COMBINATORICS

COMBINATIONS

Question 1 (**)

The Oakwood Jogging Club consists of 7 men and 6 women who go for a 5 mile run every Thursday.

It is decided that a team of 8 runners would be picked at random out of the 13 runners, to represent the club at a larger meeting.

Determine the proportion of teams of 8, which have more women than men.

$\boxed{}$, $\boxed{\frac{7}{39} \approx 17.95\%}$

6 WOMEN + 7 MEN = TOTAL 13 RUNNERS

TOTAL NUMBER OF TEAMS OF 8, OUT OF 13, REGARDLESS OF GENDER

$$\binom{13}{8} = \frac{13!}{5!\,8!} = 1287$$

NEXT THE TEAMS OF 8, WITH MORE WOMEN

- 6 WOMEN (OUT OF 6) + 2 MEN (OUT OF 7): $\binom{6}{6} \times \binom{7}{2} = 1 \times 21 = 21$
- 5 WOMEN (OUT OF 6) + 3 MEN (OUT OF 7): $\binom{6}{5} \times \binom{7}{3} = 6 \times 35 = 210$

$$\underline{\underline{231}}$$

HENCE A PROPORTION OF $\frac{231}{1287} = \frac{7}{39} = 17.95\%$

Question 2 (**)

A football manager has available for selection 3 goalkeepers, 8 defenders, 7 midfielders and 4 strikers.

a) Determine the number of possible teams of 11 he can select, assuming that all 22 players are equally likely to be picked up, and equally likely to play in any position.

b) Find the number of possible teams he can pick with 1 goalkeeper, 4 defenders, 4 midfielders and 2 strikers.

$\boxed{\quad}$, $\boxed{44100}$, $\boxed{\frac{525}{8398} \approx 0.0625}$

a) TEAMS OF 11 OUT OF 22

$$\binom{22}{11} = \frac{22!}{11!\,11!} = \underline{705432}$$

b) GOALKEEPERS (OUT OF 3) DEFENDERS (OUT OF 8) MIDFIELDERS (OUT OF 7) STRIKERS (OUT OF 4)

$$= \binom{3}{1} \times \binom{8}{4} \times \binom{7}{4} \times \binom{4}{2}$$

$$= 3 \times 70 \times 35 \times 6$$

$$= \underline{44100}$$

Question 3 (**)

A taxi which can carry at most 5 passengers on any journey, makes two journeys in transporting 8 passengers from their hotel to the airport.

Determine the number of different ways in which the people for the first journey may be selected.

$\boxed{\text{LP4-D}}$, $\boxed{182}$

FIRST JOURNEY	(SECOND JOURNEY)
5	(3)
4	(4)
3	(5)

$\rightarrow \binom{8}{5} = \frac{8!}{5!3!} = 56$

$\rightarrow \binom{8}{4} = \frac{8!}{4!4!} = 70$

$\rightarrow \binom{8}{3} = \frac{8!}{3!5!} = 56$

ADDING 182

Question 4 (+)**

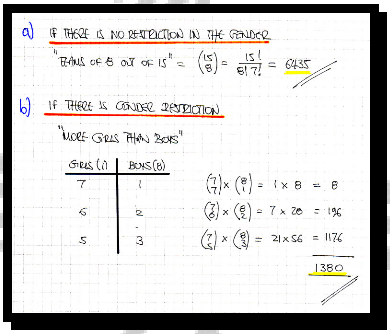
There are 8 boys and 7 girls in the student council of a school.

A committee of 8 people is to be selected from the members of this council to organize a sports day.

a) Find the number of different ways in which the committee can be selected if all the members are available.

b) Determine the number of different ways in which the committee can be selected if the committee is to have more girls than boys.

$\boxed{\text{FP4}}$, $\boxed{6435}$, $\boxed{1380}$



a) IF THERE IS NO RESTRICTION IN THE GENDER

"TEAMS OF 8 OUT OF 15" $= \binom{15}{8} = \frac{15!}{8!\,7!} = 6435$

b) IF THERE IS GENDER RESTRICTION

"MORE GIRLS THAN BOYS"

GIRLS (7)	BOYS (8)
7	1
6	2
5	3

$\binom{7}{7} \times \binom{8}{1} = 1 \times 8 = 8$

$\binom{7}{6} \times \binom{8}{2} = 7 \times 28 = 196$

$\binom{7}{5} \times \binom{8}{3} = 21 \times 56 = 1176$

1380

Question 5 (**+)

A **five** member committee is to be selected at random from a group consisting of 8 men and 4 women.

Find the number of possible committees which contain …

a) … exactly 2 women.

b) … no more than 2 women.

$\boxed{}$, $\boxed{336}$, $\boxed{672}$

8 MEN & 4 WOMEN / "TEAM" OF 5

a) TWO WOMEN & THREE MEN

$$\binom{4}{2} \times \binom{8}{3} = 6 \times 56 = 336$$

b) "NO MORE THAN 2 WOMEN"

- NO WOMAN & 5 MEN = $\binom{4}{0} \times \binom{8}{5} = 1 \times 56 = 56$
- 1 WOMAN & 4 MEN = $\binom{4}{1} \times \binom{8}{4} = 4 \times 70 = 280$
- 2 WOMEN & 3 MEN = $= 336$

672

Question 6 (+)**

A committee of 4 people is to be chosen at random from a group of 5 men and 7 women.

Determine the probability that the committee will consist …

a) … of members of the same gender.

b) … of members of both genders but at least as many women as men.

$\boxed{\frac{8}{99}}, \boxed{\frac{7}{9}}$

(a) TOTAL COMMITTEES $= \binom{12}{4} = 495$

ONLY MEN $= \binom{5}{4} = 5$

ONLY WOMEN $= \binom{7}{4} = 35$

$\Big\}$ TOTAL 40

REQUIRED PROBABILITY $= \frac{40}{495} = \frac{8}{99}$

(b) WWMM $= \binom{7}{2} \times \binom{5}{2} = 21 \times 10 = 210$

WWWM $= \binom{7}{3} \times \binom{5}{1} = 35 \times 5 = 175$

385

REQUIRED PROBABILITY $= \frac{385}{495} = \frac{7}{9}$

Question 7 (*+)**

A committee of 4 people is to be chosen at random from the members of a school council which consists of 5 pupils, 4 teachers and 3 administrators.

Determine the probability that the committee will contain ...

a) … no teachers.

b) … at least 2 pupils, no more than 1 teacher and no more than 1 administrator.

$\boxed{\frac{14}{99}}$, $\boxed{\frac{13}{33}}$

(a) TOTAL COMMITTEES $= \binom{12}{4} = 495$

COMMITTEES WITHOUT TEACHERS $= \binom{8}{4} = 70$

REQUIRED PROBABILITY $= \frac{70}{495} = \frac{14}{99}$

(b)

PUPILS	TEACHERS	ADMINISTRATORS	
4	0	0	$\leftarrow \binom{5}{4}\times\binom{4}{0}\times\binom{3}{0} = 5$
3	1	0	$\leftarrow \binom{5}{3}\times\binom{4}{1}\times\binom{3}{0} = 40$
3	0	1	$\leftarrow \binom{5}{3}\times\binom{4}{0}\times\binom{3}{1} = 30$
2	1	1	$\leftarrow \binom{5}{2}\times\binom{4}{1}\times\binom{3}{1} = 120$

$\left.\right\} 195$

$\therefore$ REQUIRED PROBABILITY $= \frac{195}{495} = \frac{13}{33}$

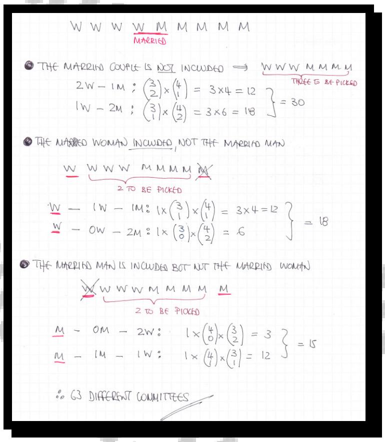
Question 8 **(***+)**

A committee of 3 people is to be picked from 9 individuals, of which 4 are women and 5 are men. One of the 4 women is married to one of the 5 men.

The selection rules state that the committee must have at least a member from each gender and no married couple can serve together in a committee.

Determine the number of possible committees which can be picked from these 9 individuals.

$\boxed{63}$



W W W $\underbrace{\text{W M}}_{\text{MARRIED}}$ M M M M

- THE MARRIED COUPLE IS NOT INCLUDED $\Rightarrow$ $\underbrace{\text{W W W M M M M}}_{\text{THREE TO BE PICKED}}$

$$\left.\begin{array}{l} 2W - 1M : \binom{3}{2}\times\binom{4}{1} = 3\times4 = 12 \\ 1W - 2M : \binom{3}{1}\times\binom{4}{2} = 3\times6 = 18 \end{array}\right\} = 30$$

- THE MARRIED WOMAN INCLUDED, NOT THE MARRIED MAN

W $\underbrace{\text{W W W M M M M}}_{\text{2 TO BE PICKED}}$ $\cancel{\text{M}}$

$$\left.\begin{array}{l} W - 1W - 1M : 1\times\binom{3}{1}\times\binom{4}{1} = 3\times4 = 12 \\ W - 0W - 2M : 1\times\binom{3}{0}\times\binom{4}{2} = 6 \end{array}\right\} = 18$$

- THE MARRIED MAN IS INCLUDED BUT NOT THE MARRIED WOMAN

$\cancel{\text{W}}$ $\underbrace{\text{W W W M M M M}}_{\text{2 TO BE PICKED}}$ M

$$\left.\begin{array}{l} M - 0M - 2W : 1\times\binom{4}{0}\times\binom{3}{2} = 3 \\ M - 1M - 1W : 1\times\binom{4}{1}\times\binom{3}{1} = 12 \end{array}\right\} = 15$$

$\therefore$ 63 DIFFERENT COMMITTEES

Question 9 (****)

From a total of 6 men, 3 women and 3 children, two teams of six people are selected at random.

Find the probability that both teams contain women.

$$\boxed{\frac{10}{11}}$$

6 MEN, 3 WOMEN, 3 CHILDREN

2 TEAMS OF 6

3 WOMEN, 9 NON WOMEN

START SELECTING THE FIRST TEAM OF 6

$$P(W'W'W'W'W'W') = \frac{9}{12} \times \frac{8}{11} \times \frac{7}{10} \times \frac{6}{9} \times \frac{5}{8} \times \frac{4}{7} = \frac{1}{11}$$

P(FIRST TEAM HAS NO WOMEN) $= \frac{1}{11}$

P(ONE OF THE TEAMS HAS 3 WOMEN) $= \frac{1}{11}$

P(WOMEN ON BOTH TEAMS) $= \frac{10}{11}$

PERMUTATIONS

Question 1 (+)**

The five letters of the word T-E-A-C-H are written on five separate pieces of card.

a) Find the number of arrangements that can be made using these five letters.

Find the proportion of five letter arrangements in which …

i. … the first letter is T.

ii. … the letters C and H are next to each other.

iii. … the first letter is T **and** the letters C and H are next to each other.

$\boxed{}$, $\boxed{120}$, $\boxed{\dfrac{1}{120}}$, $\boxed{\dfrac{1}{5}}$, $\boxed{\dfrac{2}{5}}$, $\boxed{\dfrac{1}{10}}$

a) PERMUTATION OF 5 OUT OF 5

$^5P_5 = 5! = 120$

b) I) FIXING THE "T" AT THE FRONT, LEAVES 4 LETTERS TO BE PERMUTED FOR THE REMAINING 4 SPACES

$\underline{T}\ \underset{①}{\underline{E}}\ \underset{②}{\underline{A}}\ \underset{③}{\underline{C}}\ \underset{④}{\underline{H}} \Rightarrow {}^4P_4 = 4! = 24$

$\Rightarrow$ PROPORTION $= \frac{24}{120} = \frac{1}{5}$

II) TREATING C & H AS ONE LETTER & NOTE THAT THIS CAN OCCUR TWICE (CH) OR (HC)

$\underset{①}{\underline{CH}}\ \underset{②}{\underline{T}}\ \underset{③}{\underline{E}}\ \underset{④}{\underline{A}}$ (HC) $\Rightarrow {}^4P_4 \times 2\text{ WAYS} = 24 \times 2 = 48$

$\Rightarrow$ PROPORTION $= \frac{48}{120} = \frac{2}{5}$

III) COMBINING THE IDEAS FROM (I) & (II)

$\underline{T}\ \underset{①}{\underline{CH}}\ \underset{②}{\underline{E}}\ \underset{③}{\underline{A}}$ (HC) $\Rightarrow {}^3P_3 \times 2\text{ WAYS} = 6 \times 2 = 12$

$\Rightarrow$ PROPORTION $= \frac{12}{120} = \frac{1}{10}$

Question 2 (***)

The eleven letters of the word E-X-A-M-I-N-A-T-I-O-N are written on eleven separate pieces of card.

a) Find the number of arrangements that can be made using these eleven letters.

b) Find the probability that the four letter word E-X-A-M will appear in one of these eleven letter arrangements

$\boxed{4989600}$, $\boxed{\frac{1}{990}}$

a) "EXAMINATION"

E X A M I N T O
A I N

3 DOUBLE REPEATS

$$\text{ARRANGEMENTS} = \frac{{}^{11}P_{11}}{2!\,2!\,2!} = \frac{11!}{8} = 4\,989\,600$$

b) TREAT "EXAM" AS ONE LETTER — SO 8 LETTERS NOW

(EXAM) A I N T O
I N

2 DOUBLE REPEATS NOW I & N
BUT SWAPPING THE 2 "As" MAKES
NO MORE ARRANGEMENTS, SO THE
REPEATS ARE STILL THREE.

$$\text{HENCE ARRANGEMENTS} = \frac{{}^{8}P_{8}}{2!\,2!\,2!} = \frac{8!}{8} = 7! = 5040$$

$$\text{REQUIRED PROBABILITY} = \frac{5040}{4\,989\,600} = \frac{1}{990}$$

Question 3 (***)

4 men and 4 women are going to stand next to each other for a group photograph.

Given that the way they stand next to each other is completely random, determine the number of photographs that can be taken in which no 2 men and no 2 women stand next to each other.

$\boxed{\text{E P4 B}}$, $\boxed{1152}$

$\boxed{\text{4 MEN \& 4 WOMEN, IE 8 IN TOTAL}}$

STARTING WITH THE TWO BASIC CONFIGURATIONS

M W M W M W M W

OR　　　　　　　I.E. $\times 2$ WAYS

W M W M W M W M

LOOKING AT SAY THE "TOP" OF THE TWO CONFIGURATIONS SHOWN ABOVE, WE HAVE "INTERLACED"

$\underbrace{\text{M M M M}}_{4!}$ & $\underbrace{\text{W W W W}}_{4!}$

HENCE THE REQUIRED NUMBER IS

$4! \times 4! \times 2 \text{ WAYS}$

$= 24 \times 24 \times 2$

$= 1152$

Question 4 (***+)

Six books labelled as A, B, C, D, E and F are arranged at random on a shelf.

Determine the number of arrangements in which ...

a) ... A and B are placed next to each other.

b) ... C and D are **not** placed next to each other.

c) ... A and B are placed next to each other, and C and D are **not** placed next to each other.

$\boxed{\text{P4}}$, $\boxed{240}$, $\boxed{480}$, $\boxed{144}$

a) TREATING A&B AS ONE ITEM, NOTING THIS IS "TWO WAYS" AS IT CAN BE BLOCKED AS AB OR BA

AB C D E F

$\therefore\ 5! \times 2 \text{ WAYS} = 240$

b) NOTHING SPECIAL ABOUT C & D COMPARED TO A & B

TOTAL ARRANGEMENTS: $6! = 720$

C & D NEXT TO EACH OTHER: 240 (PART a)

REQUIRED NUMBER IS $720 - 240 = 480$

c) PROCEED AS FOLLOWS

NUMBER OF ARRANGEMENTS WITH A & B NEXT TO EACH OTHER IS 240 (PART a)

NUMBER OF ARRANGEMENTS WITH A & B AND C & D NEXT TO EACH OTHER

$\underbrace{AB}_{\text{2 WAYS}}\ \underbrace{CD}_{\text{2 WAYS}}\ E\ F = 4! \times 2 \times 2 = 96$

REQUIRED NUMBER IS $240 - 96 = 144$

Question 5 (*+)**

A group of 7 pupils consists of 3 girls and 4 boys.

The names of two of the boys are Argi and Bargi.

All seven students sit in a random order on a bench.

a) Determine the number of sitting arrangements in which …

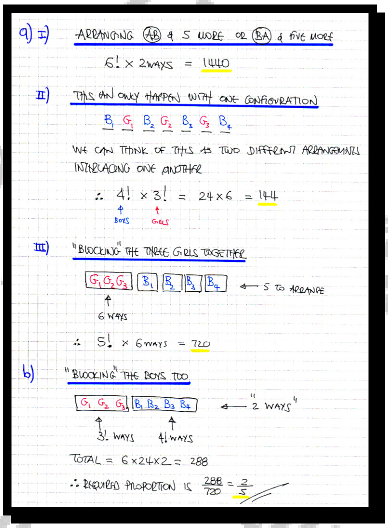
  i. … Argi and Bargi sit next to each other.

  ii. … no two boys sit next to each other.

  iii. … the three girls sit next to each other.

b) Find the proportion of the sitting arrangements in which the three girls sat next to each other which include arrangements in which the four boys **also** sat next to each other.

$\boxed{}$, $\boxed{1440}$, $\boxed{144}$, $\boxed{720}$, $\boxed{\frac{2}{5}}$


a) I) ARRANGING AB & 5 MORE OR BA & FIVE MORE

$6! \times 2\text{ WAYS} = 1440$

II) THIS CAN ONLY HAPPEN WITH ONE CONFIGURATION

$B_1\ G_1\ B_2\ G_2\ B_3\ G_3\ B_4$

WE CAN THINK OF THIS AS TWO DIFFERENT ARRANGEMENTS INTERLACING ONE ANOTHER

$\therefore \underbrace{4!}_{\text{BOYS}} \times \underbrace{3!}_{\text{GIRLS}} = 24 \times 6 = 144$

III) "BLOCKING" THE THREE GIRLS TOGETHER

$\boxed{G_1 G_2 G_3}\ \boxed{B_1}\ \boxed{B_2}\ \boxed{B_3}\ \boxed{B_4}$ ← 5 TO ARRANGE

6 WAYS

$\therefore 5! \times 6\text{ WAYS} = 720$

b) "BLOCKING" THE BOYS TOO

$\boxed{G_1\ G_2\ G_3}\ \boxed{B_1\ B_2\ B_3\ B_4}$ ← "2 WAYS"

3! WAYS   4! WAYS

TOTAL $= 6 \times 24 \times 2 = 288$

$\therefore$ REQUIRED PROPORTION IS $\frac{288}{720} = \frac{2}{5}$

Question 6 (*+)**

The 11 letters of the word *PROBABILITY* are written on 11 separate pieces of card. These cards are selected at random and arranged in a line next to each other.

a) Determine the probability that the two cards with the letter B will appear next to each other.

b) Find the probability that the two cards with the letter B will appear next to each other **and** the two cards with the letter I will appear next to each other.

c) Hence deduce the probability that the two cards with the letter B will **not** appear next to each other **and** the two cards with the letter I will **not** appear next to each other.

$\boxed{\frac{2}{11}}, \boxed{\frac{2}{55}}, \boxed{\frac{37}{55}}$

a) P R O B A B I L I T Y

TOTAL OF 11 LETTER WORDS $= \frac{11!}{2!2!} = 9979200$

Bs NEXT TO EACH OTHER (TREAT THEM AS ONE LETTER)

$\frac{10!}{2!} = 1814400$

"Is" REPEAT ONLY

REQUIRED PROBABILITY $= \frac{1814400}{9979200} = \frac{2}{11}$

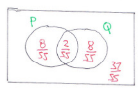
b) NOW BOTH "Is" & "Bs" NEXT TO EACH OTHER

$9!$ WAYS $= 362880$

REQUIRED PROBABILITY $= \frac{362880}{9979200} = \frac{2}{55}$

c) LET P BE THE EVENT THAT "Is" ARE NEXT TO EACH OTHER

LET Q BE THE EVENT THAT "Bs" ARE NEXT TO EACH OTHER



THUS THE REQUIRED PROBABILITY IS $\frac{37}{55}$

Question 7 (*+)**

$$S\ S\ S\ T\ T\ T\ C\ I\ I\ A$$

The 10 letters above, are written on 10 separate pieces of card. These cards are selected at random and arranged in a line next to each other.

a) Find the probability that the 10 letter arrangement will spell *STATISTICS* .

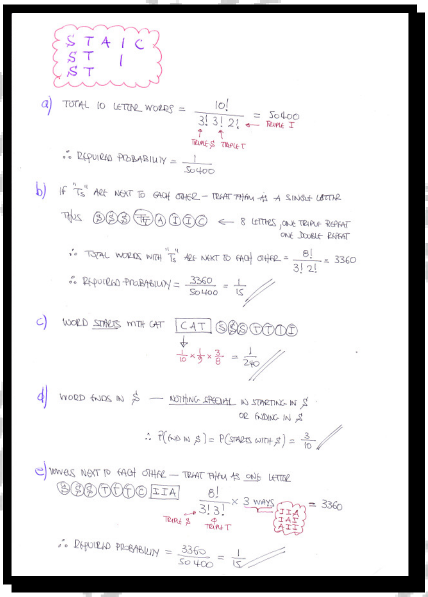
b) Determine the probability that in the 10 letter arrangement the 3 cards with the letter T will be next to one another.

c) Calculate the probability that the 10 letter arrangement will start with *CAT* , in that order.

d) Find the probability that the 10 letter arrangement will end with the letter S .

e) Determine the probability that in the 10 letter arrangement the 3 cards showing a vowel will be next to one another.

$\boxed{\frac{1}{50400}}$, $\boxed{\frac{1}{15}}$, $\boxed{\frac{1}{240}}$, $\boxed{\frac{3}{10}}$, $\boxed{\frac{1}{15}}$


S T A I C
S T I
S T

a) TOTAL 10 LETTER WORDS = $\frac{10!}{\underset{\text{TRIPLE S}}{3!}\,\underset{\text{TRIPLE T}}{3!}\,2!} = 50400$ ← TRIPLE I

∴ REQUIRED PROBABILITY = $\frac{1}{50400}$

b) IF "Ts" ARE NEXT TO EACH OTHER – TREAT THEM AS A SINGLE LETTER

THUS S S S TTT A I I C ← 8 LETTERS, ONE TRIPLE REPEAT, ONE DOUBLE REPEAT

∴ TOTAL WORDS WITH "Ts" ARE NEXT TO EACH OTHER = $\frac{8!}{3!\,2!} = 3360$

∴ REQUIRED PROBABILITY = $\frac{3360}{50400} = \frac{1}{15}$

c) WORD <u>STARTS</u> WITH CAT $\boxed{CAT}$ S S S T T I I

$\frac{1}{10} \times \frac{1}{9} \times \frac{3}{8} = \frac{1}{240}$

d) WORD ENDS IN S — <u>NOTHING SPECIAL</u> IN STARTING IN S OR ENDING IN S

∴ P(END IN S) = P(STARTS WITH S) = $\frac{3}{10}$

e) VOWELS NEXT TO EACH OTHER — TREAT THEM AS <u>ONE</u> LETTER

S S S T T T C $\boxed{IIA}$ $\frac{8!}{3!\,3!} \times \underbrace{3\text{ WAYS}}$ = 3360

TRIPLE S TRIPLE T

IIA
IAI
AII

∴ REQUIRED PROBABILITY = $\frac{3360}{50400} = \frac{1}{15}$

Question 8 (***+)

The 10 letters of the word

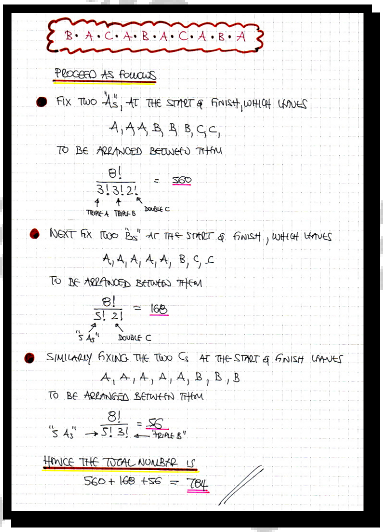
$$B \ A \ C \ A \ B \ A \ C \ A \ B \ A$$

are written on 10 separate pieces of card.

These cards are selected at random and arranged in a line next to each other.

Determine the number of arrangements which start and finish with the same letter.

$\boxed{}$, $\boxed{784}$



B · A · C · A · B · A · C · A · B · A

PROCEED AS FOLLOWS

- FIX TWO "A's" AT THE START & FINISH, WHICH LEAVES

A, A, A, B, B, B, C, C,

TO BE ARRANGED BETWEEN THEM

$$\frac{8!}{3!\,3!\,2!} = 560$$

TRIPLE A TRIPLE B DOUBLE C

- NEXT FIX TWO "B's" AT THE START & FINISH, WHICH LEAVES

A, A, A, A, A, B, C, C

TO BE ARRANGED BETWEEN THEM

$$\frac{8!}{5!\,2!} = 168$$

"5 A's" DOUBLE C

- SIMILARLY FIXING THE TWO Cs AT THE START & FINISH LEAVES

A, A, A, A, A, B, B, B

TO BE ARRANGED BETWEEN THEM

$$\text{"5 A's"} \rightarrow \frac{8!}{5!\,3!} = 56 \leftarrow \text{"TRIPLE B"}$$

HENCE THE TOTAL NUMBER IS

$$560 + 168 + 56 = 784$$

Question 9 (****)

Coloured pegs are to be placed in 4 holes which are drilled in a straight line, next to each other. These coloured pegs are identical in size and 2 of them are red, 2 of them are green, 2 of them are brown, 2 of them are orange, 2 of them are pink and 2 of them are blue.

6 pegs, one from each of the 6 colours, are picked from the 12 pegs and four are placed in the holes.

a) Determine the number of different arrangements which can be made.

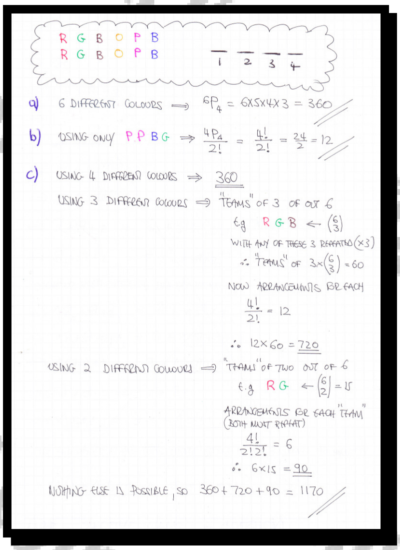
Next 4 pegs, 2 pink, 1 blue and 1 green are picked from the 12 pegs and are placed in the holes.

b) Find the number of different arrangements which can now be made.

Finally 4 pegs are picked at random from the total of 12 pegs and placed in the holes.

c) Determine the number of different arrangements which can be made on this occasion.

$\boxed{360}$, $\boxed{12}$, $\boxed{1170}$


R G B O P B
R G B O P B

1 2 3 4

a) 6 DIFFERENT COLOURS $\Rightarrow$ ${}^6P_4 = 6\times5\times4\times3 = 360$

b) USING ONLY P P B G $\Rightarrow$ $\frac{{}^4P_4}{2!} = \frac{4!}{2!} = \frac{24}{2} = 12$

c) USING 4 DIFFERENT COLOURS $\Rightarrow$ $\underline{360}$

USING 3 DIFFERENT COLOURS $\Rightarrow$ "TEAMS" OF 3 OF OUT 6

E.g R G B $\leftarrow \binom{6}{3}$

WITH ANY OF THESE 3 REPEATED $(\times 3)$

$\therefore$ "TEAMS" OF $3\times\binom{6}{3} = 60$

NOW ARRANGEMENTS FOR EACH

$\frac{4!}{2!} = 12$

$\therefore 12\times 60 = \underline{720}$

USING 2 DIFFERENT COLOURS $\Rightarrow$ "TEAMS" OF TWO OUT OF 6

E.g R G $\leftarrow \binom{6}{2} = 15$

ARRANGEMENTS FOR EACH "TEAM" (BOTH MUST REPEAT)

$\frac{4!}{2!2!} = 6$

$\therefore 6\times 15 = \underline{90}$

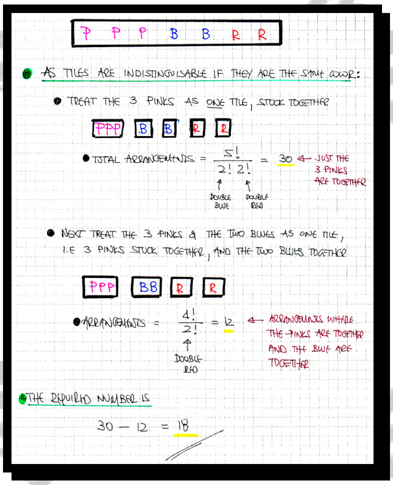
NOTHING ELSE IS POSSIBLE, SO $360 + 720 + 90 = 1170$

Question 10 (****)

Seven rectangular tiles, of which 3 are pink, 2 are blue and 2 are red, are placed in a straight line, next to each other.

Find the number of arrangements where the pink tiles are next to each other and the blue tiles are **not** next to each other.

FP4 , $\boxed{18}$



P P P B B R R

• AS TILES ARE INDISTINGUISABLE IF THEY ARE THE SAME COLOUR:

• TREAT THE 3 PINKS AS ONE TILE, STUCK TOGETHER

PPP B B R R

• TOTAL ARRANGEMENTS $= \frac{5!}{2!\,2!} = 30$ ← JUST THE 3 PINKS ARE TOGETHER

DOUBLE BLUE DOUBLE RED

• NEXT TREAT THE 3 PINKS & THE TWO BLUES AS ONE TILE, I.E 3 PINKS STUCK TOGETHER, AND THE TWO BLUES TOGETHER

PPP BB R R

• ARRANGEMENTS $= \frac{4!}{2!} = 12$ ← ARRANGEMENTS WHERE THE PINKS ARE TOGETHER AND THE BLUE ARE TOGETHER

DOUBLE RED

• THE REQUIRED NUMBER IS

$30 - 12 = 18$

Question 11 (*****)

Five 1^{st} year students and three 2^{nd} year students are standing next to each other, for a photograph to be taken.

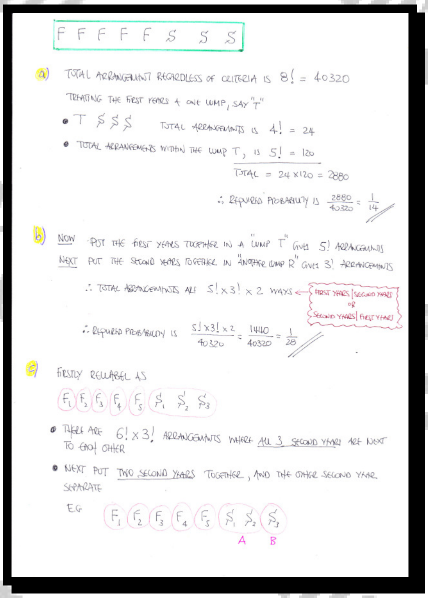
It assumed that the eight students positioned themselves at random.

a) Find the probability that all the 1^{st} year students are standing next to each other.

b) Determine the probability that all the 1^{st} year students are standing next to each other and all the 2^{nd} year students are standing next to each other.

c) Find the probability that no 2^{nd} year students are standing next to each other.

$\boxed{\frac{1}{14}}$, $\boxed{\frac{1}{28}}$, $\boxed{\frac{5}{14}}$



F F F F F S S S

a) TOTAL ARRANGEMENT REGARDLESS OF CRITERIA IS $8! = 40320$

TREATING THE FIRST YEARS AS ONE LUMP, SAY "T"

- T S S S TOTAL ARRANGEMENTS IS $4! = 24$
- TOTAL ARRANGEMENTS WITHIN THE LUMP T, IS $5! = 120$

TOTAL $= 24 \times 120 = 2880$

$\therefore$ REQUIRED PROBABILITY IS $\frac{2880}{40320} = \frac{1}{14}$

b) NOW PUT THE FIRST YEARS TOGETHER IN A "LUMP T" GIVES $5!$ ARRANGEMENTS

NEXT PUT THE SECOND YEARS TOGETHER IN "ANOTHER LUMP R" GIVES $3!$ ARRANGEMENTS

$\therefore$ TOTAL ARRANGEMENTS ARE $5! \times 3! \times 2$ WAYS ← FIRST YEARS | SECOND YEARS OR SECOND YEARS | FIRST YEARS

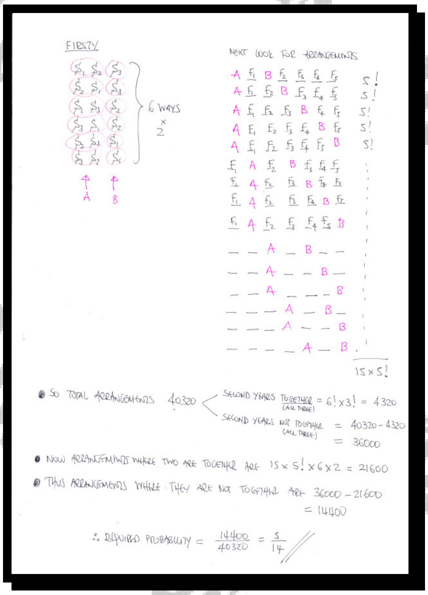
$\therefore$ REQUIRED PROBABILITY IS $\frac{5! \times 3! \times 2}{40320} = \frac{1440}{40320} = \frac{1}{28}$

c) FIRSTLY RELABEL AS

$F_1 \ F_2 \ F_3 \ F_4 \ F_5 \ S_1 \ S_2 \ S_3$

- THERE ARE $6! \times 3!$ ARRANGEMENTS WHERE ALL 3 SECOND YEARS ARE NEXT TO EACH OTHER
- NEXT PUT TWO SECOND YEARS TOGETHER, AND THE OTHER SECOND YEAR SEPARATE

E.G. $F_1 \ F_2 \ F_3 \ F_4 \ F_5 \ \underbrace{S_1 \ S_2}_{A} \ \underbrace{S_3}_{B}$



FIRSTLY

$S_1S_2 \ S_3$; $S_2S_1 \ S_3$; $S_1S_3 \ S_2$; $S_3S_1 \ S_2$; $S_2S_3 \ S_1$; $S_3S_2 \ S_1$ — 6 WAYS $\times$ 2

(A, B)

NEXT LOOK FOR ARRANGEMENTS

A F_1 B F_2 F_3 F_4 F_5 — $5!$

A F_1 F_2 B F_3 F_4 F_5 — $5!$

A F_1 F_2 F_3 B F_4 F_5 — $5!$

A F_1 F_2 F_3 F_4 B F_5 — $5!$

A F_1 F_2 F_3 F_4 F_5 B — $5!$

F_1 A F_2 B F_3 F_4 F_5

F_1 A F_2 F_3 B F_4 F_5

F_1 A F_2 F_3 F_4 B F_5

F_1 A F_2 F_3 F_4 F_5 B

_ _ A _ B _ _

_ _ A _ _ B _

_ _ A _ _ _ B

_ _ _ A _ B _

_ _ _ A _ _ B

_ _ _ _ A _ B

$15 \times 5!$

- SO TOTAL ARRANGEMENTS 40320: SECOND YEARS TOGETHER (ALL THREE) $= 6! \times 3! = 4320$; SECOND YEARS NOT TOGETHER (ALL THREE) $= 40320 - 4320 = 36000$
- NOW ARRANGEMENTS WHERE TWO ARE TOGETHER ARE $15 \times 5! \times 6 \times 2 = 21600$
- THUS ARRANGEMENTS WHERE THEY ARE NOT TOGETHER ARE $36000 - 21600 = 14400$

$\therefore$ REQUIRED PROBABILITY $= \frac{14400}{40320} = \frac{5}{14}$

Question 12 (*****)

5 adults and 6 children go to the cinema and sit next to each other, in a row which contains 11 empty consecutive seats.

a) Determine the number of ways these 11 people can sit so that no two adults sit next to each other.

Another 3 adults and 8 children go to the cinema and sit next to each other, in a row which also contains 11 empty consecutive seats.

b) Find the number of ways these 11 people can sit so that at least two of the adults sit next to each other.

$\boxed{}$, $\boxed{1,814,400}$, $\boxed{19,595,520}$

a) MODEL BY "FIXING" THE 6 CHILDREN

$\underset{1}{\_}\ \underset{2}{C}\ \underset{3}{\_}\ \underset{4}{C}\ \underset{5}{\_}\ \underset{6}{C}\ \underset{7}{\_}\ \underset{8}{C}\ \underset{9}{\_}\ \underset{10}{C}\ \underset{11}{\_}\ \underset{12}{C}\ \underset{13}{\_}$

NOW THE FIVE ADULTS CAN SIT IN ANY OF THE 7 POSITIONS WHICH ARE FREE (& THEN SHRINK OUT ANY GAPS TO 11)

HENCE THE REQUIRED NUMBER IS GIVEN BY

$$6! \times {}^7P_5 = 720 \times 2520 = 1,814,400$$

b) USING A SIMILAR APPROACH TO PART (a)

- $\underset{1}{\_}\ \underset{2}{C}\ \underset{3}{\_}\ \underset{4}{C}\ \underset{5}{\_}\ \underset{6}{C}\ \underset{7}{\_}\ \underset{8}{C}\ \underset{9}{\_}\ \underset{10}{C}\ \underset{11}{\_}\ \underset{12}{C}\ \underset{13}{\_}\ \underset{14}{C}\ \underset{15}{\_}\ \underset{16}{C}\ \underset{17}{\_}$

$$8! \times {}^9P_3 = 40320 \times 504 = 20,321,280$$

NO ADULTS NEXT TO EACH OTHER

- ALL POSSIBLE WAYS $= 11! = 39,916,800$
- AT LEAST 2 ADULTS NEXT TO EACH OTHER $= 39,916,800 - 20,321,280$

$= 19,595,520$

MIXED COUNTING

Question 1 (***+)

The numbers 1, 2, 3 and 4 are to be used to make a four digit password.

Calculate the number of the four digit passwords that can be created if …

a) ... any repetitions are allowed.

b) ... no repetitions are allowed

c) ... a digit can be repeated at most twice.

$\boxed{}$, $\boxed{4^4 = 256}$, $\boxed{4! = 24}$, $\boxed{204}$

a) EVIDENTLY THE REQUIRED ANSWER IS

$\underline{4} \times \underline{4} \times \underline{4} \times \underline{4} = 4^4 = 256$

b) NO REPETITIONS IS A STANDARD PERMUTATION

${}^4P_4 = 4! = 24$

c) A DIGIT CAN REPEATED AT MOST TWICE …

ONE DOUBLE, 2 DISTINCT … "2 PAIRS" …

EASIER TO WORK THE COMPLEMENT

- "4 THE SAME" = 4
- 3 THE SAME – 1 DISTINCT = ?
- 2 PAIRS OR ONE DOUBLE & 2 DISTINCT = ?
- 4 DISTINCT = 24

TOTAL OF ALL = 256

3 THE SAME & ONE DIFFERENT – SAY 1, 1, 1, 2

THIS GIVES 4 ARRANGEMENTS

×3 (ONE WITH 2, ONE WITH 3, ONE WITH 4)

×4 (1,1,1 / 2,2,2 / 3,3,3 / 4,4,4)

48

THE REQUIRED NUMBER IS GIVEN BY

$256 - 4 - 48 = 204$

Question 2 (****)

Alex, Beth and Cain are 3 students in a class which consists of a total of 8 students.

a) Determine the number of selections of 4 students which contain both Alex and Beth but not Cain.

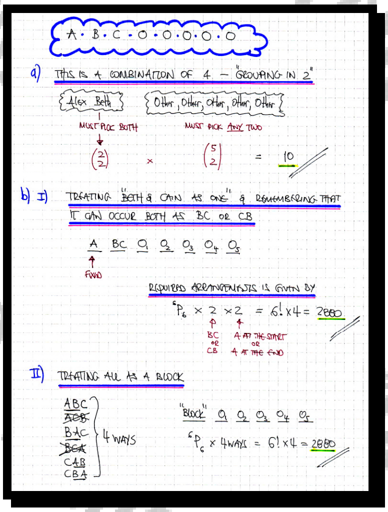
Next all 8 students are standing next to each for a group photo.

b) Determine the number of arrangements in which ...

i. ... Alex is standing at one end and Beth and Cain are standing next to each other.

ii. ... Alex and Beth are standing next to each other and Cain is standing next to them.

$\boxed{}$, $\boxed{10}$, $\boxed{2880}$, $\boxed{2880}$



A · B · C · O · O · O · O · O

a) THIS IS A COMBINATION OF 4 – "GROUPING IN 2"

{Alex, Beth} {Other, Other, Other, Other, Other}

MUST PICK BOTH MUST PICK ANY TWO

$\binom{2}{2} \times \binom{5}{2} = 10$

b) I) TREATING "BETH & CAIN AS ONE" & REMEMBERING THAT IT CAN OCCUR BOTH AS BC OR CB

A BC O_1 O_2 O_3 O_4 O_5

FIXED

REQUIRED ARRANGEMENTS IS GIVEN BY

$^6P_6 \times 2 \times 2 = 6! \times 4 = 2880$

BC OR CB; A AT THE START OR A AT THE END

II) TREATING ALL AS A BLOCK

ABC, ~~ACB~~, BAC, ~~BCA~~, CAB, CBA – 4 WAYS

"BLOCK" O_1 O_2 O_3 O_4 O_5

$^6P_6 \times 4\text{WAYS} = 6! \times 4 = 2880$

Question 3 (****+)

1, 2, 3, 4, 5, 6, 7, 8, 9

The above nine single digit numbers are written on nine separate pieces of card.

Four of these cards are picked at random and placed next to each other to form a four digit number.

Find the total different number of arrangements of …

a) … four digit numbers that can be formed.

b) … four digit **odd** numbers that can be formed.

c) … four digit numbers that can be formed, whose all four digits are **odd**.

d) … four digit numbers that can be formed which have odd and even digits.

e) … four digit numbers that can be formed which have **at least** three odd digits.

f) … four digit numbers that can be formed whose **sum** of digits is 28 .

$\boxed{}$, $\boxed{3024}$, $\boxed{1680}$, $\boxed{120}$, $\boxed{2880}$, $\boxed{1080}$, $\boxed{48}$

1 2 3 4 5 6 7 8 9

NOTE THAT ALL NINE NUMBERS/DIGITS ARE DISTINCT.

a) 4 DIGIT NUMBERS

$${}^9P_4 = \frac{9!}{(9-4)!} = \frac{9!}{5!} = 3024$$

b) 4 DIGIT NUMBERS WHICH ARE ODD

- PLACE AN ODD NUMBER AT THE END
- THIS LEAVES 8 NUMBERS, TO PICK 3 FROM, WHICH CAN GO IN THE FIRST THREE POSITIONS
- THIS CAN OCCUR 5 WAYS (DIFFERENT ODD NUMBER AT THE END)

$$\underbrace{[\_\ \_\ \_]}_{\text{ANY 3 OUT 8}}\ \underset{\text{1 ODD OUT OF 5}}{\_}$$

$\therefore\ {}^8P_3 \times 5\text{WAYS} = 1680$

c) 4 DIGIT NUMBER WITH JUST ODD DIGITS

- SIMPLY ARRANGEMENTS OF 4 OUT 5, ${}^5P_4 = 120$
- OR BY LOOKING AT THE AVAILABLE CHOICES INTO "SLOTS"

(5)ODD (4)ODD (3)ODD 2(ODD) $= 5\times4\times3\times2 = 120$

d) FIRSTLY NUMBERS WITH …

- ALL DIGITS EVEN $\Longrightarrow 4! = 24$
- ALL DIGITS ODD $\Longrightarrow 120$ (FOUND IN c)
- TOTAL FOUR DIGIT NUMBERS $\Longrightarrow 3024$ (FOUND IN a)

REQUIRED NUMBER IS $3024 - 24 - 120 = 2880$

e) 4 DIGIT NUMBERS WITH AT LEAST 3 ODD DIGITS

- ALL ODD ARE 120 (FOUND IN c)
- 3 ODD, 1 EVEN

OOOE
OOEO
OEOO
EOOO

$$\underbrace{\text{ODD}\ \ \text{ODD}\ \ \text{ODD}}_{{}^5P_3}\ \ \underset{4}{\text{EVEN}} \times 4 \text{ WAYS} = 960$$

REQUIRED TOTAL IS $960 + 120 = 1080$

f) 4 DIGIT NUMBERS WHOSE DIGITS SUM TO 28

IT BECOMES EVIDENT AFTER FEW TRIES THAT THERE ARE ONLY 2 POSSIBLE SELECTIONS

9, 8, 7, 4 & 9, 8, 6, 5

EACH PRODUCING $4!$ ARRANGEMENTS

REQUIRED TOTAL IS $2\times4! = 48$

Question 4 (**+)**

The six letters of the word ***RADIAN*** are written on six separate pieces of card.

In an experiment, four cards are selected and placed next to each other, forming a four letter arrangement.

Calculate the number of different four letter arrangement.

$\boxed{}$, $\boxed{192}$

R · A · D · I · A · N

THE ARE 3 CASES TO CONSIDER

- CASE 1 – NO "A" IS PICKED

R · D · I · N $\Longrightarrow$ $4! = 24$ ARRANGEMENTS

- CASE 2 – ONLY ONE "A" IS SELECTED

A & ANY 3 FROM $\underbrace{\text{R · D · I · N}}_{\binom{4}{3} = 4 \text{ WAYS}}$, SAY $\underbrace{\text{A, R, D, I}}_{4!}$

$\therefore\ 4! \times 4 \text{ WAYS} = 96$ ARRANGEMENTS

- CASE 3 – BOTH "As" ARE SELECTED

A, A & ANY 2 FROM $\underbrace{\text{R · D · I · N}}_{\binom{4}{2} = 6 \text{ WAYS}}$, SAY $\underbrace{\text{A, A, R, D}}_{\frac{4!}{2!}}$

$\therefore\ \frac{4!}{2!} \times 6 \text{ WAYS} = 72$ ARRANGEMENTS

THE REQUIRED TOTAL NUMBER IS

$24 + 96 + 72 = 192$

Question 5 (****+)

$\boxed{B}, \boxed{A}, \boxed{N}, \boxed{A}, \boxed{N}, \boxed{A}, \boxed{S}$

The 7 letters shown above are written on separate pieces of card.

a) Find the number of arrangements which can be made if all 7 letters are used.

b) Find the number of arrangements which can be made if all 7 letters are used and the three vowels are together.

c) Find the number of arrangements which can be made if all 7 letters are used and the three vowels are together and the four consonants are together.

d) Determine the number of ways in which 4 letters can be picked from the total of 7 letters.

e) Calculate the number of arrangements of which 4 letters are used from the total of 7 letters.

$\boxed{}, \boxed{420}, \boxed{60}, \boxed{24}, \boxed{11}, \boxed{114}$

a) REQUIRED NUMBER IS GIVEN BY

$$\frac{7!}{3!\,2!} = 420$$

TRIPLE "A" (REPEAT) DOUBLE "N" (REPEAT)

b) TREATING THE VOWELS "AAA", AS A SINGLE LETTER

$\boxed{AAA}$ $\boxed{B}$ $\boxed{N}$ $\boxed{N}$ $\boxed{S}$

1 2 3 4 5

$$\frac{5!}{2!} = 60$$

DOUBLE "N" REPEAT

c) BLOCKING THE VOWELS & CONSONANTS TOGETHER

$\boxed{A\ A\ A}$ $\boxed{B\ N\ N\ S}$

1 WAY $\quad \frac{4!}{2!}$ WAYS $= 12$

HENCE $(1 \times 12) \times 2 = 24$

3 VOWELS – 4 CONSONANTS OR 4 CONSONANTS – 3 VOWELS

d) SPLITTING IN SEPARATE CASES & NOTE IN THIS PART ORDER DOES NOT MATTER

I	AAA	WITH ONE OF B, N, S	3 WAYS
II	AA	WITH TWO OF B, N, S	3 WAYS
III	NN	WITH TWO OF A, B, S	3 WAYS
IV	AANN		1 WAY
V	ALL DIFFERENT	B A N S	1 WAY
			11 WAYS

e) USING PART (d)

CASE I, SAY A A A B $\quad 3 \times \frac{4!}{3!} = 12$

CASE II, SAY A A B N $\quad 3 \times \frac{4!}{2!} = 36$

CASE III, SAY N N A B $\quad 3 \times \frac{4!}{2!} = 36$

CASE IV, A A N N $\quad 1 \times \frac{4!}{2!\,2!} = 6$

CASE V B A N S $\quad 1 \times 4! = 24$

114

Question 6 (****+)

$$1,\ 1,\ 2,\ 2,\ 3,\ 3,\ 4,\ 4$$

The above 8 single digit numbers are written on 8 separate pieces of card.

These cards are placed next to each other at random, forming an 8 digit number.

a) Determine the number of the 8 digit numbers that can be formed, which exceed $30,000,000$.

Next 4 cards are picked at random and placed next to each other to form a 4 digit number.

b) Find the number of 4 digit numbers that can be formed, which exceed 3000.

$\boxed{}$, $\boxed{1260}$, $\boxed{102}$

a) IF ALL NUMBER ARE TO BE USED, THEN THE "REQUIRED" NUMBER OF ARRANGEMENTS MUST START WITH 3 OR 4

$\underline{3/4}\ \underbrace{\_\ \_\ \_\ \_\ \_\ \_\ \_}_{\text{7 TO CHOOSE FROM } 1,1,2,2,3,4,4 \text{ OR } 1,1,2,2,3,3,4}$

HENCE $\dfrac{7!}{2!2!2!} \times 2$ WAYS $= 1260$

(STARTING WITH 3 OR 4)

ALTERNATIVE, FIND ALL POSSIBLE ARRANGEMENTS OF 8

$$\frac{8!}{2!2!2!2!} = 2520$$

HALF OF THESE (BY SYMMETRY) WILL BE OVER 30,000,000

HENCE $\frac{1}{2} \times 2520 = 1260$

b) FIND ALL ARRANGEMENTS OF 4

(A) ALL 4 NUMBERS ARE DISTINCT

$4! = 24$ $\quad$ $\frac{1}{2}$ OF THESE WILL BE OVER 3000

IE 12

(B) ONE DOUBLE REPEAT & TWO DISTINCT

SAY $1,1,2,3$ ← CHOICE OF 2 OUT 3 (3)

CHOICE OF 1 OUT OF 4 (4)

$\dfrac{4!}{2!} \times 4 \times 3 = 144$

NO OF ARRANGEMENTS

AGAIN HALF OF THESE BY SYMMETRY WILL START WITH 3 OR 4, SO

$\frac{1}{2} \times 144 = 72$

(C) TWO DOUBLE REPEATS

SAY $1\ 1\ 2\ 2$

$\dfrac{4!}{2!2!} = 6$ ARRANGEMENTS $\times \dbinom{4}{2} = 36$

ANY TWO NUMBER OUT OF 1, 2, 3, 4

HALF OF THESE BY SYMMETRY WILL BE OVER 3000

IE 18

HENCE THE TOTAL IS $12 + 72 + 18 = 102$

Question 7 (*****)

The 6 letters of the word *BUTTER* are written on 6 separate pieces of card.

In an experiment 4 cards are selected at random, forming a 4 letter **arrangement**.

a) Determine the number of 4 letter arrangements which ...

  i. … will begin and end with a consonant.

  ii. … will begin with a vowel.

  iii. … will start with B and end with a vowel.

b) Find the total number of all 4 letter arrangements which can be formed.

$\boxed{}$, $\boxed{40}$, $\boxed{66}$, $\boxed{14}$, $\boxed{192}$

a) I) "BEGIN AND END WITH A CONSONANT"

CASE A (Two t)

T _ _ T

• still available B, U, E, R

${}^4P_2 = \frac{4!}{2!} = 12$

(TWO LETTER ARRANGEMENTS OUT OF 4 LETTERS)

CASE B (ONE t)

T _ _ B [OR T _ _ R]

... OR THE OTHER ROUND ...

• Looking at the 1st choice ... still available are U, E, R

${}^3P_2 \times 2 \times 2 = 24$

2 LETTER CHOICES OUT OF 3

R INSTEAD OF B AT THE END

T AT THE END

CASE C (No t)

B _ _ R [OR R _ _ B]

• still available U, E

– can only be arranged as U, E OR E U

TOTAL 4

SWAP B & R AT THE ENDS & SWAP U & E IN THE MIDDLE

REQUIRED NUMBER IS $12 + 24 + 4 = 40$

a) II) "BEGIN WITH A VOWEL"

CASE A (Two t)

U _ _ _ LEAVES 1 TO PICK FROM TTB × 3 WAYS

TTE × 3 WAYS

TTR × 3 WAYS

$9 \times 2 = 18$

SWAP Ⓔ & Ⓤ

CASE B (one t)

U _ _ _ LEAVES 1 TO BE PICKED FROM T, B, E × 6 WAYS

T, B, R × 6 WAYS

T, E, R × 6 WAYS

$18 \times 2 = 36$

SWAP Ⓤ & Ⓔ

CASE C (No t)

U _ _ _ LEAVES B, E, R × 6 WAYS × 2 = 12

SWAP Ⓤ & Ⓔ

∴ REQUIRED NUMBER IS $18 + 36 + 12 = 66$

a) III) "STARTS WITH Ⓑ AND FINISHES IN A VOWEL"

CASE A (two t)

B T T U OR B T T E   2

CASE B (one t)

B _ _ U LEAVES 1 TO BE PICKED FROM T, E × 2 WAYS

TR × 2 WAYS

$4 \times 2 = 8$

SWAP Ⓔ & Ⓤ

CASE C (No t)

B _ _ U LEAVES E, R TO BE PICKED, 2 WAYS

$2 \times 2 = 4$

SWAP Ⓔ & Ⓤ

REQUIRED NUMBER IS $2+8+4 = 14$

b) "ANY 4 LETTER ARRANGEMENT"

CASE A (two t)

T, T & any two from U, B, E, R,

• SAY T, T, U, B

$\frac{4!}{2!} \times \binom{4}{2} = 12 \times 6 = 72$

PERMUTATIONS OF 4, WITH A DOUBLE REPEAT

any two from U, B, E, R

CASE B (one t)

T & any three from U, B, E, R

• T U B R

$4! \times \binom{4}{3} = 24 \times 4 = 96$

CASE C (No t)

B U E R ← $4! = 24$

∴ TOTAL NUMBER IS $72+96+24 = 192$

Question 8 (*****)

The 7 letters of the word *MINIMUM* are written on 7 separate pieces of card.

Four of these cards are picked at random, one after the other, and are arranged into a four letter word in the order they were picked.

Determine the number of the four letter words which can be formed.

$\boxed{}$, $\boxed{114}$

START BY CONSIDERING THE DIFFERENT SELECTIONS OF 4 LETTERS (ORDER NOT IMPORTANT / COMBINATIONS)

① M M M with one of I, N, U — 3 WAYS

② M M with two of I, N, U — 3 WAYS

③ I I with two of M, N, U — 3 WAYS

④ M M I I (two pairs) — 1 WAY

⑤ M I N U (All 4 different) — 1 WAY

NOW WE CAN CONSIDER THE NUMBER OF ARRANGEMENTS IN EACH OF THE ABOVE CASES (PERMUTATIONS)

CASE 1 SAY M M M I $\quad \frac{4!}{3!} \times 3 \text{ WAYS} = 12$

CASE 2 SAY M M N I $\quad \frac{4!}{2!} \times 3 \text{ WAYS} = 36$

CASE 3 SAY I I M N $\quad \frac{4!}{2!} \times 3 \text{ WAYS} = 36$

CASE 4 M M I I $\quad \frac{4!}{2!2!} \times 1 \text{ WAY} = 6$

CASE 5 M I N U $\quad 4! \times 1 \text{ WAY} = 24$

ADDING TO OBTAIN A TOTAL OF 114