frac{1}{2}e^x(\sin x - \cos x)\right) = e^x \sin x$

Hence $2e^x\cos x - 3e^x \sin x = 2\frac{d}{dx}\left(\frac{1}{2}e^x(\sin x + \cos x)\right) - 3\frac{d}{dx}\left(\frac{1}{2}e^x(\sin x - \cos x)\right)$

$2e^x\cos x - 3e^x \sin x = \frac{d}{dx}\left[e^x(\sin x + \cos x) - \frac{3}{2}e^x(\sin x - \cos x)\right]$

$2e^x\cos x - 3e^x \sin x = \frac{d}{dx}\left[\frac{1}{2}e^x\left[2\sin x + 2\cos x - 3\sin x + 3\cos x\right]\right]$

$2e^x\cos x - 3e^x \sin x = \frac{d}{dx}\left[\frac{1}{2}e^x(5\cos x - \sin x)\right]$

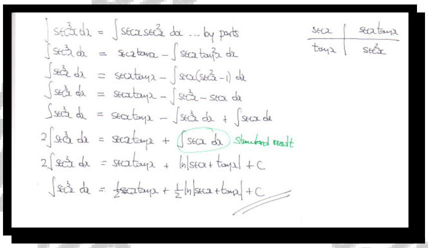
$\therefore \int e^x(2\cos x - 3\sin x)\,dx = \frac{1}{2}e^x(5\cos x - \sin x) + C$

Question 90 (*****)

Use integration by parts and suitable trigonometric identities to find

$$\int \sec^3 x\,dx\,.$$

$$\boxed{\frac{1}{2}\sec x\tan x + \frac{1}{2}\ln\left|\sec x + \tan x\right| + C}$$



$\int \sec^3 x\,dx = \int \sec x \sec^2 x\,dx$ ... by parts

$\sec x$	$\sec x \tan x$
$\tan x$	$\sec^2 x$

$\int \sec^3 x\,dx = \sec x\tan x - \int \sec x \tan^2 x\,dx$

$\int \sec^3 x\,dx = \sec x\tan x - \int \sec x(\sec^2 x - 1)\,dx$

$\int \sec^3 x\,dx = \sec x\tan x - \int \sec^3 x - \sec x\,dx$

$\int \sec^3 x\,dx = \sec x\tan x - \int \sec^3 x\,dx + \int \sec x\,dx$

$2\int \sec^3 x\,dx = \sec x\tan x + \int \sec x\,dx$ Standard result

$2\int \sec^3 x\,dx = \sec x\tan x + \ln|\sec x + \tan x| + C$

$\int \sec^3 x\,dx = \frac{1}{2}\sec x\tan x + \frac{1}{2}\ln|\sec x + \tan x| + C$

Question 91 (*****)

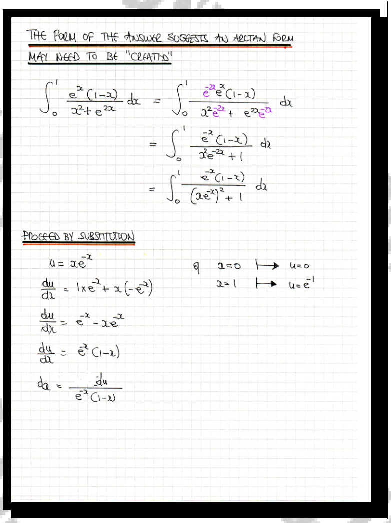
Show that the exact value of

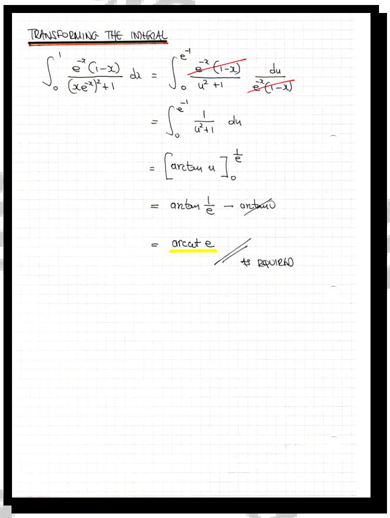
$$\int_0^1 \frac{(1-x)e^x}{x^2+e^{2x}}\,dx\,,$$

can be written as

$$\operatorname{arccot}(e)\,.$$

[ ], proof





Question 92 (*****)

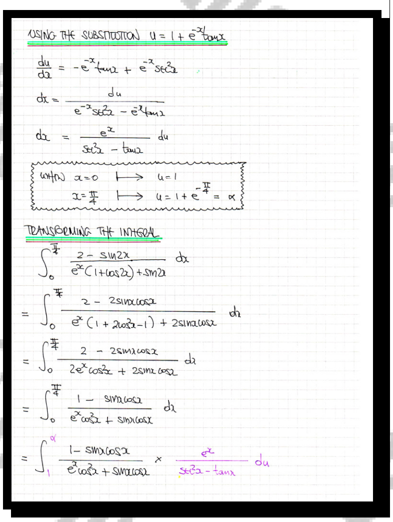
By using the substitution $u = 1 + e^{-x}\tan x$, or otherwise, show that the exact value of

$$\int_0^{\frac{1}{4}\pi} \frac{2 - \sin 2x}{(1+\cos 2x)e^x + \sin 2x}\, dx,$$

can be written as

$$\ln\left[2e^{-\frac{1}{8}\pi}\cosh\left(\tfrac{1}{8}\pi\right)\right].$$

$\boxed{}$, $\boxed{\text{proof}}$



USING THE SUBSTITUTION $u = 1 + e^{-x}\tan x$

$$\frac{du}{dx} = -e^{-x}\tan x + e^{-x}\sec^2 x$$

$$dx = \frac{du}{e^{-x}\sec^2 x - e^{-x}\tan x}$$

$$dx = \frac{e^x}{\sec^2 x - \tan x}\, du$$

WHEN $x = 0 \longmapsto u = 1$

$x = \frac{\pi}{4} \longmapsto u = 1 + e^{-\frac{\pi}{4}} = \alpha$

TRANSFORMING THE INTEGRAL

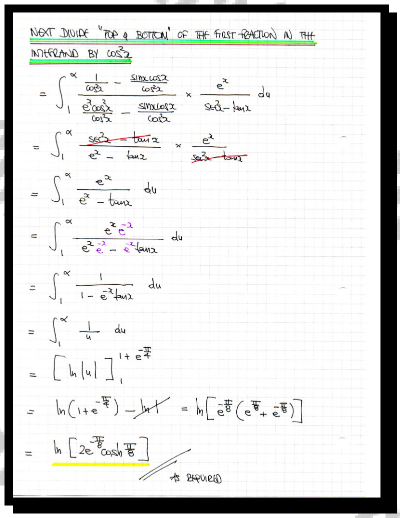
$$\int_0^{\frac{\pi}{4}} \frac{2 - \sin 2x}{e^x(1+\cos 2x) + \sin 2x}\, dx$$

$$= \int_0^{\frac{\pi}{4}} \frac{2 - 2\sin x\cos x}{e^x(1 + 2\cos^2 x - 1) + 2\sin x\cos x}\, dx$$

$$= \int_0^{\frac{\pi}{4}} \frac{2 - 2\sin x\cos x}{2e^x\cos^2 x + 2\sin x\cos x}\, dx$$

$$= \int_0^{\frac{\pi}{4}} \frac{1 - \sin x\cos x}{e^x\cos^2 x + \sin x\cos x}\, dx$$

$$= \int_1^{\alpha} \frac{1 - \sin x\cos x}{e^x\cos^2 x + \sin x\cos x} \times \frac{e^x}{\sec^2 x - \tan x}\, du$$



NEXT DIVIDE "TOP & BOTTOM" OF THE FIRST FRACTION IN THE INTEGRAND BY $\cos^2 x$

$$= \int_1^{\alpha} \frac{\frac{1}{\cos^2 x} - \frac{\sin x\cos x}{\cos^2 x}}{\frac{e^x\cos^2 x}{\cos^2 x} - \frac{\sin x\cos x}{\cos^2 x}} \times \frac{e^x}{\sec^2 x - \tan x}\, du$$

$$= \int_1^{\alpha} \frac{\sec^2 x - \tan x}{e^x - \tan x} \times \frac{e^x}{\sec^2 x - \tan x}$$

$$= \int_1^{\alpha} \frac{e^x}{e^x - \tan x}\, du$$

$$= \int_1^{\alpha} \frac{e^x e^{-x}}{e^x e^{-x} - e^{-x}\tan x}\, du$$

$$= \int_1^{\alpha} \frac{1}{1 - e^{-x}\tan x}\, du$$

$$= \int_1^{\alpha} \frac{1}{u}\, du$$

$$= \Big[\ln|u|\Big]_1^{1+e^{-\frac{\pi}{4}}}$$

$$= \ln\left(1 + e^{-\frac{\pi}{4}}\right) - \ln 1 = \ln\left[e^{-\frac{\pi}{8}}\left(e^{\frac{\pi}{8}} + e^{-\frac{\pi}{8}}\right)\right]$$

$$= \ln\left[2e^{-\frac{\pi}{8}}\cosh\frac{\pi}{8}\right]$$

AS REQUIRED

Question 93 (*****)

The function f is a continuous function and a is a real constant.

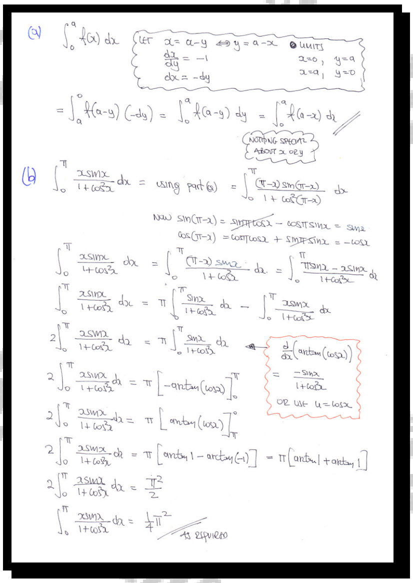
$$\int_0^a f(x)\ dx \equiv \int_0^a f(a-x)\ dx.$$

a) Prove the validity of the above identity.

b) Hence show clearly that

$$\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}\ dx = \frac{1}{4}\pi^2.$$

proof



(a) $\int_0^a f(x)\,dx$ [LET $x = a-y \Leftrightarrow y = a-x$; $\frac{dx}{dy} = -1$; $dx = -dy$; LIMITS $x=0,\ y=a$; $x=a,\ y=0$]

$= \int_a^0 f(a-y)(-dy) = \int_0^a f(a-y)\,dy = \int_0^a f(a-x)\,dx$ // NOTHING SPECIAL ABOUT x OR y

(b) $\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}\,dx$ = using part (a) $= \int_0^{\pi} \frac{(\pi-x)\sin(\pi-x)}{1+\cos^2(\pi-x)}\,dx$

NOW $\sin(\pi-x) = \sin\pi\cos x - \cos\pi\sin x = \sin x$
$\cos(\pi-x) = \cos\pi\cos x + \sin\pi\sin x = -\cos x$

$\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}\,dx = \int_0^{\pi} \frac{(\pi-x)\sin x}{1+\cos^2 x}\,dx = \int_0^{\pi} \frac{\pi\sin x - x\sin x}{1+\cos^2 x}\,dx$

$\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}\,dx = \pi\int_0^{\pi} \frac{\sin x}{1+\cos^2 x}\,dx - \int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}\,dx$

$2\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}\,dx = \pi\int_0^{\pi} \frac{\sin x}{1+\cos^2 x}\,dx$ ← $\frac{d}{dx}(\arctan(\cos x)) = \frac{-\sin x}{1+\cos^2 x}$ OR USE $u = \cos x$

$2\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}\,dx = \pi\left[-\arctan(\cos x)\right]_0^{\pi}$

$2\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}\,dx = \pi\left[\arctan(\cos x)\right]_{\pi}^{0}$

$2\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}\,dx = \pi\left[\arctan 1 - \arctan(-1)\right] = \pi\left[\arctan 1 + \arctan 1\right]$

$2\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}\,dx = \frac{\pi^2}{2}$

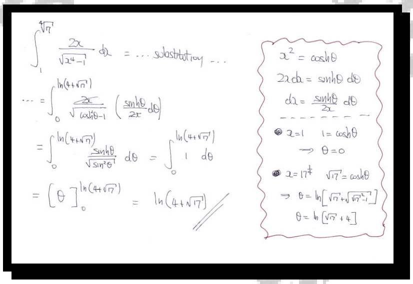
$\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}\,dx = \frac{1}{4}\pi^2$ // AS REQUIRED

Question 94 (*****)

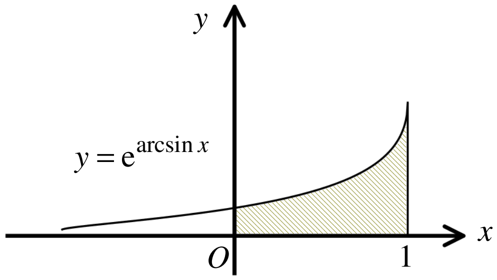
By using a suitable substitution, find the exact value of

$$\int_{1}^{\sqrt[4]{17}} \frac{2x}{\sqrt{x^4-1}}\ dx .$$

$\boxed{}$, $\boxed{\ln\left(4+\sqrt{17}\right)}$



Question 95 (*****)



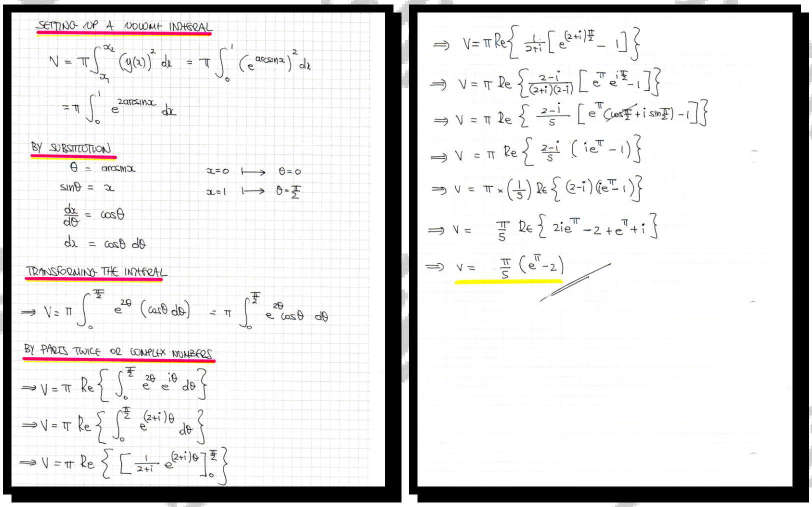
The figure above show the curve with equation

$$y = e^{\arcsin x}, \quad x \in \mathbb{R}, \quad |x| \le 1.$$

The finite region, shown shaded in the figure, bounded by the curve, the coordinate axes and the straight line with equation $x = 1$, is fully revolved about the x axis.

Find, an exact simplified value, for the volume of the solid of revolution formed.

$\boxed{}$, $\boxed{\frac{1}{5}\pi\left(e^{\pi} - 2\right)}$



SETTING UP A VOLUME INTEGRAL

$V = \pi \int_{x_1}^{x_2} (y(x))^2 \, dx = \pi \int_0^1 \left(e^{\arcsin x}\right)^2 dx$

$= \pi \int_0^1 e^{2\arcsin x} \, dx$

BY SUBSTITUTION

$\theta = \arcsin x$ $\quad x = 0 \mapsto \theta = 0$

$\sin\theta = x$ $\quad x = 1 \mapsto \theta = \frac{\pi}{2}$

$\frac{dx}{d\theta} = \cos\theta$

$dx = \cos\theta \, d\theta$

TRANSFORMING THE INTEGRAL

$\Rightarrow V = \pi \int_0^{\frac{\pi}{2}} e^{2\theta} (\cos\theta \, d\theta) = \pi \int_0^{\frac{\pi}{2}} e^{2\theta} \cos\theta \, d\theta$

BY PARTS TWICE OR COMPLEX NUMBERS

$\Rightarrow V = \pi \, \text{Re}\left\{ \int_0^{\frac{\pi}{2}} e^{2\theta} e^{i\theta} \, d\theta \right\}$

$\Rightarrow V = \pi \, \text{Re}\left\{ \int_0^{\frac{\pi}{2}} e^{(2+i)\theta} \, d\theta \right\}$

$\Rightarrow V = \pi \, \text{Re}\left\{ \left[\frac{1}{2+i} e^{(2+i)\theta} \right]_0^{\frac{\pi}{2}} \right\}$

$\Rightarrow V = \pi \, \text{Re}\left\{ \frac{1}{2+i} \left[e^{(2+i)\frac{\pi}{2}} - 1 \right] \right\}$

$\Rightarrow V = \pi \, \text{Re}\left\{ \frac{2-i}{(2+i)(2-i)} \left[e^{\pi} e^{i\frac{\pi}{2}} - 1 \right] \right\}$

$\Rightarrow V = \pi \, \text{Re}\left\{ \frac{2-i}{5} \left[e^{\pi}\left(\cos\frac{\pi}{2} + i \sin\frac{\pi}{2}\right) - 1 \right] \right\}$

$\Rightarrow V = \pi \, \text{Re}\left\{ \frac{2-i}{5} \left(i e^{\pi} - 1 \right) \right\}$

$\Rightarrow V = \pi \times \left(\frac{1}{5}\right) \text{Re}\left\{ (2-i)\left(i e^{\pi} - 1\right) \right\}$

$\Rightarrow V = \frac{\pi}{5} \text{Re}\left\{ 2ie^{\pi} - 2 + e^{\pi} + i \right\}$

$\Rightarrow V = \frac{\pi}{5}\left(e^{\pi} - 2\right)$

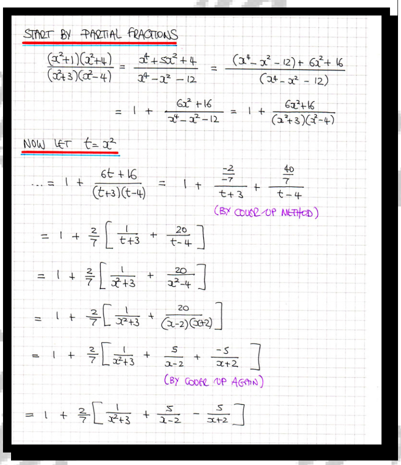
Question 96 (*****)

$$I = \int_0^1 \frac{\left(x^2+1\right)\left(x^2+4\right)}{\left(x^2+3\right)\left(x^2-4\right)}\, dx .$$

Use appropriate integration techniques to show that

$$I = 1 + \frac{2}{7}\left[\frac{\pi}{6\sqrt{3}} - 5\ln 3\right].$$

[ ] , [proof]



START BY PARTIAL FRACTIONS

$$\frac{(x^2+1)(x^2+4)}{(x^2+3)(x^2-4)} = \frac{x^4+5x^2+4}{x^4-x^2-12} = \frac{(x^4-x^2-12)+6x^2+16}{(x^4-x^2-12)}$$

$$= 1 + \frac{6x^2+16}{x^4-x^2-12} = 1 + \frac{6x^2+16}{(x^2+3)(x^2-4)}$$

NOW LET $t = x^2$

$$\ldots = 1 + \frac{6t+16}{(t+3)(t-4)} = 1 + \frac{\frac{-2}{-7}}{t+3} + \frac{\frac{40}{7}}{t-4}$$

(BY COVER UP METHOD)

$$= 1 + \frac{2}{7}\left[\frac{1}{t+3} + \frac{20}{t-4}\right]$$

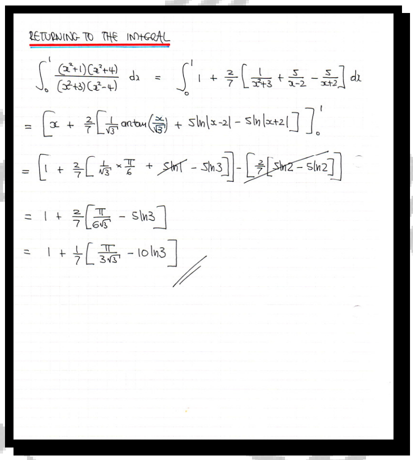
$$= 1 + \frac{2}{7}\left[\frac{1}{x^2+3} + \frac{20}{x^2-4}\right]$$

$$= 1 + \frac{2}{7}\left[\frac{1}{x^2+3} + \frac{20}{(x-2)(x+2)}\right]$$

$$= 1 + \frac{2}{7}\left[\frac{1}{x^2+3} + \frac{5}{x-2} + \frac{-5}{x+2}\right]$$

(BY COVER UP AGAIN)

$$= 1 + \frac{2}{7}\left[\frac{1}{x^2+3} + \frac{5}{x-2} - \frac{5}{x+2}\right]$$



RETURNING TO THE INTEGRAL

$$\int_0^1 \frac{(x^2+1)(x^2+4)}{(x^2+3)(x^2-4)}\, dx = \int_0^1 1 + \frac{2}{7}\left[\frac{1}{x^2+3} + \frac{5}{x-2} - \frac{5}{x+2}\right] dx$$

$$= \left[x + \frac{2}{7}\left[\frac{1}{\sqrt{3}}\arctan\left(\frac{x}{\sqrt{3}}\right) + 5\ln|x-2| - 5\ln|x+2|\right]\right]_0^1$$

$$= \left[1 + \frac{2}{7}\left[\frac{1}{\sqrt{3}} \times \frac{\pi}{6} + 5\ln 1 - 5\ln 3\right]\right] - \left[\frac{2}{7}\left[5\ln 2 - 5\ln 2\right]\right]$$

$$= 1 + \frac{2}{7}\left[\frac{\pi}{6\sqrt{3}} - 5\ln 3\right]$$

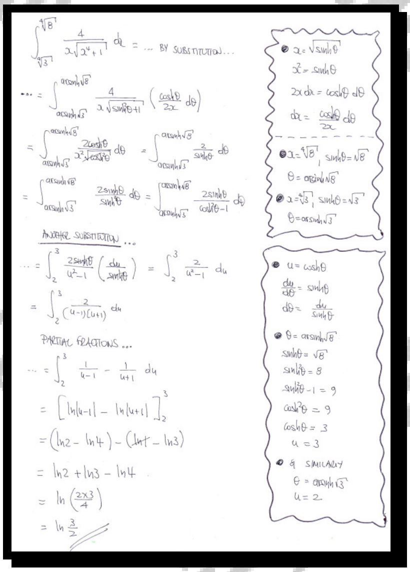
$$= 1 + \frac{1}{7}\left[\frac{\pi}{3\sqrt{3}} - 10\ln 3\right]$$

Question 97 (*****)

By using a suitable substitution, find the exact value of

$$\int_{\sqrt[4]{3}}^{\sqrt[4]{8}} \frac{2}{x\sqrt{x^4+1}}\, dx \, .$$

$\boxed{}$, $\boxed{\ln\left(\frac{3}{2}\right)}$



$\int_{\sqrt[4]{3}}^{\sqrt[4]{8}} \frac{4}{x\sqrt{x^4+1}}\,dx = \dots$ BY SUBSTITUTION...

$\dots = \int_{\text{arsinh}\sqrt{3}}^{\text{arsinh}\sqrt{8}} \frac{4}{x\sqrt{\sinh^2\theta+1}}\left(\frac{\cosh\theta}{2x}\,d\theta\right)$

$= \int_{\text{arsinh}\sqrt{3}}^{\text{arsinh}\sqrt{8}} \frac{2\cosh\theta}{x^2\sqrt{\cosh^2\theta}}\,d\theta = \int_{\text{arsinh}\sqrt{3}}^{\text{arsinh}\sqrt{8}} \frac{2}{\sinh\theta}\,d\theta$

$= \int_{\text{arsinh}\sqrt{3}}^{\text{arsinh}\sqrt{8}} \frac{2\sinh\theta}{\sinh^2\theta}\,d\theta = \int_{\text{arsinh}\sqrt{3}}^{\text{arsinh}\sqrt{8}} \frac{2\sinh\theta}{\cosh^2\theta-1}\,d\theta$

ANOTHER SUBSTITUTION...

$\dots = \int_2^3 \frac{2\sinh\theta}{u^2-1}\left(\frac{du}{\sinh\theta}\right) = \int_2^3 \frac{2}{u^2-1}\,du$

$= \int_2^3 \frac{2}{(u-1)(u+1)}\,du$

PARTIAL FRACTIONS...

$\dots = \int_2^3 \frac{1}{u-1} - \frac{1}{u+1}\,du$

$= \left[\ln|u-1| - \ln|u+1|\right]_2^3$

$= (\ln 2 - \ln 4) - (\ln 1 - \ln 3)$

$= \ln 2 + \ln 3 - \ln 4$

$= \ln\left(\frac{2\times 3}{4}\right)$

$= \ln\frac{3}{2}$

- $x = \sqrt{\sinh\theta}$
  $x^2 = \sinh\theta$
  $2x\,dx = \cosh\theta\,d\theta$
  $dx = \frac{\cosh\theta}{2x}\,d\theta$
- $x = \sqrt[4]{8}$, $\sinh\theta = \sqrt{8}$
  $\theta = \text{arsinh}\sqrt{8}$
- $x = \sqrt[4]{3}$, $\sinh\theta = \sqrt{3}$
  $\theta = \text{arsinh}\sqrt{3}$

- $u = \cosh\theta$
  $\frac{du}{d\theta} = \sinh\theta$
  $d\theta = \frac{du}{\sinh\theta}$
- $\theta = \text{arsinh}\sqrt{8}$
  $\sinh\theta = \sqrt{8}$
  $\sinh^2\theta = 8$
  $\sinh^2\theta - 1 = 9$
  $\cosh^2\theta = 9$
  $\cosh\theta = 3$
  $u = 3$
- & SIMILARLY
  $\theta = \text{arsinh}\sqrt{3}$
  $u = 2$

Question 98 (*****)

$$I = \int_{0}^{\arctan(\tanh(\ln 2))} \frac{\sec^2 x \tan 2x}{\tan x - \tan^3 x} \, dx$$

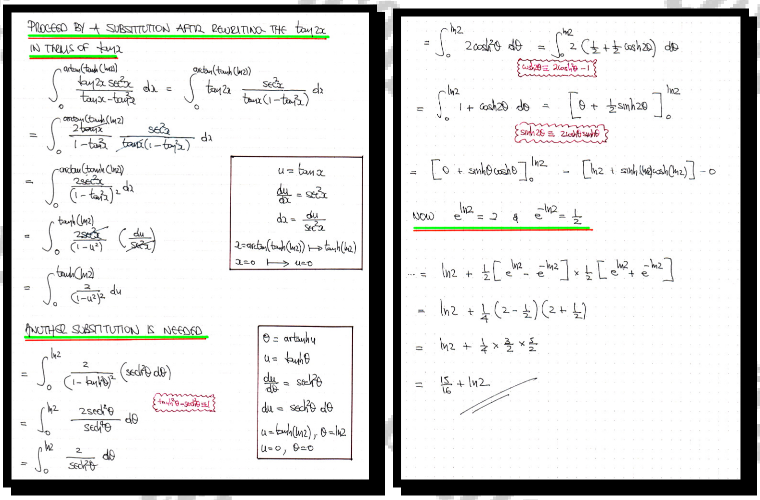
Use appropriate integration techniques to show that

$$I = k + \ln 2,$$

where k is a rational constant to be found.

You may assume that the limit of the integrand, as x tends to zero, exists.

$k = \frac{15}{16}$

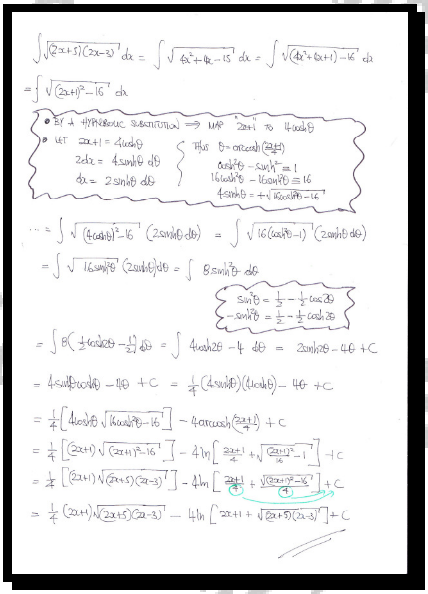


Question 99 (*****)

Use a suitable hyperbolic substitution to find a simplified expression for

$$\int \sqrt{(2x+5)(2x-3)}\, dx .$$

$$\boxed{}, \quad \boxed{\frac{1}{4}(2x+1)\sqrt{(2x+5)(2x-3)} - 4\ln\left[2x+1+\sqrt{(2x+5)(2x-3)}\right]+C}$$



Question 100 (*****)

It is given that the following integral converges to a finite value L.

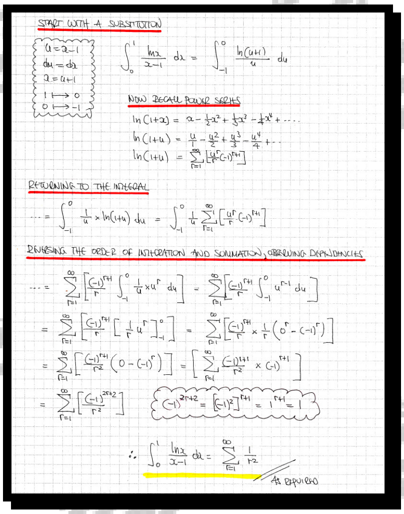
$$\int_0^1 \frac{\ln x}{x-1}\, dx \, .$$

Show, with details workings, that

$$L = \sum_{r=1}^{\infty}\left[\frac{1}{r^2}\right].$$

You may further assume that integration and summation commute.

proof



Question 101 (*****)

a) If $p \in (0, \infty)$, show that

$$\lim_{x \to 0^+} \left[x^p \ln x \right] = 0, \quad x \in (0, \infty).$$

b) Hence find a simplified expression for

$$\int_0^1 x^n \ln x \; dx, \quad n \in \mathbb{N}.$$

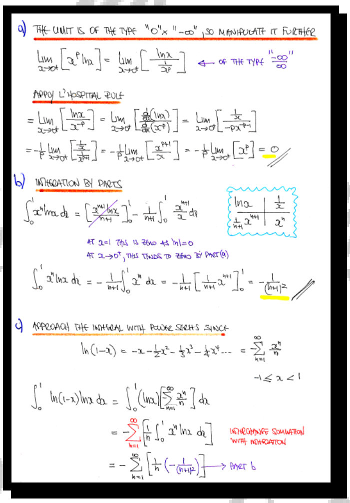
c) Hence, showing a detailed method, evaluate

$$\int_0^1 \left[\ln(1-x) \right] \ln x \; dx.$$

You may assume without proof that

- the integral converges.
- integration and summation commute.
- $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{6}\pi^2, \; n \in \mathbb{N}.$

$\square$, $\boxed{-\dfrac{1}{(n+1)^2}}$, $\boxed{2 - \dfrac{1}{6}\pi^2}$



a) THE LIMIT IS OF THE TYPE "0" × "−∞", SO MANIPULATE IT FURTHER

$$\lim_{x\to 0^+}\left[x^p \ln x\right] = \lim_{x\to 0^+}\left[\frac{\ln x}{\frac{1}{x^p}}\right] \quad \leftarrow \text{ OF THE TYPE } \text{"}\frac{-\infty}{\infty}\text{"}$$

APPLY L'HOSPITAL RULE

$$= \lim_{x\to 0^+}\left[\frac{\ln x}{x^{-p}}\right] = \lim_{x\to 0^+}\left[\frac{\frac{d}{dx}(\ln x)}{\frac{d}{dx}(x^{-p})}\right] = \lim_{x\to 0^+}\left[\frac{\frac{1}{x}}{-px^{-p-1}}\right]$$

$$= -\frac{1}{p}\lim_{x\to 0^+}\left[\frac{\frac{1}{x}}{\frac{1}{x^{p+1}}}\right] = -\frac{1}{p}\lim_{x\to 0^+}\left[\frac{x^{p+1}}{x}\right] = -\frac{1}{p}\lim_{x\to 0^+}\left[x^p\right] = 0$$

b) INTEGRATION BY PARTS

$$\int_0^1 x^n \ln x \, dx = \left[\frac{x^{n+1}\ln x}{n+1}\right]_0^1 - \frac{1}{n+1}\int_0^1 \frac{x^{n+1}}{x}\,dx$$

$\ln x$	$\frac{1}{x}$
$\frac{1}{n+1}x^{n+1}$	x^n

AT $x=1$ THIS IS ZERO AS $\ln 1 = 0$

AT $x \to 0^+$, THIS TENDS TO ZERO BY PART (a)

$$\int_0^1 x^n \ln x \, dx = -\frac{1}{n+1}\int_0^1 x^n \, dx = -\frac{1}{n+1}\left[\frac{1}{n+1}x^{n+1}\right]_0^1 = -\frac{1}{(n+1)^2}$$

c) APPROACH THE INTEGRAL WITH POWER SERIES SINCE

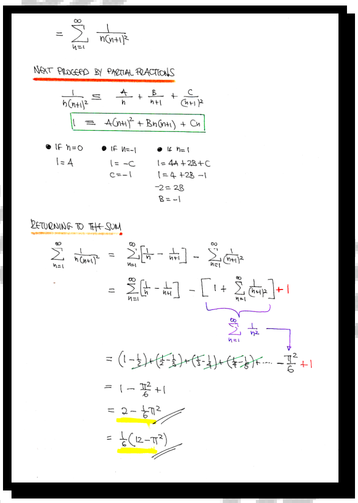
$$\ln(1-x) = -x - \frac{1}{2}x^2 - \frac{1}{3}x^3 - \frac{1}{4}x^4 \dots = -\sum_{n=1}^{\infty}\frac{x^n}{n}$$

$$-1 \leqslant x < 1$$

$$\int_0^1 \ln(1-x)\ln x \, dx = \int_0^1 (\ln x)\left[-\sum_{n=1}^{\infty}\frac{x^n}{n}\right]dx$$

$$= -\sum_{n=1}^{\infty}\left[\frac{1}{n}\int_0^1 x^n \ln x \, dx\right] \quad \text{INTERCHANGE SUMMATION WITH INTEGRATION}$$

$$= -\sum_{n=1}^{\infty}\left[\frac{1}{n}\left(-\frac{1}{(n+1)^2}\right)\right] \longrightarrow \text{PART b}$$



$$= \sum_{n=1}^{\infty}\frac{1}{n(n+1)^2}$$

NEXT PROCEED BY PARTIAL FRACTIONS

$$\frac{1}{n(n+1)^2} \equiv \frac{A}{n} + \frac{B}{n+1} + \frac{C}{(n+1)^2}$$

$$1 \equiv A(n+1)^2 + Bn(n+1) + Cn$$

- IF $n=0$: $1 = A$
- IF $n=-1$: $1 = -C$, $C = -1$
- IF $n=1$: $1 = 4A + 2B + C$, $1 = 4 + 2B - 1$, $-2 = 2B$, $B = -1$

RETURNING TO THE SUM

$$\sum_{n=1}^{\infty}\frac{1}{n(n+1)^2} = \sum_{n=1}^{\infty}\left[\frac{1}{n} - \frac{1}{n+1}\right] - \sum_{n=1}^{\infty}\frac{1}{(n+1)^2}$$

$$= \sum_{n=1}^{\infty}\left[\frac{1}{n} - \frac{1}{n+1}\right] - \left[1 + \sum_{n=1}^{\infty}\frac{1}{(n+1)^2}\right] + 1$$

$$\sum_{n=1}^{\infty}\frac{1}{n^2}$$

$$= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{5}\right) + \dots - \frac{\pi^2}{6} + 1$$

$$= 1 - \frac{\pi^2}{6} + 1$$

$$= 2 - \frac{1}{6}\pi^2$$

$$= \frac{1}{6}(12 - \pi^2)$$

Question 102 (*****)

$$I = \int \cos(\ln x)\, dx \quad \text{and} \quad J = \int \sin(\ln x)\, dx$$

a) Use an appropriate method to find expressions for I and J.

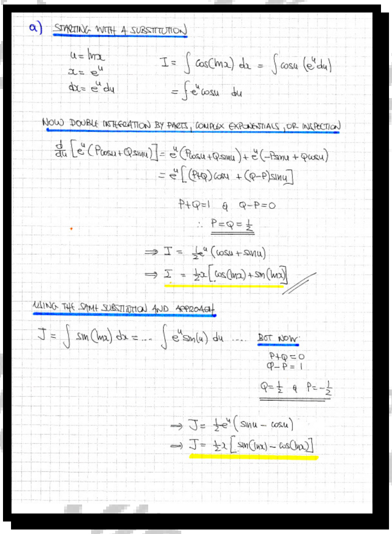
b) Use the integral $\int x^{\mathrm{i}}\, dx$, where i is the imaginary unit, to verify the answers given in part **(a)**.

c) Find an exact simplified value for

$$\int_1^{e^{\frac{\pi}{2}}} 2x^{\mathrm{i}}\, dx .$$

$\boxed{}$, $\boxed{I = \frac{1}{2}x\left[\sin(\ln x) + \cos(\ln x)\right]}$, $\boxed{J = \frac{1}{2}x\left[\sin(\ln x) - \cos(\ln x)\right]}$,

$$\boxed{\int_1^{e^{\frac{\pi}{2}}} 2x^{\mathrm{i}}\, dx = \left(e^{\frac{1}{2}\pi} - 1\right) + \left(e^{\frac{1}{2}\pi} + 1\right)\mathrm{i}}$$



a) STARTING WITH A SUBSTITUTION

$u = \ln x$, $x = e^u$, $dx = e^u\, du$

$I = \int \cos(\ln x)\, dx = \int \cos u\, (e^u du) = \int e^u \cos u\, du$

NOW DOUBLE INTEGRATION BY PARTS, COMPLEX EXPONENTIALS, OR INSPECTION

$\frac{d}{du}\left[e^u(P\cos u + Q\sin u)\right] = e^u(P\cos u + Q\sin u) + e^u(-P\sin u + Q\cos u) = e^u\left[(P+Q)\cos u + (Q-P)\sin u\right]$

$P+Q=1$ & $Q-P=0$ $\therefore P=Q=\frac{1}{2}$

$\Rightarrow I = \frac{1}{2}e^u(\cos u + \sin u)$

$\Rightarrow I = \frac{1}{2}x\left[\cos(\ln x) + \sin(\ln x)\right]$

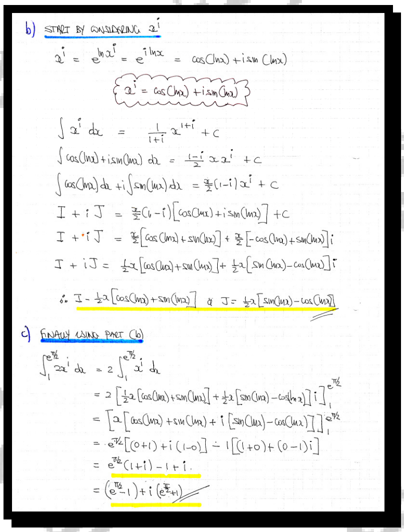
USING THE SAME SUBSTITUTION AND APPROACH

$J = \int \sin(\ln x)\, dx = \dots \int e^u \sin(u)\, du \dots$ BUT NOW

$P+Q=0$, $Q-P=1$; $Q=\frac{1}{2}$ & $P=-\frac{1}{2}$

$\Rightarrow J = \frac{1}{2}e^u(\sin u - \cos u)$

$\Rightarrow J = \frac{1}{2}x\left[\sin(\ln x) - \cos(\ln x)\right]$



b) START BY CONSIDERING x^i

$x^i = e^{\ln x^i} = e^{i\ln x} = \cos(\ln x) + i\sin(\ln x)$

$x^i = \cos(\ln x) + i\sin(\ln x)$

$\int x^i\, dx = \frac{1}{1+i}x^{1+i} + C$

$\int \cos(\ln x) + i\sin(\ln x)\, dx = \frac{1-i}{2}x x^i + C$

$\int \cos(\ln x)\, dx + i\int \sin(\ln x)\, dx = \frac{x}{2}(1-i)x^i + C$

$I + iJ = \frac{x}{2}(1-i)\left[\cos(\ln x) + i\sin(\ln x)\right] + C$

$I + iJ = \frac{x}{2}\left[\cos(\ln x) + \sin(\ln x)\right] + \frac{x}{2}\left[-\cos(\ln x) + \sin(\ln x)\right]i$

$I + iJ = \frac{1}{2}x\left[\cos(\ln x) + \sin(\ln x)\right] + \frac{1}{2}x\left[\sin(\ln x) - \cos(\ln x)\right]i$

$\therefore I = \frac{1}{2}x\left[\cos(\ln x) + \sin(\ln x)\right]$ & $J = \frac{1}{2}x\left[\sin(\ln x) - \cos(\ln x)\right]$

c) FINALLY USING PART (b)

$\int_1^{e^{\pi/2}} 2x^i\, dx = 2\int_1^{e^{\pi/2}} x^i\, dx$

$= 2\left[\frac{1}{2}x\left[\cos(\ln x) + \sin(\ln x)\right] + \frac{1}{2}x\left[\sin(\ln x) - \cos(\ln x)\right]i\right]_1^{e^{\pi/2}}$

$= \left[x\left[\cos(\ln x) + \sin(\ln x) + i\left[\sin(\ln x) - \cos(\ln x)\right]\right]\right]_1^{e^{\pi/2}}$

$= e^{\pi/2}\left[(0+1) + i(1-0)\right] - 1\left[(1+0) + (0-1)i\right]$

$= e^{\pi/2}(1+i) - 1 + i$

$= \left(e^{\pi/2} - 1\right) + i\left(e^{\frac{\pi}{2}} + 1\right)$

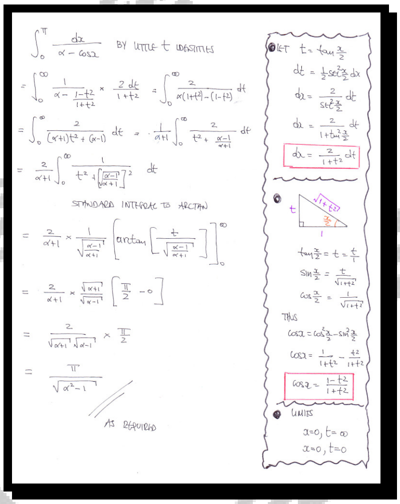
Question 103 (*****)

$$I(\alpha) = \int_0^{\pi} \frac{1}{\alpha - \cos x} \, dx \, , \ |\alpha| > 1 \, .$$

Use an appropriate method to show that

$$I(\alpha) = \frac{\pi}{\sqrt{\alpha^2 - 1}} \, .$$

[ ] , [proof]

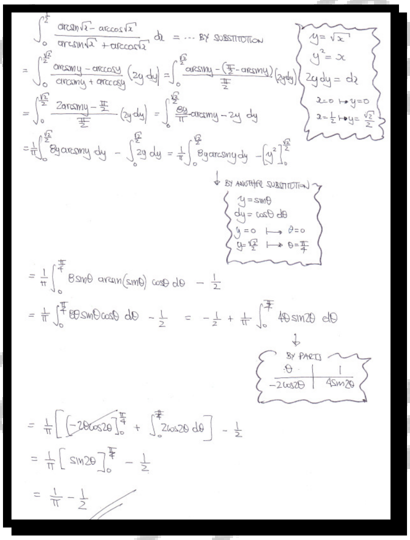


Question 104 (*****)

Use appropriate integration techniques to show that

$$\int_0^{\frac{1}{2}} \frac{\arcsin\sqrt{x} - \arccos\sqrt{x}}{\arcsin\sqrt{x} + \arccos\sqrt{x}}\, dx = \frac{1}{\pi} - \frac{1}{2}.$$

P_T, proof



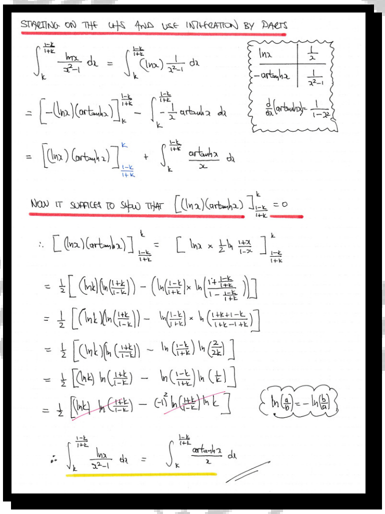
Question 105 (*****)

If $0 < k < \sqrt{2} - 1$ prove that

$$\int_{k}^{\frac{1-k}{1+k}} \frac{\ln x}{x^2 - 1}\, dx = \int_{k}^{\frac{1-k}{1+k}} \frac{\operatorname{artanh} x}{x}\, dx .$$

You need not evaluate these integrals.

[ ], [proof]

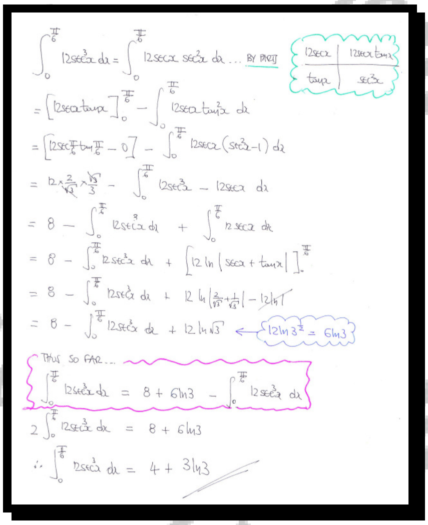


Question 106 (*****)

Use integration by parts and trigonometric identities to find the exact value of

$$\int_0^{\frac{\pi}{6}} 12\sec^3 x \ dx .$$

$\boxed{4+3\ln 3}$



$\int_0^{\frac{\pi}{6}} 12\sec^3 x \, dx = \int_0^{\frac{\pi}{6}} 12\sec x \sec^2 x \, dx \ldots$ BY PARTS

$12\sec x$	$12\sec x\tan x$
$\tan x$	$\sec^2 x$

$= \left[12\sec x\tan x\right]_0^{\frac{\pi}{6}} - \int_0^{\frac{\pi}{6}} 12\sec x\tan^2 x \, dx$

$= \left[12\sec\frac{\pi}{6}\tan\frac{\pi}{6} - 0\right] - \int_0^{\frac{\pi}{6}} 12\sec x(\sec^2 x - 1) \, dx$

$= 12 \times \frac{2}{\sqrt{3}} \times \frac{\sqrt{3}}{3} - \int_0^{\frac{\pi}{6}} 12\sec^3 x - 12\sec x \, dx$

$= 8 - \int_0^{\frac{\pi}{6}} 12\sec^3 x \, dx + \int_0^{\frac{\pi}{6}} 12\sec x \, dx$

$= 8 - \int_0^{\frac{\pi}{6}} 12\sec^3 x \, dx + \left[12\ln|\sec x + \tan x|\right]_0^{\frac{\pi}{6}}$

$= 8 - \int_0^{\frac{\pi}{6}} 12\sec^3 x \, dx + 12\ln\left|\frac{2}{\sqrt{3}} + \frac{1}{\sqrt{3}}\right| - 12\ln 1$

$= 8 - \int_0^{\frac{\pi}{6}} 12\sec^3 x \, dx + 12\ln\sqrt{3}$ ← $12\ln 3^{\frac{1}{2}} = 6\ln 3$

THUS SO FAR...

$\int_0^{\frac{\pi}{6}} 12\sec^3 x \, dx = 8 + 6\ln 3 - \int_0^{\frac{\pi}{6}} 12\sec^3 x \, dx$

$2\int_0^{\frac{\pi}{6}} 12\sec^3 x \, dx = 8 + 6\ln 3$

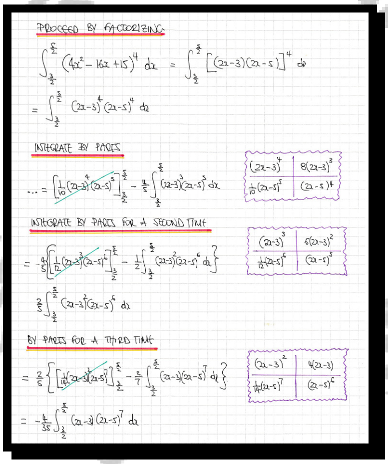
$\therefore \int_0^{\frac{\pi}{6}} 12\sec^3 x \, dx = 4 + 3\ln 3$

Question 107 (*****)

Determine, as an exact simplified fraction, the value of the following integral.

$$\int_{\frac{3}{2}}^{\frac{5}{2}} \left(4x^2 - 16x + 15\right)^4 \, dx .$$

$$\boxed{\quad} , \boxed{\frac{128}{315}}$$



PROCEED BY FACTORIZING

$$\int_{\frac{3}{2}}^{\frac{5}{2}} (4x^2-16x+15)^4\,dx = \int_{\frac{3}{2}}^{\frac{5}{2}} \left[(2x-3)(2x-5)\right]^4\,dx$$

$$= \int_{\frac{3}{2}}^{\frac{5}{2}} (2x-3)^4(2x-5)^4\,dx$$

INTEGRATE BY PARTS

$$\dots = \left[\frac{1}{10}(2x-3)^4(2x-5)^5\right]_{\frac{3}{2}}^{\frac{5}{2}} - \frac{4}{5}\int_{\frac{3}{2}}^{\frac{5}{2}} (2x-3)^3(2x-5)^5\,dx$$

$(2x-3)^4$	$8(2x-3)^3$
$\frac{1}{10}(2x-5)^5$	$(2x-5)^4$

INTEGRATE BY PARTS FOR A SECOND TIME

$$= -\frac{4}{5}\left\{\left[\frac{1}{12}(2x-3)^3(2x-5)^6\right]_{\frac{3}{2}}^{\frac{5}{2}} - \frac{1}{2}\int_{\frac{3}{2}}^{\frac{5}{2}} (2x-3)^2(2x-5)^6\,dx\right\}$$

$$\frac{2}{5}\int_{\frac{3}{2}}^{\frac{5}{2}} (2x-3)^2(2x-5)^6\,dx$$

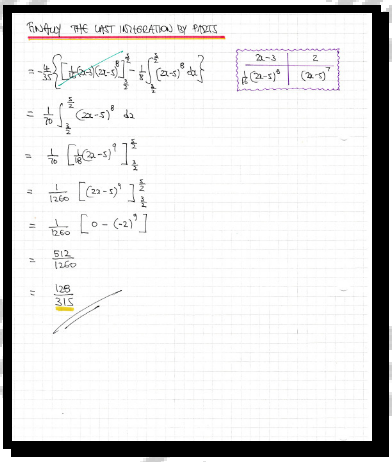
$(2x-3)^3$	$6(2x-3)^2$
$\frac{1}{12}(2x-5)^6$	$(2x-5)^5$

BY PARTS FOR A THIRD TIME

$$= \frac{2}{5}\left\{\left[\frac{1}{14}(2x-3)^2(2x-5)^7\right]_{\frac{3}{2}}^{\frac{5}{2}} - \frac{2}{7}\int_{\frac{3}{2}}^{\frac{5}{2}} (2x-3)(2x-5)^7\,dx\right\}$$

$$= -\frac{4}{35}\int_{\frac{3}{2}}^{\frac{5}{2}} (2x-3)(2x-5)^7\,dx$$

$(2x-3)^2$	$4(2x-3)$
$\frac{1}{14}(2x-5)^7$	$(2x-5)^6$



FINALLY THE LAST INTEGRATION BY PARTS

$$= -\frac{4}{35}\left\{\left[\frac{1}{16}(2x-3)(2x-5)^8\right]_{\frac{3}{2}}^{\frac{5}{2}} - \frac{1}{8}\int_{\frac{3}{2}}^{\frac{5}{2}} (2x-5)^8\,dx\right\}$$

$2x-3$	2
$\frac{1}{16}(2x-5)^8$	$(2x-5)^7$

$$= \frac{1}{70}\int_{\frac{3}{2}}^{\frac{5}{2}} (2x-5)^8\,dx$$

$$= \frac{1}{70}\left[\frac{1}{18}(2x-5)^9\right]_{\frac{3}{2}}^{\frac{5}{2}}$$

$$= \frac{1}{1260}\left[(2x-5)^9\right]_{\frac{3}{2}}^{\frac{5}{2}}$$

$$= \frac{1}{1260}\left[0-(-2)^9\right]$$

$$= \frac{512}{1260}$$

$$= \frac{128}{315}$$

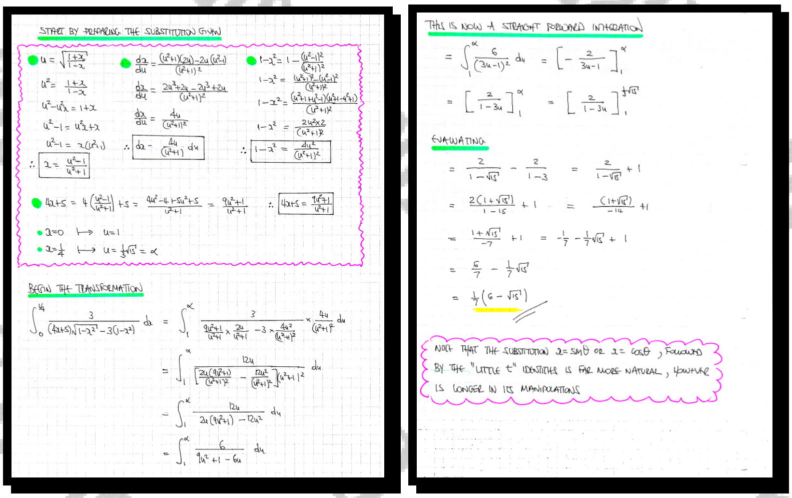
Question 108 (*****)

Use the substitution $u=\sqrt{\dfrac{1+x}{1-x}}$, to evaluate the following integral.

$$\int_{0}^{\frac{1}{4}} \frac{3}{(4x+5)\sqrt{1-x^2}-3\left(1-x^2\right)}\,dx .$$

Give the answer in the form $\frac{1}{7}\left(a+\sqrt{b}\right)$, where a and b are integers.

SPX , $\boxed{\frac{1}{7}\left(6-\sqrt{15}\right)}$



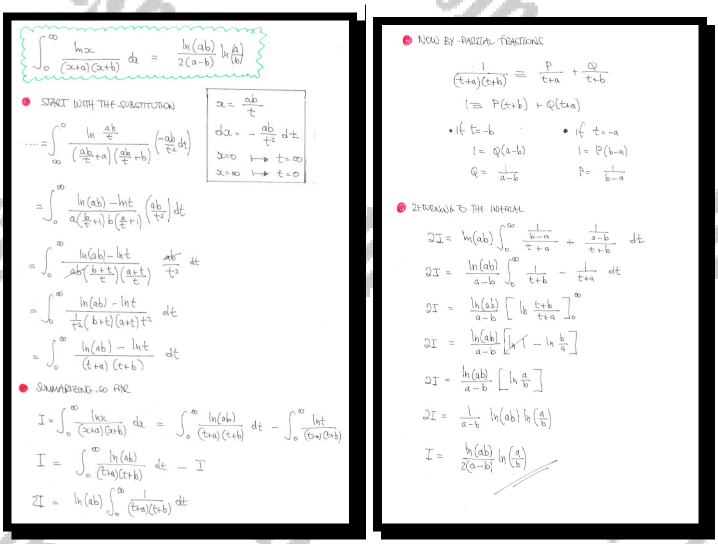
Question 109 (*****)

Use the substitution $x = \dfrac{ab}{t}$ to find the exact value of

$$\int_0^{\infty} \frac{\ln x}{(x+a)(x+b)}\, dx ,$$

where a and b are real positive constants with $a > b$.

$$\boxed{}\ , \quad \boxed{\frac{\ln(ab)}{2(a-b)}\ln\left(\frac{a}{b}\right) = \frac{(\ln a)^2 - (\ln b)^2}{2(a-b)}}$$



$$\int_0^{\infty} \frac{\ln x}{(x+a)(x+b)}\, dx = \frac{\ln(ab)}{2(a-b)} \ln\left(\frac{a}{b}\right)$$

• START WITH THE SUBSTITUTION

$x = \frac{ab}{t}$, $dx = -\frac{ab}{t^2}\,dt$, $x=0 \mapsto t=\infty$, $x=\infty \mapsto t=0$

$$\dots = \int_{\infty}^{0} \frac{\ln \frac{ab}{t}}{\left(\frac{ab}{t}+a\right)\left(\frac{ab}{t}+b\right)} \left(-\frac{ab}{t^2}\,dt\right)$$

$$= \int_0^{\infty} \frac{\ln(ab) - \ln t}{a\left(\frac{b}{t}+1\right) b\left(\frac{a}{t}+1\right)} \left(\frac{ab}{t^2}\right) dt$$

$$= \int_0^{\infty} \frac{\ln(ab) - \ln t}{ab\left(\frac{b+t}{t}\right)\left(\frac{a+t}{t}\right)} \frac{ab}{t^2}\, dt$$

$$= \int_0^{\infty} \frac{\ln(ab) - \ln t}{\frac{1}{t^2}(b+t)(a+t)t^2}\, dt$$

$$= \int_0^{\infty} \frac{\ln(ab) - \ln t}{(t+a)(t+b)}\, dt$$

• SUMMARIZING SO FAR

$$I = \int_0^{\infty} \frac{\ln x}{(x+a)(x+b)}\, dx = \int_0^{\infty} \frac{\ln(ab)}{(t+a)(t+b)}\, dt - \int_0^{\infty} \frac{\ln t}{(t+a)(t+b)}\, dt$$

$$I = \int_0^{\infty} \frac{\ln(ab)}{(t+a)(t+b)}\, dt - I$$

$$2I = \ln(ab) \int_0^{\infty} \frac{1}{(t+a)(t+b)}\, dt$$

• NOW BY PARTIAL FRACTIONS

$$\frac{1}{(t+a)(t+b)} \equiv \frac{P}{t+a} + \frac{Q}{t+b}$$

$$1 \equiv P(t+b) + Q(t+a)$$

• if $t=-b$: $1 = Q(a-b)$, $Q = \frac{1}{a-b}$

• if $t=-a$: $1 = P(b-a)$, $P = \frac{1}{b-a}$

• RETURNING TO THE INTEGRAL

$$2I = \ln(ab) \int_0^{\infty} \frac{\frac{1}{b-a}}{t+a} + \frac{\frac{1}{a-b}}{t+b}\, dt$$

$$2I = \frac{\ln(ab)}{a-b} \int_0^{\infty} \frac{1}{t+b} - \frac{1}{t+a}\, dt$$

$$2I = \frac{\ln(ab)}{a-b} \left[\ln \frac{t+b}{t+a}\right]_0^{\infty}$$

$$2I = \frac{\ln(ab)}{a-b} \left[\ln 1 - \ln \frac{b}{a}\right]$$

$$2I = \frac{\ln(ab)}{a-b} \left[\ln \frac{a}{b}\right]$$

$$2I = \frac{1}{a-b} \ln(ab) \ln\left(\frac{a}{b}\right)$$

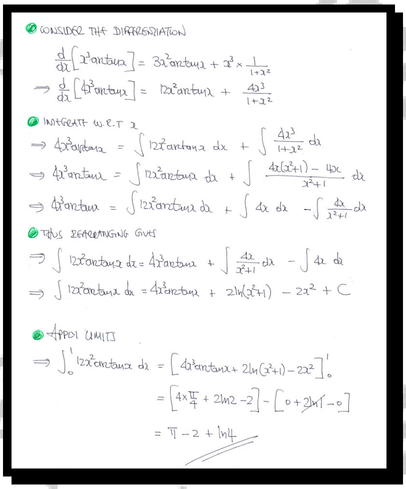
$$I = \frac{\ln(ab)}{2(a-b)} \ln\left(\frac{a}{b}\right)$$

Question 110 (*****)

Use appropriate integration methods to show that

$$\int_0^1 12x^2 \arctan x \; dx = \pi - 2 + \ln 4 .$$

SP-X-S , proof



CONSIDER THE DIFFERENTIATION

$$\frac{d}{dx}\left[x^3 \arctan x\right] = 3x^2 \arctan x + x^3 \times \frac{1}{1+x^2}$$

$$\Rightarrow \frac{d}{dx}\left[4x^3 \arctan x\right] = 12x^2 \arctan x + \frac{4x^3}{1+x^2}$$

INTEGRATE W.R.T x

$$\Rightarrow 4x^3 \arctan x = \int 12x^2 \arctan x \, dx + \int \frac{4x^3}{1+x^2} \, dx$$

$$\Rightarrow 4x^3 \arctan x = \int 12x^2 \arctan x \, dx + \int \frac{4x(x^2+1) - 4x}{x^2+1} \, dx$$

$$\Rightarrow 4x^3 \arctan x = \int 12x^2 \arctan x \, dx + \int 4x \, dx - \int \frac{4x}{x^2+1} \, dx$$

THUS REARRANGING GIVES

$$\Rightarrow \int 12x^2 \arctan x \, dx = 4x^3 \arctan x + \int \frac{4x}{x^2+1} \, dx - \int 4x \, dx$$

$$\Rightarrow \int 12x^2 \arctan x \, dx = 4x^3 \arctan x + 2\ln(x^2+1) - 2x^2 + C$$

APPLY LIMITS

$$\Rightarrow \int_0^1 12x^2 \arctan x \, dx = \left[4x^3 \arctan x + 2\ln(x^2+1) - 2x^2\right]_0^1$$

$$= \left[4 \times \frac{\pi}{4} + 2\ln 2 - 2\right] - \left[0 + 2\ln 1 - 0\right]$$

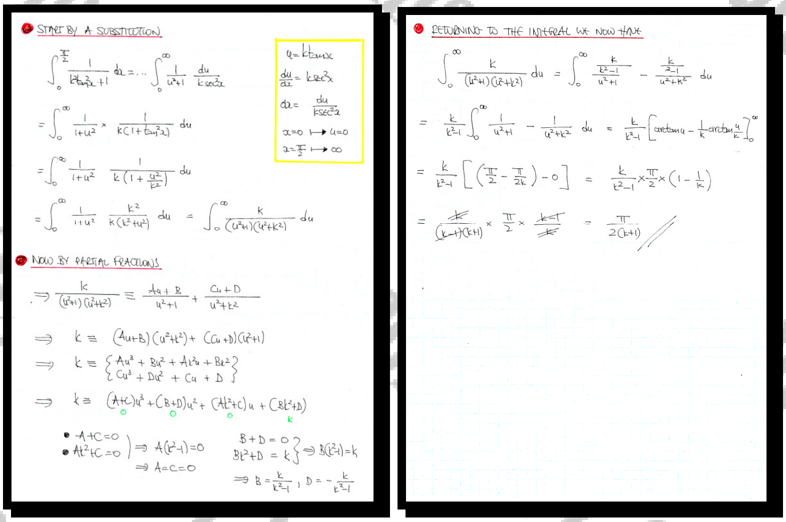
$$= \pi - 2 + \ln 4$$

Question 111 (*****)

Use appropriate integration methods to find, in terms of k, a simplified expression for

$$\int_0^{\frac{\pi}{2}} \frac{1}{1+k^2\tan^2 x}\,dx, \quad |k| \neq 1.$$

$\boxed{}$, $\boxed{\dfrac{\pi}{2(k+1)}}$


START BY A SUBSTITUTION

$$\int_0^{\frac{\pi}{2}} \frac{1}{k^2\tan^2 x+1}\,dx = \ldots \int_0^{\infty} \frac{1}{u^2+1}\,\frac{du}{k\sec^2 x}$$

$u = k\tan x$, $\frac{du}{dx} = k\sec^2 x$, $dx = \frac{du}{k\sec^2 x}$, $x=0 \mapsto u=0$, $x=\frac{\pi}{2} \mapsto \infty$

$$= \int_0^{\infty} \frac{1}{1+u^2} \times \frac{1}{k(1+\tan^2 x)}\,du$$

$$= \int_0^{\infty} \frac{1}{1+u^2}\,\frac{1}{k\left(1+\frac{u^2}{k^2}\right)}\,du$$

$$= \int_0^{\infty} \frac{1}{1+u^2}\,\frac{k^2}{k(k^2+u^2)}\,du = \int_0^{\infty} \frac{k}{(u^2+1)(u^2+k^2)}\,du$$

NOW BY PARTIAL FRACTIONS

$$\Rightarrow \frac{k}{(u^2+1)(u^2+k^2)} \equiv \frac{Au+B}{u^2+1} + \frac{Cu+D}{u^2+k^2}$$

$$\Rightarrow k \equiv (Au+B)(u^2+k^2) + (Cu+D)(u^2+1)$$

$$\Rightarrow k \equiv \begin{Bmatrix} Au^3 + Bu^2 + Ak^2u + Bk^2 \\ Cu^3 + Du^2 + Cu + D \end{Bmatrix}$$

$$\Rightarrow k \equiv (A+C)u^3 + (B+D)u^2 + (Ak^2+C)u + (Bk^2+D)$$

- $A+C=0$
- $Ak^2+C=0$

$\Rightarrow A(k^2-1)=0 \Rightarrow A=C=0$

$B+D=0$, $Bk^2+D=k$ $\Rightarrow B(k^2-1)=k$ $\Rightarrow B=\frac{k}{k^2-1}$, $D=-\frac{k}{k^2-1}$

RETURNING TO THE INTEGRAL WE NOW HAVE

$$\int_0^{\infty} \frac{k}{(u^2+1)(u^2+k^2)}\,du = \int_0^{\infty} \frac{\frac{k}{k^2-1}}{u^2+1} - \frac{\frac{k}{k^2-1}}{u^2+k^2}\,du$$

$$= \frac{k}{k^2-1}\int_0^{\infty} \frac{1}{u^2+1} - \frac{1}{u^2+k^2}\,du = \frac{k}{k^2-1}\left[\arctan u - \frac{1}{k}\arctan\frac{u}{k}\right]_0^{\infty}$$

$$= \frac{k}{k^2-1}\left[\left(\frac{\pi}{2}-\frac{\pi}{2k}\right)-0\right] = \frac{k}{k^2-1}\times\frac{\pi}{2}\times\left(1-\frac{1}{k}\right)$$

$$= \frac{k}{(k-1)(k+1)} \times \frac{\pi}{2} \times \frac{k-1}{k} = \frac{\pi}{2(k+1)}$$

Question 112 **(*****)**

$$I = \int_0^{\frac{1}{2}\ln 3} \operatorname{sech} x \ dx .$$

a) Use the substitution $u = e^x$ to show that $I = \dfrac{\pi}{k}$, where k is a positive integer.

b) Given that $t = \tanh\left(\frac{1}{2}x\right)$ show that …

i. … $\dfrac{dt}{dx} = \frac{1}{2}\left(1 - t^2\right)$.

ii. … if $x = \frac{1}{2}\ln 3$, then $t = 2 - \sqrt{3}$.

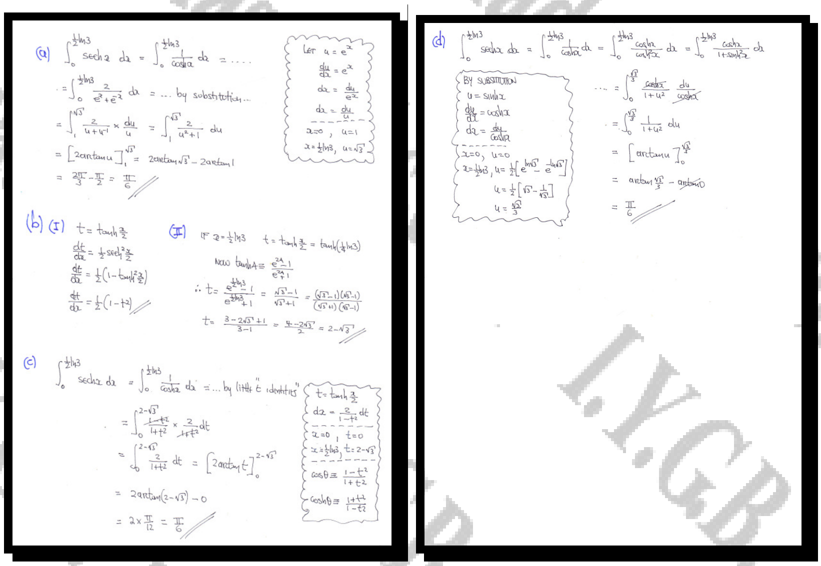
c) Use the results of part **(b)** to find again the exact value of I .

d) Show that I can be written as

$$\int_0^{\frac{1}{2}\ln 3} \frac{\cosh x}{1 + \sinh^2 x} \ dx ,$$

and hence obtain the exact value of I for a third time.

[ ] , [proof]



(a) $\int_0^{\frac{1}{2}\ln 3} \operatorname{sech} x \, dx = \int_0^{\frac{1}{2}\ln 3} \frac{1}{\cosh x} dx = \dots$

$= \int_0^{\frac{1}{2}\ln 3} \frac{2}{e^x + e^{-x}} dx = \dots$ by substitution…

$= \int_1^{\sqrt{3}} \frac{2}{u + u^{-1}} \times \frac{du}{u} = \int_1^{\sqrt{3}} \frac{2}{u^2+1} du$

$= \left[2\arctan u\right]_1^{\sqrt{3}} = 2\arctan\sqrt{3} - 2\arctan 1$

$= \frac{2\pi}{3} - \frac{\pi}{2} = \frac{\pi}{6}$

Let $u = e^x$; $\frac{du}{dx} = e^x$; $dx = \frac{du}{e^x}$; $dx = \frac{du}{u}$; $x = 0,\ u = 1$; $x = \frac{1}{2}\ln 3,\ u = \sqrt{3}$

(b) (I) $t = \tanh\frac{x}{2}$

$\frac{dt}{dx} = \frac{1}{2}\operatorname{sech}^2\frac{x}{2}$

$\frac{dt}{dx} = \frac{1}{2}\left(1 - \tanh^2\frac{x}{2}\right)$

$\frac{dt}{dx} = \frac{1}{2}\left(1 - t^2\right)$

(II) If $x = \frac{1}{2}\ln 3$, $t = \tanh\frac{x}{2} = \tanh\left(\frac{1}{4}\ln 3\right)$

Now $\tanh A \equiv \frac{e^{2A} - 1}{e^{2A} + 1}$

$\therefore t = \frac{e^{\frac{1}{2}\ln 3} - 1}{e^{\frac{1}{2}\ln 3} + 1} = \frac{\sqrt{3} - 1}{\sqrt{3} + 1} = \frac{(\sqrt{3}-1)(\sqrt{3}-1)}{(\sqrt{3}+1)(\sqrt{3}-1)}$

$t = \frac{3 - 2\sqrt{3} + 1}{3 - 1} = \frac{4 - 2\sqrt{3}}{2} = 2 - \sqrt{3}$

(c) $\int_0^{\frac{1}{2}\ln 3} \operatorname{sech} x \, dx = \int_0^{\frac{1}{2}\ln 3} \frac{1}{\cosh x} dx = \dots$ by little "t identities"

$= \int_0^{2-\sqrt{3}} \frac{1}{\frac{1+t^2}{1-t^2}} \times \frac{2}{1-t^2} dt$

$= \int_0^{2-\sqrt{3}} \frac{2}{1+t^2} dt = \left[2\arctan t\right]_0^{2-\sqrt{3}}$

$= 2\arctan(2 - \sqrt{3}) - 0$

$= 2 \times \frac{\pi}{12} = \frac{\pi}{6}$

$t = \tanh\frac{x}{2}$; $dx = \frac{2}{1-t^2} dt$; $x = 0,\ t = 0$; $x = \frac{1}{2}\ln 3,\ t = 2 - \sqrt{3}$; $\cos\theta \equiv \frac{1-t^2}{1+t^2}$; $\cosh\theta \equiv \frac{1+t^2}{1-t^2}$

(d) $\int_0^{\frac{1}{2}\ln 3} \operatorname{sech} x \, dx = \int_0^{\frac{1}{2}\ln 3} \frac{1}{\cosh x} dx = \int_0^{\frac{1}{2}\ln 3} \frac{\cosh x}{\cosh^2 x} dx = \int_0^{\frac{1}{2}\ln 3} \frac{\cosh x}{1 + \sinh^2 x} dx$

BY SUBSTITUTION: $u = \sinh x$; $\frac{du}{dx} = \cosh x$; $dx = \frac{du}{\cosh x}$; $x = 0,\ u = 0$; $x = \frac{1}{2}\ln 3,\ u = \frac{1}{2}\left[e^{\ln\sqrt{3}} - e^{-\ln\sqrt{3}}\right]$; $u = \frac{1}{2}\left[\sqrt{3} - \frac{1}{\sqrt{3}}\right]$; $u = \frac{\sqrt{3}}{3}$

$\dots = \int_0^{\frac{\sqrt{3}}{3}} \frac{\cosh x}{1+u^2} \frac{du}{\cosh x}$

$= \int_0^{\frac{\sqrt{3}}{3}} \frac{1}{1+u^2} du$

$= \left[\arctan u\right]_0^{\frac{\sqrt{3}}{3}}$

$= \arctan\frac{\sqrt{3}}{3} - \arctan 0$

$= \frac{\pi}{6}$

Question 113 (*****)

$$I = \int_0^1 2\,\text{arsinh}\sqrt{x}\ dx\,.$$

The value of I is to be found using two methods.

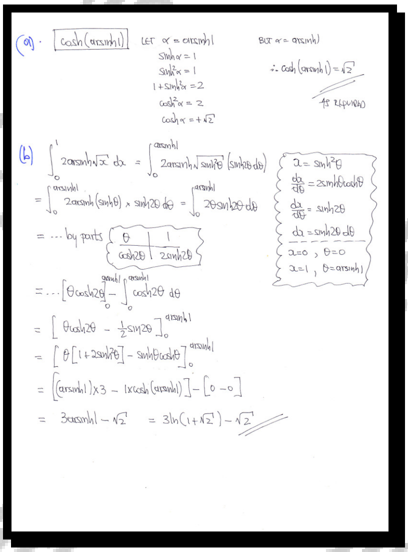
a) Use the substitution $x = \sinh^2\theta$ to show that

$$I = 3\ln\left(1+\sqrt{2}\right) - \sqrt{2}\,.$$

A different approach is to be used to find the value of I .

b) Use the substitution $u = \sqrt{x}$, followed by a suitable hyperbolic substitution to to verify the answer of part **(a)**.

[ ] , [proof]



(a) $\cosh(\text{arsinh}1)$ LET $\alpha = \text{arsinh}1$, $\sinh\alpha = 1$, $\sinh^2\alpha = 1$, $1+\sinh^2\alpha = 2$, $\cosh^2\alpha = 2$, $\cosh\alpha = +\sqrt{2}$. BUT $\alpha = \text{arsinh}1$ $\therefore \cosh(\text{arsinh}1) = \sqrt{2}$ AS REQUIRED

(b) $\int_0^1 2\text{arsinh}\sqrt{x}\,dx = \int_0^{\text{arsinh}1} 2\text{arsinh}\sqrt{\sinh^2\theta}\,(\sinh2\theta\,d\theta)$

$x = \sinh^2\theta$, $\frac{dx}{d\theta} = 2\sinh\theta\cosh\theta$, $\frac{dx}{d\theta} = \sinh2\theta$, $dx = \sinh2\theta\,d\theta$; $x=0,\ \theta=0$; $x=1,\ \theta=\text{arsinh}1$

$= \int_0^{\text{arsinh}1} 2\text{arsinh}(\sinh\theta)\times\sinh2\theta\,d\theta = \int_0^{\text{arsinh}1} 2\theta\sinh2\theta\,d\theta$

$= \dots$ by parts

θ	1
$\cosh2\theta$	$2\sinh2\theta$

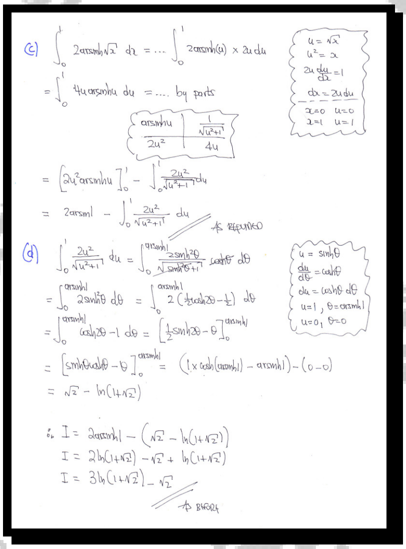
$= \dots\left[\theta\cosh2\theta\right]_0^{\text{arsinh}1} - \int_0^{\text{arsinh}1}\cosh2\theta\,d\theta$

$= \left[\theta\cosh2\theta - \frac{1}{2}\sinh2\theta\right]_0^{\text{arsinh}1}$

$= \left[\theta\left[1+2\sinh^2\theta\right] - \sinh\theta\cosh\theta\right]_0^{\text{arsinh}1}$

$= \left[(\text{arsinh}1)\times3 - 1\times\cosh(\text{arsinh}1)\right] - \left[0-0\right]$

$= 3\text{arsinh}1 - \sqrt{2} = 3\ln(1+\sqrt{2}) - \sqrt{2}$



(c) $\int_0^1 2\text{arsinh}\sqrt{x}\,dx = \dots \int_0^1 2\text{arsinh}(u)\times 2u\,du$

$u = \sqrt{x}$, $u^2 = x$, $2u\frac{du}{dx} = 1$, $dx = 2u\,du$; $x=0\ u=0$; $x=1\ u=1$

$= \int_0^1 4u\,\text{arsinh}u\,du = \dots$ by parts

$\text{arsinh}u$	$\frac{1}{\sqrt{u^2+1}}$
$2u^2$	$4u$

$= \left[2u^2\text{arsinh}u\right]_0^1 - \int_0^1 \frac{2u^2}{\sqrt{u^2+1}}\,du$

$= 2\text{arsinh}1 - \int_0^1 \frac{2u^2}{\sqrt{u^2+1}}\,du$ AS REQUIRED

(d) $\int_0^1 \frac{2u^2}{\sqrt{u^2+1}}\,du = \int_0^{\text{arsinh}1} \frac{2\sinh^2\theta}{\sqrt{\sinh^2\theta+1}}\cosh\theta\,d\theta$

$u = \sinh\theta$, $\frac{du}{d\theta} = \cosh\theta$, $du = \cosh\theta\,d\theta$, $u=1,\ \theta=\text{arsinh}1$; $u=0,\ \theta=0$

$= \int_0^{\text{arsinh}1} 2\sinh^2\theta\,d\theta = \int_0^{\text{arsinh}1} 2\left(\frac{1}{2}\cosh2\theta - \frac{1}{2}\right)d\theta$

$= \int_0^{\text{arsinh}1} \cosh2\theta - 1\,d\theta = \left[\frac{1}{2}\sinh2\theta - \theta\right]_0^{\text{arsinh}1}$

$= \left[\sinh\theta\cosh\theta - \theta\right]_0^{\text{arsinh}1} = (1\times\cosh(\text{arsinh}1) - \text{arsinh}1) - (0-0)$

$= \sqrt{2} - \ln(1+\sqrt{2})$

$\therefore I = 2\text{arsinh}1 - \left(\sqrt{2} - \ln(1+\sqrt{2})\right)$

$I = 2\ln(1+\sqrt{2}) - \sqrt{2} + \ln(1+\sqrt{2})$

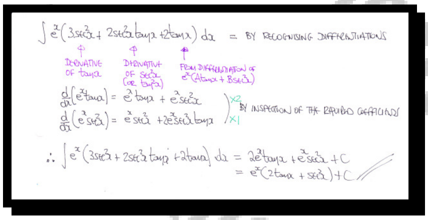
$I = 3\ln(1+\sqrt{2}) - \sqrt{2}$ AS BEFORE

Question 114 (*****)

By considering the differentiation of products of appropriate functions, find

$$\int e^{x}\left(3\sec^{2}x+2\sec^{2}x\tan x+2\tan x\right)\,dx\,.$$

$\boxed{}$, $\boxed{e^{x}\left(2\tan x+\sec^{2}x\right)+C}$

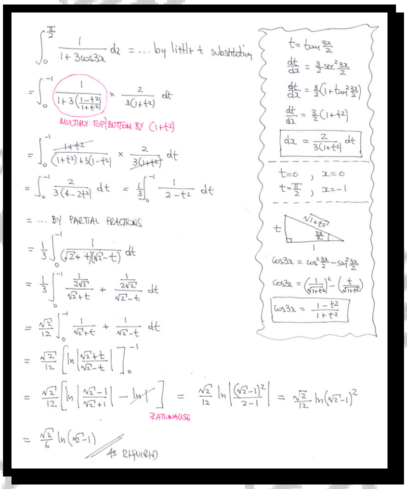


Question 115 (*****)

By using a trigonometric substitution or otherwise, find an exact simplified value for the following integral.

$$\int_{0}^{\frac{\pi}{2}}\frac{1}{1+3\cos 3x}\,dx\,.$$

$\boxed{}$, $\boxed{\frac{\sqrt{2}}{6}\ln\left(\sqrt{2}-1\right)}$



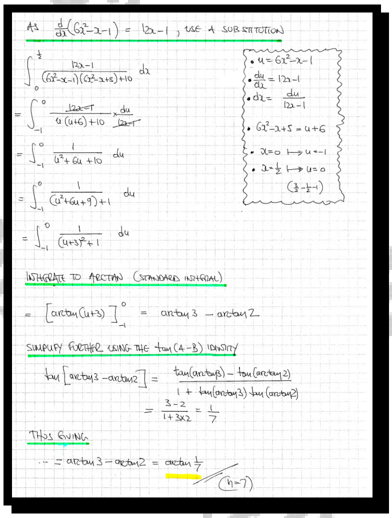
Question 116 (*****)

Find the value of the following definite integral.

$$\int_0^{\frac{1}{2}} \frac{12x-1}{\left(6x^2-x-1\right)\left(6x^2-x-5\right)+10}\, dx \ .$$

Give the answer in the form $\arctan\left(\frac{1}{n}\right)$, where n is a positive integer.

$\boxed{}$, $\boxed{n=7}$



AS $\frac{d}{dx}(6x^2-x-1) = 12x-1$, USE A SUBSTITUTION

$\int_0^{\frac{1}{2}} \frac{12x-1}{(6x^2-x-1)(6x^2-x+5)+10}\,dx$

$= \int_{-1}^{0} \frac{12x-1}{u(u+6)+10}\,\frac{du}{12x-1}$

$= \int_{-1}^{0} \frac{1}{u^2+6u+10}\,du$

$= \int_{-1}^{0} \frac{1}{(u^2+6u+9)+1}\,du$

$= \int_{-1}^{0} \frac{1}{(u+3)^2+1}\,du$

• $u = 6x^2-x-1$
• $\frac{du}{dx} = 12x-1$
• $dx = \frac{du}{12x-1}$
• $6x^2-x+5 = u+6$
• $x=0 \mapsto u=-1$
• $x=\frac{1}{2} \mapsto u=0$ $\left(\frac{3}{2}-\frac{1}{2}-1\right)$

INTEGRATE TO ARCTAN (STANDARD INTEGRAL)

$= \left[\arctan(u+3)\right]_{-1}^{0} = \arctan 3 - \arctan 2$

SIMPLIFY FURTHER USING THE $\tan(A-B)$ IDENTITY

$\tan\left[\arctan 3 - \arctan 2\right] = \frac{\tan(\arctan 3) - \tan(\arctan 2)}{1+\tan(\arctan 3)\tan(\arctan 2)}$

$= \frac{3-2}{1+3\times 2} = \frac{1}{7}$

THUS GIVING

$\therefore \ = \arctan 3 - \arctan 2 = \arctan\frac{1}{7}$ $(n=7)$

Question 117 (*****)

$$I=\int_{\frac{1}{\sqrt{3}}}^{1}\frac{\sqrt{1+x^2}}{x^4}\,dx\,.$$

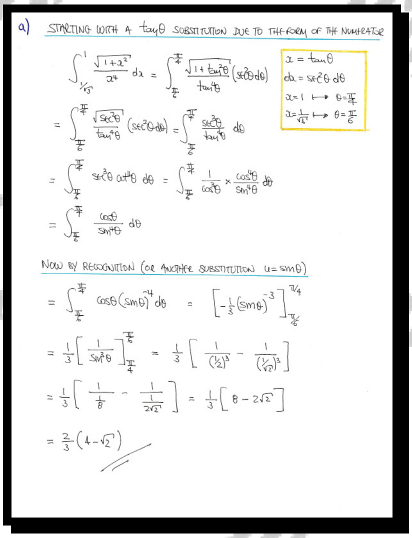
a) Use a trigonometric substitution to show that

$$I=\frac{2}{3}\left(a+b\sqrt{2}\right),$$

where a and b are integers to be found.

b) Use a hyperbolic substitution to verify the answer of part (**a**).

$\boxed{\quad}$, $\boxed{I=\frac{2}{3}\left(4-\sqrt{2}\right)}$



a) STARTING WITH A $\tan\theta$ SUBSTITUTION DUE TO THE FORM OF THE NUMERATOR

$x=\tan\theta$, $dx=\sec^2\theta\,d\theta$, $x=1\mapsto\theta=\frac{\pi}{4}$, $x=\frac{1}{\sqrt{3}}\mapsto\theta=\frac{\pi}{6}$

$$\int_{1/\sqrt{3}}^{1}\frac{\sqrt{1+x^2}}{x^4}\,dx=\int_{\frac{\pi}{6}}^{\frac{\pi}{4}}\frac{\sqrt{1+\tan^2\theta}}{\tan^4\theta}(\sec^2\theta\,d\theta)$$

$$=\int_{\frac{\pi}{6}}^{\frac{\pi}{4}}\frac{\sqrt{\sec^2\theta}}{\tan^4\theta}(\sec^2\theta\,d\theta)=\int_{\frac{\pi}{6}}^{\frac{\pi}{4}}\frac{\sec^3\theta}{\tan^4\theta}\,d\theta$$

$$=\int_{\frac{\pi}{6}}^{\frac{\pi}{4}}\sec^3\theta\cot^4\theta\,d\theta=\int_{\frac{\pi}{6}}^{\frac{\pi}{4}}\frac{1}{\cos^3\theta}\times\frac{\cos^4\theta}{\sin^4\theta}\,d\theta$$

$$=\int_{\frac{\pi}{6}}^{\frac{\pi}{4}}\frac{\cos\theta}{\sin^4\theta}\,d\theta$$

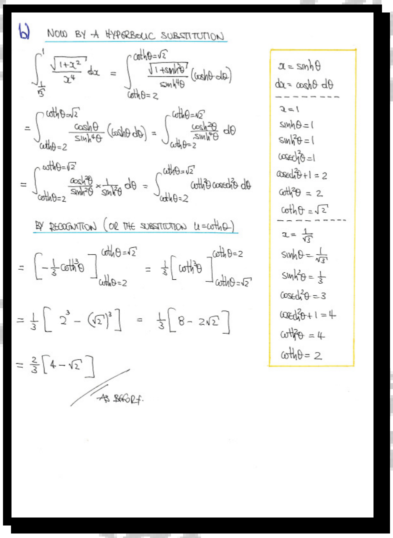
NOW BY RECOGNITION (OR ANOTHER SUBSTITUTION $u=\sin\theta$)

$$=\int_{\frac{\pi}{6}}^{\frac{\pi}{4}}\cos\theta(\sin\theta)^{-4}\,d\theta=\left[-\frac{1}{3}(\sin\theta)^{-3}\right]_{\pi/6}^{\pi/4}$$

$$=\frac{1}{3}\left[\frac{1}{\sin^3\theta}\right]_{\frac{\pi}{4}}^{\frac{\pi}{6}}=\frac{1}{3}\left[\frac{1}{(\frac{1}{2})^3}-\frac{1}{(\frac{1}{\sqrt{2}})^3}\right]$$

$$=\frac{1}{3}\left[\frac{1}{\frac{1}{8}}-\frac{1}{\frac{1}{2\sqrt{2}}}\right]=\frac{1}{3}\left[8-2\sqrt{2}\right]$$

$$=\frac{2}{3}(4-\sqrt{2})$$



b) NOW BY A HYPERBOLIC SUBSTITUTION

$x=\sinh\theta$, $dx=\cosh\theta\,d\theta$

$x=1$, $\sinh\theta=1$, $\sinh^2\theta=1$, $\text{cosech}^2\theta=1$, $\text{cosech}^2\theta+1=2$, $\coth^2\theta=2$, $\coth\theta=\sqrt{2}$

$x=\frac{1}{\sqrt{3}}$, $\sinh\theta=\frac{1}{\sqrt{3}}$, $\sinh^2\theta=\frac{1}{3}$, $\text{cosech}^2\theta=3$, $\text{cosech}^2\theta+1=4$, $\coth^2\theta=4$, $\coth\theta=2$

$$\int_{\frac{1}{\sqrt{3}}}^{1}\frac{\sqrt{1+x^2}}{x^4}\,dx=\int_{\coth\theta=2}^{\coth\theta=\sqrt{2}}\frac{\sqrt{1+\sinh^2\theta}}{\sinh^4\theta}(\cosh\theta\,d\theta)$$

$$=\int_{\coth\theta=2}^{\coth\theta=\sqrt{2}}\frac{\cosh\theta}{\sinh^4\theta}(\cosh\theta\,d\theta)=\int_{\coth\theta=2}^{\coth\theta=\sqrt{2}}\frac{\cosh^2\theta}{\sinh^4\theta}\,d\theta$$

$$=\int_{\coth\theta=2}^{\coth\theta=\sqrt{2}}\frac{\cosh^2\theta}{\sinh^2\theta}\times\frac{1}{\sinh^2\theta}\,d\theta=\int_{\coth\theta=2}^{\coth\theta=\sqrt{2}}\coth^2\theta\,\text{cosech}^2\theta\,d\theta$$

BY RECOGNITION (OR THE SUBSTITUTION $u=\coth\theta$)

$$=\left[-\frac{1}{3}\coth^3\theta\right]_{\coth\theta=2}^{\coth\theta=\sqrt{2}}=\frac{1}{3}\left[\coth^3\theta\right]_{\coth\theta=\sqrt{2}}^{\coth\theta=2}$$

$$=\frac{1}{3}\left[2^3-(\sqrt{2})^3\right]=\frac{1}{3}\left[8-2\sqrt{2}\right]$$

$$=\frac{2}{3}\left[4-\sqrt{2}\right]$$

AS BEFORE.

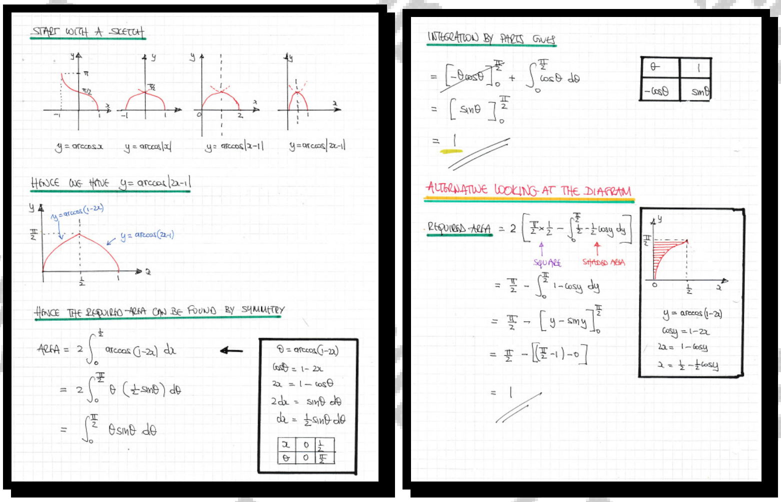
Question 118 (*****)

The function f is defined in the largest real domain by the equation

$$f(x) \equiv \arccos|2x-1|.$$

Determine the area of the finite region bounded by f and the coordinate axes.

$\boxed{}$, $\boxed{\text{area} = 1}$



Question 119 (*****)

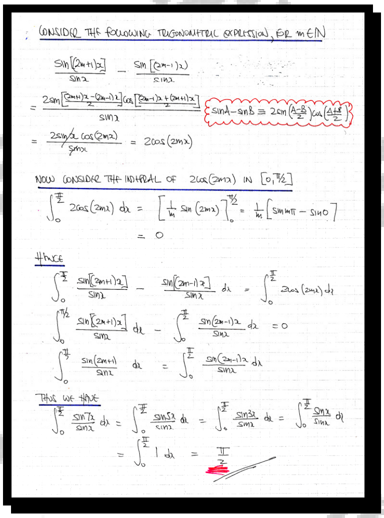
By considering

$$\frac{\sin\left[(2m+1)x\right]}{\sin x}-\frac{\sin\left[(2m-1)x\right]}{\sin x},\ m\in\mathbb{N},$$

determine the exact value of

$$\int_{0}^{\frac{1}{2}\pi}\frac{\sin 7x}{\sin x}\,dx.$$

$\boxed{}$, $\boxed{\frac{1}{2}\pi}$


CONSIDER THE FOLLOWING TRIGONOMETRIC EXPRESSION, FOR $m\in\mathbb{N}$

$$\frac{\sin[(2m+1)x]}{\sin x}-\frac{\sin[(2m-1)x]}{\sin x}$$

$$=\frac{2\sin\left[\frac{(2m+1)x-(2m-1)x}{2}\right]\cos\left[\frac{(2m-1)x+(2m+1)x}{2}\right]}{\sin x}\qquad \sin A-\sin B\equiv 2\sin\left(\frac{A-B}{2}\right)\cos\left(\frac{A+B}{2}\right)$$

$$=\frac{2\sin x\cos(2mx)}{\sin x}=2\cos(2mx)$$

NOW CONSIDER THE INTEGRAL OF $2\cos(2mx)$ IN $[0,\pi/2]$

$$\int_0^{\frac{\pi}{2}}2\cos(2mx)\,dx=\left[\frac{1}{m}\sin(2mx)\right]_0^{\pi/2}=\frac{1}{m}\left[\sin m\pi-\sin 0\right]=0$$

HENCE

$$\int_0^{\frac{\pi}{2}}\frac{\sin[(2m+1)x]}{\sin x}-\frac{\sin[(2m-1)x]}{\sin x}\,dx=\int_0^{\frac{\pi}{2}}2\cos(2mx)\,dx$$

$$\int_0^{\pi/2}\frac{\sin[(2m+1)x]}{\sin x}\,dx-\int_0^{\frac{\pi}{2}}\frac{\sin(2m-1)x}{\sin x}\,dx=0$$

$$\int_0^{\frac{\pi}{2}}\frac{\sin(2m+1)}{\sin x}\,dx=\int_0^{\frac{\pi}{2}}\frac{\sin(2m-1)x}{\sin x}\,dx$$

THUS WE HAVE

$$\int_0^{\frac{\pi}{2}}\frac{\sin 7x}{\sin x}\,dx=\int_0^{\frac{\pi}{2}}\frac{\sin 5x}{\sin x}\,dx=\int_0^{\frac{\pi}{2}}\frac{\sin 3x}{\sin x}\,dx=\int_0^{\frac{\pi}{2}}\frac{\sin x}{\sin x}\,dx$$

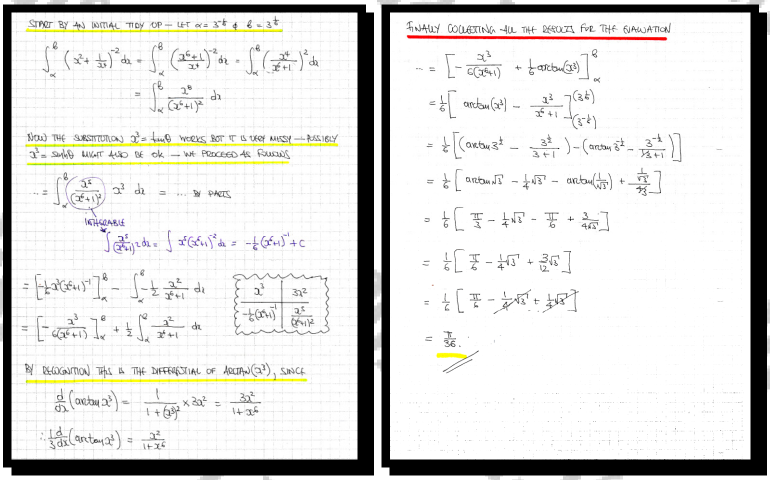
$$=\int_0^{\frac{\pi}{2}}1\,dx=\frac{\pi}{2}$$

Question 120 (*****)

Find in exact simplified form the value of the following definite integral.

$$\int_{3^{-\frac{1}{6}}}^{3^{\frac{1}{6}}} \left(x^2 + \frac{1}{x^4}\right)^{-2} \, dx \, .$$

$\boxed{}$, $\boxed{\dfrac{\pi}{36}}$



START BY AN INITIAL TIDY UP – LET $\alpha = 3^{-\frac{1}{6}}$ & $\beta = 3^{\frac{1}{6}}$

$$\int_{\alpha}^{\beta} \left(x^2 + \frac{1}{x^4}\right)^{-2} dx = \int_{\alpha}^{\beta} \left(\frac{x^6+1}{x^4}\right)^{-2} dx = \int_{\alpha}^{\beta} \left(\frac{x^4}{x^6+1}\right)^{2} dx$$

$$= \int_{\alpha}^{\beta} \frac{x^8}{(x^6+1)^2} \, dx$$

NOW THE SUBSTITUTION $x^3 = \tan\theta$ WORKS BUT IT IS VERY MESSY – POSSIBLY $x^3 = \sinh\theta$ MIGHT ALSO BE OK – WE PROCEED AS FOLLOWS

$$\dots = \int_{\alpha}^{\beta} \frac{x^5}{(x^6+1)^2} \, x^3 \, dx = \dots \text{ BY PARTS}$$

INTEGRABLE

$$\int \frac{x^5}{(x^6+1)^2} \, dx = \int x^5 (x^6+1)^{-2} \, dx = -\frac{1}{6}(x^6+1)^{-1} + C$$

$$= \left[-\frac{1}{6}x^3(x^6+1)^{-1}\right]_{\alpha}^{\beta} - \int_{\alpha}^{\beta} -\frac{1}{2}\frac{x^2}{x^6+1} \, dx$$

$$= \left[-\frac{x^3}{6(x^6+1)}\right]_{\alpha}^{\beta} + \frac{1}{2}\int_{\alpha}^{\beta} \frac{x^2}{x^6+1} \, dx$$

x^3	$3x^2$
$-\frac{1}{6}(x^6+1)^{-1}$	$\frac{x^5}{(x^6+1)^2}$

BY RECOGNITION THIS IS THE DIFFERENTIAL OF $\text{ARCTAN}(x^3)$, SINCE

$$\frac{d}{dx}(\arctan x^3) = \frac{1}{1+(x^3)^2} \times 3x^2 = \frac{3x^2}{1+x^6}$$

$$\therefore \frac{1}{3}\frac{d}{dx}(\arctan x^3) = \frac{x^2}{1+x^6}$$

FINALLY COLLECTING ALL THE RESULTS FOR THE EVALUATION

$$\dots = \left[-\frac{x^3}{6(x^6+1)} + \frac{1}{6}\arctan(x^3)\right]_{\alpha}^{\beta}$$

$$= \frac{1}{6}\left[\arctan(x^3) - \frac{x^3}{x^6+1}\right]_{(3^{-\frac{1}{6}})}^{(3^{\frac{1}{6}})}$$

$$= \frac{1}{6}\left[\left(\arctan 3^{\frac{1}{2}} - \frac{3^{\frac{1}{2}}}{3+1}\right) - \left(\arctan 3^{-\frac{1}{2}} - \frac{3^{-\frac{1}{2}}}{\frac{1}{3}+1}\right)\right]$$

$$= \frac{1}{6}\left[\arctan\sqrt{3} - \frac{1}{4}\sqrt{3} - \arctan\left(\frac{1}{\sqrt{3}}\right) + \frac{\frac{1}{\sqrt{3}}}{\frac{4}{3}}\right]$$

$$= \frac{1}{6}\left[\frac{\pi}{3} - \frac{1}{4}\sqrt{3} - \frac{\pi}{6} + \frac{3}{4\sqrt{3}}\right]$$

$$= \frac{1}{6}\left[\frac{\pi}{6} - \frac{1}{4}\sqrt{3} + \frac{3}{12}\sqrt{3}\right]$$

$$= \frac{1}{6}\left[\frac{\pi}{6} - \frac{1}{4}\sqrt{3} + \frac{1}{4}\sqrt{3}\right]$$

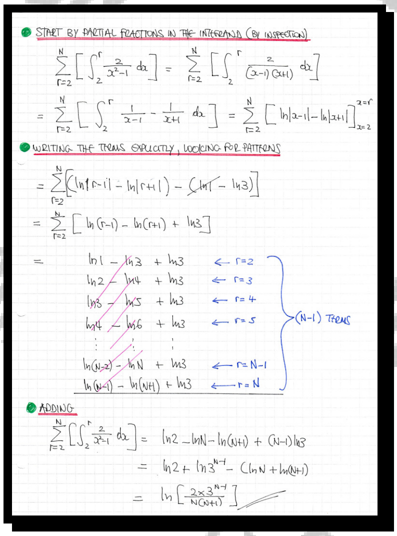
$$= \frac{\pi}{36}.$$

Question 121 (*****)

Determine a simplified expression, in the form $\ln\left[f(n)\right]$, for the following sum.

$$\sum_{r=2}^{N}\left[\int_{2}^{r}\frac{2}{x^2-1}\,dx\right].$$

$$\boxed{}\,,\quad \boxed{\ln\left[\frac{2\times 3^{N-1}}{N(N+1)}\right]}$$



Question 122 (*****)

Use appropriate integration methods to find a simplified expression for

$$\int x \arccos\left[\frac{1-x^2}{1+x^2}\right] dx .$$

$$\boxed{-x+\left(1+x^2\right)\arctan x+\text{constant}}$$

USING THE SUBSTITUTION GIVEN

$x = \tan(\tfrac{1}{2}\theta)$

$dx = \tfrac{1}{2}\sec^2(\tfrac{1}{2}\theta)\, d\theta$

OR $\left[dx = \tfrac{1}{2}\left[1+\tan^2(\tfrac{1}{2}\theta)\right] d\theta \right]$

OR $\left[dx = \tfrac{1}{2}(1+x^2)\, d\theta \right]$

(WE SHALL SEE WHICH FORM IS BETTER FOR THIS QUESTION)

$$\frac{1-x^2}{1+x^2} = \frac{1-\tan^2(\frac{1}{2}\theta)}{1+\tan^2(\frac{1}{2}\theta)} = \frac{1-\tan^2(\frac{1}{2}\theta)}{\sec^2(\frac{1}{2}\theta)} = \frac{1}{\sec^2(\frac{1}{2}\theta)} - \frac{\tan^2(\frac{1}{2}\theta)}{\sec^2(\frac{1}{2}\theta)}$$

$$= \cos^2(\tfrac{1}{2}\theta) - \frac{\sin^2(\frac{1}{2}\theta)}{\cos^2(\frac{1}{2}\theta)}\cos^2(\tfrac{1}{2}\theta) = \cos^2(\tfrac{1}{2}\theta) - \sin^2(\tfrac{1}{2}\theta) = \cos\theta$$

TRANSFORMING THE INTEGRAL WE HAVE

$$\int x \arccos\left(\frac{1-x^2}{1+x^2}\right) dx = \int \tan(\tfrac{1}{2}\theta) \arccos(\cos\theta) \left[\tfrac{1}{2}\sec^2(\tfrac{1}{2}\theta)\, d\theta\right]$$

$$= \int \tfrac{1}{2}\theta \tan(\tfrac{1}{2}\theta) \sec^2(\tfrac{1}{2}\theta)\, d\theta$$

INTEGRATION BY PARTS

$\frac{1}{2}\theta$	$\frac{1}{2}$
$\tan^2(\frac{1}{2}\theta)$	$\tan(\frac{1}{2}\theta)\sec^2(\frac{1}{2}\theta)$

$$\ldots = \tfrac{1}{2}\theta \tan^2(\tfrac{1}{2}\theta) - \int \tfrac{1}{2}\tan^2(\tfrac{1}{2}\theta)\, d\theta$$

$$= \tfrac{1}{2}\theta \tan^2(\tfrac{1}{2}\theta) - \tfrac{1}{2}\int \sec^2\tfrac{\theta}{2} - 1\, d\theta$$

$$= \tfrac{1}{2}\theta \tan^2(\tfrac{1}{2}\theta) - \tfrac{1}{2}\left[2\tan(\tfrac{1}{2}\theta) - \theta\right] + C$$

$$= \tfrac{1}{2}\theta \tan^2(\tfrac{1}{2}\theta) - \tan(\tfrac{1}{2}\theta) + \tfrac{1}{2}\theta + C$$

$$= \tfrac{1}{2}\theta\left(1+\tan^2(\tfrac{1}{2}\theta)\right) - \tan(\tfrac{1}{2}\theta) + C$$

$x = \tan\frac{1}{2}\theta$

$\arctan x = \frac{1}{2}\theta$

$$= \left[\arctan x\right]\left[1+x^2\right] - x + C$$

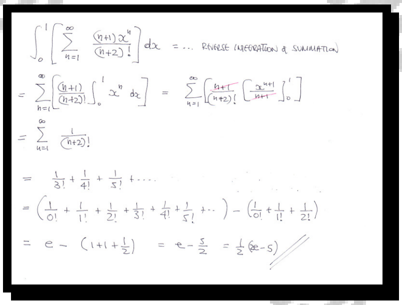
$$= -x + (1+x^2)\arctan x + C$$

Question 123 (*****)

Find the exact value of

$$\int_0^1 \left[\sum_{n=1}^{\infty} \frac{(n+1)x^n}{(n+2)!} \right] dx .$$

$\boxed{}$, $\boxed{\frac{1}{2}(2e-5)}$



Question 124 (*****)

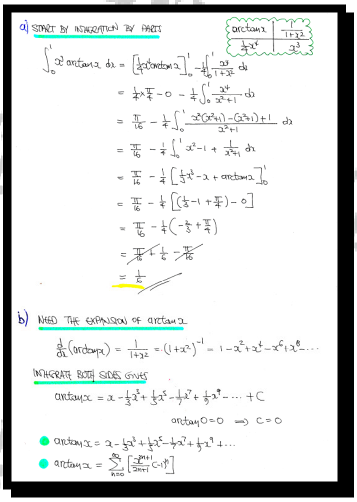
a) Use an appropriate integration method to evaluate the following integral.

$$\int_0^1 x^3 \arctan x \, dx .$$

b) Obtain an infinite series expansion for $\arctan x$ and use this series expansion to verify the answer obtained for the above integral in part **(a)**.

[*you may assume that integration and summation commute*]

$\boxed{}$, $\boxed{\frac{1}{6}}$



a) START BY INTEGRATION BY PARTS

$\arctan x$	$\frac{1}{1+x^2}$
$\frac{1}{4}x^4$	x^3

$$\int_0^1 x^3 \arctan x \, dx = \left[\tfrac{1}{4}x^4 \arctan x\right]_0^1 - \tfrac{1}{4}\int_0^1 \frac{x^4}{1+x^2}\,dx$$
$$= \tfrac{1}{4}\times\tfrac{\pi}{4} - 0 - \tfrac{1}{4}\int_0^1 \frac{x^4}{x^2+1}\,dx$$
$$= \tfrac{\pi}{16} - \tfrac{1}{4}\int_0^1 \frac{x^2(x^2+1)-(x^2+1)+1}{x^2+1}\,dx$$
$$= \tfrac{\pi}{16} - \tfrac{1}{4}\int_0^1 x^2 - 1 + \frac{1}{x^2+1}\,dx$$
$$= \tfrac{\pi}{16} - \tfrac{1}{4}\left[\tfrac{1}{3}x^3 - x + \arctan x\right]_0^1$$
$$= \tfrac{\pi}{16} - \tfrac{1}{4}\left[\left(\tfrac{1}{3} - 1 + \tfrac{\pi}{4}\right) - 0\right]$$
$$= \tfrac{\pi}{16} - \tfrac{1}{4}\left(-\tfrac{2}{3} + \tfrac{\pi}{4}\right)$$
$$= \tfrac{\pi}{16} + \tfrac{1}{6} - \tfrac{\pi}{16}$$
$$= \tfrac{1}{6}$$

b) NEED THE EXPANSION OF $\arctan x$

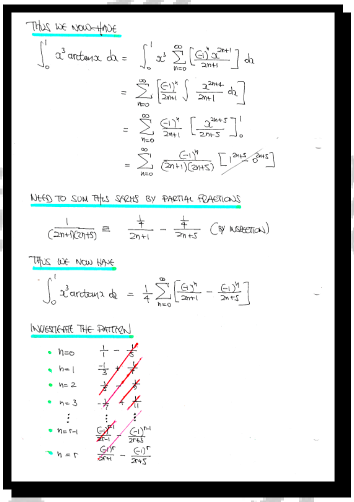
$$\frac{d}{dx}(\arctan x) = \frac{1}{1+x^2} = (1+x^2)^{-1} = 1 - x^2 + x^4 - x^6 + x^8 - \dots$$

INTEGRATE BOTH SIDES GIVES

$$\arctan x = x - \tfrac{1}{3}x^3 + \tfrac{1}{5}x^5 - \tfrac{1}{7}x^7 + \tfrac{1}{9}x^9 - \dots + C$$

$\arctan 0 = 0 \implies C = 0$

- $\arctan x = x - \tfrac{1}{3}x^3 + \tfrac{1}{5}x^5 - \tfrac{1}{7}x^7 + \tfrac{1}{9}x^9 + \dots$
- $\arctan x = \sum_{n=0}^{\infty}\left[\frac{x^{2n+1}}{2n+1}(-1)^n\right]$



THUS WE NOW HAVE

$$\int_0^1 x^3 \arctan x \, dx = \int_0^1 x^3 \sum_{n=0}^{\infty}\left[\frac{(-1)^n x^{2n+1}}{2n+1}\right]dx$$
$$= \sum_{n=0}^{\infty}\left[\frac{(-1)^n}{2n+1}\int \frac{x^{2n+4}}{2n+1}\,dx\right]$$
$$= \sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\left[\frac{x^{2n+5}}{2n+5}\right]_0^1$$
$$= \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)(2n+5)}\left[1^{2n+5} - 0^{2n+5}\right]$$

NEED TO SUM THIS SERIES BY PARTIAL FRACTIONS

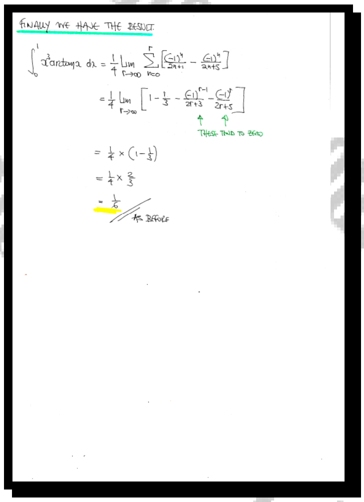
$$\frac{1}{(2n+1)(2n+5)} \equiv \frac{\frac{1}{4}}{2n+1} - \frac{\frac{1}{4}}{2n+5} \quad \text{(BY INSPECTION)}$$

THUS WE NOW HAVE

$$\int_0^1 x^3 \arctan x \, dx = \frac{1}{4}\sum_{n=0}^{\infty}\left[\frac{(-1)^n}{2n+1} - \frac{(-1)^n}{2n+5}\right]$$

INVESTIGATE THE PATTERN

- $n=0$: $\frac{1}{1} - \frac{1}{5}$
- $n=1$: $-\frac{1}{3} + \frac{1}{7}$
- $n=2$: $\frac{1}{5} - \frac{1}{9}$
- $n=3$: $-\frac{1}{7} + \frac{1}{11}$
- $\vdots$
- $n=r-1$: $\frac{(-1)^{r-1}}{2r-1} - \frac{(-1)^{r-1}}{2r+3}$
- $n=r$: $\frac{(-1)^r}{2r+1} - \frac{(-1)^r}{2r+5}$



FINALLY WE HAVE THE RESULT

$$\int_0^1 x^3 \arctan x \, dx = \frac{1}{4}\lim_{r\to\infty}\sum_{n=0}^{r}\left[\frac{(-1)^n}{2n+1} - \frac{(-1)^n}{2n+5}\right]$$
$$= \frac{1}{4}\lim_{r\to\infty}\left[1 - \frac{1}{3} - \frac{(-1)^{r-1}}{2r+3} - \frac{(-1)^r}{2r+5}\right]$$

THESE TEND TO ZERO

$$= \tfrac{1}{4}\times\left(1 - \tfrac{1}{3}\right)$$
$$= \tfrac{1}{4}\times\tfrac{2}{3}$$
$$= \tfrac{1}{6}$$

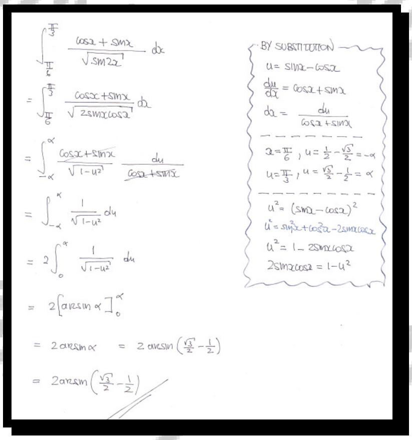
AS BEFORE

Question 125 (*****)

Find the exact value of

$$\int_{\frac{1}{6}\pi}^{\frac{1}{3}\pi} \frac{\cos x + \sin x}{\sqrt{\sin 2x}} \, dx .$$

$\boxed{}$, $\boxed{2\arcsin\left[\frac{\sqrt{3}-1}{2}\right]}$



$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\cos x + \sin x}{\sqrt{\sin 2x}} \, dx$

$= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\cos x + \sin x}{\sqrt{2\sin x \cos x}} \, dx$

$= \int_{-\alpha}^{\alpha} \frac{\cos x + \sin x}{\sqrt{1-u^2}} \frac{du}{\cos x + \sin x}$

$= \int_{-\alpha}^{\alpha} \frac{1}{\sqrt{1-u^2}} \, du$

$= 2\int_{0}^{\alpha} \frac{1}{\sqrt{1-u^2}} \, du$

$= 2\left[\arcsin \alpha\right]_{0}^{\alpha}$

$= 2\arcsin \alpha = 2\arcsin\left(\frac{\sqrt{3}}{2} - \frac{1}{2}\right)$

$= 2\arcsin\left(\frac{\sqrt{3}}{2} - \frac{1}{2}\right)$

BY SUBSTITUTION

$u = \sin x - \cos x$

$\frac{du}{dx} = \cos x + \sin x$

$dx = \frac{du}{\cos x + \sin x}$

$x = \frac{\pi}{6}, \ u = \frac{1}{2} - \frac{\sqrt{3}}{2} = -\alpha$

$x = \frac{\pi}{3}, \ u = \frac{\sqrt{3}}{2} - \frac{1}{2} = \alpha$

$u^2 = (\sin x - \cos x)^2$

$u^2 = \sin^2 x + \cos^2 x - 2\sin x \cos x$

$u^2 = 1 - 2\sin x \cos x$

$2\sin x \cos x = 1 - u^2$

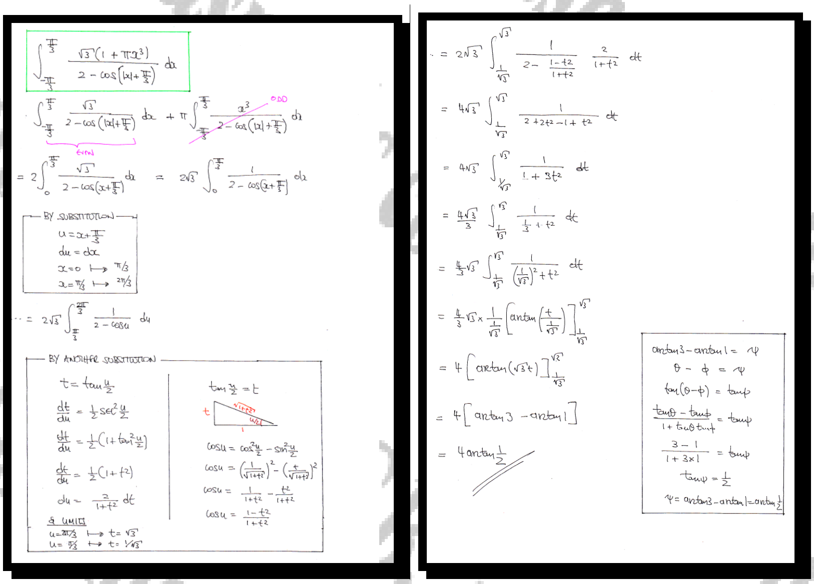
Question 126 (*****)

$$I = \int_{-\frac{1}{3}\pi}^{\frac{1}{3}\pi} \frac{\sqrt{3}\left(1+\pi x^{3}\right)}{2-\cos\left(|x|+\frac{1}{3}\pi\right)}\, dx .$$

Show that

$$I = 4\arctan\frac{1}{2}.$$

[ ] , [proof]

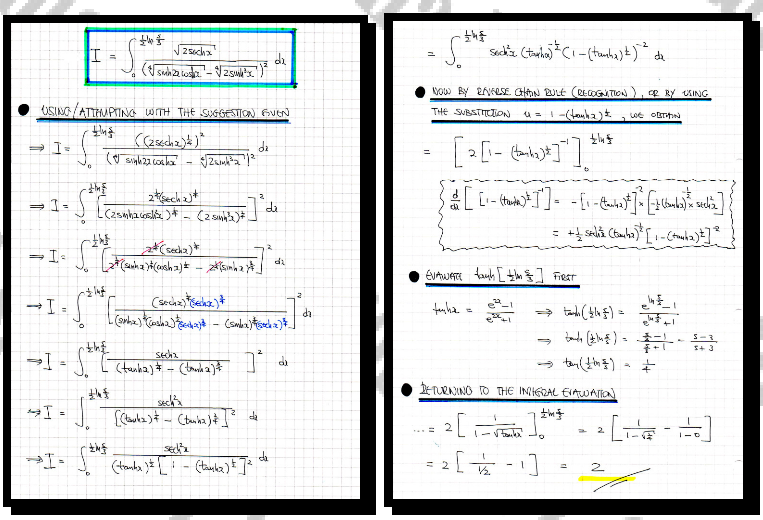


Question 127 (*****)

By expressing the integrand in the form $\text{sech}^2 x \, f(\tanh x)$, or otherwise, find the value of the following integral.

$$\int_0^{\frac{1}{2}\ln\frac{5}{3}} \frac{\sqrt{2\,\text{sech}\,x}}{\sqrt[4]{\sinh 2x \cosh x} - \sqrt[4]{2\sinh^3 x}} \, dx \, .$$

[   ], [2]



$$I = \int_0^{\frac{1}{2}\ln\frac{5}{3}} \frac{\sqrt{2\,\text{sech}\,x}}{\left(\sqrt[4]{\sinh 2x \cosh x} - \sqrt[4]{2\sinh^3 x}\right)^2} \, dx$$

- USING/ATTEMPTING WITH THE SUGGESTION GIVEN

$$\Rightarrow I = \int_0^{\frac{1}{2}\ln\frac{5}{3}} \frac{\left((2\,\text{sech}\,x)^{\frac{1}{4}}\right)^2}{\left(\sqrt[4]{\sinh 2x \cosh x} - \sqrt[4]{2\sinh^3 x}\right)^2} \, dx$$

$$\Rightarrow I = \int_0^{\frac{1}{2}\ln\frac{5}{3}} \left[\frac{2^{\frac{1}{4}}(\text{sech}\,x)^{\frac{1}{4}}}{(2\sinh x\cosh^2 x)^{\frac{1}{4}} - (2\sinh^3 x)^{\frac{1}{4}}}\right]^2 dx$$

$$\Rightarrow I = \int_0^{\frac{1}{2}\ln\frac{5}{3}} \left[\frac{2^{\frac{1}{4}}(\text{sech}\,x)^{\frac{1}{4}}}{2^{\frac{1}{4}}(\sinh x)^{\frac{1}{4}}(\cosh x)^{\frac{1}{2}} - 2^{\frac{1}{4}}(\sinh x)^{\frac{3}{4}}}\right]^2 dx$$

$$\Rightarrow I = \int_0^{\frac{1}{2}\ln\frac{5}{3}} \left[\frac{(\text{sech}\,x)^{\frac{1}{4}}(\text{sech}\,x)^{\frac{3}{4}}}{(\sinh x)^{\frac{1}{4}}(\cosh x)^{\frac{1}{2}}(\text{sech}\,x)^{\frac{3}{4}} - (\sinh x)^{\frac{3}{4}}(\text{sech}\,x)^{\frac{3}{4}}}\right]^2 dx$$

$$\Rightarrow I = \int_0^{\frac{1}{2}\ln\frac{5}{3}} \left[\frac{\text{sech}\,x}{(\tanh x)^{\frac{1}{4}} - (\tanh x)^{\frac{3}{4}}}\right]^2 dx$$

$$\Rightarrow I = \int_0^{\frac{1}{2}\ln\frac{5}{3}} \frac{\text{sech}^2 x}{\left[(\tanh x)^{\frac{1}{4}} - (\tanh x)^{\frac{3}{4}}\right]^2} \, dx$$

$$\Rightarrow I = \int_0^{\frac{1}{2}\ln\frac{5}{3}} \frac{\text{sech}^2 x}{(\tanh x)^{\frac{1}{2}}\left[1 - (\tanh x)^{\frac{1}{2}}\right]^2} \, dx$$

$$= \int_0^{\frac{1}{2}\ln\frac{5}{3}} \text{sech}^2 x \,(\tanh x)^{-\frac{1}{2}}\left(1 - (\tanh x)^{\frac{1}{2}}\right)^{-2} dx$$

- NOW BY REVERSE CHAIN RULE (RECOGNITION), OR BY USING THE SUBSTITUTION $u = 1 - (\tanh x)^{\frac{1}{2}}$, WE OBTAIN

$$= \left[2\left[1 - (\tanh x)^{\frac{1}{2}}\right]^{-1}\right]_0^{\frac{1}{2}\ln\frac{5}{3}}$$

$$\frac{d}{dx}\left[\left[1 - (\tanh x)^{\frac{1}{2}}\right]^{-1}\right] = -\left[1 - (\tanh x)^{\frac{1}{2}}\right]^{-2} \times \left[-\tfrac{1}{2}(\tanh x)^{-\frac{1}{2}} \times \text{sech}^2 x\right]$$

$$= +\tfrac{1}{2}\,\text{sech}^2 x\,(\tanh x)^{-\frac{1}{2}}\left[1 - (\tanh x)^{\frac{1}{2}}\right]^{-2}$$

- EVALUATE $\tanh\left[\frac{1}{2}\ln\frac{5}{3}\right]$ FIRST

$$\tanh x = \frac{e^{2x} - 1}{e^{2x} + 1} \Rightarrow \tanh\left(\tfrac{1}{2}\ln\tfrac{5}{3}\right) = \frac{e^{\ln\frac{5}{3}} - 1}{e^{\ln\frac{5}{3}} + 1}$$

$$\Rightarrow \tanh\left(\tfrac{1}{2}\ln\tfrac{5}{3}\right) = \frac{\frac{5}{3} - 1}{\frac{5}{3} + 1} = \frac{5 - 3}{5 + 3}$$

$$\Rightarrow \tan\left(\tfrac{1}{2}\ln\tfrac{5}{3}\right) = \frac{1}{4}$$

- RETURNING TO THE INTEGRAL EVALUATION

$$\ldots = 2\left[\frac{1}{1 - \sqrt{\tanh x}}\right]_0^{\frac{1}{2}\ln\frac{5}{3}} = 2\left[\frac{1}{1 - \sqrt{\frac{1}{4}}} - \frac{1}{1 - 0}\right]$$

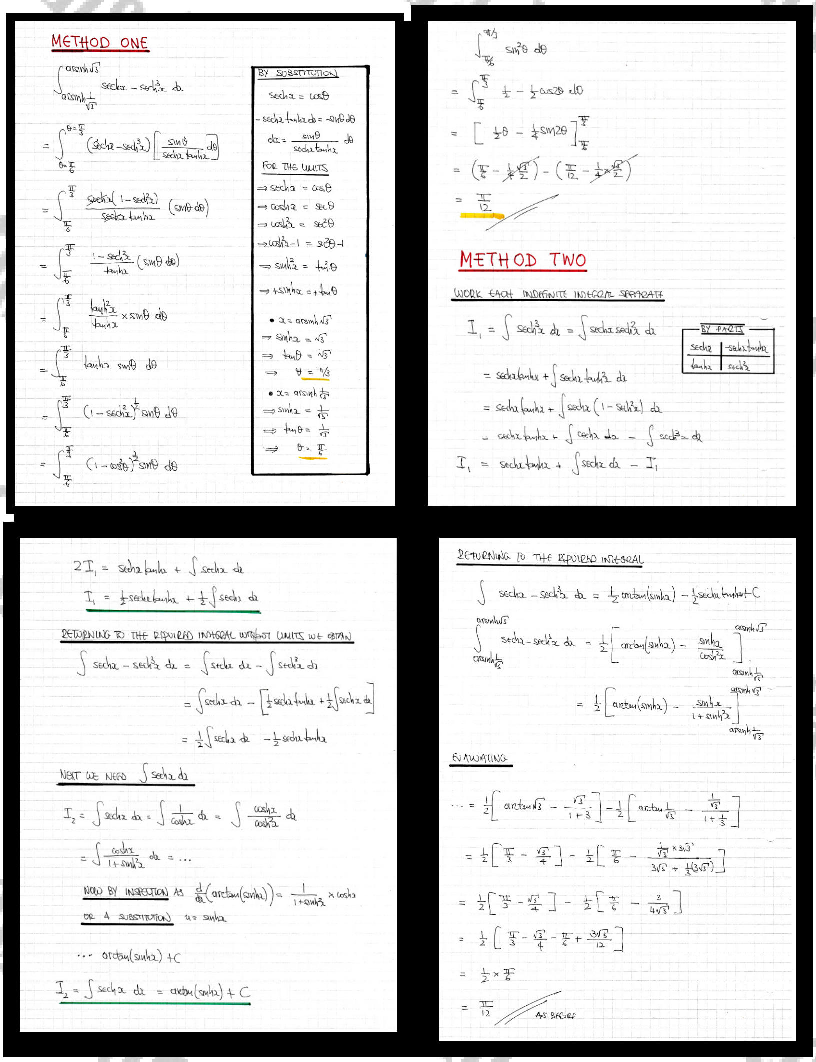
$$= 2\left[\frac{1}{\frac{1}{2}} - 1\right] = 2$$

Question 128 (*****)

Use appropriate integration techniques to show that

$$\int_{\operatorname{arsinh}\frac{1}{\sqrt{3}}}^{\operatorname{arsinh}\sqrt{3}} \operatorname{sech} x\left(1-\operatorname{sech} x\right)\,dx = \frac{\pi}{12}.$$

[ ] , [proof]



METHOD ONE

$\int_{\operatorname{arsinh}\frac{1}{\sqrt{3}}}^{\operatorname{arsinh}\sqrt{3}} \operatorname{sech}x - \operatorname{sech}^2 x \, dx$

$= \int_{\theta=\frac{\pi}{6}}^{\theta=\frac{\pi}{3}} (\operatorname{sech}x - \operatorname{sech}^2 x)\left[\frac{\sin\theta}{\operatorname{sech}x \tanh x}\,d\theta\right]$

$= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\operatorname{sech}x(1-\operatorname{sech}x)}{\operatorname{sech}x \tanh x}(\sin\theta\,d\theta)$

$= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1-\operatorname{sech}x}{\tanh x}(\sin\theta\,d\theta)$

$= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\tanh^2 x}{\tanh x} \times \sin\theta\,d\theta$

$= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \tanh x \sin\theta\,d\theta$

$= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} (1-\operatorname{sech}^2 x)^{\frac{1}{2}} \sin\theta\,d\theta$

$= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} (1-\cos^2\theta)^{\frac{1}{2}} \sin\theta\,d\theta$

BY SUBSTITUTION

$\operatorname{sech}x = \cos\theta$

$-\operatorname{sech}x\tanh x\,dx = -\sin\theta\,d\theta$

$dx = \frac{\sin\theta}{\operatorname{sech}x\tanh x}\,d\theta$

FOR THE LIMITS

$\Rightarrow \operatorname{sech}x = \cos\theta$

$\Rightarrow \cosh x = \sec\theta$

$\Rightarrow \cosh^2 x = \sec^2\theta$

$\Rightarrow \cosh^2 x - 1 = \sec^2\theta - 1$

$\Rightarrow \sinh^2 x = \tan^2\theta$

$\Rightarrow +\sinh x = +\tan\theta$

- $x = \operatorname{arsinh}\sqrt{3}$

$\Rightarrow \sinh x = \sqrt{3}$

$\Rightarrow \tan\theta = \sqrt{3}$

$\Rightarrow \theta = \frac{\pi}{3}$

- $x = \operatorname{arsinh}\frac{1}{\sqrt{3}}$

$\Rightarrow \sinh x = \frac{1}{\sqrt{3}}$

$\Rightarrow \tan\theta = \frac{1}{\sqrt{3}}$

$\Rightarrow \theta = \frac{\pi}{6}$

$\int_{\pi/6}^{\pi/3} \sin^2\theta\,d\theta$

$= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1}{2} - \frac{1}{2}\cos 2\theta\,d\theta$

$= \left[\frac{1}{2}\theta - \frac{1}{4}\sin 2\theta\right]_{\frac{\pi}{6}}^{\frac{\pi}{3}}$

$= \left(\frac{\pi}{6} - \frac{1}{4}\times\frac{\sqrt{3}}{2}\right) - \left(\frac{\pi}{12} - \frac{1}{4}\times\frac{\sqrt{3}}{2}\right)$

$= \frac{\pi}{12}$

METHOD TWO

WORK EACH INDEFINITE INTEGRAL SEPARATELY

$I_1 = \int \operatorname{sech}^3 x\,dx = \int \operatorname{sech}x \operatorname{sech}^2 x\,dx$

BY PARTS

$\operatorname{sech}x$	$-\operatorname{sech}x\tanh x$
$\tanh x$	$\operatorname{sech}^2 x$

$= \operatorname{sech}x\tanh x + \int \operatorname{sech}x\tanh^2 x\,dx$

$= \operatorname{sech}x\tanh x + \int \operatorname{sech}x(1-\operatorname{sech}^2 x)\,dx$

$= \operatorname{sech}x\tanh x + \int \operatorname{sech}x\,dx - \int \operatorname{sech}^3 x\,dx$

$I_1 = \operatorname{sech}x\tanh x + \int \operatorname{sech}x\,dx - I_1$

$2I_1 = \operatorname{sech}x\tanh x + \int \operatorname{sech}x\,dx$

$I_1 = \frac{1}{2}\operatorname{sech}x\tanh x + \frac{1}{2}\int \operatorname{sech}x\,dx$

RETURNING TO THE REQUIRED INTEGRAL WITHOUT LIMITS WE OBTAIN

$\int \operatorname{sech}x - \operatorname{sech}^2 x\,dx = \int \operatorname{sech}x\,dx - \int \operatorname{sech}^3 x\,dx$

$= \int \operatorname{sech}x\,dx - \left[\frac{1}{2}\operatorname{sech}x\tanh x + \frac{1}{2}\int \operatorname{sech}x\,dx\right]$

$= \frac{1}{2}\int \operatorname{sech}x\,dx - \frac{1}{2}\operatorname{sech}x\tanh x$

NEXT WE NEED $\int \operatorname{sech}x\,dx$

$I_2 = \int \operatorname{sech}x\,dx = \int \frac{1}{\cosh x}\,dx = \int \frac{\cosh x}{\cosh^2 x}\,dx$

$= \int \frac{\cosh x}{1+\sinh^2 x}\,dx = \dots$

NOW BY INSPECTION AS $\frac{d}{dx}\left(\arctan(\sinh x)\right) = \frac{1}{1+\sinh^2 x}\times\cosh x$

OR A SUBSTITUTION $u = \sinh x$

$\dots \arctan(\sinh x) + C$

$I_2 = \int \operatorname{sech}x\,dx = \arctan(\sinh x) + C$

RETURNING TO THE REQUIRED INTEGRAL

$\int \operatorname{sech}x - \operatorname{sech}^2 x\,dx = \frac{1}{2}\arctan(\sinh x) - \frac{1}{2}\operatorname{sech}x\tanh x + C$

$\int_{\operatorname{arsinh}\frac{1}{\sqrt{3}}}^{\operatorname{arsinh}\sqrt{3}} \operatorname{sech}x - \operatorname{sech}^2 x\,dx = \frac{1}{2}\left[\arctan(\sinh x) - \frac{\sinh x}{\cosh^2 x}\right]_{\operatorname{arsinh}\frac{1}{\sqrt{3}}}^{\operatorname{arsinh}\sqrt{3}}$

$= \frac{1}{2}\left[\arctan(\sinh x) - \frac{\sinh x}{1+\sinh^2 x}\right]_{\operatorname{arsinh}\frac{1}{\sqrt{3}}}^{\operatorname{arsinh}\sqrt{3}}$

EVALUATING

$\dots = \frac{1}{2}\left[\arctan\sqrt{3} - \frac{\sqrt{3}}{1+3}\right] - \frac{1}{2}\left[\arctan\frac{1}{\sqrt{3}} - \frac{\frac{1}{\sqrt{3}}}{1+\frac{1}{3}}\right]$

$= \frac{1}{2}\left[\frac{\pi}{3} - \frac{\sqrt{3}}{4}\right] - \frac{1}{2}\left[\frac{\pi}{6} - \frac{\frac{1}{\sqrt{3}}\times 3\sqrt{3}}{3\sqrt{3}+\frac{1}{3}(3\sqrt{3})}\right]$

$= \frac{1}{2}\left[\frac{\pi}{3} - \frac{\sqrt{3}}{4}\right] - \frac{1}{2}\left[\frac{\pi}{6} - \frac{3}{4\sqrt{3}}\right]$

$= \frac{1}{2}\left[\frac{\pi}{3} - \frac{\sqrt{3}}{4} - \frac{\pi}{6} + \frac{3\sqrt{3}}{12}\right]$

$= \frac{1}{2}\times\frac{\pi}{6}$

$= \frac{\pi}{12}$ AS BEFORE

Question 129 (*****)

The function f is defined as.

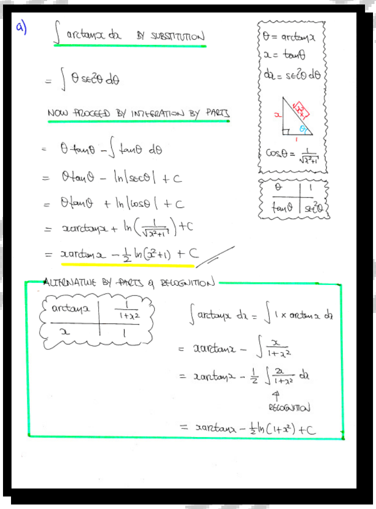
$$f(x) = \arctan x\,, \quad x \in \left(-\tfrac{1}{2}\pi, \tfrac{1}{2}\pi\right).$$

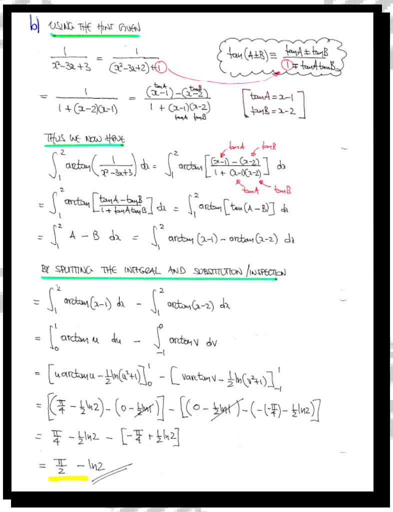
a) Find a simplified expression for $\int f(x)\,dx$.

b) By considering the tangent compound angle identity, or otherwise, find an exact simplified value for

$$\int_1^2 \arctan\left[\frac{1}{x^2 - 3x + 3}\right] dx\,.$$

$\boxed{x \arctan x - \frac{1}{2}\ln\left(x^2+1\right)}$, $\boxed{\frac{1}{2}\pi - \ln 2}$





Question 130 **(*****)**

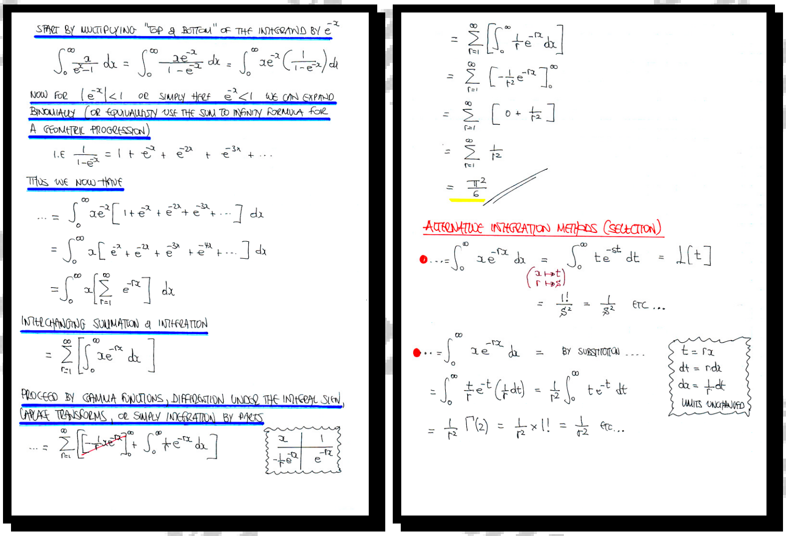
Use appropriate integration techniques to find an exact simplified value for the following improper integral.

$$\int_0^{\infty} \frac{x}{e^x - 1}\, dx\,.$$

You may assume without proof that

$$\sum_{r=1}^{\infty} \frac{1}{r^2} = \frac{\pi^2}{6}\,.$$

SP-X, $\boxed{\dfrac{\pi^2}{6}}$



START BY MULTIPLYING "TOP & BOTTOM" OF THE INTEGRAND BY e^{-x}

$$\int_0^{\infty} \frac{x}{e^x-1}\,dx = \int_0^{\infty} \frac{xe^{-x}}{1-e^{-x}}\,dx = \int_0^{\infty} xe^{-x}\left(\frac{1}{1-e^{-x}}\right)dx$$

NOW FOR $|e^{-x}|<1$ OR SIMPLY THAT $e^{-x}<1$ WE CAN EXPAND BINOMIALLY (OR EQUIVALENTLY USE THE SUM TO INFINITY FORMULA FOR A GEOMETRIC PROGRESSION)

I.E $\frac{1}{1-e^{-x}} = 1 + e^{-x} + e^{-2x} + e^{-3x} + \dots$

THUS WE NOW HAVE

$$\dots = \int_0^{\infty} xe^{-x}\left[1 + e^{-x} + e^{-2x} + e^{-3x} + \dots\right] dx$$

$$= \int_0^{\infty} x\left[e^{-x} + e^{-2x} + e^{-3x} + e^{-4x} + \dots\right] dx$$

$$= \int_0^{\infty} x\left[\sum_{r=1}^{\infty} e^{-rx}\right] dx$$

INTERCHANGING SUMMATION & INTEGRATION

$$= \sum_{r=1}^{\infty}\left[\int_0^{\infty} xe^{-rx}\,dx\right]$$

PROCEED BY GAMMA FUNCTIONS, DIFFERENTIATION UNDER THE INTEGRAL SIGN, LAPLACE TRANSFORMS, OR SIMPLY INTEGRATION BY PARTS

$$\dots = \sum_{r=1}^{\infty}\left[\left[-\frac{1}{r}xe^{-rx}\right]_0^{\infty} + \int_0^{\infty} \frac{1}{r}e^{-rx}\,dx\right]$$

x	1
$-\frac{1}{r}e^{-rx}$	e^{-rx}

$$= \sum_{r=1}^{\infty}\left[\int_0^{\infty} \frac{1}{r}e^{-rx}\,dx\right]$$

$$= \sum_{r=1}^{\infty}\left[-\frac{1}{r^2}e^{-rx}\right]_0^{\infty}$$

$$= \sum_{r=1}^{\infty}\left[0 + \frac{1}{r^2}\right]$$

$$= \sum_{r=1}^{\infty} \frac{1}{r^2}$$

$$= \frac{\pi^2}{6}$$

ALTERNATIVE INTEGRATION METHODS (SELECTION)

- $\dots = \int_0^{\infty} xe^{-rx}\,dx = \int_0^{\infty} te^{-st}\,dt = \mathcal{L}[t]$ $\left(\begin{matrix} x \mapsto t \\ r \mapsto s\end{matrix}\right)$

  $= \frac{1!}{s^2} = \frac{1}{s^2}$ ETC …

- $\dots = \int_0^{\infty} xe^{-rx}\,dx =$ BY SUBSTITUTION ….

  $t = rx$, $dt = r\,dx$, $dx = \frac{1}{r}dt$, LIMITS UNCHANGED

  $= \int_0^{\infty} \frac{t}{r}e^{-t}\left(\frac{1}{r}dt\right) = \frac{1}{r^2}\int_0^{\infty} te^{-t}\,dt$

  $= \frac{1}{r^2}\Gamma(2) = \frac{1}{r^2}\times 1! = \frac{1}{r^2}$ ETC…

Question 131 (*****)

The positive solution of the quadratic equation $x^2 - x - 1 = 0$ is denoted by ϕ, and is commonly known as the golden section or golden number.

This implies that $\phi^2 - \phi - 1 = 0$, $\phi = \frac{1}{2}\left(1+\sqrt{5}\right) \approx 1.62$.

a) Show, with a detailed method, that

$$\frac{d}{dx}\left[x\left(x^{\phi}+1\right)^{1-\phi}\right] = \left(x^{\phi}+1\right)^{-\phi}.$$

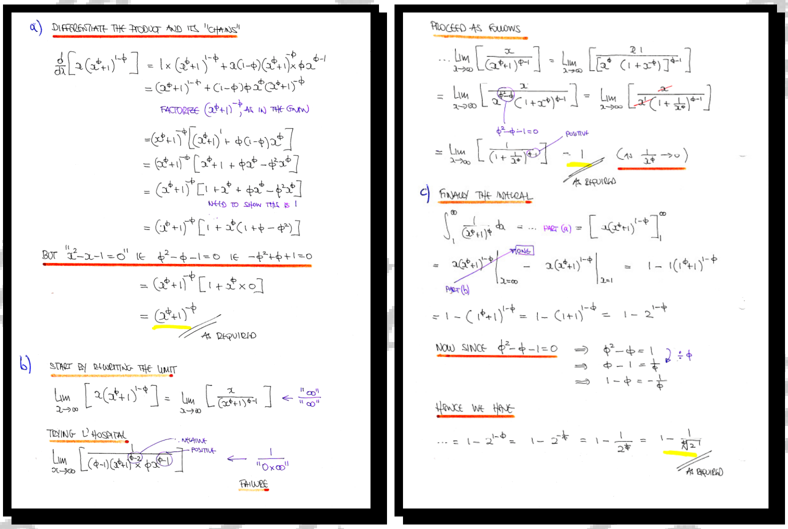
b) Show, with full justification, that

$$\lim_{x \to \infty}\left[x\left(x^{\phi}+1\right)^{1-\phi}\right] = 1.$$

c) Show further that

$$1 - \frac{1}{\sqrt[\phi]{2}} = \int_{1}^{\infty} \frac{1}{\left(x^{\phi}+1\right)^{\phi}}\, dx .$$

[ ], proof



Question 132 (*****)

It is given that

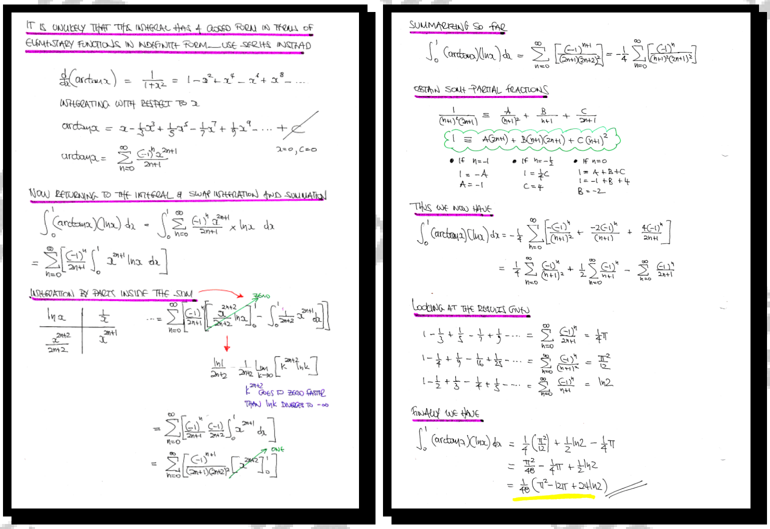
- $1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\frac{1}{11}+\ldots=\frac{1}{4}\pi$
- $1-\frac{1}{4}+\frac{1}{9}-\frac{1}{16}+\frac{1}{25}-\ \ldots\ =\frac{1}{12}\pi^2$
- $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\ldots\ =\ln 2$

Assuming the following integral converges find its exact value.

$$\int_0^1 (\ln x)(\arctan x)\ dx\,.$$

[*you may assume that integration and summation commute*]

$\boxed{}$, $\boxed{\frac{1}{48}\left[\pi^2-12\pi+24\ln 2\right]}$



IT IS UNLIKELY THAT THE INTEGRAL HAS A CLOSED FORM IN TERMS OF ELEMENTARY FUNCTIONS IN INDEFINITE FORM — USE SERIES INSTEAD

$\frac{d}{dx}(\arctan x) = \frac{1}{1+x^2} = 1-x^2+x^4-x^6+x^8-\ldots$

INTEGRATING WITH RESPECT TO x

$\arctan x = x-\frac{1}{3}x^3+\frac{1}{5}x^5-\frac{1}{7}x^7+\frac{1}{9}x^9-\ldots+C$ ($x=0, C=0$)

$\arctan x = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{2n+1}$

NOW RETURNING TO THE INTEGRAL & SWAP INTEGRATION AND SUMMATION

$\int_0^1(\arctan x)(\ln x)\,dx = \int_0^1\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{2n+1}\times\ln x\,dx$

$= \sum_{n=0}^{\infty}\left[\frac{(-1)^n}{2n+1}\int_0^1 x^{2n+1}\ln x\,dx\right]$

INTEGRATION BY PARTS INSIDE THE SUM

$\ln x$	$\frac{1}{x}$
$\frac{x^{2n+2}}{2n+2}$	x^{2n+1}

$\ldots = \sum_{n=0}^{\infty}\left[\frac{(-1)^n}{2n+1}\left[\left[\frac{x^{2n+2}}{2n+2}\ln x\right]_0^1-\int_0^1\frac{1}{2n+2}x^{2n+1}\,dx\right]\right]$ (ZERO)

$\frac{\ln 1}{2n+2}-\frac{1}{2n+2}\lim_{k\to 0}\left[k^{2n+2}\ln k\right]$

k^{2n+2} GOES TO ZERO FASTER THAN $\ln k$ DIVERGES TO $-\infty$

$= \sum_{n=0}^{\infty}\left[\frac{(-1)^n}{2n+1}\frac{(-1)}{2n+2}\int_0^1 x^{2n+1}\,dx\right]$

$= \sum_{n=0}^{\infty}\left[\frac{(-1)^{n+1}}{(2n+1)(2n+2)^2}\left[x^{2n+2}\right]_0^1\right]$ (ONE)

SUMMARIZING SO FAR

$\int_0^1(\arctan x)(\ln x)\,dx = \sum_{n=0}^{\infty}\left[\frac{(-1)^{n+1}}{(2n+1)(2n+2)^2}\right] = -\frac{1}{4}\sum_{n=0}^{\infty}\left[\frac{(-1)^n}{(n+1)^2(2n+1)}\right]$

OBTAIN SOME PARTIAL FRACTIONS

$\frac{1}{(n+1)^2(2n+1)} \equiv \frac{A}{(n+1)^2}+\frac{B}{n+1}+\frac{C}{2n+1}$

$1 \equiv A(2n+1)+B(n+1)(2n+1)+C(n+1)^2$

- IF $n=-1$: $1=-A$, $A=-1$
- IF $n=-\frac{1}{2}$: $1=\frac{1}{4}C$, $C=4$
- IF $n=0$: $1=A+B+C$, $1=-1+B+4$, $B=-2$

THUS WE NOW HAVE

$\int_0^1(\arctan x)(\ln x)\,dx = -\frac{1}{4}\sum_{n=0}^{\infty}\left[\frac{-(-1)^n}{(n+1)^2}+\frac{-2(-1)^n}{(n+1)}+\frac{4(-1)^n}{2n+1}\right]$

$= \frac{1}{4}\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)^2}+\frac{1}{2}\sum_{n=0}^{\infty}\frac{(-1)^n}{n+1}-\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}$

LOOKING AT THE RESULTS GIVEN

$1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\ldots = \sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1} = \frac{1}{4}\pi$

$1-\frac{1}{4}+\frac{1}{9}-\frac{1}{16}+\frac{1}{25}-\ldots = \sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)^2} = \frac{\pi^2}{12}$

$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\ldots = \sum_{n=0}^{\infty}\frac{(-1)^n}{n+1} = \ln 2$

FINALLY WE HAVE

$\int_0^1(\arctan x)(\ln x)\,dx = \frac{1}{4}\left(\frac{\pi^2}{12}\right)+\frac{1}{2}\ln 2-\frac{1}{4}\pi$

$= \frac{\pi^2}{48}-\frac{1}{4}\pi+\frac{1}{2}\ln 2$

$= \frac{1}{48}\left(\pi^2-12\pi+24\ln 2\right)$

Question 133 (*****)

Find the value of

$$\frac{1}{\pi}\int_{-\pi}^{\pi} \frac{\sin\frac{9}{2}x}{\sin\frac{1}{2}x}\, dx .$$

You may assume that the integrand is continuous at $x = 0$.

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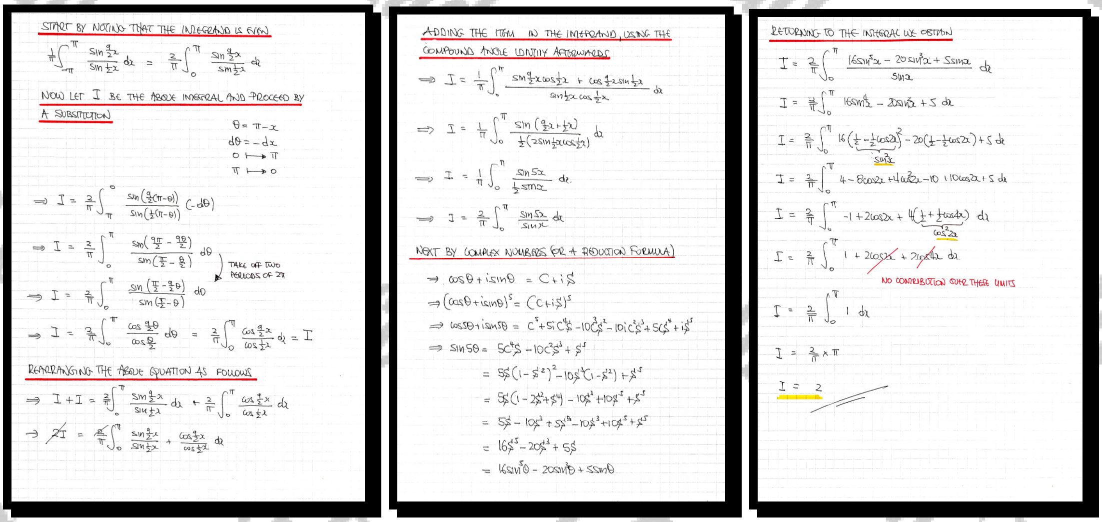


Question 134 (*****)

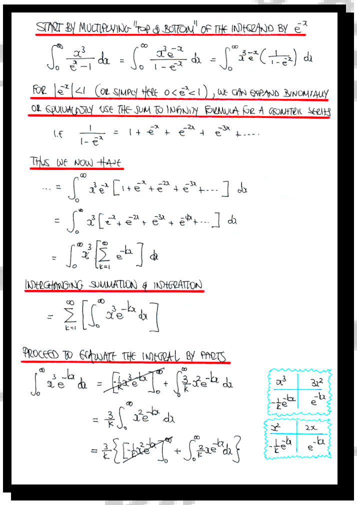
Use appropriate integration techniques to find an exact simplified value for the following improper integral.

$$\int_0^\infty \frac{x^3}{e^x - 1}\, dx .$$

You may assume without proof that

$$\sum_{r=1}^{\infty} \frac{1}{r^4} = \frac{\pi^4}{90} .$$

$$\boxed{\frac{\pi^4}{15}}$$



START BY MULTIPLYING "TOP & BOTTOM" OF THE INTEGRAND BY e^{-x}

$$\int_0^\infty \frac{x^3}{e^x-1}\,dx = \int_0^\infty \frac{x^3 e^{-x}}{1-e^{-x}}\,dx = \int_0^\infty x^3 e^{-x}\left(\frac{1}{1-e^{-x}}\right)dx$$

FOR $|e^{-x}|<1$ (OR SIMPLY HERE $0<e^{-x}<1$), WE CAN EXPAND BINOMIALLY OR EQUIVALENTLY USE THE SUM TO INFINITY FORMULA FOR A GEOMETRIC SERIES

$$\text{i.e. } \frac{1}{1-e^{-x}} = 1 + e^{-x} + e^{-2x} + e^{-3x} + \dots$$

THUS WE NOW HAVE

$$\dots = \int_0^\infty x^3 e^{-x}\left[1 + e^{-x} + e^{-2x} + e^{-3x} + \dots\right] dx$$

$$= \int_0^\infty x^3\left[e^{-x} + e^{-2x} + e^{-3x} + e^{-4x} + \dots\right] dx$$

$$= \int_0^\infty x^3\left[\sum_{k=1}^{\infty} e^{-kx}\right] dx$$

INTERCHANGING SUMMATION & INTEGRATION

$$= \sum_{k=1}^{\infty}\left[\int_0^\infty x^3 e^{-kx}\,dx\right]$$

PROCEED TO EVALUATE THE INTEGRAL BY PARTS

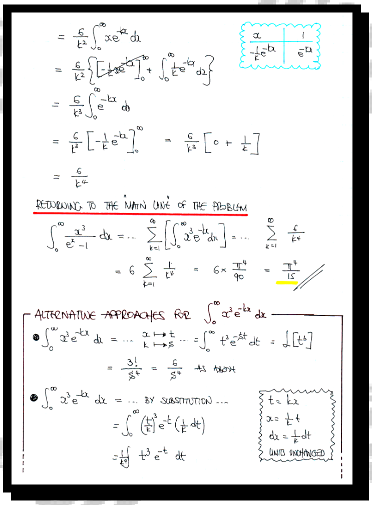
$$\int_0^\infty x^3 e^{-kx}\,dx = \left[-\frac{1}{k}x^3 e^{-kx}\right]_0^\infty + \int_0^\infty \frac{3}{k}x^2 e^{-kx}\,dx$$

$$= \frac{3}{k}\int_0^\infty x^2 e^{-kx}\,dx$$

$$= \frac{3}{k}\left\{\left[-\frac{1}{k}x^2 e^{-kx}\right]_0^\infty + \int_0^\infty \frac{2}{k}x e^{-kx}\,dx\right\}$$

x^3	$3x^2$
$-\frac{1}{k}e^{-kx}$	e^{-kx}

x^2	$2x$
$-\frac{1}{k}e^{-kx}$	e^{-kx}



$$= \frac{6}{k^2}\int_0^\infty x e^{-kx}\,dx$$

$$= \frac{6}{k^2}\left\{\left[-\frac{1}{k}x e^{-kx}\right]_0^\infty + \int_0^\infty \frac{1}{k}e^{-kx}\,dx\right\}$$

$$= \frac{6}{k^3}\int_0^\infty e^{-kx}\,dx$$

$$= \frac{6}{k^3}\left[-\frac{1}{k}e^{-kx}\right]_0^\infty = \frac{6}{k^3}\left[0 + \frac{1}{k}\right]$$

$$= \frac{6}{k^4}$$

x	1
$-\frac{1}{k}e^{-kx}$	e^{-kx}

RETURNING TO THE "MAIN LINE" OF THE PROBLEM

$$\int_0^\infty \frac{x^3}{e^x-1}\,dx = \dots \sum_{k=1}^{\infty}\left[\int_0^\infty x^3 e^{-kx}\,dx\right] = \dots \sum_{k=1}^{\infty}\frac{6}{k^4}$$

$$= 6\sum_{k=1}^{\infty}\frac{1}{k^4} = 6 \times \frac{\pi^4}{90} = \frac{\pi^4}{15}$$

ALTERNATIVE APPROACHES FOR $\int_0^\infty x^3 e^{-kx}\,dx$

- $\int_0^\infty x^3 e^{-kx}\,dx = \dots \begin{matrix} x \mapsto t \\ k \mapsto s \end{matrix} \dots = \int_0^\infty t^3 e^{-st}\,dt = \mathcal{L}[t^3]$

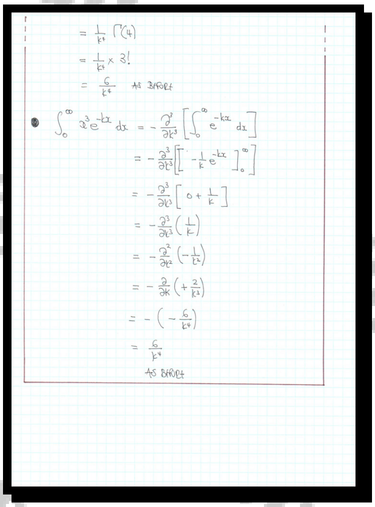
$$= \frac{3!}{s^4} = \frac{6}{s^4} \text{ AS ABOVE}$$

- $\int_0^\infty x^3 e^{-kx}\,dx = \dots$ BY SUBSTITUTION $\dots$

$$= \int_0^\infty \left(\frac{t}{k}\right)^3 e^{-t}\left(\frac{1}{k}\,dt\right)$$

$$= \frac{1}{k^4}\int_0^\infty t^3 e^{-t}\,dt$$

$t = kx$
$x = \frac{1}{k}t$
$dx = \frac{1}{k}dt$
LIMITS UNCHANGED



$$= \frac{1}{k^4}\Gamma(4)$$

$$= \frac{1}{k^4} \times 3!$$

$$= \frac{6}{k^4} \text{ AS BEFORE}$$

- $\int_0^\infty x^3 e^{-kx}\,dx = -\frac{\partial^3}{\partial k^3}\left[\int_0^\infty e^{-kx}\,dx\right]$

$$= -\frac{\partial^3}{\partial k^3}\left[\left[-\frac{1}{k}e^{-kx}\right]_0^\infty\right]$$

$$= -\frac{\partial^3}{\partial k^3}\left[0 + \frac{1}{k}\right]$$

$$= -\frac{\partial^3}{\partial k^3}\left(\frac{1}{k}\right)$$

$$= -\frac{\partial^2}{\partial k^2}\left(-\frac{1}{k^2}\right)$$

$$= -\frac{\partial}{\partial k}\left(+\frac{2}{k^3}\right)$$

$$= -\left(-\frac{6}{k^4}\right)$$

$$= \frac{6}{k^4}$$

AS BEFORE

Question 135 (*****)

The function f is defined as

$$f(x) = \arctan\left(\frac{1}{2x^2}\right), \quad x \in (-\infty, \infty).$$

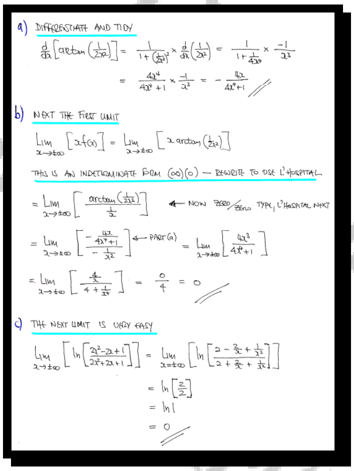
a) Find a simplified expression for $f'(x)$.

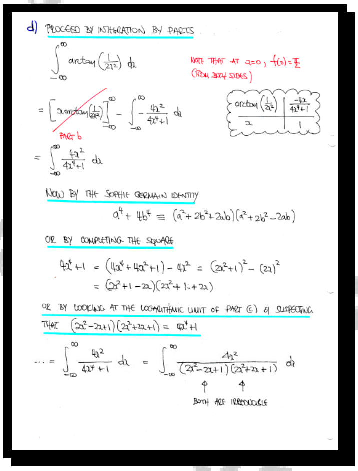
b) Show that $\lim_{x\to\pm\infty}\left[x f(x)\right] = 0$.

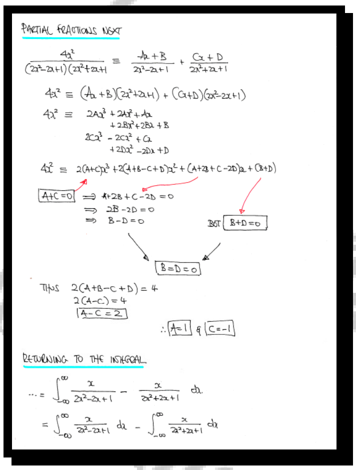
c) Determine the value of $\lim_{x\to\pm\infty}\left[\ln\left[\frac{2x^2-2x+1}{2x^2+2x+1}\right]\right]$.

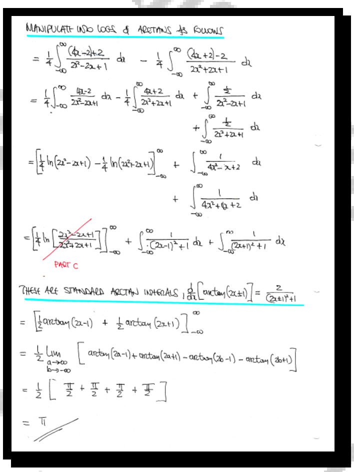
d) Hence find the value of $\int_{-\infty}^{\infty} f(x)\ dx$.

☐, $f'(x) = -\frac{4x}{4x^4+1}$, $\lim_{x\to\pm\infty}\left[\frac{2x^2-2x+1}{2x^2+2x+1}\right] = 0$, $\int_0^{\infty} f(x)\ dx = \frac{1}{2}\pi$









Question 136 (*****)

A family of functions $f_n(x)$, where $n = 0, 1, 2, 3, 4, \ldots$, satisfies the equation

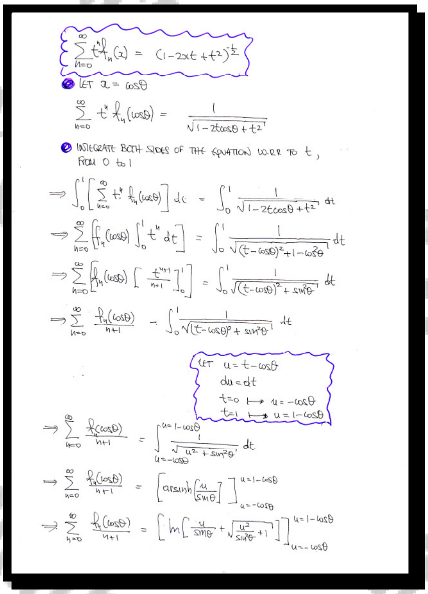
$$\sum_{n=0}^{\infty}\left[t^n f_n(x)\right] = \left(1-2xt+t^2\right)^{-\frac{1}{2}}.$$

By integrating both sides of the above equation with respect to t, from 0 to 1, show that

$$\sum_{n=0}^{\infty}\left[\frac{f_n(\cos\theta)}{n+1}\right] = \ln\left[1+\text{cosec}\left(\frac{1}{2}\theta\right)\right].$$

You may assume in this question that integration and summation commute.

[ ], [proof]



$$\sum_{n=0}^{\infty} t^n f_n(x) = (1-2xt+t^2)^{-\frac{1}{2}}$$

- LET $x = \cos\theta$

$$\sum_{n=0}^{\infty} t^n f_n(\cos\theta) = \frac{1}{\sqrt{1-2t\cos\theta+t^2}}$$

- INTEGRATE BOTH SIDES OF THE EQUATION W.R.T TO t, FROM 0 to 1

$$\Rightarrow \int_0^1\left[\sum_{n=0}^{\infty} t^n f_n(\cos\theta)\right]dt = \int_0^1 \frac{1}{\sqrt{1-2t\cos\theta+t^2}}\,dt$$

$$\Rightarrow \sum_{n=0}^{\infty}\left[f_n(\cos\theta)\int_0^1 t^n\,dt\right] = \int_0^1 \frac{1}{\sqrt{(t-\cos\theta)^2+1-\cos^2\theta}}\,dt$$

$$\Rightarrow \sum_{n=0}^{\infty}\left[f_n(\cos\theta)\left[\frac{t^{n+1}}{n+1}\right]_0^1\right] = \int_0^1 \frac{1}{\sqrt{(t-\cos\theta)^2+\sin^2\theta}}\,dt$$

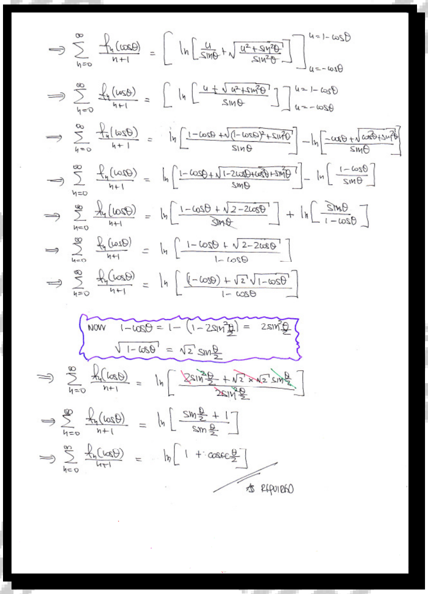
$$\Rightarrow \sum_{n=0}^{\infty}\frac{f_n(\cos\theta)}{n+1} = \int_0^1 \frac{1}{\sqrt{(t-\cos\theta)^2+\sin^2\theta}}\,dt$$

LET $u = t-\cos\theta$
$du = dt$
$t=0 \mapsto u=-\cos\theta$
$t=1 \mapsto u=1-\cos\theta$

$$\Rightarrow \sum_{n=0}^{\infty}\frac{f_n(\cos\theta)}{n+1} = \int_{u=-\cos\theta}^{u=1-\cos\theta} \frac{1}{\sqrt{u^2+\sin^2\theta}}\,dt$$

$$\Rightarrow \sum_{n=0}^{\infty}\frac{f_n(\cos\theta)}{n+1} = \left[\text{arsinh}\left(\frac{u}{\sin\theta}\right)\right]_{u=-\cos\theta}^{u=1-\cos\theta}$$

$$\Rightarrow \sum_{n=0}^{\infty}\frac{f_n(\cos\theta)}{n+1} = \left[\ln\left[\frac{u}{\sin\theta}+\sqrt{\frac{u^2}{\sin^2\theta}+1}\right]\right]_{u=-\cos\theta}^{u=1-\cos\theta}$$



$$\Rightarrow \sum_{n=0}^{\infty}\frac{f_n(\cos\theta)}{n+1} = \left[\ln\left[\frac{u}{\sin\theta}+\sqrt{\frac{u^2+\sin^2\theta}{\sin^2\theta}}\right]\right]_{u=-\cos\theta}^{u=1-\cos\theta}$$

$$\Rightarrow \sum_{n=0}^{\infty}\frac{f_n(\cos\theta)}{n+1} = \left[\ln\left[\frac{u+\sqrt{u^2+\sin^2\theta}}{\sin\theta}\right]\right]_{u=-\cos\theta}^{u=1-\cos\theta}$$

$$\Rightarrow \sum_{n=0}^{\infty}\frac{f_n(\cos\theta)}{n+1} = \ln\left[\frac{1-\cos\theta+\sqrt{(1-\cos\theta)^2+\sin^2\theta}}{\sin\theta}\right] - \ln\left[\frac{-\cos\theta+\sqrt{\cos^2\theta+\sin^2\theta}}{\sin\theta}\right]$$

$$\Rightarrow \sum_{n=0}^{\infty}\frac{f_n(\cos\theta)}{n+1} = \ln\left[\frac{1-\cos\theta+\sqrt{1-2\cos\theta+\cos^2\theta+\sin^2\theta}}{\sin\theta}\right] - \ln\left[\frac{1-\cos\theta}{\sin\theta}\right]$$

$$\Rightarrow \sum_{n=0}^{\infty}\frac{f_n(\cos\theta)}{n+1} = \ln\left[\frac{1-\cos\theta+\sqrt{2-2\cos\theta}}{\sin\theta}\right] + \ln\left[\frac{\sin\theta}{1-\cos\theta}\right]$$

$$\Rightarrow \sum_{n=0}^{\infty}\frac{f_n(\cos\theta)}{n+1} = \ln\left[\frac{1-\cos\theta+\sqrt{2-2\cos\theta}}{1-\cos\theta}\right]$$

$$\Rightarrow \sum_{n=0}^{\infty}\frac{f_n(\cos\theta)}{n+1} = \ln\left[\frac{(1-\cos\theta)+\sqrt{2}\sqrt{1-\cos\theta}}{1-\cos\theta}\right]$$

NOW $1-\cos\theta = 1-\left(1-2\sin^2\frac{\theta}{2}\right) = 2\sin^2\frac{\theta}{2}$

$\sqrt{1-\cos\theta} = \sqrt{2}\sin\frac{\theta}{2}$

$$\Rightarrow \sum_{n=0}^{\infty}\frac{f_n(\cos\theta)}{n+1} = \ln\left[\frac{2\sin^2\frac{\theta}{2}+\sqrt{2}\times\sqrt{2}\sin\frac{\theta}{2}}{2\sin^2\frac{\theta}{2}}\right]$$

$$\Rightarrow \sum_{n=0}^{\infty}\frac{f_n(\cos\theta)}{n+1} = \ln\left[\frac{\sin\frac{\theta}{2}+1}{\sin\frac{\theta}{2}}\right]$$

$$\Rightarrow \sum_{n=0}^{\infty}\frac{f_n(\cos\theta)}{n+1} = \ln\left[1+\text{cosec}\frac{\theta}{2}\right]$$

AS REQUIRED