

INTEGRATION STRUCTURED EXAM QUESTIONS PART II

Question 1 (**)

$$\frac{4}{(x-1)(x^2+1)} \equiv \frac{A}{x-1} + \frac{Bx+C}{x^2+1}.$$

- a) Find the values of A , B and C in the above identity.
b) Hence find

$$\int \frac{4}{(x-1)(x^2+1)} dx.$$

$$\boxed{A=2}, \boxed{B=-2}, \boxed{C=-2}, \boxed{2\ln|x-1| - \ln(x^2+1) - 2\arctan x + C}$$

(a) $\frac{4}{(x-1)(x^2+1)} \equiv \frac{A}{x-1} + \frac{Bx+C}{x^2+1}$
 $4 \equiv A(x^2+1) + (x-1)(Bx+C)$
 $\begin{aligned} \text{If } x=1, & \quad 4=2A \Rightarrow A=2 \\ \text{If } x=0, & \quad 4=A-C \Rightarrow C=-2 \\ \text{If } x=2, & \quad 4=5A+2B+C \\ & \quad 4=10+2B-2 \\ & \quad -4=2B \\ & \quad B=-2 \end{aligned}$
 (b) $\int \frac{4}{(x-1)(x^2+1)} dx = \int \frac{2}{x-1} - \frac{2x+2}{x^2+1} dx$
 $= \int \frac{2}{x-1} - \frac{2x}{x^2+1} - \frac{2}{x^2+1} dx$
 $= 2\ln|x-1| - \ln|x^2+1| - 2\arctan x + C$

Question 2 (**)

Find an exact value for

$$\int_0^{\sqrt{3}} \frac{3}{\sqrt{4-x^2}} dx.$$

π

$$\int_0^{\sqrt{3}} \frac{3}{\sqrt{4-x^2}} dx = \int_0^{\sqrt{3}} \frac{3}{\sqrt{2^2-x^2}} dx = \left[3 \arcsin \frac{x}{2} \right]_0^{\sqrt{3}}$$

$$= 3 \arcsin \frac{\sqrt{3}}{2} - 3 \arcsin 0 = 3 \times \frac{\pi}{3} = \pi$$

Question 3 ()**

By using the substitution $u = \arctan x$, or otherwise, find an exact value for

$$\int_0^1 \frac{\arctan x}{1+x^2} dx.$$

$$\frac{\pi^2}{32}$$

Handwritten solution for Question 3:

$$\int_0^1 \frac{\arctan x}{1+x^2} dx = \dots \text{change the substitution } (u = \arctan x)$$

$$= \int_0^{\frac{\pi}{4}} \frac{u}{1+u^2} du = \int_0^{\frac{\pi}{4}} u \, du = \left[\frac{1}{2} u^2 \right]_0^{\frac{\pi}{4}}$$

$$= \frac{\pi^2}{32} - 0 = \frac{\pi^2}{32}$$

Side notes in the handwritten solution:

- $u = \arctan x$
- $\frac{du}{dx} = \frac{1}{1+x^2}$
- $dx = (1+x^2) du$
- $x=0, u=0$
- $x=1, u=\frac{\pi}{4}$

Question 4 ()**

Find an expression for

$$\int \frac{x+2}{\sqrt{1-4x^2}} dx.$$

$$\arcsin 2x - \frac{1}{4} \sqrt{1-4x^2} + C$$

Handwritten solution for Question 4:

$$\int \frac{x+2}{\sqrt{1-4x^2}} dx = \int \frac{x}{\sqrt{1-4x^2}} dx + \int \frac{2}{\sqrt{1-4x^2}} dx = \int \frac{x(-4x)^{-\frac{1}{2}}}{2\sqrt{1-4x^2}} dx + \int \frac{2}{\sqrt{1-4x^2}} dx$$

Annotations in the handwritten solution:

- For the first integral, $\frac{x}{\sqrt{1-4x^2}}$ is noted as $\frac{1}{2} \sqrt{1-4x^2}$.
- For the second integral, $\frac{2}{\sqrt{1-4x^2}}$ is noted as $\frac{1}{2} \sqrt{1-4x^2}$.
- The final result is $\frac{1}{2} \sqrt{1-4x^2} + \arcsin\left(\frac{2x}{1}\right) + C = \arcsin(2x) - \frac{1}{4} \sqrt{1-4x^2} + C$.

Question 5 (**)

$$\frac{x^2 + x + 5}{(x+1)(x^2 + 4)} \equiv \frac{A}{x+1} + \frac{Bx+C}{x^2 + 4}.$$

- a) Find the values of A , B and C in the above identity.
- b) Hence find the exact value of

$$\int_0^2 \frac{x^2 + x + 5}{(x+1)(x^2 + 4)} dx.$$

$$\boxed{A=1}, \boxed{B=0}, \boxed{C=1}, \boxed{\frac{\pi}{8} + \ln 3}$$

(a) $\frac{x^2+x+5}{(x+1)(x^2+4)} \equiv \frac{A}{x+1} + \frac{Bx+C}{x^2+4}$
 $x^2+x+5 \equiv A(x^2+4) + (x+1)(Bx+C)$
 $\text{If } x=-1 \Rightarrow 5 = 5A \Rightarrow A=1$
 $\text{If } x=0 \Rightarrow 5 = 4A+C \Rightarrow 5 = 4+C \Rightarrow C=1$
 $\text{If } x=1 \Rightarrow 7 = 5A + 2(B+C)$
 $7 = 5 + 2(B+1)$
 $2 = 2(B+1)$
 $1 = B+1$
 $B=0$

(b) $\int_0^2 \frac{x^2+x+5}{(x+1)(x^2+4)} dx = \int_0^2 \left(\frac{1}{x+1} + \frac{1}{x^2+4} \right) dx = \left(\ln|x+1| + \frac{1}{2} \arctan \frac{x}{2} \right) \Big|_0^2$
 $= \left(\ln 3 + \frac{1}{2} \arctan(1) \right) - \left(\ln 1 + \frac{1}{2} \arctan(0) \right)$
 $= \ln 3 + \frac{\pi}{8}$

Question 6 ()**

Find an exact value for

$$\int_0^{\frac{1}{2}} \frac{6x+1}{\sqrt{1-x^2}} dx.$$

$$\frac{\pi}{6} - 3\sqrt{3} + 6$$

Handwritten solution for Question 6:

$$\begin{aligned} \int_0^{\frac{1}{2}} \frac{6x+1}{\sqrt{1-x^2}} dx &= \int_0^{\frac{1}{2}} \frac{6x}{\sqrt{1-x^2}} + \frac{1}{\sqrt{1-x^2}} dx = \int_0^{\frac{1}{2}} \frac{6x(-x)^{-\frac{1}{2}}}{(-x)^{-\frac{1}{2}}} + \frac{1}{(-x)^{-\frac{1}{2}}} dx \\ &= \left[-6(-x)^{\frac{1}{2}} + \arcsin(-x) \right]_0^{\frac{1}{2}} \\ &= \left[-6\sqrt{1-x^2} - \arcsin(x) \right]_0^{\frac{1}{2}} = \left(-6\sqrt{1-\frac{1}{4}} - \arcsin\left(\frac{1}{2}\right) \right) - \left(-6\sqrt{1-0} - \arcsin(0) \right) \\ &= \frac{\pi}{6} - 3\sqrt{3} + 6 \end{aligned}$$

Question 7 ()**Use the substitution $u = \sin x$ to find an exact value in terms of natural logarithms for

$$\int_0^{\frac{\pi}{2}} \frac{\cos x}{\sqrt{1+\sin^2 x}} dx.$$

$$\ln(1+\sqrt{2})$$

Handwritten solution for Question 7:

$$\begin{aligned} \int_0^{\frac{\pi}{2}} \frac{\cos x}{\sqrt{1+\sin^2 x}} dx &\quad \text{using the substitution } u = \sin x \\ &= \int_0^1 \frac{1}{\sqrt{1+u^2}} du = \int_0^1 \frac{1}{\sqrt{1+u^2}} du \\ &= \left[\operatorname{arcsinh} u \right]_0^1 = \operatorname{arcsinh}(1) - \operatorname{arcsinh}(0) \\ &= \ln(1+\sqrt{2}) \end{aligned}$$

Side note: $u = \sin x$
 $\frac{du}{dx} = \cos x$
 $dx = \frac{du}{\cos x}$
 $x=0, u=0$
 $x=\frac{\pi}{2}, u=1$

Question 8 (**)The function f is defined as

$$f(x) \equiv \tanh^2 x, \quad x \in \mathbb{R}, \quad 0 \leq x \leq \ln 3.$$

Determine the mean value of f , in its entire domain.

$$\boxed{}, \quad 1 - \frac{4}{5 \ln 3} \approx 0.272$$

Handwritten solution for the mean value of $f(x) = \tanh^2(x)$ from 0 to $\ln 3$.

Using Hyperbolic Identities:

$$1 + \tanh x = \operatorname{sech}^2 x$$

$$1 - \tanh x = \operatorname{sech}^2 x$$

$$1 - \operatorname{sech}^2 x = \tanh^2 x$$

$$\int_0^{\ln 3} \tanh^2 x \, dx = \int_0^{\ln 3} (1 - \operatorname{sech}^2 x) \, dx = \left[x - \tanh x \right]_0^{\ln 3}$$

$$= \left[\ln 3 - \tanh(\ln 3) \right] - \left[0 - 0 \right]$$

$$= \ln 3 - \frac{e^{\ln 3} - 1}{e^{\ln 3} + 1} \quad (\text{or calculator})$$

$$= \ln 3 - \frac{3-1}{3+1}$$

$$= \ln 3 - \frac{2}{4}$$

$$= \ln 3 - \frac{1}{2}$$

\therefore Mean Value $= \frac{1}{b-a} \int_a^b f(x) \, dx$

$$= \frac{1}{\ln 3 - 0} \left(\ln 3 - \frac{1}{2} \right)$$

$$= \frac{\ln 3 - \frac{1}{2}}{\ln 3}$$

$$= \frac{5 \ln 3 - 1}{5 \ln 3} \quad \text{or} \quad 1 - \frac{1}{5 \ln 3}$$

Question 9 (+)**

Find an exact value for

$$\int_0^{\frac{3}{4}} \frac{6}{\sqrt{3-4x^2}} dx.$$

 π

Handwritten solution for Question 9:

$$\int_0^{\frac{3}{4}} \frac{6}{\sqrt{3-4x^2}} dx = \int_0^{\frac{3}{4}} \frac{6}{\sqrt{4(\frac{3}{4}-x^2)}} dx = \int_0^{\frac{3}{4}} \frac{6}{2\sqrt{\frac{3}{4}-x^2}} dx$$

$$= \int_0^{\frac{3}{4}} \frac{3}{\sqrt{\frac{3}{4}-x^2}} dx = \text{SOMEONE ASKING: INVERSE} = \left[3 \arcsin\left(\frac{\sqrt{x}}{\sqrt{\frac{3}{4}}}\right) \right]_0^{\frac{3}{4}}$$

$$= 3 \left[\arcsin\left(\frac{\sqrt{\frac{3}{4}}}{\sqrt{\frac{3}{4}}}\right) - \arcsin(0) \right] = 3 \arcsin\left(\frac{\sqrt{\frac{3}{4}}}{\sqrt{\frac{3}{4}}}\right)$$

$$= 3 \times \frac{\pi}{2} = \frac{3\pi}{2}$$

Question 10 (+)**By using a suitable substitution, find in terms of π , the value of

$$\int_0^1 \frac{1}{(x+1)\sqrt{x}} dx.$$

 $\frac{\pi}{2}$

Handwritten solution for Question 10:

$$\int_0^1 \frac{1}{(x+1)\sqrt{x}} dx = \dots \text{by substitution} \dots$$

$$= \int_0^1 \frac{1}{u(u^2+1)} (2u du) = \int_0^1 \frac{2}{u^2+1} du$$

$$= \left[2 \arctan u \right]_0^1 = 2 \arctan 1 - 2 \arctan 0$$

$$= 2 \times \frac{\pi}{4} = \frac{\pi}{2}$$

Boxed notes:

$$\begin{aligned} u &= \sqrt{x} \\ u^2 &= x \\ 2u \frac{du}{dx} &= 1 \\ 2u du &= dx \\ x=0 &\rightarrow u=0 \\ x=1 &\rightarrow u=1 \end{aligned}$$

Question 11 (**+)

Show clearly that

$$\int \frac{4x+1}{\sqrt{4x^2-9}} dx = f(x) + \frac{1}{2} \ln[2x + f(x)] + C,$$

where $f(x)$ is a function to be found.

$$f(x) = \sqrt{4x^2 - 9}$$

$$\begin{aligned} \int \frac{4x+1}{\sqrt{4x^2-9}} dx &= \int \frac{4x}{\sqrt{4x^2-9}} + \frac{1}{\sqrt{4x^2-9}} dx \\ &= \int \frac{4x}{\sqrt{4(x^2-\frac{9}{4})}} + \frac{1}{\sqrt{4(x^2-\frac{9}{4})}} dx \\ &= \int \frac{4x}{2\sqrt{x^2-\frac{9}{4}}} + \frac{1}{2\sqrt{x^2-\frac{9}{4}}} dx \\ &= \frac{4x}{2} \cdot \frac{1}{\sqrt{x^2-\frac{9}{4}}} + \frac{1}{2} \ln\left(\frac{2x + \sqrt{4x^2-9}}{2}\right) + C \\ &= \sqrt{4x^2-9} + \frac{1}{2} \ln\left(\frac{2x + \sqrt{4x^2-9}}{2}\right) + C \\ &= \sqrt{4x^2-9} + \frac{1}{2} \ln\left(\frac{2x + \sqrt{4x^2-9}}{2}\right) + C \\ &= \sqrt{4x^2-9} + \frac{1}{2} \ln(2x + \sqrt{4x^2-9}) + C \end{aligned}$$

Question 12 (**+)

By using a suitable substitution, or otherwise, find

$$\int \frac{1}{(1+x^2) \arctan x} dx$$

$$\ln|\arctan x| + C$$

$$\begin{aligned} \int \frac{1}{(1+x^2) \arctan x} dx &= \text{by recognizing or substitution} \\ &= \int \frac{1}{(1+x^2) u} du = \int \frac{1}{u} du = \ln|u| + C \\ &= \ln|\arctan x| + C \end{aligned}$$

Question 13 (**+)

$$f(x) \equiv \frac{x^2 + 3x + 36}{(x+9)(x^2+9)}$$

a) Express $f(x)$ into partial fractions.

b) Hence find

$$\int f(x) dx.$$

$$\frac{1}{x+9} + \frac{3}{x^2+9}, \quad \ln|x+9| + \arctan \frac{x}{3} + C$$

Handwritten solution for Question 13:

(a) $\frac{x^2+3x+36}{(x+9)(x^2+9)} \equiv \frac{A}{x+9} + \frac{Bx+C}{x^2+9}$
 $x^2+3x+36 \equiv A(x^2+9) + (x+9)(Bx+C)$
 If $x = -9$, $90 = 90A \Rightarrow A = 1$
 If $x = 0$, $36 = 9A + 9C$
 $4 = A + C$
 $C = 3$
 If $x = -8$, $76 = 73A - 68B + 3$
 $73 = 73 - 68B + 3$
 $B = 0$
 $\therefore f(x) = \frac{1}{x+9} + \frac{3}{x^2+9}$
 (b) $\int f(x) dx = \int \frac{1}{x+9} + \frac{3}{x^2+9} dx = \ln|x+9| + \frac{3}{3} \arctan \frac{x}{3} + C$
 $= \ln|x+9| + \arctan \frac{x}{3} + C$

Question 14 (***)

Use the substitution $t = x - 8$ to find the exact value of

$$\int_8^{8.75} \frac{1}{\sqrt{x^2 - 16x + 65}} dx,$$

giving the answer as a single natural logarithm.

 $\ln 2$

Handwritten solution for Question 14:

$$\int_8^{8.75} \frac{1}{\sqrt{x^2 - 16x + 65}} dx = \dots \text{ USING THE SUBSTITUTION GIVEN}$$

$$= \int_0^{0.75} \frac{1}{\sqrt{(t+8)^2 - 16(t+8) + 65}} dt$$

$$= \int_0^{0.75} \frac{1}{\sqrt{t^2 + 16t + 64 - 16t - 128 + 65}} dt = \int_0^{0.75} \frac{1}{\sqrt{t^2 + 1}} dt$$

$$= \left[\operatorname{arcsinh} t \right]_0^{0.75} = \operatorname{arcsinh} \frac{3}{4} - \operatorname{arcsinh} 0 = \ln \left(\frac{3}{4} + \sqrt{\frac{9}{16} + 1} \right)$$

$$= \ln \left(\frac{3}{4} + \sqrt{\frac{25}{16}} \right) = \ln \left(\frac{3}{4} + \frac{5}{4} \right) = \ln 2$$

Side notes: $t = x - 8$, $\frac{dt}{dx} = 1$, $dx = dt$, $x = 8 \Rightarrow t = 0$, $x = 8.75 \Rightarrow t = 0.75$

Question 15 (***)

$$f(x) = \sinh x \cos x + \sin x \cosh x, \quad x \in \mathbb{R}.$$

a) Find a simplified expression for $f'(x)$.

b) Use the answer to part (a) to find

$$\int \frac{2}{\tanh x + \tan x} dx.$$

$$\boxed{\ln 2}, \quad \boxed{f'(x) = 2 \cosh x \cos x}, \quad \boxed{\ln |\sinh x \cos x + \sin x \cosh x| + C}$$

Handwritten solution for Question 15:

a) $f(x) = \sinh x \cos x + \sin x \cosh x$
 $f'(x) = \cosh x \cos x + \sinh x (-\sin x) + \cos x \cosh x + \sin x \sinh x$
 $f'(x) = 2 \cosh x \cos x$

b) $\int \frac{2}{\tanh x + \tan x} dx = \int \frac{2}{\frac{\sinh x}{\cosh x} + \frac{\sin x}{\cos x}} dx$
 Multiply top & bottom of the fraction by $\cosh x \cos x$
 $= \int \frac{2 \cosh x \cos x}{\sinh x \cos x + \sin x \cosh x} dx$
 which is of the form $\int \frac{f'(x)}{f(x)} dx$
 $= \ln |\sinh x \cos x + \sin x \cosh x| + C$

Question 16 (*)**

Find the exact value of

$$\int_{2.5}^{7.5} \frac{15\sqrt{3}}{4x^2 + 75} dx.$$

$$\frac{\pi}{4}$$

$$\int_{2.5}^{7.5} \frac{15\sqrt{3}}{4x^2 + 75} dx = \text{Divide top \& bottom of the integrand by 4} \dots$$

$$= \int_{2.5}^{7.5} \frac{\frac{15\sqrt{3}}{4}}{x^2 + \frac{75}{4}} dx = \frac{15\sqrt{3}}{4} \int_{2.5}^{7.5} \frac{1}{x^2 + \left(\frac{\sqrt{75}}{2}\right)^2} dx = \dots \text{standard integral} \dots$$

$$= \frac{15\sqrt{3}}{4} \times \frac{1}{\frac{\sqrt{75}}{2}} \times \left[\arctan\left(\frac{2x}{\sqrt{75}}\right) \right]_{2.5}^{7.5} = \frac{15\sqrt{3}}{4} \times \frac{2}{\sqrt{75}} \times \left[\arctan\left(\frac{2x}{\sqrt{75}}\right) \right]_{2.5}^{7.5}$$

$$= \frac{15\sqrt{3}}{4} \times \frac{2}{\sqrt{75}} \times \left[\arctan\left(\frac{15}{\sqrt{75}}\right) - \arctan\left(\frac{5}{\sqrt{75}}\right) \right] = \frac{3}{2} \left[\arctan\sqrt{3} - \arctan\frac{1}{\sqrt{3}} \right]$$

$$= \frac{3}{2} \left[\frac{\pi}{3} - \frac{\pi}{6} \right] = \frac{3}{2} \times \frac{\pi}{6} = \frac{\pi}{4}$$

Question 17 (*)**

Find the exact value of

$$\int_0^1 \frac{2\sqrt{3}}{\sqrt{4\pi^2 - 3\pi^2 x^2}} dx.$$

$$\frac{2}{3}$$

$$\int_0^1 \frac{2\sqrt{3}}{\sqrt{4\pi^2 - 3\pi^2 x^2}} dx = \int_0^1 \frac{2\sqrt{3}}{\sqrt{\pi^2 \left(\frac{4}{3} - x^2\right)}} dx = \int_0^1 \frac{2\sqrt{3}}{\sqrt{\frac{4}{3} - x^2}} dx$$

$$= \frac{2}{\sqrt{3}} \int_0^1 \frac{1}{\sqrt{\left(\frac{2}{3}\right)^2 - x^2}} dx = \left[\frac{2}{\sqrt{3}} \arcsin\left(\frac{x}{\frac{2}{3}}\right) \right]_0^1$$

$$= \frac{2}{\sqrt{3}} \left[\arcsin\left(\frac{\sqrt{3}}{2}\right) - \arcsin 0 \right]$$

$$= \frac{2}{\sqrt{3}} \times \frac{\pi}{3} = \frac{2}{3}$$

Question 18 (***)

Find the exact value of each of the following integrals.

a) $\int_{-3}^{-2} \frac{1}{\sqrt{-x^2 - 6x - 5}} dx.$

b) $\int_{-3}^{-2} \frac{x}{\sqrt{-x^2 - 6x - 5}} dx.$

$$\frac{\pi}{6}, \quad \sqrt{3} + \frac{\pi}{2} - 2$$

(a) $\int_{-3}^{-2} \frac{1}{\sqrt{-x^2 - 6x - 5}} dx = \int_{-3}^{-2} \frac{1}{\sqrt{-(x^2 + 6x + 5)}} dx = \int_{-3}^{-2} \frac{1}{\sqrt{-(x+3)^2 + 4}} dx$
 $= \int_{-3}^{-2} \frac{1}{\sqrt{4 - (x+3)^2}} dx = \dots$ substitution $u = x+3 \quad \frac{du}{dx} = 1 \quad du = dx$
 $= \int_0^1 \frac{1}{\sqrt{4 - u^2}} du = \left[\arcsin \frac{u}{2} \right]_0^1$
 $= \arcsin \frac{1}{2} - \arcsin 0 = \frac{\pi}{6}$

(b) $\int_{-3}^{-2} \frac{x}{\sqrt{-x^2 - 6x - 5}} dx = \dots$ as in part (a) $= \dots$ substitution
 $= \int_0^1 \frac{u-3}{\sqrt{4 - u^2}} du = \int_0^1 \frac{u}{\sqrt{4 - u^2}} du - 3 \int_0^1 \frac{1}{\sqrt{4 - u^2}} du$
 $= \int_0^1 u(4 - u^2)^{-\frac{1}{2}} du - 3 \int_0^1 \frac{1}{\sqrt{4 - u^2}} du$
 $= \left[-\frac{1}{3}(4 - u^2)^{\frac{1}{2}} - 3 \arcsin \frac{u}{2} \right]_0^1 = \left[\sqrt{4 - u^2} + 3 \arcsin \frac{u}{2} \right]_0^1$
 $= (\sqrt{3} + 3 \arcsin \frac{1}{2}) - (2 + 3 \arcsin 0) = \sqrt{3} + \frac{\pi}{2} - 2$

Question 19 (***)

By using the substitution $x = 9 \sin^2 \theta$, or otherwise, find the exact value of

$$\int_0^{\frac{9}{4}} \frac{1}{\sqrt{x(9-x)}} dx.$$

$$\frac{\pi}{3}$$

$\int_0^{\frac{9}{4}} \frac{1}{\sqrt{x(9-x)}} dx = \dots$ by substitution
 $= \int_0^{\frac{\pi}{6}} \frac{1}{\sqrt{9 \sin^2 \theta (1 - \sin^2 \theta)}} \cdot 18 \sin \theta \cos \theta d\theta$
 $= \int_0^{\frac{\pi}{6}} \frac{18 \sin \theta \cos \theta}{\sqrt{81 \sin^2 \theta \cos^2 \theta}} d\theta = \int_0^{\frac{\pi}{6}} \frac{18 \sin \theta \cos \theta}{9 \sin \theta \cos \theta} d\theta$
 $= \int_0^{\frac{\pi}{6}} 2 d\theta = \left[2\theta \right]_0^{\frac{\pi}{6}} = \frac{\pi}{3}$

$x = 9 \sin^2 \theta$
 $\frac{dx}{d\theta} = 18 \sin \theta \cos \theta$
 $x = 0, \theta = 0$
 $x = \frac{9}{4}, \frac{9}{4} = 9 \sin^2 \theta$
 $\theta = \frac{\pi}{6}$

Question 20 (***)

Find the exact value of

$$\int_{\frac{4}{3}}^{\frac{5}{3}} \frac{x+1}{\sqrt{9x^2-16}} dx.$$

$$\boxed{\frac{1}{3}(1-\ln 2)}$$

$$\begin{aligned} \int_{\frac{4}{3}}^{\frac{5}{3}} \frac{x+1}{\sqrt{9x^2-16}} dx &= \int_{\frac{4}{3}}^{\frac{5}{3}} \frac{x}{\sqrt{9x^2-16}} dx + \int_{\frac{4}{3}}^{\frac{5}{3}} \frac{1}{\sqrt{9x^2-16}} dx \\ &= \int_{\frac{4}{3}}^{\frac{5}{3}} \frac{x(3x^2-16)^{-\frac{1}{2}}}{dx} + \int_{\frac{4}{3}}^{\frac{5}{3}} \frac{1}{3\sqrt{x^2-\frac{16}{9}}} dx \\ &= \left[\frac{1}{3}(3x^2-16)^{\frac{1}{2}} - \frac{1}{3} \operatorname{arccosh}\left(\frac{3x}{4}\right) \right]_{\frac{4}{3}}^{\frac{5}{3}} \\ &= \left[\frac{1}{3}\sqrt{9x^2-16} - \frac{1}{3} \operatorname{arccosh}\left(\frac{3x}{4}\right) \right]_{\frac{4}{3}}^{\frac{5}{3}} \\ &= \left(\frac{1}{3} - \frac{1}{3} \operatorname{arccosh}\left(\frac{5}{4}\right) \right) - \left(0 - \frac{1}{3} \operatorname{arccosh}(1) \right) \\ &= \frac{1}{3} \left(1 - \operatorname{arccosh}\left(\frac{5}{4}\right) \right) = \frac{1}{3} \left[1 - \ln\left(\frac{5}{4} + \sqrt{\frac{25}{16} - 1}\right) \right] \\ &= \frac{1}{3} \left[1 - \ln\left(\frac{5}{4} + \frac{3}{4}\right) \right] = \frac{1}{3} [1 - \ln 2] // \end{aligned}$$

Question 21 (***)

Find the exact value of each of the following integrals.

a) $\int_5^7 \frac{1}{x^2-10x+29} dx.$

b) $\int_5^7 \frac{x}{\sqrt{x^2-10x+29}} dx.$

$$\boxed{\frac{\pi}{8}}, \quad \boxed{\ln(1+\sqrt{2})}$$

$$\begin{aligned} \text{a) } \int_5^7 \frac{1}{x^2-10x+29} dx &= \int_5^7 \frac{1}{(x-5)^2-25+29} dx = \int_5^7 \frac{1}{(x-5)^2+4} dx \\ &= \int_0^2 \frac{1}{u^2+4} du = \int_0^2 \frac{1}{4(u^2+1)} du = \left[\frac{1}{4} \operatorname{arctan} u \right]_0^2 \\ &= \frac{1}{4} \operatorname{arctan} 2 - \frac{1}{4} \operatorname{arctan} 0 = \frac{1}{4} \times \frac{\pi}{4} = \frac{\pi}{16} // \\ \text{b) } \int_5^7 \frac{x}{\sqrt{x^2-10x+29}} dx &= \dots \text{ as in part (a) involving the substitution} \\ &= \int_0^2 \frac{1}{\sqrt{u^2+1}} du = \left[\operatorname{arcsinh} u \right]_0^2 \\ &= \operatorname{arcsinh}(2) - \operatorname{arcsinh}(0) = \ln(1+\sqrt{2}) // \end{aligned}$$

Question 22 (***)

Use the substitution $u = e^x$ to find

$$\int \frac{\sqrt{e^x}}{\sqrt{e^x + e^{-x}}} dx.$$

$$\boxed{\operatorname{arsinh}(e^x) + C}$$

$$\begin{aligned} \int \frac{\sqrt{e^x}}{\sqrt{e^x + e^{-x}}} dx &= \dots \text{by substitution} \dots = \int \frac{\sqrt{u}}{\sqrt{u + u^{-1}}} \frac{du}{u} \\ &= \int \frac{\sqrt{u}}{\sqrt{u + \frac{1}{u}}} \frac{du}{u} = \int \frac{\sqrt{u}}{\sqrt{\frac{u^2 + 1}{u}}} \frac{du}{u} = \int \frac{\sqrt{u}}{\sqrt{u^2 + 1}} \frac{du}{u} \\ &= \int \frac{\sqrt{u}}{\sqrt{u^2 + 1}} \frac{du}{u} = \operatorname{arsinh}\left(\frac{1}{u}\right) + C = \operatorname{arsinh}(e^x) + C \end{aligned}$$

$u = e^x$
 $\frac{du}{dx} = e^x$
 $\frac{du}{dx} = u$

Question 23 (***)

Find in exact simplified form in terms of natural logarithms

$$\int_3^6 \frac{1}{2x+6} \sqrt{\frac{x+3}{x-2}} dx.$$

$$\boxed{\frac{1}{2} \ln(2 + \sqrt{3})}$$

$$\begin{aligned} \int_3^6 \frac{1}{2x+6} \sqrt{\frac{x+3}{x-2}} dx &= \frac{1}{2} \int_3^6 \frac{1}{x+3} \left(\frac{x+3}{x-2}\right)^{\frac{1}{2}} dx \\ &= \frac{1}{2} \int_3^6 \frac{1}{\sqrt{(x+3)(x-2)}} dx = \frac{1}{2} \int_3^6 \frac{1}{\sqrt{x^2 - 9}} dx \\ &= \frac{1}{2} \left[\operatorname{arccosh} \frac{x}{3} \right]_3^6 = \frac{1}{2} \left[\operatorname{arccosh} 2 - \frac{1}{2} \operatorname{arccosh} 1 \right] \\ &= \frac{1}{2} \ln(2 + \sqrt{2^2 - 1}) = \frac{1}{2} \ln(2 + \sqrt{3}) \end{aligned}$$

Question 24 (***)

Show that the exact value of the following integral

$$\int_1^2 \frac{3x^2 + 1}{2x^3 + x} dx$$

is $\frac{1}{4} \ln 48$., proof

PROCEED TO TRY

$$\int_1^2 \frac{3x^2 + 1}{2x^3 + x} dx = \int_1^2 \frac{3x^2 + 1}{x(2x^2 + 1)} dx$$

WRITE WITH PARTIAL FRACTIONS

$$\frac{3x^2 + 1}{x(2x^2 + 1)} = \frac{A}{x} + \frac{Bx + C}{2x^2 + 1}$$

$$3x^2 + 1 = A(2x^2 + 1) + (Bx + C)x$$

$$3x^2 + 1 = 2Ax^2 + A + Bx^2 + Cx$$

$$3x^2 + 1 = (2A + B)x^2 + Cx + A$$

- $A = 1$
- $C = 0$
- $2A + B = 3$
- $B = 1$

DEPEND ON THE INTEGRAL

$$\int_1^2 \left(\frac{1}{x} + \frac{x}{2x^2 + 1} \right) dx = \left[\ln|x| + \frac{1}{2} \ln(2x^2 + 1) \right]_1^2$$

$$= \left(\ln 2 + \frac{1}{2} \ln 5 \right) - \left(\ln 1 + \frac{1}{2} \ln 3 \right)$$

$$= \ln 2 + \frac{1}{2} \ln 5 - \frac{1}{2} \ln 3$$

$$= \ln 2 + \frac{1}{2} \ln 3$$

$$= \frac{1}{2} [\ln 4 + \ln 3]$$

$$= \frac{1}{2} \ln 12$$

Question 25 (***)

$$\int_0^1 \frac{x \arcsin x}{\sqrt{1-x^2}} dx.$$

Show that value of the above definite integral is 1.

proof

Handwritten proof for Question 25. The main calculation shows integration by parts with $u = \arcsin x$ and $v = -\sqrt{1-x^2}$. A table on the right lists derivatives: $\frac{d}{dx} \arcsin x = \frac{1}{\sqrt{1-x^2}}$ and $\frac{d}{dx} \sqrt{1-x^2} = \frac{-x}{\sqrt{1-x^2}}$. The final result is $[-\arcsin x \sqrt{1-x^2} + \frac{1}{2} \ln|1-x^2|]_0^1 = 1$.

Question 26 (***)

$$f(x) = x \arctan x, \quad x \in \mathbb{R}.$$

- Find an expression for $f'(x)$.
- Use the answer to part (a) to find the exact value of

$$\int_0^1 4 \arctan x \, dx.$$

You may **not** use standard integration by parts to obtain the answer to part (b).

$$f'(x) = \frac{x}{1+x^2} + \arctan x, \quad \pi - \ln 4$$

Handwritten solution for Question 26 part (b). It defines $I(a) = \int_0^1 x \arctan(ax) dx$ and differentiates with respect to a to get $I'(a) = \int_0^1 \frac{x^2}{1+a^2x^2} dx$. This is then evaluated using partial fractions and limits as $a \rightarrow \infty$ to find $I(1) = \frac{\pi}{4} - \frac{\ln 4}{4}$.

Question 27 (***)

By using the substitution $x^2 = 3 \tan \theta$, or otherwise, find the exact value of

$$\int_0^{\sqrt{3}} \frac{x}{x^4 + 9} dx.$$

$$\frac{\pi}{24}$$

$$\begin{aligned} & \int \frac{x}{x^2+9} dx = \text{by substituting or magnitude} \\ & = \int \frac{x}{(\frac{1}{2} \ln(x^2+9))^{1/2}} \cdot \frac{3x}{2x} dx = \int \frac{3x^2}{2(9+x^2)^{1/2}} dx \\ & = \int \frac{3x^2}{16 \sec^2 \theta} d\theta = \int \frac{1}{6} d\theta = \left[\frac{\theta}{6} \right]_0^{\frac{\pi}{4}} \\ & = \frac{\frac{\pi}{4}}{6} - 0 = \frac{\pi}{24} \end{aligned}$$

Question 28 (***)

Use an appropriate substitution to find an exact value for the following integral.

$$\int_1^{\sqrt{e}} \frac{1}{x\sqrt{1-(\ln x)^2}} dx.$$

You may assume that the integral converges.

$$\boxed{}, \boxed{\frac{1}{6}\pi}$$

$$\begin{aligned} u &= \ln x \\ \frac{du}{dx} &= \frac{1}{x} & \int \frac{1}{x} dx &= \ln|x| + C \\ \frac{du}{dx} &= \frac{1}{x} & \int \frac{1}{x^2} dx &= -\frac{1}{x} + C \end{aligned}$$

Question 29 (***)

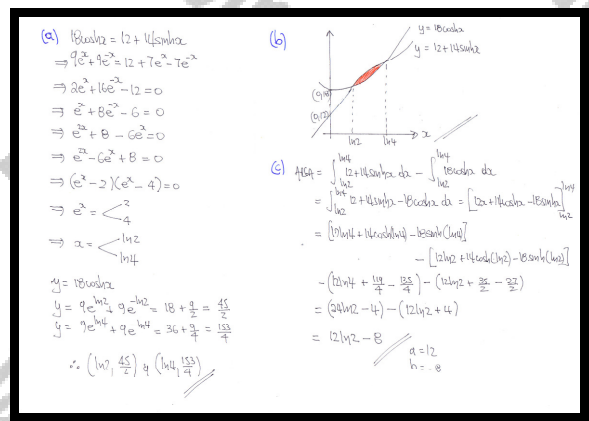
The curves C_1 and C_2 have respective equations

$$y = 18 \cosh x, \quad x \in \mathbb{R} \quad \text{and} \quad y = 12 + 14 \sinh x, \quad x \in \mathbb{R}.$$

- Find the exact coordinates of the points of intersection between C_1 and C_2 .
- Sketch in the same diagram the graph of C_1 and the graph of C_2 .
- Show that the finite region bounded by the graphs of C_1 and C_2 has an area of $a \ln 2 + b$,

where a and b are integers to be found.

$$\left(\ln 2, \frac{45}{2} \right) \text{ \& \; } \left(\ln 4, \frac{153}{4} \right), \quad 12 \ln 2 - 8$$



Question 30 (***)

$$f(x) \equiv \frac{4x}{1-x^4}.$$

- a) Express $f(x)$ into partial fractions.
- b) Hence find, as a single natural logarithm, the value of

$$\int_0^{\frac{1}{2}} f(x) \, dx.$$

$$\boxed{}, \quad \boxed{f(x) = \frac{1}{1-x} - \frac{1}{1+x} + \frac{2x}{1+x^2}}, \quad \boxed{\ln \frac{5}{3}}$$

a) STEP BY FACTORISING THE DENOMINATOR EQUALLY

$$f(x) = \frac{4x}{1-x^4} = \frac{4x}{(1-x^2)(1+x^2)} = \frac{4x}{(1-x)(1+x)(1+x^2)}$$

$$f(x) \equiv \frac{A}{1-x} + \frac{B}{1+x} + \frac{Cx+D}{1+x^2}$$

$$\frac{4x}{1-x^4} \equiv \frac{A(1+x)(1+x^2) + B(1-x)(1+x^2) + (Cx+D)(1-x)(1+x)}{(1-x)(1+x)(1+x^2)}$$

• IF $3=1$ • IF $3=-1$ • IF $3=0$

$$\begin{array}{l} 4 = A(2)(2) \\ 4 = A(4) \\ A = 1 \end{array} \quad \begin{array}{l} -4 = (2)(2)B \\ -4 = 4B \\ B = -1 \end{array} \quad \begin{array}{l} 0 = A+B+D \\ 0 = 1-1+D \\ D = 0 \end{array}$$

• COMPARING COEFFICIENTS OF x^2 ON BOTH SIDES

$$\begin{array}{l} 0 = A^2 - B^2 - C^2 \\ 0 = 1^2 - (-1)^2 - C^2 \\ \therefore C = 2 \end{array}$$

$$\therefore f(x) = \frac{1}{1-x} - \frac{1}{1+x} + \frac{2x}{1+x^2}$$

b) USING PART (a)

$$\int_0^{\frac{1}{2}} f(x) \, dx = \int_0^{\frac{1}{2}} \left(\frac{1}{1-x} - \frac{1}{1+x} + \frac{2x}{1+x^2} \right) dx$$

$$= \left[-\ln|1-x| - \ln|1+x| + \ln|1+x^2| \right]_0^{\frac{1}{2}}$$

$$= \left(-\ln \frac{1}{2} - \ln \frac{3}{2} + \ln \frac{5}{4} \right) - \left(-\ln 1 - \ln 1 + \ln 1 \right)$$

$$= \ln \frac{5}{4} - \ln \frac{3}{2} - \ln \frac{1}{2}$$

$$= \ln \left(\frac{5}{4} \cdot \frac{2}{3} \cdot 2 \right)$$

$$= \ln \left(\frac{5}{3} \right)$$

Question 31 (***)

$$f(x) = x \operatorname{arsinh}\left(\frac{1}{2}x\right), \quad x \in \mathbb{R}.$$

- a) Find a simplified expression for $f'(x)$.
- b) Use the answer to part (a) to show that

$$\int_0^{\sqrt{12}} \operatorname{arsinh}\left(\frac{1}{2}x\right) dx = 2\sqrt{3} \ln(2 + \sqrt{3}) - 2.$$

$$f'(x) = \operatorname{arsinh}\left(\frac{1}{2}x\right) + \frac{x}{\sqrt{x^2 + 4}}$$

(a) $f(x) = x \operatorname{arsinh}\left(\frac{1}{2}x\right)$
 $f'(x) = 1 \times \operatorname{arsinh}\left(\frac{1}{2}x\right) + x \times \frac{1}{\sqrt{\frac{1}{4}x^2 + 1}} \times \frac{1}{2}$
 $f'(x) = \operatorname{arsinh}\left(\frac{1}{2}x\right) + \frac{x}{\sqrt{\frac{1}{4}x^2 + 4}} \times \frac{1}{2}$
 $f'(x) = \operatorname{arsinh}\left(\frac{1}{2}x\right) + \frac{x}{\sqrt{x^2 + 4}}$

(b) Now $\frac{d}{dx} [x \operatorname{arsinh}\left(\frac{1}{2}x\right)] = \operatorname{arsinh}\left(\frac{1}{2}x\right) + \frac{x}{\sqrt{x^2 + 4}}$
 INTEGRATE USING SUMS
 $\int_0^{\sqrt{12}} \frac{d}{dx} [x \operatorname{arsinh}\left(\frac{1}{2}x\right)] dx = \int_0^{\sqrt{12}} \operatorname{arsinh}\left(\frac{1}{2}x\right) dx + \int_0^{\sqrt{12}} \frac{x}{\sqrt{x^2 + 4}} dx$
 $[x \operatorname{arsinh}\left(\frac{1}{2}x\right)]_0^{\sqrt{12}} = \int_0^{\sqrt{12}} \operatorname{arsinh}\left(\frac{1}{2}x\right) dx + \int_0^{\sqrt{12}} \frac{1}{2} (x^2 + 4)^{-\frac{1}{2}} dx$
 $\sqrt{12} \operatorname{arsinh}\left(\frac{1}{2}\sqrt{12}\right) = \int_0^{\sqrt{12}} \operatorname{arsinh}\left(\frac{1}{2}x\right) dx + (x-2)$
 $2\sqrt{3} \operatorname{arsinh}\left(\frac{\sqrt{3}}{2}\right) = \int_0^{\sqrt{12}} \operatorname{arsinh}\left(\frac{1}{2}x\right) dx + 2$
 $\therefore \int_0^{\sqrt{12}} \operatorname{arsinh}\left(\frac{1}{2}x\right) dx = 2\sqrt{3} \ln(2 + \sqrt{3}) - 2$
 (b) (14 marks)

Question 32 (***)

$$I = \int_1^4 \frac{3}{(x+9)\sqrt{x}} dx.$$

- a) By using a suitable substitution find an exact value for I .
- b) Show clearly that the answer to part (a) can be written as $2 \arctan \frac{3}{11}$.

$$\boxed{}, I = 2 \left(\arctan \frac{2}{3} - \arctan \frac{1}{3} \right)$$

a) USING THE SUBSTITUTION $u = \sqrt{x}$

$u = \sqrt{x}$	or	limits
$x = u^2$		$x=1 \rightarrow u=1$
$dx = 2u \, du$		$x=4 \rightarrow u=2$

$$\int_1^4 \frac{3}{(x+9)\sqrt{x}} dx = \int_1^2 \frac{3}{(u^2+9)u} (2u \, du) = \int_1^2 \frac{6}{u^2+9} du$$

★ SIMPLIFIED INTEGRAL

$$= \frac{1}{3} \times 6 \times \left[\arctan\left(\frac{u}{3}\right) \right]_1^2 = 2 \left[\arctan \frac{2}{3} - \arctan \frac{1}{3} \right]$$

b) USING THE IDENTITY $\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$

$$\tan \left(\arctan \frac{2}{3} - \arctan \frac{1}{3} \right) = \frac{\tan \left(\arctan \frac{2}{3} \right) - \tan \left(\arctan \frac{1}{3} \right)}{1 + \tan \left(\arctan \frac{2}{3} \right) \tan \left(\arctan \frac{1}{3} \right)}$$

$$= \frac{\frac{2}{3} - \frac{1}{3}}{1 + \frac{2}{3} \times \frac{1}{3}} = \frac{\frac{1}{3}}{1 + \frac{2}{9}} = \frac{\frac{1}{3}}{\frac{11}{9}} = \frac{3}{11}$$

$\therefore \arctan \frac{2}{3} - \arctan \frac{1}{3} = \arctan \frac{3}{11}$

$\therefore I = 2 \arctan \frac{3}{11}$

4 MARKS

Question 33 (***)

$$I = \int_0^{\frac{\pi}{3}} \frac{1}{1 + 8 \cos^2 x} dx.$$

- a) By using the substitution $t = \tan x$, or otherwise, show clearly that

$$I = \int_0^{\sqrt{3}} \frac{1}{9+t^2} dt.$$

- b) Hence find the exact value of I .

$$\frac{\pi}{18}$$

(a) $\int_0^{\frac{\pi}{2}} \frac{1}{8\omega \omega^4 + 1} dx = \dots$ by substitution

$$t = \tan x$$

$$\frac{dx}{dt} = \frac{1}{1+t^2}$$

$$dx = \frac{dt}{1+t^2}$$

$$2 = 0, \quad t = 0$$

$$2 = \frac{\pi}{2}, \quad t = \infty$$

$= \int_0^{\frac{\pi}{2}} \frac{1}{8\omega \omega^4 + 1} \cdot \frac{dt}{1+t^2}$

$= \int_0^{\frac{\pi}{2}} \frac{1}{8 + 16t^2} dt = \int_0^{\frac{\pi}{2}} \frac{1}{8(1+t^2)} dt$

$= \int_0^{\frac{\pi}{2}} \frac{1}{8(1+t^2)} dt = \int_0^{\frac{\pi}{2}} \frac{1}{8(1+t^2)} dt$

$= \int_0^{\frac{\pi}{2}} \frac{1}{8} \frac{dt}{1+t^2}$

~~do~~ $\frac{1}{8} \arctan t$

(b) $\dots = \frac{1}{8} \left[\arctan \frac{t}{1} \right]_0^{\frac{\pi}{2}} = \frac{1}{8} \left[\arctan \frac{\infty}{1} - \arctan 0 \right] = \frac{1}{8} \times \frac{\pi}{2} = \frac{\pi}{16}$

Question 34 (***)

By using the substitution $u = \cosh x - 1$, or otherwise, find the value of

$$\int_{\ln 2}^{\ln 3} \frac{\cosh x + 1}{\sinh x (\cosh x - 1)} dx.$$

$$\boxed{\frac{5}{2}}$$

$$\begin{aligned}
 & \int_{\ln 2}^{\ln 3} \frac{\cosh u + 1}{\sinh u (\cosh u - 1)} du \quad \dots \text{by substituting } \dots \\
 & \begin{array}{l} u = \cosh u - 1 \\ du = \sinh u \\ \frac{du}{du} = du \end{array} \\
 & = \int_{\frac{1}{2}}^{\frac{3}{2}} \frac{\cosh u + 1}{\sinh u} \cdot u \cdot \frac{du}{\sinh u} = \int_{\frac{1}{2}}^{\frac{3}{2}} \frac{\cosh u + 1}{u \sinh u} du \\
 & = \int_{\frac{1}{2}}^{\frac{3}{2}} \frac{\cosh u + 1}{u (\cosh u - 1)} du = \int_{\frac{1}{2}}^{\frac{3}{2}} \frac{\cosh u + 1}{u (\sinh u + 1) (\cosh u - 1)} du \\
 & \begin{array}{l} 2 = \ln 2 \rightarrow u = \frac{1}{2} \\ 3 = \ln 3 \rightarrow u = \frac{3}{2} \end{array} \\
 & = \int_{\frac{1}{2}}^{\frac{3}{2}} \frac{1}{u (\cosh u - 1)} du = \int_{\frac{1}{2}}^{\frac{3}{2}} \frac{1}{u^2} du \\
 & = \left[-\frac{1}{u} \right]_{\frac{1}{2}}^{\frac{3}{2}} = \left[-\frac{1}{u} \right]_{\frac{3}{2}}^{\frac{1}{2}} = -4 - \frac{-2}{3} = \frac{8}{3}
 \end{aligned}$$

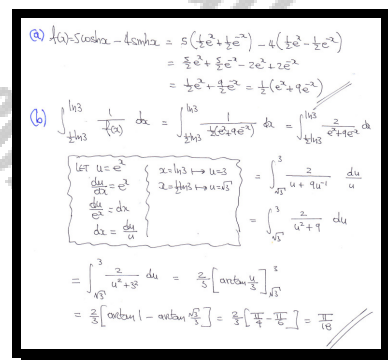
Question 35 (***)

$$f(x) = 5 \cosh x - 4 \sinh x, \quad x \in \mathbb{R}.$$

- a) Find a simplified expression for $f(x)$ in terms of e^x .
- b) Hence by using the substitution $u = e^x$, or otherwise, show that

$$\int_{\frac{1}{2} \ln 3}^{\ln 3} \frac{1}{f(x)} dx = \frac{\pi}{18}.$$

$$f(x) = \frac{1}{2}e^x + \frac{9}{2}e^{-x}$$



(a) $f(x) = 5 \cosh x - 4 \sinh x = 5 \left(\frac{e^x + e^{-x}}{2} \right) - 4 \left(\frac{e^x - e^{-x}}{2} \right)$
 $= \frac{5}{2}e^x + \frac{5}{2}e^{-x} - 2e^x + 2e^{-x}$
 $= \frac{1}{2}e^x + \frac{9}{2}e^{-x}$

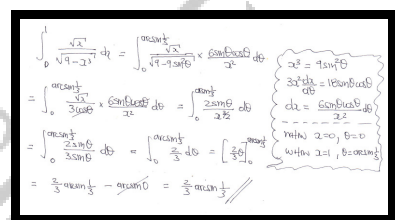
(b) $\int_{\frac{1}{2} \ln 3}^{\ln 3} \frac{1}{f(x)} dx = \int_{\frac{1}{2} \ln 3}^{\ln 3} \frac{1}{\frac{1}{2}e^x + \frac{9}{2}e^{-x}} dx = \int_{\frac{1}{2} \ln 3}^{\ln 3} \frac{2}{e^x + 9e^{-x}} dx$
 Let $u = e^x$, then $\frac{du}{dx} = e^x = u$, so $dx = \frac{du}{u}$.
 When $x = \frac{1}{2} \ln 3$, $u = \sqrt{3}$. When $x = \ln 3$, $u = 3$.
 $\int_{\sqrt{3}}^3 \frac{2}{u^2 + 9} \frac{du}{u} = \int_{\sqrt{3}}^3 \frac{2}{u^3 + 9u} du = \int_{\sqrt{3}}^3 \frac{2}{u^2(u^2 + 9)} du$
 $= \int_{\sqrt{3}}^3 \frac{2}{u^2(u^2 + 9)} du = \frac{2}{9} \left[\arctan \frac{u}{3} \right]_{\sqrt{3}}^3$
 $= \frac{2}{9} \left[\arctan 1 - \arctan \frac{\sqrt{3}}{3} \right] = \frac{2}{9} \left[\frac{\pi}{4} - \frac{\pi}{6} \right] = \frac{\pi}{18}$

Question 36 (***)

By using the substitution $x^3 = 9 \sin^2 \theta$, or otherwise, find the exact value of

$$\int_0^1 \frac{\sqrt{x}}{\sqrt{9-x^3}} dx.$$

$$\frac{2}{3} \arcsin\left(\frac{1}{3}\right)$$



$\int_0^1 \frac{\sqrt{x}}{\sqrt{9-x^3}} dx = \int_0^{\arcsin(1/3)} \frac{\sqrt{9 \sin^2 \theta}}{\sqrt{9-9 \sin^2 \theta}} \cdot \frac{3 \sin \theta \cos \theta}{2} d\theta$
 $= \int_0^{\arcsin(1/3)} \frac{3 \sin \theta}{2 \cos \theta} \cdot \frac{3 \sin \theta \cos \theta}{2} d\theta = \int_0^{\arcsin(1/3)} \frac{9 \sin^2 \theta}{4} d\theta$
 $= \frac{9}{4} \int_0^{\arcsin(1/3)} \sin^2 \theta d\theta = \frac{9}{4} \int_0^{\arcsin(1/3)} \frac{1 - \cos 2\theta}{2} d\theta$
 $= \frac{9}{8} \left[\theta - \frac{\sin 2\theta}{2} \right]_0^{\arcsin(1/3)} = \frac{9}{8} \left[\arcsin\left(\frac{1}{3}\right) - \frac{\sin(2 \arcsin(1/3))}{2} \right]$
 Note: $\sin(2 \arcsin(1/3)) = 2 \sin(\arcsin(1/3)) \cos(\arcsin(1/3)) = \frac{2}{3} \cdot \frac{\sqrt{8}}{3} = \frac{2\sqrt{8}}{9}$
 $= \frac{9}{8} \left[\arcsin\left(\frac{1}{3}\right) - \frac{\sqrt{8}}{3} \right]$

Question 37 (***)

$$f(x) \equiv (2x^2 - 1)\arcsin x + x\sqrt{1-x^2}, \quad -1 \leq x \leq 1.$$

- a) Find a simplified expression for $f'(x)$.
- b) Hence find

$$\int_0^{\frac{\sqrt{2}}{2}} x \arcsin x \, dx.$$

$$\boxed{}, \quad \boxed{f'(x) = 4x \arcsin x}, \quad \boxed{\frac{1}{8}}$$

a) DIFFERENTIATE THE SUM OF THE TWO PRODUCTS

$$f(x) = (2x^2 - 1)\arcsin x + x\sqrt{1-x^2}$$

$$f'(x) = 4x \arcsin x + (2x^2 - 1) \cdot \frac{1}{\sqrt{1-x^2}} + 1 \cdot (-x)^{-\frac{1}{2}} + x \cdot \frac{1}{2}(-x^2)^{-\frac{1}{2}}(-2x)$$

$$= 4x \arcsin x + \frac{2x^2 - 1}{\sqrt{1-x^2}} + (-x)^{-\frac{1}{2}} - \frac{x^2}{\sqrt{1-x^2}}$$

$$= 4x \arcsin x + \frac{2x^2 - 1}{\sqrt{1-x^2}} + \frac{(1-x^2)^{-\frac{1}{2}}}{1} - \frac{x^2}{\sqrt{1-x^2}}$$

$$= 4x \arcsin x + \frac{2x^2 - 1 + 1 - x^2}{\sqrt{1-x^2}}$$

$$= 4x \arcsin x + \frac{x^2}{\sqrt{1-x^2}}$$

b) USING PART (a)

$$\int_0^{\frac{\sqrt{2}}{2}} x \arcsin x \, dx = \frac{1}{4} \int_0^{\frac{\sqrt{2}}{2}} 4x \arcsin x \, dx$$

$$= \frac{1}{4} \left[(2x^2 - 1)\arcsin x + x\sqrt{1-x^2} \right]_0^{\frac{\sqrt{2}}{2}}$$

$$= \frac{1}{4} \left[\left(0 + \frac{1}{2}\right) - (0 - 0) \right]$$

$$= \frac{1}{8}$$

Question 38 (***)

By using the substitution $u = \sqrt{e^x - 1}$, or otherwise, find the exact value of

$$\int_0^{\ln 2} \sqrt{e^x - 1} \, dx.$$

$$2 - \frac{\pi}{2}$$

Question 39 (***)

Find in exact simplified form the value of

$$\int_0^{\ln 2} \frac{e^x}{\cosh x} \, dx.$$

$$\boxed{}, \ln \frac{5}{2}$$

Question 40 (***)

$$f(x) = \frac{2}{(x-1)^2(x^2+1)}, \quad x \neq 0.$$

Use partial fractions to show that

$$\int_2^3 f(x) \, dx = \frac{1}{2}(1 - \ln 2).$$

proof

$f(x) = \frac{2}{(x-1)^2(x^2+1)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{Cx+D}{x^2+1}$
 $\therefore 2 = A(x-1)(x^2+1) + B(x^2+1) + (x-1)^2(Cx+D)$
 • If $x=1 \Rightarrow 2 = 2B \Rightarrow B=1$
 • If $x=0 \Rightarrow 2 = A+B+D \Rightarrow A+D=1$ (i)
 • If $x=2 \Rightarrow 2 = 5A+5B+2C+D \Rightarrow 2 = 5A+5+2C+D \Rightarrow 5A+2C+D=-3$ (ii)
 • If $x=-1 \Rightarrow 2 = -4A+2B+4(D-C) \Rightarrow 2 = -4A+2+4D-4C \Rightarrow 4A+4C-4D=0 \Rightarrow A+C-D=0$ (iii)
 (i) $\Rightarrow D=1+A$
 (ii) $5A+2C+(1+A)=-3 \Rightarrow 6A+2C=-4$
 (iii) $A+C-(1+A)=0 \Rightarrow C=1$
 $6A+2(1)=-4 \Rightarrow 6A=-6 \Rightarrow A=-1$
 $D=1+A=0$
 $\therefore f(x) = \frac{1}{x-1} - \frac{1}{(x-1)^2} + \frac{x}{x^2+1}$
 $\int_2^3 \left(\frac{1}{x-1} - \frac{1}{(x-1)^2} + \frac{x}{x^2+1} \right) dx$
 $= \left[\ln|x-1| + \frac{1}{x-1} + \frac{1}{2} \ln(x^2+1) \right]_2^3$
 $= \left[\ln 2 + \frac{1}{2} + \frac{1}{2} \ln 5 \right] - \left[\ln 1 + \frac{1}{1} + \frac{1}{2} \ln 2 \right]$
 $= \ln 2 + \frac{1}{2} + \frac{1}{2} \ln 5 - 1 - \frac{1}{2} \ln 2$
 $= \frac{1}{2} - \ln 2 + \frac{1}{2} \ln 5$
 $= \frac{1}{2} (1 - \ln 2)$

Question 41 (***)

Use an appropriate substitution to find an exact value for the following integral.

$$\int_{\frac{1}{2}}^{\frac{1}{2}\sqrt{3}} \frac{1 - \sqrt{\arcsin x}}{\sqrt{1-x^2} \arcsin x} dx.$$

$$\boxed{}, \ln 2 - 2\left(\sqrt{\frac{1}{3}}\pi - \sqrt{\frac{1}{6}}\pi\right)$$

By substituting $u = \arcsin x$

$$u = \arcsin x$$

$$2u \frac{du}{dx} = \frac{1}{\sqrt{1-x^2}}$$

$$dx = 2u \sqrt{1-x^2} du$$

$$x = \frac{1}{2} \rightarrow u = \frac{\pi}{6}$$

$$x = \frac{1}{2}\sqrt{3} \rightarrow u = \frac{\pi}{3}$$

By transforming the integral:

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1-u}{\sqrt{1-x^2} \arcsin x} (2u \sqrt{1-x^2} du)$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{2-u}{u} du = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{2}{u} - 2 du$$

$$= [2 \ln |u| - 2u]_{\frac{\pi}{6}}^{\frac{\pi}{3}} = (2 \ln \sqrt{3} - 2(\frac{\pi}{3})) - (2 \ln \frac{\pi}{6} - 2(\frac{\pi}{6}))$$

$$= \ln 3 - 2\sqrt{3} - \ln \frac{\pi}{6} + 2\sqrt{\frac{\pi}{6}}$$

$$= (\ln 3 + \ln \frac{6}{\pi}) - 2\left(\sqrt{3} - \sqrt{\frac{\pi}{6}}\right)$$

$$= \ln 2 - 2\left(\sqrt{\frac{1}{3}}\pi - \sqrt{\frac{1}{6}}\pi\right)$$

Question 42 (****)

Find the exact value of

$$\int_0^1 \frac{1-3x^3}{\sqrt{1-x^2}} dx.$$

$$\frac{1}{2}(\pi-4)$$

Handwritten solution for Question 42:

$$\begin{aligned} \int_0^1 \frac{1-3x^3}{\sqrt{1-x^2}} dx &= \int_0^1 \frac{1}{\sqrt{1-x^2}} dx - \int_0^1 \frac{3x^3}{\sqrt{1-x^2}} dx \\ &= \left[\arcsin x \right]_0^1 - \int_1^0 \frac{3x^3}{\sqrt{1-x^2}} dx \\ &= \left(\frac{\pi}{2} - 0 \right) + \int_1^0 \frac{3x^3}{\sqrt{1-x^2}} dx \\ &= \frac{\pi}{2} + \int_1^0 \frac{3(-u^2)}{\sqrt{1-u^2}} du \\ &= \frac{\pi}{2} + \int_0^1 \frac{3-u^3}{\sqrt{1-u^2}} du \\ &= \frac{\pi}{2} + \left[3u - u^3 \right]_0^1 \\ &= \frac{\pi}{2} + (0) - (2) \\ &= \frac{\pi}{2} - 2 = \frac{1}{2}(\pi-4) \end{aligned}$$

Substitution: $u = \sqrt{1-x^2}$
 $u^2 = 1-x^2$
 $2u \frac{du}{dx} = -2x$
 $\frac{du}{dx} = -\frac{x}{u}$
 $x = 1 \quad u = 0$
 $x = 0 \quad u = 1$

Question 43 (****)

Use a suitable trigonometric substitution to find an integrated expression for

$$\int \frac{9}{(9-x^2)^{\frac{3}{2}}} dx.$$

$$\frac{x}{\sqrt{9-x^2}} + C$$

Handwritten solution for Question 43:

$$\begin{aligned} \int \frac{9}{(9-x^2)^{\frac{3}{2}}} dx &= \dots \text{ trig substitution} \\ &= \int \frac{9}{[9-(3\sin\theta)^2]^{\frac{3}{2}}} (3\cos\theta d\theta) = \int \frac{27\cos\theta}{(9-9\sin^2\theta)^{\frac{3}{2}}} d\theta \\ &= \int \frac{27\cos\theta}{[9(1-\sin^2\theta)]^{\frac{3}{2}}} d\theta = \int \frac{27\cos\theta}{(9\cos^2\theta)^{\frac{3}{2}}} d\theta \\ &= \int \frac{27\cos\theta}{27\cos^3\theta} d\theta = \int \frac{1}{\cos^2\theta} d\theta = \int \sec^2\theta d\theta \\ &= \tan\theta + C = \frac{x}{\sqrt{9-x^2}} + C \end{aligned}$$

Substitution: $x = 3\sin\theta$
 $\frac{dx}{d\theta} = 3\cos\theta$
 $\frac{dx}{3} = \cos\theta d\theta$
 $\frac{dx}{3} = \cos\theta d\theta$
 $x = 3\sin\theta$
 $\theta = \arcsin\left(\frac{x}{3}\right)$
 $\cos\theta = \frac{\sqrt{9-x^2}}{3}$

Question 44 (****)

Use the substitution $t = \tan\left(\frac{x}{2}\right)$ to find the value of

$$\int_0^{\frac{2\pi}{3}} \frac{1}{5+4\cos x} dx.$$

$$\boxed{}, \frac{\pi}{9}$$

using the substitution (min)

$$t = \tan\left(\frac{x}{2}\right) \Rightarrow \frac{dt}{dx} = \frac{1}{2} \sec^2\left(\frac{x}{2}\right)$$

$$\frac{dt}{dx} = \frac{1}{2} [1 + \tan^2\left(\frac{x}{2}\right)]$$

$$\frac{dt}{dx} = \frac{1}{2} (1+t^2)$$

$$2 \frac{dt}{dx} = 1+t^2$$

$$dx = \frac{2}{1+t^2} dt$$

Also using the cosine double angle identity

$$\Rightarrow \cos x = \cos^2\left(\frac{x}{2}\right) - \sin^2\left(\frac{x}{2}\right)$$

$$\Rightarrow \cos x = \left(\frac{1}{1+t^2}\right)^2 - \left(\frac{t}{1+t^2}\right)^2$$

$$\Rightarrow \cos x = \frac{1-t^2}{1+t^2}$$

$$\Rightarrow \cos x = \frac{1-t^2}{1+t^2}$$

$$\Rightarrow 5 + \cos x = 5 + \frac{1-t^2}{1+t^2}$$

$$= \frac{5(1+t^2) + 1 - t^2}{1+t^2}$$

$$= \frac{6+4t^2}{1+t^2}$$

$\tan \frac{x}{2} = \frac{t}{1}$
 $\cos \frac{x}{2} = \frac{1}{\sqrt{1+t^2}}$
 $\sin \frac{x}{2} = \frac{t}{\sqrt{1+t^2}}$

finally the limits if $t = \tan\left(\frac{x}{2}\right)$

$$x=0 \mapsto t=0$$

$$x=\frac{2\pi}{3} \mapsto t=\sqrt{3}$$

Transforming the integral

$$\Rightarrow \int_0^{\frac{2\pi}{3}} \frac{1}{5+4\cos x} dx = \int_0^{\sqrt{3}} \frac{1}{\frac{6+4t^2}{1+t^2}} \times \frac{2}{1+t^2} dt$$

$$= \int_0^{\sqrt{3}} \frac{2}{6+4t^2} dt$$

This is a standard 'arctan type' integral

$$= \int_0^{\sqrt{3}} \frac{2}{4\left(\frac{3}{2} + t^2\right)} dt$$

$$= \left[\frac{2}{4} \arctan\left(\frac{t}{\sqrt{3/2}}\right) \right]_0^{\sqrt{3}}$$

$$= \frac{1}{2} \left[\arctan\left(\frac{\sqrt{3}}{\sqrt{3/2}}\right) - \arctan(0) \right]$$

$$= \frac{1}{2} \times \frac{\pi}{4}$$

$$= \frac{\pi}{8}$$

Question 45 (****)

Show that the exact value of the following integral

$$\int_0^1 \frac{x+3}{(x+1)(x^2+4x+5)} dx$$

is $\frac{1}{2} \ln 2$.

□, proof

START WITH PARTIAL FRACTIONS. NOTE: THAT x^2+4x+5 IS IRREDUCIBLE

$$\frac{x+3}{(x+1)(x^2+4x+5)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+4x+5}$$

$$x+3 = A(x^2+4x+5) + (x+1)(Bx+C)$$

$$x+3 = (A+B)x^2 + (4A+B+C)x + (5A+C)$$

$A+B=0$	$4A+B+C=1$	$5A+C=3$
$A=-B$	$-4B+B+C=1$	$-5B+C=3$
	$-3B+C=1$	$C=3+5B$
	$C=1+3B$	

$$1+3B=3+5B$$

$$-2=2B$$

$$B=-1$$

$$A=1$$

$$C=-2$$

RETURNING TO THE INTEGRAL

$$\int_0^1 \frac{1}{x+1} + \frac{-2x-2}{x^2+4x+5} dx = \int_0^1 \frac{1}{x+1} - \frac{2x+2}{x^2+4x+5} dx$$

$$= \int_0^1 \frac{1}{x+1} - \frac{2x+4}{x^2+4x+5} dx = \left[\ln|x+1| - \frac{1}{2} \ln|x^2+4x+5| \right]_0^1$$

$$= (\ln 2 - \frac{1}{2} \ln 10) - (\ln 1 - \frac{1}{2} \ln 5) = \ln 2 - \frac{1}{2} \ln 10 + \frac{1}{2} \ln 5$$

$$= \frac{1}{2} [\ln 2 - \ln 10 + \ln 5] = \frac{1}{2} [\ln 2 - \ln 10 + \ln 5]$$

$$= \frac{1}{2} \ln \left(\frac{4 \times 5}{10} \right) = \frac{1}{2} \ln 2$$

As Required

Question 46 (****)

Use the substitution $t = \tan\left(\frac{1}{2}x\right)$ to find an exact simplified value for

$$\int_0^{\frac{\pi}{2}} \frac{1}{2 - \cos x} dx.$$

Any trigonometric identities to convert $\cos x$ in terms of t must be derived.

$$\boxed{}, \quad \boxed{\frac{2\pi\sqrt{3}}{9}}$$

STRIC BY MANIPULATIONS & IDENTITIES

- $t = \tan\left(\frac{x}{2}\right)$
- $\frac{dt}{dx} = \frac{1}{2} \sec^2\left(\frac{x}{2}\right)$
- $\frac{dx}{dt} = \frac{1}{\frac{1}{2} [1 + \tan^2(\frac{x}{2})]}$
- $\frac{dx}{dt} = \frac{1}{\frac{1}{2} [1 + t^2]}$
- $\frac{dx}{dt} = \frac{2}{1+t^2}$
- $dx = \frac{2}{1+t^2} dt$
- $x = \frac{\pi}{2} \rightarrow t = 1$
- $x = 0 \rightarrow t = 0$

TRANSFORMING THE INTEGRAL

$$\int_0^{\frac{\pi}{2}} \frac{1}{2 - \cos x} dx = \int_0^1 \frac{1}{2 - \frac{1-t^2}{1+t^2}} \times \frac{2}{1+t^2} dt$$

$$= \int_0^1 \frac{2}{2(1+t^2) - (1-t^2)} dt$$

$$= \int_0^1 \frac{2}{1+t^2} dt$$

MANIPULATE INTO A STANDARD INTEGRAL FORM

$$= \frac{2}{3} \int_0^1 \frac{1}{t^2 + \frac{1}{3}} dt$$

$$= \frac{2}{3} \times \frac{1}{\frac{1}{3}} \left[\arctan\left(\frac{\frac{1}{3}t}{\frac{1}{3}}\right) \right]_0^1$$

$$= \frac{2}{3} \sqrt{3} \left[\arctan(\sqrt{3}t) - \arctan(0) \right]$$

$$= \frac{2}{3} \sqrt{3} \times \frac{\pi}{3}$$

$$= \frac{2\pi\sqrt{3}}{9}$$

Question 47 (****)

$$I = \int \frac{18}{3\cos^2 x + \sin^2 x} dx.$$

- a) By using the substitution $t = \tan x$, or otherwise, show clearly that

$$I = 6\sqrt{3} \arctan\left(\frac{\sqrt{3}}{3} \tan x\right) + \text{constant}.$$

- b) Hence find the exact value of $\int_0^{\frac{\pi}{4}} \frac{18}{3\cos^2 x + \sin^2 x} dx.$

$$\boxed{\pi\sqrt{3}}$$

a) $\int \frac{18}{3\cos^2 x + \sin^2 x} dx = \int \frac{\frac{18}{\cos^2 x}}{\frac{3\cos^2 x}{\cos^2 x} + \frac{\sin^2 x}{\cos^2 x}} dx$ $t = \tan x$
 $\frac{dt}{dx} = \sec^2 x$
 $dx = \frac{dt}{\sec^2 x}$
 $= \int \frac{18 \sec^2 x}{3 + \tan^2 x} dx = \dots$ by substitution
 $= \int \frac{18 \sec^2 x}{3 + t^2} \frac{dt}{\sec^2 x} = \int \frac{18}{3 + t^2} dt = \int \frac{18}{4 + t^2 - 1} dt$
 $= \frac{18}{\sqrt{3}} \arctan \frac{t}{\sqrt{3}} + C = \frac{18}{\sqrt{3}} \arctan \left(\frac{\sqrt{3}}{3} \tan x \right) + C$
 $= 6\sqrt{3} \arctan \left(\frac{\sqrt{3}}{3} \tan x \right) + C$
 b) $\int_0^{\frac{\pi}{4}} \frac{18}{3\cos^2 x + \sin^2 x} dx = \left[6\sqrt{3} \arctan \left(\frac{\sqrt{3}}{3} \tan x \right) \right]_0^{\frac{\pi}{4}}$
 $= 6\sqrt{3} \left[\arctan \frac{\sqrt{3}}{3} - \arctan 0 \right] = 6\sqrt{3} \times \frac{\pi}{6} = \pi\sqrt{3}$

Question 48 (****)

By using the substitution $u = \sqrt{x}$, or otherwise, find an exact value for

$$\int_0^1 \frac{\sqrt{x}}{x+1} dx.$$

$$2 - \frac{\pi}{2}$$

$$\int_0^1 \frac{\sqrt{x+1}}{x+1} dx = \dots \text{ by substitution}$$

$$= \int_0^1 \frac{1}{u^{\frac{1}{2}}} (2u du) = \int_0^1 \frac{2u^{\frac{1}{2}}}{u^{\frac{1}{2}}} du = 2 \left[\frac{2u^{\frac{1}{2}+1}}{\frac{1}{2}+1} \right]_0^1 = 2 \left[\frac{4u^{\frac{3}{2}}}{3} \right]_0^1$$

$$= \left[2 - \frac{2}{u^{\frac{1}{2}}} \right]_0^1 = \left[2 - 2\sqrt{u} \right]_0^1$$

$$= \left(2 - 2\sqrt{\frac{1}{4}} \right) - (0 - 0) = 2 - \frac{1}{2}$$

$u = \sqrt{x+1}$
 $u^{\frac{1}{2}} = \sqrt{x}$
 $\frac{2u du}{dx} = 1$
 $dx = 2u du$
 $2u = \sqrt{x+1}$
 $x = 1 \Rightarrow u = \frac{1}{2}$

$\frac{1}{2} \rightarrow 0$

NOTE THIS CAN ALSO BE DONE BY THE SUBSTITUTION $x = \tan^2 \theta$

$$\int_{x=0}^{x=1} \frac{\sqrt{x+1}}{x+1} dx = \dots = \int_0^{\frac{\pi}{4}} \frac{\sec \theta}{\sec^2 \theta} (2 \tan \theta \sec \theta d\theta) = \int_0^{\frac{\pi}{4}} \frac{2 \sec \theta \tan \theta}{\sec \theta} d\theta$$

$$= \int_0^{\frac{\pi}{4}} 2 \tan \theta d\theta = \left[2 \ln |\sec \theta| \right]_0^{\frac{\pi}{4}} = \left[2 \ln \sqrt{2} - 2 \ln 1 \right] = 2 \ln \sqrt{2} = \ln 2$$

$x = \tan^2 \theta$
 $\frac{dx}{d\theta} = 2 \tan \theta \sec \theta$
 $dx = 2 \tan \theta \sec \theta d\theta$
 $2 = \tan^2 \theta + 1$
 $2 = 1 + \tan^2 \theta$
 $1 = \tan^2 \theta$
 $\tan \theta = 1$
 $\theta = \frac{\pi}{4}$

Question 49 (****)

By using the substitution $u = \sqrt{3 - \sec^2 x}$, or otherwise, find the exact value of

$$\int_0^{\frac{\pi}{4}} \frac{\sec^2 x}{\sqrt{3 - \sec^2 x}} dx.$$

$$\frac{\pi}{4}$$

$$\begin{aligned}
 & \int_{\sqrt{2}}^{\sqrt{3}} \frac{\sec^2 u}{\sqrt{3-\sec^2 u}} \, du \\
 &= \int_{\sqrt{2}}^1 \frac{\sec^2 u}{\sec} \left(-\frac{-u}{\sec^2 \tan u} \, du \right) \\
 &= \int_1^{\sqrt{2}} \frac{1}{\tan u} \, du = \int_1^{\sqrt{2}} \frac{1}{\sqrt{2-u^2}} \, du \\
 &= \int_1^{\sqrt{2}} \frac{1}{(\sqrt{2})^2 - u^2} \, du = \left[\arcsin \frac{u}{\sqrt{2}} \right]_1^{\sqrt{2}} \\
 &= \arcsin 1 - \arcsin \frac{1}{\sqrt{2}} = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}
 \end{aligned}$$

Question 50 (****)

By using a suitable trigonometric substitution, show clearly that

$$\int_0^{\frac{1}{2}} \sqrt{\frac{16x}{1-x}} dx = \pi - 2.$$

proof

Handwritten proof for Question 50:

$$\begin{aligned} \int_0^{\frac{1}{2}} \sqrt{\frac{16x}{1-x}} dx &= \text{by substitution} \\ &= \int_0^{\frac{\pi}{6}} \sqrt{\frac{16 \sin^2 \theta}{1 - \sin^2 \theta}} (2 \sin \theta \cos \theta) d\theta \\ &= \int_0^{\frac{\pi}{6}} \frac{16 \sin^2 \theta}{\cos^2 \theta} (2 \sin \theta \cos \theta) d\theta \\ &= \int_0^{\frac{\pi}{6}} \frac{4 \sin^3 \theta}{\cos \theta} d\theta = \int_0^{\frac{\pi}{6}} 4 \sin^2 \theta d\theta = \int_0^{\frac{\pi}{6}} 4 \left(\frac{1}{2} - \frac{1}{2} \cos 2\theta \right) d\theta \\ &= \left[4 \left(\frac{1}{2} \theta - \frac{1}{4} \sin 2\theta \right) \right]_0^{\frac{\pi}{6}} = \left(\pi - 2 \right) - (0 - 0) = \pi - 2 \end{aligned}$$

Boxed notes on the right:

$$\begin{aligned} x &= \sin^2 \theta \\ \frac{dx}{d\theta} &= 2 \sin \theta \cos \theta \\ x=0, \theta &= 0 \\ x=\frac{1}{2}, \theta &= \frac{\pi}{6} \end{aligned}$$

Question 51 (****)By using the substitution $u = \tan x$, or otherwise, show clearly that

$$\int_0^{\frac{\pi}{4}} \frac{1}{\cos^2 x + 25 \sin^2 x} dx = \frac{1}{5} \arctan 5.$$

proof

Handwritten proof for Question 51:

$$\begin{aligned} \int_0^{\frac{\pi}{4}} \frac{1}{\cos^2 x + 25 \sin^2 x} dx &= \dots \text{by substitution} \\ &= \int_0^1 \frac{1}{\cos^2 x + 25 \sin^2 x} \frac{dx}{du} du = \int_0^1 \frac{1}{1 + 25 \sin^2 x} du \\ &= \int_0^1 \frac{1}{1 + 25 \frac{u^2}{1+u^2}} du = \int_0^1 \frac{1}{1 + 25u^2} du = \frac{1}{25} \int_0^1 \frac{du}{\frac{1}{25} + u^2} \\ &= \frac{1}{25} \int_0^1 \frac{1}{\left(\frac{1}{5}\right)^2 + u^2} du = \frac{1}{25} \left[\frac{1}{\frac{1}{5}} \arctan \left(\frac{u}{\frac{1}{5}} \right) \right]_0^1 = \frac{1}{25} \times 5 \left[\arctan 5u \right]_0^1 \\ &= \frac{1}{5} \left[\arctan 5 - \arctan 0 \right] = \frac{1}{5} \arctan 5 \end{aligned}$$

Boxed notes on the right:

$$\begin{aligned} u &= \tan x \\ \frac{du}{dx} &= \sec^2 x \\ dx &= \frac{du}{\sec^2 x} \\ x=0, u &= 0 \\ x=\frac{\pi}{4}, u &= 1 \end{aligned}$$

Question 52 (****)

By using the substitution $x = \cosh^2 u$, or otherwise, show that

$$\int \sqrt{\frac{x}{x-1}} dx = \ln(\sqrt{x} + \sqrt{x-1}) + \sqrt{x^2 - x} + \text{constant}.$$

proof

Handwritten proof showing the integration of $\sqrt{\frac{x}{x-1}}$ using the substitution $x = \cosh^2 u$.

$$\int \sqrt{\frac{x}{x-1}} dx = \int \sqrt{\frac{\cosh^2 u}{\cosh^2 u - 1}} 2 \cosh u \sinh u du$$

$$= \int \frac{\cosh u}{\sinh u} 2 \cosh u \sinh u du = \int 2 \cosh^2 u du$$

$$= \int 2 \left(\frac{1}{2} + \frac{1}{2} \cosh 2u \right) du = \int (1 + \cosh 2u) du$$

$$= u + \frac{1}{2} \sinh 2u + C = u + \sinh u \cosh u + C$$

$$= \operatorname{arccosh} \sqrt{x} + \sqrt{x-1} \sqrt{x} + C$$

$$= \operatorname{arccosh} \sqrt{x} + \sqrt{x^2 - x} + C$$

$$= \ln(\sqrt{x} + \sqrt{x-1}) + \sqrt{x^2 - x} + C$$

Notes:
 $x = \cosh^2 u$
 $\frac{dx}{du} = 2 \cosh u \sinh u$
 $\cosh^2 u - 1 = \sinh^2 u$
 $\sinh u = \sqrt{x-1}$
 $\cosh u = \sqrt{x}$
 $u = \operatorname{arccosh} \sqrt{x}$

Question 53 (****)

$$\sin 2x \equiv \frac{2 \tan x}{1 + \tan^2 x}$$

a) Prove the validity of the above trigonometric identity.

b) Express $\frac{8}{(3t+1)(t+3)}$ into partial fractions.

c) Hence use the substitution $t = \tan x$ to show that

$$\int_0^{\frac{\pi}{4}} \frac{8}{3+5\sin 2x} dx = \ln 3.$$

$$\frac{8}{(3t+1)(t+3)} = \frac{3}{3t+1} - \frac{1}{t+3}$$

(a) $\frac{1}{1+\tan^2 x} = \frac{\sin x \cos x}{1+\tan^2 x} = \frac{\sin x \cos x}{\sec^2 x} = \sin x \cos x = \frac{1}{2} \sin 2x$
 $\therefore \sin 2x = \frac{2 \tan x}{1+\tan^2 x}$

(b) $\frac{8}{(3t+1)(t+3)} = \frac{A}{3t+1} + \frac{B}{t+3}$
 $8 = A(t+3) + B(3t+1)$
 $\text{If } t = -3, 8 = -6B \Rightarrow B = -\frac{4}{3}$
 $\text{If } t = -\frac{1}{3}, 8 = 2A \Rightarrow A = 4$

(c) $\int_0^{\frac{\pi}{4}} \frac{8}{3+5\sin 2x} dx = \dots$ BY SUBSTITUTION
 $t = \tan x$
 $\frac{dt}{dx} = \sec^2 x$
 $\frac{dx}{dt} = \frac{1}{1+t^2}$
 $x=0 \Rightarrow t=0$
 $x=\frac{\pi}{4} \Rightarrow t=1$

$= \int_0^1 \frac{8}{3+5\left(\frac{2t}{1+t^2}\right)} \times \frac{dt}{1+t^2}$
 $= \int_0^1 \frac{8}{3 + \frac{10t}{1+t^2}} \times \frac{dt}{1+t^2}$
 $= \int_0^1 \frac{8}{3(1+t^2) + 10t} dt$
 $= \int_0^1 \frac{8}{3t^2 + 10t + 3} dt = \int_0^1 \frac{8}{3t^2 + 10t + 3} dt$
 $= \int_0^1 \frac{8}{(3t+1)(t+3)} dt = \int_0^1 \left(\frac{3}{3t+1} - \frac{1}{t+3} \right) dt$
 $= \left(\ln 4 - \ln 1 \right) - \left(\ln 4 - \ln 3 \right) = \ln 3$

Question 54 (****)

$$\frac{2t}{(t+1)(t^2+1)} \equiv \frac{A}{t+1} + \frac{Bt+C}{t^2+1}.$$

- a) Determine the values of A , B and C in the above identity.
- b) Hence find an value for

$$\int_0^{\frac{\pi}{2}} \sqrt{\frac{1 - \cos x}{1 + \sin x}} \, dx.$$

$$\boxed{\text{ME}}, \boxed{A = -1, B = 1, C = 1}, \boxed{\frac{\sqrt{2}}{2}(\pi - 2\ln 2)}$$

(c) $\frac{2t}{t^4 + 1} \equiv \frac{A}{t} + \frac{Bt+C}{t^2+1}$
 $2t \equiv A(t^2+1) + (Bt+C)(t^2+1)$

• If $t=1$, $2 = 2A + B + C$
 • If $t=0$, $0 = A + C$
 • If $t=i$, $2 = 2A + 2(B+C)$

$A = -1$ $C = 1$ $B = 1$

(b) $\int_0^{\infty} \frac{1-\cos x}{1+\sin x} dx$

$= \int_0^{\infty} \frac{1 - \frac{e^{ix} + e^{-ix}}{2}}{1 + \frac{e^{ix} - e^{-ix}}{2i}} \cdot \frac{2}{1+it} dt$

$= \int_0^{\infty} \frac{(1-t^2) - (1+t^2)}{(1+it)(1-t^2)} \cdot \frac{2}{1+it} dt$

$= \sqrt{2} \int_0^{\infty} \frac{(1-t^2)}{(1+it)(1+it^2)} \cdot \frac{2}{1+it} dt$

$= \sqrt{2} \int_0^{\infty} \frac{1}{(1+it)(1+it^2)} dt$

$= \sqrt{2} \int_0^{\infty} \left[\frac{\frac{1}{1+it}}{1+it^2} + \frac{\frac{1}{1+it^2}}{1+it} \right] dt$

$= \sqrt{2} \left[\frac{1}{2} \ln|1+it| + \frac{1}{2} \ln|1+it^2| \right]_0^{\infty}$

$= \sqrt{2} \left[\frac{1}{2} \ln 2 + \frac{1}{2} \ln 2 \right]$

$= \sqrt{2} \ln 2$

Question 55 (****)

Use the substitution $t = \tan\left(\frac{x}{2}\right)$ to find the value of

$$\int_0^{\frac{\pi}{2}} \frac{1}{1 + \sin x} dx.$$

,

FOUND THE SUBSTITUTION FIRST

$$t = \tan \frac{x}{2}$$

$$\frac{dt}{dx} = \frac{1}{2} \sec^2 \frac{x}{2}$$

$$\frac{dt}{dx} = \frac{1}{2} (1 + \tan^2 \frac{x}{2})$$

$$\frac{dt}{dx} = \frac{1}{2} (1 + t^2)$$

$$dx = \frac{dt}{\frac{1}{2}(1+t^2)}$$

$$dx = \frac{2}{1+t^2} dt$$

CHANGE THE LIMITS

$$x=0 \rightarrow t=0$$

$$x=\frac{\pi}{2} \rightarrow t=1$$

REWRITE AND EXPRESSED THE SIN

IN TERMS OF T. BY ANY SORTER

WATER / MANIPULATION

$$\sin \frac{x}{2} = \frac{2 \tan \frac{x}{2}}{1 + \tan^2 \frac{x}{2}}$$

$$\sin x = \frac{2 \sin \frac{x}{2} \cos \frac{x}{2}}{\cos^2 \frac{x}{2} + \sin^2 \frac{x}{2}}$$

$$\sin x = \frac{2 \tan \frac{x}{2} \times \frac{1}{1 + \tan^2 \frac{x}{2}}}{1 + \tan^2 \frac{x}{2}}$$

$$\sin x = \frac{2t}{1+t^2}$$

THUS REWRITE THE INTEGRAL USING THE SUBSTITUTION FROM ABOVE

$$\int_0^{\frac{\pi}{2}} \frac{1}{1 + \sin x} dx = \dots = \int_0^1 \frac{1}{1 + \frac{2t}{1+t^2}} \times \frac{2}{1+t^2} dt = \int_0^1 \frac{2}{1+t^2+2t} dt$$

$$= \int_0^1 \frac{2}{(t+1)^2} dt = \int_0^1 \frac{2}{(t+1)^2} dt$$

$$= \left[-\frac{2}{t+1} \right]_0^1 = \left[-\frac{2}{1+1} \right]_0^1 = 2 - 1 = 1$$

Question 56 (****)

Use suitable substitution to find the exact value of

$$\int_0^{\frac{\pi}{2}} \frac{\sin 2x}{\sqrt{4 - \sin^4 x}} dx.$$

FOUND THE SUBSTITUTION FIRST

$$u = \sin^2 x$$

$$\frac{du}{dx} = 2 \sin x \cos x$$

$$du = 2 \sin x \cos x dx$$

$$dx = \frac{du}{2 \sin x \cos x}$$

CHANGE THE LIMITS

$$x=0 \rightarrow u=0$$

$$x=\frac{\pi}{2} \rightarrow u=1$$

REWRITE AND EXPRESSED THE SIN

IN TERMS OF U. BY ANY SORTER

WATER / MANIPULATION

$$\int_0^{\frac{\pi}{2}} \frac{\sin 2x}{\sqrt{4 - \sin^4 x}} dx = \dots = \int_0^1 \frac{1}{\sqrt{4 - u^2}} du$$

$$= \int_0^1 \frac{1}{\sqrt{4 - u^2}} du = \int_0^1 \frac{1}{\sqrt{2^2 - u^2}} du$$

$$= \left[\arcsin \frac{u}{2} \right]_0^1 = \arcsin \frac{1}{2} - \arcsin 0$$

$$= \frac{\pi}{6}$$

Question 57 (****)

$$I = \int \sqrt{\frac{x}{1-x}} dx$$

- a) Use the substitution $\sqrt{x} = \sin \theta$ to show that

$$I = \int 2 \sin^2 \theta d\theta.$$

- b) Hence show further that

$$I = \arcsin \sqrt{x} - \sqrt{x-x^2} + \text{constant}$$

proof

(a) $\int \sqrt{\frac{x}{1-x}} dx = \int \frac{\sqrt{x}}{\sqrt{1-x}} dx = \dots$ by the substitution $\sqrt{x} = \sin \theta$
 $\frac{dx}{d\theta} = 2 \sin \theta \cos \theta$
 $dx = 2 \sin \theta \cos \theta d\theta$
 $\theta = \arcsin \sqrt{x}$

(b) $\int 2 \sin^2 \theta d\theta = \int 1 - \cos 2\theta d\theta$
 $= \theta - \frac{1}{2} \sin 2\theta + C = \theta - \frac{1}{2} (2 \sin \theta \cos \theta) + C$
 $= \theta - \sin \theta \cos \theta + C = \arcsin \sqrt{x} - \sqrt{x} \sqrt{1-x} + C$
 $= \arcsin \sqrt{x} - \sqrt{x-x^2} + C$

By Pythagoras
 $\therefore \cos \theta = \sqrt{1-x^2}$

Question 58 (***)

The curve with the following equation is defined in the largest real domain.

$$y = (4x - 3)\sqrt{-8(2x^2 - 3x + 1)} + \arcsin(4x - 3).$$

a) Show that

$$\frac{dy}{dx} = k\sqrt{-2x^2 + 3x - 1},$$

where k is an exact constant to be found.

b) Hence find the exact value of the following integral.

$$\int_{\frac{1}{2}}^1 \sqrt{-2x^2 + 3x - 1} \, dx.$$

$$\boxed{\frac{1}{2}}, \quad \boxed{k = 16\sqrt{2}}, \quad \boxed{\frac{\pi}{16\sqrt{2}}}$$

$$q) \quad y = \frac{1}{\sqrt{(x-3)}} + \frac{1}{\sqrt{(4-x)}} - \left[\frac{-8(x^2+3x+1)}{\sqrt{x}} \right]^{\frac{1}{2}}$$

$$\Rightarrow \frac{dA}{dx} = 4 \times \frac{1}{\sqrt{x}} \times 16 \times (-2x^2 + 3x - 1)^{\frac{1}{2}}$$

$$\Rightarrow \frac{dA}{dx} = \frac{64\sqrt{x}}{8} (-2x^2 + 3x - 1)^{\frac{1}{2}}$$

$$\Rightarrow \frac{dA}{dx} = 8\sqrt{x} (-2x^2 + 3x - 1)^{\frac{1}{2}} \quad // \quad k = 8\sqrt{x}$$

b) 4x25 PROB (4)

$$\int_{\frac{1}{2}}^1 \sqrt{-2x^2 + 3x - 1} \, dx = \frac{1}{16\sqrt{2}} \left[\int_{\frac{1}{2}}^1 4\sqrt{2} (2x^2 + 3x - 1)^{\frac{1}{2}} dx \right]$$

THUS WE HAVE

$$= \frac{1}{16\sqrt{2}} \left[\arcsin(4x-3) + \frac{1}{2} (x-3) \left[-\theta(-2x^2+3x-1) \right]^{\frac{1}{2}} \right]_{\frac{1}{2}}^1$$

$$- \frac{1}{16\sqrt{2}} \left[\arcsin(1) + \sqrt{\frac{1}{2}(1-4)} \right] - \left[\arcsin(-1) - \sqrt{\frac{1}{2}(-1+4)} \right]$$

$$= \frac{1}{16\sqrt{2}} \left\{ \frac{\pi}{2} + \frac{\pi}{2} \right\}$$

$$= \frac{\pi}{16\sqrt{2}}$$

Question 59 (****)

Use the substitution $t = \tan\left(\frac{x}{2}\right)$ to find the exact value of

$$\int_0^{\frac{\pi}{2}} \frac{3\sqrt{3}}{2 - \cos x} dx.$$

$$\boxed{}, \boxed{2\pi}$$

DOING THE SUBSTITUTION FIRST

$\rightarrow t = \tan \frac{x}{2}$
 $\rightarrow \frac{dt}{dx} = \frac{1}{2} \sec^2 \frac{x}{2}$
 $\rightarrow \frac{dx}{dt} = \frac{1}{\frac{1}{2} \sec^2 \frac{x}{2}}$
 $\rightarrow \frac{dx}{dt} = \frac{1}{\frac{1}{2}(1+t^2)}$
 $\rightarrow dx = \frac{2}{1+t^2} dt$
 $\rightarrow \boxed{dx = \frac{2}{1+t^2} dt}$

CHANGING THE LIMITS

$x=0 \rightarrow t=0$
 $x=\frac{\pi}{2} \rightarrow t=1$

GETTING AN EXPRESSION FOR COS IN TERMS OF t

$\cos x = 2\cos^2 \frac{x}{2} - 1$
 $\cos x = \frac{2}{1+t^2} - 1$
 $\cos x = \frac{2}{1+t^2} - 1$
 $\cos x = \frac{2 - (1+t^2)}{1+t^2}$
 $\cos x = \frac{1-t^2}{1+t^2}$

TRANSFORMING THE GIVEN INTEGRAL USING THE ABOVE RESULTS

$\int_0^{\frac{\pi}{2}} \frac{3\sqrt{3}}{2 - \cos x} dx = \dots = \int_0^1 \frac{3\sqrt{3}}{2 - \frac{1-t^2}{1+t^2}} \times \frac{2}{1+t^2} dt$
 $= \int_0^1 \frac{6\sqrt{3}}{2(1+t^2) - (1-t^2)} dt = \int_0^1 \frac{6\sqrt{3}}{1+3t^2} dt = \int_0^1 \frac{2\sqrt{3}}{\frac{1}{3} + t^2} dt$

THIS IS A STANDARD ARCTAN INTEGRAL

$= \left[\frac{2\sqrt{3}}{1/3} \arctan\left(\frac{t}{1/3}\right) \right]_0^1 = \left[6 \arctan(3t) \right]_0^1$
 $= 6 \arctan(3) - 6 \arctan(0) = 6 \times \frac{\pi}{2} = 3\pi$

Question 60 (****)

Use the substitution $u = \frac{1}{x}$ to find

$$\int \frac{1}{x\sqrt{3x^2 + 2x - 1}} dx.$$

$$-\arcsin\left(\frac{1-x}{2x}\right) + C$$

Handwritten solution for the integral using the substitution $u = \frac{1}{x}$:

$$\int \frac{1}{x\sqrt{3x^2 + 2x - 1}} dx \quad \dots \text{by substitution } \dots$$

Let $u = \frac{1}{x} \Rightarrow x = \frac{1}{u}$
 $\frac{du}{dx} = -\frac{1}{x^2}$
 $dx = -\frac{1}{u^2} du$

$$\int \frac{1}{\frac{1}{u} \left(\frac{3}{u^2} + \frac{2}{u} - 1 \right)^{\frac{1}{2}}} \left(-\frac{1}{u^2} du \right) = \int \frac{u}{\left(\frac{3+2u-u^2}{u^2} \right)^{\frac{1}{2}}} \left(-\frac{1}{u^2} du \right)$$

$$= - \int \frac{u}{\frac{(3+2u-u^2)^{\frac{1}{2}}}{u}} \left(\frac{1}{u^2} du \right) = - \int \frac{u^2}{(3+2u-u^2)^{\frac{1}{2}}} \left(\frac{1}{u^2} du \right)$$

$$= - \int \frac{1}{(-u^2 + 2u + 3)^{\frac{1}{2}}} du = - \int \frac{1}{\sqrt{4 - (u-2)^2}} du$$

By substituting $v = u-2$
 $\frac{dv}{du} = 1$
 $du = dv$

$$= - \int \frac{1}{\sqrt{4 - v^2}} dv = -\arcsin \frac{v}{2} + C$$

$$= -\arcsin \left(\frac{u-2}{2} \right) + C = -\arcsin \left(\frac{\frac{1}{x} - 2}{2} \right) + C$$

$$= -\arcsin \left(\frac{1-x}{2x} \right) + C$$

Question 61 (****+)

Use the substitution $t = \tan\left(\frac{x}{2}\right)$ to find the value of

$$\int_0^{\frac{\pi}{2}} \frac{1}{5 + 3\sin x + 4\cos x} dx.$$

All relevant results used in this evaluation must be carefully derived.

$$\boxed{}, \frac{1}{6}$$

USING THE SUBSTITUTION GIVEN

- $t = \tan \frac{\theta}{2}$
- $\sin \theta = \frac{2 \tan \frac{\theta}{2}}{1 + \tan^2 \frac{\theta}{2}}$
- $\cos \theta = \frac{1 - \tan^2 \frac{\theta}{2}}{1 + \tan^2 \frac{\theta}{2}}$
- $d\theta = \frac{2}{1 + \tan^2 \frac{\theta}{2}} dt$

EX. 1. Find $\int_0^{\pi} \frac{1}{3 + 5 \cos \theta + 4 \cos^2 \theta} d\theta$

Solution: Let $t = \tan \frac{\theta}{2}$

- $\sin \theta = \frac{2t}{1+t^2}$
- $\cos \theta = \frac{1-t^2}{1+t^2}$
- $d\theta = \frac{2}{1+t^2} dt$

When $\theta = 0$, $t = \tan \frac{0}{2} = 0$

When $\theta = \pi$, $t = \tan \frac{\pi}{2} = \infty$

Substituting in the integral

$$\int_0^{\pi} \frac{1}{3 + 5 \cos \theta + 4 \cos^2 \theta} d\theta = \int_0^{\infty} \frac{\frac{2}{1+t^2}}{3 + 5 \left(\frac{1-t^2}{1+t^2} \right) + 4 \left(\frac{1-t^2}{1+t^2} \right)^2} \cdot \frac{2}{1+t^2} dt$$

$$= \int_0^{\infty} \frac{1}{\frac{3(1+t^2) + 5(1-t^2) + 4(1-t^2)^2}{(1+t^2)^2}} dt = \int_0^{\infty} \frac{2}{(1+t^2)^2} dt$$

$$= \int_0^{\infty} \frac{2}{(t^2 + 1)^2} dt = \int_0^{\infty} \frac{2}{(t^2 + 1)^2} dt$$

$$= \left[-\frac{2}{t+1} \right]_0^{\infty} = \left[-\frac{2}{t+1} \right]_0^{\infty} = \frac{2}{\infty} - \frac{2}{1} = \frac{2}{\infty} - 2 = -2$$

Question 62 (****+)

Use the substitution $t = \tan\left(\frac{1}{2}x\right)$ to find the exact value for the integral

$$\int_0^{\frac{1}{2}\pi} \frac{2}{1 + \sin x + 2 \cos x} dx$$

All relevant results used in this evaluation must be carefully derived.

, $\ln 3$

START BY TESTING INFORMATION BASED ON THE GIVEN SUBSTITUTION

$\bullet \quad t = \tan\left(\frac{x}{2}\right)$
 $\frac{dt}{dx} = \frac{1}{2} \sec^2\left(\frac{x}{2}\right)$
 $\frac{dt}{dx} = \frac{1}{2} (1 + \tan^2\left(\frac{x}{2}\right))$
 $\frac{dt}{dx} = \frac{1}{2} (1 + t^2)$
 $\frac{dx}{dt} = \frac{2}{1+t^2} dt$

$\bullet \quad \sin x = 2 \sin\left(\frac{x}{2}\right) \cos\left(\frac{x}{2}\right) = \frac{2 \sin\left(\frac{x}{2}\right) \cos\left(\frac{x}{2}\right)}{\cos^2\left(\frac{x}{2}\right)} \cdot \cos^2\left(\frac{x}{2}\right)$
 $= \frac{2 \tan\left(\frac{x}{2}\right) \cos^2\left(\frac{x}{2}\right)}{\cos^2\left(\frac{x}{2}\right)} = \frac{2 \tan\left(\frac{x}{2}\right)}{1 + \tan^2\left(\frac{x}{2}\right)} = \frac{2t}{1+t^2}$
 $\bullet \quad \cos x = 2 \cos^2\left(\frac{x}{2}\right) - 1 = \frac{2 \cos^2\left(\frac{x}{2}\right)}{\cos^2\left(\frac{x}{2}\right)} - 1$
 $= \frac{2}{1 + \tan^2\left(\frac{x}{2}\right)} - 1 = \frac{2}{1+t^2} - 1$
 $= \frac{2 - (1+t^2)}{1+t^2} = \frac{1-t^2}{1+t^2}$

TURN THE LIMITS

$3.00 \rightarrow t = 0$
 $3.00 \rightarrow t = 1$

TRANSFORMING THE INTEGRAL

$$\int_0^{\frac{1}{2}\pi} \frac{2}{1 + \sin x + 2 \cos x} dx = \int_0^1 \frac{\frac{2}{1+t^2}}{1 + \frac{2t}{1+t^2} + 2\left(\frac{1-t^2}{1+t^2}\right)} \left(\frac{2}{1+t^2}\right) dt$$

$$= \int_0^1 \frac{\frac{4}{1+t^2}}{1 + \frac{2t}{1+t^2} + 2\left(\frac{1-t^2}{1+t^2}\right)} dt$$

$$= \int_0^1 \frac{4}{1+t^2 + 2t + 2 - 2t^2} dt$$

$$= \int_0^1 \frac{4}{-t^2 + 2t + 3} dt$$

$$= \int_0^1 \frac{4}{t^2 - 2t - 3} dt$$

PROCEED BY PARTIAL FRACTIONS (BY INSPECTION)

$$\int_0^1 \frac{4}{t^2 - 2t - 3} dt = \int_0^1 \frac{4}{(t-3)(t+1)} dt$$

$$= \int_0^1 \left(\frac{1}{t-3} - \frac{1}{t+1} \right) dt$$

$$= [\ln|t-3| - \ln|t+1|]_0^1$$

$$= [\ln|-2| - \ln|1|] - [\ln|-3| - \ln|1|]$$

$$= \ln 2 - \ln 3 + \ln 3$$

$$= \ln 2$$

Question 63 (****+)

$$y = \arcsin x, \quad -1 \leq x \leq 1.$$

a) Show clearly that

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}.$$

b) Use the substitution $x = \sin \theta$ to find

$$\int \frac{2x^2}{\sqrt{1-x^2}} dx.$$

c) Hence find an exact value for

$$\int_0^1 4x \arcsin x \, dx$$

$$\boxed{\arcsin x - x\sqrt{1-x^2} + C}, \quad \boxed{\frac{\pi}{2}}$$

[illegible]

Question 64 (****+)

It is given that

$$x = -2 + \sqrt{3} \cosh \theta, \quad \theta \geq 0.$$

a) Show clearly that ...

$$\text{i.} \quad \dots \sinh \theta = \frac{\sqrt{x^2 + 4x + 1}}{\sqrt{3}}.$$

$$\text{ii.} \quad \dots \int \frac{x+2}{(x^2 + 4x + 1)^{\frac{3}{2}}} dx = \frac{\sqrt{3}}{3} \int \frac{\cosh \theta}{\sinh^2 \theta} d\theta.$$

b) By considering the derivative of cosech θ find

$$\int \frac{x+2}{(x^2 + 4x + 1)^{\frac{3}{2}}} dx$$

$$-\left(x^2 + 4x + 1\right)^{-\frac{1}{2}} + C$$

(a) $x = -2 + \sqrt{3} \cosh \theta$
 $x+2 = \sqrt{3} \cosh \theta$
 $(x+2)^2 = 3 \cosh^2 \theta$
 $\frac{x^2+4x+1}{3} = \cosh^2 \theta$
 $\frac{x^2+4x+1}{3} - 1 = \cosh^2 \theta - 1$
 $\frac{x^2+4x+1}{3} - 1 = \sinh^2 \theta$
 $\sinh \theta = \pm \frac{\sqrt{x^2+4x+1}}{\sqrt{3}}$
 (b) $x = -2 + \sqrt{3} \cosh \theta$
 $\frac{dx}{d\theta} = \sqrt{3} \sinh \theta$
 $dx = \sqrt{3} \sinh \theta d\theta$
 $\int \frac{x+2}{(x^2+4x+1)^{\frac{3}{2}}} dx = \int \frac{\sqrt{3} \cosh \theta}{(\sqrt{3} \cosh^2 \theta)^{\frac{3}{2}}} \sqrt{3} \sinh \theta d\theta$
 $= \int \frac{\sqrt{3} \cosh \theta}{(\sqrt{3} \sinh \theta)^2} d\theta = \int \frac{\cosh \theta}{\sinh^2 \theta} d\theta$
 $= \int \frac{\cosh \theta}{\sinh \theta} \cdot \frac{1}{\sinh \theta} d\theta = \int \frac{1}{\sinh \theta} d\theta$
 $= \int \frac{1}{\sinh \theta} d\theta = \int \frac{1}{\cosh^2 \theta} d\theta = \int \text{sech}^2 \theta d\theta = \tanh \theta + C$
 $\tanh \theta = \frac{\sinh \theta}{\cosh \theta} = \frac{\frac{\sqrt{x^2+4x+1}}{\sqrt{3}}}{\frac{x+2}{\sqrt{3}}} = \frac{\sqrt{x^2+4x+1}}{x+2}$
 $\therefore \int \frac{x+2}{(x^2+4x+1)^{\frac{3}{2}}} dx = \frac{\sqrt{x^2+4x+1}}{x+2} + C$
 (c) $\frac{d}{dx} \left(\frac{1}{\sqrt{x^2+4x+1}} \right) = -\frac{1}{2} (x^2+4x+1)^{-\frac{3}{2}} (2x+4)$
 $= -\frac{1}{2} \cdot \frac{2(x+2)}{(x^2+4x+1)^{\frac{3}{2}}} = -\frac{x+2}{(x^2+4x+1)^{\frac{3}{2}}}$
 $\therefore \int \frac{x+2}{(x^2+4x+1)^{\frac{3}{2}}} dx = -\frac{1}{\sqrt{x^2+4x+1}} + C$

Question 65 (****+)

Use the substitution $u = \sqrt{x}$ to find

$$\int \frac{1}{(x^2 - 1)\sqrt{x}} dx.$$

$$\frac{1}{2} \ln \left| \frac{\sqrt{x} - 1}{\sqrt{x} + 1} \right| - \arctan \sqrt{x} + C$$

Handwritten solution for the integral using the substitution $u = \sqrt{x}$:

$$\begin{aligned} \int \frac{1}{(x^2 - 1)\sqrt{x}} dx &= \dots \text{substitution} = \int \frac{1}{(u^2 - 1)u} 2u du = \int \frac{2}{u^2 - 1} du \\ &= \int \frac{2}{(u - 1)(u + 1)} du = \dots \text{partial fractions by inspection} \\ &= \int \frac{1}{u - 1} - \frac{1}{u + 1} du = \int \frac{1}{(u - 1)u} - \frac{1}{u + 1} du \\ &= \dots \text{partial fractions again by inspection} \dots \\ &= \int \frac{\frac{1}{2}}{u - 1} - \frac{\frac{1}{2}}{u + 1} - \frac{1}{u + 1} du \\ &= \frac{1}{2} \ln |u - 1| - \frac{1}{2} \ln |u + 1| - \arctan u + C \\ &= \frac{1}{2} \ln \left| \frac{u - 1}{u + 1} \right| - \arctan u + C = \frac{1}{2} \ln \left| \frac{\sqrt{x} - 1}{\sqrt{x} + 1} \right| - \arctan \sqrt{x} + C \end{aligned}$$

Question 66 (****+)

- a) Find a simplified expression for

$$\frac{d}{dx} \left[\arctan \frac{2}{x} \right].$$

- b) Hence show that

$$\int_{\frac{2}{3}\sqrt{3}}^2 9x \arctan \left(\frac{2}{x} \right) dx = \pi + 18 - 6\sqrt{3}$$

$$\frac{d}{dx} \left[\arctan \frac{2}{x} \right] = -\frac{2}{x^2 + 4}$$

(a) $\frac{d}{dx} \left[\arctan \left(\frac{2}{x} \right) \right] = \frac{1}{1 + \left(\frac{2}{x} \right)^2} \cdot \left(-\frac{2}{x^2} \right) = \frac{-2/x^2}{1 + 4/x^2} = \frac{-2/x^2}{(x^2 + 4)/x^2} = \frac{-2}{x^2 + 4}$
 $= -\frac{2}{x^2 + 4}$ ✓
 (b) $\int_{\frac{2}{3}\sqrt{3}}^2 9x \arctan \left(\frac{2}{x} \right) dx = \dots$ by parts ...
 $\frac{d}{dx} \left[\arctan \left(\frac{2}{x} \right) \right] = \frac{d}{dx} \left[\frac{2}{x} - \arctan \left(\frac{x}{2} \right) \right] = -\frac{2}{x^2} - \frac{1}{1 + \frac{x^2}{4}} \cdot \frac{1}{2} = -\frac{2}{x^2} - \frac{2}{x^2 + 4}$
 $= -\frac{2}{x^2 + 4}$ ✓ by part
 $\int_{\frac{2}{3}\sqrt{3}}^2 9x \arctan \left(\frac{2}{x} \right) dx = \dots$ by parts ...
 $= \left[\frac{9}{2} x^2 \arctan \left(\frac{2}{x} \right) - \int \frac{9x^2}{x^2 + 4} dx \right]_{\frac{2}{3}\sqrt{3}}^2$
 $= 18 \arctan 1 - 6 \arctan \frac{1}{2} + 9 \int_{\frac{2}{3}\sqrt{3}}^2 \frac{2x^2 + 4 - 4}{x^2 + 4} dx$
 $= 18 \left(\frac{\pi}{4} \right) - 6 \left(\frac{\pi}{6} \right) + 9 \left[\frac{2}{3} \ln \left| \frac{x^2 + 4}{x^2} \right| - \frac{4}{x^2 + 4} \right]_{\frac{2}{3}\sqrt{3}}^2$
 $= \frac{9}{2} \pi - 2\pi + 9 \left[2 - \frac{4}{x^2 + 4} - \frac{4}{x^2} \right]_{\frac{2}{3}\sqrt{3}}^2$
 $= \frac{5}{2} \pi + 9 \left[(2 - 2 \arctan 1) - (2 - 2 \arctan \frac{1}{2}) \right]$
 $= \frac{5}{2} \pi + 9 \left[2 - 2 \arctan 1 - \frac{2}{3} \ln 3 + 2 \arctan \frac{1}{2} \right]$
 $= \frac{5}{2} \pi + 9 \left[2 - \frac{\pi}{2} - \frac{2}{3} \ln 3 + \frac{\pi}{3} \right]$
 $= \frac{5}{2} \pi + 18 - \frac{6\pi}{2} - 6 \ln 3 + 3\pi$
 $= 18 - 6\sqrt{3} + \pi$ ✓ by simplification

Question 67 (****+)

$$\int_0^4 \frac{16}{3(3x^2 + 16)^{\frac{5}{2}}} dx.$$

- a) By using a suitable trigonometric substitution in terms of θ , show that the above integral can be transformed to

$$\frac{\sqrt{3}}{144} \int_0^{\frac{\pi}{3}} \cos^3 \theta \, d\theta.$$

- b) Hence evaluate the original integral.

$$\boxed{\frac{1}{128}}$$

Handwritten solution for Question 67:

(a) $\int_0^4 \frac{16}{3(3x^2 + 16)^{\frac{5}{2}}} dx = \dots$

Let $x = \sqrt{\frac{16}{3}} \tan \theta$ then $dx = \sqrt{\frac{16}{3}} \sec^2 \theta \, d\theta$

When $x = 0$, $\theta = 0$
 When $x = 4$, $\theta = \frac{\pi}{3}$

$3x^2 + 16 = 3 \left(\frac{16}{3} \tan^2 \theta \right) + 16 = 16(\tan^2 \theta + 1) = 16 \sec^2 \theta$

$\therefore \int_0^4 \frac{16}{3(3x^2 + 16)^{\frac{5}{2}}} dx = \int_0^{\frac{\pi}{3}} \frac{16}{3 \times (16 \sec^2 \theta)^{\frac{5}{2}}} \times \sqrt{\frac{16}{3}} \sec^2 \theta \, d\theta$

$= \int_0^{\frac{\pi}{3}} \frac{16}{3 \times 128 \sec^5 \theta} \times \frac{4}{\sqrt{3}} \sec^2 \theta \, d\theta = \frac{16}{3 \times 128} \times \frac{4}{\sqrt{3}} \int_0^{\frac{\pi}{3}} \frac{1}{\sec^3 \theta} \, d\theta$

$= \frac{16}{3 \times 128} \times \frac{4}{\sqrt{3}} \int_0^{\frac{\pi}{3}} \cos^3 \theta \, d\theta = \frac{\sqrt{3}}{144} \int_0^{\frac{\pi}{3}} \cos^3 \theta \, d\theta$

(b) $\therefore \frac{\sqrt{3}}{144} \int_0^{\frac{\pi}{3}} \cos^3 \theta \, d\theta = \frac{\sqrt{3}}{144} \int_0^{\frac{\pi}{3}} \cos \theta (1 - \sin^2 \theta) \, d\theta$

$= \frac{\sqrt{3}}{144} \left[\sin \theta - \frac{1}{3} \sin^3 \theta \right]_0^{\frac{\pi}{3}} = \frac{\sqrt{3}}{144} \left[\frac{\sqrt{3}}{2} - \frac{1}{3} \left(\frac{\sqrt{3}}{2} \right)^3 \right]$

$= \frac{\sqrt{3}}{144} \left[\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{6} \right] = \frac{\sqrt{3}}{144} \times \frac{2\sqrt{3}}{3} = \frac{2}{1152} = \frac{1}{576}$

Question 68 (****+)

$$I = \int_0^{\frac{\pi}{8}} \frac{\sqrt{3}}{2 + \sin 4x} dx.$$

a) Show that the substitution $u = \tan 2x$ transforms I into

$$J = \int_0^1 \frac{\sqrt{3}}{(2u+1)^2 + 3} du.$$

b) Hence find the exact value of I , giving the answer in terms of π .

$$I = \frac{\pi}{12}$$

Handwritten solution for Question 68:

(a)
$$I = \int_0^{\frac{\pi}{8}} \frac{\sqrt{3}}{2 + \sin 4x} dx$$

$$u = \tan 2x$$

$$\frac{du}{dx} = 2 \sec^2 2x$$

$$dx = \frac{du}{2 \sec^2 2x}$$

$$2 = \frac{2}{1 + \tan^2 2x} \Rightarrow 1 + \tan^2 2x = 1$$

$$2 = \frac{2}{1 + u^2} \Rightarrow 1 + u^2 = 1$$

$$u^2 = 0 \Rightarrow u = 0$$

$$u = 1 \Rightarrow 2x = \frac{\pi}{4} \Rightarrow x = \frac{\pi}{8}$$

$$J = \int_0^1 \frac{\sqrt{3}}{(2u+1)^2 + 3} du$$

(b)
$$J = \int_0^1 \frac{\sqrt{3}}{(2u+1)^2 + 3} du$$

$$v = 2u+1$$

$$\frac{dv}{du} = 2$$

$$du = \frac{dv}{2}$$

$$u=0 \Rightarrow v=1$$

$$u=1 \Rightarrow v=3$$

$$J = \int_1^3 \frac{\sqrt{3}}{v^2 + 3} \cdot \frac{dv}{2}$$

$$= \frac{\sqrt{3}}{2} \int_1^3 \frac{1}{v^2 + 3} dv$$

$$= \frac{\sqrt{3}}{2} \left[\frac{1}{\sqrt{3}} \arctan \left(\frac{v}{\sqrt{3}} \right) \right]_1^3$$

$$= \frac{1}{2} \left[\arctan \sqrt{3} - \arctan \frac{1}{\sqrt{3}} \right]$$

$$= \frac{1}{2} \left(\frac{\pi}{3} - \frac{\pi}{6} \right)$$

$$= \frac{\pi}{12}$$

Question 69 (****+)

$$\sec x \equiv \frac{1 + \tan^2\left(\frac{x}{2}\right)}{1 - \tan^2\left(\frac{x}{2}\right)}$$

a) Prove the validity of the above trigonometric identity.

b) Express $\frac{2}{1-t^2}$ into partial fractions.

c) Hence use the substitution $t = \tan\left(\frac{x}{2}\right)$ to show that

$$\int \sec x \, dx = \ln \left| \tan \left(\frac{x}{2} + \frac{\pi}{4} \right) \right| + C.$$

$$\boxed{}, \quad \frac{2}{1-t^2} = \frac{1}{1+t} + \frac{1}{1-t}$$

a) WORKING AS FOLLOWS

$$\begin{aligned} \text{L.H.S} &= \sec x = \frac{1}{\cos x} = \frac{1}{\cos^2 \frac{x}{2} - \sin^2 \frac{x}{2}} = \frac{\cos^2 \frac{x}{2} + \sin^2 \frac{x}{2}}{\cos^2 \frac{x}{2} - \sin^2 \frac{x}{2}} \\ &= \frac{\frac{\cos^2 \frac{x}{2}}{\cos^2 \frac{x}{2}} + \frac{\sin^2 \frac{x}{2}}{\cos^2 \frac{x}{2}}}{\frac{\cos^2 \frac{x}{2}}{\cos^2 \frac{x}{2}} - \frac{\sin^2 \frac{x}{2}}{\cos^2 \frac{x}{2}}} = \frac{1 + \tan^2 \frac{x}{2}}{1 - \tan^2 \frac{x}{2}} = \text{R.H.S} \end{aligned}$$

[We obtained working to change from the R.H.S to L.H.S]

b) BY INSPECTION / GUESS OR ANY SIMILAR METHOD

$$\frac{2}{1-t^2} = \frac{2}{(1-t)(1+t)} = \frac{1}{1-t} + \frac{1}{1+t}$$

c) USING THE SUBSTITUTION GIVEN

• $t = \tan \frac{x}{2} \Rightarrow \frac{dt}{dx} = \frac{1}{2} \sec^2 \frac{x}{2}$

$$\begin{aligned} \Rightarrow \frac{dt}{dx} &= \frac{1}{2} (1 + \tan^2 \frac{x}{2}) \\ \Rightarrow \frac{dt}{dx} &= \frac{1}{2} (1 + t^2) \\ \Rightarrow \frac{dt}{1+t^2} &= \frac{1}{2} dx \\ \Rightarrow \int \frac{dt}{1+t^2} &= \frac{1}{2} \int dx \end{aligned}$$

USING PARTS (a) & (b)

$$\begin{aligned} \int \sec x \, dx &= \int \frac{1 + \tan^2 \frac{x}{2}}{1 - \tan^2 \frac{x}{2}} dx = \int \frac{1+t^2}{1-t^2} \times \frac{2}{1+t^2} dt \\ &= \int \frac{2}{1-t^2} dt = \int \frac{1}{1-t} + \frac{1}{1+t} dt \\ &= \ln|1-t| - \ln|1+t| + C \\ &= \ln \left| \frac{1-t}{1+t} \right| + C \end{aligned}$$

NOW REMEMBER THAT $\tan \frac{x}{2} = t$ & $\tan \frac{\pi}{4} = 1$

$$\begin{aligned} \dots &= \ln \left| \frac{\tan \frac{x}{2} - \tan \frac{\pi}{4}}{1 + \tan \frac{x}{2} \tan \frac{\pi}{4}} \right| + C \\ &= \ln \left| \frac{\tan \left(\frac{x}{2} + \frac{\pi}{4} \right)}{1 + \tan \frac{x}{2} \cdot 1} \right| + C \end{aligned}$$

At $x=0$, $t=0$

Question 70 (****+)

Find an exact value for

$$\int_0^{\infty} \frac{16}{(1+x^2)^3} dx.$$

3π

$$\begin{aligned} \int_0^{\infty} \frac{16}{(1+x^2)^3} dx &= \dots = \int_0^{\frac{\pi}{2}} \frac{16}{(\sec^2 \theta)^3} \sec^2 \theta d\theta = \int_0^{\frac{\pi}{2}} \frac{16 \cos^6 \theta}{(\sec^2 \theta)^3} d\theta = \int_0^{\frac{\pi}{2}} 16 \cos^6 \theta d\theta = \int_0^{\frac{\pi}{2}} 16 (3 \cos^4 \theta - 3 \cos^2 \theta + \cos^2 \theta) d\theta \\ &= \int_0^{\frac{\pi}{2}} 16 (2 \cos^4 \theta - 2 \cos^2 \theta + \cos^2 \theta) d\theta = \int_0^{\frac{\pi}{2}} 16 (2 \cos^4 \theta - \cos^2 \theta) d\theta = \int_0^{\frac{\pi}{2}} 32 \cos^4 \theta d\theta - \int_0^{\frac{\pi}{2}} 16 \cos^2 \theta d\theta \\ &= \left[32 \cos^3 \theta + 24 \cos \theta \right]_0^{\frac{\pi}{2}} - \left[16 \cos \theta + \frac{16}{3} \sin^3 \theta \right]_0^{\frac{\pi}{2}} = (32(0) + 24(0)) - (16(0) + \frac{16}{3}(1)) = -\frac{16}{3} \end{aligned}$$

Question 71 (****+)

$$I = \int_0^{\frac{\pi}{2}} \frac{1}{1 + \sin x + \cos x} dx.$$

By using the substitution $t = \tan\left(\frac{x}{2}\right)$, or otherwise, show that

$$I = \ln 2.$$

proof

$$\begin{aligned} \int_0^{\frac{\pi}{2}} \frac{1}{1 + \sin x + \cos x} dx &= \int_0^1 \frac{1}{1 + \frac{2t}{1+t^2} + \frac{1-t^2}{1+t^2}} \times \frac{2}{1+t^2} dt \\ &= \int_0^1 \frac{2}{1+t^2+2t+1-t^2} dt = \int_0^1 \frac{2}{(t+1)^2} dt \\ &= \left[-\frac{2}{t+1} \right]_0^1 = -\frac{2}{2} + \frac{2}{1} = 1 \end{aligned}$$

Question 72 (****+)

Use the substitution $x = 2 \cosh \theta$, to find a simplified expression for

$$\int \frac{6}{(x^2 - 4)^{\frac{3}{2}}} dx.$$

$$-\frac{x}{\sqrt{x^2 - 4}} + C$$

Handwritten solution for the integral using the substitution $x = 2 \cosh \theta$.

Left side (main calculation):

$$\begin{aligned} \int \frac{6}{(x^2 - 4)^{\frac{3}{2}}} dx &= \dots \text{HYPERBOLIC SUBSTITUTION} \\ &= \int \frac{6}{(4 \cosh^2 \theta - 4)^{\frac{3}{2}}} (2 \sinh \theta d\theta) \\ &= \int \frac{6 \sinh \theta}{[4(\cosh^2 \theta - 1)]^{\frac{3}{2}}} d\theta \\ &= \int \frac{6 \sinh \theta}{(4 \sinh^2 \theta)^{\frac{3}{2}}} d\theta \\ &= \int \frac{6 \sinh \theta}{8 \sinh^3 \theta} d\theta \\ &= \int \frac{1}{\sinh^2 \theta} d\theta \\ &= \int \operatorname{cosech}^2 \theta d\theta \\ &= -\operatorname{coth} \theta + C \\ &= -\frac{\cosh \theta}{\sinh \theta} + C \\ &= -\frac{x}{\sqrt{x^2 - 4}} + C \end{aligned}$$

Right side (trigonometric identities):

$$\begin{aligned} x &= 2 \cosh \theta \\ dx &= 2 \sinh \theta d\theta \\ \cosh \theta &= \frac{x}{2} \\ \sinh \theta &= \frac{\sqrt{x^2 - 4}}{2} \\ \cosh^2 \theta - 1 &= \frac{x^2}{4} - 1 \\ \sinh^2 \theta &= \frac{x^2 - 4}{4} \\ \sinh \theta &= \frac{\sqrt{x^2 - 4}}{2} \\ \operatorname{coth} \theta &= \frac{\cosh \theta}{\sinh \theta} = \frac{\frac{x}{2}}{\frac{\sqrt{x^2 - 4}}{2}} = \frac{x}{\sqrt{x^2 - 4}} \end{aligned}$$

Question 73 (****+)

Use the substitution $u = \frac{1}{x+1}$ to find

$$\int \frac{1}{(x+1)\sqrt{x^2+4x+2}} dx.$$

$$\arcsin \left[\frac{x}{(x+1)\sqrt{2}} \right] + C$$

Handwritten solution for the integral using the substitution $u = \frac{1}{x+1}$.

Original integral: $\int \frac{1}{(x+1)\sqrt{x^2+4x+2}} dx$

Substitution: $u = \frac{1}{x+1}$
 $x+1 = \frac{1}{u}$
 $x = \frac{1}{u} - 1$
 $dx = -\frac{1}{u^2} du$

Substituting into the integral:

$$= \int \frac{1}{\left(\frac{1}{u}\right)\sqrt{\left(\frac{1}{u}-1\right)^2 + 4\left(\frac{1}{u}-1\right) + 2}} \left(-\frac{1}{u^2} du\right)$$

$$= \int \frac{-\frac{1}{u^2} du}{\frac{1}{u} \sqrt{\frac{1}{u^2} - \frac{2}{u} + 2}}$$

$$= \int \frac{-1}{\sqrt{1 - 2u + 2u^2}} \cdot \frac{1}{u} du$$

$$= \int \frac{-1}{\sqrt{2 - 2u + u^2}} du$$

$$= \int \frac{-1}{\sqrt{(u-1)^2 + 1}} du$$

Note: $v = u-1$
 $dv = du$

$$= \int \frac{-1}{\sqrt{v^2 + 1}} dv = -\arcsin\left(\frac{v}{\sqrt{v^2+1}}\right) + C = -\arcsin\left(\frac{u-1}{\sqrt{u^2}}\right) + C$$

$$= -\arcsin\left(\frac{\frac{1}{x+1}-1}{\sqrt{\frac{1}{x+1}}}\right) + C = -\arcsin\left(\frac{\frac{1-(x+1)}{x+1}}{\sqrt{\frac{1}{x+1}}}\right) + C$$

$$= -\arcsin\left(\frac{1-(x+1)}{\sqrt{x+1}}\right) + C = -\arcsin\left(\frac{-x}{\sqrt{x+1}}\right) + C = \arcsin\left(\frac{x}{(x+1)\sqrt{2}}\right) + C$$

Question 74 (****+)

$$I = \int_0^{\frac{\pi}{2}} \frac{2 \cos x}{1 + \cos x} dx.$$

By using the substitution $t = \tan\left(\frac{x}{2}\right)$, or otherwise, show that

$$I = \pi - 2.$$

proof

Handwritten solution for the integral $I = \int_0^{\frac{\pi}{2}} \frac{2 \cos x}{1 + \cos x} dx$ using the substitution $t = \tan\left(\frac{x}{2}\right)$.

Left side of the work:

$$\begin{aligned} \int_0^{\frac{\pi}{2}} \frac{2 \cos x}{1 + \cos x} dx &= \text{By using } t \text{ theorem} \\ &= \int_0^1 \frac{2 \left(\frac{1-t^2}{1+t^2} \right) \times \frac{2}{1+t^2} dt}{1 + \frac{1-t^2}{1+t^2}} \\ &= \int_0^1 \frac{2(1-t^2)}{(1+t^2)(1+t^2)} \times \frac{2}{1+t^2} dt \\ &= \int_0^1 \frac{2(1-t^2)}{(1+t^2)^2} \times \frac{2}{1+t^2} dt = 2 \int_0^1 \frac{1-t^2}{(1+t^2)^2} dt \\ &= 2 \int_0^1 \frac{2 - (1+t^2)}{(1+t^2)^2} dt = 2 \int_0^1 \frac{1-t^2}{(1+t^2)^2} dt \\ &= 2 \left[2 \arctan t - \frac{t}{1+t^2} \right]_0^1 = 2 \left(2 \arctan 1 - \frac{1}{2} \right) - 0 \\ &= 2 \left(2 \times \frac{\pi}{4} - \frac{1}{2} \right) = \pi - 2 \end{aligned}$$

Right side of the work (limits and differentials):

$$\begin{aligned} t &= \tan \frac{x}{2} \\ \frac{dt}{dx} &= \frac{1}{2} \sec^2 \frac{x}{2} \\ dx &= \frac{2 dt}{\sec^2 \frac{x}{2}} \\ dx &= \frac{2 dt}{1 + \tan^2 \frac{x}{2}} \\ dx &= \frac{2}{1+t^2} dt \\ \text{When } x=0, t=0 \\ \text{When } x=\frac{\pi}{2}, t=1 \end{aligned}$$

Question 75 (****+)

Find an exact value for

$$\int_{-1.5}^{1.5} 8x \arcsin\left(\frac{1}{3}x\right) dx.$$

, $9\sqrt{3} - 3\pi$

As the integrand is even we may rewrite as follows

$$\int_{-1.5}^{1.5} 8x \arcsin\left(\frac{1}{3}x\right) dx = 2 \int_0^{1.5} 8x \arcsin\left(\frac{1}{3}x\right) dx$$

$$= 16 \int_0^{1.5} x \arcsin\left(\frac{1}{3}x\right) dx$$

Using a substitution

$$= 16 \int_0^{\frac{\pi}{6}} (3 \sin \theta) \theta (3 \cos \theta d\theta)$$

$$= 144 \int_0^{\frac{\pi}{6}} \theta \cos \theta d\theta$$

$$= 72 \int_0^{\frac{\pi}{6}} \theta \sin \theta d\theta$$

Proceed by integration by parts

$$= 72 \left[-\frac{1}{2} \theta \cos \theta \right]_0^{\frac{\pi}{6}} - \left[-\frac{1}{2} \cos \theta d\theta \right]$$

$$= 72 \left[-\frac{1}{2} \theta \cos \theta \right]_0^{\frac{\pi}{6}} + \left[\frac{1}{2} \sin \theta d\theta \right]$$

$$= 72 \left[-\frac{1}{2} \theta \cos \theta + \frac{1}{2} \sin \theta \right]_0^{\frac{\pi}{6}}$$

$$= 72 \left[\left(-\frac{1}{2} \times \frac{\pi}{6} \times \frac{1}{2} + \frac{1}{2} \times \frac{\sqrt{3}}{2} \right) - (0) \right]$$

$$= 72 \left[\frac{\sqrt{3}}{4} - \frac{\pi}{12} \right]$$

$$= 9\sqrt{3} - 3\pi$$

$\theta = \arcsin\left(\frac{x}{3}\right)$
 $\sin \theta = \frac{x}{3}$
 $x = 3 \sin \theta$
 $\frac{dx}{d\theta} = 3 \cos \theta$
 $dx = 3 \cos \theta d\theta$
 $x=0 \rightarrow \theta=0$
 $x=1.5 \rightarrow \theta=\frac{\pi}{6}$

θ	1
$-\frac{1}{2} \theta \cos \theta$	$\sin \theta$

Question 76 (****+)

$$I = \int \frac{\operatorname{sech} x}{\cosh x - \sinh x} dx.$$

- a) By multiplying the numerator and denominator of the integrand by $\operatorname{sech} x$, show that

$$I = -\ln(1 - \tanh x) + C,$$

where C is an arbitrary constant.

- b) By multiplying the numerator and denominator of the integrand by $(\cosh x - \sinh x)$, show that

$$I = x + \ln(\cosh x) + K,$$

where K is an arbitrary constant.

- c) Show clearly that $C = K$.

proof

Handwritten mathematical proof for Question 76c, showing the equivalence of constants C and K from parts (a) and (b).

(a) $\int \frac{\operatorname{sech} x}{\cosh x - \sinh x} dx = \int \frac{\operatorname{sech} x \operatorname{sech} x}{\operatorname{sech} x (\cosh x - \sinh x)} dx = \int \frac{\operatorname{sech}^2 x}{1 - \tanh x} dx$
 $= -\ln(1 - \tanh x) + C$ (as required)

(b) $\int \frac{\operatorname{sech} x}{\cosh x - \sinh x} dx = \int \frac{\operatorname{sech} x (\cosh x + \sinh x)}{(\cosh x - \sinh x)(\cosh x + \sinh x)} dx = \int \frac{1 + \tanh x}{\cosh^2 x - \sinh^2 x} dx$
 $= \int \frac{1 + \tanh x}{1} dx = x + \ln(\cosh x) + C$ (as required)

(c) LHS $= -\ln(1 - \tanh x) = \ln\left[\frac{1}{1 - \tanh x}\right] = \ln\left[\frac{1}{1 - \frac{\cosh x - \sinh x}{\cosh x + \sinh x}}\right]$
 $= \ln\left[\frac{\cosh x + \sinh x}{\cosh x - \sinh x}\right] = \ln\left[\frac{\frac{e^x + e^{-x}}{2} + \frac{e^x - e^{-x}}{2}}{\frac{e^x + e^{-x}}{2} - \frac{e^x - e^{-x}}{2}}\right] = \ln\left[\frac{e^x + e^{-x} + e^x - e^{-x}}{e^x + e^{-x} - e^x + e^{-x}}\right]$
 $= \ln\left[\frac{2e^x}{2e^{-x}}\right] = \ln(e^{2x}) = \ln(e^2) + \ln(e^{2x-2})$
 $= 2 + \ln(\cosh x) = RHS$
 $\therefore C = K$ (as required)

Question 77 (****+)

$$\sec x \equiv \frac{\cos x}{1 - \sin^2 x}.$$

- a)** Prove the validity of the above trigonometric identity.
- b)** Use the substitution $u = \sin x$ to show that

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sec x \, dx = \frac{1}{2} \ln \left(\frac{7+4\sqrt{3}}{3} \right).$$

- c) Show clearly that

$$\frac{1}{2} \ln \left(\frac{7+4\sqrt{3}}{3} \right) = \ln \left(1 + \frac{2}{3} \sqrt{3} \right)$$

proof

(a) $\frac{1}{\cos x} = \sec x = \frac{1}{\cos x} = \frac{\cos x}{\cos^2 x} = \frac{\cos x}{1 - \sin^2 x} = 24.5$
 $\frac{1}{\cos x} = \sec x = \frac{1}{\cos x} = \frac{\cos x}{\cos^2 x} = \frac{\cos x}{1 - \sin^2 x} = 24.5$
 (b) $\int \frac{1}{\cos x} dx = \int \frac{\cos x}{1 - \sin^2 x} dx = \dots$ substitution $u = \sin x$
 $\frac{du}{dx} = \cos x$
 $dx = \frac{du}{\cos x}$
 $\frac{1}{\cos x} \cdot \frac{du}{\cos x} = \frac{1}{1 - u^2} du = \frac{1}{(1-u)(1+u)} du$
 $= \dots$ by partial fractions
 $= \int \frac{\frac{1}{2}}{1-u} + \frac{-\frac{1}{2}}{1+u} du = \left[\frac{1}{2} \ln|1-u| - \frac{1}{2} \ln|1+u| \right] + C$
 $= \frac{1}{2} \ln \left| \frac{1-u}{1+u} \right| + C = \frac{1}{2} \left[\ln \left| \frac{1 - \sin x}{1 + \sin x} \right| - \ln \left| \frac{1 - \cos x}{1 + \cos x} \right| \right]$
 $= \frac{1}{2} \left[\ln \left| \frac{1 + \cos x}{1 - \cos x} \right| - \ln 3 \right] = \frac{1}{2} \left[\ln(7 + 4\sqrt{3}) - \ln 3 \right]$
 $= \frac{1}{2} \ln \left(\frac{7 + 4\sqrt{3}}{3} \right)$
 (c) $\frac{1}{2} \ln \left(\frac{7 + 4\sqrt{3}}{3} \right) = \frac{1}{2} \ln \left(\frac{21 + 12\sqrt{3}}{9} \right) = \frac{1}{2} \ln \left[q \cdot \frac{2 \times 3 \times 2\sqrt{3} + 12}{9} \right]$
 $= \frac{1}{2} \ln \left(\frac{3^2 + 2 \times 3 \times 2\sqrt{3} + (2\sqrt{3})^2}{9} \right)$
 $= \frac{1}{2} \ln \left(\frac{(3 + 2\sqrt{3})^2}{9} \right) = \ln \left(\frac{3 + 2\sqrt{3}}{3} \right)$
 $= \ln \left(1 + \frac{2\sqrt{3}}{3} \right)$

Question 78 (****+)

$$\frac{9}{x^3+1} \equiv \frac{A}{x+1} + \frac{Bx+C}{x^2-x+1}.$$

- a) Find the value of each of the constants A , B and C in the above identity.
- b) Hence find the exact value of

$$\int_0^1 \frac{9}{x^3 - 1} \, dx.$$

$$\boxed{A=3}, \boxed{B=-3}, \boxed{C=6}, \boxed{3\ln 2 + \pi\sqrt{3}}$$

$$\begin{aligned}
 (a) \quad \frac{9}{x^3+1} &= \frac{9}{(x+1)(x^2-x+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2-x+1} \\
 &\quad \swarrow \text{Partial Fraction} \\
 \boxed{9} &= A(x^2-x+1) + (x+1)(Bx+C) \\
 9 &= Ax^2 + Ax + 1 + Bx^2 + Bx + Cx + C \\
 9 &= (A+B)x^2 + (A+B+C)x + (1+C) \\
 \bullet \quad 1+C &= 9, \quad C=8 \\
 \bullet \quad A+B &= 0, \quad B=-A \\
 \bullet \quad A+C &= 9, \quad C=8 \\
 &\quad \therefore \quad B=-1 \\
 &\quad \quad \quad C=8
 \end{aligned}$$

Question 79 (****+)

Show that

$$\int_0^{\frac{1}{3}\pi} \sec^2 x \operatorname{artanh}(\sin x) dx = -1 + \sqrt{3} \ln(2 + \sqrt{3}).$$

proof

$\int_0^{\frac{\pi}{3}} \sec^2 x \operatorname{artanh}(\sin x) dx \dots$ BY PARTS, since $\frac{d}{dx}(\operatorname{artanh} u) = \frac{1}{1-u^2}$

$\operatorname{artanh}(\sin x)$	$\frac{1}{1-\sin^2 x} \times \sec^2 x$
$\frac{1}{4} \pi$	$\sec^2 x$

$= \left[\operatorname{artanh}(\sin x) \times \tan x \right]_0^{\frac{\pi}{3}} - \int_0^{\frac{\pi}{3}} \frac{\sec x}{1-\sin^2 x} \tan x dx$
 $= \left[\tan x \operatorname{artanh}(\sin x) \right]_0^{\frac{\pi}{3}} - \int_0^{\frac{\pi}{3}} \frac{\tan x}{\cos x} dx$
 $= \left[\tan x \operatorname{artanh}(\sin x) \right]_0^{\frac{\pi}{3}} - \int_0^{\frac{\pi}{3}} \sec x \tan x dx$
 $= \left[\tan x \operatorname{artanh}(\sin x) - \sec x \right]_0^{\frac{\pi}{3}}$
 $= \left[\sqrt{3} \operatorname{artanh}\left(\frac{\sqrt{3}}{2}\right) - 2 \right] - [0 - 1]$
 $= \sqrt{3} \operatorname{artanh}\left(\frac{\sqrt{3}}{2}\right) - 1$
 $= \frac{\sqrt{3}}{2} \ln \left[\frac{1 + \frac{\sqrt{3}}{2}}{1 - \frac{\sqrt{3}}{2}} \right] - 1 = \frac{\sqrt{3}}{2} \ln \left[\frac{2 + \sqrt{3}}{2 - \sqrt{3}} \right] - 1$
 $= \frac{\sqrt{3}}{2} \ln \left[\frac{(2 + \sqrt{3})^2}{4 - 3} \right] - 1 = \frac{\sqrt{3}}{2} \ln(2 + \sqrt{3})^2 - 1$
 $= \sqrt{3} \ln(2 + \sqrt{3}) - 1$

Question 80 (****+)

Use appropriate integration techniques to find an exact simplified value for

$$\int_0^{\frac{1}{4}\pi} \frac{10}{2 - \tan x} dx.$$

$$\boxed{}, \boxed{\pi + 3\ln 2}$$

• SOLVE BY A SUBSTITUTION

$$\int_0^{\frac{1}{4}\pi} \frac{10}{2 - \tan x} dx = \int_0^1 \frac{10}{2-u} \left(\frac{1}{1+u^2} \right) du$$

$$= \int_0^1 \frac{10}{(2-u)(1+u^2)} du$$

• BY PARTIAL FRACTIONS

$$\frac{10}{(2-u)(1+u^2)} = \frac{A}{2-u} + \frac{B}{1+u^2}$$

$$10 = (2-u)(A+Bu)$$

• If $u=2$: $10 = 5C$
 $C = 2$

• If $u=0$: $10 = 2B + C$
 $10 = 2B + 2$
 $8 = 2B$
 $B = 4$

• If $u=1$: $10 = A + B + C$
 $10 = A + 4 + 2$
 $A = 2$

• RETURNING TO THE ORIGINAL

$$\dots = \int_0^1 \frac{2u+8}{u^2+1} + \frac{2}{2-u} du = \int_0^1 \frac{4}{u^2+1} + \frac{2u}{u^2+1} + \frac{2}{2-u} du$$

$$= \left[4 \arctan u + \ln(u^2+1) - 2 \ln|2-u| \right]_0^1$$

$$= (4 \arctan 1 + \ln 2 - 2 \ln 1) - (0 + \ln 1 - 2 \ln 2)$$

$$= 4 \times \frac{\pi}{4} + 3 \ln 2$$

$$= \pi + 3 \ln 2$$

Question 81 (****)

Use appropriate integration techniques to find an exact simplified value for

$$\int_0^{\infty} \frac{1}{\left(x+x^{-1}\right)^2} dx.$$

$$\frac{\pi}{4}$$

$$\begin{aligned} \int_0^{\infty} \frac{1}{(x+\frac{1}{x})^2} dx &= \int_0^{\infty} \frac{1}{(\frac{x^2+1}{x})^2} dx = \int_0^{\infty} \frac{x}{(x^2+1)^2} dx \\ &= \int_0^{\infty} \frac{x^2}{(x^2+1)^2} dx \\ \text{Now using a trigonometric substitution} \\ x &= \tan \theta \quad (\theta = \arctan x) \\ dx &= \sec^2 \theta d\theta \\ x=0 &\mapsto \theta=0 \\ x=\infty &\mapsto \theta=\frac{\pi}{2} \end{aligned}$$

Question 82 (*****)

Use the substitution $t = \tan\left(\frac{3}{2}x\right)$ to find, in terms of π , the exact value of

$$\int_{\frac{\pi}{6}}^{\frac{2\pi}{9}} \frac{1}{2 + \cos 3x} dx.$$

$$\frac{\pi\sqrt{3}}{54}$$

Handwritten solution for the integral problem using the substitution $t = \tan\left(\frac{3}{2}x\right)$.

Left side (main calculation):

$$\int_{\frac{\pi}{6}}^{\frac{2\pi}{9}} \frac{1}{2 + \cos 3x} dx$$

$$= \int_{\frac{\pi}{6}}^{\frac{2\pi}{9}} \frac{1}{2 + \frac{1-t^2}{1+t^2}} \times \frac{2}{3(1+t^2)} dt$$

Annotation: Multiply top & bottom by $(1+t^2)$

$$= \int_{\frac{\pi}{6}}^{\frac{2\pi}{9}} \frac{1+t^2}{2(1+t^2) + (1-t^2)} \times \frac{2}{3(1+t^2)} dt$$

$$= \int_{\frac{\pi}{6}}^{\frac{2\pi}{9}} \frac{1+t^2}{t^2 + 3} dt$$

Annotation: Which is a standard arctan type integral

$$= \frac{2}{3} \int_{\frac{\pi}{6}}^{\frac{2\pi}{9}} \frac{1}{t^2 + 3} dt$$

$$= \frac{2}{3} \times \frac{1}{3} \left[\arctan\left(\frac{t}{\sqrt{3}}\right) \right]_{\frac{\pi}{6}}^{\frac{2\pi}{9}}$$

$$= \frac{2\sqrt{3}}{9} \left[\arctan\left(\frac{t}{\sqrt{3}}\right) - \arctan\left(\frac{t}{\sqrt{3}}\right) \right]$$

$$= \frac{2}{3\sqrt{3}} \left[\frac{\pi}{4} - \frac{\pi}{6} \right]$$

$$= \frac{2}{3\sqrt{3}} \times \frac{\pi}{12} = \frac{\pi\sqrt{3}}{54}$$

Right side (substitution and triangle):

Let $t = \tan\frac{3}{2}x$

$$\frac{dt}{dx} = \frac{3}{2} \sec^2 \frac{3}{2}x$$

$$dx = \frac{2}{3} \frac{dt}{\sec^2 \frac{3}{2}x}$$

$$dx = \frac{2}{3} \frac{dt}{1 + \tan^2 \frac{3}{2}x}$$

$$dx = \frac{2}{3(1+t^2)} dt$$

When $x = \frac{\pi}{6}$, $t = 1$

When $x = \frac{2\pi}{9}$, $t = \sqrt{3}$

Triangle diagram showing $\tan \frac{3}{2}x = t = \frac{t}{1}$

Using the triangle:

$$\cos \frac{3}{2}x = \frac{1}{\sqrt{1+t^2}}$$

$$\cos 3x = \left(\frac{1}{\sqrt{1+t^2}} \right)^2 = \frac{1}{1+t^2}$$

$$\cos 3x = \frac{1-t^2}{1+t^2}$$

Question 83 (*****)

Use the substitution $u = \sqrt{x+2}$ to find

$$\int \frac{16}{(x+6)(x-2)\sqrt{x+2}} dx.$$

$$\ln \left| \frac{\sqrt{x+2}-2}{\sqrt{x+2}+2} \right| - 2 \arctan \left(\frac{\sqrt{x+2}}{2} \right) + C$$

Handwritten solution for the integral problem:

$$\int \frac{16}{(x+6)(x-2)\sqrt{x+2}} dx = \dots \text{by substitution } \dots$$

$u = \sqrt{x+2}$
 $u^2 = x+2$
 $2u \frac{du}{dx} = 1$
 $dx = 2u du$
 $x = u^2 - 2$

$$= \int \frac{16}{(u^2+4)(u^2-2)u} (2u du) = \int \frac{32}{(u^2+4)(u^2-2)} du$$

... PARTIAL FRACTIONS BY INSPECTION

$$= \int \frac{4}{u^2-4} - \frac{4}{u^2+4} du = \int \frac{d}{(u-2)(u+2)} - \frac{4}{u^2+4} du$$

... PARTIAL FRACTIONS BY INSPECTION

$$= \int \frac{1}{u-2} - \frac{1}{u+2} - \frac{4}{u^2+4} du$$

$$= \ln|u-2| - \ln|u+2| - 4 \times \frac{1}{4} \arctan\left(\frac{u}{2}\right) + C$$

$$= \ln \left| \frac{u-2}{u+2} \right| - \arctan\left(\frac{u}{2}\right) + C$$

$$= \ln \left| \frac{\sqrt{x+2}-2}{\sqrt{x+2}+2} \right| - \arctan\left(\frac{\sqrt{x+2}}{2}\right) + C$$

Question 84 (*****)

By using the substitution $t = \tan\left(\frac{x}{2}\right)$, or otherwise, show that

$$\int \frac{5}{4 \cos x + 3 \sin x} dx = \ln \left| \frac{2 + \sin x - 2 \cos x}{2 \sin x + \cos x - 1} \right| + C.$$

proof

Handwritten solution for Question 84 using the substitution $t = \tan\left(\frac{x}{2}\right)$. The solution shows the conversion of the integral into a rational function of t , followed by partial fraction decomposition and integration. A box labeled "proof" is present.

Question 85 (*****)

By using the substitution $x = e^{-\frac{1}{2}u}$, or otherwise, find a simplified expression for

$$\int \frac{x^4 - 1}{x^2 \sqrt{x^4 + 1}} dx.$$

$$\frac{\sqrt{x^4 + 1}}{x} + C$$

Handwritten solution for Question 85 using the substitution $x = e^{-\frac{1}{2}u}$. The solution shows the substitution, differentiation, and integration steps, leading to the final result.

Question 86 (****)

Use a suitable hyperbolic substitution to find the exact value of

$$\int_2^{3.75} \sqrt{(2x+5)(2x-3)} \, dx.$$

$$\frac{195}{16} - 4 \ln 2$$

$$\int_2^{3.75} \sqrt{(2x+5)(2x-3)} \, dx = \int_2^{3.75} \sqrt{4x^2 + 4x - 15} \, dx = \int_2^{3.75} \sqrt{(2x+1)^2 - 16} \, dx$$

Make the integrand to a HYPERBOLIC SUBSTITUTION i.e. $2x+1 = 4 \cosh \theta$
 Then $2x+1 = 4 \cosh \theta$ $2x = 4 \cosh \theta - 1$ $2dx = 4 \sinh \theta \, d\theta$ $dx = 2 \sinh \theta \, d\theta$

$$\int_2^{3.75} \sqrt{(2x+1)^2 - 16} \, dx = \int_{\cosh^{-1} \frac{3}{4}}^{\cosh^{-1} \frac{9}{4}} \sqrt{16(\cosh^2 \theta - 1)} (2 \sinh \theta) \, d\theta$$

$$= \int_{\cosh^{-1} \frac{3}{4}}^{\cosh^{-1} \frac{9}{4}} 16 \sinh \theta (2 \sinh \theta) \, d\theta = \int_{\cosh^{-1} \frac{3}{4}}^{\cosh^{-1} \frac{9}{4}} 32 \sinh^2 \theta \, d\theta$$

$$= \int_{\cosh^{-1} \frac{3}{4}}^{\cosh^{-1} \frac{9}{4}} 16 (2 \sinh^2 \theta) \, d\theta = \int_{\cosh^{-1} \frac{3}{4}}^{\cosh^{-1} \frac{9}{4}} 16 (\cosh 2\theta - 1) \, d\theta$$

$$= [8 \sinh 2\theta - 16\theta]_{\cosh^{-1} \frac{3}{4}}^{\cosh^{-1} \frac{9}{4}}$$

$$= \left[8 \left(\frac{9}{4} \right) - 16 \cosh^{-1} \frac{9}{4} \right] - \left[8 \left(\frac{3}{4} \right) - 16 \cosh^{-1} \frac{3}{4} \right]$$

$$= \frac{255}{16} - 4 \ln \left[\frac{9}{4} + \sqrt{\left(\frac{9}{4} \right)^2 - 1} \right] - \left[\frac{15}{4} + 4 \ln \left[\frac{3}{4} + \sqrt{\left(\frac{3}{4} \right)^2 - 1} \right] \right]$$

$$= \frac{195}{16} - 4 \ln 2 - 4 \ln \frac{1}{2} = \frac{195}{16} - 4 \ln 2$$

SIMPLIFY:
 $\cosh \theta = \frac{9}{4}$
 $\sinh \theta = \frac{7}{4}$
 $\cosh \theta = \frac{3}{4}$
 $\sinh \theta = \frac{1}{4}$

Question 87 (****)

$$\tan 3\theta \equiv \frac{3 \tan \theta - \tan^3 \theta}{1 - 3 \tan^2 \theta}.$$

- a) Prove the validity of the above trigonometric identity by writing $\tan 3\theta$ as $\tan(2\theta + \theta)$.
- b) Hence, show clearly that

$$\int_0^{2-\sqrt{3}} \frac{6x(3-x^2)}{1-2x^2-3x^4} dx = \ln 2.$$

proof

(a) $\tan 3\theta = \tan(2\theta + \theta) = \frac{\tan 2\theta + \tan \theta}{1 - \tan 2\theta \tan \theta}$
 $= \frac{\frac{2 \tan \theta}{1 - \tan^2 \theta} + \tan \theta}{1 - \frac{2 \tan \theta}{1 - \tan^2 \theta} \tan \theta} = \dots$ *{MULTIPLY TOP & BOTTOM OF THE FRACTION BY $1 - \tan^2 \theta$ }*
 $= \frac{2 \tan \theta + \tan \theta(1 - \tan^2 \theta)}{1 - \tan^2 \theta - 2 \tan^2 \theta} = \frac{3 \tan \theta - \tan^3 \theta}{1 - 3 \tan^2 \theta}$ *As required*

(b) $\int_0^{2-\sqrt{3}} \frac{6x(3-x^2)}{1-2x^2-3x^4} dx = \int_0^{2-\sqrt{3}} \frac{6x(3-x^2)}{(3x^2-1)(x^2+1)} dx = \dots$ *Factorize*
 $= \int_0^{2-\sqrt{3}} \frac{6x(3-x^2)}{(3x^2-1)(x^2+1)} dx = \int_0^{2-\sqrt{3}} \frac{6x(3-x^2)}{(1-3x^2)(x^2+1)} dx = \dots$ *by substituting*
 $\begin{cases} u = \tan \theta \\ \frac{du}{d\theta} = \sec^2 \theta \\ u = 0, \theta = 0 \\ x = 2-\sqrt{3}, \theta = \frac{\pi}{12} \end{cases}$
 $\dots = \int_0^{\frac{\pi}{12}} \frac{6 \tan \theta \sec^2 \theta}{\tan^2 \theta + 1} d\theta = \int_0^{\frac{\pi}{12}} 6 \tan \theta d\theta = \left[2 \ln |\sec \theta| \right]_0^{\frac{\pi}{12}}$
 $= 2 \ln(\sec \frac{\pi}{12}) - 2 \ln(\sec 0)$
 $= 2 \ln(\sqrt{3}) = \ln 2$

Question 88 (****)

$$J = \int_0^{\frac{\pi}{3}} \frac{6\sqrt{3} \cos x}{4 + \sin 2x \tan\left(\frac{1}{2}x\right)} dx.$$

Use the substitution $t = \tan\left(\frac{1}{2}x\right)$ to show that

$$J = \pi - 1.$$

, proof

Handwritten solution for Question 88:

(a) $\int_0^{\frac{\pi}{3}} \frac{6\sqrt{3} \cos x}{4 + \sin 2x \tan\left(\frac{1}{2}x\right)} dx$ = Weierstrass Substitution

Let $t = \tan\left(\frac{1}{2}x\right)$

$\frac{dt}{dx} = \frac{1}{2} \sec^2\left(\frac{1}{2}x\right)$

$\frac{dx}{dt} = \frac{2}{1+t^2}$

$x=0, t=0$

$x=\frac{\pi}{3}, t=\tan\left(\frac{\pi}{6}\right) = \frac{1}{\sqrt{3}}$

Substituting into the integral:

$\int_0^{\frac{\pi}{3}} \frac{6\sqrt{3} \cos x}{4 + \sin 2x \tan\left(\frac{1}{2}x\right)} dx = \int_0^{\frac{1}{\sqrt{3}}} \frac{6\sqrt{3} \cos x}{4 + 2 \sin x \cos x \tan\left(\frac{1}{2}x\right)} \cdot \frac{2}{1+t^2} dt$

Multiplying top & bottom by $(1+t^2)^2$

$= \int_0^{\frac{1}{\sqrt{3}}} \frac{6\sqrt{3} (1+t^2)^2 \cos x}{4(1+t^2)^2 + 2 \sin x \cos x \tan\left(\frac{1}{2}x\right) (1+t^2)^2} dt$

$= \int_0^{\frac{1}{\sqrt{3}}} \frac{6\sqrt{3} (1-t^2)}{4(1+t^2)^2 + 2(1-t^2)(1+t^2)} dt$

$= \int_0^{\frac{1}{\sqrt{3}}} \frac{3\sqrt{3} (1-t^2)}{2t^4 + 2t^2 + 1} dt$

As before

$= \int_0^{\frac{1}{\sqrt{3}}} \frac{3\sqrt{3} (1-t^2)}{2t^4 + 2t^2 + 1} dt$

(b) $\dots = \int_0^{\frac{1}{\sqrt{3}}} \frac{3\sqrt{3} (1-t^2)}{2t^4 + 2t^2 + 1} dt = \sqrt{3} \int_0^{\frac{1}{\sqrt{3}}} \frac{3t^2 - 3}{2t^4 + 1} dt = \sqrt{3} \int_0^{\frac{1}{\sqrt{3}}} \frac{3t^2 + 1}{2t^4 + 1} dt$

$= \sqrt{3} \int_0^{\frac{1}{\sqrt{3}}} \frac{4t^2}{t^4 + 1} dt = \sqrt{3} \int_0^{\frac{1}{\sqrt{3}}} \frac{4t^2}{t^4 + 1} dt$

$= \sqrt{3} \left[\frac{1}{2} \arctan\left(\frac{t}{1}\right) - \frac{1}{2} \arctan\left(\frac{t}{1}\right) \right]_0^{\frac{1}{\sqrt{3}}} = \sqrt{3} \left[\frac{1}{2} \arctan\left(\frac{1}{\sqrt{3}}\right) - \frac{1}{2} \arctan\left(\frac{1}{\sqrt{3}}\right) \right]$

$= \sqrt{3} \left[\frac{1}{2} \arctan\left(\frac{1}{\sqrt{3}}\right) - \frac{1}{2} \arctan\left(\frac{1}{\sqrt{3}}\right) \right] = 4 \arctan\left(\frac{1}{\sqrt{3}}\right) - 1$

$= 4 \times \frac{\pi}{6} - 1 = \pi - 1$

Question 89 (*****)

By considering the derivatives of $e^x \sin x$ and $e^x \cos x$, find

$$\int e^x (2 \cos x - 3 \sin x) dx.$$

$$\boxed{}, \quad \boxed{\frac{1}{2} e^x (5 \cos x - \sin x) + C}$$

Handwritten solution for Question 89:

$$\begin{aligned} \frac{d}{dx}(e^x \sin x) &= e^x \sin x + e^x \cos x \\ \frac{d}{dx}(e^x \cos x) &= e^x \cos x - e^x \sin x \end{aligned} \quad \text{Add & subtract gives}$$

$$\begin{aligned} \frac{d}{dx}(e^x \sin x + e^x \cos x) &= 2e^x \cos x \\ \frac{d}{dx}(e^x \sin x - e^x \cos x) &= 2e^x \sin x \end{aligned} \quad \Rightarrow \quad \begin{aligned} \frac{d}{dx}\left(\frac{1}{2}(e^x \sin x + e^x \cos x)\right) &= e^x \cos x \\ \frac{d}{dx}\left(\frac{1}{2}(e^x \sin x - e^x \cos x)\right) &= e^x \sin x \end{aligned}$$

Therefore

$$\begin{aligned} 2e^x \cos x - 3e^x \sin x &= 2 \frac{d}{dx}\left(\frac{1}{2}(e^x \sin x + e^x \cos x)\right) - 3 \frac{d}{dx}\left(\frac{1}{2}(e^x \sin x - e^x \cos x)\right) \\ 2e^x \cos x - 3e^x \sin x &= \frac{d}{dx}\left[e^x(\sin x + \cos x) - \frac{3}{2}e^x(\sin x - \cos x)\right] \\ 2e^x \cos x - 3e^x \sin x &= \frac{d}{dx}\left[\frac{1}{2}e^x(2\cos x + 2\sin x - 3\sin x + 3\cos x)\right] \\ 2e^x \cos x - 3e^x \sin x &= \frac{d}{dx}\left[\frac{1}{2}e^x(5\cos x - \sin x)\right] \\ \therefore \int (2\cos x - 3\sin x) dx &= \frac{1}{2}e^x(5\cos x - \sin x) + C \end{aligned}$$

Question 90 (*****)

Use integration by parts and suitable trigonometric identities to find

$$\int \sec^3 x dx.$$

$$\boxed{\frac{1}{2} \sec x \tan x + \frac{1}{2} \ln |\sec x + \tan x| + C}$$

Handwritten solution for Question 90:

$$\begin{aligned} \int \sec^3 x dx &= \int \sec x \sec^2 x dx \dots \text{by parts} \\ \int \sec^3 x dx &= \sec x \tan x - \int \sec x \tan^2 x dx \\ \int \sec^3 x dx &= \sec x \tan x - \int \sec x (\sec^2 x - 1) dx \\ \int \sec^3 x dx &= \sec x \tan x - \int \sec^3 x dx + \int \sec x dx \\ 2 \int \sec^3 x dx &= \sec x \tan x + \int \sec x dx \quad \text{Simultaneous result} \\ 2 \int \sec^3 x dx &= \sec x \tan x + \ln |\sec x + \tan x| + C \\ \int \sec^3 x dx &= \frac{1}{2} \sec x \tan x + \frac{1}{2} \ln |\sec x + \tan x| + C \end{aligned}$$

Question 91 (****)

Show that the exact value of

$$\int_0^1 \frac{(1-x)e^x}{x^2 + e^{2x}} dx,$$

can be written as

$$\operatorname{arccot}(e).$$

, proof

THE FORM OF THE ANSWER SUGGESTS AN ARCTAN FORM
MAY NEED TO BE "CREATED"

$$\int_0^1 \frac{e^x(1-x)}{x^2 + e^{2x}} dx = \int_0^1 \frac{e^x(1-x)}{x^2 e^{2x} + e^{2x} e^{2x}} dx$$

$$= \int_0^1 \frac{e^x(1-x)}{x^2 e^{4x} + e^{4x}} dx$$

$$= \int_0^1 \frac{e^x(1-x)}{(x^2 + 1)e^{4x}} dx$$

PROCEED BY SUBSTITUTION

$u = xe^{2x}$ $q = 2=0 \rightarrow u=0$
 $\frac{du}{dx} = 1 \cdot e^{2x} + x \cdot (2e^{2x})$ $2=1 \rightarrow u=e^1$
 $\frac{du}{dx} = e^{2x} + 2xe^{2x}$
 $\frac{du}{dx} = e^{2x}(1+2x)$
 $\frac{du}{dx} = \frac{du}{e^{2x}(1+2x)}$

TRANSFORMING THE INTEGRAL

$$\int_0^1 \frac{e^x(1-x)}{(x^2 + 1)e^{4x}} dx = \int_0^1 \frac{e^x(1-x)}{x^2 + 1} \cdot \frac{du}{e^{4x}} dx$$

$$= \int_0^1 \frac{1}{u^2 + 1} du$$

$$= \left[\arctan u \right]_0^1$$

$$= \arctan \frac{1}{e} - \arctan 0$$

$$= \arctan e$$

to reverse

Question 92 (****)

By using the substitution $u = 1 + e^{-x} \tan x$, or otherwise, show that the exact value of

$$\int_0^{\frac{1}{4}\pi} \frac{2 - \sin 2x}{(1 + \cos 2x)e^x + \sin 2x} dx,$$

can be written as

$$\ln \left[2e^{-\frac{1}{8}\pi} \cosh \left(\frac{1}{8}\pi \right) \right].$$

, proof

USING THE SUBSTITUTION $u = 1 + e^{-x} \tan x$

$$\frac{du}{dx} = -e^{-x} \tan x + e^{-x} \sec^2 x$$

$$dx = \frac{du}{e^{-x} \sec^2 x - e^{-x} \tan x}$$

$$dx = \frac{e^x}{\sec^2 x - \tan x} du$$

when $x=0 \rightarrow u=1$
 $x=\frac{\pi}{4} \rightarrow u = 1 + e^{-\frac{\pi}{4}} = \alpha$

TRANSFORMING THE INTEGRAL

$$\int_0^{\frac{\pi}{4}} \frac{2 - \sin 2x}{e^x(1 + \cos 2x) + \sin 2x} dx$$

$$= \int_1^{\alpha} \frac{2 - 2 \sin x \cos x}{e^x(1 + 2 \cos^2 x - 1) + 2 \sin x \cos x} du$$

$$= \int_1^{\alpha} \frac{2 - 2 \sin x \cos x}{2e^x \cos^2 x + 2 \sin x \cos x} du$$

$$= \int_1^{\alpha} \frac{1 - \sin x \cos x}{e^x \cos^2 x + \sin x \cos x} du$$

$$= \int_1^{\alpha} \frac{1 - \sin x \cos x}{e^x \cos^2 x + \sin x \cos x} \times \frac{e^x}{\sec^2 x - \tan x} du$$

NEXT, DIVIDE 'TOP & BOTTOM' OF THE FIRST FRACTION BY $\cos^2 x$

$$= \int_1^{\alpha} \frac{\frac{1}{\cos^2 x} - \frac{\sin x \cos x}{\cos^2 x}}{\frac{e^x \cos^2 x}{\cos^2 x} - \frac{\sin x \cos x}{\cos^2 x}} \times \frac{e^x}{\sec^2 x - \tan x} du$$

$$= \int_1^{\alpha} \frac{\sec^2 x - \tan x}{e^x - \tan x} \times \frac{e^x}{\sec^2 x - \tan x} du$$

$$= \int_1^{\alpha} \frac{e^x}{e^x - \tan x} du$$

$$= \int_1^{\alpha} \frac{e^x}{e^x - e^{-x} \tan x} du$$

$$= \int_1^{\alpha} \frac{1}{1 - e^{-2x} \tan x} du$$

$$= \int_1^{\alpha} \frac{1}{u} du$$

$$= [\ln |u|]_1^{\alpha} = \ln \left(\frac{\alpha}{1} \right) = \ln \left(\frac{e^{\frac{\pi}{4}}}{e^{-\frac{\pi}{4}}} (e^{\frac{\pi}{4}} + e^{-\frac{\pi}{4}}) \right)$$

$$= \ln \left[2e^{\frac{\pi}{8}} \cosh \left(\frac{\pi}{8} \right) \right]$$

✓

Question 93 (*****)

The function f is a continuous function and a is a real constant.

$$\int_0^a f(x) \, dx \equiv \int_0^a f(a-x) \, dx.$$

a) Prove the validity of the above identity.

b) Hence show clearly that

$$\int_0^{\pi} \frac{x \sin x}{1 + \cos^2 x} \, dx = \frac{1}{4} \pi^2.$$

proof

(b) $\int_0^{\pi} \frac{x \sin x}{1 + \cos^2 x} \, dx$
 Let $x = a - y \Rightarrow y = a - x$ • LIMITS
 $\frac{dx}{dy} = -1$ $x=0, y=a$
 $dx = -dy$ $x=a, y=0$
 $= \int_a^0 \frac{f(a-y)}{1 + \cos^2(a-y)} (-dy) = \int_0^a \frac{f(a-y)}{1 + \cos^2(a-y)} dy$
 (Nothing shorter about a neg)
 Now $\sin(\pi-x) = \sin x$
 $\cos(\pi-x) = -\cos x$
 $\int_0^{\pi} \frac{x \sin x}{1 + \cos^2 x} \, dx = \int_0^{\pi} \frac{(\pi-x) \sin(\pi-x)}{1 + \cos^2(\pi-x)} \, dx$
 $\int_0^{\pi} \frac{x \sin x}{1 + \cos^2 x} \, dx = \int_0^{\pi} \frac{(\pi-x) \sin x}{1 + \cos^2 x} \, dx$
 $\int_0^{\pi} \frac{x \sin x}{1 + \cos^2 x} \, dx = \int_0^{\pi} \frac{\pi \sin x}{1 + \cos^2 x} \, dx - \int_0^{\pi} \frac{x \sin x}{1 + \cos^2 x} \, dx$
 $2 \int_0^{\pi} \frac{x \sin x}{1 + \cos^2 x} \, dx = \pi \int_0^{\pi} \frac{\sin x}{1 + \cos^2 x} \, dx$
 $2 \int_0^{\pi} \frac{x \sin x}{1 + \cos^2 x} \, dx = \pi \left[-\arctan(\cos x) \right]_0^{\pi}$
 $2 \int_0^{\pi} \frac{x \sin x}{1 + \cos^2 x} \, dx = \pi \left[\arctan(1) - \arctan(-1) \right] = \pi \left[\arctan(1) + \arctan(1) \right]$
 $2 \int_0^{\pi} \frac{x \sin x}{1 + \cos^2 x} \, dx = \frac{\pi^2}{2}$
 $\int_0^{\pi} \frac{x \sin x}{1 + \cos^2 x} \, dx = \frac{1}{2} \pi^2$

Question 94 (****)

By using a suitable substitution, find the exact value of

$$\int_1^{\sqrt[4]{17}} \frac{2x}{\sqrt{x^4-1}} dx.$$

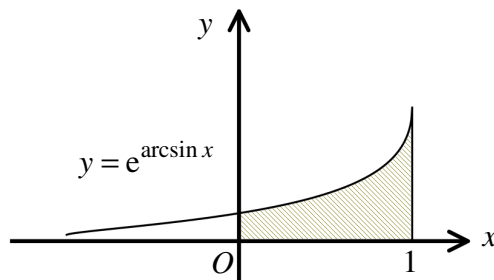
$$\boxed{}, \ln(4 + \sqrt{17})$$

Handwritten solution for the integral problem:

$\int_1^{\sqrt[4]{17}} \frac{2x}{\sqrt{x^4-1}} dx = \dots$ Substitution \dots
 $\dots = \int_0^{\ln(4+\sqrt{17})} \frac{\frac{2x}{\sqrt{x^4-1}}}{\frac{\sinh \theta}{2x}} \left(\frac{\sinh \theta}{2x} d\theta \right)$
 $= \int_0^{\ln(4+\sqrt{17})} \frac{\sinh \theta}{\sqrt{\sinh^2 \theta}} d\theta = \int_0^{\ln(4+\sqrt{17})} 1 d\theta$
 $= \left[\theta \right]_0^{\ln(4+\sqrt{17})} = \ln(4+\sqrt{17}) //$

$x^2 = \cosh \theta$
 $2x dx = \sinh \theta d\theta$
 $\frac{2x}{\sqrt{x^4-1}} = \frac{\sinh \theta}{\sqrt{\cosh^2 \theta - 1}} = \frac{\sinh \theta}{\sinh \theta} = 1$
 $\bullet \ 2 = 1 \Rightarrow \cosh \theta$
 $\Rightarrow \theta = 0$
 $\bullet \ 2 = 17^{\frac{1}{4}} \Rightarrow \cosh \theta$
 $\Rightarrow \theta = \ln \left[\frac{17^{\frac{1}{4}} + \sqrt{17^{\frac{1}{4}} - 1}}{2} \right]$
 $\theta = \ln(4 + \sqrt{17})$

Question 95 (****)



The figure above show the curve with equation

$$y = e^{\arcsin x}, \quad x \in \mathbb{R}, \quad |x| \leq 1.$$

The finite region, shown shaded in the figure, bounded by the curve, the coordinate axes and the straight line with equation $x = 1$, is fully revolved about the x axis.

Find, an exact simplified value, for the volume of the solid of revolution formed.

$$\boxed{\frac{1}{5}\pi(e^\pi - 2)}$$

SETTING UP A DOUBLE INTEGRAL

$$V = \pi \int_{-1}^1 (y(x))^2 dx = \pi \int_{-1}^1 (e^{\arcsin x})^2 dx$$

$$= \pi \int_{-1}^1 e^{2\arcsin x} dx$$

BY SUBSTITUTION

$$\theta = \arcsin x \quad x = 0 \rightarrow \theta = 0$$

$$\sin \theta = x \quad x = 1 \rightarrow \theta = \frac{\pi}{2}$$

$$\frac{dx}{d\theta} = \cos \theta$$

$$dx = \cos \theta d\theta$$

TRANSFORMING THE INTEGRAL

$$\rightarrow V = \pi \int_0^{\frac{\pi}{2}} e^{2\theta} (\cos \theta d\theta) = \pi \int_0^{\frac{\pi}{2}} e^{2\theta} \cos \theta d\theta$$

BY PARTS TWICE OR COMPLEX NUMBERS

$$\Rightarrow V = \pi \operatorname{Re} \left\{ \int_0^{\frac{\pi}{2}} e^{2\theta} e^{i\theta} d\theta \right\}$$

$$\Rightarrow V = \pi \operatorname{Re} \left\{ \int_0^{\frac{\pi}{2}} e^{(2+i)\theta} d\theta \right\}$$

$$\Rightarrow V = \pi \operatorname{Re} \left\{ \left[\frac{1}{2+i} e^{(2+i)\theta} \right]_0^{\frac{\pi}{2}} \right\}$$

$$\Rightarrow V = \pi \operatorname{Re} \left\{ \frac{1}{2+i} [e^{(2+i)\frac{\pi}{2}} - 1] \right\}$$

$$\Rightarrow V = \pi \operatorname{Re} \left\{ \frac{2-i}{(2+i)(2-i)} [e^{2\frac{\pi}{2}} e^{i\frac{\pi}{2}} - 1] \right\}$$

$$\Rightarrow V = \pi \operatorname{Re} \left\{ \frac{2-i}{5} [e^\pi (i) - 1] \right\}$$

$$\Rightarrow V = \pi \operatorname{Re} \left\{ \frac{2-i}{5} (ie^\pi - 1) \right\}$$

$$\Rightarrow V = \frac{\pi}{5} \operatorname{Re} \{ 2ie^\pi - 2 + e^\pi + i \}$$

$$\Rightarrow V = \frac{\pi}{5} (e^\pi - 2)$$

Question 96 (*****)

$$I = \int_0^1 \frac{(x^2 + 1)(x^2 + 4)}{(x^2 + 3)(x^2 - 4)} dx.$$

Use appropriate integration techniques to show that

$$I = 1 + \frac{2}{7} \left[\frac{\pi}{6\sqrt{3}} - 5 \ln 3 \right].$$

, proof

SPLIT BY PARTIAL FRACTIONS

$$\frac{(x^2+1)(x^2+4)}{(x^2+3)(x^2-4)} = \frac{x^4+5x^2+4}{x^4-x^2-12} = \frac{(x^2-2)(x^2+6)}{(x^2-4)(x^2-3)}$$

$$= 1 + \frac{6x^2+16}{x^4-x^2-12} = 1 + \frac{6x^2+16}{(x^2+3)(x^2-4)}$$

NOW LET $t = x^2$

$$\dots = 1 + \frac{6t+16}{(t+3)(t-4)} = 1 + \frac{\frac{-2}{7}}{t+3} + \frac{\frac{40}{7}}{t-4}$$

(BY COVER-UP METHOD)

$$= 1 + \frac{2}{7} \left[\frac{1}{t+3} + \frac{20}{t-4} \right]$$

$$= 1 + \frac{2}{7} \left[\frac{1}{x^2+3} + \frac{20}{x^2-4} \right]$$

$$= 1 + \frac{2}{7} \left[\frac{1}{x^2+3} + \frac{20}{(x-2)(x+2)} \right]$$

(BY COVER-UP METHOD)

$$= 1 + \frac{2}{7} \left[\frac{1}{x^2+3} + \frac{5}{x-2} - \frac{5}{x+2} \right]$$

RETURNING TO THE INTEGRAL

$$\int_0^1 \frac{(x^2+1)(x^2+4)}{(x^2+3)(x^2-4)} dx = \int_0^1 \left[1 + \frac{2}{7} \left[\frac{1}{x^2+3} + \frac{5}{x-2} - \frac{5}{x+2} \right] \right] dx$$

$$= \left[x + \frac{2}{7} \left[\frac{1}{\sqrt{3}} \arctan\left(\frac{x}{\sqrt{3}}\right) + 5 \ln|x-2| - 5 \ln|x+2| \right] \right]_0^1$$

$$= \left[1 + \frac{2}{7} \left[\frac{1}{\sqrt{3}} \times \frac{\pi}{6} + 5 \ln|1-2| - 5 \ln|1+2| \right] \right]$$

$$= 1 + \frac{2}{7} \left[\frac{\pi}{6\sqrt{3}} - 5 \ln 3 \right]$$

$$= 1 + \frac{2}{7} \left[\frac{\pi}{3\sqrt{3}} - 10 \ln 3 \right]$$

Question 97 (*****)

By using a suitable substitution, find the exact value of

$$\int_{\sqrt[4]{3}}^{\sqrt[4]{8}} \frac{2}{x\sqrt{x^4+1}} dx.$$

$$\boxed{}, \ln\left(\frac{3}{2}\right)$$

Handwritten solution for the integral problem:

Let $u = \sqrt{x^4 + 1}$
 $\frac{du}{dx} = \frac{4x^3}{2u} = \frac{2x^3}{u}$
 $du = \frac{2x^3}{u} dx$
 $u du = x^3 dx$
 $\frac{1}{2} u^2 = \frac{1}{4} x^4 + \frac{1}{2}$
 $u^2 = \frac{1}{2} x^4 + 1$
 $u = \sqrt{\frac{1}{2} x^4 + 1}$
 $\frac{du}{dx} = \frac{x^3}{\sqrt{\frac{1}{2} x^4 + 1}}$
 $dx = \frac{\sqrt{\frac{1}{2} x^4 + 1}}{x^3} du$
 $\int \frac{2}{x\sqrt{x^4+1}} dx = \int \frac{2}{x\sqrt{\frac{1}{2} x^4 + 1}} \cdot \frac{\sqrt{\frac{1}{2} x^4 + 1}}{x^3} du$
 $= \int \frac{2}{x^4} du$
 $= \int \frac{2}{u^2} du$
 $= -\frac{2}{u} + C$
 $= -\frac{2}{\sqrt{\frac{1}{2} x^4 + 1}} + C$
 $= -\frac{2}{\sqrt{\frac{1}{2} (\sqrt[4]{8})^4 + 1}} + \frac{2}{\sqrt{\frac{1}{2} (\sqrt[4]{3})^4 + 1}}$
 $= -\frac{2}{\sqrt{2 + 1}} + \frac{2}{\sqrt{1 + 1}}$
 $= -\frac{2}{\sqrt{3}} + \frac{2}{\sqrt{2}}$
 $= \frac{2}{\sqrt{2}} - \frac{2}{\sqrt{3}}$
 $= \frac{2\sqrt{2}}{2} - \frac{2\sqrt{3}}{2}$
 $= \sqrt{2} - \sqrt{3}$

Question 98 (****)

$$I = \int_0^{\arctan(\tanh(\ln 2))} \frac{\sec^2 x \tan 2x}{\tan x - \tan^3 x} dx$$

Use appropriate integration techniques to show that

$$I = k + \ln 2,$$

where k is a rational constant to be found.

You may assume that the limit of the integrand, as x tends to zero, exists.

$$\boxed{}, k = \frac{15}{16}$$

PROCEED BY A SUBSTITUTION AFTER REVEALING THE $\tan x$
IN TERMS OF $\tan x$

$$\int_0^{\arctan(\tan(h_2))} \frac{\tan x \sec^2 x}{\tan x - \tan^3 x} dx = \int_0^{\arctan(\tan(h_2))} \frac{\sec^2 x}{\tan x (1 - \tan^2 x)} dx$$

$$= \int_0^{\arctan(\tan(h_2))} \frac{\sec^2 x}{(1 - \tan^2 x) \tan x} dx$$

$$= \int_0^{\arctan(\tan(h_2))} \frac{\sec^2 x}{(1 - \tan^2 x)^2} dx$$

$$= \int_0^{\arctan(h_2)} \frac{\sec^2 x}{(1 - u^2)^2} \left(\frac{du}{\sec^2 x} \right)$$

$$= \int_0^{\arctan(h_2)} \frac{du}{(1 - u^2)^2} du$$

ANOTHER SUBSTITUTION IS NEEDED

$$= \int_0^{h_2} \frac{2}{(1 - u^2)^2} (\sec^2 \theta) d\theta$$

$$= \int_0^{h_2} \frac{2 \sec^2 \theta}{\sec^4 \theta} d\theta$$

$$= \int_0^{h_2} \frac{2}{\sec^2 \theta} d\theta$$

$$\begin{aligned} \theta &= \arctan u \\ u &= \tan \theta \\ \frac{du}{d\theta} &= \sec^2 \theta \\ du &= \sec^2 \theta d\theta \\ u &= \tan(h_2), \theta = h_2 \\ u=0, \theta &= 0 \end{aligned}$$

$$= \int_0^{h_2} 2 \cos^2 \theta d\theta = \int_0^{h_2} 2 \left(\frac{1}{2} + \frac{1}{2} \cos(2\theta) \right) d\theta$$

$$= \int_0^{h_2} (1 + \cos(2\theta)) d\theta = \left[\theta + \frac{1}{2} \sin(2\theta) \right]_0^{h_2}$$

$$= \left[\theta + \sin \theta \cos \theta \right]_0^{h_2} = \left[h_2 + \sin(h_2) \cos(h_2) - 0 \right]$$

$$= h_2 + \frac{1}{2} [e^{h_2} - e^{-h_2}] + \frac{1}{2} [e^{h_2} + e^{-h_2}]$$

$$= h_2 + \frac{1}{4} (2 - \frac{1}{2}) (2 + \frac{1}{2})$$

$$= h_2 + \frac{1}{4} \times \frac{9}{2} \times \frac{5}{2}$$

$$= \frac{15}{16} + h_2$$

Question 99 (*****)

Use a suitable hyperbolic substitution to find a simplified expression for

$$\int \sqrt{(2x+5)(2x-3)} \, dx.$$

$$\boxed{}, \left[\frac{1}{4}(2x+1)\sqrt{(2x+5)(2x-3)} - 4 \ln \left[2x+1 + \sqrt{(2x+5)(2x-3)} \right] + C \right]$$

Handwritten solution for the integral:

$$\begin{aligned} \int \sqrt{(2x+5)(2x-3)} \, dx &= \int \sqrt{4x^2 + 4x - 15} \, dx = \int \sqrt{4x^2 + 4x + 1 - 16} \, dx \\ &= \int \sqrt{(2x+1)^2 - 16} \, dx \\ &\bullet \text{ Use a hyperbolic substitution } \Rightarrow \text{ let } 2x+1 = 4 \cosh u \\ &\quad 2dx = 4 \sinh u \, du \quad \text{This } u = \cosh^{-1}\left(\frac{2x+1}{4}\right) \\ &\quad dx = 2 \sinh u \, du \quad \cosh^2 u - \sinh^2 u = 1 \\ &\quad \quad \quad 16 \cosh^2 u - 16 \sinh^2 u = 16 \\ &\quad \quad \quad 4 \sinh u + 16 \cosh u = 16 \\ &\therefore \int \sqrt{(4 \cosh u)^2 - 16} (2 \sinh u \, du) = \int \sqrt{16(\cosh^2 u - 1)} (2 \sinh u \, du) \\ &= \int \sqrt{16 \sinh^2 u} (2 \sinh u \, du) = \int 8 \sinh^2 u \, du \\ &\quad \sinh u = \frac{1}{2} - \frac{1}{2} \cosh 2u \\ &\quad -\cosh u = \frac{1}{2} - \frac{1}{2} \cosh 2u \\ &= \int 8 \left(\frac{1}{4} \cosh 2u - \frac{1}{4} \right) du = \int 2 \cosh 2u - 2 \, du = 2 \sinh 2u - 2u + C \\ &= 4 \sinh u \cosh u - 2u + C = \frac{1}{4} (4 \sinh u)(4 \cosh u) - 2u + C \\ &= \frac{1}{4} \left[4 \sinh u \sqrt{16 \cosh^2 u - 16} \right] - 2 \ln \left[\frac{2x+1}{4} + \sqrt{\left(\frac{2x+1}{4}\right)^2 - 1} \right] + C \\ &= \frac{1}{4} \left[(2x+1) \sqrt{(2x+5)(2x-3)} \right] - 4 \ln \left[\frac{2x+1}{4} + \sqrt{\frac{(2x+1)^2 - 16}{16}} \right] + C \\ &= \frac{1}{4} (2x+1) \sqrt{(2x+5)(2x-3)} - 4 \ln \left[2x+1 + \sqrt{(2x+5)(2x-3)} \right] + C \end{aligned}$$

Question 100 (*****)

It is given that the following integral converges to a finite value L .

$$\int_0^1 \frac{\ln x}{x-1} dx.$$

Show, with details workings, that

$$L = \sum_{r=1}^{\infty} \left[\frac{1}{r^2} \right].$$

You may further assume that integration and summation commute.

SPY, proof

START WITH A SUBSTITUTION

$$\begin{aligned} u &= x-1 \\ du &= dx \\ 0 &\rightarrow -1 \\ 1 &\rightarrow 0 \end{aligned}$$

$$\int_0^1 \frac{\ln x}{x-1} dx = \int_{-1}^0 \frac{\ln(u+1)}{u} du$$

NOW RECALL POWER SERIES

$$\begin{aligned} \ln(1+u) &= u - \frac{1}{2}u^2 + \frac{1}{3}u^3 - \frac{1}{4}u^4 + \dots \\ \ln(1+u) &= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} u^n \end{aligned}$$

PREPARE TO THE INTEGRAL

$$\dots \rightarrow \int_{-1}^0 \frac{1}{u} \ln(1+u) du = \int_{-1}^0 \frac{1}{u} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} u^n du$$

REVERSING THE ORDER OF INTEGRATION AND SUMMATION, GIVING CONVERGENCE

$$\begin{aligned} \dots &= \sum_{n=1}^{\infty} \left[\frac{(-1)^{n+1}}{n} \int_{-1}^0 \frac{1}{u} u^n du \right] = \sum_{n=1}^{\infty} \left[\frac{(-1)^{n+1}}{n} \int_{-1}^0 u^{n-1} du \right] \\ &= \sum_{n=1}^{\infty} \left[\frac{(-1)^{n+1}}{n} \left[\frac{1}{n} u^n \right]_{-1}^0 \right] = \sum_{n=1}^{\infty} \left[\frac{(-1)^{n+1}}{n} \times \frac{1}{n} (0 - (-1)^n) \right] \\ &= \sum_{n=1}^{\infty} \left[\frac{(-1)^{n+1}}{n^2} (0 - (-1)^n) \right] = \sum_{n=1}^{\infty} \left[\frac{(-1)^{2n+1}}{n^2} \right] = \sum_{n=1}^{\infty} \left[\frac{(-1)^{2n}}{n^2} \right] \\ &= \sum_{n=1}^{\infty} \left[\frac{1}{n^2} \right] \end{aligned}$$

$\therefore \int_0^1 \frac{\ln x}{x-1} dx = \sum_{n=1}^{\infty} \frac{1}{n^2}$ At QED

Question 101 (*****)

a) If $p \in (0, \infty)$, show that

$$\lim_{x \rightarrow 0^+} [x^p \ln x] = 0, \quad x \in (0, \infty).$$

b) Hence find a simplified expression for

$$\int_0^1 x^n \ln x \, dx, \quad n \in \mathbb{N}.$$

c) Hence, showing a detailed method, evaluate

$$\int_0^1 [\ln(1-x)] \ln x \, dx.$$

You may assume without proof that

- the integral converges.
- integration and summation commute.

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{6} \pi^2, \quad n \in \mathbb{N}.$$

$$\square, \quad -\frac{1}{(n+1)^2}, \quad 2 - \frac{1}{6} \pi^2$$

1) THE LIMIT IS OF THE TYPE $0 \times \infty$, SO MANIPULATE IT FURTHER

$$\lim_{x \rightarrow 0^+} [x^p \ln x] = \lim_{x \rightarrow 0^+} \left[\frac{\ln x}{x^{-p}} \right] \leftarrow \text{OF THE TYPE } \frac{\infty}{\infty}$$

PARTIAL DIFFERENTIAL RULE

$$= \lim_{x \rightarrow 0^+} \left[\frac{\frac{1}{x}}{-p x^{-p-1}} \right] = \lim_{x \rightarrow 0^+} \left[\frac{1}{-p x^p} \right]$$

$$= -\frac{1}{p} \lim_{x \rightarrow 0^+} \left[\frac{1}{x^p} \right] = -\frac{1}{p} \lim_{x \rightarrow 0^+} \left[\frac{1}{x^p} \right] = -\frac{1}{p} \lim_{x \rightarrow 0^+} \left[\frac{1}{x^p} \right] = 0$$

2) INTEGRATION BY PARTS

$$\int_0^1 x^n \ln x \, dx = \left[\frac{x^{n+1} \ln x}{n+1} - \int_0^1 \frac{x^{n+1}}{n+1} \cdot \frac{1}{x} \, dx \right] = \left[\frac{x^{n+1} \ln x}{n+1} - \frac{1}{(n+1)^2} x^{n+1} \right]_0^1$$

AT $x=1$, $\ln x = 0$ AS $\ln 1 = 0$
 AT $x=0$, THE TERM $\frac{1}{x}$ TENDS TO ∞ (DIVERGES)

$$\int_0^1 x^n \ln x \, dx = -\frac{1}{(n+1)^2} x^{n+1} \Big|_0^1 = -\frac{1}{(n+1)^2}$$

3) APPROACH THE NORMAL WAY, BUT SEEMS SLIGHT

$$\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} \dots = -\sum_{n=1}^{\infty} \frac{x^n}{n}$$

$$\int_0^1 \ln(1-x) \ln x \, dx = \int_0^1 \left[-\sum_{n=1}^{\infty} \frac{x^n}{n} \right] \ln x \, dx$$

$$= -\sum_{n=1}^{\infty} \left[\frac{1}{n} \int_0^1 x^n \ln x \, dx \right]$$

INTEGRATION SUMMATION
 WITH INTEGRATION

$$= -\sum_{n=1}^{\infty} \left[\frac{1}{n} \left(-\frac{1}{(n+1)^2} \right) \right] \rightarrow \text{WELL BE}$$

$$= \sum_{n=1}^{\infty} \frac{1}{n(n+1)^2}$$

NEXT, PROCEED BY PARTIAL FRACTIONS

$$\frac{1}{n(n+1)^2} = \frac{A}{n} + \frac{B}{n+1} + \frac{C}{(n+1)^2}$$

$$1 = A(n+1)^2 + Bn(n+1) + Cn$$

IF $n=0$, $1 = A$
 IF $n=-1$, $0 = -C$
 IF $n=1$, $1 = 4A + 2B + C$
 $1 = 4 + 2B - 1$
 $2 = 2B$
 $B = 1$

RETURNING TO THE SUM

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)^2} = \sum_{n=1}^{\infty} \left[\frac{1}{n} - \frac{1}{n+1} + \frac{1}{(n+1)^2} \right]$$

$$= \sum_{n=1}^{\infty} \left[\frac{1}{n} - \frac{1}{n+1} \right] + \sum_{n=1}^{\infty} \frac{1}{(n+1)^2}$$

$$= \left(1 - \frac{1}{2} \right) + \left(\frac{1}{2} - \frac{1}{3} \right) + \left(\frac{1}{3} - \frac{1}{4} \right) + \dots = 1$$

$$= 1 - \frac{1}{2} + 1$$

$$= 2 - \frac{1}{2}$$

$$= \frac{3}{2}$$

Question 102 (****)

$$I = \int \cos(\ln x) dx \quad \text{and} \quad J = \int \sin(\ln x) dx$$

- a) Use an appropriate method to find expressions for I and J .
- b) Use the integral $\int x^i dx$, where i is the imaginary unit, to verify the answers given in part (a).
- c) Find an exact simplified value for

$$\int_1^{e^{\frac{\pi}{2}}} 2x^i dx.$$

$$\boxed{}, \quad I = \frac{1}{2} x [\sin(\ln x) + \cos(\ln x)], \quad J = \frac{1}{2} x [\sin(\ln x) - \cos(\ln x)],$$

$$\int_1^{e^{\frac{\pi}{2}}} 2x^i dx = \left(e^{\frac{1}{2}\pi} - 1\right) + \left(e^{\frac{1}{2}\pi} + 1\right)i$$

a) SPLITTING WITH A SUBSTITUTION

$$u = \ln x \quad \frac{du}{dx} = \frac{1}{x} \quad dx = x du$$

$$I = \int \cos(\ln x) dx = \int \cos(u) x du = \int \cos(u) e^u du$$

NOW DOUBLE INTEGRATION BY PARTS, (SOME EXACT, SOME BY INSPECTION)

$$\frac{1}{2} \left[e^u (\cos(u) + \sin(u)) \right] + \frac{1}{2} \left[e^u (\cos(u) - \sin(u)) \right]$$

$$= \frac{1}{2} e^u [\cos(u) + \sin(u) + \cos(u) - \sin(u)]$$

$$= \frac{1}{2} e^u [2\cos(u)]$$

$$= e^u \cos(u)$$

$$\Rightarrow I = \frac{1}{2} x [\cos(\ln x) + \sin(\ln x)]$$

USING THE OTHER SUBSTITUTION AND APPROACH

$$J = \int \sin(\ln x) dx = \dots = \int e^u \sin(u) du$$

BUT NOW

$$P+Q=0 \quad P-Q=1$$

$$Q=\frac{1}{2} \quad P=-\frac{1}{2}$$

$$\Rightarrow J = \frac{1}{2} x [\sin(\ln x) - \cos(\ln x)]$$

b) SIMPLY BY CONSIDERING z^i

$$z^i = e^{i \ln z} = \cos(\ln z) + i \sin(\ln z)$$

$$z^i = \cos(\ln z) + i \sin(\ln z)$$

$$\int z^i dz = \frac{1}{i+1} z^{i+1} + C$$

$$\int \cos(\ln z) + i \sin(\ln z) dz = \frac{1}{i+1} z^{i+1} + C$$

$$\int \cos(\ln z) dz + i \int \sin(\ln z) dz = \frac{1}{i+1} z^{i+1} + C$$

$$I + iJ = \frac{1}{i+1} z^{i+1} + C$$

$$I + iJ = \frac{1}{i+1} [\cos(\ln z) + i \sin(\ln z)] z^{i+1} + C$$

$$I + iJ = \frac{1}{i+1} [\cos(\ln z) + i \sin(\ln z)] z^{i+1} + C$$

$$I + iJ = \frac{1}{i+1} [\cos(\ln z) + i \sin(\ln z)] z^{i+1} + C$$

$$\therefore I = \frac{1}{2} x [\cos(\ln x) + \sin(\ln x)] \quad \text{and} \quad J = \frac{1}{2} x [\sin(\ln x) - \cos(\ln x)]$$

c) FIND THE EXACT PART (b)

$$\int_1^{e^{\frac{\pi}{2}}} z^i dz = \frac{1}{i+1} z^{i+1} \Big|_1^{e^{\frac{\pi}{2}}}$$

$$= \frac{1}{i+1} [\cos(\ln z) + i \sin(\ln z)] z^{i+1} \Big|_1^{e^{\frac{\pi}{2}}}$$

$$= \frac{1}{i+1} [\cos(\ln z) + i \sin(\ln z)] z^{i+1} \Big|_1^{e^{\frac{\pi}{2}}}$$

$$= \frac{1}{i+1} [(0+1) + i(1-0)] = \frac{1}{i+1} [1 + i]$$

$$= \frac{1}{i+1} (1+i) = \frac{1}{i+1} (1+i)$$

$$= \frac{1}{i+1} (1+i) = \frac{1}{i+1} (1+i)$$

Question 103 (****)

$$I(\alpha) = \int_0^{\pi} \frac{1}{\alpha - \cos x} dx, \quad |\alpha| > 1.$$

Use an appropriate method to show that

$$I(\alpha) = \frac{\pi}{\sqrt{\alpha^2 - 1}}.$$

, proof

Handwritten solution for Question 103 using the Weierstrass substitution $t = \tan \frac{x}{2}$.

Left side of the solution:

$$\begin{aligned} \int_0^{\pi} \frac{dx}{\alpha - \cos x} & \text{ BY LET } t = \tan \frac{x}{2} \\ & = \int_0^{\infty} \frac{1}{\alpha - \frac{1-t^2}{1+t^2}} \times \frac{2 dt}{1+t^2} = \int_0^{\infty} \frac{2}{\alpha(1+t^2) - (1-t^2)} dt \\ & = \int_0^{\infty} \frac{2}{(\alpha+1)t^2 + (\alpha-1)} dt = \frac{1}{\alpha+1} \int_0^{\infty} \frac{2}{t^2 + \frac{\alpha-1}{\alpha+1}} dt \\ & = \frac{2}{\alpha+1} \int_0^{\infty} \frac{1}{t^2 + \left(\frac{\sqrt{\alpha^2-1}}{\alpha+1}\right)^2} dt \\ & \text{ STANDARD INTEGRAL TO ARCTAN} \\ & = \frac{2}{\alpha+1} \times \frac{1}{\sqrt{\alpha^2-1}} \left[\arctan \left[\frac{t}{\frac{\sqrt{\alpha^2-1}}{\alpha+1}} \right] \right]_0^{\infty} \\ & = \frac{2}{\alpha+1} \times \frac{\alpha+1}{\sqrt{\alpha^2-1}} \left[\frac{\pi}{2} - 0 \right] \\ & = \frac{2}{\sqrt{\alpha^2-1}} \times \frac{\pi}{2} \\ & = \frac{\pi}{\sqrt{\alpha^2-1}} \end{aligned}$$

As required

Right side of the solution:

Let $t = \tan \frac{x}{2}$

$$\begin{aligned} dt &= \frac{1}{2} \sec^2 \frac{x}{2} dx \\ dx &= \frac{2}{\sec^2 \frac{x}{2}} dt \\ dx &= \frac{2}{1+\tan^2 \frac{x}{2}} dt \\ dx &= \frac{2}{1+t^2} dt \end{aligned}$$

Diagram of a right-angled triangle with angle $\frac{x}{2}$, opposite side t , adjacent side 1 , and hypotenuse $\sqrt{1+t^2}$.

$$\begin{aligned} \tan \frac{x}{2} &= t = \frac{t}{1} \\ \sin \frac{x}{2} &= \frac{t}{\sqrt{1+t^2}} \\ \cos \frac{x}{2} &= \frac{1}{\sqrt{1+t^2}} \end{aligned}$$

THIS

$$\begin{aligned} \cos x &= \cos^2 \frac{x}{2} - \sin^2 \frac{x}{2} \\ \cos x &= \frac{1}{1+t^2} - \frac{t^2}{1+t^2} \\ \cos x &= \frac{1-t^2}{1+t^2} \end{aligned}$$

LIMITS

$$\begin{aligned} x=0, t &= 0 \\ x=\pi, t &= \infty \end{aligned}$$

Question 104 (****)

Use appropriate integration techniques to show that

$$\int_0^{\frac{1}{2}} \frac{\arcsin \sqrt{x} - \arccos \sqrt{x}}{\arcsin \sqrt{x} + \arccos \sqrt{x}} dx = \frac{1}{\pi} - \frac{1}{2}.$$

IP

,

proof

Handwritten solution for the integral problem:

$$\int_0^{\frac{1}{2}} \frac{\arcsin \sqrt{x} - \arccos \sqrt{x}}{\arcsin \sqrt{x} + \arccos \sqrt{x}} dx = \dots$$

By substitution $y = \sqrt{x}$, $dy = \frac{1}{2\sqrt{x}} dx$, $dx = 2y dy$. Limits: $x=0 \rightarrow y=0$, $x=\frac{1}{2} \rightarrow y=\frac{1}{\sqrt{2}}$.

$$= \int_0^{\frac{1}{\sqrt{2}}} \frac{\arcsin y - \arccos y}{\arcsin y + \arccos y} \cdot 2y dy = \int_0^{\frac{1}{\sqrt{2}}} \frac{2y(\arcsin y - \arccos y)}{\arcsin y + \arccos y} dy$$

By another substitution $u = \arcsin y + \arccos y$, $du = \frac{1}{\sqrt{1-y^2}} + \frac{-1}{\sqrt{1-y^2}} dy = 0$. So u is constant. $u = \frac{\pi}{2}$.

$$= \int_0^{\frac{1}{\sqrt{2}}} \frac{2y(\arcsin y - \arccos y)}{\frac{\pi}{2}} dy = \frac{4}{\pi} \int_0^{\frac{1}{\sqrt{2}}} y(\arcsin y - \arccos y) dy$$

By parts: $u = \arcsin y$, $dv = y dy$. $du = \frac{1}{\sqrt{1-y^2}} dy$, $v = \frac{1}{2} y^2$.

$$= \frac{4}{\pi} \left[\frac{1}{2} y^2 \arcsin y - \int \frac{1}{2} y^2 \cdot \frac{1}{\sqrt{1-y^2}} dy \right]_0^{\frac{1}{\sqrt{2}}}$$

By parts: $u = \arccos y$, $dv = y dy$. $du = \frac{-1}{\sqrt{1-y^2}} dy$, $v = \frac{1}{2} y^2$.

$$= \frac{4}{\pi} \left[\frac{1}{2} y^2 \arccos y - \int \frac{1}{2} y^2 \cdot \frac{-1}{\sqrt{1-y^2}} dy \right]_0^{\frac{1}{\sqrt{2}}}$$

By partial fractions: $\frac{y^2}{\sqrt{1-y^2}} = \frac{-1}{\sqrt{1-y^2}} + \frac{1}{\sqrt{1-y^2}}$.

$$= \frac{4}{\pi} \left[\frac{1}{2} y^2 \arcsin y - \frac{1}{2} \arcsin y + \frac{1}{2} y^2 \arccos y - \frac{1}{2} \arccos y \right]_0^{\frac{1}{\sqrt{2}}}$$

$$= \frac{4}{\pi} \left[\frac{1}{2} y^2 (\arcsin y + \arccos y) - \frac{1}{2} (\arcsin y - \arccos y) \right]_0^{\frac{1}{\sqrt{2}}}$$

$$= \frac{4}{\pi} \left[\frac{1}{2} y^2 \cdot \frac{\pi}{2} - \frac{1}{2} \left(\arcsin y - \arccos y \right) \right]_0^{\frac{1}{\sqrt{2}}}$$

$$= \frac{4}{\pi} \left[\frac{\pi}{4} y^2 - \frac{1}{2} (\arcsin y - \arccos y) \right]_0^{\frac{1}{\sqrt{2}}}$$

$$= \frac{4}{\pi} \left[\frac{\pi}{4} \cdot \frac{1}{2} - \frac{1}{2} \left(\arcsin \frac{1}{\sqrt{2}} - \arccos \frac{1}{\sqrt{2}} \right) \right]$$

$$= \frac{4}{\pi} \left[\frac{\pi}{8} - \frac{1}{2} \left(\frac{\pi}{4} - \frac{\pi}{4} \right) \right] = \frac{4}{\pi} \cdot \frac{\pi}{8} = \frac{1}{2}$$

Question 105 (****)If $0 < k < \sqrt{2} - 1$ prove that

$$\int_k^{\frac{1-k}{1+k}} \frac{\ln x}{x^2 - 1} dx = \int_k^{\frac{1-k}{1+k}} \frac{\operatorname{artanh} x}{x} dx.$$

You need not evaluate these integrals.

☐, **proof**

SYNOPSIS ON THE LEFT AND USE ITERATION BY PARTS

$$\begin{aligned} \int_k^{\frac{1-k}{1+k}} \frac{\ln x}{x^2 - 1} dx &= \int_k^{\frac{1-k}{1+k}} \left(\frac{\ln x}{x} \right) \frac{1}{x^2 - 1} dx \\ &= \left[-(\ln x)(\operatorname{artanh} x) \right]_k^{\frac{1-k}{1+k}} - \int_k^{\frac{1-k}{1+k}} \frac{1}{x} \operatorname{artanh} x dx \\ &= \left[(\ln x)(\operatorname{artanh} x) \right]_{\frac{1-k}{1+k}}^k + \int_k^{\frac{1-k}{1+k}} \frac{\operatorname{artanh} x}{x} dx \end{aligned}$$

Now it suffices to show that $\left[(\ln x)(\operatorname{artanh} x) \right]_{\frac{1-k}{1+k}}^k = 0$

$$\begin{aligned} \therefore \left[(\ln x)(\operatorname{artanh} x) \right]_{\frac{1-k}{1+k}}^k &= \left[\ln x \times \frac{1}{2} \ln \frac{1+x}{1-x} \right]_{\frac{1-k}{1+k}}^k \\ &= \frac{1}{2} \left[\left(\ln k \right) \ln \left(\frac{1+k}{1-k} \right) - \left(\ln \left(\frac{1-k}{1+k} \right) \right) \ln \left(\frac{1+\frac{1-k}{1+k}}{1-\frac{1-k}{1+k}} \right) \right] \\ &= \frac{1}{2} \left[\left(\ln k \right) \ln \left(\frac{1+k}{1-k} \right) - \ln \left(\frac{1-k}{1+k} \right) \cdot \ln \left(\frac{1+\frac{1-k}{1+k}}{1-\frac{1-k}{1+k}} \right) \right] \\ &= \frac{1}{2} \left[\left(\ln k \right) \ln \left(\frac{1+k}{1-k} \right) - \ln \left(\frac{1-k}{1+k} \right) \ln \left(\frac{2}{1-k} \right) \right] \\ &= \frac{1}{2} \left[\left(\ln k \right) \ln \left(\frac{1+k}{1-k} \right) - \ln \left(\frac{1-k}{1+k} \right) \ln \left(\frac{2}{1-k} \right) \right] \\ &= \frac{1}{2} \left[\left(\ln k \right) \ln \left(\frac{1+k}{1-k} \right) - \left(\ln \left(\frac{1-k}{1+k} \right) \right) \ln \left(\frac{2}{1-k} \right) \right] \end{aligned}$$

$\ln \left(\frac{2}{1-k} \right) = \ln \left(\frac{2}{1-k} \right)$

$$\therefore \int_k^{\frac{1-k}{1+k}} \frac{\ln x}{x^2 - 1} dx = \int_k^{\frac{1-k}{1+k}} \frac{\operatorname{artanh} x}{x} dx$$

Question 106 (****)

Use integration by parts and trigonometric identities to find the exact value of

$$\int_0^{\frac{\pi}{6}} 12 \sec^3 x \, dx.$$

$$4 + 3 \ln 3$$

Handwritten solution for the integral:

$$\begin{aligned} \int_0^{\frac{\pi}{6}} 12 \sec^3 x \, dx &= \int_0^{\frac{\pi}{6}} 12 \sec x \sec^2 x \, dx \quad \text{By part} \quad \begin{array}{|l} 12 \sec x \\ \hline \tan x \quad \sec x \end{array} \\ &= \left[12 \sec x \tan x \right]_0^{\frac{\pi}{6}} - \int_0^{\frac{\pi}{6}} 12 \sec x \tan^2 x \, dx \\ &= \left[12 \sec \frac{\pi}{6} \tan \frac{\pi}{6} - 0 \right] - \int_0^{\frac{\pi}{6}} 12 \sec x (\sec^2 x - 1) \, dx \\ &= 12 \times \frac{2}{\sqrt{3}} \times \frac{\sqrt{3}}{3} - \int_0^{\frac{\pi}{6}} 12 \sec^3 x \, dx - 12 \sec x \, dx \\ &= 8 - \int_0^{\frac{\pi}{6}} 12 \sec^3 x \, dx + \int_0^{\frac{\pi}{6}} 12 \sec x \, dx \\ &= 8 - \int_0^{\frac{\pi}{6}} 12 \sec^3 x \, dx + \left[12 \ln |\sec x + \tan x| \right]_0^{\frac{\pi}{6}} \\ &= 8 - \int_0^{\frac{\pi}{6}} 12 \sec^3 x \, dx + 12 \ln \left(\frac{2}{\sqrt{3}} + \frac{\sqrt{3}}{3} \right) - 12 \ln 1 \\ &= 8 - \int_0^{\frac{\pi}{6}} 12 \sec^3 x \, dx + 12 \ln 3 \quad \leftarrow 12 \ln 3 = 6 \ln 3 \\ \text{Thus so far...} \\ \int_0^{\frac{\pi}{6}} 12 \sec^3 x \, dx &= 8 + 6 \ln 3 - \int_0^{\frac{\pi}{6}} 12 \sec^3 x \, dx \\ 2 \int_0^{\frac{\pi}{6}} 12 \sec^3 x \, dx &= 8 + 6 \ln 3 \\ \therefore \int_0^{\frac{\pi}{6}} 12 \sec^3 x \, dx &= 4 + 3 \ln 3 \end{aligned}$$

Question 107 (****)

Determine, as an exact simplified fraction, the value of the following integral.

$$\int_{\frac{3}{2}}^{\frac{5}{2}} (4x^2 - 16x + 15)^4 dx.$$

$$\boxed{}, \frac{128}{315}$$

INTEGRATE BY PARTS

$$\int_{\frac{3}{2}}^{\frac{5}{2}} (4x^2 - 16x + 15)^4 dx = \int_{\frac{3}{2}}^{\frac{5}{2}} [(2x-3)(2x-5)]^4 dx$$

$$= \int_{\frac{3}{2}}^{\frac{5}{2}} (2x-3)^4 (2x-5)^4 dx$$

INTEGRATE BY PARTS

$$= \left[\frac{1}{5} (2x-3)^4 (2x-5)^5 - \frac{4}{5} \int_{\frac{3}{2}}^{\frac{5}{2}} (2x-3)^4 (2x-5)^4 dx \right]_{\frac{3}{2}}^{\frac{5}{2}}$$

INTEGRATE BY PARTS FOR A SECOND TIME

$$= \left[\frac{1}{5} \left(\frac{1}{5} (2x-3)^4 (2x-5)^5 - \frac{4}{5} \int_{\frac{3}{2}}^{\frac{5}{2}} (2x-3)^4 (2x-5)^4 dx \right) - \frac{4}{5} \left(\frac{1}{5} (2x-3)^4 (2x-5)^5 - \frac{4}{5} \int_{\frac{3}{2}}^{\frac{5}{2}} (2x-3)^4 (2x-5)^4 dx \right) \right]_{\frac{3}{2}}^{\frac{5}{2}}$$

BY PARTS FOR A THIRD TIME

$$= \left[\frac{1}{5} \left(\frac{1}{5} \left(\frac{1}{5} (2x-3)^4 (2x-5)^5 - \frac{4}{5} \int_{\frac{3}{2}}^{\frac{5}{2}} (2x-3)^4 (2x-5)^4 dx \right) - \frac{4}{5} \left(\frac{1}{5} (2x-3)^4 (2x-5)^5 - \frac{4}{5} \int_{\frac{3}{2}}^{\frac{5}{2}} (2x-3)^4 (2x-5)^4 dx \right) \right) - \frac{4}{5} \left(\frac{1}{5} (2x-3)^4 (2x-5)^5 - \frac{4}{5} \int_{\frac{3}{2}}^{\frac{5}{2}} (2x-3)^4 (2x-5)^4 dx \right) \right]_{\frac{3}{2}}^{\frac{5}{2}}$$

FINALLY THE LAST INTEGRATION BY PARTS

$$= \left[\frac{1}{5} \left(\frac{1}{5} \left(\frac{1}{5} \left(\frac{1}{5} (2x-3)^4 (2x-5)^5 - \frac{4}{5} \int_{\frac{3}{2}}^{\frac{5}{2}} (2x-3)^4 (2x-5)^4 dx \right) - \frac{4}{5} \left(\frac{1}{5} (2x-3)^4 (2x-5)^5 - \frac{4}{5} \int_{\frac{3}{2}}^{\frac{5}{2}} (2x-3)^4 (2x-5)^4 dx \right) \right) - \frac{4}{5} \left(\frac{1}{5} (2x-3)^4 (2x-5)^5 - \frac{4}{5} \int_{\frac{3}{2}}^{\frac{5}{2}} (2x-3)^4 (2x-5)^4 dx \right) \right) - \frac{4}{5} \left(\frac{1}{5} (2x-3)^4 (2x-5)^5 - \frac{4}{5} \int_{\frac{3}{2}}^{\frac{5}{2}} (2x-3)^4 (2x-5)^4 dx \right) \right]_{\frac{3}{2}}^{\frac{5}{2}}$$

$$= \frac{128}{315}$$

Question 108 (****)

Use the substitution $u = \sqrt{\frac{1+x}{1-x}}$, to evaluate the following integral.

$$\int_0^{\frac{1}{4}} \frac{3}{(4x+5)\sqrt{1-x^2} - 3(1-x^2)} dx.$$

Give the answer in the form $\frac{1}{7}(a + \sqrt{b})$, where a and b are integers.

$$\boxed{\frac{1}{7}(6 - \sqrt{15})}$$

STRATEGY: BY REPAIRING THE SUBSTITUTION GIVE

$u = \sqrt{\frac{1+x}{1-x}}$
 $u^2 = \frac{1+x}{1-x}$
 $u^2(1-x) = 1+x$
 $u^2 - u^2x = 1+x$
 $u^2 - 1 = x(u^2 + 1)$
 $x = \frac{u^2 - 1}{u^2 + 1}$
 $\therefore \frac{dx}{du} = \frac{2u}{(u^2 + 1)^2}$

$1-x^2 = 1 - \left(\frac{u^2 - 1}{u^2 + 1}\right)^2 = \frac{(u^2 + 1)^2 - (u^2 - 1)^2}{(u^2 + 1)^2} = \frac{4u^2}{(u^2 + 1)^2}$
 $\sqrt{1-x^2} = \frac{2u}{u^2 + 1}$
 $1-x^2 = \frac{4u^2}{(u^2 + 1)^2}$
 $\therefore \frac{dx}{du} = \frac{2u}{(u^2 + 1)^2}$

$4x+5 = 4\left(\frac{u^2 - 1}{u^2 + 1}\right) + 5 = \frac{4u^2 - 4 + 5u^2 + 5}{u^2 + 1} = \frac{9u^2 + 1}{u^2 + 1}$
 $\therefore \frac{dx}{du} = \frac{2u}{(u^2 + 1)^2}$

$3 = 0 \rightarrow u = 1$
 $2 = \frac{1}{4} \rightarrow u = \frac{1}{2}\sqrt{15}$

SEEN THE TRANSFORMATION

$$\int_0^{\frac{1}{4}} \frac{3}{(4x+5)\sqrt{1-x^2} - 3(1-x^2)} dx = \int_1^{\frac{1}{2}\sqrt{15}} \frac{2}{\frac{9u^2 + 1}{u^2 + 1} \cdot \frac{2u}{u^2 + 1} - 3 \cdot \frac{4u^2}{(u^2 + 1)^2}} \cdot \frac{2u}{(u^2 + 1)^2} du$$

$$= \int_1^{\frac{1}{2}\sqrt{15}} \frac{2u}{\frac{9u^2 + 1}{u^2 + 1} \cdot \frac{2u}{u^2 + 1} - \frac{12u^2}{(u^2 + 1)^2}} du$$

$$= \int_1^{\frac{1}{2}\sqrt{15}} \frac{2u}{\frac{9u^2 + 1}{u^2 + 1} - \frac{12u^2}{(u^2 + 1)^2}} du$$

$$= \int_1^{\frac{1}{2}\sqrt{15}} \frac{2u}{\frac{9u^2 + 1}{u^2 + 1} - \frac{12u^2}{(u^2 + 1)^2}} du$$

THIS IS NOW A STANDARD REDUCED INTEGRATION

$$= \int_1^{\frac{1}{2}\sqrt{15}} \frac{2}{\frac{9u^2 + 1}{u^2 + 1} - \frac{12u^2}{(u^2 + 1)^2}} du = \left[-\frac{2}{3u-1} \right]_1^{\frac{1}{2}\sqrt{15}}$$

$$= \left[-\frac{2}{3u-1} \right]_1^{\frac{1}{2}\sqrt{15}} = \left[-\frac{2}{3 \cdot \frac{1}{2}\sqrt{15} - 1} \right] - \left[-\frac{2}{3 \cdot 1 - 1} \right]$$

EVALUATING

$$= \frac{2}{1 - \sqrt{15}} - \frac{2}{1-3} = \frac{2}{1 - \sqrt{15}} + 1$$

$$= \frac{2(1 + \sqrt{15})}{1 - 15} + 1 = \frac{2(1 + \sqrt{15})}{-14} + 1$$

$$= \frac{1 + \sqrt{15}}{-7} + 1 = -\frac{1}{7} - \frac{1}{7}\sqrt{15} + 1$$

$$= \frac{6}{7} - \frac{1}{7}\sqrt{15}$$

NOTE THAT THE SUBSTITUTION $u = \sqrt{\frac{1+x}{1-x}}$ OR $u = \sqrt{\frac{1-x}{1+x}}$, FURTHER BY THE "LITTLE t " IDENTICAL IS FAR MORE NATURAL, HOWEVER IS LONGER IN ITS MANIPULATIONS

Question 109 (****)

Use the substitution $x = \frac{ab}{t}$ to find the exact value of

$$\int_0^{\infty} \frac{\ln x}{(x+a)(x+b)} dx,$$

where a and b are real positive constants with $a > b$.

$$\boxed{}, \quad \frac{\ln(ab)}{2(a-b)} \ln\left(\frac{a}{b}\right) = \frac{(\ln a)^2 - (\ln b)^2}{2(a-b)}$$

$\int_0^{\infty} \frac{\ln x}{(x+a)(x+b)} dx = \frac{\ln(ab)}{2(a-b)} \ln\left(\frac{a}{b}\right)$

• START WITH THE SUBSTITUTION

$x = \frac{ab}{t}$
$dx = -\frac{ab}{t^2} dt$
$x=0 \mapsto t=\infty$
$x=\infty \mapsto t=0$

$= \int_{\infty}^0 \frac{\ln\left(\frac{ab}{t}\right) - \ln t}{a\left(\frac{ab}{t}+a\right)b\left(\frac{ab}{t}+b\right)} \left(-\frac{ab}{t^2}\right) dt$

$= \int_0^{\infty} \frac{\ln(ab) - \ln t}{ab\left(\frac{b+t}{t}\right)\left(\frac{a+t}{t}\right)} \frac{ab}{t^2} dt$

$= \int_0^{\infty} \frac{\ln(ab) - \ln t}{\frac{1}{t^2}(b+t)(a+t)t^2} dt$

$= \int_0^{\infty} \frac{\ln(ab) - \ln t}{(t+a)(t+b)} dt$

• SPLIT INTO TWO PARTS

$I = \int_0^{\infty} \frac{\ln(ab)}{(x+a)(x+b)} dx = \int_0^{\infty} \frac{\ln(ab)}{(t+a)(t+b)} dt - \int_0^{\infty} \frac{\ln t}{(t+a)(t+b)} dt$

$I = \int_0^{\infty} \frac{\ln(ab)}{(t+a)(t+b)} dt - I$

$2I = \ln(ab) \int_0^{\infty} \frac{1}{(t+a)(t+b)} dt$

• NOW BY PARTIAL FRACTIONS

$\frac{1}{(t+a)(t+b)} = \frac{P}{t+a} + \frac{Q}{t+b}$

$1 = P(t+b) + Q(t+a)$

\bullet If $t=-b$ \bullet If $t=-a$

$1 = Q(a-b)$ $1 = P(b-a)$

$Q = \frac{1}{a-b}$ $P = \frac{1}{b-a}$

• RETURN TO THE INTEGRAL

$2I = \ln(ab) \int_0^{\infty} \left(\frac{1}{t+a} - \frac{1}{t+b} \right) dt$

$2I = \frac{\ln(ab)}{a-b} \left[\ln\left(\frac{t+b}{t+a}\right) \right]_0^{\infty}$

$2I = \frac{\ln(ab)}{a-b} \left[\ln 1 - \ln \frac{b}{a} \right]$

$2I = \frac{\ln(ab)}{a-b} \left[\ln \frac{a}{b} \right]$

$I = \frac{\ln(ab)}{2(a-b)} \ln\left(\frac{a}{b}\right)$

Question 110 (****)

Use appropriate integration methods to show that

$$\int_0^1 12x^2 \arctan x \, dx = \pi - 2 + \ln 4.$$

$\frac{d}{dx} [x^3 \arctan x]$

, proof

• CONSIDER THE DIFFERENTIATION
 $\frac{d}{dx} [x^3 \arctan x] = 3x^2 \arctan x + x^3 \times \frac{1}{1+x^2}$
 $\Rightarrow \frac{d}{dx} [4x^3 \arctan x] = 12x^2 \arctan x + \frac{4x^3}{1+x^2}$

• INTEGRATE w.r.t x
 $\Rightarrow 4x^3 \arctan x = \int 12x^2 \arctan x \, dx + \int \frac{4x^3}{1+x^2} \, dx$
 $\Rightarrow 4x^3 \arctan x = \int 12x^2 \arctan x \, dx + \int \frac{4x(x^2+1) - 4x}{x^2+1} \, dx$
 $\Rightarrow 4x^3 \arctan x = \int 12x^2 \arctan x \, dx + \int 4x \, dx - \int \frac{4x}{x^2+1} \, dx$

• THIS REARRANGING GIVES
 $\Rightarrow \int 12x^2 \arctan x \, dx = 4x^3 \arctan x + \int \frac{4x}{x^2+1} \, dx - \int 4x \, dx$
 $\Rightarrow \int 12x^2 \arctan x \, dx = 4x^3 \arctan x + 2\ln(x^2+1) - 2x^2 + C$

• APPLY LIMITS
 $\Rightarrow \int_0^1 12x^2 \arctan x \, dx = [4x^3 \arctan x + 2\ln(x^2+1) - 2x^2]_0^1$
 $= \left[4 \times \frac{\pi}{4} + 2\ln 2 - 2 \right] - [0 + 2\ln 1 - 0]$
 $= \pi - 2 + \ln 4$

Question 111 (****)

Use appropriate integration methods to find, in terms of k , a simplified expression for

$$\int_0^{\frac{\pi}{2}} \frac{1}{1+k^2 \tan^2 x} dx, \quad |k| \neq 1.$$

$$\boxed{}, \quad \frac{\pi}{2(k+1)}$$

• SING BY A SUBSTITUTION

$$\int_0^{\frac{\pi}{2}} \frac{1}{1+k^2 \tan^2 x} dx = \dots = \int_0^{\infty} \frac{1}{1+u^2} \cdot \frac{du}{k \sec^2 x}$$

$$= \int_0^{\infty} \frac{1}{1+u^2} \cdot \frac{1}{k(1+\frac{u^2}{k^2})} du$$

$$= \int_0^{\infty} \frac{1}{1+u^2} \cdot \frac{k^2}{k(k^2+u^2)} du = \int_0^{\infty} \frac{k}{(k^2+u^2)(k^2+u^2)} du$$

• NEW BY PARTIAL FRACTIONS

$$\Rightarrow \frac{k}{(k^2+u^2)(k^2+u^2)} = \frac{A}{u^2+1} + \frac{C}{u^2+k^2}$$

$$\Rightarrow k = (A+B)(u^2+1) + (C+B)(k^2+u^2)$$

$$\Rightarrow k = \begin{cases} A+B \\ C+B \end{cases} \begin{cases} u^2+1 \\ k^2+u^2 \end{cases}$$

$$\Rightarrow k = \begin{cases} A+B \\ C+B \end{cases} \begin{cases} u^2+1 \\ k^2+u^2 \end{cases}$$

- $A+B=0 \Rightarrow A=-B$
- $A+C=0 \Rightarrow A=-C$

$$\Rightarrow B=C$$

• RETURNING TO THE INTEGRAL WE NOW HAVE

$$\int_0^{\infty} \frac{k}{(k^2+u^2)(k^2+u^2)} du = \int_0^{\infty} \frac{k}{u^2+1} - \frac{k}{u^2+k^2} du$$

$$= \frac{k}{2} \int_0^{\infty} \left(\frac{1}{u^2+1} - \frac{1}{u^2+k^2} \right) du = \frac{k}{2} \left[\arctan u - \frac{1}{k} \arctan \frac{u}{k} \right]_0^{\infty}$$

$$= \frac{k}{2} \left[\left(\frac{\pi}{2} - \frac{\pi}{2k} \right) - 0 \right] = \frac{k}{2} \times \frac{\pi}{2} \times \left(1 - \frac{1}{k} \right)$$

$$= \frac{k}{2} \times \frac{\pi}{2} \times \frac{k-1}{k} = \frac{\pi}{2} \times \frac{k-1}{2}$$

Question 112 (****)

$$I = \int_0^{\frac{1}{2}\ln 3} \operatorname{sech} x \, dx.$$

a) Use the substitution $u = e^x$ to show that $I = \frac{\pi}{k}$, where k is a positive integer.

b) Given that $t = \tanh\left(\frac{1}{2}x\right)$ show that ...

i. ... $\frac{dt}{dx} = \frac{1}{2}(1-t^2).$

ii. ... if $x = \frac{1}{2}\ln 3$, then $t = 2 - \sqrt{3}$.

c) Use the results of part (b) to find again the exact value of I .

d) Show that I can be written as

$$\int_0^{\frac{1}{2}\ln 3} \frac{\cosh x}{1 + \sinh^2 x} \, dx,$$

and hence obtain the exact value of I for a third time.

, proof

(a) $\int_0^{\frac{1}{2}\ln 3} \operatorname{sech} x \, dx = \int_0^{\frac{1}{2}\ln 3} \frac{1}{\cosh x} \, dx = \dots$
 $\therefore \int_0^{\frac{1}{2}\ln 3} \frac{2}{e^x + e^{-x}} \, dx = \dots$ by substitution $u = e^x$
 $\frac{du}{dx} = e^x = u$
 $\frac{dx}{u} = \frac{du}{u^2}$
 $\int \frac{2}{u + \frac{1}{u}} \times \frac{du}{u} = \int \frac{2}{u^2 + 1} \, du$
 $= [2 \arctan u]_1^{2-\sqrt{3}}$
 $= 2 \arctan(2-\sqrt{3}) - 2 \arctan(1)$
 $= \frac{\pi}{6} - \frac{\pi}{4} = \frac{\pi}{12}$

(b) (i) $t = \tanh \frac{x}{2}$
 $\frac{dt}{dx} = \frac{1}{2}(1-t^2)$
 $\frac{dx}{1-t^2} = \frac{dt}{1-t^2}$
 $\frac{dx}{1-t^2} = \frac{dt}{1-t^2}$
 $\frac{dx}{1-t^2} = \frac{dt}{1-t^2}$

(ii) $x = \frac{1}{2}\ln 3$ then $t = 2 - \sqrt{3}$
 $t = \tanh \frac{x}{2} = \frac{e^{\frac{x}{2}} - e^{-\frac{x}{2}}}{e^{\frac{x}{2}} + e^{-\frac{x}{2}}}$
 $\therefore \frac{e^{\frac{x}{2}} - e^{-\frac{x}{2}}}{e^{\frac{x}{2}} + e^{-\frac{x}{2}}} = \frac{3^{\frac{1}{4}} - 3^{-\frac{1}{4}}}{3^{\frac{1}{4}} + 3^{-\frac{1}{4}}} = \frac{3^{\frac{1}{4}}(1 - 3^{-\frac{1}{2}})}{3^{\frac{1}{4}}(1 + 3^{-\frac{1}{2}})}$
 $= \frac{3^{\frac{1}{4}}(1 - \frac{1}{\sqrt{3}})}{3^{\frac{1}{4}}(1 + \frac{1}{\sqrt{3}})} = \frac{1 - \frac{1}{\sqrt{3}}}{1 + \frac{1}{\sqrt{3}}} = \frac{\sqrt{3} - 1}{\sqrt{3} + 1} = 2 - \sqrt{3}$

(c) $\int_0^{\frac{1}{2}\ln 3} \operatorname{sech} x \, dx = \int_0^{\frac{1}{2}\ln 3} \frac{1}{\cosh x} \, dx = \dots$ by little t identity
 $\frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}} = \frac{2}{e^x + \frac{1}{e^x}} = \frac{2e^x}{e^{2x} + 1}$
 $\int \frac{2e^x}{e^{2x} + 1} \, dx = \int \frac{2}{t^2 + 1} \, dt = [2 \arctan t]_1^{2-\sqrt{3}}$
 $= 2 \arctan(2-\sqrt{3}) - 2 \arctan(1) = \frac{\pi}{6} - \frac{\pi}{4} = \frac{\pi}{12}$

(d) $\int_0^{\frac{1}{2}\ln 3} \frac{\cosh x}{1 + \sinh^2 x} \, dx = \int_0^{\frac{1}{2}\ln 3} \frac{\cosh x}{\cosh^2 x} \, dx = \int_0^{\frac{1}{2}\ln 3} \frac{1}{\cosh x} \, dx = \dots$
 $\frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}} = \frac{2e^x}{e^{2x} + 1}$
 $\int \frac{2e^x}{e^{2x} + 1} \, dx = \int \frac{2}{t^2 + 1} \, dt = [2 \arctan t]_1^{2-\sqrt{3}}$
 $= 2 \arctan(2-\sqrt{3}) - 2 \arctan(1) = \frac{\pi}{6} - \frac{\pi}{4} = \frac{\pi}{12}$

Question 113 (****)

$$I = \int_0^1 2 \operatorname{arsinh} \sqrt{x} \, dx.$$

The value of I is to be found using two methods.

- a)** Use the substitution $x = \sinh^2 \theta$ to show that

$$I = 3 \ln(1 + \sqrt{2}) - \sqrt{2}.$$

A different approach is to be used to find the value of I .

- b)** Use the substitution $u = \sqrt{x}$, followed by a suitable hyperbolic substitution to verify the answer of part **(a)**.

, proof

(a) $\cosh(\operatorname{arcsinh} x)$ Let $x = \sinh u$ | $\sinh x = \cosh u$
 $\sinh u = x$
 $1 + \sinh^2 u = 2$
 $\cosh^2 u = 2$
 $\cosh u = +\sqrt{2}$
 $\forall x \in \mathbb{R} \quad \cosh(\operatorname{arcsinh} x) = \sqrt{2}$

(b) $\int_0^1 2 \operatorname{arcsinh} \sqrt{x} \, dx = \int_0^{\operatorname{arcsinh} 1} 2 \operatorname{arcsinh} u \cdot \sinh u \, du$
 $= \int_0^{\operatorname{arcsinh} 1} 2 \operatorname{arcsinh}(\sinh u) \times \sinh 2u \, du = \int_0^{\operatorname{arcsinh} 1} 2 \sinh^2 2u \, du$
 $= \dots$ by parts

0	1
$\cosh 2u$	$2 \sinh 2u$

 $= \dots \left[\theta \cosh 2u \right]_0^{\operatorname{arcsinh} 1} - \int_0^{\operatorname{arcsinh} 1} \cosh 2u \, du$
 $= \left[\theta \cosh 2u - \frac{1}{2} \sinh 2u \right]_0^{\operatorname{arcsinh} 1}$
 $= \left[\theta [1 + 2 \sinh^2] - \sinh \theta \cosh \theta \right]_0^{\operatorname{arcsinh} 1}$
 $= [\operatorname{arcsinh} 1] \times 3 - \sinh(\operatorname{arcsinh} 1) - [0 - 0]$
 $= 3 \operatorname{arcsinh} 1 - \sqrt{2} = 3 \ln(1 + \sqrt{2}) - \sqrt{2}$

(c) $\int_0^1 2 \arcsinh(x) dx = \dots = \int_0^1 2 \arcsinh(u) \times 2 du$

$= \int_0^1 4u \arcsinh(u) du = \dots$ by parts

$$\begin{array}{|c|c|} \hline \arcsinh(u) & 4u \\ \hline \frac{1}{2u} & \frac{4}{4u} \\ \hline \end{array}$$

$= [2u^2 \arcsinh(u)]_0^1 - \int_0^1 \frac{2u^2}{u^2+1} du$

$= 2 \arcsinh(1) - \int_0^1 \frac{2u^2}{u^2+1} du$ \rightarrow by inspection

(d) $\int_0^1 \frac{2u^2}{u^2+1} du = \int_0^1 \frac{2u^2+2-2}{u^2+1} du = \int_0^1 \frac{2u^2+2}{u^2+1} du - \int_0^1 \frac{2}{u^2+1} du$

$= \int_0^1 2 \arcsinh(u) du = \int_0^1 2 \left(\frac{1}{2} \ln(u^2+1) - \frac{1}{2} \right) du$

$= \int_0^1 \arcsinh(u) du - \int_0^1 1 du = \left(\frac{1}{2} \ln(u^2+1) - u \right) \Big|_0^1$

$= \left(\frac{1}{2} \ln(u^2+1) - u \right) \Big|_0^1 = \left(\frac{1}{2} \ln(u^2+1) - \arcsinh(u) \right) \Big|_0^1 - (0-0)$

$= \sqrt{2} - \ln(1+\sqrt{2})$

4. $I = 2 \arcsinh(1) - (\sqrt{2} - \ln(1+\sqrt{2}))$

$I = 2 \ln(1+\sqrt{2}) - \sqrt{2} + \ln(1+\sqrt{2})$

$I = 3 \ln(1+\sqrt{2}) - \sqrt{2}$

\rightarrow by inspection

$u = \sinh t$
 $\frac{du}{dt} = \cosh t$
 $du = \cosh t dt$
 $u=1, 0 \rightarrow \arcsinh(1)$
 $u=0, 0 \rightarrow 0$

Question 114 (****)

By considering the differentiation of products of appropriate functions, find

$$\int e^x (3 \sec^2 x + 2 \sec^2 x \tan x + 2 \tan x) dx.$$

, $e^x (2 \tan x + \sec^2 x) + C$

$$\int \underbrace{(3e^{2x} + 2e^{2x} \ln x)}_{\text{Derivative of } 2e^{2x}} \underbrace{(2e^{2x})}_{\text{Derivative of } e^{2x} \text{ (or } \ln x)} dx = \text{BY RECOGNISING DIFFERENTIATIONS}$$

$$\frac{d}{dx}(2e^{2x}) = 2e^{2x} + 2e^{2x} \ln x$$

$$\frac{d}{dx}(e^{2x}) = e^{2x}$$

BY INSPECTION OF THE ABOVE DIFFERENTIALS

$$\therefore \int (3e^{2x} + 2e^{2x} \ln x + 2e^{2x}) dx = \frac{1}{2}e^{2x} + e^{2x} \ln x + C$$

Question 115 (****)

By using a trigonometric substitution or otherwise, find an exact simplified value for the following integral.

$$\int_0^{\frac{\pi}{2}} \frac{1}{1+3\cos 3x} \, dx.$$

$$\boxed{}, \frac{\sqrt{2}}{6} \ln(\sqrt{2}-1)$$

$\int_0^{\frac{1}{2}} \frac{1}{1+3\cos 3x} dx = \dots$ by little & substituting
 $= \int_0^1 \frac{1}{1 + 3 \frac{1-t^2}{1+t^2}} \times \frac{2}{3(1+t^2)} dt$
 (Circled part)
 Applying partial fraction (C14)
 $= \int_0^1 \frac{1+t^2}{(1+t^2)(3(1-t^2))} \times \frac{2}{3(1+t^2)} dt$
 $= \int_0^1 \frac{1}{3(1-t^2)} dt = \frac{1}{3} \int_0^1 \frac{1}{1-t^2} dt$
 $= \dots$ by PARTIAL FRACTIONS
 $= \frac{1}{3} \int_0^1 \left(\frac{1}{\sqrt{2} + \sqrt{2-1}t} + \frac{1}{\sqrt{2}-\sqrt{2-1}t} \right) dt$
 $= \frac{1}{3} \left[\frac{\sqrt{2}}{\sqrt{2-1}} \ln \left| \frac{\sqrt{2} + \sqrt{2-1}t}{\sqrt{2}-\sqrt{2-1}t} \right| \right]_0^1$
 $= \frac{\sqrt{2}}{3} \left[\ln \left| \frac{\sqrt{2}-1}{\sqrt{2}+1} \right| - \ln 1 \right] = \frac{\sqrt{2}}{3} \ln \left| \frac{(\sqrt{2}-1)^2}{2-1} \right| = \frac{\sqrt{2}}{3} \ln (\sqrt{2}-1)^2$
 (C14)

Question 116 (****)

Find the value of the following definite integral.

$$\int_0^{\frac{1}{2}} \frac{12x-1}{(6x^2-x-1)(6x^2-x-5)+10} dx$$

Give the answer in the form $\arctan\left(\frac{1}{n}\right)$, where n is a positive integer.
 , $n=7$

As $\frac{d}{dx}(6x^2-x-1) = 12x-1$, take a substitution

$$u = 6x^2 - x - 1$$

$$\frac{du}{dx} = 12x - 1$$

$$dx = \frac{du}{12x-1}$$

$$6x^2 - x - 5 = u - 4$$

$$3x - \frac{1}{2} \rightarrow u = 0$$

$$\left(3 - \frac{1}{2}\right)$$

$$\int_0^{\frac{1}{2}} \frac{12x-1}{(6x^2-x-1)(6x^2-x-5)+10} dx$$

$$= \int_{-1}^0 \frac{12x-1}{u(u-4)+10} \frac{du}{12x-1}$$

$$= \int_{-1}^0 \frac{1}{u^2-4u+10} du$$

$$= \int_{-1}^0 \frac{1}{(u^2-4u+4)+2} du$$

$$= \int_{-1}^0 \frac{1}{(u-2)^2+2} du$$

INTEGRATE TO ARCTAN (SOMEONE ASKED)

$$= \left[\arctan\left(\frac{u-2}{\sqrt{2}}\right) \right]_{-1}^0 = \arctan 3 - \arctan 2$$

SIMPLY FURTHER USING THE $\tan(A-B)$ IDENTITY

$$\tan[\arctan 3 - \arctan 2] = \frac{\tan(\arctan 3) - \tan(\arctan 2)}{1 + \tan(\arctan 3)\tan(\arctan 2)}$$

$$= \frac{3-2}{1+3 \times 2} = \frac{1}{7}$$

THIS GIVES

$$\therefore \arctan 3 - \arctan 2 = \arctan \frac{1}{7}$$

(1=7)

Question 117 (****)

$$I = \int_{\frac{1}{\sqrt{3}}}^1 \frac{\sqrt{1+x^2}}{x^4} dx.$$

- a) Use a trigonometric substitution to show that

$$I = \frac{2}{3}(a + b\sqrt{2}),$$

where a and b are integers to be found.

- b) Use a hyperbolic substitution to verify the answer of part (a).

$$\boxed{}, \quad I = \frac{2}{3}(4 - \sqrt{2})$$

a) STARTING WITH A $\tan \theta$ SUBSTITUTION DUE TO THE FORM OF THE RADICAND

$$\int_{\frac{1}{\sqrt{3}}}^1 \frac{\sqrt{1+x^2}}{x^4} dx = \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{\sqrt{1+\tan^2 \theta}}{\tan^4 \theta} (\sec^2 \theta d\theta)$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{\sec^2 \theta}{\tan^4 \theta} (\sec^2 \theta d\theta) = \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{\sec^4 \theta}{\tan^4 \theta} d\theta$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{\sec^2 \theta \cos^4 \theta}{\sin^4 \theta} d\theta = \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{1}{\cos^2 \theta} \times \frac{\cos^4 \theta}{\sin^4 \theta} d\theta$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{\cos^2 \theta}{\sin^4 \theta} d\theta$$

Now by RECOGNITION (or ALGEBRAIC SUBSTITUTION $u = \sin \theta$)

$$= \int_{\frac{1}{2}}^{\frac{\sqrt{2}}{2}} \cos^2(\sin^{-1} u) \frac{1}{u^4} du = \left[-\frac{1}{3} (\sin \theta)^{-3} \right]_{\frac{\pi}{6}}^{\frac{\pi}{4}}$$

$$= \frac{1}{3} \left[\frac{1}{\sin^3 \theta} \right]_{\frac{\pi}{6}}^{\frac{\pi}{4}} = \frac{1}{3} \left[\frac{1}{(\frac{\sqrt{2}}{2})^3} - \frac{1}{(\frac{1}{2})^3} \right]$$

$$= \frac{1}{3} \left[\frac{1}{\frac{\sqrt{2}}{2}} - \frac{1}{\frac{1}{2}} \right] = \frac{1}{3} [2 - 2\sqrt{2}]$$

$$= \frac{2}{3}(4 - \sqrt{2})$$

$z = \tan \theta$
 $dz = \sec^2 \theta d\theta$
 $z = 1 \rightarrow \theta = \frac{\pi}{4}$
 $z = \frac{1}{\sqrt{3}} \rightarrow \theta = \frac{\pi}{6}$

b) Now BY A HYPERBOLIC SUBSTITUTION

$$\int_{\frac{1}{\sqrt{3}}}^1 \frac{\sqrt{1+x^2}}{x^4} dx = \int_{\frac{1}{\sqrt{3}}}^1 \frac{\sqrt{1+\sinh^2 u}}{\sinh^4 u} (\cosh u du)$$

$$= \int_{\frac{1}{\sqrt{3}}}^1 \frac{\cosh u}{\sinh^4 u} (\cosh u du) = \int_{\frac{1}{\sqrt{3}}}^1 \frac{\cosh^2 u}{\sinh^4 u} du$$

$$= \int_{\frac{1}{\sqrt{3}}}^1 \frac{\cosh^2 u}{\sinh^4 u} du = \int_{\frac{1}{\sqrt{3}}}^1 \frac{\cosh^2 u}{\sinh^4 u} du$$

By RECOGNITION (or the substitution $u = \sinh v$)

$$= \left[-\frac{1}{3} \cosh^3 v \right]_{\frac{1}{\sqrt{3}}}^1 = \frac{1}{3} \left[\cosh^3 v \right]_{\frac{1}{\sqrt{3}}}^1$$

$$= \frac{1}{3} [2^3 - (\sqrt{2})^3] = \frac{1}{3} [8 - 2\sqrt{2}]$$

$$= \frac{2}{3}(4 - \sqrt{2})$$

$z = \sinh u$
 $dz = \cosh u du$
 $z = 1 \rightarrow \sinh u = 1 \rightarrow \cosh u = \sqrt{2}$
 $z = \frac{1}{\sqrt{3}} \rightarrow \sinh u = \frac{1}{\sqrt{3}} \rightarrow \cosh u = \frac{2}{\sqrt{3}}$
 $z = \frac{1}{\sqrt{3}}$
 $\sinh u = \frac{1}{\sqrt{3}}$
 $\cosh u = \frac{2}{\sqrt{3}}$
 $\cosh^2 u = \frac{4}{3}$
 $\sinh^2 u = \frac{1}{3}$
 $\sinh^4 u = \frac{1}{9}$
 $\cosh^2 u = \frac{4}{3}$
 $\sinh^4 u = \frac{1}{9}$
 $\cosh^2 u = \frac{4}{3}$
 $\sinh^4 u = \frac{1}{9}$
 $\cosh^2 u = \frac{4}{3}$
 $\sinh^4 u = \frac{1}{9}$

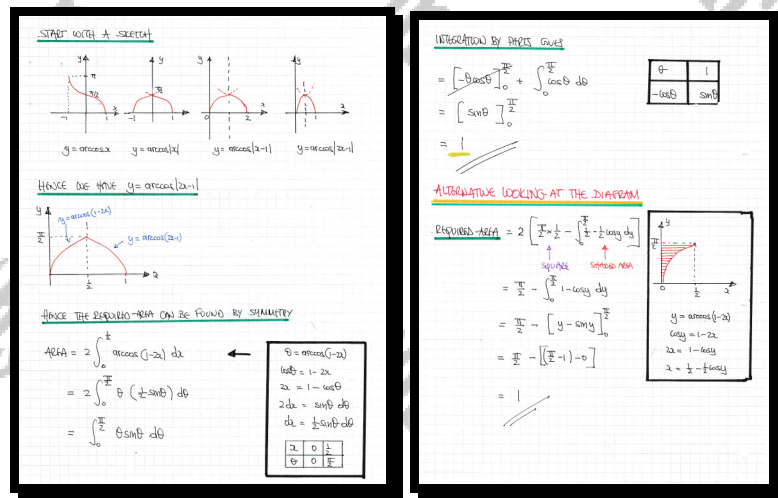
Question 118 (**)**

The function f is defined in the largest real domain by the equation

$$f(x) \equiv \arccos|2x-1|.$$

Determine the area of the finite region bounded by f and the coordinate axes.

, area = 1



Question 119 (****)

By considering

$$\frac{\sin[(2m+1)x]}{\sin x} - \frac{\sin[(2m-1)x]}{\sin x}, \quad m \in \mathbb{N},$$

determine the exact value of

$$\int_0^{\frac{1}{2}\pi} \frac{\sin 7x}{\sin x} dx.$$

$$\boxed{}, \quad \boxed{\frac{1}{2}\pi}$$

CONSIDER THE FOLLOWING TRIGONOMETRIC EXPRESSION, $m \in \mathbb{N}$

$$\frac{\sin[(2m+1)x]}{\sin x} - \frac{\sin[(2m-1)x]}{\sin x}$$

$$= \frac{2 \sin\left(\frac{(2m+1)x + (2m-1)x}{2}\right) \cos\left(\frac{(2m+1)x - (2m-1)x}{2}\right)}{\sin x}$$

$$= \frac{2 \sin(2mx) \cos(x)}{\sin x} = 2 \cos(x) \cos(2mx)$$

NEW CONSIDER THE INTEGRAL OF $2 \cos(x) \cos(2mx)$ IN $[0, \frac{1}{2}\pi]$

$$\int_0^{\frac{1}{2}\pi} 2 \cos(x) \cos(2mx) dx = \left[\frac{1}{m} \sin(2mx) \right]_0^{\frac{1}{2}\pi} = \frac{1}{m} [\sin m\pi - \sin 0]$$

$$= 0$$

4 marks

$$\int_0^{\frac{1}{2}\pi} \frac{\sin[(2m+1)x]}{\sin x} - \frac{\sin[(2m-1)x]}{\sin x} dx = \int_0^{\frac{1}{2}\pi} 2 \cos(x) \cos(2mx) dx$$

$$= \int_0^{\frac{1}{2}\pi} \frac{\sin[(2m+1)x]}{\sin x} dx - \int_0^{\frac{1}{2}\pi} \frac{\sin[(2m-1)x]}{\sin x} dx = 0$$

$$\int_0^{\frac{1}{2}\pi} \frac{\sin[(2m+1)x]}{\sin x} dx = \int_0^{\frac{1}{2}\pi} \frac{\sin[(2m-1)x]}{\sin x} dx$$

THIS USES THE

$$\int_0^{\frac{1}{2}\pi} \frac{\sin 7x}{\sin x} dx = \int_0^{\frac{1}{2}\pi} \frac{\sin 5x}{\sin x} dx = \int_0^{\frac{1}{2}\pi} \frac{\sin 3x}{\sin x} dx = \int_0^{\frac{1}{2}\pi} \frac{\sin x}{\sin x} dx$$

$$= \int_0^{\frac{1}{2}\pi} 1 dx = \frac{1}{2}\pi$$

Question 120 (****)

Find in exact simplified form the value of the following definite integral.

$$\int_{3^{-\frac{1}{6}}}^{\frac{1}{36}} \left(x^2 + \frac{1}{x^4} \right)^{-2} dx$$

$$\boxed{}, \quad \frac{\pi}{36}$$

START BY AN INITIAL TRY / CF - LET $u = x^2$ & $du = 2x$

$$\int_{3^{-\frac{1}{6}}}^{\frac{1}{36}} \left(x^2 + \frac{1}{x^4} \right)^{-2} dx = \int_{3^{-\frac{1}{6}}}^{\frac{1}{36}} \left(\frac{x^2 + 1}{x^4} \right)^{-2} dx = \int_{3^{-\frac{1}{6}}}^{\frac{1}{36}} \frac{x^8}{(x^2 + 1)^2} dx$$

KNOW THE SUBSTITUTION $x^2 = u$ WORKS, BUT IT IS VERY MESSY - REVIEW

$x^2 = 36$ MIGHT ALSO BE OK - WE PROCEED AS FOLLOWS

WE CAN WRITE $\frac{x^8}{(x^2 + 1)^2} = \frac{x^6}{(x^2 + 1)^2}$ IN PARTS

INTEGRABLE

$$\int \frac{x^6}{(x^2 + 1)^2} dx = \int \frac{x^4(x^2 + 1) - x^4}{(x^2 + 1)^2} dx = \int \frac{x^4(x^2 + 1)}{(x^2 + 1)^2} dx - \int \frac{x^4}{(x^2 + 1)^2} dx$$

$$= \int \frac{x^4}{x^2 + 1} dx - \int \frac{x^4}{(x^2 + 1)^2} dx$$

BY RECOGNITION THIS IS THE DIFFERENTIAL OF $\arctan(x)$, SINCE

$$\frac{d}{dx}(\arctan(x)) = \frac{1}{1 + x^2} \times 2x^2 = \frac{2x^2}{1 + x^2}$$

$$\therefore \frac{1}{2} \frac{d}{dx}(\arctan(x^2)) = \frac{x^2}{1 + x^2}$$

FINALLY COLLECTING ALL THE RESULTS FOR THE EVALUATION

$$= \left[-\frac{x^3}{3(x^2 + 1)} + \frac{1}{2} \arctan(x^2) \right]_{3^{-\frac{1}{6}}}^{\frac{1}{36}}$$

$$= \frac{1}{6} \left[\arctan(x^2) - \frac{x^3}{x^2 + 1} \right]_{3^{-\frac{1}{6}}}^{\frac{1}{36}}$$

$$= \frac{1}{6} \left[\arctan(3^{\frac{1}{6}}) - \frac{3^{\frac{1}{6}}}{3^{\frac{1}{6}} + 1} \right] - \left[\arctan(3^{-\frac{1}{6}}) - \frac{3^{-\frac{1}{6}}}{3^{-\frac{1}{6}} + 1} \right]$$

$$= \frac{1}{6} \left[\arctan(3^{\frac{1}{6}}) - \frac{1}{4} \sqrt{3} - \arctan\left(\frac{1}{\sqrt{3}}\right) + \frac{1}{\sqrt{3} + 1} \right]$$

$$= \frac{1}{6} \left[\frac{\pi}{6} - \frac{1}{4} \sqrt{3} - \frac{\pi}{6} + \frac{3}{4\sqrt{3}} \right]$$

$$= \frac{1}{6} \left[\frac{\pi}{6} - \frac{1}{4} \sqrt{3} + \frac{3}{4\sqrt{3}} \right]$$

$$= \frac{1}{36}$$

Question 121 (****)

Determine a simplified expression, in the form $\ln[f(n)]$, for the following sum.

$$\sum_{r=2}^N \left[\int_2^r \frac{2}{x^2-1} dx \right].$$

$$\boxed{}, \ln \left[\frac{2 \times 3^{N-1}}{N(N+1)} \right]$$

• STRIKE BY PARTIAL FRACTIONS IN THE INTEGRAND (BY INSPECTION)

$$\sum_{r=2}^N \left[\int_2^r \frac{2}{x^2-1} dx \right] = \sum_{r=2}^N \left[\int_2^r \frac{2}{(x-1)(x+1)} dx \right]$$

$$= \sum_{r=2}^N \left[\int_2^r \frac{1}{x-1} - \frac{1}{x+1} dx \right] = \sum_{r=2}^N \left[\ln|2-1| - \ln|2+1| \right]_{2=1}^{2=r}$$

• WRITING THE TERMS EXPLICITLY, LOOKING FOR PATTERNS

$$= \sum_{r=2}^N \left[\ln|r-1| - \ln|r+1| \right] - \left[\ln|1| - \ln|3| \right]$$

$$= \sum_{r=2}^N \left[\ln(r-1) - \ln(r+1) + \ln 3 \right]$$

$$= \begin{array}{l} \ln 1 - \ln 3 + \ln 3 \quad \leftarrow r=2 \\ \ln 2 - \ln 4 + \ln 3 \quad \leftarrow r=3 \\ \ln 3 - \ln 5 + \ln 3 \quad \leftarrow r=4 \\ \ln 4 - \ln 6 + \ln 3 \quad \leftarrow r=5 \\ \vdots \\ \ln(N-2) - \ln N + \ln 3 \quad \leftarrow r=N-1 \\ \ln(N-1) - \ln(N+1) + \ln 3 \quad \leftarrow r=N \end{array} \quad \left. \vphantom{\sum_{r=2}^N} \right\} (N-1) \text{ TERMS}$$

• ADDING:

$$\sum_{r=2}^N \left[\int_2^r \frac{2}{x^2-1} dx \right] = \ln 2 - \ln N - \ln(N+1) + (N-1)\ln 3$$

$$= \ln 2 + (N-1)\ln 3 - (\ln N + \ln(N+1))$$

$$= \ln \left[\frac{2 \times 3^{N-1}}{N(N+1)} \right]$$

Question 122 (****)

Use appropriate integration methods to find a simplified expression for

$$\int x \arccos \left[\frac{1-x^2}{1+x^2} \right] dx.$$

$$\boxed{}, -x + (1+x^2) \arctan x + \text{constant}$$

CONSIDER THE SUBSTITUTION $\theta = \alpha$

$\alpha = \arctan\left(\frac{1}{x}\right)$
 $d\alpha = \frac{1}{1+x^2} dx$
 or $\left[d\alpha = \frac{1}{1+x^2} dx \right]$
 or $\left[dx = \frac{1}{1+x^2} d\alpha \right]$
 (we should see which form is better for the question)

$\frac{1-x^2}{1+x^2} = \frac{1 - \tan^2(\frac{\pi}{2} - \alpha)}{1 + \tan^2(\frac{\pi}{2} - \alpha)}$
 $= \frac{1 - \tan^2(\frac{\pi}{2} - \alpha)}{\sec^2(\frac{\pi}{2} - \alpha)}$
 $= \frac{1}{\sec^2(\frac{\pi}{2} - \alpha)} - \frac{\tan^2(\frac{\pi}{2} - \alpha)}{\sec^2(\frac{\pi}{2} - \alpha)}$
 $= \cos^2(\frac{\pi}{2} - \alpha) - \sin^2(\frac{\pi}{2} - \alpha)$
 $= \cos(2\alpha)$

TRANSFORMING THE INTEGRAL WE HAVE

$$\int x \arccos \left(\frac{1-x^2}{1+x^2} \right) dx = \int \tan(\frac{\pi}{2} - \alpha) \arccos(\cos \alpha) \left[\frac{1}{\sec^2(\frac{\pi}{2} - \alpha)} d\alpha \right]$$

$$= \int \frac{1}{2} \theta \tan(\theta) \sec^2(\theta) d\theta$$

INTEGRATION BY PARTS

$\frac{1}{2} \theta$	$\frac{1}{2}$
$\tan^2(\frac{\pi}{2} - \alpha)$	$\tan(\frac{\pi}{2} - \alpha) \sec^2(\frac{\pi}{2} - \alpha)$

$\dots = \frac{1}{2} \theta \tan^2(\frac{\pi}{2} - \alpha) - \int \frac{1}{2} \tan^2(\frac{\pi}{2} - \alpha) d\theta$
 $= \frac{1}{2} \theta \tan^2(\frac{\pi}{2} - \alpha) - \frac{1}{2} \int \sec^2 \theta - 1 d\theta$

$= \frac{1}{2} \theta \tan^2(\frac{\pi}{2} - \alpha) - \frac{1}{2} [2 \tan(\frac{\pi}{2} - \alpha) - \theta] + C$
 $= \frac{1}{2} \theta \tan^2(\frac{\pi}{2} - \alpha) - \tan(\frac{\pi}{2} - \alpha) + \frac{1}{2} \theta + C$
 $= \frac{1}{2} \theta (1 + \tan^2(\frac{\pi}{2} - \alpha)) - \tan(\frac{\pi}{2} - \alpha) + C$

$\alpha = \arctan \frac{1}{x}$
 $\arctan \frac{1}{x} = \frac{\pi}{2} - \theta$

$= [\arctan \frac{1}{x}] [1 + x^2] - \arctan \frac{1}{x} + C$
 $= -x + (1+x^2) \arctan x + C$

Question 123 (****)

Find the exact value of

$$\int_0^1 \left[\sum_{n=1}^{\infty} \frac{(n+1)x^n}{(n+2)!} \right] dx.$$

$$\boxed{}, \frac{1}{2}(2e-5)$$

Handwritten solution for Question 123:

$$\begin{aligned} \int_0^1 \left[\sum_{n=1}^{\infty} \frac{(n+1)x^n}{(n+2)!} \right] dx &= \dots \text{ RECALL: (APPROXIMATION) \& SUMMATION } \\ &= \sum_{n=1}^{\infty} \left[\frac{(n+1)}{(n+2)!} \int_0^1 x^n dx \right] = \sum_{n=1}^{\infty} \left[\frac{(n+1)}{(n+2)!} \left(\frac{x^{n+1}}{n+1} \right) \Big|_0^1 \right] \\ &= \sum_{n=1}^{\infty} \left[\frac{1}{(n+2)!} \right] \\ &= \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \dots \\ &= \left(\frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \dots \right) - \left(\frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} \right) \\ &= e - \left(1 + 1 + \frac{1}{2} \right) = e - \frac{5}{2} = \frac{1}{2}(2e-5) \end{aligned}$$

Question 124 (****)

- a) Use an appropriate integration method to evaluate the following integral.

$$\int_0^1 x^3 \arctan x \, dx.$$

- b) Obtain an infinite series expansion for $\arctan x$ and use this series expansion to verify the answer obtained for the above integral in part (a).

[you may assume that integration and summation commute]

$$\boxed{}, \boxed{\frac{1}{6}}$$

a) SIMPLY BY INTEGRATION BY PARTS

$\int_0^1 x^3 \arctan x \, dx = \left[\frac{1}{4} x^4 \arctan x \right]_0^1 - \int_0^1 \frac{1}{4} x^4 \cdot \frac{1}{1+x^2} \, dx$

$= \frac{1}{4} \cdot \frac{\pi}{4} - \frac{1}{4} \int_0^1 \frac{x^4}{1+x^2} \, dx$

$= \frac{\pi}{16} - \frac{1}{4} \int_0^1 \frac{x^2(x^2+1) - x^2}{1+x^2} \, dx$

$= \frac{\pi}{16} - \frac{1}{4} \int_0^1 \left(x^2 - \frac{x^2}{1+x^2} \right) \, dx$

$= \frac{\pi}{16} - \frac{1}{4} \left[\frac{1}{3} x^3 - \arctan x \right]_0^1$

$= \frac{\pi}{16} - \frac{1}{4} \left(\frac{1}{3} - \frac{\pi}{4} \right) = \frac{\pi}{16} - \frac{1}{12} + \frac{\pi}{16} = \frac{\pi}{8} - \frac{1}{12}$

b) FIND THE EXPANSION OF $\arctan x$

$\frac{d}{dx} (\arctan x) = \frac{1}{1+x^2} = (1+x^2)^{-1} = 1 - x^2 + x^4 - x^6 + \dots$

INTEGRATE BOTH SIDES

$\arctan x = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \frac{1}{9}x^9 - \dots + C$

$\arctan 0 = 0 \implies C = 0$

$\arctan x = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \frac{1}{9}x^9 - \dots$

$\arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$

THIS IS NOW TRUE

$\int_0^1 x^3 \arctan x \, dx = \int_0^1 x^3 \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} \, dx$

INTEGRATE THE ENTIRE

$\int_0^1 x^3 \arctan x \, dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \int_0^1 x^{2n+4} \, dx$

$= \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \left[\frac{x^{2n+5}}{2n+5} \right]_0^1$

$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)(2n+5)}$

NEED TO SUM THIS SERIES BY PARTIAL FRACTIONS

$\frac{1}{(2n+1)(2n+5)} = \frac{1}{4} \left(\frac{1}{2n+1} - \frac{1}{2n+5} \right)$ (BY INSPECTION)

FINALLY WE HAVE THE RESULT

$\int_0^1 x^3 \arctan x \, dx = \frac{1}{4} \sum_{n=0}^{\infty} \left(\frac{(-1)^n}{2n+1} - \frac{(-1)^n}{2n+5} \right)$

$= \frac{1}{4} \lim_{N \rightarrow \infty} \left[1 - \frac{1}{2} - \frac{(-1)^N}{2N+3} + \frac{(-1)^N}{2N+5} \right]$

$= \frac{1}{4} \times \left(1 - \frac{1}{2} \right) = \frac{1}{8}$

$= \frac{1}{8} \times \frac{3}{3} = \frac{3}{8}$

$= \frac{3}{8}$

Question 125 (****)

Find the exact value of

$$\int_{\frac{1}{6}\pi}^{\frac{1}{3}\pi} \frac{\cos x + \sin x}{\sqrt{\sin 2x}} dx.$$

$$\boxed{}, \left[2 \arcsin \left[\frac{\sqrt{3}-1}{2} \right] \right]$$

Handwritten solution for the integral problem:

$$\begin{aligned} & \int_{\frac{1}{6}\pi}^{\frac{1}{3}\pi} \frac{\cos x + \sin x}{\sqrt{\sin 2x}} dx \\ &= \int_{\frac{1}{6}\pi}^{\frac{1}{3}\pi} \frac{\cos x + \sin x}{\sqrt{2 \sin x \cos x}} dx \\ &= \int_{\frac{1}{6}\pi}^{\frac{1}{3}\pi} \frac{\cos x + \sin x}{\sqrt{1-u^2}} \frac{du}{\cos x + \sin x} \\ &= \int_{\frac{1}{6}\pi}^{\frac{1}{3}\pi} \frac{1}{\sqrt{1-u^2}} du \\ &= 2 \int_0^{\frac{1}{2}} \frac{1}{\sqrt{1-u^2}} du \\ &= 2 \left[\arcsin u \right]_0^{\frac{1}{2}} \\ &= 2 \arcsin \frac{1}{2} = 2 \arcsin \left(\frac{\sqrt{3}-1}{2} \right) \\ &= 2 \arcsin \left(\frac{\sqrt{3}-1}{2} \right) \end{aligned}$$

BY SUBSTITUTION

$$\begin{aligned} u &= \sin x - \cos x \\ \frac{du}{dx} &= \cos x + \sin x \\ dx &= \frac{du}{\cos x + \sin x} \\ x = \frac{\pi}{6}, u = \frac{1}{2} - \frac{\sqrt{3}}{2} = -\frac{1}{2} \\ x = \frac{\pi}{3}, u = \frac{\sqrt{3}}{2} - \frac{1}{2} = \frac{1}{2} \\ u^2 &= (\sin x - \cos x)^2 \\ u^2 &= \sin^2 x + \cos^2 x - 2 \sin x \cos x \\ u^2 &= 1 - \sin 2x \\ 2 \sin x \cos x &= 1 - u^2 \end{aligned}$$

Question 126 (****)

$$I = \int_{-\frac{1}{3}\pi}^{\frac{1}{3}\pi} \frac{\sqrt{3}(1+\pi x^3)}{2 - \cos\left(|x| + \frac{1}{3}\pi\right)} dx.$$

Show that

$$I = 4 \arctan \frac{1}{2}.$$

□, proof

The image shows two pages of handwritten work for Question 126. The left page shows the integral $I = \int_{-\frac{1}{3}\pi}^{\frac{1}{3}\pi} \frac{\sqrt{3}(1+\pi x^3)}{2 - \cos(|x| + \frac{1}{3}\pi)} dx$ and uses the substitution $u = 2 + \frac{x}{\sqrt{3}}$ to transform it into $2\sqrt{3} \int_{\frac{1}{2}}^{\frac{3}{2}} \frac{1}{2 - \cos u} du$. It then uses the trigonometric substitution $\tan \frac{u}{2} = t$ to evaluate the integral. The right page shows the same integral evaluated using a different method, resulting in $4 \arctan \frac{1}{2}$. A small box on the right page contains the identity $\arctan 3 - \arctan 1 = \frac{\pi}{4}$ and other related identities.

Question 127 (****)

By expressing the integrand in the form $\operatorname{sech}^2 x f(\tanh x)$, or otherwise, find the value of the following integral.

$$\int_0^{\frac{1}{2} \ln \frac{5}{3}} \frac{\sqrt{2 \operatorname{sech} x}}{\sqrt[4]{\sinh 2x \cosh x} - \sqrt[4]{2 \sinh^3 x}} dx.$$

 ,

Using/Assuming with the suggestion given

$$I = \int_0^{\frac{1}{2} \ln \frac{5}{3}} \frac{\sqrt{2 \operatorname{sech} x}}{(\sqrt[4]{\sinh 2x \cosh x} - \sqrt[4]{2 \sinh^3 x})^2} dx$$

$$\Rightarrow I = \int_0^{\frac{1}{2} \ln \frac{5}{3}} \frac{(2 \operatorname{sech} x)^{\frac{1}{2}}}{\left((\sinh 2x \cosh x)^{\frac{1}{4}} - (2 \sinh^3 x)^{\frac{1}{4}} \right)^2} dx$$

$$\Rightarrow I = \int_0^{\frac{1}{2} \ln \frac{5}{3}} \frac{2^{\frac{1}{2}} \operatorname{sech} x}{\left((\sinh 2x)^{\frac{1}{4}} (\cosh x)^{\frac{1}{4}} - (2 \sinh^3 x)^{\frac{1}{4}} \right)^2} dx$$

$$\Rightarrow I = \int_0^{\frac{1}{2} \ln \frac{5}{3}} \frac{2^{\frac{1}{2}} \operatorname{sech} x}{\left((\sinh 2x)^{\frac{1}{4}} (\cosh x)^{\frac{1}{4}} - (\cosh x)^{\frac{1}{4}} (2 \sinh^3 x)^{\frac{1}{4}} \right)^2} dx$$

$$\Rightarrow I = \int_0^{\frac{1}{2} \ln \frac{5}{3}} \frac{\operatorname{sech} x}{\left((\tanh x)^{\frac{1}{4}} - (\tanh x)^{\frac{3}{4}} \right)^2} dx$$

$$\Rightarrow I = \int_0^{\frac{1}{2} \ln \frac{5}{3}} \frac{\operatorname{sech} x}{(\tanh x)^{\frac{1}{2}} \left(1 - (\tanh x)^{\frac{1}{2}} \right)^2} dx$$

Now by reverse chain rule (recognition), or by using the substitution $u = 1 - (\tanh x)^{\frac{1}{2}}$, we obtain

$$= \int_0^{\frac{1}{2} \ln \frac{5}{3}} \operatorname{sech} x (\tanh x)^{-\frac{1}{2}} (1 - (\tanh x)^{\frac{1}{2}})^{-2} dx$$

THE SUBSTITUTION $u = 1 - (\tanh x)^{\frac{1}{2}}$, we obtain

$$= \left[2 \left[1 - (\tanh x)^{\frac{1}{2}} \right]^{-1} \right]_0^{\frac{1}{2} \ln \frac{5}{3}}$$

Evaluate $\tanh \left[\frac{1}{2} \ln \frac{5}{3} \right]$ First

$$\tanh x = \frac{e^x - 1}{e^x + 1} \Rightarrow \tanh \left(\frac{1}{2} \ln \frac{5}{3} \right) = \frac{\frac{e^{\frac{1}{2} \ln \frac{5}{3}} - 1}{e^{\frac{1}{2} \ln \frac{5}{3}} + 1}}{\frac{e^{\frac{1}{2} \ln \frac{5}{3}} - 1}{e^{\frac{1}{2} \ln \frac{5}{3}} + 1} + 1} = \frac{\frac{\sqrt{\frac{5}{3}} - 1}{\sqrt{\frac{5}{3}} + 1}}{\frac{\sqrt{\frac{5}{3}} - 1}{\sqrt{\frac{5}{3}} + 1} + 1} = \frac{\sqrt{\frac{5}{3}} - 1}{\sqrt{\frac{5}{3}} + 1 + \sqrt{\frac{5}{3}} + 1} = \frac{\sqrt{\frac{5}{3}} - 1}{2\sqrt{\frac{5}{3}} + 2} = \frac{\sqrt{\frac{5}{3}} - 1}{2(\sqrt{\frac{5}{3}} + 1)}$$

RETURNING TO THE INTEGRAL EVALUATION

$$\dots = 2 \left[\frac{1}{1 - \sqrt{\tanh x}} \right]_0^{\frac{1}{2} \ln \frac{5}{3}} = 2 \left[\frac{1}{1 - \sqrt{\frac{\sqrt{\frac{5}{3}} - 1}{2(\sqrt{\frac{5}{3}} + 1)}}} - \frac{1}{1 - 0} \right]$$

$$= 2 \left[\frac{1}{1 - \frac{1}{2}} - 1 \right] = 2$$

Question 128 (****)

Use appropriate integration techniques to show that

$$\int_{\operatorname{arsinh} \frac{1}{\sqrt{3}}}^{\operatorname{arsinh} \sqrt{3}} \operatorname{sech} x (1 - \operatorname{sech} x) dx = \frac{\pi}{12}.$$

, proof

METHOD ONE

$\frac{\operatorname{arsinh} \sqrt{3}}{\operatorname{arsinh} \frac{1}{\sqrt{3}}} \operatorname{sech} x - \operatorname{sech}^2 x \, dx$
 $= \int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} \frac{(1 - \operatorname{sech}^2 x)}{\operatorname{sech}^2 x} \left(\frac{\sinh x}{\operatorname{sech}^2 x} dx \right)$
 $= \int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} \frac{\operatorname{sech} x (1 - \operatorname{sech}^2 x)}{\operatorname{sech}^2 x \sinh x} (\sinh x dx)$
 $= \int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} \frac{1 - \operatorname{sech}^2 x}{\sinh x} (\sinh x dx)$
 $= \int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} \frac{\sinh^2 x}{\sinh x} dx$
 $= \int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} \sinh x dx$
 $= \int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} (1 - \operatorname{sech}^2 x) dx$
 $= \int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} (1 - \operatorname{sech}^2 x) dx$

BY SUBSTITUTION

$\operatorname{sech} x = \cosh u$
 $-\operatorname{sech}^2 x dx = -\sinh u du$
 $dx = \frac{\sinh u}{\operatorname{sech}^2 x} du$
FOR THE LIMITS
 $\Rightarrow \operatorname{sech} x = \cosh u$
 $\Rightarrow \cosh x = \sec \theta$
 $\Rightarrow \cosh^2 x = \sec^2 \theta$
 $\Rightarrow \cosh^2 x - 1 = \sec^2 \theta - 1$
 $\Rightarrow \sinh^2 x = \tan^2 \theta$
 $\Rightarrow \sinh x = \tan \theta$

$\bullet x = \operatorname{arsinh} \sqrt{3}$
 $\Rightarrow \sinh x = \sqrt{3}$
 $\Rightarrow \tan \theta = \sqrt{3}$
 $\Rightarrow \theta = \frac{\pi}{3}$

$\bullet x = \operatorname{arsinh} \frac{1}{\sqrt{3}}$
 $\Rightarrow \sinh x = \frac{1}{\sqrt{3}}$
 $\Rightarrow \tan \theta = \frac{1}{\sqrt{3}}$
 $\Rightarrow \theta = \frac{\pi}{6}$

METHOD TWO

WORK EACH INDEFINITE INTEGRAL SEPARATELY

$I_1 = \int \operatorname{sech}^2 x dx = \int \operatorname{sech} x \operatorname{sech} x dx$
 $= \operatorname{sech} x \tanh x + \int \operatorname{sech} x \tanh^2 x dx$
 $= \operatorname{sech} x \tanh x + \int \operatorname{sech} x (1 - \operatorname{sech}^2 x) dx$
 $= \operatorname{sech} x \tanh x + \int \operatorname{sech} x dx - \int \operatorname{sech}^3 x dx$
 $I_1 = \operatorname{sech} x \tanh x + \int \operatorname{sech} x dx - I_1$

$2I_1 = \operatorname{sech} x \tanh x + \int \operatorname{sech} x dx$
 $I_1 = \frac{1}{2} \operatorname{sech} x \tanh x + \frac{1}{2} \int \operatorname{sech} x dx$

RETURNING TO THE REQUIRED INVERSE TRIGONOMETRIC UNITS WE OBTAIN

$\int \operatorname{sech} x - \operatorname{sech}^3 x dx = \int \operatorname{sech} x dx - \int \operatorname{sech}^3 x dx$
 $= \int \operatorname{sech} x dx - \left[\frac{1}{2} \operatorname{sech} x \tanh x + \frac{1}{2} \int \operatorname{sech} x dx \right]$
 $= \frac{1}{2} \int \operatorname{sech} x dx - \frac{1}{2} \operatorname{sech} x \tanh x$

NEXT WE NEED $\int \operatorname{sech} x dx$

$I_2 = \int \operatorname{sech} x dx = \int \frac{1}{\cosh x} dx = \int \frac{\cosh x}{\cosh^2 x} dx$
 $= \int \frac{\cosh x}{1 + \sinh^2 x} dx = \dots$

NOW BY INSPECTION AS $\frac{d}{dx}(\operatorname{arctan}(\sinh x)) = \frac{1}{1 + \sinh^2 x} \times \cosh x$

OR A SUBSTITUTION $u = \sinh x$

$\dots \operatorname{arctan}(\sinh x) + C$
 $I_2 = \int \operatorname{sech} x dx = \operatorname{arctan}(\sinh x) + C$

RETURNING TO THE REQUIRED INTEGRAL

$\int \operatorname{sech} x - \operatorname{sech}^3 x dx = \frac{1}{2} \operatorname{arctan}(\sinh x) - \frac{1}{2} \operatorname{sech} x \tanh x + C$
 $\int \operatorname{sech} x - \operatorname{sech}^3 x dx = \frac{1}{2} \left[\operatorname{arctan}(\sinh x) - \frac{\sinh x}{\cosh^2 x} \right]_{\frac{1}{\sqrt{3}}}^{\sqrt{3}}$
 $= \frac{1}{2} \left[\operatorname{arctan}(\sinh x) - \frac{\sinh x}{1 + \sinh^2 x} \right]_{\frac{1}{\sqrt{3}}}^{\sqrt{3}}$

EVALUATING

$\dots = \frac{1}{2} \left[\operatorname{arctan} \sqrt{3} - \frac{\sqrt{3}}{1 + 3} \right] - \frac{1}{2} \left[\operatorname{arctan} \frac{1}{\sqrt{3}} - \frac{\frac{1}{\sqrt{3}}}{1 + \frac{1}{3}} \right]$
 $= \frac{1}{2} \left[\frac{\pi}{3} - \frac{\sqrt{3}}{4} \right] - \frac{1}{2} \left[\frac{\pi}{6} - \frac{\frac{1}{\sqrt{3}} \times 3/4}{\frac{4}{3} + \frac{1}{3}} \right]$
 $= \frac{1}{2} \left[\frac{\pi}{3} - \frac{\sqrt{3}}{4} \right] - \frac{1}{2} \left[\frac{\pi}{6} - \frac{\frac{3}{4\sqrt{3}}}{\frac{5}{3}} \right]$
 $= \frac{1}{2} \left[\frac{\pi}{3} - \frac{\sqrt{3}}{4} - \frac{\pi}{6} + \frac{3\sqrt{3}}{10} \right]$
 $= \frac{1}{2} \times \frac{\pi}{6}$
 $= \frac{\pi}{12}$

Question 129 (****)

The function f is defined as.

$$f(x) = \arctan x, \quad x \in \left(-\frac{1}{2}\pi, \frac{1}{2}\pi\right).$$

- a) Find a simplified expression for $\int f(x) dx$.
- b) By considering the tangent compound angle identity, or otherwise, find an exact simplified value for

$$\int_1^2 \arctan \left[\frac{1}{x^2 - 3x + 3} \right] dx.$$

$$\boxed{}, \quad \boxed{x \arctan x - \frac{1}{2} \ln(x^2 + 1)}, \quad \boxed{\frac{1}{2}\pi - \ln 2}$$

a) Integrate $\arctan x$ by substitution

$\theta = \arctan x$
 $x = \tan \theta$
 $dx = \sec^2 \theta d\theta$

Now proceed by integration by parts

$\int \theta \sec^2 \theta d\theta$
 $= \theta \tan \theta - \int \tan \theta d\theta$
 $= \theta \tan \theta - \ln |\sec \theta| + C$
 $= \theta \tan \theta + \ln |\cos \theta| + C$
 $= x \arctan x + \ln \left(\frac{1}{\sqrt{1+x^2}} \right) + C$
 $= x \arctan x - \frac{1}{2} \ln(1+x^2) + C$

Alternative by direct integration

$\int \arctan x dx = \int x \arctan x dx$
 $= x \arctan x - \int \frac{x}{1+x^2} dx$
 $= x \arctan x - \frac{1}{2} \int \frac{2x}{1+x^2} dx$
 $= x \arctan x - \frac{1}{2} \ln(1+x^2) + C$

b) Using the first form

$\frac{1}{x^2 - 3x + 3} = \frac{1}{(x-3/2)^2 + 3/4}$

$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$
 $\tan A = x-1$
 $\tan B = x-2$

Thus we have here

$\int_1^2 \arctan \left(\frac{1}{x^2 - 3x + 3} \right) dx = \int_1^2 \arctan \left(\frac{(x-1) - (x-2)}{1 + (x-1)(x-2)} \right) dx$
 $= \int_1^2 \arctan \left(\frac{\tan A - \tan B}{1 + \tan A \tan B} \right) dx = \int_1^2 \arctan [\tan(A-B)] dx$
 $= \int_1^2 A - B dx = \int_1^2 \arctan(x-1) - \arctan(x-2) dx$

By splitting the integral and substitution

$= \int_1^2 \arctan(x-1) dx - \int_1^2 \arctan(x-2) dx$
 $= \int_0^1 \arctan u du - \int_{-1}^0 \arctan v dv$
 $= \left[u \arctan u - \frac{1}{2} \ln(u^2 + 1) \right]_0^1 - \left[v \arctan v - \frac{1}{2} \ln(v^2 + 1) \right]_{-1}^0$
 $= \left[\left(\frac{\pi}{4} - \frac{1}{2} \ln 2 \right) - (0 - \frac{1}{2} \ln 1) \right] - \left[(0 - \frac{1}{2} \ln 1) - \left(-\frac{\pi}{4} - \frac{1}{2} \ln 2 \right) \right]$
 $= \frac{\pi}{4} - \frac{1}{2} \ln 2 - \left[-\frac{\pi}{4} + \frac{1}{2} \ln 2 \right]$
 $= \frac{\pi}{2} - \ln 2$

Question 130 (*****)

Use appropriate integration techniques to find an exact simplified value for the following improper integral.

$$\int_0^{\infty} \frac{x}{e^x - 1} dx.$$

You may assume without proof that

$$\sum_{r=1}^{\infty} \frac{1}{r^2} = \frac{\pi^2}{6}.$$

$$\boxed{\frac{\pi^2}{6}},$$

START BY MULTIPLYING "TOP & BOTTOM" OF THE INTEGRAND BY e^x

$$\int_0^{\infty} \frac{x}{e^x - 1} dx = \int_0^{\infty} \frac{x e^x}{e^x - 1} dx = \int_0^{\infty} x e^x \left(\frac{1}{1 - e^{-x}} \right) dx$$

NOW FOR $|e^{-x}| < 1$ OR SIMPLY THAT $e^{-x} < 1$ WE CAN EXPAND
RECOGNISE / OR EQUIVALENTLY USE THE SUM TO INFINITY FORMULA FOR
A GEOMETRIC PROGRESSION

$$\text{i.e. } \frac{1}{1 - e^{-x}} = 1 + e^{-x} + e^{-2x} + e^{-3x} + \dots$$

THIS USE BRACKET

$$\dots = \int_0^{\infty} x e^x [1 + e^{-x} + e^{-2x} + e^{-3x} + \dots] dx$$

$$= \int_0^{\infty} x [e^x + e^x e^{-x} + e^x e^{-2x} + e^x e^{-3x} + \dots] dx$$

$$= \int_0^{\infty} x \left[\sum_{r=1}^{\infty} e^{-rx} \right] dx$$

INTERCHANGING SUMMATION & INTEGRATION

$$= \sum_{r=1}^{\infty} \left[\int_0^{\infty} x e^{-rx} dx \right]$$

PROCEED BY GRADIENT FUNCTIONS, DIFFERENTIATION UNLESS THE INTEGRAL SIMILAR
(PLEASE RECOGNISE, OR SIMPLY INTEGRATION BY PARTS)

$$\dots = \sum_{r=1}^{\infty} \left[-\frac{1}{r} e^{-rx} \right]_0^{\infty} + \int_0^{\infty} e^{-rx} dx$$

APPROXIMATE INTEGRATION METHODS (SELECTION)

$$\bullet \dots = \int_0^{\infty} x e^{-rx} dx = \int_0^{\infty} t e^{-t} dt = 1(t)$$

$$= \frac{1!}{r^2} = \frac{1}{r^2} \text{ etc. ...}$$

$$\bullet \dots = \int_0^{\infty} x e^{-rx} dx = \text{BY SUBSTITUTION ...}$$

$t = rx$
 $dt = r dx$
 $dx = \frac{1}{r} dt$
 LIMITS CALCULATED

$$= \int_0^{\infty} \frac{t}{r} e^{-t} \left(\frac{1}{r} dt \right) = \frac{1}{r^2} \int_0^{\infty} t e^{-t} dt$$

$$= \frac{1}{r^2} \Gamma(2) = \frac{1}{r^2} \times 1! = \frac{1}{r^2} \text{ etc. ...}$$

Question 131 (****)

The positive solution of the quadratic equation $x^2 - x - 1 = 0$ is denoted by ϕ , and is commonly known as the golden section or golden number.

This implies that $\phi^2 - \phi - 1 = 0$, $\phi = \frac{1}{2}(1 + \sqrt{5}) \approx 1.62$.

a) Show, with a detailed method, that

$$\frac{d}{dx} \left[x(x^\phi + 1)^{1-\phi} \right] = (x^\phi + 1)^{-\phi}.$$

b) Show, with full justification, that

$$\lim_{x \rightarrow \infty} \left[x(x^\phi + 1)^{1-\phi} \right] = 1.$$

c) Show further that

$$1 - \frac{1}{\sqrt[2]{2}} = \int_1^\infty \frac{1}{(x^\phi + 1)^\phi} dx.$$

 , proof

a) DIFFERENTIATE THE PRODUCT AND USE 'CHAIN'

$$\frac{d}{dx} \left[x(x^\phi + 1)^{1-\phi} \right] = 1 \times (x^\phi + 1)^{1-\phi} + x(1-\phi)(x^\phi + 1)^{-\phi} \times \phi x^{\phi-1}$$

FACTORISE $(x^\phi + 1)^{-\phi}$ AND USE THE GRIN

$$= (x^\phi + 1)^{1-\phi} + (1-\phi)\phi x^\phi (x^\phi + 1)^{-\phi}$$

$$= (x^\phi + 1)^{-\phi} \left[(x^\phi + 1) + \phi(1-\phi)x^\phi \right]$$

$$= (x^\phi + 1)^{-\phi} \left[x^\phi + 1 + \phi x^\phi - \phi^2 x^\phi \right]$$

NEED TO SHOW THIS IS 1

$$= (x^\phi + 1)^{-\phi} \left[1 + x^\phi(1 + \phi - \phi^2) \right]$$

BUT $x^2 - x - 1 = 0 \Rightarrow \phi^2 - \phi - 1 = 0 \Rightarrow \phi^2 - \phi + 1 = 0$

$$= (x^\phi + 1)^{-\phi} \left[1 + x^\phi \times 0 \right]$$

$$= (x^\phi + 1)^{-\phi}$$

AS REQUIRED

b) EITHER BY REWRITING THE LIMIT

$$\lim_{x \rightarrow \infty} \left[x(x^\phi + 1)^{1-\phi} \right] = \lim_{x \rightarrow \infty} \left[\frac{x}{(x^\phi + 1)^\phi} \right] \leftarrow \frac{\infty}{\infty}$$

TECHNIQUE: L'HOSPITAL

$$\lim_{x \rightarrow \infty} \left[\frac{1}{\phi(x^\phi + 1)^{\phi-1}} \right] \leftarrow \frac{1}{\infty}$$

FAILURE

PROCEED AS FOLLOWS

$$\dots \lim_{x \rightarrow \infty} \left[\frac{x}{(x^\phi + 1)^\phi} \right] = \lim_{x \rightarrow \infty} \left[\frac{x^{\frac{1}{\phi}}}{(1 + x^{\frac{1}{\phi}})^\phi} \right]$$

$$= \lim_{x \rightarrow \infty} \left[\frac{x^{\frac{1}{\phi}}}{2^\phi} \right] = \lim_{x \rightarrow \infty} \left[\frac{x^{\frac{1}{\phi}}}{2^\phi} \right]$$

As $\phi^2 - \phi - 1 = 0 \Rightarrow \phi^2 - \phi + 1 = 0 \Rightarrow \phi^2 - \phi = -1 \Rightarrow \phi = \frac{1}{\phi}$

c) FINISH THE ARGUMENT

$$\int_1^\infty \frac{1}{(x^\phi + 1)^\phi} dx = \dots$$

PROVE (b) = $\left[-\frac{1}{\phi} (x^\phi + 1)^{1-\phi} \right]_1^\infty$

$$= -\frac{1}{\phi} (x^\phi + 1)^{1-\phi} \Big|_1^\infty = -\frac{1}{\phi} (1 + 1)^{1-\phi} - \left(-\frac{1}{\phi} (1 + 1)^{1-\phi} \right)$$

$$= -\frac{1}{\phi} (2)^{1-\phi} + \frac{1}{\phi} (2)^{1-\phi} = 0$$

NOW SINCE $\phi^2 - \phi - 1 = 0 \Rightarrow \phi^2 - \phi = 1 \Rightarrow \phi - 1 = \frac{1}{\phi} \Rightarrow 1 - \phi = -\frac{1}{\phi}$

HENCE WE HAVE

$$\dots = 1 - 2^{-\phi} = 1 - 2^{-\frac{1}{\phi}} = 1 - \frac{1}{2^{\frac{1}{\phi}}} = 1 - \frac{1}{\sqrt[2]{2}}$$

AS REQUIRED

Question 132 (****)

It is given that

$$\begin{aligned} \diamond 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \dots &= \frac{1}{4}\pi \\ \diamond 1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \frac{1}{25} - \dots &= \frac{1}{12}\pi^2 \\ \diamond 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \dots &= \ln 2 \end{aligned}$$

Assuming the following integral converges find its exact value.

$$\int_0^1 (\ln x)(\arctan x) dx.$$

[you may assume that integration and summation commute]

$$\boxed{}, \quad \frac{1}{48} [\pi^2 - 12\pi + 24\ln 2]$$

IT IS CRUCIAL THAT THE INTEGRAL HAS A CLOSED FORM IN TERMS OF ELEMENTARY FUNCTIONS IN INDEFINITE FORM... USE SERIES EXPANSION

$$\frac{d}{dx}(\arctan x) = \frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + x^8 - \dots$$

INTEGRATING WITH RESPECT TO x

$$\arctan x = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \frac{1}{9}x^9 - \dots + C \quad x=0, C=0$$

NOW RETURNING TO THE INTEGRAL & SUMMATION AND EVALUATION

$$\int_0^1 (\arctan x)(\ln x) dx = \int_0^1 \left(x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \frac{1}{9}x^9 - \dots \right) \ln x dx$$

$$= \sum_{n=0}^{\infty} \left[\frac{(-1)^n}{(2n+1)} \int_0^1 x^{2n+1} \ln x dx \right]$$

ANALYSING BY PARTS USING THE FORM

$u = \ln x$	$\frac{1}{x}$
$\frac{du}{dx} = \frac{1}{x}$	dx

$$\int_0^1 x^{2n+1} \ln x dx = \left[\frac{1}{2n+2} x^{2n+2} \ln x - \int \frac{1}{2n+2} x^{2n+1} dx \right]_0^1$$

$$= \left[\frac{1}{2n+2} x^{2n+2} \ln x - \frac{1}{2n+2} x^{2n+2} \right]_0^1$$

$$= \left[\frac{1}{2n+2} (1 \ln 1 - 1) - \left(\lim_{x \rightarrow 0} \frac{1}{2n+2} x^{2n+2} \ln x - \frac{1}{2n+2} x^{2n+2} \right) \right]$$

$$= \left[\frac{1}{2n+2} (-1) - \left(0 - \frac{1}{2n+2} \right) \right] = -\frac{1}{(2n+2)^2}$$

SUMMATION SERIES

$$\int_0^1 (\arctan x)(\ln x) dx = \sum_{n=0}^{\infty} \left[\frac{(-1)^n}{(2n+1)} \left(-\frac{1}{(2n+2)^2} \right) \right] = -\frac{1}{4} \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)^3}$$

EXPANDED PARTIAL FRACTIONS

$$\frac{1}{(n+1)^3} = \frac{A}{n+1} + \frac{B}{(n+1)^2} + \frac{C}{(n+1)^3}$$

IF $n=0$ $\frac{1}{1} = \frac{A}{1} + \frac{B}{1} + \frac{C}{1} \Rightarrow 1 = A + B + C$
 IF $n=1$ $\frac{1}{8} = \frac{A}{2} + \frac{B}{4} + \frac{C}{8} \Rightarrow 1 = 4A + 2B + C$
 IF $n=2$ $\frac{1}{27} = \frac{A}{3} + \frac{B}{9} + \frac{C}{27} \Rightarrow 1 = 3A + B + C$

THIS WE CAN USE

$$\int_0^1 (\arctan x)(\ln x) dx = -\frac{1}{4} \sum_{n=0}^{\infty} \left[\frac{(-1)^n}{(n+1)^3} \right]$$

$$= -\frac{1}{4} \left[\sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)^3} \right]$$

LOOKING AT THE SERIES FORM

$$1 - \frac{1}{8} + \frac{1}{27} - \frac{1}{64} + \dots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)^3} = \frac{7}{8}$$

$$1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \dots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)^2} = \frac{\pi^2}{6}$$

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)} = \ln 2$$

FINALLY USE THIS

$$\int_0^1 (\arctan x)(\ln x) dx = -\frac{1}{4} \left(\frac{7}{8} + \frac{1}{2} \ln 2 - \frac{1}{12} \pi^2 \right)$$

$$= -\frac{7}{32} - \frac{1}{8} \ln 2 + \frac{1}{48} \pi^2$$

$$= \frac{1}{48} (\pi^2 - 12\ln 2 + 24 \times \frac{7}{8})$$

Question 133 (****)

Find the value of

$$\frac{1}{\pi} \int_{-\pi}^{\pi} \frac{\sin \frac{9}{2}x}{\sin \frac{1}{2}x} dx.$$

You may assume that the integrand is continuous at $x = 0$.
, 2

START BY NOTING THAT THE INTEGRAND IS EVEN

$$\frac{1}{\pi} \int_{-\pi}^{\pi} \frac{\sin \frac{9}{2}x}{\sin \frac{1}{2}x} dx = \frac{2}{\pi} \int_0^{\pi} \frac{\sin \frac{9}{2}x}{\sin \frac{1}{2}x} dx$$

NOW LET I BE THE ABOVE INTEGRAL AND PROCEED BY A SUBSTITUTION

$\theta = \pi - x$
 $d\theta = -dx$
 $0 \rightarrow \pi$
 $\pi \rightarrow 0$

$$\Rightarrow I = \frac{2}{\pi} \int_{\pi}^0 \frac{\sin(\frac{3}{2}(\pi - \theta))}{\sin(\frac{1}{2}(\pi - \theta))} (-d\theta)$$

$$\Rightarrow I = \frac{2}{\pi} \int_0^{\pi} \frac{\sin(\frac{3}{2}\pi - \frac{3}{2}\theta)}{\sin(\frac{\pi}{2} - \frac{\theta}{2})} d\theta$$

TABLE OF TWO
REDUCES OF 2π

$$\Rightarrow I = \frac{2}{\pi} \int_0^{\pi} \frac{\sin(\frac{3}{2}\pi - \frac{3}{2}\theta)}{\sin(\frac{\pi}{2} - \frac{\theta}{2})} d\theta$$

$$\Rightarrow I = \frac{2}{\pi} \int_0^{\pi} \frac{\cos \frac{3}{2}\theta}{\cos \frac{\theta}{2}} d\theta = \frac{2}{\pi} \int_0^{\pi} \frac{\cos \frac{3}{2}\theta}{\cos \frac{\theta}{2}} d\theta = I$$

RENDERING THE ABOVE EQUATION AS FOLLOWS

$$\Rightarrow I + I = \frac{2}{\pi} \int_0^{\pi} \frac{\sin \frac{3}{2}\theta}{\sin \frac{\theta}{2}} d\theta + \frac{2}{\pi} \int_0^{\pi} \frac{\cos \frac{3}{2}\theta}{\cos \frac{\theta}{2}} d\theta$$

$$\Rightarrow \cancel{I} = \frac{2}{\pi} \int_0^{\pi} \frac{\sin \frac{3}{2}\theta}{\sin \frac{\theta}{2}} + \frac{\cos \frac{3}{2}\theta}{\cos \frac{\theta}{2}} d\theta$$

ADDING THE ITEM IN THE INTEGRAND, USING THE COMPOUND ANGLE IDENTITY APPROACH

$$\Rightarrow I = \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{\sin \frac{9}{2}x \cos \frac{1}{2}x + \cos \frac{1}{2}x \sin \frac{9}{2}x}{\sin \frac{1}{2}x \cos \frac{1}{2}x} dx$$

$$\Rightarrow I = \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{\sin(\frac{9}{2}x + \frac{1}{2}x)}{\frac{1}{2} \sin(2 \cdot \frac{1}{2}x)} dx$$

$$\Rightarrow I = \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{\sin 5x}{\frac{1}{2} \sin x} dx$$

$$\Rightarrow I = \frac{2}{\pi} \int_0^{\pi} \frac{\sin 5x}{\sin x} dx$$

NEXT BY COMPLEX NUMBERS FOR A REDUCTION FORMULA

$$\Rightarrow \cos \theta + i \sin \theta = C + iS$$

$$\Rightarrow (\cos \theta + i \sin \theta)^5 = (C + iS)^5$$

$$\Rightarrow \cos 5\theta + i \sin 5\theta = C^5 + 5C^4 iS - 10C^3 S^2 - 10iC^2 S^3 + 5C S^4 + iS^5$$

$$\Rightarrow \sin 5\theta = 5C^4 S - 10C^2 S^3 + S^5$$

$$= 5S(1 - S^2)^2 - 10S^2 C(1 - S^2) + S^5$$

$$= 5S(1 - 2S^2 + S^4) - 10S^2 + 10S^2 + S^5$$

$$= 5S - 10S^3 + 5S^5 - 10S^2 + 10S^2 + S^5$$

$$= 16S^5 - 20S^3 + 5S$$

$$= 16 \sin^5 \theta - 20 \sin^3 \theta + 5 \sin \theta$$

RETURNING TO THE INTEGRAL WE OBTAIN

$$I = \frac{2}{\pi} \int_0^{\pi} \frac{16 \sin^5 x - 20 \sin^3 x + 5 \sin x}{\sin x} dx$$

$$I = \frac{2}{\pi} \int_0^{\pi} 16 \sin^4 x - 20 \sin^2 x + 5 dx$$

$$I = \frac{2}{\pi} \int_0^{\pi} 16 \left(\frac{1}{2} - \frac{1}{2} \cos 2x\right)^2 - 20 \left(\frac{1}{2} - \frac{1}{2} \cos 2x\right) + 5 dx$$

$$I = \frac{2}{\pi} \int_0^{\pi} 4 - 8 \cos 2x + 4 \cos^2 2x - 10 + 10 \cos 2x + 5 dx$$

$$I = \frac{2}{\pi} \int_0^{\pi} -1 + 2 \cos 2x + 4 \left(\frac{1}{2} + \frac{\cos 4x}{2}\right) dx$$

$$I = \frac{2}{\pi} \int_0^{\pi} 1 + 2 \cos 2x + 2 \cos 4x dx$$

NO COMBINATION SOME THESE TERMS

$$I = \frac{2}{\pi} \int_0^{\pi} 1 dx$$

$$I = \frac{2}{\pi} \times \pi$$

$$I = 2$$

Question 134 (*****)

Use appropriate integration techniques to find an exact simplified value for the following improper integral.

$$\int_0^{\infty} \frac{x^3}{e^x - 1} dx.$$

You may assume without proof that

$$\sum_{r=1}^{\infty} \frac{1}{r^4} = \frac{\pi^4}{90}.$$

$$\boxed{}, \quad \frac{\pi^4}{15}$$

SOLVE BY MULTIPLYING TOP & BOTTOM OF THE INTEGRAND BY e^{-x}

$$\int_0^{\infty} \frac{x^3}{e^x - 1} dx = \int_0^{\infty} \frac{x^3 e^{-x}}{1 - e^{-x}} dx = \int_0^{\infty} x^3 e^{-x} \left(\frac{1}{1 - e^{-x}} \right) dx$$

FOR $|e^{-x}| < 1$ (OR SIMPLY TRUE $0 < e^{-x} < 1$), WE CAN EXPAND BINOMIALLY
OR SIMPLY USE THE SUM TO INFINITY FORMULA FOR A GEOMETRIC SERIES

$$\text{I.E. } \frac{1}{1 - e^{-x}} = 1 + e^{-x} + e^{-2x} + e^{-3x} + \dots$$

THIS WE KNOW THAT

$$\begin{aligned} \dots &= \int_0^{\infty} x^3 e^{-x} [1 + e^{-x} + e^{-2x} + e^{-3x} + \dots] dx \\ &= \int_0^{\infty} x^3 [e^{-x} + e^{-2x} + e^{-3x} + e^{-4x} + \dots] dx \\ &= \int_0^{\infty} x^3 \left[\sum_{k=1}^{\infty} e^{-kx} \right] dx \end{aligned}$$

INTERCHANGING SUMMATION & INTEGRATION

$$= \sum_{k=1}^{\infty} \left[\int_0^{\infty} x^3 e^{-kx} dx \right]$$

PROCEED TO EVALUATE THE INTEGRAL BY PARTS

$$\begin{aligned} \int_0^{\infty} x^3 e^{-kx} dx &= \left[-\frac{1}{k} x^3 e^{-kx} \right]_0^{\infty} + \int_0^{\infty} \frac{3}{k} x^2 e^{-kx} dx \\ &= \frac{3}{k} \int_0^{\infty} x^2 e^{-kx} dx \\ &= \frac{3}{k} \left\{ \left[-\frac{1}{k} x^2 e^{-kx} \right]_0^{\infty} + \int_0^{\infty} \frac{2}{k} x e^{-kx} dx \right\} \end{aligned}$$

$$\begin{aligned} &= \frac{6}{k^3} \int_0^{\infty} x e^{-kx} dx \\ &= \frac{6}{k^3} \left\{ \left[-\frac{1}{k} x e^{-kx} \right]_0^{\infty} + \int_0^{\infty} \frac{1}{k} e^{-kx} dx \right\} \\ &= \frac{6}{k^3} \left\{ 0 + \left[-\frac{1}{k^2} e^{-kx} \right]_0^{\infty} \right\} = \frac{6}{k^3} \left[0 + \frac{1}{k^2} \right] \\ &= \frac{6}{k^5} \end{aligned}$$

RETURNING TO THE MAIN LINE OF THE PROBLEM

$$\begin{aligned} \int_0^{\infty} \frac{x^3}{e^x - 1} dx &= \dots = \sum_{k=1}^{\infty} \left[\int_0^{\infty} x^3 e^{-kx} dx \right] = \dots = \sum_{k=1}^{\infty} \frac{6}{k^5} \\ &= 6 \sum_{k=1}^{\infty} \frac{1}{k^5} = 6 \times \frac{\pi^5}{315} = \frac{\pi^5}{15} \end{aligned}$$

ALTERNATIVE APPROACHES FOR $\int_0^{\infty} x^3 e^{-kx} dx$

- $\int_0^{\infty} x^3 e^{-kx} dx = \dots = \frac{3}{k^4} \int_0^{\infty} t^3 e^{-kt} dt = \frac{3}{k^4} \int_0^{\infty} t^3 dt$
 $= \frac{3}{k^4} \cdot \frac{1}{4} = \frac{3}{4k^4}$ AS ABOVE
- $\int_0^{\infty} x^3 e^{-kx} dx = \dots$ BY SUBSTITUTION \dots
 $= \int_0^{\infty} \left(\frac{t}{k} \right)^3 e^{-t} \left(\frac{1}{k} dt \right)$
 $= \frac{1}{k^4} \int_0^{\infty} t^3 e^{-t} dt$

$t = kx$
 $x = \frac{1}{k} t$
 $dx = \frac{1}{k} dt$
AND INFINITELY

$$\begin{aligned} &= \frac{1}{k^4} \Gamma(4) \\ &= \frac{1}{k^4} \times 3! \\ &= \frac{6}{k^4} \text{ AS ABOVE} \end{aligned}$$

- $\int_0^{\infty} x^3 e^{-kx} dx = -\frac{\partial^3}{\partial k^3} \left[\int_0^{\infty} e^{-kx} dx \right]$
 $= -\frac{\partial^3}{\partial k^3} \left[\left[-\frac{1}{k} e^{-kx} \right]_0^{\infty} \right]$
 $= -\frac{\partial^3}{\partial k^3} \left[0 + \frac{1}{k} \right]$
 $= -\frac{\partial^3}{\partial k^3} \left(\frac{1}{k} \right)$
 $= -\frac{\partial^2}{\partial k^2} \left(-\frac{1}{k^2} \right)$
 $= -\frac{\partial}{\partial k} \left(+\frac{2}{k^3} \right)$
 $= - \left(-\frac{6}{k^4} \right)$
 $= \frac{6}{k^4}$
 AS ABOVE

Question 135 (****)

The function f is defined as

$$f(x) = \arctan\left(\frac{1}{2x^2}\right), \quad x \in (-\infty, \infty).$$

- a)** Find a simplified expression for $f'(x)$.

- b)** Show that $\lim_{x \rightarrow \pm\infty} [x f(x)] = 0$.

- c) Determine the value of $\lim_{x \rightarrow \pm\infty} \ln \left[\frac{2x^2 - 2x + 1}{2x^2 + 2x + 1} \right]$.

- d)** Hence find the value of $\int_{-\infty}^{\infty} f(x) \, dx$.

$$\square, \quad \boxed{f'(x) = -\frac{4x}{4x^4 + 1}}, \quad \boxed{\lim_{x \rightarrow \pm\infty} \left[\frac{2x^2 - 2x + 1}{2x^2 + 2x + 1} \right] = 0}, \quad \boxed{\int_0^\infty f(x) \, dx = \frac{1}{2}\pi}$$

4) DIFFERENTIATE AND FIND

$$\frac{d}{dx} \left[\arctan \left(\frac{1}{2x} \right) \right] = \frac{1}{1 + \frac{1}{4x^2}} \times \frac{d}{dx} \left(\frac{1}{2x} \right) = \frac{1}{1 + \frac{1}{4x^2}} \times -\frac{1}{2x^2}$$
$$= \frac{\frac{4x^2}{4x^2 + 1}}{2x^2} = -\frac{2x}{4x^2 + 1}$$

b) NEXT THE FIRST LIMIT

$$\lim_{x \rightarrow -\infty} \left[2 \arctan \left(\frac{1}{2x} \right) \right] = \lim_{x \rightarrow -\infty} \left[2 \arctan \left(\frac{1}{2x} \right) \right]$$

THIS IS AN INDETERMINATE FORM $(\infty)(0)$ - NEEDS TO USE L'HOSPITAL

$$= \lim_{x \rightarrow -\infty} \left[\frac{\arctan \left(\frac{1}{2x} \right)}{\frac{1}{2}} \right] \quad \leftarrow \text{NOW NEED TO FIND TYPE OF INDETERMINATE}$$
$$= \lim_{x \rightarrow -\infty} \left[\frac{-\frac{\frac{1}{2x^2}}{1 + \frac{1}{4x^2}}}{-\frac{1}{2x^2}} \right] \quad \leftarrow \text{FIND } f'(x) = \lim_{x \rightarrow -\infty} \left[\frac{\frac{4x^2}{4x^2 + 1}}{\frac{1}{x^2}} \right]$$
$$= \lim_{x \rightarrow -\infty} \left[\frac{4}{1 + \frac{1}{4x^2}} \right] = \frac{4}{1} = 4$$

c) THE NEXT LIMIT IS EASY ENOUGH

$$\lim_{x \rightarrow \pm\infty} \left[\ln \left(\frac{2 - \frac{1}{2x} + \frac{1}{4x^2}}{2 + \frac{1}{2x} + \frac{1}{4x^2}} \right) \right] = \lim_{x \rightarrow \pm\infty} \left[\ln \left(\frac{2 - \frac{1}{2x} + \frac{1}{4x^2}}{2 + \frac{1}{2x} + \frac{1}{4x^2}} \right) \right]$$
$$= \ln \left[\frac{2}{2} \right]$$
$$= \ln 1$$
$$= 0$$

d) PROCESSED BY INTEGRATION BY PARTS

$$\int_0^{\infty} \arctan\left(\frac{1}{x}\right) dx$$

NOT THAT AT $x=0$, $\frac{1}{x} \rightarrow \infty$
(ON A SINGLE SIDE)

$$= \int_0^{\infty} \cancel{\arctan\left(\frac{1}{x}\right)} \cdot \frac{1}{x^2} dx$$

NOT b

$$= \int_0^{\infty} \frac{1}{x^2} dx$$

NO! BY THE SAME GRADIENT IDENTITY

$$x^2 + \frac{1}{x^2} \equiv (x^2 + 2x + 1) + (x^2 + 2/x + 1/x^2) - 2/x$$

OR BY COMPUTING THE SQUARE

$$x^2 + 1 = (x^2 + 2x + 1) - 2x = (x^2 + 1)^2 - (2x)^2$$

$$= (x^2 + 1 + 2x)(x^2 + 1 - 2x)$$

WE'RE LOOKING AT THE LOGARITHMIC UNIT OF PART (c) IS SUFFICIENT

THAT $(x^2 - 2x + 1)(x^2 + 2x + 1) = x^4 + 1$

$$\dots = \int_0^{\infty} \frac{1}{x^2} dx = \int_0^{\infty} \frac{1}{(x^2 - 2x + 1)(x^2 + 2x + 1)} dx$$

BY THE RESIDUE CALC

[illegible]

$$\begin{aligned} & \text{MANIPULATE INTO LOSS OF A CONSTANT TO EXISTENCE} \\ & = \int_0^{\infty} \frac{(x^2 - \frac{1}{2}x + 2)}{2x^2 + 2x + 1} dx = \frac{1}{2} \int_0^{\infty} \frac{(2x^2 - x + 2)}{2x^2 + 2x + 1} dx \\ & = \frac{1}{2} \int_0^{\infty} \frac{(2x^2 + 2x + 1) - 3x + 1}{2x^2 + 2x + 1} dx = \frac{1}{2} \int_0^{\infty} \frac{(2x^2 + 2x + 1)}{2x^2 + 2x + 1} dx \\ & \quad + \frac{1}{2} \int_0^{\infty} \frac{(-3x + 1)}{2x^2 + 2x + 1} dx \\ & = \frac{1}{2} \left[\ln(2x^2 + 2x + 1) - \frac{1}{2} \ln(2x^2 + 2x + 1) \right]_0^{\infty} + \frac{1}{2} \int_0^{\infty} \frac{(-3x + 1)}{2x^2 + 2x + 1} dx \\ & \quad + \frac{1}{2} \int_0^{\infty} \frac{1}{2x^2 + 2x + 1} dx \\ & = \left[\frac{1}{2} \ln \frac{2x^2 + 2x + 1}{2x^2 + 2x + 1} \right]_0^{\infty} + \left[\frac{1}{2} \ln(2x^2 + 2x + 1) + \frac{1}{2} \int_0^{\infty} \frac{1}{2x^2 + 2x + 1} dx \right] \\ & \quad \text{DNE C} \\ & \text{THESE ARE STANDARDISED ACTING) NUMERICALS} \quad \frac{d}{dx} (\ln(x^2 + 2x + 1)) = \frac{2}{2x^2 + 2x + 1} \\ & = \left[\frac{1}{2} \ln(2x^2 + 2x + 1) + \frac{1}{2} \ln(2x^2 + 2x + 1) \right]_{-\infty}^{\infty} \\ & = \frac{1}{2} \lim_{x \rightarrow \infty} \left[\ln(2x^2 + 2x + 1) + \ln(2x^2 + 2x + 1) - \ln(2x^2 + 2x + 1) - \ln(2x^2 + 2x + 1) \right] \\ & = \frac{1}{2} \left[\frac{1}{2} \ln \frac{1}{2} + \frac{1}{2} \ln \frac{1}{2} + \frac{1}{2} \ln \frac{1}{2} \right] \\ & = \frac{\pi}{2} \end{aligned}$$

Question 136 (*****)

A family of functions $f_n(x)$, where $n = 0, 1, 2, 3, 4, \dots$, satisfies the equation

$$\sum_{n=0}^{\infty} [t^n f_n(x)] = (1 - 2xt + t^2)^{-\frac{1}{2}}.$$

By integrating both sides of the above equation with respect to t , from 0 to 1, show that

$$\sum_{n=0}^{\infty} \left[\frac{f_n(\cos \theta)}{n+1} \right] = \ln \left[1 + \operatorname{cosec} \left(\frac{1}{2} \theta \right) \right].$$

You may assume in this question that integration and summation commute.

 , proof

The handwritten solution is divided into two columns. The left column shows the initial steps: defining $f_n(x) = (1 - 2xt + t^2)^{-\frac{1}{2}}$, substituting $x = \cos \theta$, and integrating both sides of the equation with respect to t from 0 to 1. It uses the identity $1 - 2\cos \theta t + t^2 = (t - \cos \theta)^2 + \sin^2 \theta$ and the substitution $u = t - \cos \theta$ to simplify the integral. The right column continues the derivation, showing the integral of $\frac{1}{u^2 + \sin^2 \theta}$ and the final result $\ln \left[1 + \operatorname{cosec} \left(\frac{1}{2} \theta \right) \right]$.