COMPLEX NUMBERS

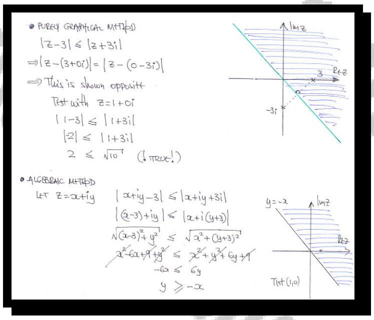
(Exam Questions II)

Question 1 (**)

By finding a suitable Cartesian locus for the complex z plane, shade the region R that satisfies the inequality

$$|z-3| \le |z+3\mathrm{i}|.$$

$$\boxed{x+y \ge 0}$$



- PURELY GRAPHICAL METHOD

$|z-3| \leqslant |z+3i|$

$\Rightarrow |z-(3+0i)| = |z-(0-3i)|$

$\Rightarrow$ This is shown opposite

Test with $z=1+0i$

$|1-3| \leqslant |1+3i|$

$|-2| \leqslant |1+3i|$

$2 \leqslant \sqrt{10}$ (TRUE!)

- ALGEBRAIC METHOD

LET $z=x+iy$

$|x+iy-3| \leqslant |x+iy+3i|$

$|(x-3)+iy| \leqslant |x+i(y+3)|$

$\sqrt{(x-3)^2+y^2} \leqslant \sqrt{x^2+(y+3)^2}$

$\cancel{x^2}-6x+\cancel{9}+\cancel{y^2} \leqslant \cancel{x^2}+\cancel{y^2}+6y+\cancel{9}$

$-6x \leqslant 6y$

$y \geqslant -x$

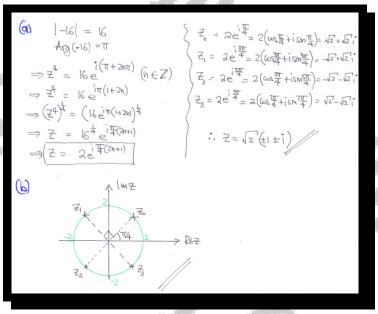
Question 2 (**)

$$z^4 = -16, \ z \in \mathbb{C}.$$

a) Determine the solutions of the above equation, giving the answers in the form $a+b\mathrm{i}$, where a and b are real numbers.

b) Plot the roots of the equation as points in an Argand diagram.

$$\boxed{z=\sqrt{2}(\pm 1 \pm \mathrm{i})}$$



(a) $|-16| = 16$

$\text{Arg}(-16) = \pi$

$\Rightarrow z^4 = 16e^{i(\pi+2n\pi)}$ $(n \in \mathbb{Z})$

$\Rightarrow z^4 = 16e^{i\pi(1+2n)}$

$\Rightarrow (z^4)^{\frac{1}{4}} = (16e^{i\pi(1+2n)})^{\frac{1}{4}}$

$\Rightarrow z = 16^{\frac{1}{4}}e^{i\frac{\pi}{4}(2n+1)}$

$\Rightarrow \boxed{z = 2e^{i\frac{\pi}{4}(2n+1)}}$

$z_0 = 2e^{i\frac{\pi}{4}} = 2(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}) = \sqrt{2}+\sqrt{2}i$

$z_1 = 2e^{i\frac{3\pi}{4}} = 2(\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4}) = -\sqrt{2}+\sqrt{2}i$

$z_2 = 2e^{i\frac{5\pi}{4}} = 2(\cos\frac{5\pi}{4}+i\sin\frac{5\pi}{4}) = -\sqrt{2}-\sqrt{2}i$

$z_3 = 2e^{i\frac{7\pi}{4}} = 2(\cos\frac{7\pi}{4}+i\sin\frac{7\pi}{4}) = \sqrt{2}-\sqrt{2}i$

$\therefore z = \sqrt{2}(\pm 1 \pm i)$

(b)

Question 3 **(**)**

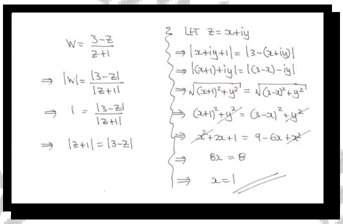
A transformation from the z plane to the w plane is defined by the complex function

$$w = \frac{3-z}{z+1}, \quad z \neq -1.$$

The locus of the points represented by the complex number $z = x + \mathrm{i}y$ is transformed to the circle with equation $|w| = 1$ in the w plane.

Find, in Cartesian form, an equation of the locus of the points represented by the complex number z.

$$\boxed{x = 1}$$



$W = \frac{3-Z}{Z+1}$

$\Rightarrow |W| = \frac{|3-Z|}{|Z+1|}$

$\Rightarrow 1 = \frac{|3-Z|}{|Z+1|}$

$\Rightarrow |Z+1| = |3-Z|$

LET $Z = x+iy$

$\Rightarrow |x+iy+1| = |3-(x+iy)|$

$\Rightarrow |(x+1)+iy| = |(3-x)-iy|$

$\Rightarrow \sqrt{(x+1)^2+y^2} = \sqrt{(3-x)^2+y^2}$

$\Rightarrow (x+1)^2+\cancel{y^2} = (3-x)^2+\cancel{y^2}$

$\Rightarrow \cancel{x^2}+2x+1 = 9-6x+\cancel{x^2}$

$\Rightarrow 8x = 8$

$\Rightarrow x = 1$

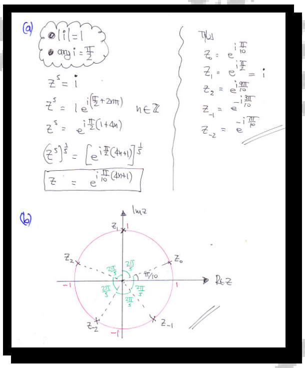
Question 4 **(**)**

$$z^5 = \mathrm{i}, \quad z \in \mathbb{C}.$$

a) Solve the equation, giving the roots in the form $r\mathrm{e}^{\mathrm{i}\theta}, r > 0, -\pi < \theta \leq \pi$.

b) Plot the roots of the equation as points in an Argand diagram.

$$\boxed{z = \mathrm{e}^{\mathrm{i}\frac{\pi}{10}}, \quad z = \mathrm{e}^{\mathrm{i}\frac{\pi}{2}}, \quad z = \mathrm{e}^{\mathrm{i}\frac{9\pi}{10}}, \quad z = \mathrm{e}^{-\mathrm{i}\frac{3\pi}{10}}, \quad z = \mathrm{e}^{-\mathrm{i}\frac{7\pi}{10}}}$$



(a)

- $|i| = 1$
- $\arg i = \frac{\pi}{2}$

$Z^5 = i$

$Z^5 = 1e^{i\left(\frac{\pi}{2}+2n\pi\right)} \quad n \in \mathbb{Z}$

$Z^5 = e^{i\frac{\pi}{2}(1+4n)}$

$\left(Z^5\right)^{\frac{1}{5}} = \left[e^{i\frac{\pi}{2}(4n+1)}\right]^{\frac{1}{5}}$

$\boxed{Z = e^{i\frac{\pi}{10}(4n+1)}}$

Thus

$Z_0 = e^{i\frac{\pi}{10}}$

$Z_1 = e^{i\frac{\pi}{2}} = i$

$Z_2 = e^{i\frac{9\pi}{10}}$

$Z_{-1} = e^{-i\frac{3\pi}{10}}$

$Z_{-2} = e^{-i\frac{7\pi}{10}}$

(b)

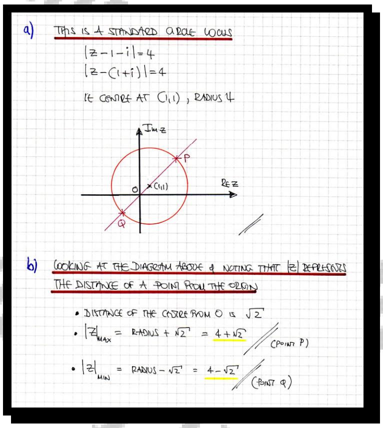
Question 5 (**)

$$|z-1-\mathrm{i}|=4\ ,\ z\in\mathbb{C}\ .$$

a) Sketch, in a standard Argand diagram, the locus of the points that satisfy the above equation.

b) Find the minimum and maximum value of $|z|$ for points that lie on this locus.

$\boxed{}$, $\boxed{z_{\min}=4-\sqrt{2}}$, $\boxed{z_{\min}=4+\sqrt{2}}$



a) THIS IS A STANDARD CIRCLE LOCUS

$|z-1-i|=4$

$|z-(1+i)|=4$

IE CENTRE AT $(1,1)$, RADIUS 4

b) LOOKING AT THE DIAGRAM ABOVE & NOTING THAT $|z|$ REPRESENTS THE DISTANCE OF A POINT FROM THE ORIGIN

- DISTANCE OF THE CENTRE FROM O IS $\sqrt{2}$
- $|z|_{MAX}$ = RADIUS + $\sqrt{2}$ = $4+\sqrt{2}$ (POINT P)
- $|z|_{MIN}$ = RADIUS − $\sqrt{2}$ = $4-\sqrt{2}$ (POINT Q)

Question 6 (**)

The complex number z represents the point $P(x, y)$ in the Argand diagram.

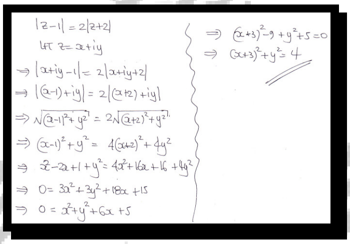
Given that

$$|z-1| = 2|z+2|,$$

show that the locus of P is given by

$$(x+3)^2 + y^2 = 4.$$

$\boxed{\text{proof}}$



$|z-1| = 2|z+2|$

LET $z = x+iy$

$\Rightarrow |x+iy-1| = 2|x+iy+2|$

$\Rightarrow |(x-1)+iy| = 2|(x+2)+iy|$

$\Rightarrow \sqrt{(x-1)^2+y^2} = 2\sqrt{(x+2)^2+y^2}$

$\Rightarrow (x-1)^2+y^2 = 4(x+2)^2+4y^2$

$\Rightarrow x^2-2x+1+y^2 = 4x^2+16x+16+4y^2$

$\Rightarrow 0 = 3x^2+3y^2+18x+15$

$\Rightarrow 0 = x^2+y^2+6x+5$

$\Rightarrow (x+3)^2-9+y^2+5=0$

$\Rightarrow (x+3)^2+y^2=4$

Question 7 (**)

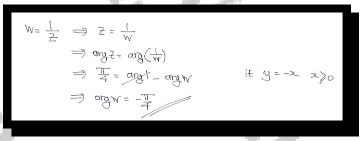
Find an equation of the locus of the points which lie on the half line with equation

$$\arg z = \frac{\pi}{4}, \ z \neq 0$$

after it has been transformed by the complex function

$$w = \frac{1}{z}.$$

$$\boxed{\arg w = -\frac{\pi}{4}}$$



$w = \frac{1}{z} \Rightarrow z = \frac{1}{w}$

$\Rightarrow \arg z = \arg\left(\frac{1}{w}\right)$

$\Rightarrow \frac{\pi}{4} = \arg 1 - \arg w$

$\Rightarrow \arg w = -\frac{\pi}{4}$

IE $y = -x \quad x \geqslant 0$

Question 8 (**)

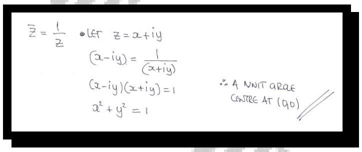
The complex number $z = x + \mathrm{i}y$ represents the point P in the complex plane.

Given that

$$\overline{z} = \frac{1}{z}, \; z \neq 0$$

determine a Cartesian equation for the locus of P.

$$\boxed{x^2 + y^2 = 1}$$



$\overline{z} = \frac{1}{z}$ • LET $z = x + iy$

$(x - iy) = \frac{1}{(x+iy)}$

$(x - iy)(x + iy) = 1$

$x^2 + y^2 = 1$

∴ A UNIT CIRCLE CENTRE AT (0,0) //

Question 9 (**)

Sketch, on the same Argand diagram, the locus of the points satisfying each of the following equations.

a) $|z-3+\mathrm{i}|=3$.

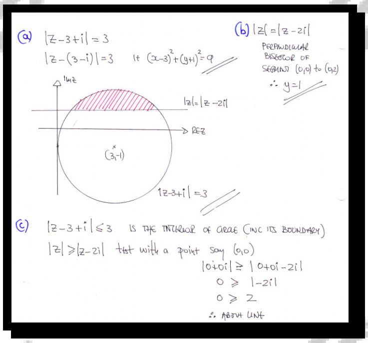
b) $|z|=|z-2\mathrm{i}|$.

Give in each case a Cartesian equation for the locus.

c) Shade in the sketch the region that is satisfied by both these inequalities

$$|z-3+\mathrm{i}|\le 3$$
$$|z|\ge|z-2\mathrm{i}|$$

$\boxed{(x-3)^2+(y+1)^2=9}$, $\boxed{y=1}$



(a) $|z-3+i|=3$

$|z-(3-i)|=3 \quad \therefore (x-3)^2+(y+1)^2=9$

(b) $|z|=|z-2i|$

PERPENDICULAR BISECTOR OF SEGMENT (0,0) to (0,2)

$\therefore y=1$

(c) $|z-3+i|\le 3$ IS THE INTERIOR OF CIRCLE (INC ITS BOUNDARY)

$|z|\ge|z-2i|$ test with a point say (0,0)

$|0+0i|\ge|0+0i-2i|$

$0\ge|-2i|$

$0\ge 2$

∴ ABOVE LINE

Question 10 (**)

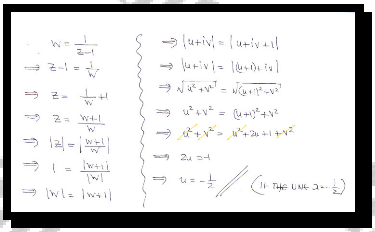
The complex function

$$w = \frac{1}{z-1}, \quad z \neq 1, z \in \mathbb{C}, \quad z \neq 1$$

transforms the point represented by $z = x + \mathrm{i}y$ in the z plane into the point represented by $w = u + \mathrm{i}v$ in the w plane.

Given that z satisfies the equation $|z| = 1$, find a Cartesian locus for w.

$$\boxed{u = -\frac{1}{2}}$$



$w = \frac{1}{z-1}$

$\Rightarrow z-1 = \frac{1}{w}$

$\Rightarrow z = \frac{1}{w}+1$

$\Rightarrow z = \frac{w+1}{w}$

$\Rightarrow |z| = \left|\frac{w+1}{w}\right|$

$\Rightarrow 1 = \frac{|w+1|}{|w|}$

$\Rightarrow |w| = |w+1|$

$\Rightarrow |u+iv| = |u+iv+1|$

$\Rightarrow |u+iv| = |(u+1)+iv|$

$\Rightarrow \sqrt{u^2+v^2} = \sqrt{(u+1)^2+v^2}$

$\Rightarrow u^2+v^2 = (u+1)^2+v^2$

$\Rightarrow \cancel{u^2}+\cancel{v^2} = \cancel{u^2}+2u+1\cancel{+v^2}$

$\Rightarrow 2u = -1$

$\Rightarrow u = -\frac{1}{2}$ // (I.E THE LINE $x = -\frac{1}{2}$)

Question 11 (**)

a) Sketch on the same Argand diagram the locus of the points satisfying each of the following equations.

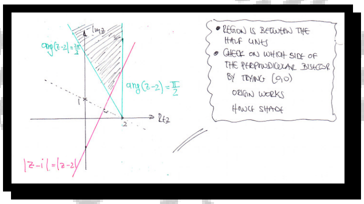
i. $|z - \mathrm{i}| = |z - 2|$.

ii. $\arg(z - 2) = \dfrac{\pi}{2}$.

b) Shade in the sketch the region that is satisfied by both these inequalities

$$|z - \mathrm{i}| \le |z - 2| \qquad \text{and} \qquad \frac{\pi}{2} \le \arg(z - 2) \le \frac{2\pi}{3}.$$

sketch



- REGION IS BETWEEN THE HALF LINES
- CHECK ON WHICH SIDE OF THE PERPENDICULAR BISECTOR BY TRYING (0,0)

ORIGIN WORKS

HENCE SHADE

Question 12 (**)

The complex function $w = f(z)$ is given by

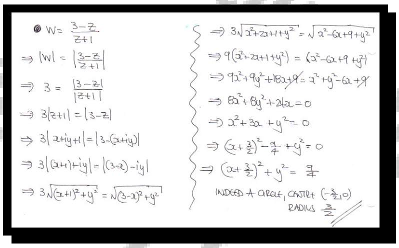
$$w = \frac{3-z}{z+1} \quad \text{where } z \in \mathbb{C}, \quad z \neq -1 .$$

A point P in the z plane gets mapped onto a point Q in the w plane.

The point Q traces the circle with equation $|w| = 3$.

Show that the locus of P in the z plane is also a circle, stating its centre and its radius.

$$\boxed{\text{centre } \left(-\frac{3}{2}, 0\right), \text{ radius } = \frac{3}{2}}$$



$$w = \frac{3-z}{z+1}$$

$$\Rightarrow |w| = \left|\frac{3-z}{z+1}\right|$$

$$\Rightarrow 3 = \frac{|3-z|}{|z+1|}$$

$$\Rightarrow 3|z+1| = |3-z|$$

$$\Rightarrow 3|x+iy+1| = |3-(x+iy)|$$

$$\Rightarrow 3|(x+1)+iy| = |(3-x)-iy|$$

$$\Rightarrow 3\sqrt{(x+1)^2+y^2} = \sqrt{(3-x)^2+y^2}$$

$$\Rightarrow 3\sqrt{x^2+2x+1+y^2} = \sqrt{x^2-6x+9+y^2}$$

$$\Rightarrow 9(x^2+2x+1+y^2) = (x^2-6x+9+y^2)$$

$$\Rightarrow 9x^2+9y^2+18x+\cancel{9} = x^2+y^2-6x+\cancel{9}$$

$$\Rightarrow 8x^2+8y^2+24x = 0$$

$$\Rightarrow x^2+3x+y^2 = 0$$

$$\Rightarrow \left(x+\frac{3}{2}\right)^2 - \frac{9}{4} + y^2 = 0$$

$$\Rightarrow \left(x+\frac{3}{2}\right)^2 + y^2 = \frac{9}{4}$$

INDEED A CIRCLE, CENTRE $\left(-\frac{3}{2}, 0\right)$

RADIUS $\frac{3}{2}$

Question 13 (**)

The general point $P(x, y)$ which is represent by the complex number $z = x + \mathrm{i}y$ in the z plane, lies on the locus of

$$|z| = 1.$$

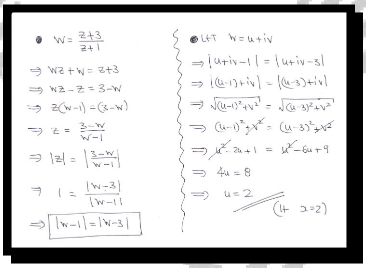
A transformation from the z plane to the w plane is defined by

$$w = \frac{z+3}{z+1}, \ \ z \neq -1,$$

and maps the point $P(x, y)$ onto the point $Q(u, v)$.

Find, in Cartesian form, the equation of the locus of the point Q in the w plane.

$\boxed{u = 2}$



● $w = \frac{z+3}{z+1}$

$\Rightarrow wz + w = z + 3$

$\Rightarrow wz - z = 3 - w$

$\Rightarrow z(w-1) = (3-w)$

$\Rightarrow z = \frac{3-w}{w-1}$

$\Rightarrow |z| = \left|\frac{3-w}{w-1}\right|$

$\Rightarrow 1 = \frac{|w-3|}{|w-1|}$

$\Rightarrow \boxed{|w-1| = |w-3|}$

● LET $w = u + \mathrm{i}v$

$\Rightarrow |u + \mathrm{i}v - 1| = |u + \mathrm{i}v - 3|$

$\Rightarrow |(u-1) + \mathrm{i}v| = |(u-3) + \mathrm{i}v|$

$\Rightarrow \sqrt{(u-1)^2 + v^2} = \sqrt{(u-3)^2 + v^2}$

$\Rightarrow (u-1)^2 \cancel{+v^2} = (u-3)^2 \cancel{+v^2}$

$\Rightarrow \cancel{u^2} - 2u + 1 = \cancel{u^2} - 6u + 9$

$\Rightarrow 4u = 8$

$\Rightarrow u = 2$

(LT $x = 2$)

Question 14 **(**)**

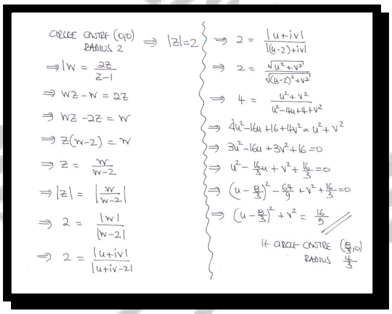
The point P represented by $z = x + \mathrm{i}y$ in the z plane is transformed into the point Q represented by $w = u + \mathrm{i}v$ in the w plane, by the complex transformation

$$w = \frac{2z}{z-1},\ z \neq 1.$$

The point P traces a circle of radius 2, centred at the origin O.

Find a Cartesian equation of the locus of the point Q.

$$\boxed{\left(u - \frac{8}{3}\right)^2 + v^2 = \frac{16}{9}}$$



CIRCLE CENTRE $(0,0)$ RADIUS 2 $\Rightarrow |z| = 2$

$\Rightarrow w = \frac{2z}{z-1}$

$\Rightarrow wz - w = 2z$

$\Rightarrow wz - 2z = w$

$\Rightarrow z(w-2) = w$

$\Rightarrow z = \frac{w}{w-2}$

$\Rightarrow |z| = \left|\frac{w}{w-2}\right|$

$\Rightarrow 2 = \frac{|w|}{|w-2|}$

$\Rightarrow 2 = \frac{|u+iv|}{|u+iv-2|}$

$\Rightarrow 2 = \frac{|u+iv|}{|(u-2)+iv|}$

$\Rightarrow 2 = \frac{\sqrt{u^2+v^2}}{\sqrt{(u-2)^2+v^2}}$

$\Rightarrow 4 = \frac{u^2+v^2}{u^2-4u+4+v^2}$

$\Rightarrow 4u^2 - 16u + 16 + 4v^2 = u^2 + v^2$

$\Rightarrow 3u^2 - 16u + 3v^2 + 16 = 0$

$\Rightarrow u^2 - \frac{16}{3}u + v^2 + \frac{16}{3} = 0$

$\Rightarrow \left(u - \frac{8}{3}\right)^2 - \frac{64}{9} + v^2 + \frac{16}{3} = 0$

$\Rightarrow \left(u - \frac{8}{3}\right)^2 + v^2 = \frac{16}{9}$

IE CIRCLE CENTRE $\left(\frac{8}{3}, 0\right)$ RADIUS $\frac{4}{3}$

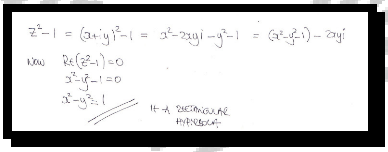
Question 15 **(**)**

The point P represents the complex number $z = x + \mathrm{i}y$ in an Argand diagram.

It is further given that $z^2 - 1$ is purely imaginary for all values of z.

Find a Cartesian equation of the locus that P is tracing in the Argand diagram.

$$\boxed{x^2 - y^2 = 1}$$



$z^2 - 1 = (x+iy)^2 - 1 = x^2 - 2xyi - y^2 - 1 = (x^2 - y^2 - 1) - 2xyi$

NOW $\text{Re}(z^2-1) = 0$

$x^2 - y^2 - 1 = 0$

$x^2 - y^2 = 1$

IE A RECTANGULAR HYPERBOLA

Question 16 **(**+)**

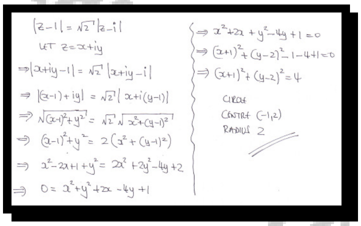
The complex number z represents the point $P(x, y)$ in the Argand diagram.

Given that

$$|z-1| = \sqrt{2}\,|z-\mathrm{i}|,$$

show that the locus of P is a circle, stating its centre and radius.

$$\boxed{(x+1)^2 + (y-2)^2 = 4}, \quad \boxed{(-1,2), r = 2}$$



$|z-1| = \sqrt{2}\,|z-i|$

LET $z = x+iy$

$\Rightarrow |x+iy-1| = \sqrt{2}\,|x+iy-i|$

$\Rightarrow |(x-1)+iy| = \sqrt{2}\,|x+i(y-1)|$

$\Rightarrow \sqrt{(x-1)^2+y^2} = \sqrt{2}\sqrt{x^2+(y-1)^2}$

$\Rightarrow (x-1)^2+y^2 = 2(x^2+(y-1)^2)$

$\Rightarrow x^2-2x+1+y^2 = 2x^2+2y^2-4y+2$

$\Rightarrow 0 = x^2+y^2+2x-4y+1$

$\Rightarrow x^2+2x+y^2-4y+1=0$

$\Rightarrow (x+1)^2+(y-2)^2-1-4+1=0$

$\Rightarrow (x+1)^2+(y-2)^2=4$

CIRCLE

CENTRE $(-1,2)$

RADIUS 2

Question 17 (+)**

The complex numbers $z = x + \mathrm{i}y$ and $w = u + \mathrm{i}v$ are represented by the points P and Q, respectively, in separate Argand diagrams.

The two numbers are related by the equation

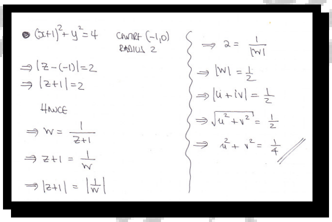
$$w = \frac{1}{z+1}, \quad z \neq -1.$$

If P is moving along the circle with equation

$$(x+1)^2 + y^2 = 4,$$

find in Cartesian form an equation of the locus of the point Q.

$$\boxed{u^2 + v^2 = \frac{1}{4}}$$



$(x+1)^2 + y^2 = 4$ CENTRE $(-1,0)$ RADIUS 2

$\Rightarrow |z-(-1)| = 2$

$\Rightarrow |z+1| = 2$

HENCE

$\Rightarrow w = \frac{1}{z+1}$

$\Rightarrow z+1 = \frac{1}{w}$

$\Rightarrow |z+1| = \left|\frac{1}{w}\right|$

$\Rightarrow 2 = \frac{1}{|w|}$

$\Rightarrow |w| = \frac{1}{2}$

$\Rightarrow |u+\mathrm{i}v| = \frac{1}{2}$

$\Rightarrow \sqrt{u^2+v^2} = \frac{1}{2}$

$\Rightarrow u^2+v^2 = \frac{1}{4}$ //

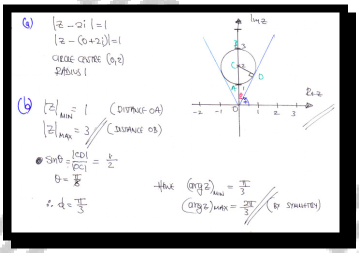
Question 18 **(**+)**

$$|z-2\mathrm{i}|=1,\ z\in\mathbb{C}.$$

a) In the Argand diagram, sketch the locus of the points that satisfy the above equation.

b) Find the minimum value and the maximum value of $|z|$, and the minimum value and the maximum of $\arg z$, for points that lie on this locus.

$$\boxed{|z|_{\text{min}}=1},\ \boxed{|z|_{\text{max}}=3},\ \boxed{\arg z_{\text{min}}=\frac{\pi}{3}},\ \boxed{\arg z_{\text{max}}=\frac{2\pi}{3}}$$


(a) $|z-2i|=1$

$|z-(0+2i)|=1$

CIRCLE CENTRE $(0,2)$
RADIUS 1

(b) $|z|_{\text{MIN}}=1$ (DISTANCE OA)

$|z|_{\text{MAX}}=3$ // (DISTANCE OB)

● $\sin\theta=\frac{|CD|}{|OC|}=\frac{1}{2}$

$\theta=\frac{\pi}{6}$

$\therefore\ \phi=\frac{\pi}{3}$

HENCE $(\arg z)_{\text{MIN}}=\frac{\pi}{3}$

$(\arg z)_{\text{MAX}}=\frac{2\pi}{3}$ // (BY SYMMETRY)

Question 19 **(**+)**

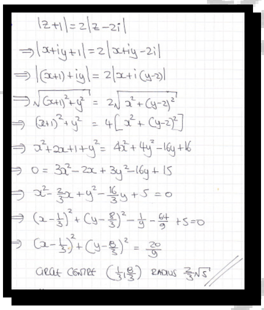
The complex number z represents the point $P(x,y)$ in the Argand diagram.

Given that

$$|z+1|=2|z-2\mathrm{i}|,$$

show that the locus of P is a circle and state its radius and the coordinates of its centre.

$$\boxed{\left(\frac{1}{3},\frac{8}{3}\right),\quad r=\frac{2}{3}\sqrt{5}}$$


$|z+1|=2|z-2i|$

$\Rightarrow |x+iy+1|=2|x+iy-2i|$

$\Rightarrow |(x+1)+iy|=2|x+i(y-2)|$

$\Rightarrow \sqrt{(x+1)^2+y^2}=2\sqrt{x^2+(y-2)^2}$

$\Rightarrow (x+1)^2+y^2=4\left[x^2+(y-2)^2\right]$

$\Rightarrow x^2+2x+1+y^2=4x^2+4y^2-16y+16$

$\Rightarrow 0=3x^2-2x+3y^2-16y+15$

$\Rightarrow x^2-\frac{2}{3}x+y^2-\frac{16}{3}y+5=0$

$\Rightarrow \left(x-\frac{1}{3}\right)^2+\left(y-\frac{8}{3}\right)^2-\frac{1}{9}-\frac{64}{9}+5=0$

$\Rightarrow \left(x-\frac{1}{3}\right)^2+\left(y-\frac{8}{3}\right)^2=\frac{20}{9}$

CIRCLE CENTRE $\left(\frac{1}{3},\frac{8}{3}\right)$ RADIUS $\frac{2}{3}\sqrt{5}$ //

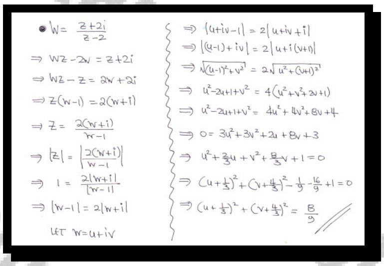
Question 20 **(**+)**

A transformation from the z plane to the w plane is defined by the equation

$$w = \frac{z+2\mathrm{i}}{z-2}, \ z \neq 2 .$$

Find in the w plane, in Cartesian form, the equation of the image of the circle with equation $|z| = 1, \ z \in \mathbb{C}$.

$$\boxed{\left(u+\frac{1}{3}\right)^2 + \left(v+\frac{4}{3}\right)^2 = \frac{8}{9}}$$



$$\bullet \ w = \frac{z+2i}{z-2}$$
$$\Rightarrow wz - 2w = z + 2i$$
$$\Rightarrow wz - z = 2w + 2i$$
$$\Rightarrow z(w-1) = 2(w+i)$$
$$\Rightarrow z = \frac{2(w+i)}{w-1}$$
$$\Rightarrow |z| = \left|\frac{2(w+i)}{w-1}\right|$$
$$\Rightarrow 1 = \frac{2|w+i|}{|w-1|}$$
$$\Rightarrow |w-1| = 2|w+i|$$

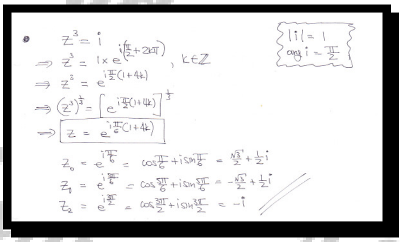
LET $w = u + iv$

$$\Rightarrow |u+iv-1| = 2|u+iv+i|$$
$$\Rightarrow |(u-1)+iv| = 2|u+i(v+1)|$$
$$\Rightarrow \sqrt{(u-1)^2+v^2} = 2\sqrt{u^2+(v+1)^2}$$
$$\Rightarrow u^2-2u+1+v^2 = 4(u^2+v^2+2v+1)$$
$$\Rightarrow u^2-2u+1+v^2 = 4u^2+4v^2+8v+4$$
$$\Rightarrow 0 = 3u^2+3v^2+2u+8v+3$$
$$\Rightarrow u^2+\tfrac{2}{3}u+v^2+\tfrac{8}{3}v+1=0$$
$$\Rightarrow (u+\tfrac{1}{3})^2+(v+\tfrac{4}{3})^2-\tfrac{1}{9}-\tfrac{16}{9}+1=0$$
$$\Rightarrow (u+\tfrac{1}{3})^2+(v+\tfrac{4}{3})^2 = \tfrac{8}{9} \ //$$

Question 21 **(**+)**

Find the cube roots of the imaginary unit i, giving the answers in the form $a + b\mathrm{i}$, where a and b are real numbers.

$$\boxed{z_1 = \frac{\sqrt{3}}{2} + \frac{1}{2}\mathrm{i}, \ \ z_2 = -\frac{\sqrt{3}}{2} + \frac{1}{2}\mathrm{i}, \ \ z_3 = -\mathrm{i}}$$



$$\bullet \ z^3 = i$$
$$\Rightarrow z^3 = 1 \times e^{i\left(\frac{\pi}{2}+2k\pi\right)}, \ k \in \mathbb{Z}$$
$$\Rightarrow z^3 = e^{i\frac{\pi}{2}(1+4k)}$$
$$\Rightarrow (z^3)^{\frac{1}{3}} = \left[e^{i\frac{\pi}{2}(1+4k)}\right]^{\frac{1}{3}}$$
$$\Rightarrow \boxed{z = e^{i\frac{\pi}{6}(1+4k)}}$$

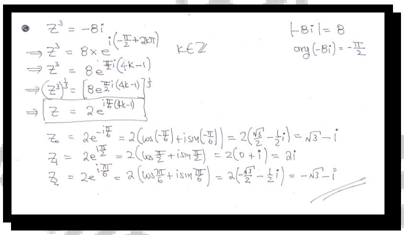
$|i| = 1$
$\arg i = \frac{\pi}{2}$

$$z_0 = e^{i\frac{\pi}{6}} = \cos\tfrac{\pi}{6} + i\sin\tfrac{\pi}{6} = \tfrac{\sqrt{3}}{2} + \tfrac{1}{2}i$$
$$z_1 = e^{i\frac{5\pi}{6}} = \cos\tfrac{5\pi}{6} + i\sin\tfrac{5\pi}{6} = -\tfrac{\sqrt{3}}{2} + \tfrac{1}{2}i$$
$$z_2 = e^{i\frac{3\pi}{2}} = \cos\tfrac{3\pi}{2} + i\sin\tfrac{3\pi}{2} = -i \ //$$

Question 22 **(**+)**

Find the cube roots of the complex number -8i, giving the answers in the form $a+b\text{i}$, where a and b are real numbers.

$$\boxed{z_1=\sqrt{3}-\text{i},\quad z_2=-\sqrt{3}-\text{i},\quad z_3=2\text{i}}$$



- $z^3=-8i$ $\qquad |-8i|=8$, $\arg(-8i)=-\frac{\pi}{2}$

$\Rightarrow z^3=8\times e^{i\left(-\frac{\pi}{2}+2k\pi\right)}$ $\quad k\in\mathbb{Z}$

$\Rightarrow z^3=8e^{\frac{\pi}{2}i(4k-1)}$

$\Rightarrow (z^3)^{\frac{1}{3}}=\left[8e^{\frac{\pi}{2}i(4k-1)}\right]^{\frac{1}{3}}$

$\Rightarrow \boxed{z=2e^{i\frac{\pi}{6}(4k-1)}}$

$z_0=2e^{-i\frac{\pi}{6}}=2\left(\cos\left(-\frac{\pi}{6}\right)+i\sin\left(-\frac{\pi}{6}\right)\right)=2\left(\frac{\sqrt{3}}{2}-\frac{1}{2}i\right)=\sqrt{3}-i$

$z_1=2e^{i\frac{\pi}{2}}=2\left(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right)=2(0+i)=2i$

$z_2=2e^{i\frac{7\pi}{6}}=2\left(\cos\frac{7\pi}{6}+i\sin\frac{7\pi}{6}\right)=2\left(-\frac{\sqrt{3}}{2}-\frac{1}{2}i\right)=-\sqrt{3}-i$

Question 23 **(**+)**

The complex number z satisfies the relationship

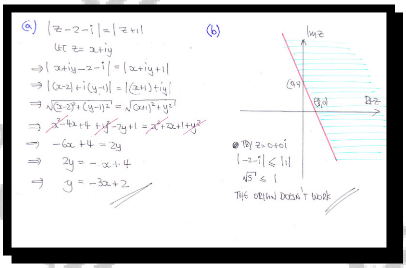
$$|z-2-\text{i}|=|z+1|.$$

a) Find a Cartesian equation for the locus of z.

b) Shade in an Argand diagram the region that satisfy the inequality

$$|z-2-\text{i}|\le|z+1|.$$

$$\boxed{y=2-3x}$$



(a) $|z-2-i|=|z+1|$

LET $z=x+iy$

$\Rightarrow |x+iy-2-i|=|x+iy+1|$

$\Rightarrow |(x-2)+i(y-1)|=|(x+1)+iy|$

$\Rightarrow \sqrt{(x-2)^2+(y-1)^2}=\sqrt{(x+1)^2+y^2}$

$\Rightarrow \cancel{x^2}-4x+4+\cancel{y^2}-2y+1=\cancel{x^2}+2x+1+\cancel{y^2}$

$\Rightarrow -6x+4=2y$

$\Rightarrow 2y=-x+4$

$\Rightarrow y=-3x+2$

(b)

- TRY $z=0+0i$

$|-2-i|\leqslant|1|$

$\sqrt{5}\leqslant 1$

THE ORIGIN DOESN'T WORK

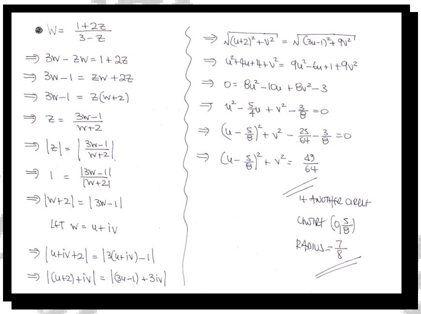
Question 24 **(**+)**

A transformation from the z plane to the w plane is given by the equation

$$w = \frac{1+2z}{3-z}, \ \ z \neq 3 .$$

Show that the in the w plane, the image of the circle with equation $|z| = 1$, $z \in \mathbb{C}$, is another circle, stating its centre and its radius .

$$\boxed{\left(u-\frac{5}{8}\right)^2 + v^2 = \frac{49}{64}}, \ \boxed{\text{centre}\left(\frac{5}{8},0\right)}, \ \boxed{r = \frac{7}{8}}$$



$\bullet \ w = \frac{1+2z}{3-z}$

$\Rightarrow 3w - zw = 1 + 2z$

$\Rightarrow 3w - 1 = zw + 2z$

$\Rightarrow 3w - 1 = z(w+2)$

$\Rightarrow z = \frac{3w-1}{w+2}$

$\Rightarrow |z| = \left|\frac{3w-1}{w+2}\right|$

$\Rightarrow 1 = \frac{|3w-1|}{|w+2|}$

$\Rightarrow |w+2| = |3w-1|$

LET $w = u + iv$

$\Rightarrow |u+iv+2| = |3(u+iv)-1|$

$\Rightarrow |(u+2)+iv| = |(3u-1)+3iv|$

$\Rightarrow \sqrt{(u+2)^2+v^2} = \sqrt{(3u-1)^2+9v^2}$

$\Rightarrow u^2+4u+4+v^2 = 9u^2-6u+1+9v^2$

$\Rightarrow 0 = 8u^2-10u+8v^2-3$

$\Rightarrow u^2 - \frac{5}{4}u + v^2 - \frac{3}{8} = 0$

$\Rightarrow \left(u-\frac{5}{8}\right)^2 + v^2 - \frac{25}{64} - \frac{3}{8} = 0$

$\Rightarrow \left(u-\frac{5}{8}\right)^2 + v^2 = \frac{49}{64}$

∴ ANOTHER CIRCLE

CENTRE $\left(0, \frac{5}{8}\right)$

RADIUS $= \frac{7}{8}$

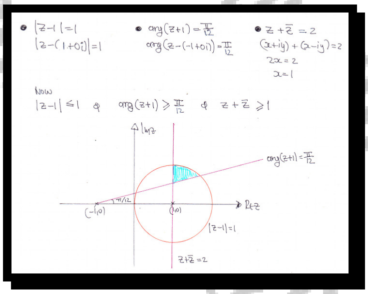
Question 25 **(**+)**

The complex number z satisfies all three relationships

$$|z-1| \leq 1, \quad \arg(z+1) \geq \frac{\pi}{12} \quad \text{and} \quad z + \bar{z} \geq 1 .$$

Shade in an Argand diagram the region of the locus of z .

$\boxed{\text{sketch}}$



$\bullet \ |z-1| = 1$

$|z-(1+0i)| = 1$

$\bullet \ \arg(z+1) = \frac{\pi}{12}$

$\arg(z-(-1+0i)) = \frac{\pi}{12}$

$\bullet \ z + \bar{z} = 2$

$(x+iy)+(x-iy) = 2$

$2x = 2$

$x = 1$

Now

$|z-1| \leq 1$ & $\arg(z+1) \geq \frac{\pi}{12}$ & $z + \bar{z} \geq 1$

Question 26 (+)**

In separate Argand diagrams, the complex numbers $z = x + iy$ and $w = u + iv$ are represented by the points P and Q, respectively.

The two numbers are related by the equation

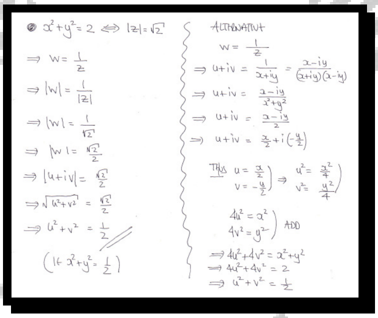
$$w = \frac{1}{z}, \quad z \neq 0 .$$

If P is moving along the circle with equation

$$x^2 + y^2 = 2 ,$$

find in Cartesian form an equation for the locus of the point Q.

$$\boxed{u^2 + v^2 = \frac{1}{2}}$$



● $x^2+y^2=2 \iff |z|=\sqrt{2}$

$\Rightarrow w=\frac{1}{z}$

$\Rightarrow |w|=\frac{1}{|z|}$

$\Rightarrow |w|=\frac{1}{\sqrt{2}}$

$\Rightarrow |w|=\frac{\sqrt{2}}{2}$

$\Rightarrow |u+iv|=\frac{\sqrt{2}}{2}$

$\Rightarrow \sqrt{u^2+v^2}=\frac{\sqrt{2}}{2}$

$\Rightarrow u^2+v^2=\frac{1}{2}$ //

$\left(1+ x^2+y^2=\frac{1}{2}\right)$

ALTERNATIVE

$w=\frac{1}{z}$

$\Rightarrow u+iv=\frac{1}{x+iy}=\frac{x-iy}{(x+iy)(x-iy)}$

$\Rightarrow u+iv=\frac{x-iy}{x^2+y^2}$

$\Rightarrow u+iv=\frac{x-iy}{2}$

$\Rightarrow u+iv=\frac{x}{2}+i\left(-\frac{y}{2}\right)$

THUS $\left.\begin{array}{l} u=\frac{x}{2} \\ v=-\frac{y}{2} \end{array}\right\} \Rightarrow \left.\begin{array}{l} u^2=\frac{x^2}{4} \\ v^2=\frac{y^2}{4} \end{array}\right\}$

$\left.\begin{array}{l} 4u^2=x^2 \\ 4v^2=y^2 \end{array}\right\}$ ADD

$\Rightarrow 4u^2+4v^2=x^2+y^2$

$\Rightarrow 4u^2+4v^2=2$

$\Rightarrow u^2+v^2=\frac{1}{2}$

Question 27 (+)**

The complex conjugate of z is denoted by $\overline{z}$.

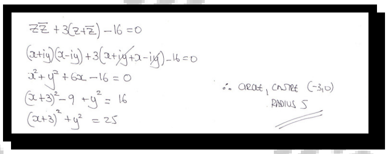
The point P represents the complex number $z = x + \mathrm{i}y$ in an Argand diagram.

Given further that

$$z\overline{z} + 3\left(z + \overline{z}\right) - 16 = 0$$

describe mathematically the locus of P.

$\boxed{\text{circle, centre at } \left(-3,0\right), \text{ radius } 5}$



$z\overline{z} + 3(z+\overline{z}) - 16 = 0$

$(x+iy)(x-iy) + 3(x+\cancel{iy}+x-\cancel{iy}) - 16 = 0$

$x^2 + y^2 + 6x - 16 = 0$

$(x+3)^2 - 9 + y^2 = 16$

$(x+3)^2 + y^2 = 25$

∴ CIRCLE, CENTRE $(-3,0)$
RADIUS 5

Question 28 (***)

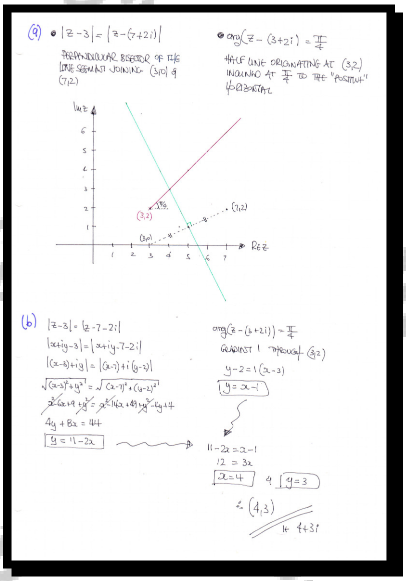
Two loci are defined in the complex plane by the relationships

$$|z-3| = |z-7-2\mathrm{i}| \quad \text{and} \quad \arg(z-3-2\mathrm{i}) = \frac{\pi}{4}.$$

a) Sketch the two loci in the same Argand diagram.

b) Determine algebraically the complex number which lies on both loci.

$\boxed{4+3\mathrm{i}}$



(a) • $|z-3| = |z-(7+2i)|$

PERPENDICULAR BISECTOR OF THE LINE SEGMENT JOINING (3,0) & (7,2)

• $\arg(z-(3+2i) = \frac{\pi}{4}$

HALF LINE ORIGINATING AT (3,2) INCLINED AT $\frac{\pi}{4}$ TO THE "POSITIVE" HORIZONTAL

(b) $|z-3| = |z-7-2i|$

$|x+iy-3| = |x+iy-7-2i|$

$|(x-3)+iy| = |(x-7)+i(y-2)|$

$\sqrt{(x-3)^2+y^2} = \sqrt{(x-7)^2+(y-2)^2}$

$\cancel{x^2}-6x+9+\cancel{y^2} = \cancel{x^2}-14x+49+\cancel{y^2}-4y+4$

$4y+8x = 44$

$\boxed{y = 11-2x}$

$\arg(z-(3+2i)) = \frac{\pi}{4}$

GRADIENT 1 THROUGH (3,2)

$y-2 = 1(x-3)$

$\boxed{y = x-1}$

$11-2x = x-1$

$12 = 3x$

$\boxed{x=4}$ & $\boxed{y=3}$

$\therefore (4,3)$

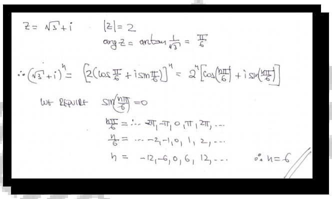
ie $4+3i$

Question 29 **(***)**

Consider the expression $\left(\sqrt{3}+\mathrm{i}\right)^n$, where n is a positive integer.

Find the smallest positive value for n so that the expression is real.

$\boxed{n=6}$



$z = \sqrt{3}+i$ $\quad |z| = 2$

$\arg z = \arctan\frac{1}{\sqrt{3}} = \frac{\pi}{6}$

$\therefore (\sqrt{3}+i)^n = \left[2\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)\right]^n = 2^n\left[\cos\left(\frac{n\pi}{6}\right)+i\sin\left(\frac{n\pi}{6}\right)\right]$

WE REQUIRE $\sin\left(\frac{n\pi}{6}\right) = 0$

$\frac{n\pi}{6} = \dots -2\pi, -\pi, 0, \pi, 2\pi, \dots$

$\frac{n}{6} = \dots -2, -1, 0, 1, 2, \dots$

$n = -12, -6, 0, 6, 12, \dots$ $\quad \therefore n = 6$

Question 30 **(***)**

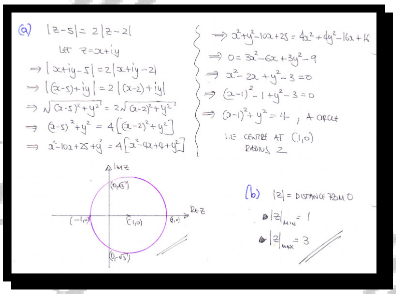
The complex number z satisfies the relationship

$$|z-5| = 2|z-2|.$$

a) Sketch in an Argand diagram the locus of z.

b) State the minimum value of $|z|$ and maximum value of $|z|$, for points which lie on this locus.

$\boxed{|z|_{\text{min}}=1}$, $\boxed{|z|_{\text{max}}=3}$



(a) $|z-5| = 2|z-2|$

LET $z = x+iy$

$\Rightarrow |x+iy-5| = 2|x+iy-2|$

$\Rightarrow |(x-5)+iy| = 2|(x-2)+iy|$

$\Rightarrow \sqrt{(x-5)^2+y^2} = 2\sqrt{(x-2)^2+y^2}$

$\Rightarrow (x-5)^2+y^2 = 4\left[(x-2)^2+y^2\right]$

$\Rightarrow x^2-10x+25+y^2 = 4\left[x^2-4x+4+y^2\right]$

$\Rightarrow x^2+y^2-10x+25 = 4x^2+4y^2-16x+16$

$\Rightarrow 0 = 3x^2-6x+3y^2-9$

$\Rightarrow x^2-2x+y^2-3=0$

$\Rightarrow (x-1)^2-1+y^2-3=0$

$\Rightarrow (x-1)^2+y^2=4$, A CIRCLE

I.E CENTRE AT $(1,0)$

RADIUS 2

(b) $|z|$ = DISTANCE FROM O

$|z|_{\text{MIN}} = 1$

$|z|_{\text{MAX}} = 3$

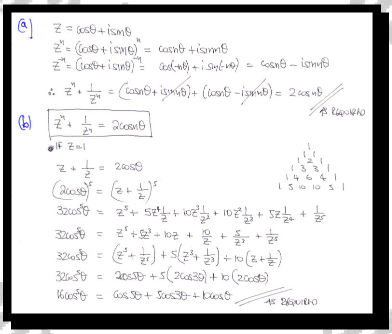
Question 31 (***)

If $z = \cos\theta + \mathrm{i}\sin\theta$, show clearly that …

a) … $z^n + \dfrac{1}{z^n} \equiv 2\cos n\theta$.

b) … $16\cos^5\theta \equiv \cos 5\theta + 5\cos 3\theta + 10\cos\theta$.

proof



(a) $Z = \cos\theta + i\sin\theta$

$Z^n = (\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$

$Z^{-n} = (\cos\theta + i\sin\theta)^{-n} = \cos(-n\theta) + i\sin(-n\theta) = \cos n\theta - i\sin n\theta$

$\therefore\ Z^n + \frac{1}{Z^n} = (\cos n\theta + \cancel{i\sin n\theta}) + (\cos n\theta - \cancel{i\sin n\theta}) = 2\cos n\theta$ // AS REQUIRED

(b) $\boxed{Z^n + \frac{1}{Z^n} = 2\cos n\theta}$

• If $Z = 1$

$Z + \frac{1}{Z} = 2\cos\theta$

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1

$(2\cos\theta)^5 = \left(Z + \frac{1}{Z}\right)^5$

$32\cos^5\theta = Z^5 + 5Z^4\frac{1}{Z} + 10Z^3\frac{1}{Z^2} + 10Z^2\frac{1}{Z^3} + 5Z\frac{1}{Z^4} + \frac{1}{Z^5}$

$32\cos^5\theta = Z^5 + 5Z^3 + 10Z + \frac{10}{Z} + \frac{5}{Z^3} + \frac{1}{Z^5}$

$32\cos^5\theta = \left(Z^5 + \frac{1}{Z^5}\right) + 5\left(Z^3 + \frac{1}{Z^3}\right) + 10\left(Z + \frac{1}{Z}\right)$

$32\cos^5\theta = 2\cos 5\theta + 5(2\cos 3\theta) + 10(2\cos\theta)$

$16\cos^5\theta = \cos 5\theta + 5\cos 3\theta + 10\cos\theta$ // AS REQUIRED

Question 32 (***)

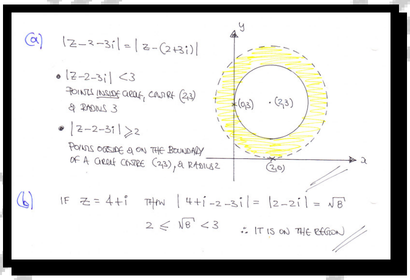
The complex number $z = x + \mathrm{i}y$ satisfies the relationship

$$2 \le |z - 2 - 3\mathrm{i}| < 3.$$

a) Shade **accurately** in an Argand diagram the region represented by the above relationship.

b) Determine algebraically whether the point that represents the number $4 + \mathrm{i}$ lies inside or outside this region.

inside the region



(a) $|Z - 2 - 3i| = |Z - (2+3i)|$

• $|Z - 2 - 3i| < 3$

POINTS INSIDE CIRCLE, CENTRE (2,3) & RADIUS 3

• $|Z - 2 - 3i| \geqslant 2$

POINTS OUTSIDE & ON THE BOUNDARY OF A CIRCLE CENTRE (2,3), & RADIUS 2

(b) IF $Z = 4 + i$ THEN $|4 + i - 2 - 3i| = |2 - 2i| = \sqrt{8}$

$2 \leqslant \sqrt{8} < 3$ ∴ IT IS ON THE REGION //

Question 33 (***)

The complex number is defined as $z = \cos\theta + \mathrm{i}\sin\theta$, $-\pi < \theta \le \pi$.

a) Show clearly that …

i. … $z^n + \dfrac{1}{z^n} = 2\cos\theta$.

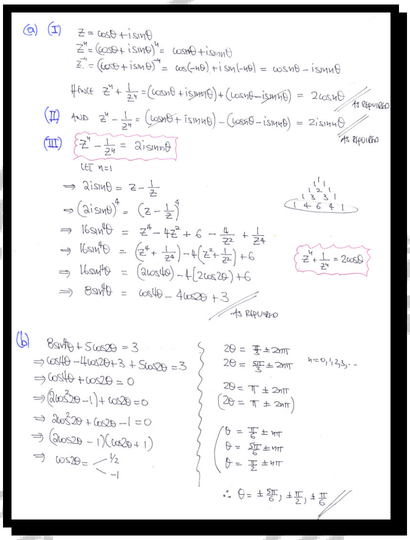
ii. … $z^n - \dfrac{1}{z^n} = 2\mathrm{i}\sin\theta$.

iii. … $8\sin^4\theta = \cos 4\theta - 4\cos 2\theta + 3$.

b) Hence solve the equation

$$8\sin^4\theta + 5\cos 2\theta = 3, \ -\pi < \theta \le \pi.$$

$$\boxed{\theta = \pm\frac{5\pi}{6}, \pm\frac{\pi}{2}, \pm\frac{\pi}{6}}$$



(a) (I) $z = \cos\theta + i\sin\theta$

$z^n = (\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$

$z^{-n} = (\cos\theta + i\sin\theta)^{-n} = \cos(-n\theta) + i\sin(-n\theta) = \cos n\theta - i\sin n\theta$

HENCE $z^n + \frac{1}{z^n} = (\cos n\theta + i\cancel{\sin n\theta}) + (\cos n\theta - i\cancel{\sin n\theta}) = 2\cos n\theta$ // AS REQUIRED

(II) AND $z^n - \frac{1}{z^n} = (\cancel{\cos n\theta} + i\sin n\theta) - (\cancel{\cos n\theta} - i\sin n\theta) = 2i\sin n\theta$ // AS REQUIRED

(III) $z^n - \frac{1}{z^n} = 2i\sin n\theta$

LET $n=1$

$\Rightarrow 2i\sin\theta = z - \frac{1}{z}$

$\Rightarrow (2i\sin\theta)^4 = \left(z - \frac{1}{z}\right)^4$

$\Rightarrow 16\sin^4\theta = z^4 - 4z^2 + 6 - \frac{4}{z^2} + \frac{1}{z^4}$

$\Rightarrow 16\sin^4\theta = \left(z^4 + \frac{1}{z^4}\right) - 4\left(z^2 + \frac{1}{z^2}\right) + 6$

$\Rightarrow 16\sin^4\theta = (2\cos 4\theta) - 4(2\cos 2\theta) + 6$

$\Rightarrow 8\sin^4\theta = \cos 4\theta - 4\cos 2\theta + 3$ // AS REQUIRED

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1

$z^n + \frac{1}{z^n} = 2\cos n\theta$

(b) $8\sin^4\theta + 5\cos 2\theta = 3$

$\Rightarrow \cos 4\theta - 4\cos 2\theta + 3 + 5\cos 2\theta = 3$

$\Rightarrow \cos 4\theta + \cos 2\theta = 0$

$\Rightarrow (2\cos^2 2\theta - 1) + \cos 2\theta = 0$

$\Rightarrow 2\cos^2 2\theta + \cos 2\theta - 1 = 0$

$\Rightarrow (2\cos 2\theta - 1)(\cos 2\theta + 1)$

$\Rightarrow \cos 2\theta = \begin{cases} 1/2 \\ -1 \end{cases}$

$2\theta = \frac{\pi}{3} \pm 2n\pi$

$2\theta = \frac{5\pi}{3} \pm 2n\pi$ $\quad n = 0, 1, 2, 3, \ldots$

$2\theta = \pi \pm 2n\pi$

$(2\theta = \pi \pm 2n\pi)$

$\theta = \frac{\pi}{6} \pm n\pi$

$\theta = \frac{5\pi}{6} \pm n\pi$

$\theta = \frac{\pi}{2} \pm n\pi$

$\therefore \theta = \pm\frac{5\pi}{6}, \pm\frac{\pi}{2}, \pm\frac{\pi}{6}$ //

Question 34 **(***)**

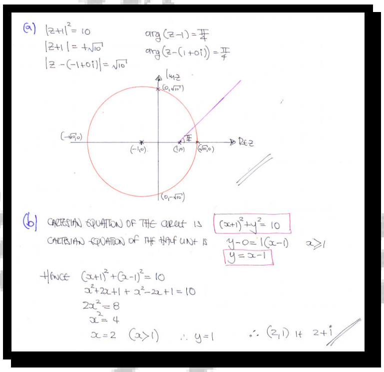
It is given that for $z \in \mathbb{C}$ the loci L_1 and L_2 have respective equations,

$$|z+1|^2 = 10 \qquad \text{and} \qquad \arg(z-1) = \frac{\pi}{4}.$$

a) Sketch L_1 and L_2 in the same Argand diagram.

b) Find the complex number that lies on both L_1 and L_2.

$\boxed{2+i}$



(a) $|z+1|^2 = 10$

$|z+1| = +\sqrt{10}$

$|z-(-1+0i)| = \sqrt{10}$

$\arg(z-1) = \frac{\pi}{4}$

$\arg(z-(1+0i)) = \frac{\pi}{4}$

(b) CARTESIAN EQUATION OF THE CIRCLE IS $\boxed{(x+1)^2+y^2 = 10}$

CARTESIAN EQUATION OF THE HALF LINE IS $y-0 = 1(x-1) \quad x \geqslant 1$

$\boxed{y = x-1}$

HENCE $(x+1)^2+(x-1)^2 = 10$

$x^2+2x+1+x^2-2x+1 = 10$

$2x^2 = 8$

$x^2 = 4$

$x = 2 \quad (x>1) \qquad \therefore y = 1 \qquad \therefore (2,1) \text{ is } 2+i$

Question 35 (***)

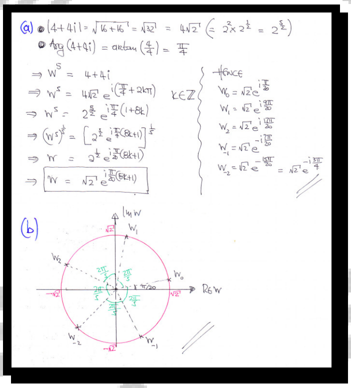
$$z = 4 + 4\text{i}\,.$$

a) Find the fifth roots of z.

Give the answers in the form $r\,\text{e}^{\text{i}\theta},\ r > 0,\ -\pi < \theta \le \pi$.

b) Plot the roots as points in an Argand diagram.

$$\boxed{\sqrt{2}\,\text{e}^{\text{i}\frac{\pi}{20}}, \sqrt{2}\,\text{e}^{\text{i}\frac{9\pi}{20}}, \sqrt{2}\,\text{e}^{\text{i}\frac{17\pi}{20}}, \sqrt{2}\,\text{e}^{-\text{i}\frac{7\pi}{20}}, \sqrt{2}\,\text{e}^{-\text{i}\frac{3\pi}{4}}}$$



(a) ● $|4+4i| = \sqrt{16+16} = \sqrt{32} = 4\sqrt{2}\ \left(= 2^2 \times 2^{\frac{1}{2}} = 2^{\frac{5}{2}}\right)$

● $\text{Arg}(4+4i) = \arctan\left(\frac{4}{4}\right) = \frac{\pi}{4}$

$\Rightarrow w^5 = 4+4i$

$\Rightarrow w^5 = 4\sqrt{2}\,e^{i\left(\frac{\pi}{4}+2k\pi\right)}\quad k \in \mathbb{Z}$

$\Rightarrow w^5 = 2^{\frac{5}{2}}\,e^{i\frac{\pi}{4}(1+8k)}$

$\Rightarrow (w^5)^{\frac{1}{5}} = \left[2^{\frac{5}{2}}\,e^{i\frac{\pi}{4}(8k+1)}\right]^{\frac{1}{5}}$

$\Rightarrow w = 2^{\frac{1}{2}}\,e^{i\frac{\pi}{20}(8k+1)}$

$\Rightarrow \boxed{w = \sqrt{2}\,e^{i\frac{\pi}{20}(8k+1)}}$

Hence

$w_0 = \sqrt{2}\,e^{i\frac{\pi}{20}}$

$w_1 = \sqrt{2}\,e^{i\frac{9\pi}{20}}$

$w_2 = \sqrt{2}\,e^{i\frac{17\pi}{20}}$

$w_{-1} = \sqrt{2}\,e^{-i\frac{7\pi}{20}}$

$w_{-2} = \sqrt{2}\,e^{-\frac{15\pi}{20}} = \sqrt{2}\,e^{-i\frac{3\pi}{4}}$

(b)

Question 36 (***)

A straight line L and a circle C are to be drawn on a standard Argand diagram.

The equation of L is $\arg z = \frac{\pi}{3}$.

The centre of C lies on L and its radius is 3 units. The line with equation $\operatorname{Im} z = 0$ is a tangent to C.

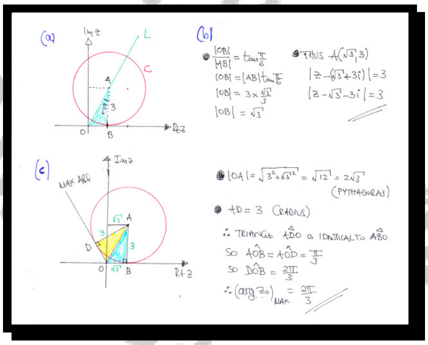
a) Sketch L and C on the same Argand diagram.

b) Determine an equation for C, giving the answer in the form $|z-\alpha| = k$, where α and k are constants.

The point that represents the complex number z_0 lies on C.

c) Determine the maximum value of $\arg z_0$, fully justifying the answer.

$\boxed{|z-\sqrt{3}-3\mathrm{i}| = 3}$, $\boxed{\arg z_0 = \frac{2\pi}{3}}$



(b)

- $\frac{|OB|}{|AB|} = \tan\frac{\pi}{6}$

$|OB| = |AB|\tan\frac{\pi}{6}$

$|OB| = 3 \times \frac{\sqrt{3}}{3}$

$|OB| = \sqrt{3}$

- THUS $A(\sqrt{3}, 3)$

$|z-(\sqrt{3}+3i)| = 3$

$|z-\sqrt{3}-3i| = 3$

- $|OA| = \sqrt{3^2+\sqrt{3}^2} = \sqrt{12} = 2\sqrt{3}$ (PYTHAGORAS)

- $AD = 3$ (RADIUS)

$\therefore$ TRIANGLE $\overset{\triangle}{ADO}$ IS IDENTICAL TO $\overset{\triangle}{ABO}$

SO $A\hat{O}B = A\hat{O}D = \frac{\pi}{3}$

SO $D\hat{O}B = \frac{2\pi}{3}$

$\therefore (\arg z_0)_{\text{MAX}} = \frac{2\pi}{3}$

Question 37 (***)

The complex numbers $z = x + \mathrm{i}y$ and $w = u + \mathrm{i}v$ are represented by the points P and Q on separate Argand diagrams.

In the z plane, the point P is tracing the line with equation $y = x$.

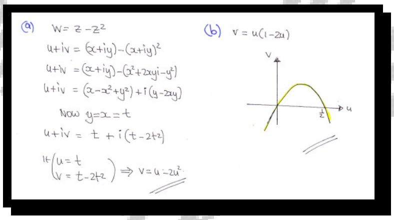
The complex numbers z and w are related by

$$w = z - z^2.$$

a) Find, in Cartesian form, the equation of the locus of Q in the w plane.

b) Sketch the locus traced by Q.

$$\boxed{v = u - 2u^2 \text{ or } y = x - 2x^2}$$



(a) $W = z - z^2$

$u + iv = (x+iy) - (x+iy)^2$

$u + iv = (x+iy) - (x^2 + 2xyi - y^2)$

$u + iv = (x - x^2 + y^2) + i(y - 2xy)$

Now $y = x = t$

$u + iv = t + i(t - 2t^2)$

$\left.\begin{matrix} u = t \\ v = t - 2t^2 \end{matrix}\right) \Rightarrow v = u - 2u^2$

(b) $v = u(1 - 2u)$

Question 38 (***)

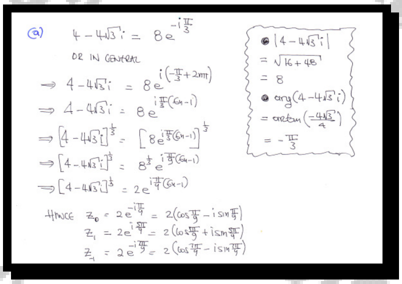
$$z = 4 - 4\sqrt{3}\mathrm{i}\,.$$

a) Find the cube roots of z.

Give the answers in polar form $r(\cos\theta + \mathrm{i}\sin\theta)$, $r > 0$, $-\pi < \theta \le \pi$.

b) Plot the roots as points in an Argand diagram.

$$\boxed{z = 2\left(\cos\frac{\pi}{9} - \mathrm{i}\sin\frac{\pi}{9}\right),\ z = 2\left(\cos\frac{5\pi}{9} + \mathrm{i}\sin\frac{5\pi}{9}\right),\ z = 2\left(\cos\frac{7\pi}{9} - \mathrm{i}\sin\frac{7\pi}{9}\right)}$$



(a) $4 - 4\sqrt{3}i = 8e^{-i\frac{\pi}{3}}$

OR IN GENERAL

$\Rightarrow 4 - 4\sqrt{3}i = 8e^{i\left(-\frac{\pi}{3} + 2n\pi\right)}$

$\Rightarrow 4 - 4\sqrt{3}i = 8e^{i\frac{\pi}{3}(6n-1)}$

$\Rightarrow \left[4 - 4\sqrt{3}i\right]^{\frac{1}{3}} = \left[8e^{i\frac{\pi}{3}(6n-1)}\right]^{\frac{1}{3}}$

$\Rightarrow \left[4 - 4\sqrt{3}i\right]^{\frac{1}{3}} = 8^{\frac{1}{3}}e^{i\frac{\pi}{9}(6n-1)}$

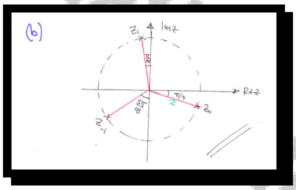
$\Rightarrow \left[4 - 4\sqrt{3}i\right]^{\frac{1}{3}} = 2e^{i\frac{\pi}{9}(6n-1)}$

HENCE $z_0 = 2e^{-i\frac{\pi}{9}} = 2\left(\cos\frac{\pi}{9} - i\sin\frac{\pi}{9}\right)$

$z_1 = 2e^{i\frac{5\pi}{9}} = 2\left(\cos\frac{5\pi}{9} + i\sin\frac{5\pi}{9}\right)$

$z_{-1} = 2e^{-i\frac{7\pi}{9}} = 2\left(\cos\frac{7\pi}{9} - i\sin\frac{7\pi}{9}\right)$

- $|4 - 4\sqrt{3}i|$
$= \sqrt{16 + 48}$
$= 8$
- $\arg(4 - 4\sqrt{3}i)$
$= \arctan\left(\frac{-4\sqrt{3}}{4}\right)$
$= -\frac{\pi}{3}$



Question 39 **(***)**

The following complex number relationships are given

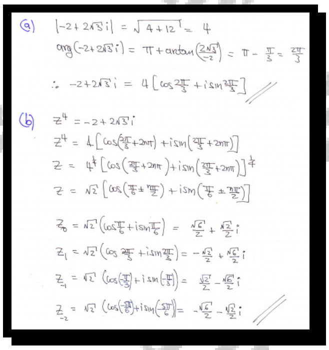
$$w = -2 + 2\sqrt{3}\mathrm{i}, \quad z^4 = w.$$

a) Express w in the form $r(\cos\theta + \mathrm{i}\sin\theta)$, where $r > 0$ and $-\pi < \theta \le \pi$.

b) Find the possible values of z, giving the answers in the form $x + \mathrm{i}y$, where x and y are real numbers.

$$\boxed{w = 2\left[\cos\left(\frac{2\pi}{3}\right) + \mathrm{i}\sin\left(\frac{2\pi}{3}\right)\right]},$$

$$\boxed{z = \tfrac{1}{2}\left(\sqrt{6} + \mathrm{i}\sqrt{2}\right),\ z = \tfrac{1}{2}\left(-\sqrt{2} + \mathrm{i}\sqrt{6}\right),\ z = \tfrac{1}{2}\left(\sqrt{2} - \mathrm{i}\sqrt{6}\right),\ z = \tfrac{1}{2}\left(-\sqrt{6} - \mathrm{i}\sqrt{2}\right)}$$



(a) $|-2+2\sqrt{3}i| = \sqrt{4+12} = 4$

$\arg(-2+2\sqrt{3}i) = \pi + \arctan\left(\frac{2\sqrt{3}}{-2}\right) = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$

$\therefore\ -2+2\sqrt{3}i = 4\left[\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right]$ //

(b) $z^4 = -2+2\sqrt{3}i$

$z^4 = 4\left[\cos\left(\frac{2\pi}{3}+2n\pi\right) + i\sin\left(\frac{2\pi}{3}+2n\pi\right)\right]$

$z = 4^{\frac{1}{4}}\left[\cos\left(\frac{2\pi}{3}+2n\pi\right) + i\sin\left(\frac{2\pi}{3}+2n\pi\right)\right]^{\frac{1}{4}}$

$z = \sqrt{2}\left[\cos\left(\frac{\pi}{6} \pm \frac{n\pi}{2}\right) + i\sin\left(\frac{\pi}{6} \pm \frac{n\pi}{2}\right)\right]$

$z_0 = \sqrt{2}\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right) = \frac{\sqrt{6}}{2} + \frac{\sqrt{2}}{2}i$

$z_1 = \sqrt{2}\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right) = -\frac{\sqrt{2}}{2} + \frac{\sqrt{6}}{2}i$

$z_{-1} = \sqrt{2}\left(\cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)\right) = \frac{\sqrt{2}}{2} - \frac{\sqrt{6}}{2}i$

$z_{-2} = \sqrt{2}\left(\cos\left(-\frac{5\pi}{6}\right) + i\sin\left(-\frac{5\pi}{6}\right)\right) = -\frac{\sqrt{6}}{2} - \frac{\sqrt{2}}{2}i$ //

Question 40 (***)

Two sets of loci in the Argand diagram are given by the following equations

$$|z| = |z+2| \quad \text{and} \quad |z| = 2, \quad z \in \mathbb{C}.$$

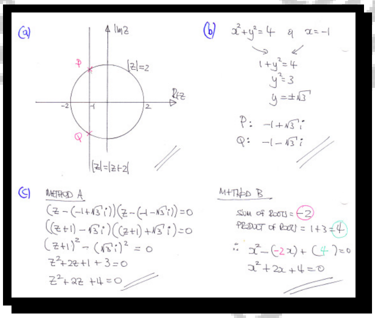
a) Sketch both these loci in the same Argand diagram.

The points P and Q in the Argand diagram satisfy both loci equations.

b) Write the complex numbers represented by P and Q, in the form $a+ib$, where a and b are real numbers.

c) Find a quadratic equation with real coefficients, whose solutions are the complex numbers represented by the points P and Q.

$\boxed{z=-1\pm\sqrt{3}}$, $\boxed{z^2+2z+4=0}$



(b) $x^2+y^2=4$ & $x=-1$

$1+y^2=4$

$y^2=3$

$y=\pm\sqrt{3}$

P: $-1+\sqrt{3}i$

Q: $-1-\sqrt{3}i$

(c) METHOD A

$(z-(-1+\sqrt{3}i))(z-(-1-\sqrt{3}i))=0$

$((z+1)-\sqrt{3}i)((z+1)+\sqrt{3}i)=0$

$(z+1)^2-(\sqrt{3}i)^2=0$

$z^2+2z+1+3=0$

$z^2+2z+4=0$

METHOD B

SUM OF ROOTS = -2

PRODUCT OF ROOTS = $1+3=4$

$\therefore\ x^2-(-2x)+(4)=0$

$x^2+2x+4=0$

Question 41 (***)

The complex numbers $z = x + \mathrm{i}y$ and $w = u + \mathrm{i}v$ are represented by the points P and Q on separate Argand diagrams.

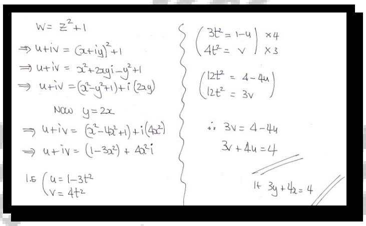
In the z plane, the point P is tracing the line with equation $y = 2x$.

Given that he complex numbers z and w are related by

$$w = z^2 + 1$$

find, in Cartesian form, the locus of Q in the w plane.

$$\boxed{4u + 3v = 4 \ \text{ or } \ 4x + 3y = 4}$$



$w = z^2 + 1$

$\Rightarrow u + iv = (x+iy)^2 + 1$

$\Rightarrow u + iv = x^2 + 2xyi - y^2 + 1$

$\Rightarrow u + iv = (x^2 - y^2 + 1) + i(2xy)$

NOW $y = 2x$

$\Rightarrow u + iv = (x^2 - 4x^2 + 1) + i(4x^2)$

$\Rightarrow u + iv = (1 - 3x^2) + 4x^2 i$

I.E $\begin{pmatrix} u = 1 - 3t^2 \\ v = 4t^2 \end{pmatrix}$

$\begin{pmatrix} 3t^2 = 1 - u \\ 4t^2 = v \end{pmatrix} \begin{matrix} \times 4 \\ \times 3 \end{matrix}$

$\begin{pmatrix} 12t^2 = 4 - 4u \\ 12t^2 = 3v \end{pmatrix}$

$\therefore 3v = 4 - 4u$

$3v + 4u = 4$

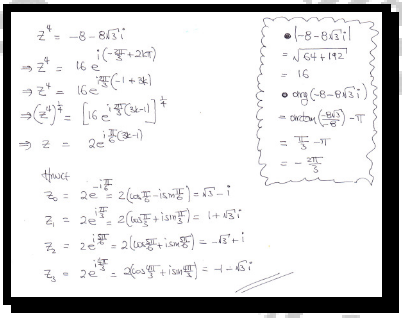
IE $3y + 4x = 4$

Question 42 **(***)**

$$z^4 = -8 - 8\sqrt{3}\,\mathrm{i}\,,\ z \in \mathbb{C}\,.$$

Solve the above equation, giving the answers in the form $a + b\mathrm{i}$, where a and b are real numbers.

$\boxed{z = \sqrt{3} - \mathrm{i}}$, $\boxed{z = 1 + \sqrt{3}\,\mathrm{i}}$, $\boxed{z = -\sqrt{3} + \mathrm{i}}$, $\boxed{z = -1 - \sqrt{3}\,\mathrm{i}}$



$z^4 = -8 - 8\sqrt{3}i$

$\Rightarrow z^4 = 16e^{i(-\frac{2\pi}{3}+2k\pi)}$

$\Rightarrow z^4 = 16e^{i\frac{2\pi}{3}(-1+3k)}$

$\Rightarrow (z^4)^{\frac{1}{4}} = \left[16e^{i\frac{2\pi}{3}(3k-1)}\right]^{\frac{1}{4}}$

$\Rightarrow z = 2e^{i\frac{\pi}{6}(3k-1)}$

Hence

$z_0 = 2e^{-i\frac{\pi}{6}} = 2(\cos\frac{\pi}{6} - i\sin\frac{\pi}{6}) = \sqrt{3} - i$

$z_1 = 2e^{i\frac{\pi}{3}} = 2(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}) = 1 + \sqrt{3}i$

$z_2 = 2e^{i\frac{5\pi}{6}} = 2(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}) = -\sqrt{3} + i$

$z_3 = 2e^{i\frac{4\pi}{3}} = 2(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}) = -1 - \sqrt{3}i$

- $|-8-8\sqrt{3}i| = \sqrt{64+192} = 16$
- $\arg(-8-8\sqrt{3}i) = \arctan\left(\frac{-8\sqrt{3}}{-8}\right) - \pi = \frac{\pi}{3} - \pi = -\frac{2\pi}{3}$

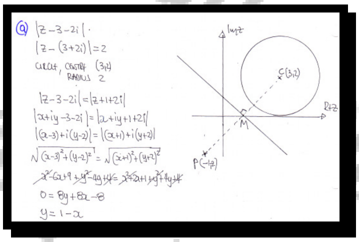
Question 43 **(***)**

a) Sketch in the same Argand diagram the locus of the points satisfying each of the following equations

i. $|z - 3 - 2\mathrm{i}| = 2$.

ii. $|z - 3 - 2\mathrm{i}| = |z + 1 + 2\mathrm{i}|$.

b) Show by a **geometric** calculation that no points lie on both loci.



ⓐ $|z-3-2i|$ ·

$|z-(3+2i)| = 2$

CIRCLE, CENTRE (3,2)
RADIUS 2

$|z-3-2i| = |z+1+2i|$

$|x+iy-3-2i| = |x+iy+1+2i|$

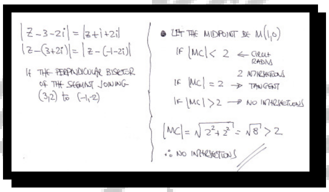
$|(x-3)+i(y-2)| = |(x+1)+i(y+2)|$

$\sqrt{(x-3)^2+(y-2)^2} = \sqrt{(x+1)^2+(y+2)^2}$

$\cancel{x^2}-6x+9+\cancel{y^2}-4y+\cancel{4} = \cancel{x^2}+2x+1+\cancel{y^2}+4y+\cancel{4}$

$0 = 8y+8x-8$

$y = 1-x$



$|z-3-2i| = |z+1+2i|$

$|z-(3+2i)| = |z-(-1-2i)|$

IS THE PERPENDICULAR BISECTOR OF THE SEGMENT JOINING (3,2) to (-1,-2)

- LET THE MIDPOINT BE M(1,0)

IF $|MC| < 2$ ← CIRCLE RADIUS
2 INTERSECTIONS

IF $|MC| = 2$ → TANGENT

IF $|MC| > 2$ → NO INTERSECTIONS

$|MC| = \sqrt{2^2+2^2} = \sqrt{8} > 2$

∴ NO INTERSECTIONS

Question 44 **(***)**

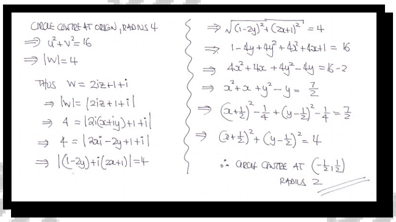
A circle C_1 in the z plane is mapped onto another circle C_2 in the w plane.

The mapping is defined by the relationship

$$w = 2\mathrm{i}z + 1 + \mathrm{i}\ .$$

Given C_2 has its centre at the origin and its radius is 4, find the coordinates of the centre of C_1 and the size of its radius.

$\boxed{\left(-\frac{1}{2}, \frac{1}{2}\right)}$, $\boxed{r = 2}$



CIRCLE CENTRE AT ORIGIN, RADIUS 4

$\Rightarrow u^2 + v^2 = 16$

$\Rightarrow |w| = 4$

THUS $w = 2iz + 1 + i$

$\Rightarrow |w| = |2iz + 1 + i|$

$\Rightarrow 4 = |2i(x+iy) + 1 + i|$

$\Rightarrow 4 = |2xi - 2y + 1 + i|$

$\Rightarrow |(1-2y) + i(2x+1)| = 4$

$\Rightarrow \sqrt{(1-2y)^2 + (2x+1)^2} = 4$

$\Rightarrow 1 - 4y + 4y^2 + 4x^2 + 4x + 1 = 16$

$\Rightarrow 4x^2 + 4x + 4y^2 - 4y = 16 - 2$

$\Rightarrow x^2 + x + y^2 - y = \frac{7}{2}$

$\Rightarrow \left(x + \frac{1}{2}\right)^2 - \frac{1}{4} + \left(y - \frac{1}{2}\right)^2 - \frac{1}{4} = \frac{7}{2}$

$\Rightarrow \left(x + \frac{1}{2}\right)^2 + \left(y - \frac{1}{2}\right)^2 = 4$

$\therefore$ CIRCLE CENTRE AT $\left(-\frac{1}{2}, \frac{1}{2}\right)$

RADIUS 2

Question 45 (***)

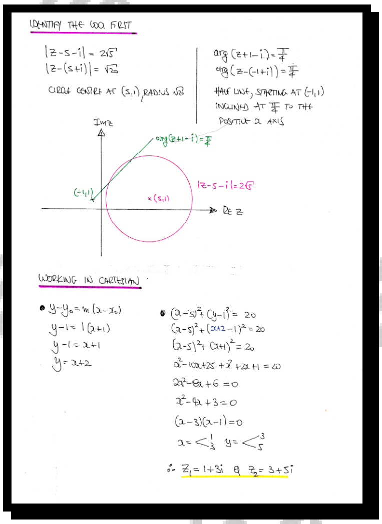
Sketch on a single Argand diagram the locus of the points z which satisfy

$$|z-5-\mathrm{i}| = 2\sqrt{5} \quad \text{and} \quad \arg(z+1-\mathrm{i}) = \tfrac{1}{4}\pi,$$

and hence find the complex numbers which lie on both of these loci.

No credit will be given to solutions based on a scale drawing.

$\boxed{\text{[illegible]}}$, $\boxed{z_1 = 1+3\mathrm{i}}$, $\boxed{z_2 = 3+5\mathrm{i}}$



IDENTIFY THE LOCI FIRST

$|z-5-i| = 2\sqrt{5}$

$|z-(5+i)| = \sqrt{20}$

CIRCLE CENTRE AT $(5,1)$, RADIUS $\sqrt{20}$

$\arg(z+1-i) = \frac{\pi}{4}$

$\arg(z-(-1+i)) = \frac{\pi}{4}$

HALF LINE, STARTING AT $(-1,1)$ INCLINED AT $\frac{\pi}{4}$ TO THE POSITIVE x AXIS

WORKING IN CARTESIAN

- $y-y_0 = m(x-x_0)$
  $y-1 = 1(x+1)$
  $y-1 = x+1$
  $y = x+2$

- $(x-5)^2+(y-1)^2 = 20$
  $(x-5)^2+(x+2-1)^2 = 20$
  $(x-5)^2+(x+1)^2 = 20$
  $x^2-10x+25+x^2+2x+1 = 20$
  $2x^2-8x+6 = 0$
  $x^2-4x+3 = 0$
  $(x-3)(x-1) = 0$
  $x = <^{1}_{3} \quad y = <^{3}_{5}$

$\therefore z_1 = 1+3i$ & $z_2 = 3+5i$

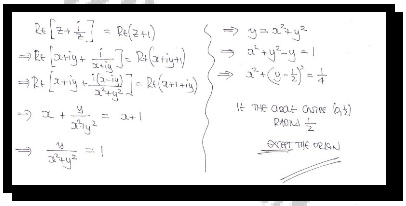
Question 46 (***)

The point P represents the complex number $z = x + \mathrm{i}y$ in an Argand diagram and satisfies the relationship

$$\operatorname{Re}\left(z + \frac{\mathrm{i}}{z}\right) = \operatorname{Re}(z+1), \quad z \neq 0 .$$

Describe mathematically the locus that P is tracing in the Argand diagram.

$\boxed{\text{circle, centre at } \left(0,\frac{1}{2}\right), \text{ radius } \frac{1}{2}, \text{ except the origin}}$



$\operatorname{Re}\left[z+\frac{i}{z}\right] = \operatorname{Re}(z+1)$

$\Rightarrow \operatorname{Re}\left[x+iy+\frac{i}{x+iy}\right] = \operatorname{Re}(x+iy+1)$

$\Rightarrow \operatorname{Re}\left[x+iy+\frac{i(x-iy)}{x^2+y^2}\right] = \operatorname{Re}(x+1+iy)$

$\Rightarrow x + \frac{y}{x^2+y^2} = x+1$

$\Rightarrow \frac{y}{x^2+y^2} = 1$

$\Rightarrow y = x^2+y^2$

$\Rightarrow x^2+y^2-y = 1$

$\Rightarrow x^2+\left(y-\frac{1}{2}\right)^2 = \frac{1}{4}$

IE THE CIRCLE CENTRE $(0,\frac{1}{2})$ RADIUS $\frac{1}{2}$

EXCEPT THE ORIGIN

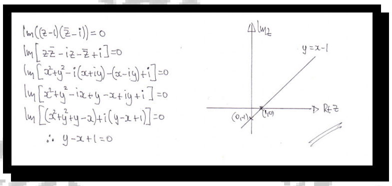
Question 47 (***)

The complex conjugate of z is denoted by $\overline{z}$.

The point P represents the complex number $z = x + \mathrm{i}y$ in an Argand diagram.

Given that $(z-1)(\overline{z}-\mathrm{i})$ is always real, sketch the locus of P.

$\boxed{y = x - 1}$



$\operatorname{Im}((z-1)(\bar{z}-i)) = 0$

$\operatorname{Im}[z\bar{z} - iz - \bar{z} + i] = 0$

$\operatorname{Im}[x^2+y^2 - i(x+iy) - (x-iy) + i] = 0$

$\operatorname{Im}[x^2+y^2 - ix + y - x + iy + i] = 0$

$\operatorname{Im}[(x^2+y^2+y-x) + i(y-x+1)] = 0$

$\therefore y - x + 1 = 0$

Question 48 **(***)**

The complex number z satisfies the equation

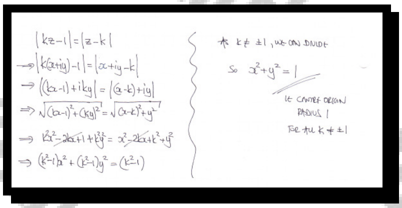
$$|kz-1| = |z-k|,$$

where k is a real constant such that $k \neq \pm 1$.

Show that for all the allowable values of the constant k, the point represented by z in an Argand diagram traces the circle with Cartesian equation

$$x^2 + y^2 = 1.$$

$\boxed{\text{proof}}$



$|kz-1| = |z-k|$

$\Rightarrow |k(x+iy)-1| = |x+iy-k|$

$\Rightarrow |(kx-1)+iky| = |(x-k)+iy|$

$\Rightarrow \sqrt{(kx-1)^2+(ky)^2} = \sqrt{(x-k)^2+y^2}$

$\Rightarrow k^2x^2 \cancel{-2kx}+1+k^2y^2 = x^2 \cancel{-2kx}+k^2+y^2$

$\Rightarrow (k^2-1)x^2+(k^2-1)y^2 = (k^2-1)$

As $k \neq \pm 1$, we can divide

So $x^2+y^2=1$

ie centre origin radius 1 for all $k \neq \pm 1$

Question 49 (***+)

It is given that

$$\sin 5\theta \equiv 16\sin^5\theta - 20\sin^3\theta + 5\sin\theta .$$

a) Use de Moivre’s theorem to prove the validity of the above trigonometric identity.

It is further given that

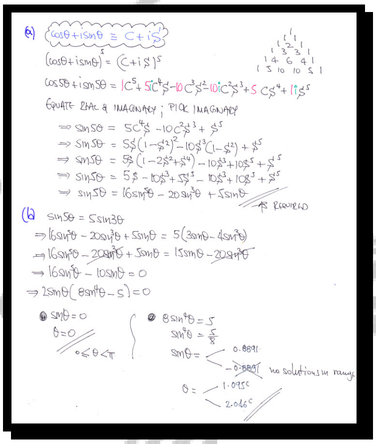
$$\sin 3\theta \equiv 3\sin\theta - 4\sin^3\theta .$$

b) Solve the equation

$$\sin 5\theta = 5\sin 3\theta \quad \text{for} \quad 0 \le \theta < \pi ,$$

giving the solutions correct to 3 decimal places.

$$\boxed{\theta = 0,\ \ 1.095^{\text{c}},\ \ 2.046^{\text{c}}}$$



(a) $\cos\theta + i\sin\theta \equiv C + iS$

$$\begin{array}{c} 1 \\ 1\ \ 1 \\ 1\ \ 2\ \ 1 \\ 1\ \ 3\ \ 3\ \ 1 \\ 1\ \ 4\ \ 6\ \ 4\ \ 1 \\ 1\ \ 5\ \ 10\ \ 10\ \ 5\ \ 1 \end{array}$$

$(\cos\theta + i\sin\theta)^5 = (C+iS)^5$

$\cos 5\theta + i\sin 5\theta = 1C^5 + 5iC^4S - 10C^3S^2 - 10iC^2S^3 + 5CS^4 + 1iS^5$

EQUATE REAL & IMAGINARY; PICK IMAGINARY

$\Rightarrow \sin 5\theta = 5C^4S - 10C^2S^3 + S^5$

$\Rightarrow \sin 5\theta = 5S(1-S^2)^2 - 10S^3(1-S^2) + S^5$

$\Rightarrow \sin 5\theta = 5S(1-2S^2+S^4) - 10S^3 + 10S^5 + S^5$

$\Rightarrow \sin 5\theta = 5S - 10S^3 + 5S^5 - 10S^3 + 10S^5 + S^5$

$\Rightarrow \sin 5\theta = 16\sin^5\theta - 20\sin^3\theta + 5\sin\theta$ // AS REQUIRED

(b) $\sin 5\theta = 5\sin 3\theta$

$\Rightarrow 16\sin^5\theta - 20\sin^3\theta + 5\sin\theta = 5(3\sin\theta - 4\sin^3\theta)$

$\Rightarrow 16\sin^5\theta - \cancel{20\sin^3\theta} + 5\sin\theta = 15\sin\theta - \cancel{20\sin^3\theta}$

$\Rightarrow 16\sin^5\theta - 10\sin\theta = 0$

$\Rightarrow 2\sin\theta(8\sin^4\theta - 5) = 0$

- $\sin\theta = 0$
  $\theta = 0$ //
  $0 \le \theta < \pi$

- $8\sin^4\theta = 5$
  $\sin^4\theta = \frac{5}{8}$
  $\sin\theta = \begin{cases} 0.8891 \\ \cancel{-0.8891} \text{ no solutions in range} \end{cases}$
  $\theta = \begin{cases} 1.095^c \\ 2.046^c \end{cases}$ //

Question 50 (***)

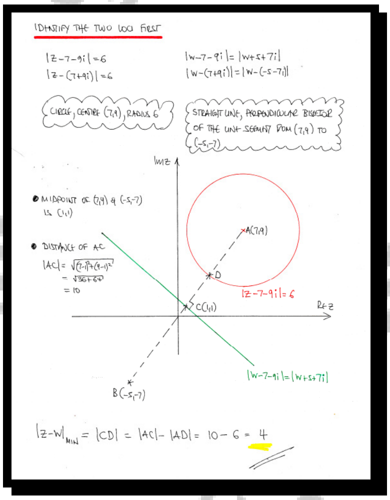
Sketch on a single Argand diagram the locus of the points z and w which satisfy

$$|z-7-9\mathrm{i}|=6 \quad \text{and} \quad |w-7-9\mathrm{i}|=|w+5+7\mathrm{i}|,$$

and hence find minimum value for $|z-w|$.

No credit will be given to solutions based on a scale drawing.

$\boxed{}$, $\boxed{|z-w|_{\min}=4}$



IDENTIFY THE TWO LOCI FIRST

$|z-7-9i|=6$
$|z-(7+9i)|=6$

CIRCLE, CENTRE (7,9), RADIUS 6

$|w-7-9i|=|w+5+7i|$
$|w-(7+9i)|=|w-(-5-7i)|$

STRAIGHT LINE, PERPENDICULAR BISECTOR OF THE LINE SEGMENT FROM (7,9) TO (-5,-7)

- MIDPOINT OF (7,9) & (-5,-7) IS (1,1)
- DISTANCE OF AC

$|AC|=\sqrt{(7-1)^2+(9-1)^2}$
$=\sqrt{36+64}$
$=10$

$|z-w|_{\text{MIN}}=|CD|=|AC|-|AD|=10-6=4$

Question 51 (*+)**

The complex number z is defined as

$$z = e^{i\theta}, \ -\pi < \theta \le \pi .$$

a) Show clearly that …

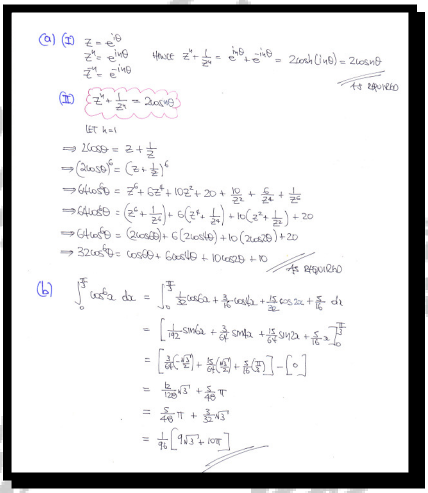
i. … $z^n + \dfrac{1}{z^n} = 2\cos\theta$.

ii. … $32\cos^6\theta = \cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10$.

b) Hence find an exact value for the integral

$$\int_0^{\frac{\pi}{3}} \cos^6 x \ dx .$$

$$\boxed{\frac{1}{96}\left(10\pi + 9\sqrt{3}\right)}$$



(a) (I) $z = e^{i\theta}$

$z^n = e^{in\theta}$ HENCE $z^n + \frac{1}{z^n} = e^{in\theta} + e^{-in\theta} = 2\cosh(in\theta) = 2\cos n\theta$

$z^{-n} = e^{-in\theta}$

AS REQUIRED

(II) $z^n + \frac{1}{z^n} = 2\cos n\theta$

LET $n=1$

$\Rightarrow 2\cos\theta = z + \frac{1}{z}$

$\Rightarrow (2\cos\theta)^6 = \left(z + \frac{1}{z}\right)^6$

$\Rightarrow 64\cos^6\theta = z^6 + 6z^4 + 10z^2 + 20 + \frac{10}{z^2} + \frac{6}{z^4} + \frac{1}{z^6}$

$\Rightarrow 64\cos^6\theta = \left(z^6 + \frac{1}{z^6}\right) + 6\left(z^4 + \frac{1}{z^4}\right) + 10\left(z^2 + \frac{1}{z^2}\right) + 20$

$\Rightarrow 64\cos^6\theta = (2\cos6\theta) + 6(2\cos4\theta) + 10(2\cos2\theta) + 20$

$\Rightarrow 32\cos^6\theta = \cos6\theta + 6\cos4\theta + 10\cos2\theta + 10$

AS REQUIRED

(b) $\int_0^{\frac{\pi}{3}} \cos^6 x \, dx = \int_0^{\frac{\pi}{3}} \frac{1}{32}\cos6x + \frac{3}{16}\cos4x + \frac{15}{32}\cos2x + \frac{5}{16} \, dx$

$= \left[\frac{1}{192}\sin6x + \frac{3}{64}\sin4x + \frac{15}{64}\sin2x + \frac{5}{16}x\right]_0^{\frac{\pi}{3}}$

$= \left[\frac{3}{64}\left(-\frac{\sqrt{3}}{2}\right) + \frac{15}{64}\left(\frac{\sqrt{3}}{2}\right) + \frac{5}{16}\left(\frac{\pi}{3}\right)\right] - [0]$

$= \frac{12}{128}\sqrt{3} + \frac{5}{48}\pi$

$= \frac{5}{48}\pi + \frac{3}{32}\sqrt{3}$

$= \frac{1}{96}\left[9\sqrt{3} + 10\pi\right]$

Question 52 (***+)

$$z_1 = 2\cos\frac{\pi}{4} + \mathrm{i}\sin\frac{\pi}{4}.$$

a) Verify that z_1 is a solution of the equation

$$z^4 + 16 = 0,$$

and plot the four roots of the equation in an Argand diagram.

b) Find the values of the real constants a and b so that

$$(z - z_1)(z - \overline{z_1}) \equiv z^2 + az + b,$$

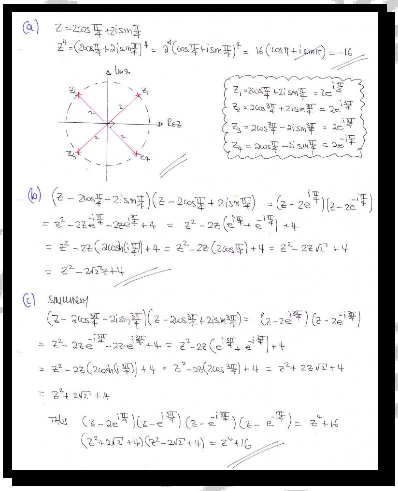
where $\overline{z_1}$ denotes the complex conjugate of z_1.

c) Hence show that

$$z^4 + 16 \equiv \left(z^2 + az + b\right)\left(z^2 + cz + d\right),$$

for some real constants c and d.

$\boxed{a = -2\sqrt{2}}$, $\boxed{b = 4}$, $\boxed{c = 2\sqrt{2}}$, $\boxed{d = 4}$,



(a) $z = 2\cos\frac{\pi}{4} + 2i\sin\frac{\pi}{4}$

$z^4 = \left(2\cos\frac{\pi}{4} + 2i\sin\frac{\pi}{4}\right)^4 = 2^4\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)^4 = 16(\cos\pi + i\sin\pi) = -16$

$z_1 = 2\cos\frac{\pi}{4} + 2i\sin\frac{\pi}{4} = 2e^{i\frac{\pi}{4}}$

$z_2 = 2\cos\frac{3\pi}{4} + 2i\sin\frac{3\pi}{4} = 2e^{i\frac{3\pi}{4}}$

$z_3 = 2\cos\frac{3\pi}{4} - 2i\sin\frac{3\pi}{4} = 2e^{-i\frac{3\pi}{4}}$

$z_4 = 2\cos\frac{\pi}{4} - 2i\sin\frac{\pi}{4} = 2e^{-i\frac{\pi}{4}}$

(b) $\left(z - 2\cos\frac{\pi}{4} - 2i\sin\frac{\pi}{4}\right)\left(z - 2\cos\frac{\pi}{4} + 2i\sin\frac{\pi}{4}\right) = \left(z - 2e^{i\frac{\pi}{4}}\right)\left(z - 2e^{-i\frac{\pi}{4}}\right)$

$= z^2 - 2ze^{-i\frac{\pi}{4}} - 2ze^{i\frac{\pi}{4}} + 4 = z^2 - 2z\left(e^{i\frac{\pi}{4}} + e^{-i\frac{\pi}{4}}\right) + 4$

$= z^2 - 2z\left(2\cosh\left(i\frac{\pi}{4}\right)\right) + 4 = z^2 - 2z\left(2\cos\frac{\pi}{4}\right) + 4 = z^2 - 2z\sqrt{2} + 4$

$= z^2 - 2\sqrt{2}z + 4$

(c) SIMILARLY

$\left(z - 2\cos\frac{3\pi}{4} - 2i\sin\frac{3\pi}{4}\right)\left(z - 2\cos\frac{3\pi}{4} + 2i\sin\frac{3\pi}{4}\right) = \left(z - 2e^{i\frac{3\pi}{4}}\right)\left(z - 2e^{-i\frac{3\pi}{4}}\right)$

$= z^2 - 2ze^{-i\frac{3\pi}{4}} - 2ze^{i\frac{3\pi}{4}} + 4 = z^2 - 2z\left(e^{i\frac{3\pi}{4}} + e^{-i\frac{3\pi}{4}}\right) + 4$

$= z^2 - 2z\left(2\cosh\left(i\frac{3\pi}{4}\right)\right) + 4 = z^2 - 2z\left(2\cos\frac{3\pi}{4}\right) + 4 = z^2 + 2z\sqrt{2} + 4$

$= z^2 + 2\sqrt{2} + 4$

THUS $\left(z - 2e^{i\frac{\pi}{4}}\right)\left(z - e^{i\frac{3\pi}{4}}\right)\left(z - e^{-i\frac{3\pi}{4}}\right)\left(z - e^{-i\frac{\pi}{4}}\right) = z^4 + 16$

$\left(z^2 + 2\sqrt{2} + 4\right)\left(z^2 - 2\sqrt{2} + 4\right) = z^4 + 16$

Question 53 **(***+)**

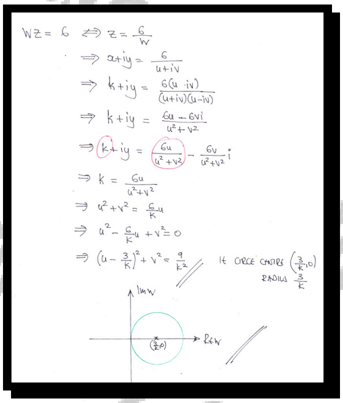
A transformation T maps points of the form $z = x + \text{i}y$ from the z plane onto points of the form $v = u + \text{i}v$ in the w plane, and is defined as

$$zw = 6 \ , \ z \neq 0 .$$

The line with equation $x = k$, $k \in \mathbb{R}$, is mapped by T onto a circle C in the w plane.

Determine a Cartesian equation for C and sketch its graph in an Argand diagram.

$$\boxed{\left(u - \frac{3}{k}\right)^2 + v^2 = \frac{9}{k^2}}$$



$$WZ = 6 \iff z = \frac{6}{W}$$

$$\Rightarrow x+iy = \frac{6}{u+iv}$$

$$\Rightarrow k+iy = \frac{6(u \ iv)}{(u+iv)(u-iv)}$$

$$\Rightarrow k+iy = \frac{6u - 6vi}{u^2+v^2}$$

$$\Rightarrow k+iy = \frac{6u}{u^2+v^2} - \frac{6v}{u^2+v^2}i$$

$$\Rightarrow k = \frac{6u}{u^2+v^2}$$

$$\Rightarrow u^2+v^2 = \frac{6}{k}u$$

$$\Rightarrow u^2 - \frac{6}{k}u + v^2 = 0$$

$$\Rightarrow \left(u - \frac{3}{k}\right)^2 + v^2 = \frac{9}{k^2} \ //$$

IE CIRCLE CENTRE $\left(\frac{3}{k}, 0\right)$

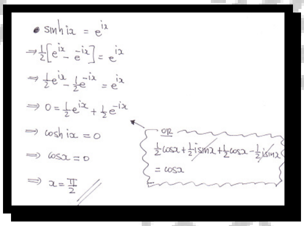
RADIUS $\frac{3}{k}$

Question 54 **(***+)**

Find a solution for the following equation

$$\sinh(\text{i}x) = \text{e}^{\text{i}x}, \ x \in \mathbb{R} .$$

$$\boxed{x = \frac{\pi}{2}}$$



- $\sinh ix = e^{ix}$

$$\Rightarrow \frac{1}{2}\left[e^{ix} - e^{-ix}\right] = e^{ix}$$

$$\Rightarrow \frac{1}{2}e^{ix} - \frac{1}{2}e^{-ix} = e^{ix}$$

$$\Rightarrow 0 = \frac{1}{2}e^{ix} + \frac{1}{2}e^{-ix}$$

$$\Rightarrow \cosh ix = 0$$

$$\Rightarrow \cos x = 0$$

$$\Rightarrow x = \frac{\pi}{2} \ //$$

OR

$$\frac{1}{2}\cos x + \cancel{\frac{1}{2}i\sin x} + \frac{1}{2}\cos x - \cancel{\frac{1}{2}i\sin x}$$

$$= \cos x$$

Question 55 **(***+)**

Sketch on a standard Argand diagram the locus of the points z which satisfy

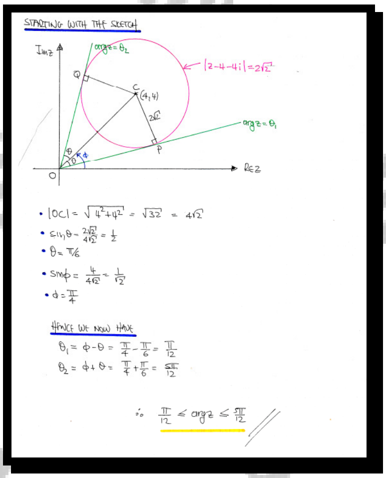
$$|z-4-4i| = 2\sqrt{2},$$

and use it to prove that

$$\frac{1}{12}\pi \le \arg z \le \frac{5}{12}\pi .$$

No credit will be given to solutions based on a scale drawings.

$\boxed{}$, $\boxed{\text{proof}}$



STARTING WITH THE SKETCH

- $|OC| = \sqrt{4^2+4^2} = \sqrt{32} = 4\sqrt{2}$
- $\sin\theta = \frac{2\sqrt{2}}{4\sqrt{2}} = \frac{1}{2}$
- $\theta = \pi/6$
- $\sin\phi = \frac{4}{4\sqrt{2}} = \frac{1}{\sqrt{2}}$
- $\phi = \frac{\pi}{4}$

HENCE WE NOW HAVE

$\theta_1 = \phi - \theta = \frac{\pi}{4} - \frac{\pi}{6} = \frac{\pi}{12}$

$\theta_2 = \phi + \theta = \frac{\pi}{4} + \frac{\pi}{6} = \frac{5\pi}{12}$

$\therefore \frac{\pi}{12} \le \arg z \le \frac{5\pi}{12}$ //

Question 56 (***+)

It is given that

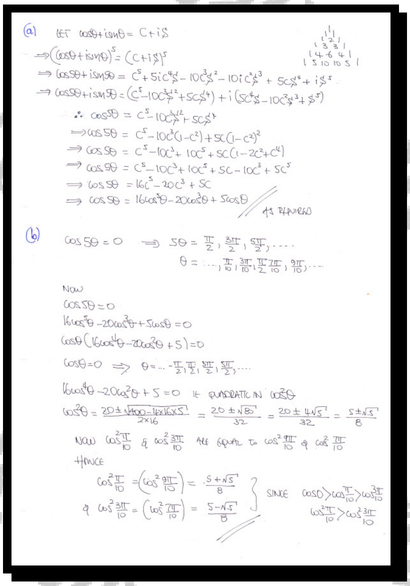
$$\cos 5\theta = 16\cos^5\theta - 20\cos^2\theta + 5\cos\theta .$$

a) Use de Moivre's theorem to prove the above trigonometric identity.

b) By considering the solution of the equation $\cos 5\theta = 0$, show that

$$\cos^2\left(\frac{3\pi}{10}\right) = \frac{5-\sqrt{5}}{8} .$$

proof



(a) LET $\cos\theta + i\sin\theta = C + iS$

$$\begin{matrix} & & & & & 1 & & & & & \\ & & & & 1 & & 1 & & & & \\ & & & 1 & & 2 & & 1 & & & \\ & & 1 & & 3 & & 3 & & 1 & & \\ & 1 & & 4 & & 6 & & 4 & & 1 & \\ 1 & & 5 & & 10 & & 10 & & 5 & & 1 \end{matrix}$$

$\Rightarrow (\cos\theta + i\sin\theta)^5 = (C+iS)^5$

$\Rightarrow \cos5\theta + i\sin5\theta = C^5 + 5iC^4S - 10C^3S^2 - 10iC^2S^3 + 5CS^4 + iS^5$

$\Rightarrow \cos5\theta + i\sin5\theta = (C^5 - 10C^3S^2 + 5CS^4) + i(5C^4S - 10C^2S^3 + S^5)$

$\therefore \cos5\theta = C^5 - 10C^3S^2 + 5CS^4$

$\Rightarrow \cos5\theta = C^5 - 10C^3(1-C^2) + 5C(1-C^2)^2$

$\Rightarrow \cos5\theta = C^5 - 10C^3 + 10C^5 + 5C(1-2C^2+C^4)$

$\Rightarrow \cos5\theta = C^5 - 10C^3 + 10C^5 + 5C - 10C^3 + 5C^5$

$\Rightarrow \cos5\theta = 16C^5 - 20C^3 + 5C$

$\Rightarrow \cos5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$ // AS REQUIRED

(b) $\cos5\theta = 0 \Rightarrow 5\theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \dots$

$\theta = \dots, \frac{\pi}{10}, \frac{3\pi}{10}, \frac{\pi}{2}, \frac{7\pi}{10}, \frac{9\pi}{10}, \dots$

NOW

$\cos5\theta = 0$

$16\cos^5\theta - 20\cos^3\theta + 5\cos\theta = 0$

$\cos\theta(16\cos^4\theta - 20\cos^2\theta + 5) = 0$

$\cos\theta = 0 \Rightarrow \theta = \dots -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \dots$

$16\cos^4\theta - 20\cos^2\theta + 5 = 0$ IT QUADRATIC IN $\cos^2\theta$

$\cos^2\theta = \frac{20 \pm \sqrt{400 - 4\times16\times5}}{2\times16} = \frac{20\pm\sqrt{80}}{32} = \frac{20\pm4\sqrt{5}}{32} = \frac{5\pm\sqrt{5}}{8}$

NOW $\cos^2\frac{\pi}{10}$ & $\cos^2\frac{3\pi}{10}$ ARE EQUAL TO $\cos^2\frac{9\pi}{10}$ & $\cos^2\frac{7\pi}{10}$

HENCE

$\cos^2\frac{\pi}{10} = \left(\cos^2\frac{9\pi}{10}\right) = \frac{5+\sqrt{5}}{8}$

& $\cos^2\frac{3\pi}{10} = \left(\cos^2\frac{7\pi}{10}\right) = \frac{5-\sqrt{5}}{8}$ //

SINCE $\cos0 > \cos\frac{\pi}{10} > \cos\frac{3\pi}{10}$

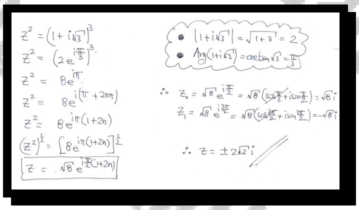
$\cos^2\frac{\pi}{10} > \cos^2\frac{3\pi}{10}$

Question 57 (***+)

$$z^2 = \left(1+\mathrm{i}\sqrt{3}\right)^3, \ z \in \mathbb{C} .$$

Solve the above equation, giving the answers in the form $a+b\mathrm{i}$, where a and b are real numbers.

$$\boxed{z = \pm \mathrm{i}2\sqrt{2}}$$



$$z^2 = \left(1+i\sqrt{3}\right)^3$$
$$z^2 = \left(2e^{i\frac{\pi}{3}}\right)^3$$
$$z^2 = 8e^{i\pi}$$
$$z^2 = 8e^{i(\pi+2n\pi)}$$
$$z^2 = 8e^{i\pi(1+2n)}$$
$$\left(z^2\right)^{\frac{1}{2}} = \left[8e^{i\pi(1+2n)}\right]^{\frac{1}{2}}$$
$$\boxed{z = \sqrt{8}e^{i\frac{\pi}{2}(1+2n)}}$$

- $\left|1+i\sqrt{3}\right| = \sqrt{1+3} = 2$
- $\mathrm{Arg}\left(1+i\sqrt{3}\right) = \arctan\sqrt{3} = \frac{\pi}{3}$

$$\therefore \ z_0 = \sqrt{8}e^{i\frac{\pi}{2}} = \sqrt{8}\left(\cancel{\cos\frac{\pi}{2}} + i\sin\frac{\pi}{2}\right) = \sqrt{8}i$$
$$z_1 = \sqrt{8}e^{i\frac{3\pi}{2}} = \sqrt{8}\left(\cancel{\cos\frac{3\pi}{2}} + i\sin\frac{3\pi}{2}\right) = -\sqrt{8}i$$

$$\therefore \ z = \pm 2\sqrt{2}i$$

Question 58 (***+)

A transformation of the z plane to the w plane is given by

$$w = \frac{1+3z}{1-z}, \; z \in \mathbb{C}, \; z \neq 1,$$

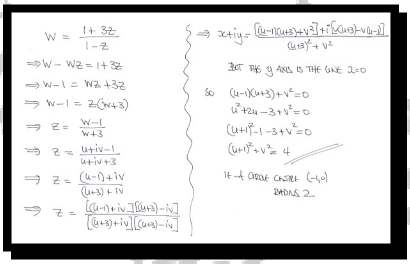
where $z = x + \mathrm{i}y$ and $w = u + \mathrm{i}v$.

The set of points that lie on the y axis of the z plane, are mapped in the w plane onto a curve C.

Show that a Cartesian equation of C is

$$(u+1)^2 + v^2 = 4.$$

proof



$$w = \frac{1+3z}{1-z}$$

$$\Rightarrow w - wz = 1+3z$$

$$\Rightarrow w-1 = wz+3z$$

$$\Rightarrow w-1 = z(w+3)$$

$$\Rightarrow z = \frac{w-1}{w+3}$$

$$\Rightarrow z = \frac{u+iv-1}{u+iv+3}$$

$$\Rightarrow z = \frac{(u-1)+iv}{(u+3)+iv}$$

$$\Rightarrow z = \frac{[(u-1)+iv][(u+3)-iv]}{[(u+3)+iv][(u+3)-iv]}$$

$$\Rightarrow x+iy = \frac{[(u-1)(u+3)+v^2]+i[v(u+3)-v(u-1)]}{(u+3)^2+v^2}$$

BUT THE y AXIS IS THE LINE $x=0$

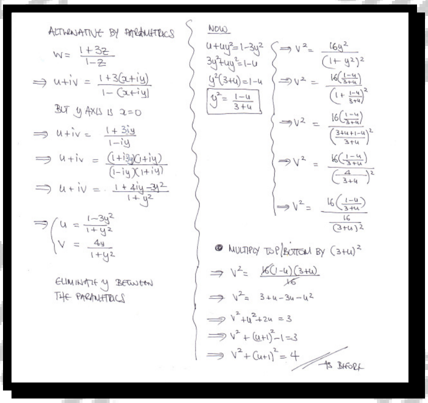
SO $(u-1)(u+3)+v^2=0$

$$u^2+2u-3+v^2=0$$

$$(u+1)^2-1-3+v^2=0$$

$$(u+1)^2+v^2=4$$

IE A CIRCLE CENTRE $(-1,0)$ RADIUS 2



ALTERNATIVE BY PARAMETRICS

$$w = \frac{1+3z}{1-z}$$

$$\Rightarrow u+iv = \frac{1+3(x+iy)}{1-(x+iy)}$$

BUT y AXIS IS $x=0$

$$\Rightarrow u+iv = \frac{1+3iy}{1-iy}$$

$$\Rightarrow u+iv = \frac{(1+i3y)(1+iy)}{(1-iy)(1+iy)}$$

$$\Rightarrow u+iv = \frac{1+4iy-3y^2}{1+y^2}$$

$$\Rightarrow \begin{cases} u = \dfrac{1-3y^2}{1+y^2} \\ v = \dfrac{4y}{1+y^2} \end{cases}$$

ELIMINATE y BETWEEN THE PARAMETRICS

NOW

$$u+uy^2=1-3y^2$$

$$3y^2+uy^2=1-u$$

$$y^2(3+u)=1-u$$

$$\boxed{y^2 = \frac{1-u}{3+u}}$$

$$\Rightarrow v^2 = \frac{16y^2}{(1+y^2)^2}$$

$$\Rightarrow v^2 = \frac{16\left(\frac{1-u}{3+u}\right)}{\left(1+\frac{1-u}{3+u}\right)^2}$$

$$\Rightarrow v^2 = \frac{16\left(\frac{1-u}{3+u}\right)}{\left(\frac{3+u+1-u}{3+u}\right)^2}$$

$$\Rightarrow v^2 = \frac{16\left(\frac{1-u}{3+u}\right)}{\left(\frac{4}{3+u}\right)^2}$$

$$\Rightarrow v^2 = \frac{16\left(\frac{1-u}{3+u}\right)}{\frac{16}{(3+u)^2}}$$

● MULTIPLY TOP/BOTTOM BY $(3+u)^2$

$$\Rightarrow v^2 = \frac{\cancel{16}(1-u)(3+u)}{\cancel{16}}$$

$$\Rightarrow v^2 = 3+u-3u-u^2$$

$$\Rightarrow v^2+u^2+2u = 3$$

$$\Rightarrow v^2+(u+1)^2-1=3$$

$$\Rightarrow v^2+(u+1)^2=4$$

AS BEFORE

Question 59 (***+)

The point A represents the complex number on the z plane such that

$$|z-6\mathrm{i}| = 2|z-3|,$$

and the point B represents the complex number on the z plane such that

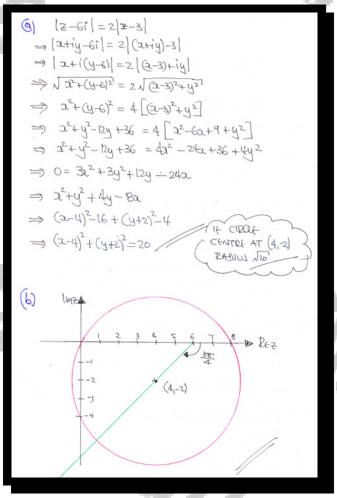
$$\arg(z-6) = -\frac{3\pi}{4}.$$

a) Show that the locus of A as z varies is a circle, stating its radius and the coordinates of its centre.

b) Sketch, on the same z plane, the locus of A and B as z varies.

c) Find the complex number z, so that the point A coincides with the point B.

$\boxed{C(4,-2),\ r=\sqrt{20}}$, $\boxed{z=\left(4-\sqrt{10}\right)+\mathrm{i}\left(-2-\sqrt{10}\right)}$



(a) $|z-6i| = 2|z-3|$

$\Rightarrow |x+iy-6i| = 2|(x+iy)-3|$

$\Rightarrow |x+i(y-6)| = 2|(x-3)+iy|$

$\Rightarrow \sqrt{x^2+(y-6)^2} = 2\sqrt{(x-3)^2+y^2}$

$\Rightarrow x^2+(y-6)^2 = 4\left[(x-3)^2+y^2\right]$

$\Rightarrow x^2+y^2-12y+36 = 4\left[x^2-6x+9+y^2\right]$

$\Rightarrow x^2+y^2-12y+36 = 4x^2-24x+36+4y^2$

$\Rightarrow 0 = 3x^2+3y^2+12y-24x$

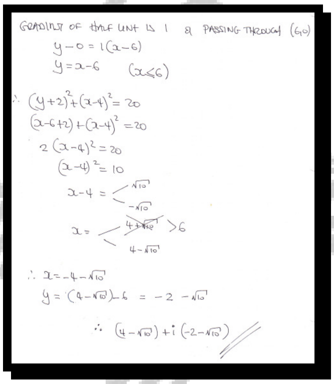
$\Rightarrow x^2+y^2+4y-8x$

$\Rightarrow (x-4)^2-16+(y+2)^2-4$

$\Rightarrow (x-4)^2+(y+2)^2=20$

IT CIRCLE CENTRE AT $(4,-2)$ RADIUS $\sqrt{20}$

(b)



GRADIENT OF HALF LINE IS 1 & PASSING THROUGH (6,0)

$y-0 = 1(x-6)$

$y = x-6 \quad (x \leqslant 6)$

$\therefore (y+2)^2+(x-4)^2 = 20$

$(x-6+2)+(x-4)^2 = 20$

$2(x-4)^2 = 20$

$(x-4)^2 = 10$

$x-4 = \sqrt{10}$, $-\sqrt{10}$

$x = \cancel{4+\sqrt{10}} > 6$, $4-\sqrt{10}$

$\therefore x = -4-\sqrt{10}$

$y = (4-\sqrt{10})-6 = -2-\sqrt{10}$

$\therefore (4-\sqrt{10})+i(-2-\sqrt{10})$

Question 60 (***+)

The complex number z is given by

$$z = \mathrm{e}^{\mathrm{i}\theta}, \ -\pi < \theta \leq \pi .$$

a) Show clearly that

$$z^n + \frac{1}{z^n} \equiv 2\cos n\theta .$$

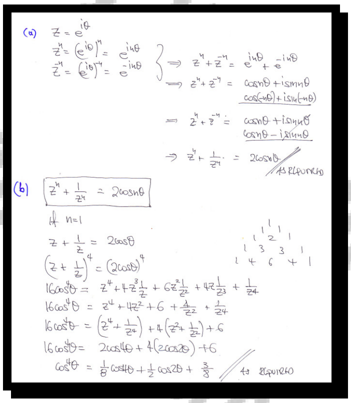
b) Hence show further that

$$\cos^4\theta \equiv \frac{1}{8}\cos 4\theta + \frac{1}{2}\cos 2\theta + \frac{3}{8} .$$

c) Solve the equation

$$2\cos 4\theta + 8\cos 2\theta + 5 = 0, \ 0 \leq \theta < 2\pi .$$

$$\boxed{\theta = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}}$$



(a)

$$z = e^{i\theta}$$

$$\left.\begin{array}{l} z^n = (e^{i\theta})^n = e^{in\theta} \\ z^{-n} = (e^{i\theta})^{-n} = e^{-in\theta} \end{array}\right\} \Rightarrow z^n + z^{-n} = e^{in\theta} + e^{-in\theta}$$

$$\Rightarrow z^n + z^{-n} = \begin{array}{l} \cos n\theta + i\sin n\theta \\ \underline{\cos(-n\theta) + i\sin(-n\theta)} \end{array}$$

$$\Rightarrow z^n + z^{-n} = \begin{array}{l} \cos n\theta + \cancel{i\sin n\theta} \\ \underline{\cos n\theta - \cancel{i\sin n\theta}} \end{array}$$

$$\Rightarrow z^n + \frac{1}{z^n} = 2\cos n\theta \ //$$

AS REQUIRED

(b) $\boxed{z^n + \frac{1}{z^n} = 2\cos n\theta}$

if $n=1$

$$z + \frac{1}{z} = 2\cos\theta$$

$$\left(z + \frac{1}{z}\right)^4 = (2\cos\theta)^4$$

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 1 & & 1 & & & \\ & & 1 & & 2 & & 1 & & \\ & 1 & & 3 & & 3 & & 1 & \\ 1 & & 4 & & 6 & & 4 & & 1 \end{array}$$

$$16\cos^4\theta = z^4 + 4z^3\frac{1}{z} + 6z^2\frac{1}{z^2} + 4z\frac{1}{z^3} + \frac{1}{z^4}$$

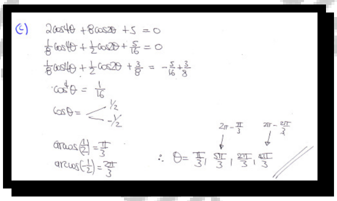
$$16\cos^4\theta = z^4 + 4z^2 + 6 + \frac{4}{z^2} + \frac{1}{z^4}$$

$$16\cos^4\theta = \left(z^4 + \frac{1}{z^4}\right) + 4\left(z^2 + \frac{1}{z^2}\right) + 6$$

$$16\cos^4\theta = 2\cos 4\theta + 4(2\cos 2\theta) + 6$$

$$\cos^4\theta = \frac{1}{8}\cos 4\theta + \frac{1}{2}\cos 2\theta + \frac{3}{8} \ //$$

AS REQUIRED



(c)

$$2\cos 4\theta + 8\cos 2\theta + 5 = 0$$

$$\frac{1}{8}\cos 4\theta + \frac{1}{2}\cos 2\theta + \frac{5}{16} = 0$$

$$\frac{1}{8}\cos 4\theta + \frac{1}{2}\cos 2\theta + \frac{3}{8} = -\frac{5}{16} + \frac{3}{8}$$

$$\cos^4\theta = \frac{1}{16}$$

$$\cos\theta = \begin{cases} \frac{1}{2} \\ -\frac{1}{2} \end{cases}$$

$$\arccos\left(\frac{1}{2}\right) = \frac{\pi}{3}$$

$$\arccos\left(-\frac{1}{2}\right) = \frac{2\pi}{3}$$

$$\therefore \ \theta = \frac{\pi}{3}, \underset{\uparrow}{\frac{5\pi}{3}}, \frac{2\pi}{3}, \underset{\uparrow}{\frac{4\pi}{3}} \ //$$

$2\pi - \frac{\pi}{3}$ $\quad$ $2\pi - \frac{2\pi}{3}$

Question 61 (***+)

The complex number $z = -9\mathrm{i}$ is given.

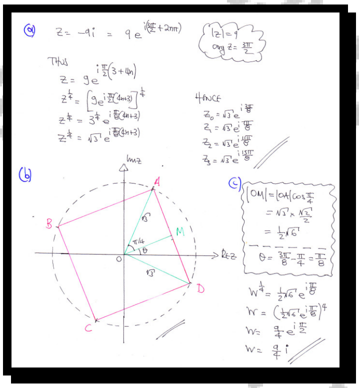
a) Determine the fourth roots of z, giving the answers in the form $r\mathrm{e}^{\mathrm{i}\theta}$, where $r > 0$ and $0 \le \theta < 2\pi$.

b) Plot the points represented by these roots in Argand diagram, and join them in order of increasing argument, labelled as A, B, C and D.

The midpoints of the sides of the quadrilateral $ABCD$ represent the 4th roots of another complex number w.

c) Find w, giving the answer in the form $x + \mathrm{i}y$, where $x \in \mathbb{R}$, $y \in \mathbb{R}$.

$$\boxed{z = \sqrt{3}\,\mathrm{e}^{\mathrm{i}\theta}, \theta = \frac{3\pi}{8}, \frac{7\pi}{8}, \frac{11\pi}{8}, \frac{15\pi}{8}}, \quad \boxed{w = \frac{9}{4}\mathrm{i}}$$



(a) $z = -9i = 9e^{i\left(\frac{3\pi}{2} + 2n\pi\right)}$

$|z| = 9$
$\arg z = \frac{3\pi}{2}$

THUS

$z = 9e^{i\frac{\pi}{2}(3+4n)}$

$z^{\frac{1}{4}} = \left[9e^{i\frac{\pi}{2}(4n+3)}\right]^{\frac{1}{4}}$

$z^{\frac{1}{4}} = 3^{\frac{1}{2}}e^{i\frac{\pi}{8}(4n+3)}$

$z^{\frac{1}{4}} = \sqrt{3}\,e^{i\frac{\pi}{8}(4n+3)}$

HENCE

$z_0 = \sqrt{3}\,e^{i\frac{3\pi}{8}}$

$z_1 = \sqrt{3}\,e^{i\frac{7\pi}{8}}$

$z_2 = \sqrt{3}\,e^{i\frac{11\pi}{8}}$

$z_3 = \sqrt{3}\,e^{i\frac{15\pi}{8}}$

(b)

(c)

$|OM| = |OA|\cos\frac{\pi}{4}$

$= \sqrt{3} \times \frac{\sqrt{2}}{2}$

$= \frac{1}{2}\sqrt{6}$

$\theta = \frac{3\pi}{8} - \frac{\pi}{4} = \frac{\pi}{8}$

$w^{\frac{1}{4}} = \frac{1}{2}\sqrt{6}\,e^{i\frac{\pi}{8}}$

$w = \left(\frac{1}{2}\sqrt{6}\,e^{i\frac{\pi}{8}}\right)^4$

$w = \frac{9}{4}e^{i\frac{\pi}{2}}$

$w = \frac{9}{4}i$

Question 62 **(***+)**

The complex numbers z and w, satisfy the relationship

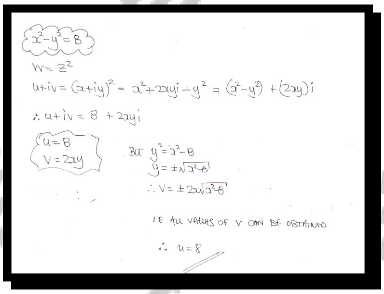
$$w = z^2.$$

Given that in an Argand diagram, z is tracing the curve with equation

$$x^2 - y^2 = 8,$$

determine a Cartesian equation of the locus that w is tracing.

$$\boxed{u = 8 \text{ or } x = 8}$$



$x^2-y^2=8$

$w=z^2$

$u+iv=(x+iy)^2=x^2+2xyi-y^2=(x^2-y^2)+(2xy)i$

$\therefore u+iv=8+2xyi$

$u=8$
$v=2xy$

BUT $y^2=x^2-8$
$y=\pm\sqrt{x^2-8}$
$\therefore v=\pm 2x\sqrt{x^2-8}$

IE ALL VALUES OF V CAN BE OBTAINED

$\therefore u=8$

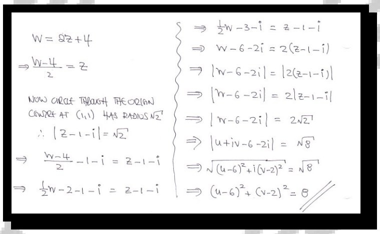
Question 63 **(***+)**

The complex numbers z and w, satisfy the relationship

$$w = 2z + 4, \quad z \neq -2.$$

Given that z is tracing a circle with centre at $(1,1)$ and radius $\sqrt{2}$ in an Argand diagram, determine a Cartesian equation of the locus that w is tracing.

$$\boxed{(u-6)^2 + (v-2)^2 = 8 \text{ or } (x-6)^2 + (y-2)^2 = 8}$$



$w=2z+4$

$\Rightarrow \frac{w-4}{2}=z$

NOW CIRCLE THROUGH THE ORIGIN
CENTRE AT (1,1) HAS RADIUS $\sqrt{2}$

$\therefore |z-1-i|=\sqrt{2}$

$\Rightarrow \frac{w-4}{2}-1-i=z-1-i$

$\Rightarrow \frac{1}{2}w-2-1-i=z-1-i$

$\Rightarrow \frac{1}{2}w-3-i=z-1-i$

$\Rightarrow w-6-2i=2(z-1-i)$

$\Rightarrow |w-6-2i|=|2(z-1-i)|$

$\Rightarrow |w-6-2i|=2|z-1-i|$

$\Rightarrow |w-6-2i|=2\sqrt{2}$

$\Rightarrow |u+iv-6-2i|=\sqrt{8}$

$\Rightarrow \sqrt{(u-6)^2+i(v-2)^2}=\sqrt{8}$

$\Rightarrow (u-6)^2+(v-2)^2=8$

Question 64 (***+)

The complex number is defined as $z = \mathrm{e}^{\mathrm{i}\theta}$, $-\pi < \theta \le \pi$.

a) Show that …

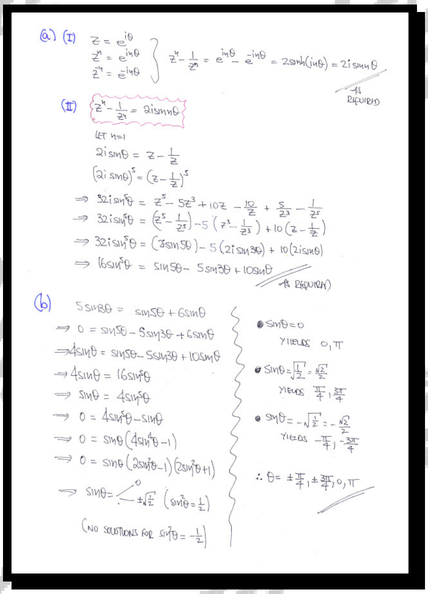
i. … $z^n - \dfrac{1}{z^n} = 2\mathrm{i}\sin\theta$.

ii. … $16\sin^5\theta = \sin 5\theta - 5\sin 3\theta + 10\sin\theta$.

b) Hence solve the equation

$$5\sin 3\theta = \sin 5\theta + 6\sin\theta\,,\ -\pi < \theta \le \pi\,.$$

$$\boxed{\theta = 0, \pm\frac{\pi}{4}, \pm\frac{3\pi}{4}, \pi}$$



(a) (I) $z = e^{i\theta}$, $z^n = e^{in\theta}$, $z^{-n} = e^{-in\theta}$ $\Big\}$ $z^n - \frac{1}{z^n} = e^{in\theta} - e^{-in\theta} = 2\sinh(in\theta) = 2i\sin n\theta$

AS REQUIRED

(II) $\boxed{z^n - \frac{1}{z^n} = 2i\sin n\theta}$

LET $n=1$

$2i\sin\theta = z - \frac{1}{z}$

$(2i\sin\theta)^5 = \left(z - \frac{1}{z}\right)^5$

$\Rightarrow 32i\sin^5\theta = z^5 - 5z^3 + 10z - \frac{10}{z} + \frac{5}{z^3} - \frac{1}{z^5}$

$\Rightarrow 32i\sin^5\theta = \left(z^5 - \frac{1}{z^5}\right) - 5\left(z^3 - \frac{1}{z^3}\right) + 10\left(z - \frac{1}{z}\right)$

$\Rightarrow 32i\sin^5\theta = (2i\sin 5\theta) - 5(2i\sin 3\theta) + 10(2i\sin\theta)$

$\Rightarrow 16\sin^5\theta = \sin 5\theta - 5\sin 3\theta + 10\sin\theta$

AS REQUIRED

(b) $5\sin 3\theta = \sin 5\theta + 6\sin\theta$

$\Rightarrow 0 = \sin 5\theta - 5\sin 3\theta + 6\sin\theta$

$\Rightarrow 4\sin\theta = \sin 5\theta - 5\sin 3\theta + 10\sin\theta$

$\Rightarrow 4\sin\theta = 16\sin^5\theta$

$\Rightarrow \sin\theta = 4\sin^5\theta$

$\Rightarrow 0 = 4\sin^5\theta - \sin\theta$

$\Rightarrow 0 = \sin\theta(4\sin^4\theta - 1)$

$\Rightarrow 0 = \sin\theta(2\sin^2\theta - 1)(2\sin^2\theta + 1)$

$\Rightarrow \sin\theta = 0,\ \pm\sqrt{\frac{1}{2}}\ \left(\sin^2\theta = \frac{1}{2}\right)$

(NO SOLUTIONS FOR $\sin^2\theta = -\frac{1}{2}$)

- $\sin\theta = 0$ YIELDS $0, \pi$
- $\sin\theta = \sqrt{\frac{1}{2}} = \frac{\sqrt{2}}{2}$ YIELDS $\frac{\pi}{4}, \frac{3\pi}{4}$
- $\sin\theta = -\sqrt{\frac{1}{2}} = -\frac{\sqrt{2}}{2}$ YIELDS $-\frac{\pi}{4}, -\frac{3\pi}{4}$

$\therefore \theta = \pm\frac{\pi}{4}, \pm\frac{3\pi}{4}, 0, \pi$

Question 65 (***+)

$$z^3 = 32 + 32\sqrt{3}\,\mathrm{i}\,,\ z \in \mathbb{C}\,.$$

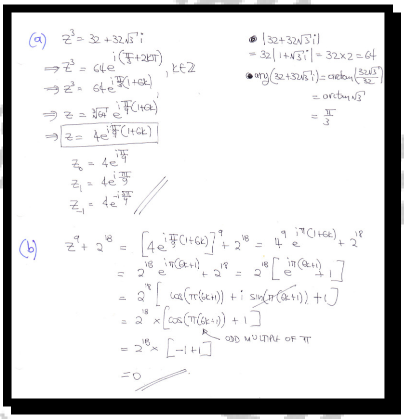
a) Solve the above equation.

Give the answers in exponential form $z = r\mathrm{e}^{\mathrm{i}\theta}$, $r > 0$, $-\pi < \theta \le \pi$.

b) Show that these roots satisfy the equation

$$w^9 + 2^{18} = 0\,.$$

$$\boxed{z = 4\mathrm{e}^{\mathrm{i}\frac{\pi}{9}},\ 4\mathrm{e}^{\mathrm{i}\frac{7\pi}{9}},\ 4\mathrm{e}^{-\mathrm{i}\frac{5\pi}{9}}}$$



(a) $z^3 = 32 + 32\sqrt{3}i$

$\Rightarrow z^3 = 64e^{i\left(\frac{\pi}{3}+2k\pi\right)}, k \in \mathbb{Z}$

$\Rightarrow z^3 = 64e^{i\frac{\pi}{3}(1+6k)},$

$\Rightarrow z = \sqrt[3]{64}\, e^{i\frac{\pi}{9}(1+6k)}$

$\Rightarrow \boxed{z = 4e^{i\frac{\pi}{9}(1+6k)}}$

$z_0 = 4e^{i\frac{\pi}{9}}$

$z_1 = 4e^{i\frac{7\pi}{9}}$

$z_{-1} = 4e^{-i\frac{5\pi}{9}}$ //

● $|32+32\sqrt{3}i|$
$= 32|1+\sqrt{3}i| = 32\times 2 = 64$

● $\arg(32+32\sqrt{3}i) = \arctan\left(\frac{32\sqrt{3}}{32}\right)$
$= \arctan\sqrt{3}$
$= \frac{\pi}{3}$

(b) $z^9 + 2^{18} = \left[4e^{i\frac{\pi}{9}(1+6k)}\right]^9 + 2^{18} = 4^9 e^{i\pi(1+6k)} + 2^{18}$

$= 2^{18} e^{i\pi(6k+1)} + 2^{18} = 2^{18}\left[e^{i\pi(6k+1)} + 1\right]$

$= 2^{18}\left[\cos(\pi(6k+1)) + \cancel{i\sin(\pi(6k+1))} + 1\right]$

$= 2^{18} \times \left[\cos(\pi(6k+1)) + 1\right]$ ← ODD MULTIPLE OF π

$= 2^{18} \times [-1+1]$

$= 0$ //

Question 66 (*+)**

The complex function $w = f(z)$ is given by

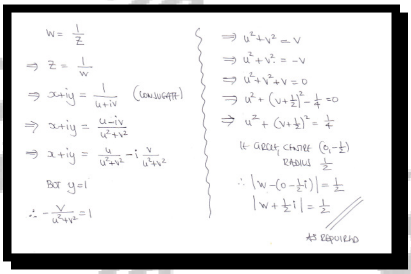
$$w = \frac{1}{z}, \ z \in \mathbb{C}, \ z \neq 0 .$$

This function maps a general point $P(x, y)$ in the z plane onto the point $Q(u, v)$ in the w plane.

Given that P lies on the line with Cartesian equation $y = 1$, show that the locus of Q is given by

$$\left| w + \frac{1}{2}\mathrm{i} \right| = \frac{1}{2} .$$

$\boxed{\text{proof}}$



$w = \frac{1}{z}$

$\Rightarrow z = \frac{1}{w}$

$\Rightarrow x + iy = \frac{1}{u+iv}$ (CONJUGATE)

$\Rightarrow x + iy = \frac{u-iv}{u^2+v^2}$

$\Rightarrow x + iy = \frac{u}{u^2+v^2} - i\frac{v}{u^2+v^2}$

BUT $y = 1$

$\therefore -\frac{v}{u^2+v^2} = 1$

$\Rightarrow u^2+v^2 = v$

$\Rightarrow u^2+v^2 = -v$

$\Rightarrow u^2+v^2+v = 0$

$\Rightarrow u^2+\left(v+\frac{1}{2}\right)^2-\frac{1}{4} = 0$

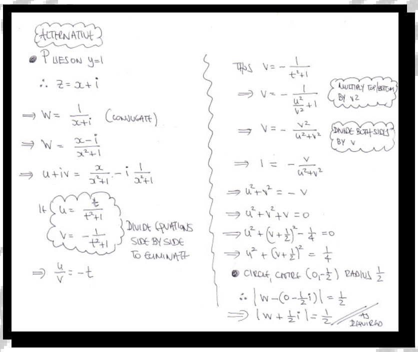
$\Rightarrow u^2+\left(v+\frac{1}{2}\right)^2 = \frac{1}{4}$

IE CIRCLE, CENTRE $(0,-\frac{1}{2})$ RADIUS $\frac{1}{2}$

$\therefore \left|w-\left(0-\frac{1}{2}i\right)\right| = \frac{1}{2}$

$\left|w+\frac{1}{2}i\right| = \frac{1}{2}$ //

AS REQUIRED



ALTERNATIVE

- P LIES ON $y=1$

$\therefore z = x + i$

$\Rightarrow w = \frac{1}{x+i}$ (CONJUGATE)

$\Rightarrow w = \frac{x-i}{x^2+1}$

$\Rightarrow u+iv = \frac{x}{x^2+1} - i\frac{1}{x^2+1}$

IE $u = \frac{t}{t^2+1}$, $v = -\frac{1}{t^2+1}$ DIVIDE EQUATIONS SIDE BY SIDE TO ELIMINATE

$\Rightarrow \frac{u}{v} = -t$

THUS $v = -\frac{1}{t^2+1}$

$\Rightarrow v = -\frac{1}{\frac{u^2}{v^2}+1}$ MULTIPLY TOP/BOTTOM BY v^2

$\Rightarrow v = -\frac{v^2}{u^2+v^2}$ DIVIDE BOTH SIDES BY v

$\Rightarrow 1 = -\frac{v}{u^2+v^2}$

$\Rightarrow u^2+v^2 = -v$

$\Rightarrow u^2+v^2+v = 0$

$\Rightarrow u^2+\left(v+\frac{1}{2}\right)^2-\frac{1}{4} = 0$

$\Rightarrow u^2+\left(v+\frac{1}{2}\right)^2 = \frac{1}{4}$

- CIRCLE, CENTRE $(0,-\frac{1}{2})$ RADIUS $\frac{1}{2}$

$\therefore \left|w-\left(0-\frac{1}{2}i\right)\right| = \frac{1}{2}$

$\Rightarrow \left|w+\frac{1}{2}i\right| = \frac{1}{2}$ // AS REQUIRED

Question 67 **(***+)**

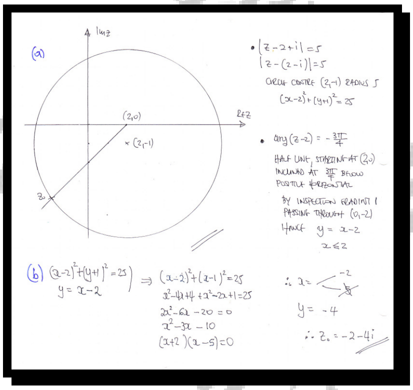
$$|z-2+\mathrm{i}|=5.$$

$$\arg(z-2)=-\frac{3\pi}{4}.$$

a) Sketch the above complex loci in the same Argand diagram.

b) Determine, in the form $x+\mathrm{i}y$, the complex number z_0 represented by the intersection of the two loci of part (a).

$$\boxed{z_0=-2-4\mathrm{i}}$$



- $|z-2+i|=5$
  $|z-(2-i)|=5$
  CIRCLE CENTRE (2,−1) RADIUS 5
  $(x-2)^2+(y+1)^2=25$

- $\arg(z-2)=-\frac{3\pi}{4}$
  HALF LINE, STARTING AT (2,0)
  INCLINED AT $\frac{3\pi}{4}$ BELOW
  POSITIVE HORIZONTAL
  BY INSPECTION GRADIENT 1
  PASSING THROUGH (0,−2)
  HENCE $y=x-2$
  $x\leq 2$

(b) $\left.\begin{array}{l}(x-2)^2+(y+1)^2=25\\ y=x-2\end{array}\right\} \Rightarrow (x-2)^2+(x-1)^2=25$

$x^2-4x+4+x^2-2x+1=25$

$2x^2-6x-20=0$

$x^2-3x-10$

$(x+2)(x-5)=0$

$\therefore x = \begin{cases}-2\\ \cancel{5}\end{cases}$

$y=-4$

$\therefore z_0=-2-4i$

Question 68 **(***+)**

The complex number z is given in polar form as

$$\cos\left(\tfrac{2}{5}\pi\right) + \mathrm{i}\sin\left(\tfrac{2}{5}\pi\right).$$

a) Write z^2, z^3 and z^4 in polar form, each with argument θ, so that $0 \le \theta < 2\pi$.

In an Argand diagram the points A, B, C, D and E represent, in respective order, the complex numbers

$$1, \quad 1+z, \quad 1+z+z^2, \quad 1+z+z^2+z^3, \quad 1+z+z^2+z^3+z^4.$$

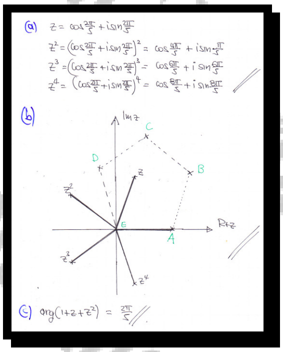
b) Sketch these points, in the sequential order given, in a standard Argand diagram.

c) State the exact argument of

$$1+z+z^2.$$

, $\boxed{z^2 = \cos\frac{4\pi}{5} + \mathrm{i}\sin\frac{4\pi}{5}}$, $\boxed{z^3 = \cos\frac{6\pi}{5} + \mathrm{i}\sin\frac{6\pi}{5}}$, $\boxed{z^4 = \cos\frac{8\pi}{5} + \mathrm{i}\sin\frac{8\pi}{5}}$,

$\boxed{\arg\left(1+z+z^2\right) = \frac{2\pi}{5}}$



(a) $z = \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}$

$z^2 = \left(\cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}\right)^2 = \cos\frac{4\pi}{5} + i\sin\frac{\pi}{5}$

$z^3 = \left(\cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}\right)^3 = \cos\frac{6\pi}{5} + i\sin\frac{6\pi}{5}$

$z^4 = \left(\cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}\right)^4 = \cos\frac{8\pi}{5} + i\sin\frac{8\pi}{5}$ //

(b)

(c) $\arg(1+z+z^2) = \frac{2\pi}{5}$ //

Question 69 (***+)

The complex number z satisfies the following equation.

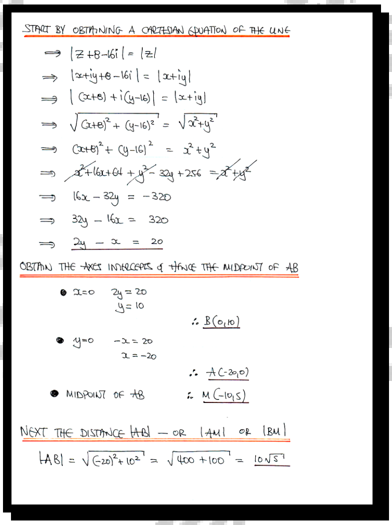
$$|z+8-16\text{i}|=|z|.$$

In a standard Argand diagram, the complex numbers represented by the points A and B lie on the real and imaginary axes, respectively.

Given further that A and B satisfy the above equation, determine an equation for the circle which passes though the points A, B and O, where O is the origin of the Argand diagram.

Give the answer in the form $|z-z_0|=r$, where $z_0 \in \mathbb{C}$ and $r \in \mathbb{R}$.

$$\boxed{}, \quad \boxed{|z+10-5\text{i}|=5\sqrt{5}}$$



START BY OBTAINING A CARTESIAN EQUATION OF THE LINE

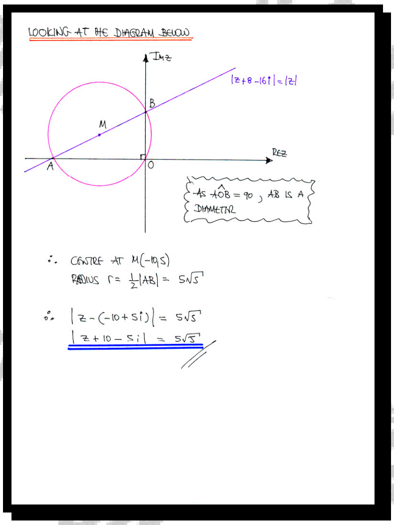
$$\Rightarrow |z+8-16i| = |z|$$
$$\Rightarrow |x+iy+8-16i| = |x+iy|$$
$$\Rightarrow |(x+8)+i(y-16)| = |x+iy|$$
$$\Rightarrow \sqrt{(x+8)^2+(y-16)^2} = \sqrt{x^2+y^2}$$
$$\Rightarrow (x+8)^2+(y-16)^2 = x^2+y^2$$
$$\Rightarrow \cancel{x^2}+16x+64+\cancel{y^2}-32y+256 = \cancel{x^2}+\cancel{y^2}$$
$$\Rightarrow 16x-32y = -320$$
$$\Rightarrow 32y-16x = 320$$
$$\Rightarrow \underline{2y-x = 20}$$

OBTAIN THE AXES INTERCEPTS & HENCE THE MIDPOINT OF AB

- $x=0$ $\quad 2y=20$, $y=10$ $\quad \therefore \underline{B(0,10)}$
- $y=0$ $\quad -x=20$, $x=-20$ $\quad \therefore \underline{A(-20,0)}$
- MIDPOINT OF AB $\quad \therefore \underline{M(-10,5)}$

NEXT THE DISTANCE |AB| — OR |AM| OR |BM|

$$|AB| = \sqrt{(-20)^2+10^2} = \sqrt{400+100} = \underline{10\sqrt{5}}$$



LOOKING AT THE DIAGRAM BELOW

As $A\hat{O}B = 90$, AB IS A DIAMETER

$\therefore$ CENTRE AT $M(-10,5)$

RADIUS $r = \frac{1}{2}|AB| = 5\sqrt{5}$

$$\therefore |z-(-10+5i)| = 5\sqrt{5}$$
$$\underline{\underline{|z+10-5i| = 5\sqrt{5}}}$$

Question 70 **(***+)**

The following convergent series C and S are given by

$$C = 1 + \frac{1}{2}\cos\theta + \frac{1}{4}\cos 2\theta + \frac{1}{8}\cos 3\theta ...$$
$$S = \quad \frac{1}{2}\sin\theta + \frac{1}{4}\sin 2\theta + \frac{1}{8}\sin 3\theta ...$$

a) Show clearly that

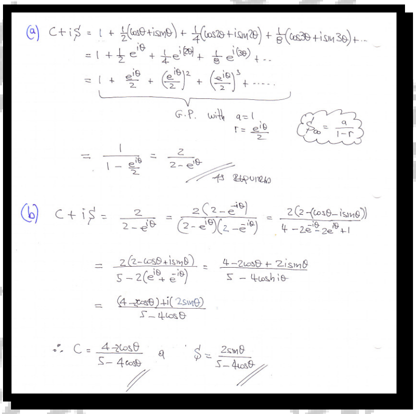
$$C + \mathrm{i}S = \frac{2}{2 - \mathrm{e}^{\mathrm{i}\theta}} .$$

b) Hence show further that

$$C = \frac{4 - 2\cos\theta}{5 - 4\cos\theta},$$

and find a similar expression for S.

$$\boxed{S = \frac{2\sin\theta}{5 - 4\cos\theta}}$$



(a) $C + iS = 1 + \frac{1}{2}(\cos\theta + i\sin\theta) + \frac{1}{4}(\cos 2\theta + i\sin 2\theta) + \frac{1}{8}(\cos 3\theta + i\sin 3\theta) + ...$

$= 1 + \frac{1}{2}e^{i\theta} + \frac{1}{4}e^{i(2\theta)} + \frac{1}{8}e^{i(3\theta)} + ..$

$= \underbrace{1 + \frac{e^{i\theta}}{2} + \left(\frac{e^{i\theta}}{2}\right)^2 + \left(\frac{e^{i\theta}}{2}\right)^3 + ...}_{\text{G.P. with } a=1,\ r=\frac{e^{i\theta}}{2}}$

$S_\infty = \frac{a}{1-r}$

$= \frac{1}{1 - \frac{e^{i\theta}}{2}} = \frac{2}{2 - e^{i\theta}}$ // AS REQUIRED

(b) $C + iS = \frac{2}{2 - e^{i\theta}} = \frac{2(2 - e^{-i\theta})}{(2 - e^{i\theta})(2 - e^{-i\theta})} = \frac{2(2 - (\cos\theta - i\sin\theta))}{4 - 2e^{-i\theta} - 2e^{i\theta} + 1}$

$= \frac{2(2 - \cos\theta + i\sin\theta)}{5 - 2(e^{i\theta} + e^{-i\theta})} = \frac{4 - 2\cos\theta + 2i\sin\theta}{5 - 4\cosh i\theta}$

$= \frac{(4 - 2\cos\theta) + i(2\sin\theta)}{5 - 4\cos\theta}$

$\therefore C = \frac{4 - 2\cos\theta}{5 - 4\cos\theta}$ // a $S = \frac{2\sin\theta}{5 - 4\cos\theta}$ //

Question 71 **(***+)**

The complex number z is given by

$$z = e^{i\theta}, \ -\pi < \theta \le \pi .$$

a) Show clearly that

$$z^n + \frac{1}{z^n} \equiv 2\cos n\theta .$$

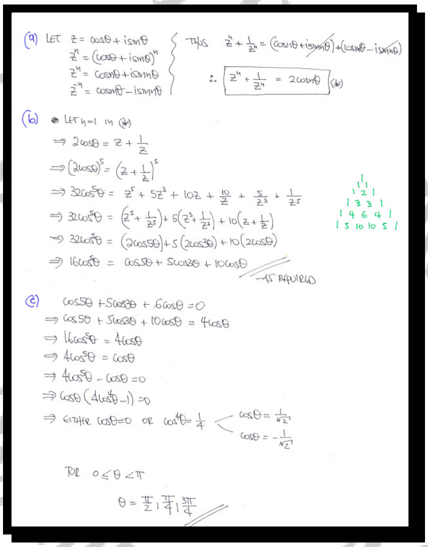
b) Hence show further that

$$16\cos^5\theta \equiv \cos 5\theta + 5\cos 3\theta + 10\cos\theta .$$

c) Use the results of parts **(a)** and **(b)** to solve the equation

$$\cos 5\theta + 5\cos 3\theta + 6\cos\theta = 0, \ 0 \le \theta < \pi .$$

$$\boxed{\theta = \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}}$$



(a) LET $z = \cos\theta + i\sin\theta$

$z^n = (\cos\theta + i\sin\theta)^n$

$z^n = \cos n\theta + i\sin n\theta$

$z^{-n} = \cos n\theta - i\sin n\theta$

THUS $z^n + \frac{1}{z^n} = (\cos n\theta + \cancel{i\sin n\theta}) + (\cos n\theta - \cancel{i\sin n\theta})$

$\therefore \boxed{z^n + \frac{1}{z^n} = 2\cos n\theta}$ (*)

(b) ● LET $n=1$ IN (*)

$\Rightarrow 2\cos\theta = z + \frac{1}{z}$

$\Rightarrow (2\cos\theta)^5 = \left(z + \frac{1}{z}\right)^5$

$\Rightarrow 32\cos^5\theta = z^5 + 5z^3 + 10z + \frac{10}{z} + \frac{5}{z^3} + \frac{1}{z^5}$

$\Rightarrow 32\cos^5\theta = \left(z^5 + \frac{1}{z^5}\right) + 5\left(z^3 + \frac{1}{z^3}\right) + 10\left(z + \frac{1}{z}\right)$

$\Rightarrow 32\cos^5\theta = (2\cos 5\theta) + 5(2\cos 3\theta) + 10(2\cos\theta)$

$\Rightarrow 16\cos^5\theta = \cos 5\theta + 5\cos 3\theta + 10\cos\theta$

AS REQUIRED

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1

(c) $\cos 5\theta + 5\cos 3\theta + 6\cos\theta = 0$

$\Rightarrow \cos 5\theta + 5\cos 3\theta + 10\cos\theta = 4\cos\theta$

$\Rightarrow 16\cos^5\theta = 4\cos\theta$

$\Rightarrow 4\cos^5\theta = \cos\theta$

$\Rightarrow 4\cos^5\theta - \cos\theta = 0$

$\Rightarrow \cos\theta(4\cos^4\theta - 1) = 0$

$\Rightarrow$ EITHER $\cos\theta = 0$ OR $\cos^4\theta = \frac{1}{4}$ $\begin{cases} \cos\theta = \frac{1}{\sqrt{2}} \\ \cos\theta = -\frac{1}{\sqrt{2}} \end{cases}$

FOR $0 \le \theta < \pi$

$\theta = \frac{\pi}{2}, \frac{\pi}{4}, \frac{3\pi}{4}$

Question 72 (***+)

The complex number z lies on the curve with equation

$$|z+5-12\mathrm{i}| = 6, \; z \in \mathbb{C}.$$

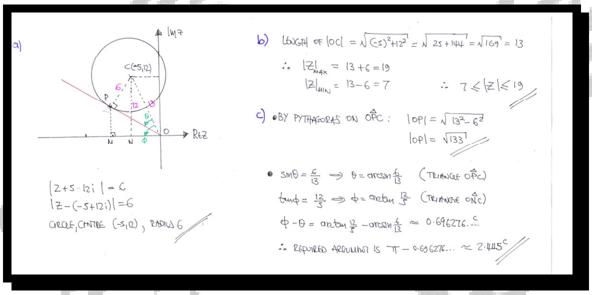
a) Sketch this curve in a standard Argand diagram.

b) Show that $a \le |z| \le b$, where a and b are integers.

The complex number z_0 lies on this curve so that its argument is the largest for all complex numbers which lie on this curve.

c) Determine the value of $|z_0|$ and the value of $\arg z_0$

$\boxed{|z_0| = \sqrt{133}}$, $\boxed{\arg z_0 \approx 2.445^{\text{c}}}$



$|z+5-12i| = 6$
$|z-(-5+12i)| = 6$
CIRCLE, CENTRE $(-5,12)$, RADIUS 6

b) LENGTH OF $|OC| = \sqrt{(-5)^2+12^2} = \sqrt{25+144} = \sqrt{169} = 13$

$\therefore \; |z|_{MAX} = 13+6 = 19$
$|z|_{MIN} = 13-6 = 7$ $\quad \therefore \; 7 \leqslant |z| \leqslant 19$

c) • BY PYTHAGORAS ON $O\hat{P}C$: $|OP| = \sqrt{13^2-6^2}$
$|OP| = \sqrt{133}$

• $\sin\theta = \frac{6}{13} \Rightarrow \theta = \arcsin\frac{6}{13}$ (TRIANGLE $O\hat{P}C$)
$\tan\phi = \frac{12}{5} \Rightarrow \phi = \arctan\frac{12}{5}$ (TRIANGLE $O\hat{N}C$)
$\phi - \theta = \arctan\frac{12}{5} - \arcsin\frac{6}{13} \approx 0.696276...^{c}$
$\therefore$ REQUIRED ARGUMENT IS $\pi - 0.696276... \approx 2.445^{c}$

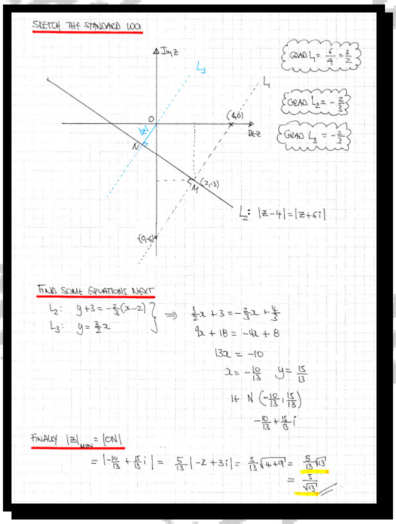
Question 73 (***+)

The complex number z satisfies

$$|z-4| = |z+6\text{i}|.$$

Determine, as an exact simplified surd, the minimum value of $|z|$.

$\boxed{}$, $\boxed{|z|_{\text{min}} = \frac{5}{\sqrt{13}}}$



SKETCH THE STANDARD LOCI

GRAD $L_1 = \frac{6}{4} = \frac{3}{2}$

GRAD $L_2 = -\frac{2}{3}$

GRAD $L_3 = -\frac{2}{3}$

L_2: $|z-4| = |z+6i|$

FIND SOME EQUATIONS NEXT

L_2: $y+3 = -\frac{2}{3}(x-2)$

L_3: $y = \frac{3}{2}x$

$\Rightarrow \frac{3}{2}x + 3 = -\frac{2}{3}x + \frac{4}{3}$

$9x + 18 = -4x + 8$

$13x = -10$

$x = -\frac{10}{13} \quad y = \frac{15}{13}$

ie $N\left(-\frac{10}{13}, \frac{15}{13}\right)$

$-\frac{10}{13} + \frac{15}{13}i$

FINALLY $|z|_{\text{MIN}} = |ON|$

$= \left|-\frac{10}{13} + \frac{15}{13}i\right| = \frac{5}{13}|-2+3i| = \frac{5}{13}\sqrt{4+9} = \frac{5}{13}\sqrt{13}$

$= \frac{5}{\sqrt{13}}$

Question 74 (****)

A transformation of the z plane onto the w plane is given by

$$w = \frac{az+b}{z+c}, \ z \in \mathbb{C}, z \neq -c$$

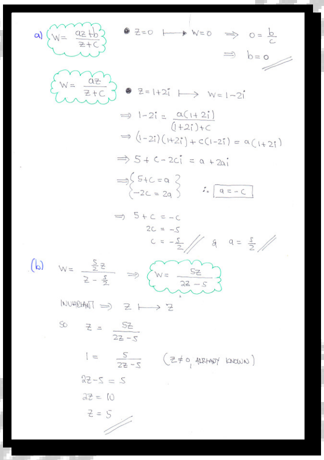
where a, b and c are real constants.

Under this transformation the point represented by the number $1+2\text{i}$ gets mapped to its complex conjugate and the origin remains invariant.

a) Find the value of a, the value of b and the value of c.

b) Find the number, other than the number represented by the origin, which remains invariant under this transformation.

$\boxed{a=\frac{5}{2}}$, $\boxed{b=0}$, $\boxed{c=-\frac{5}{2}}$, $\boxed{z=5}$



a) $w = \frac{az+b}{z+c}$

- $z=0 \longmapsto w=0 \Rightarrow 0=\frac{b}{c}$

$\Rightarrow b=0$

$w=\frac{az}{z+c}$

- $z=1+2i \longmapsto w=1-2i$

$\Rightarrow 1-2i = \frac{a(1+2i)}{(1+2i)+c}$

$\Rightarrow (1-2i)(1+2i)+c(1-2i) = a(1+2i)$

$\Rightarrow 5+c-2ci = a+2ai$

$\Rightarrow \begin{cases} 5+c=a \\ -2c=2a \end{cases} \quad \therefore \boxed{a=-c}$

$\Rightarrow 5+c=-c$

$2c=-5$

$c=-\frac{5}{2}$ & $a=\frac{5}{2}$

(b) $w=\frac{\frac{5}{2}z}{z-\frac{5}{2}} \Rightarrow w=\frac{5z}{2z-5}$

INVARIANT $\Rightarrow z \longmapsto z$

SO $z=\frac{5z}{2z-5}$

$1=\frac{5}{2z-5}$ ($z\neq 0$, ALREADY KNOWN)

$2z-5=5$

$2z=10$

$z=5$

Question 75 (****)

$$z^7 - 1 = 0, \; z \in \mathbb{C}.$$

One of the roots of the above equation is denoted by ω, where $0 < \arg \omega < \frac{\pi}{3}$.

a) Find ω in the form $\omega = r\mathrm{e}^{\mathrm{i}\theta}, \quad r > 0, \quad 0 < \theta \le \frac{\pi}{3}$.

b) Show clearly that

$$1 + \omega + \omega^2 + \omega^3 + \omega^4 + \omega^5 + \omega^6 = 0.$$

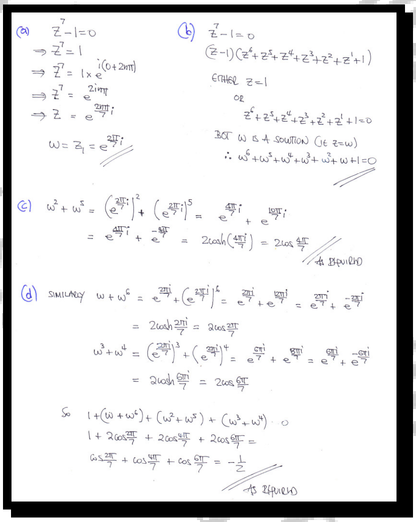
c) Show further that

$$\omega^2 + \omega^5 = 2\cos\left(\frac{4\pi}{7}\right).$$

d) Hence, using the results from the previous parts, deduce that

$$\cos\left(\frac{2\pi}{7}\right) + \cos\left(\frac{4\pi}{7}\right) + \cos\left(\frac{6\pi}{7}\right) = -\frac{1}{2}.$$

$$\boxed{\omega = \mathrm{e}^{\mathrm{i}\frac{2\pi}{7}}}$$



(a) $z^7 - 1 = 0$

$\Rightarrow z^7 = 1$

$\Rightarrow z^7 = 1 \times e^{i(0+2n\pi)}$

$\Rightarrow z^7 = e^{2in\pi}$

$\Rightarrow z = e^{\frac{2n\pi}{7}i}$

$\omega = z_1 = e^{\frac{2\pi}{7}i}$

(b) $z^7 - 1 = 0$

$(z-1)(z^6+z^5+z^4+z^3+z^2+z^1+1)$

EITHER $z=1$

OR

$z^6+z^5+z^4+z^3+z^2+z^1+1=0$

BUT ω IS A SOLUTION (IE $z=\omega$)

$\therefore \omega^6+\omega^5+\omega^4+\omega^3+\omega^2+\omega+1=0$

(c) $\omega^2+\omega^5 = \left(e^{\frac{2\pi}{7}i}\right)^2 + \left(e^{\frac{2\pi}{7}i}\right)^5 = e^{\frac{4\pi}{7}i} + e^{\frac{10\pi}{7}i}$

$= e^{\frac{4\pi}{7}i} + e^{-\frac{4\pi}{7}} = 2\cosh\left(\frac{4\pi i}{7}\right) = 2\cos\frac{4\pi}{7}$

AS REQUIRED

(d) SIMILARLY $\omega + \omega^6 = e^{\frac{2\pi i}{7}} + \left(e^{\frac{2\pi i}{7}}\right)^6 = e^{\frac{2\pi i}{7}} + e^{\frac{12\pi i}{7}} = e^{\frac{2\pi i}{7}} + e^{\frac{-2\pi i}{7}}$

$= 2\cosh\frac{2\pi i}{7} = 2\cos\frac{2\pi}{7}$

$\omega^3+\omega^4 = \left(e^{\frac{2\pi i}{7}}\right)^3 + \left(e^{\frac{2\pi i}{7}}\right)^4 = e^{\frac{6\pi i}{7}} + e^{\frac{8\pi i}{7}} = e^{\frac{6\pi i}{7}} + e^{\frac{-6\pi i}{7}}$

$= 2\cosh\frac{6\pi i}{7} = 2\cos\frac{6\pi}{7}$

So $1+(\omega+\omega^6)+(\omega^2+\omega^5)+(\omega^3+\omega^4) \cdot 0$

$1 + 2\cos\frac{2\pi}{7} + 2\cos\frac{4\pi}{7} + 2\cos\frac{6\pi}{7} =$

$\cos\frac{2\pi}{7} + \cos\frac{4\pi}{7} + \cos\frac{6\pi}{7} = -\frac{1}{2}$

AS REQUIRED

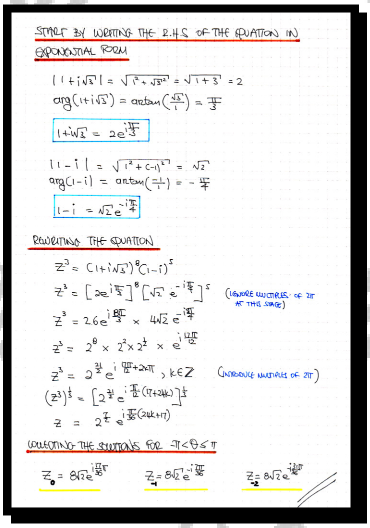
Question 76 (****)

$$z^3 = \left(1 + \mathrm{i}\sqrt{3}\right)^8 (1 - \mathrm{i})^5 \, , \ z \in \mathbb{C} \, .$$

Determine the three roots of the above equation.

Give the answers in the form $k\sqrt{2}\,\mathrm{e}^{\mathrm{i}\theta}$, where $-\pi < \theta \le \pi$, $k \in \mathbb{Z}$.

$\boxed{}$, $\boxed{z = 8\sqrt{2}\,\mathrm{e}^{\mathrm{i}\theta}, \quad \theta = -\frac{31\pi}{36}, \ -\frac{7\pi}{36}, \ \frac{17\pi}{36}}$



START BY WRITING THE R.H.S OF THE EQUATION IN EXPONENTIAL FORM

$$|1+i\sqrt{3}| = \sqrt{1^2+\sqrt{3}^2} = \sqrt{1+3} = 2$$

$$\arg(1+i\sqrt{3}) = \arctan\left(\frac{\sqrt{3}}{1}\right) = \frac{\pi}{3}$$

$$\boxed{1+i\sqrt{3} = 2e^{i\frac{\pi}{3}}}$$

$$|1-i| = \sqrt{1^2+(-1)^2} = \sqrt{2}$$

$$\arg(1-i) = \arctan\left(\frac{-1}{1}\right) = -\frac{\pi}{4}$$

$$\boxed{1-i = \sqrt{2}e^{-i\frac{\pi}{4}}}$$

REWRITING THE EQUATION

$$z^3 = (1+i\sqrt{3})^8(1-i)^5$$

$$z^3 = \left[2e^{i\frac{\pi}{3}}\right]^8\left[\sqrt{2}e^{-i\frac{\pi}{4}}\right]^5$$

(IGNORE MULTIPLES OF 2π AT THIS STAGE)

$$z^3 = 2^8 e^{i\frac{8\pi}{3}} \times 4\sqrt{2}\,e^{-i\frac{5\pi}{4}}$$

$$z^3 = 2^8 \times 2^2 \times 2^{\frac{1}{2}} \times e^{i\frac{17\pi}{12}}$$

$$z^3 = 2^{\frac{21}{2}} e^{i\frac{17\pi}{12}+2k\pi}, \ k \in \mathbb{Z}$$

(INTRODUCE MULTIPLES OF 2π)

$$(z^3)^{\frac{1}{3}} = \left[2^{\frac{21}{2}} e^{i\frac{\pi}{12}(17+24k)}\right]^{\frac{1}{3}}$$

$$z = 2^{\frac{7}{2}} e^{i\frac{\pi}{36}(24k+17)}$$

COLLECTING THE SOLUTIONS FOR $-\pi < \theta \le \pi$

$z_0 = 8\sqrt{2}e^{i\frac{17}{36}\pi}$ $z_{-1} = 8\sqrt{2}e^{-i\frac{7\pi}{36}}$ $z_{-2} = 8\sqrt{2}e^{-i\frac{31\pi}{36}}$

Question 77 **(****)**

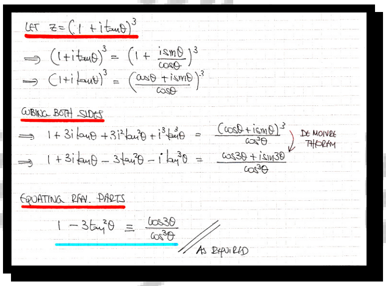
The complex number is defined as

$$z = \left(1 + \mathrm{i}\tan\theta\right)^3, \ -\frac{\pi}{2} < \theta < \frac{\pi}{2}.$$

By considering the real part of z, or otherwise, prove the validity of the following trigonometric identity

$$1 - 3\tan^2\theta \equiv \frac{\cos 3\theta}{\cos^3\theta}.$$

$\boxed{\quad\quad}$, $\boxed{\text{proof}}$



LET $z = (1 + i\tan\theta)^3$

$\Rightarrow (1 + i\tan\theta)^3 = \left(1 + \frac{i\sin\theta}{\cos\theta}\right)^3$

$\Rightarrow (1 + i\tan\theta)^3 = \left(\frac{\cos\theta + i\sin\theta}{\cos\theta}\right)^3$

CUBING BOTH SIDES

$\Rightarrow 1 + 3i\tan\theta + 3i^2\tan^2\theta + i^3\tan^3\theta = \frac{(\cos\theta + i\sin\theta)^3}{\cos^3\theta}$ DE MOIVRE THEOREM

$\Rightarrow 1 + 3i\tan\theta - 3\tan^2\theta - i\tan^3\theta = \frac{\cos 3\theta + i\sin 3\theta}{\cos^3\theta}$

EQUATING REAL PARTS

$1 - 3\tan^2\theta \equiv \frac{\cos 3\theta}{\cos^3\theta}$ // AS REQUIRED

Question 78 **(****)**

Consider the following expression

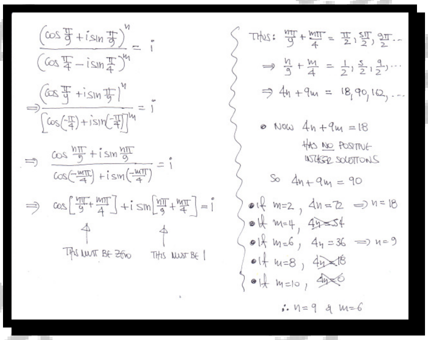
$$\frac{\left(\cos\frac{\pi}{9}+\mathrm{i}\sin\frac{\pi}{9}\right)^{n}}{\left(\cos\frac{\pi}{4}-\mathrm{i}\sin\frac{\pi}{4}\right)^{m}}=\mathrm{i}$$

The values of n and m are such so that

$$\{m\in\mathbb{N}:1\le m\le 9\} \text{ and } \{m\in\mathbb{N}:1\le m\le 9\}.$$

Determine, by a full mathematical method, the value of n and the value of m.

$\boxed{m=6}$, $\boxed{n=9}$



$$\frac{\left(\cos\frac{\pi}{9}+i\sin\frac{\pi}{9}\right)^{n}}{\left(\cos\frac{\pi}{4}-i\sin\frac{\pi}{4}\right)^{m}}=i$$

$$\Rightarrow \frac{\left(\cos\frac{\pi}{9}+i\sin\frac{\pi}{9}\right)^{n}}{\left[\cos\left(-\frac{\pi}{4}\right)+i\sin\left(-\frac{\pi}{4}\right)\right]^{m}}=i$$

$$\Rightarrow \frac{\cos\frac{n\pi}{9}+i\sin\frac{n\pi}{9}}{\cos\left(-\frac{m\pi}{4}\right)+i\sin\left(-\frac{m\pi}{4}\right)}=i$$

$$\Rightarrow \cos\left[\frac{n\pi}{9}+\frac{m\pi}{4}\right]+i\sin\left[\frac{n\pi}{9}+\frac{m\pi}{4}\right]=i$$

THIS MUST BE ZERO THIS MUST BE 1

Thus: $\frac{n\pi}{9}+\frac{m\pi}{4}=\frac{\pi}{2},\frac{5\pi}{2},\frac{9\pi}{2}\dots$

$\Rightarrow \frac{n}{9}+\frac{m}{4}=\frac{1}{2},\frac{5}{2},\frac{9}{2},\dots$

$\Rightarrow 4n+9m=18,90,162,\dots$

- Now $4n+9m=18$ HAS NO POSITIVE INTEGER SOLUTIONS

So $4n+9m=90$

- if $m=2$, $4n=72 \Rightarrow n=18$
- if $m=4$, $\cancel{4n=54}$
- if $m=6$, $4n=36 \Rightarrow n=9$
- if $m=8$, $\cancel{4n=18}$
- if $m=10$, $\cancel{4n=0}$

$\therefore n=9$ & $m=6$

Question 79 (****)

A transformation T maps points of the form $z = x + \mathrm{i}y$ from the z plane onto points of the form $v = u + \mathrm{i}v$ in the w plane, and is defined as

$$w = \frac{2}{\overline{z} - 1}, \ z \in \mathbb{C}, \ z \neq 1,$$

where $\overline{z}$ is the complex conjugate of z.

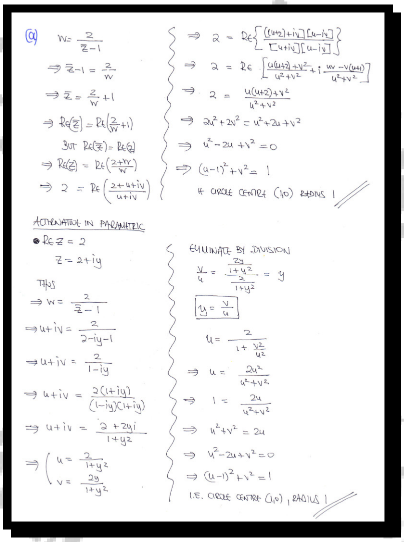
The line with equation $\operatorname{Re} z = 2$ is mapped by T onto a circle C in the w plane.

a) Determine the coordinates of the centre of C and the length of its radius.

b) Find an equation of the image in the w plane of the half line with equation

$$\arg(z-1) = \frac{\pi}{3}.$$

$\boxed{(1,0), \quad r = 1}$, $\boxed{\arg w = \frac{\pi}{3}}$



(a) $w = \frac{2}{\overline{z}-1}$

$\Rightarrow \overline{z} - 1 = \frac{2}{w}$

$\Rightarrow \overline{z} = \frac{2}{w} + 1$

$\Rightarrow \operatorname{Re}(\overline{z}) = \operatorname{Re}\left(\frac{2}{w}+1\right)$

BUT $\operatorname{Re}(\overline{z}) = \operatorname{Re}(z)$

$\Rightarrow \operatorname{Re}(z) = \operatorname{Re}\left(\frac{2+w}{w}\right)$

$\Rightarrow 2 = \operatorname{Re}\left(\frac{2+u+iv}{u+iv}\right)$

$\Rightarrow 2 = \operatorname{Re}\left\{\frac{[(u+2)+iv][u-iv]}{[u+iv][u-iv]}\right\}$

$\Rightarrow 2 = \operatorname{Re}\left[\frac{u(u+2)+v^2}{u^2+v^2} + i\,\frac{uv - v(u+1)}{u^2+v^2}\right]$

$\Rightarrow 2 = \frac{u(u+2)+v^2}{u^2+v^2}$

$\Rightarrow 2u^2 + 2v^2 = u^2 + 2u + v^2$

$\Rightarrow u^2 - 2u + v^2 = 0$

$\Rightarrow (u-1)^2 + v^2 = 1$

∴ CIRCLE CENTRE $(1,0)$ RADIUS 1

ALTERNATIVE IN PARAMETRIC

- $\operatorname{Re} z = 2$

$z = 2 + iy$

THUS

$\Rightarrow w = \frac{2}{\overline{z}-1}$

$\Rightarrow u + iv = \frac{2}{2 - iy - 1}$

$\Rightarrow u + iv = \frac{2}{1 - iy}$

$\Rightarrow u + iv = \frac{2(1+iy)}{(1-iy)(1+iy)}$

$\Rightarrow u + iv = \frac{2 + 2yi}{1+y^2}$

$\Rightarrow \begin{cases} u = \frac{2}{1+y^2} \\ v = \frac{2y}{1+y^2} \end{cases}$

ELIMINATE BY DIVISION

$\frac{v}{u} = \frac{\frac{2y}{1+y^2}}{\frac{2}{1+y^2}} = y$

$\boxed{y = \frac{v}{u}}$

$u = \frac{2}{1 + \frac{v^2}{u^2}}$

$\Rightarrow u = \frac{2u^2}{u^2+v^2}$

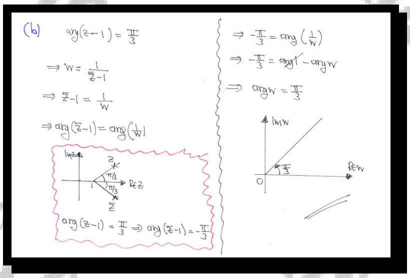
$\Rightarrow 1 = \frac{2u}{u^2+v^2}$

$\Rightarrow u^2 + v^2 = 2u$

$\Rightarrow u^2 - 2u + v^2 = 0$

$\Rightarrow (u-1)^2 + v^2 = 1$

I.E. CIRCLE CENTRE $(1,0)$, RADIUS 1



(b) $\arg(z-1) = \frac{\pi}{3}$

$\Rightarrow w = \frac{1}{\overline{z}-1}$

$\Rightarrow \overline{z} - 1 = \frac{1}{w}$

$\Rightarrow \arg(\overline{z}-1) = \arg\left(\frac{1}{w}\right)$

$\arg(z-1) = \frac{\pi}{3} \Rightarrow \arg(\overline{z}-1) = -\frac{\pi}{3}$

$\Rightarrow -\frac{\pi}{3} = \arg\left(\frac{1}{w}\right)$

$\Rightarrow -\frac{\pi}{3} = \cancel{\arg 1} - \arg w$

$\Rightarrow \arg w = \frac{\pi}{3}$

Question 80 (****)

A complex function $w = f(z)$ is defined as

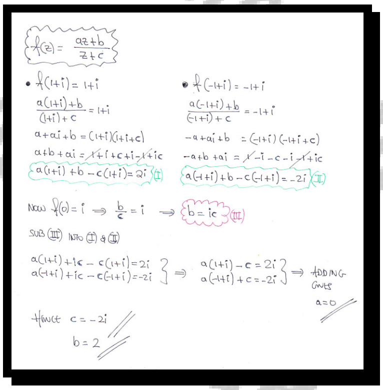
$$w = \frac{az+b}{z+c}, \ z \in \mathbb{C}, \ z \neq -c .$$

The constants a, b and c are complex.

Under the function f the points $1+\text{i}$ and $-1+\text{i}$ are invariant, while the origin is mapped onto i.

Determine the values of the constants a, b and c.

$\boxed{a=0}$, $\boxed{b=2}$, $\boxed{c=-2\text{i}}$



$f(z) = \frac{az+b}{z+c}$

- $f(1+i) = 1+i$

$\frac{a(1+i)+b}{(1+i)+c} = 1+i$

$a+ai+b = (1+i)(1+i+c)$

$a+b+ai = \cancel{1}+i+c+i-\cancel{1}+ic$

$a(1+i)+b-c(1+i) = 2i$ (I)

- $f(-1+i) = -1+i$

$\frac{a(-1+i)+b}{(-1+i)+c} = -1+i$

$-a+ai+b = (-1+i)(-1+i+c)$

$-a+b+ai = \cancel{1}-i-c-i-\cancel{1}+ic$

$a(-1+i)+b-c(-1+i) = -2i$ (II)

NOW $f(0) = i \Rightarrow \frac{b}{c} = i \Rightarrow b = ic$ (III)

SUB (III) INTO (I) & (II)

$\left.\begin{array}{l} a(1+i)+ic-c(1+i)=2i \\ a(-1+i)+ic-c(-1+i)=-2i \end{array}\right\} \Rightarrow \left.\begin{array}{l} a(1+i)-c=2i \\ a(-1+i)+c=-2i \end{array}\right\} \Rightarrow$ ADDING GIVES

$a=0$

HENCE $c=-2i$

$b=2$

Question 81 (****)

De Moivre's theorem asserts that

$$(\cos\theta + \mathrm{i}\sin\theta)^n \equiv \cos n\theta + \mathrm{i}\sin n\theta,\ \theta \in \mathbb{R},\ n \in \mathbb{Q}.$$

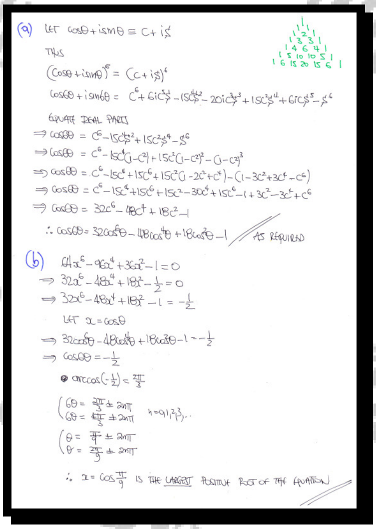
a) Use the theorem to prove the validity of the following trigonometric identity.

$$\cos 6\theta \equiv 32\cos^6\theta - 48\cos^4\theta + 18\cos^2\theta - 1.$$

b) Use the result of part (a) to find, in exact form, the largest positive root of the equation

$$64x^6 - 96x^4 + 36x^2 - 1 = 0.$$

$$\boxed{x = \cos\left(\frac{\pi}{9}\right)}$$



(a) LET $\cos\theta + i\sin\theta \equiv c + is$

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1

THUS

$(\cos\theta + i\sin\theta)^6 = (c+is)^6$

$\cos 6\theta + i\sin 6\theta = c^6 + 6ic^5s - 15c^4s^2 - 20ic^3s^3 + 15c^2s^4 + 6ics^5 - s^6$

EQUATE REAL PARTS

$\Rightarrow \cos 6\theta = c^6 - 15c^4s^2 + 15c^2s^4 - s^6$

$\Rightarrow \cos 6\theta = c^6 - 15c^4(1-c^2) + 15c^2(1-c^2)^2 - (1-c^2)^3$

$\Rightarrow \cos 6\theta = c^6 - 15c^4 + 15c^6 + 15c^2(1-2c^2+c^4) - (1-3c^2+3c^4-c^6)$

$\Rightarrow \cos 6\theta = c^6 - 15c^4 + 15c^6 + 15c^2 - 30c^4 + 15c^6 - 1 + 3c^2 - 3c^4 + c^6$

$\Rightarrow \cos 6\theta = 32c^6 - 48c^4 + 18c^2 - 1$

$\therefore \cos 6\theta = 32\cos^6\theta - 48\cos^4\theta + 18\cos^2\theta - 1$ // AS REQUIRED

(b) $64x^6 - 96x^4 + 36x^2 - 1 = 0$

$\Rightarrow 32x^6 - 48x^4 + 18x^2 - \frac{1}{2} = 0$

$\Rightarrow 32x^6 - 48x^4 + 18x^2 - 1 = -\frac{1}{2}$

LET $x = \cos\theta$

$\Rightarrow 32\cos^6\theta - 48\cos^4\theta + 18\cos^2\theta - 1 = -\frac{1}{2}$

$\Rightarrow \cos 6\theta = -\frac{1}{2}$

● $\arccos(-\frac{1}{2}) = \frac{2\pi}{3}$

$\begin{cases} 6\theta = \frac{2\pi}{3} \pm 2n\pi \\ 6\theta = \frac{4\pi}{3} \pm 2n\pi \end{cases}$ $n = 0, 1, 2, 3, \ldots$

$\begin{cases} \theta = \frac{\pi}{9} \pm 2n\pi \\ \theta = \frac{2\pi}{9} \pm 2n\pi \end{cases}$

$\therefore x = \cos\frac{\pi}{9}$ IS THE LARGEST POSITIVE ROOT OF THE EQUATION //

Question 82 **(****)**

A transformation of the z plane to the w plane is given by

$$w = \frac{1}{z-2}, \ z \in \mathbb{C}, \ z \neq 2$$

where $z = x + \mathrm{i}y$ and $w = u + \mathrm{i}v$.

The line with equation

$$2x + y = 3$$

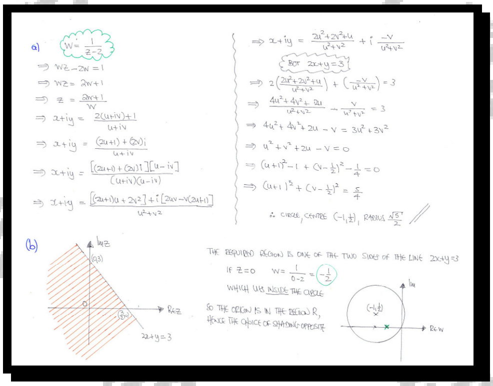
is mapped in the w plane onto a curve C.

- **a)** Show that C represents a circle and determine the coordinates of its centre and the size of its radius.

The points of a region R in the z plane are mapped onto the points which lie inside C in the w plane.

- **b)** Sketch and shade R in a suitable labelled Argand diagram, fully justifying the choice of region.

$$\boxed{\text{centre at } \left(-1, \tfrac{1}{2}\right), \ \text{radius} = \frac{\sqrt{5}}{2}}$$



a) $w = \frac{1}{z-2}$

$\Rightarrow wz - 2w = 1$

$\Rightarrow wz = 2w + 1$

$\Rightarrow z = \frac{2w+1}{w}$

$\Rightarrow x + iy = \frac{2(u+iv)+1}{u+iv}$

$\Rightarrow x + iy = \frac{(2u+1) + (2v)i}{u+iv}$

$\Rightarrow x + iy = \frac{[(2u+1)+(2v)i][u-iv]}{(u+iv)(u-iv)}$

$\Rightarrow x + iy = \frac{[(2u+1)u + 2v^2] + i[2uv - v(2u+1)]}{u^2+v^2}$

$\Rightarrow x + iy = \frac{2u^2+2v^2+u}{u^2+v^2} + i\frac{-v}{u^2+v^2}$

BUT $2x+y=3$

$\Rightarrow 2\left(\frac{2u^2+2v^2+u}{u^2+v^2}\right) + \left(\frac{-v}{u^2+v^2}\right) = 3$

$\Rightarrow \frac{4u^2+4v^2+2u}{u^2+v^2} - \frac{v}{u^2+v^2} = 3$

$\Rightarrow 4u^2+4v^2+2u-v = 3u^2+3v^2$

$\Rightarrow u^2+v^2+2u-v = 0$

$\Rightarrow (u+1)^2 - 1 + (v-\frac{1}{2})^2 - \frac{1}{4} = 0$

$\Rightarrow (u+1)^2 + (v-\frac{1}{2})^2 = \frac{5}{4}$

$\therefore$ CIRCLE, CENTRE $(-1, \frac{1}{2})$, RADIUS $\frac{\sqrt{5}}{2}$

(b)

THE REQUIRED REGION IS ONE OF THE TWO SIDES OF THE LINE $2x+y=3$

IF $z=0$ $\quad w = \frac{1}{0-2} = -\frac{1}{2}$

WHICH LIES INSIDE THE CIRCLE

SO THE ORIGIN IS IN THE REGION R, HENCE THE CHOICE OF SHADING OPPOSITE

Question 83 (****)

The locus of the point z in the Argand diagram, satisfy the equation

$$|z-2+\mathrm{i}|=\sqrt{3}\,.$$

a) Sketch the locus represented by the above equation.

The half line L with equation

$$y=mx-1, \quad x \geq 0, \;\; m>0,$$

touches the locus described in part (a) at the point P.

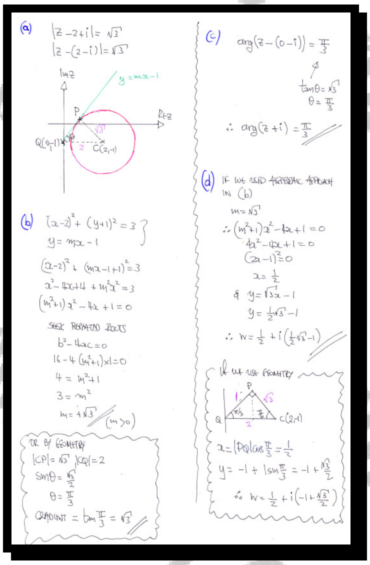
b) Find the value of m.

c) Write the equation of L, in the form

$$\arg\left(z-z_0\right)=\theta, \;\; z_0 \in \mathbb{C}, \; -\pi<\theta \leq \pi\,.$$

d) Find the complex number w, represented by the point P.

$\boxed{m=\sqrt{3}}$, $\boxed{\arg\left(z+\mathrm{i}\right)=\frac{\pi}{3}}$, $\boxed{w=\frac{1}{2}+\mathrm{i}\left(\frac{\sqrt{3}}{2}-1\right)}$



(a) $|z-2+i|=\sqrt{3}$

$|z-(2-i)|=\sqrt{3}$

(b) $(x-2)^2+(y+1)^2=3$

$y=mx-1$

$(x-2)^2+(mx-1+1)^2=3$

$x^2-4x+4+m^2x^2=3$

$(m^2+1)x^2-4x+1=0$

SEEK REPEATED ROOTS

$b^2-4ac=0$

$16-4(m^2+1)\times 1=0$

$4=m^2+1$

$3=m^2$

$m=+\sqrt{3}$ $(m>0)$

OR BY GEOMETRY

$|CP|=\sqrt{3}$, $|CQ|=2$

$\sin\theta=\frac{\sqrt{3}}{2}$

$\theta=\frac{\pi}{3}$

GRADIENT $=\tan\frac{\pi}{3}=\sqrt{3}$

(c) $\arg(z-(0-i))=\frac{\pi}{3}$

$\tan\theta=\sqrt{3}$

$\theta=\frac{\pi}{3}$

$\therefore \arg(z+i)=\frac{\pi}{3}$

(d) IF WE USED ALGEBRAIC APPROACH IN (b)

$m=\sqrt{3}$

$\therefore (m^2+1)x^2-4x+1=0$

$4x^2-4x+1=0$

$(2x-1)^2=0$

$x=\frac{1}{2}$

& $y=\sqrt{3}x-1$

$y=\frac{1}{2}\sqrt{3}-1$

$\therefore w=\frac{1}{2}+i\left(\frac{1}{2}\sqrt{3}-1\right)$

If WE USE GEOMETRY

$x=|PQ|\cos\frac{\pi}{3}=\frac{1}{2}$

$y=-1+1\sin\frac{\pi}{3}=-1+\frac{\sqrt{3}}{2}$

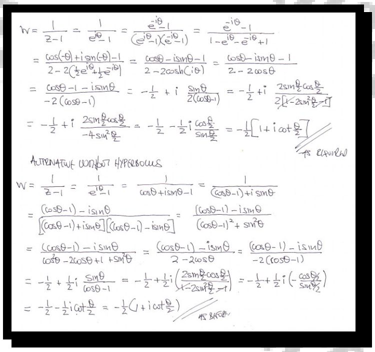
$\therefore w=\frac{1}{2}+i\left(-1+\frac{\sqrt{3}}{2}\right)$

Question 84 **(****)**

If $z = e^{i\theta}$, $-\pi < \theta \le \pi$, and $w = \dfrac{1}{z-1}$ show clearly that

$$w = -\frac{1}{2}\left[1 + i\cot\left(\frac{\theta}{2}\right)\right].$$

$\boxed{\text{proof}}$



$$w = \frac{1}{z-1} = \frac{1}{e^{i\theta}-1} = \frac{e^{-i\theta}-1}{(e^{i\theta}-1)(e^{-i\theta}-1)} = \frac{e^{-i\theta}-1}{1-e^{i\theta}-e^{-i\theta}+1}$$

$$= \frac{\cos(-\theta)+i\sin(-\theta)-1}{2-2\left(\frac{1}{2}e^{i\theta}+\frac{1}{2}e^{-i\theta}\right)} = \frac{\cos\theta - i\sin\theta - 1}{2-2\cosh(i\theta)} = \frac{\cos\theta - i\sin\theta - 1}{2-2\cos\theta}$$

$$= \frac{\cos\theta - 1 - i\sin\theta}{-2(\cos\theta-1)} = -\frac{1}{2} + i\,\frac{\sin\theta}{2(\cos\theta-1)} = -\frac{1}{2} + i\,\frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\left[\cancel{1}-2\sin^2\frac{\theta}{2}\cancel{-1}\right]}$$

$$= -\frac{1}{2} + i\,\frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{-4\sin^2\frac{\theta}{2}} = -\frac{1}{2} - \frac{1}{2}i\,\frac{\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}} = -\frac{1}{2}\left[1 + i\cot\frac{\theta}{2}\right]$$

AS REQUIRED

ALTERNATIVE WITHOUT HYPERBOLICS

$$w = \frac{1}{z-1} = \frac{1}{e^{i\theta}-1} = \frac{1}{\cos\theta + i\sin\theta - 1} = \frac{1}{(\cos\theta-1)+i\sin\theta}$$

$$= \frac{(\cos\theta-1)-i\sin\theta}{[(\cos\theta-1)+i\sin\theta][(\cos\theta-1)-i\sin\theta]} = \frac{(\cos\theta-1)-i\sin\theta}{(\cos\theta-1)^2+\sin^2\theta}$$

$$= \frac{(\cos\theta-1)-i\sin\theta}{\cos^2\theta - 2\cos\theta + 1 + \sin^2\theta} = \frac{(\cos\theta-1)-i\sin\theta}{2-2\cos\theta} = \frac{(\cos\theta-1)-i\sin\theta}{-2(\cos\theta-1)}$$

$$= -\frac{1}{2} + \frac{1}{2}i\,\frac{\sin\theta}{\cos\theta-1} = -\frac{1}{2} + \frac{1}{2}i\left(\frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{\cancel{1}-2\sin^2\frac{\theta}{2}\cancel{-1}}\right) = -\frac{1}{2} + \frac{1}{2}i\left(-\frac{\cos\theta/2}{\sin\theta/2}\right)$$

$$= -\frac{1}{2} - \frac{1}{2}i\cot\frac{\theta}{2} = -\frac{1}{2}\left(1 + i\cot\frac{\theta}{2}\right)$$

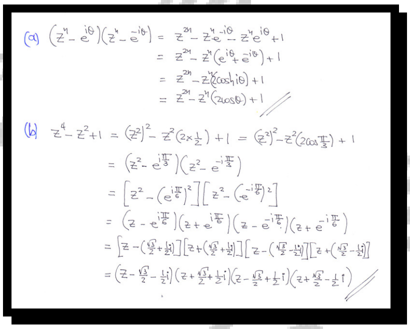
AS BEFORE

Question 85 **(****)**

a) Simplify fully $\left(z^n - e^{i\theta}\right)\left(z^n - e^{-i\theta}\right)$.

b) Hence factorize $z^4 - z^2 + 1$ into 4 linear complex factors.

$\boxed{z^{2n} - z^n(2\cos\theta) + 1}$, $\boxed{\left(z+\frac{\sqrt{3}}{2}+\frac{1}{2}i\right)\left(z+\frac{\sqrt{3}}{2}-\frac{1}{2}i\right)\left(z-\frac{\sqrt{3}}{2}+\frac{1}{2}i\right)\left(z-\frac{\sqrt{3}}{2}-\frac{1}{2}i\right)}$



(a) $\left(z^n - e^{i\theta}\right)\left(z^n - e^{-i\theta}\right) = z^{2n} - z^n e^{-i\theta} - z^n e^{i\theta} + 1$

$= z^{2n} - z^n\left(e^{i\theta} + e^{-i\theta}\right) + 1$

$= z^{2n} - z^n(2\cosh i\theta) + 1$

$= z^{2n} - z^n(2\cos\theta) + 1$

(b) $z^4 - z^2 + 1 = (z^2)^2 - z^2\left(2\times\frac{1}{2}\right) + 1 = (z^2)^2 - z^2\left(2\cos\frac{\pi}{3}\right) + 1$

$= \left(z^2 - e^{i\frac{\pi}{3}}\right)\left(z^2 - e^{-i\frac{\pi}{3}}\right)$

$= \left[z^2 - \left(e^{i\frac{\pi}{6}}\right)^2\right]\left[z^2 - \left(e^{-i\frac{\pi}{6}}\right)^2\right]$

$= \left(z - e^{i\frac{\pi}{6}}\right)\left(z + e^{i\frac{\pi}{6}}\right)\left(z - e^{-i\frac{\pi}{6}}\right)\left(z + e^{-i\frac{\pi}{6}}\right)$

$= \left[z - \left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right)\right]\left[z + \left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right)\right]\left[z - \left(\frac{\sqrt{3}}{2} - \frac{1}{2}i\right)\right]\left[z + \left(\frac{\sqrt{3}}{2} - \frac{1}{2}i\right)\right]$

$= \left(z - \frac{\sqrt{3}}{2} - \frac{1}{2}i\right)\left(z + \frac{\sqrt{3}}{2} + \frac{1}{2}i\right)\left(z - \frac{\sqrt{3}}{2} + \frac{1}{2}i\right)\left(z + \frac{\sqrt{3}}{2} - \frac{1}{2}i\right)$

Question 86 (****)

Let $z = \cos\theta + \mathrm{i}\sin\theta = C + \mathrm{i}S$, $-\pi < \theta \le \pi$.

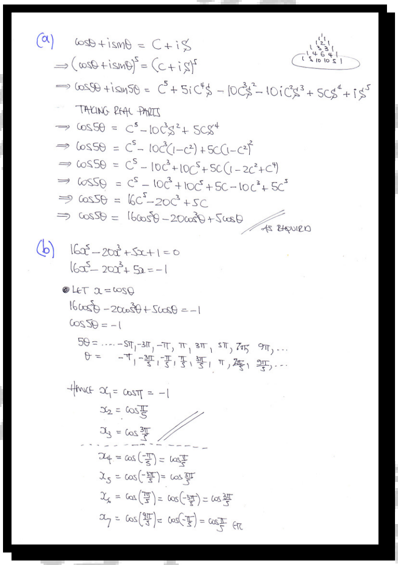
a) Use De Moivre's theorem to show that

$$\cos 5\theta \equiv 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta .$$

b) Hence or otherwise find, in exact form where appropriate, 3 distinct solutions of the quintic equation

$$16x^5 - 20x^3 + 5x + 1 = 0 .$$

$$\boxed{x = -1,\ \cos\frac{\pi}{5},\ \cos\frac{3\pi}{5}}$$



(a) $\cos\theta + i\sin\theta = C + iS$

$\begin{matrix} 1 \\ 1\ 1 \\ 1\ 2\ 1 \\ 1\ 3\ 3\ 1 \\ 1\ 4\ 6\ 4\ 1 \\ 1\ 5\ 10\ 10\ 5\ 1 \end{matrix}$

$\Rightarrow (\cos\theta + i\sin\theta)^5 = (C + iS)^5$

$\Rightarrow \cos 5\theta + i\sin 5\theta = C^5 + 5iC^4S - 10C^3S^2 - 10iC^2S^3 + 5CS^4 + iS^5$

TAKING REAL PARTS

$\Rightarrow \cos 5\theta = C^5 - 10C^3S^2 + 5CS^4$

$\Rightarrow \cos 5\theta = C^5 - 10C^3(1-C^2) + 5C(1-C^2)^2$

$\Rightarrow \cos 5\theta = C^5 - 10C^3 + 10C^5 + 5C(1 - 2C^2 + C^4)$

$\Rightarrow \cos 5\theta = C^5 - 10C^3 + 10C^5 + 5C - 10C^3 + 5C^5$

$\Rightarrow \cos 5\theta = 16C^5 - 20C^3 + 5C$

$\Rightarrow \cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$ // AS REQUIRED

(b) $16x^5 - 20x^3 + 5x + 1 = 0$

$16x^5 - 20x^3 + 5x = -1$

● LET $x = \cos\theta$

$16\cos^5\theta - 20\cos^3\theta + 5\cos\theta = -1$

$\cos 5\theta = -1$

$5\theta = \ldots -5\pi, -3\pi, -\pi, \pi, 3\pi, 5\pi, 7\pi, 9\pi, \ldots$

$\theta = -\pi, -\frac{3\pi}{5}, -\frac{\pi}{5}, \frac{\pi}{5}, \frac{3\pi}{5}, \pi, \frac{7\pi}{5}, \frac{9\pi}{5}, \ldots$

HENCE $x_1 = \cos\pi = -1$

$x_2 = \cos\frac{\pi}{5}$

$x_3 = \cos\frac{3\pi}{5}$ //

- - - - - - - - - - - - -

$x_4 = \cos\left(-\frac{\pi}{5}\right) = \cos\frac{\pi}{5}$

$x_5 = \cos\left(-\frac{3\pi}{5}\right) = \cos\frac{3\pi}{5}$

$x_6 = \cos\left(\frac{7\pi}{5}\right) = \cos\left(-\frac{3\pi}{5}\right) = \cos\frac{3\pi}{5}$

$x_7 = \cos\left(\frac{9\pi}{5}\right) = \cos\left(-\frac{\pi}{5}\right) = \cos\frac{\pi}{5}$ ETC

Question 87 (****)

Euler's identity states

$$e^{i\theta} \equiv \cos\theta + i\sin\theta, \ \theta \in \mathbb{R}.$$

a) Use the identity to show that

$$e^{in\theta} + e^{-in\theta} \equiv 2\cos n\theta.$$

b) Hence show further that

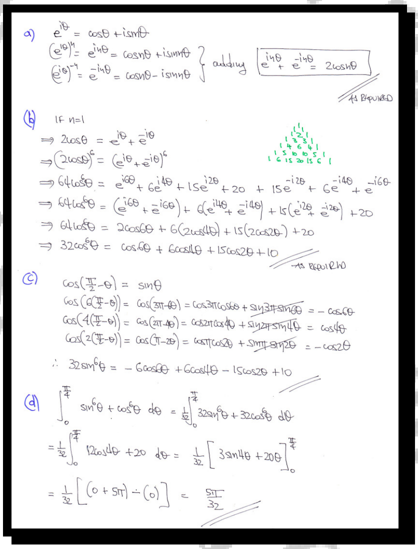
$$32\cos^6\theta \equiv \cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10.$$

c) Use the fact that $\cos\left(\frac{\pi}{2} - \theta\right) \equiv \sin\theta$ to find a similar expression for $32\sin^6\theta$.

d) Determine the exact value of

$$\int_0^{\frac{\pi}{4}} \sin^6\theta + \cos^6\theta \ d\theta.$$

$\boxed{32\sin^6\theta = -\cos 6\theta + 6\cos 4\theta - 15\cos 2\theta + 10}$, $\boxed{\frac{5\pi}{32}}$



a) $e^{i\theta} = \cos\theta + i\sin\theta$

$(e^{i\theta})^n = e^{in\theta} = \cos n\theta + i\sin n\theta$

$(e^{i\theta})^{-n} = e^{-in\theta} = \cos n\theta - i\sin n\theta$

adding $\boxed{e^{in\theta} + e^{-in\theta} = 2\cos n\theta}$

AS REQUIRED

b) IF $n=1$

$\Rightarrow 2\cos\theta = e^{i\theta} + e^{-i\theta}$

$\Rightarrow (2\cos\theta)^6 = (e^{i\theta} + e^{-i\theta})^6$

$\Rightarrow 64\cos^6\theta = e^{i6\theta} + 6e^{i4\theta} + 15e^{i2\theta} + 20 + 15e^{-i2\theta} + 6e^{-i4\theta} + e^{-i6\theta}$

$\Rightarrow 64\cos^6\theta = (e^{i6\theta} + e^{-i6\theta}) + 6(e^{i4\theta} + e^{-i4\theta}) + 15(e^{i2\theta} + e^{-i2\theta}) + 20$

$\Rightarrow 64\cos^6\theta = 2\cos 6\theta + 6(2\cos 4\theta) + 15(2\cos 2\theta) + 20$

$\Rightarrow 32\cos^6\theta = \cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10$

AS REQUIRED

1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1

c) $\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$

$\cos\left(6\left(\frac{\pi}{2} - \theta\right)\right) = \cos(3\pi - 6\theta) = \cos 3\pi\cos 6\theta + \cancel{\sin 3\pi \sin 6\theta} = -\cos 6\theta$

$\cos\left(4\left(\frac{\pi}{2} - \theta\right)\right) = \cos(2\pi - 4\theta) = \cos 2\pi\cos 4\theta + \cancel{\sin 2\pi \sin 4\theta} = \cos 4\theta$

$\cos\left(2\left(\frac{\pi}{2} - \theta\right)\right) = \cos(\pi - 2\theta) = \cos\pi\cos 2\theta + \cancel{\sin\pi \sin 2\theta} = -\cos 2\theta$

$\therefore 32\sin^6\theta = -6\cos 6\theta + 6\cos 4\theta - 15\cos 2\theta + 10$

d) $\int_0^{\frac{\pi}{4}} \sin^6\theta + \cos^6\theta \ d\theta = \frac{1}{32}\int_0^{\frac{\pi}{4}} 32\sin^6\theta + 32\cos^6\theta \ d\theta$

$= \frac{1}{32}\int_0^{\frac{\pi}{4}} 12\cos 4\theta + 20 \ d\theta = \frac{1}{32}\left[3\sin 4\theta + 20\theta\right]_0^{\frac{\pi}{4}}$

$= \frac{1}{32}\left[(0 + 5\pi) - (0)\right] = \frac{5\pi}{32}$

Question 88 **(****)**

A transformation of the z plane to the w plane is given by

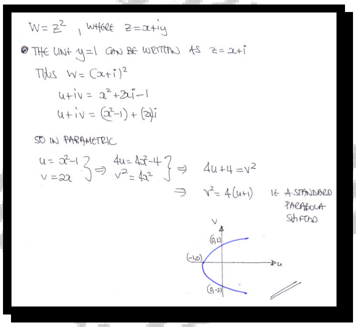
$$w = z^2, \; z \in \mathbb{C},$$

where $z = x + \mathrm{i}y$ and $w = u + \mathrm{i}v$.

The straight line with equation $y = 1$ is mapped in the w plane onto a curve C.

Sketch the graph of C, marking clearly the coordinates of all points where the graph of C meets the coordinate axes.

$\boxed{\text{sketch}}$



$W = Z^2$, WHERE $Z = x+iy$

- THE LINE $y=1$ CAN BE WRITTEN AS $z = x+i$

THUS $W = (x+i)^2$

$u+iv = x^2+2xi-1$

$u+iv = (x^2-1)+(2x)i$

SO IN PARAMETRIC

$$\left.\begin{array}{l} u = x^2-1 \\ v = 2x \end{array}\right\} \Rightarrow \left.\begin{array}{l} 4u = 4x^2-4 \\ v^2 = 4x^2 \end{array}\right\} \Rightarrow 4u+4 = v^2$$

$\Rightarrow v^2 = 4(u+1)$ IE A STANDARD PARABOLA SHIFTED

Question 89 (****)

De Moivre’s theorem asserts that

$$(\cos\theta + \mathrm{i}\sin\theta)^n \equiv \cos n\theta + \mathrm{i}\sin n\theta,\ \theta \in \mathbb{R},\ n \in \mathbb{Q}.$$

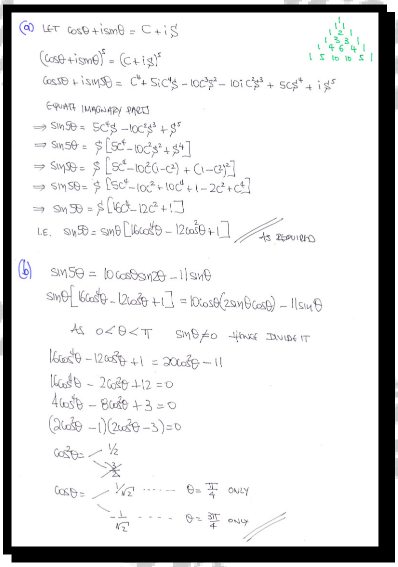
a) Use the theorem to prove validity of the following trigonometric identity

$$\sin 5\theta = \sin\theta\left(16\cos^4\theta - 12\cos^2\theta + 1\right).$$

b) Hence, or otherwise, solve the equation

$$\sin 5\theta = 10\cos\theta\sin 2\theta - 11\sin\theta,\ 0 < \theta < \pi.$$

$$\boxed{\theta = \frac{\pi}{4}, \frac{3\pi}{4}}$$



(a) LET $\cos\theta + i\sin\theta = C + iS$

$(\cos\theta + i\sin\theta)^5 = (C+iS)^5$

$\cos 5\theta + i\sin 5\theta = C^4 + 5iC^4S - 10C^3S^2 - 10iC^2S^3 + 5CS^4 + iS^5$

EQUATE IMAGINARY PARTS

$\Rightarrow \sin 5\theta = 5C^4S - 10C^2S^3 + S^5$

$\Rightarrow \sin 5\theta = S\left[5C^4 - 10C^2S^2 + S^4\right]$

$\Rightarrow \sin 5\theta = S\left[5C^4 - 10C^2(1-C^2) + (1-C^2)^2\right]$

$\Rightarrow \sin 5\theta = S\left[5C^4 - 10C^2 + 10C^4 + 1 - 2C^2 + C^4\right]$

$\Rightarrow \sin 5\theta = S\left[16C^4 - 12C^2 + 1\right]$

I.E. $\sin 5\theta = \sin\theta\left[16\cos^4\theta - 12\cos^2\theta + 1\right]$ // AS REQUIRED

$$
\begin{array}{cccccc}
 & & & 1 & & \\
 & & 1 & & 1 & \\
 & 1 & & 2 & & 1 \\
1 & & 3 & & 3 & 1 \\
1 & 4 & 6 & & 4 & 1 \\
1 & 5 & 10 & 10 & 5 & 1
\end{array}
$$

(b) $\sin 5\theta = 10\cos\theta\sin 2\theta - 11\sin\theta$

$\sin\theta\left[16\cos^4\theta - 12\cos^2\theta + 1\right] = 10\cos\theta(2\sin\theta\cos\theta) - 11\sin\theta$

AS $0 < \theta < \pi$ $\sin\theta \neq 0$ HENCE DIVIDE IT

$16\cos^4\theta - 12\cos^2\theta + 1 = 20\cos^2\theta - 11$

$16\cos^4\theta - 2\cos^2\theta + 12 = 0$

$4\cos^4\theta - 8\cos^2\theta + 3 = 0$

$(2\cos^2\theta - 1)(2\cos^2\theta - 3) = 0$

$\cos^2\theta = \frac{1}{2}$, $\cancel{\frac{3}{2}}$

$\cos\theta = \frac{1}{\sqrt{2}}$ …… $\theta = \frac{\pi}{4}$ ONLY

$\cos\theta = -\frac{1}{\sqrt{2}}$ …… $\theta = \frac{3\pi}{4}$ ONLY //

Question 90 (****)

A transformation of points from the z plane onto points in the w plane is given by the complex relationship

$$w = z^2, \quad z \in \mathbb{C},$$

where $z = x + \mathrm{i}y$ and $w = u + \mathrm{i}v$.

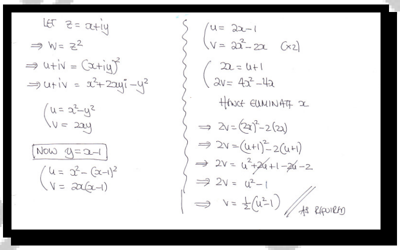
Show that if the point P in the z plane lies on the line with equation

$$y = x - 1,$$

the locus of this point in the w plane satisfies the equation

$$v = \frac{1}{2}\left(u^2 - 1\right).$$

$\boxed{\text{proof}}$



LET $z = x+iy$

$\Rightarrow w = z^2$

$\Rightarrow u+iv = (x+iy)^2$

$\Rightarrow u+iv = x^2+2xyi-y^2$

$\begin{cases} u = x^2-y^2 \\ v = 2xy \end{cases}$

$\boxed{\text{NOW } y = x-1}$

$\begin{cases} u = x^2-(x-1)^2 \\ v = 2x(x-1) \end{cases}$

$\begin{cases} u = 2x-1 \\ v = 2x^2-2x \quad (\times 2) \end{cases}$

$\begin{cases} 2x = u+1 \\ 2v = 4x^2-4x \end{cases}$

HENCE ELIMINATE x

$\Rightarrow 2v = (2x)^2-2(2x)$

$\Rightarrow 2v = (u+1)^2-2(u+1)$

$\Rightarrow 2v = u^2+\cancel{2u}+1-\cancel{2u}-2$

$\Rightarrow 2v = u^2-1$

$\Rightarrow v = \frac{1}{2}(u^2-1)$ // AS REQUIRED

Question 91 (****)

It is given that

$$\sin 5\theta \equiv \sin\theta\left(16\cos^4\theta - 12\cos^2\theta + 1\right).$$

a) Use de Moivre's theorem to prove the validity of the above trigonometric identity.

Consider the general solution of the trigonometric equation

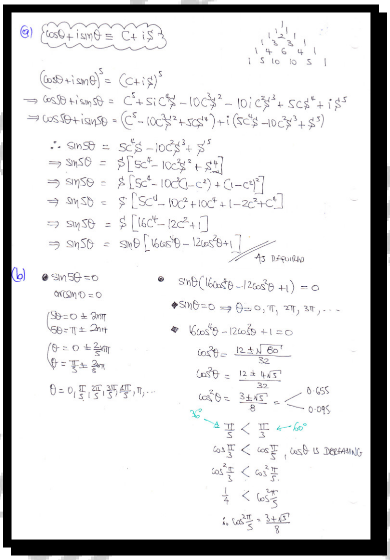
$$\sin 5\theta = 0.$$

b) Find exact simplified expressions for

$$\cos^2\left(\frac{\pi}{5}\right) \text{ and } \cos^2\left(\frac{2\pi}{5}\right),$$

fully justifying each step in the workings.

$$\boxed{\cos^2\left(\frac{\pi}{5}\right) = \frac{3+\sqrt{5}}{8}}, \quad \boxed{\cos^2\left(\frac{2\pi}{5}\right) = \frac{3-\sqrt{5}}{8}}$$



(a) $\cos\theta + i\sin\theta \equiv C + iS$

$$\begin{matrix} &&&&&1&&&&& \\ &&&&1&&1&&&& \\ &&&1&&2&&1&&& \\ &&1&&3&&3&&1&& \\ &1&&4&&6&&4&&1& \\ 1&&5&&10&&10&&5&&1 \end{matrix}$$

$(\cos\theta + i\sin\theta)^5 = (C+iS)^5$

$\Rightarrow \cos 5\theta + i\sin 5\theta = C^5 + 5iC^4S - 10C^3S^2 - 10iC^2S^3 + 5CS^4 + iS^5$

$\Rightarrow \cos 5\theta + i\sin 5\theta = (C^5 - 10C^3S^2 + 5CS^4) + i(5C^4S - 10C^2S^3 + S^5)$

$\therefore \sin 5\theta = 5C^4S - 10C^2S^3 + S^5$

$\Rightarrow \sin 5\theta = S[5C^4 - 10C^2S^2 + S^4]$

$\Rightarrow \sin 5\theta = S[5C^4 - 10C^2(1-C^2) + (1-C^2)^2]$

$\Rightarrow \sin 5\theta = S[5C^4 - 10C^2 + 10C^4 + 1 - 2C^2 + C^4]$

$\Rightarrow \sin 5\theta = S[16C^4 - 12C^2 + 1]$

$\Rightarrow \sin 5\theta = \sin\theta[16\cos^4\theta - 12\cos^2\theta + 1]$ // AS REQUIRED

(b)

- $\sin 5\theta = 0$

  $\arcsin 0 = 0$

  $\begin{cases} 5\theta = 0 \pm 2n\pi \\ 5\theta = \pi \pm 2n\pi \end{cases}$

  $\begin{cases} \theta = 0 \pm \frac{2}{5}n\pi \\ \theta = \frac{\pi}{5} \pm \frac{2n}{5}\pi \end{cases}$

  $\theta = 0, \frac{\pi}{5}, \frac{2\pi}{5}, \frac{3\pi}{5}, \frac{4\pi}{5}, \pi, \dots$

- $\sin\theta(16\cos^4\theta - 12\cos^2\theta + 1) = 0$

- $\sin\theta = 0 \Rightarrow \theta = 0, \pi, 2\pi, 3\pi, \dots$

- $16\cos^4\theta - 12\cos^2\theta + 1 = 0$

  $\cos^2\theta = \frac{12 \pm \sqrt{80}}{32}$

  $\cos^2\theta = \frac{12 \pm 4\sqrt{5}}{32}$

  $\cos^2\theta = \frac{3 \pm \sqrt{5}}{8} = \begin{cases} 0.655 \\ 0.095 \end{cases}$

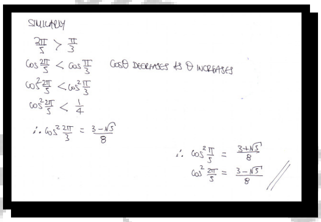
  $36° \rightarrow \frac{\pi}{5} < \frac{\pi}{3} \leftarrow 60°$

  $\cos\frac{\pi}{3} < \cos\frac{\pi}{5}$, $\cos\theta$ IS DECREASING

  $\cos^2\frac{\pi}{3} < \cos^2\frac{\pi}{5}$

  $\frac{1}{4} < \cos^2\frac{\pi}{5}$

  $\therefore \cos^2\frac{\pi}{5} = \frac{3+\sqrt{5}}{8}$



SIMILARLY

$\frac{2\pi}{5} > \frac{\pi}{3}$

$\cos\frac{2\pi}{5} < \cos\frac{\pi}{3}$ $\cos\theta$ DECREASES AS θ INCREASES

$\cos^2\frac{2\pi}{5} < \cos^2\frac{\pi}{3}$

$\cos^2\frac{2\pi}{5} < \frac{1}{4}$

$\therefore \cos^2\frac{2\pi}{3} = \frac{3-\sqrt{5}}{8}$

$\therefore \cos^2\frac{\pi}{5} = \frac{3+\sqrt{5}}{8}$

$\cos^2\frac{2\pi}{5} = \frac{3-\sqrt{5}}{8}$ //

Question 92 (****)

The complex number z is given by

$$z = \cos\theta + \mathrm{i}\sin\theta , \; -\pi < \theta \le \pi .$$

a) Show clearly that

$$z^n + \frac{1}{z^n} \equiv 2\cos n\theta .$$

b) Hence show further that if $z = \cos\theta + \mathrm{i}\sin\theta$, the equation

$$3z^4 - 5z^3 + 8z^2 - 5z + 3 = 0$$

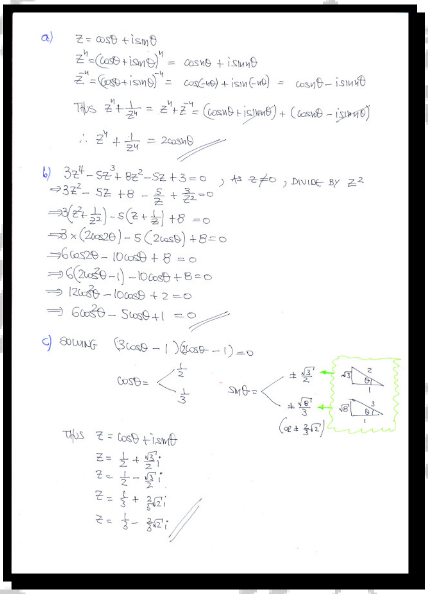
transforms into the equation

$$6\cos^2\theta - 5\cos\theta + 1 = 0 .$$

c) Hence find in exact surd form the four roots of the equation

$$3z^4 - 5z^3 + 8z^2 - 5z + 3 = 0 .$$

$$\boxed{z = \frac{1}{2} \pm \frac{\sqrt{3}}{2}\mathrm{i}, \; z = \frac{1}{3} \pm \frac{2}{3}\sqrt{2}\mathrm{i},}$$



a) $z = \cos\theta + i\sin\theta$

$z^n = (\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$

$z^{-n} = (\cos\theta + i\sin\theta)^{-n} = \cos(-n\theta) + i\sin(-n\theta) = \cos n\theta - i\sin n\theta$

THUS $z^n + \frac{1}{z^n} = z^n + z^{-n} = (\cos n\theta + \cancel{i\sin n\theta}) + (\cos n\theta - \cancel{i\sin n\theta})$

$\therefore z^n + \frac{1}{z^n} = 2\cos n\theta$

b) $3z^4 - 5z^3 + 8z^2 - 5z + 3 = 0$, AS $z \neq 0$, DIVIDE BY z^2

$\Rightarrow 3z^2 - 5z + 8 - \frac{5}{z} + \frac{3}{z^2} = 0$

$\Rightarrow 3\left(z^2 + \frac{1}{z^2}\right) - 5\left(z + \frac{1}{z}\right) + 8 = 0$

$\Rightarrow 3 \times (2\cos 2\theta) - 5(2\cos\theta) + 8 = 0$

$\Rightarrow 6\cos 2\theta - 10\cos\theta + 8 = 0$

$\Rightarrow 6(2\cos^2\theta - 1) - 10\cos\theta + 8 = 0$

$\Rightarrow 12\cos^2\theta - 10\cos\theta + 2 = 0$

$\Rightarrow 6\cos^2\theta - 5\cos\theta + 1 = 0$

c) SOLVING $(3\cos\theta - 1)(2\cos\theta - 1) = 0$

$\cos\theta = \frac{1}{2}, \; \frac{1}{3}$ $\qquad$ $\sin\theta = \pm\frac{\sqrt{3}}{2}, \; \pm\frac{\sqrt{8}}{3}$

$\left(\text{or } \pm \frac{2}{3}\sqrt{2}\right)$

THUS $z = \cos\theta + i\sin\theta$

$z = \frac{1}{2} + \frac{\sqrt{3}}{2}i$

$z = \frac{1}{2} - \frac{\sqrt{3}}{2}i$

$z = \frac{1}{3} + \frac{2}{3}\sqrt{2}i$

$z = \frac{1}{3} - \frac{2}{3}\sqrt{2}i$

Question 93 (****)

A complex transformation from the z plane to the w plane is defined by

$$w = \frac{z+\mathrm{i}}{3+\mathrm{i}z}, \ z \in \mathbb{C}, \ z \neq 3\mathrm{i}.$$

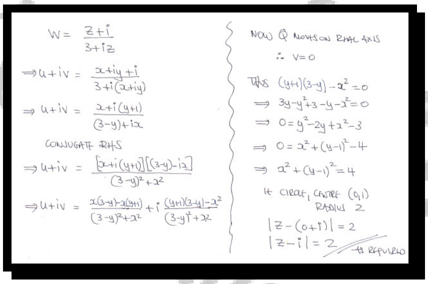
The point $P(x, y)$ is mapped by this transformation into the point $Q(u, v)$.

It is further given that Q lies on the real axis for all the possible positions of P.

Show that the P traces the curve with equation

$$|z - \mathrm{i}| = 2.$$

$\boxed{\text{proof}}$



$w = \frac{z+i}{3+iz}$

$\Rightarrow u+iv = \frac{x+iy+i}{3+i(x+iy)}$

$\Rightarrow u+iv = \frac{x+i(y+1)}{(3-y)+ix}$

CONJUGATE RHS

$\Rightarrow u+iv = \frac{[x+i(y+1)][(3-y)-ix]}{(3-y)^2+x^2}$

$\Rightarrow u+iv = \frac{x(3-y)-x(y+1)}{(3-y)^2+x^2} + i\frac{(y+1)(3-y)-x^2}{(3-y)^2+x^2}$

NOW Q MOVES ON REAL AXIS

$\therefore v=0$

THUS $(y+1)(3-y)-x^2=0$

$\Rightarrow 3y-y^2+3-y-x^2=0$

$\Rightarrow 0=y^2-2y+x^2-3$

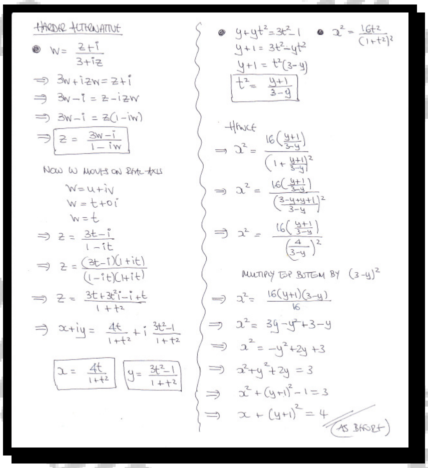
$\Rightarrow 0=x^2+(y-1)^2-4$

$\Rightarrow x^2+(y-1)^2=4$

IE CIRCLE, CENTRE $(0,1)$ RADIUS 2

$|z-(0+i)|=2$

$|z-i|=2$ // AS REQUIRED



HARDER ALTERNATIVE

- $w = \frac{z+i}{3+iz}$

$\Rightarrow 3w+izw=z+i$

$\Rightarrow 3w-i=z-izw$

$\Rightarrow 3w-i=z(1-iw)$

$\Rightarrow \boxed{z=\frac{3w-i}{1-iw}}$

NOW W MOVES ON REAL AXIS

$w=u+iv$

$w=t+0i$

$w=t$

$\Rightarrow z=\frac{3t-i}{1-it}$

$\Rightarrow z=\frac{(3t-i)(1+it)}{(1-it)(1+it)}$

$\Rightarrow z=\frac{3t+3t^2i-i+t}{1+t^2}$

$\Rightarrow x+iy=\frac{4t}{1+t^2}+i\frac{3t^2-1}{1+t^2}$

$\boxed{x=\frac{4t}{1+t^2}}$ $\boxed{y=\frac{3t^2-1}{1+t^2}}$

- $y+yt^2=3t^2-1$

$y+1=3t^2-yt^2$

$y+1=t^2(3-y)$

$\boxed{t^2=\frac{y+1}{3-y}}$

- $x^2=\frac{16t^2}{(1+t^2)^2}$

HENCE

$\Rightarrow x^2=\frac{16\left(\frac{y+1}{3-y}\right)}{\left(1+\frac{y+1}{3-y}\right)^2}$

$\Rightarrow x^2=\frac{16\left(\frac{y+1}{3-y}\right)}{\left(\frac{3-y+y+1}{3-y}\right)^2}$

$\Rightarrow x^2=\frac{16\left(\frac{y+1}{3-y}\right)}{\left(\frac{4}{3-y}\right)^2}$

MULTIPLY TOP BOTTOM BY $(3-y)^2$

$\Rightarrow x^2=\frac{16(y+1)(3-y)}{16}$

$\Rightarrow x^2=3y-y^2+3-y$

$\Rightarrow x^2=-y^2+2y+3$

$\Rightarrow x^2+y^2+2y=3$

$\Rightarrow x^2+(y+1)^2-1=3$

$\Rightarrow x+(y+1)^2=4$ // (AS BEFORE)

Question 94 (****)

The complex number z is given by $z = e^{i\theta}$, $-\pi < \theta \le \pi$

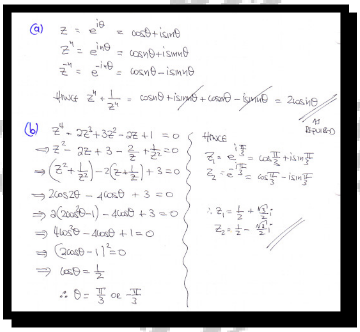
a) Show clearly that

$$z^n + \frac{1}{z^n} = 2\cos n\theta .$$

b) Hence solve the equation

$$z^4 - 2z^3 + 3z^2 - 2z + 1 = 0 .$$

$$\boxed{z = \frac{1}{2} \pm \frac{\sqrt{3}}{2}i}$$



(a) $z = e^{i\theta} = \cos\theta + i\sin\theta$

$z^n = e^{in\theta} = \cos n\theta + i\sin n\theta$

$z^{-n} = e^{-in\theta} = \cos n\theta - i\sin n\theta$

HENCE $z^n + \frac{1}{z^n} = \cos n\theta + \cancel{i\sin n\theta} + \cos n\theta - \cancel{i\sin n\theta} = 2\cos n\theta$

AS REQUIRED

(b) $z^4 - 2z^3 + 3z^2 - 2z + 1 = 0$

$\Rightarrow z^2 - 2z + 3 - \frac{2}{z} + \frac{1}{z^2} = 0$

$\Rightarrow \left(z^2 + \frac{1}{z^2}\right) - 2\left(z + \frac{1}{z}\right) + 3 = 0$

$\Rightarrow 2\cos 2\theta - 4\cos\theta + 3 = 0$

$\Rightarrow 2(2\cos^2\theta - 1) - 4\cos\theta + 3 = 0$

$\Rightarrow 4\cos^2\theta - 4\cos\theta + 1 = 0$

$\Rightarrow (2\cos\theta - 1)^2 = 0$

$\Rightarrow \cos\theta = \frac{1}{2}$

$\therefore \theta = \frac{\pi}{3}$ or $-\frac{\pi}{3}$

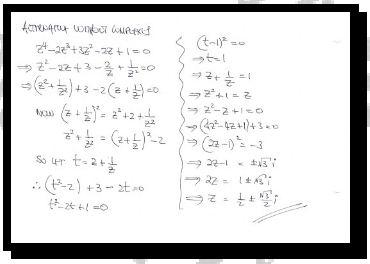
HENCE

$z_1 = e^{i\frac{\pi}{3}} = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3}$

$z_2 = e^{-i\frac{\pi}{3}} = \cos\frac{\pi}{3} - i\sin\frac{\pi}{3}$

$\therefore z_1 = \frac{1}{2} + \frac{\sqrt{3}}{2}i$

$z_2 = \frac{1}{2} - \frac{\sqrt{3}}{2}i$



ALTERNATIVE WITHOUT COMPLEXES

$z^4 - 2z^3 + 3z^2 - 2z + 1 = 0$

$\Rightarrow z^2 - 2z + 3 - \frac{2}{z} + \frac{1}{z^2} = 0$

$\Rightarrow \left(z^2 + \frac{1}{z^2}\right) + 3 - 2\left(z + \frac{1}{z}\right) = 0$

NOW $\left(z + \frac{1}{z}\right)^2 = z^2 + 2 + \frac{1}{z^2}$

$z^2 + \frac{1}{z^2} = \left(z + \frac{1}{z}\right)^2 - 2$

SO LET $t = z + \frac{1}{z}$

$\therefore (t^2 - 2) + 3 - 2t = 0$

$t^2 - 2t + 1 = 0$

$(t-1)^2 = 0$

$\Rightarrow t = 1$

$\Rightarrow z + \frac{1}{z} = 1$

$\Rightarrow z^2 + 1 = z$

$\Rightarrow z^2 - z + 1 = 0$

$\Rightarrow (4z^2 - 4z + 1) + 3 = 0$

$\Rightarrow (2z - 1)^2 = -3$

$\Rightarrow 2z - 1 = \pm\sqrt{3}i$

$\Rightarrow 2z = 1 \pm \sqrt{3}i$

$\Rightarrow z = \frac{1}{2} \pm \frac{\sqrt{3}}{2}i$

Question 95 (****)

A transformation of the z plane to the w plane is given by

$$w = \frac{2z+1}{z}, \quad z \in \mathbb{C}, \quad z \neq 0$$

where $z = x + \mathrm{i}y$ and $w = u + \mathrm{i}v$.

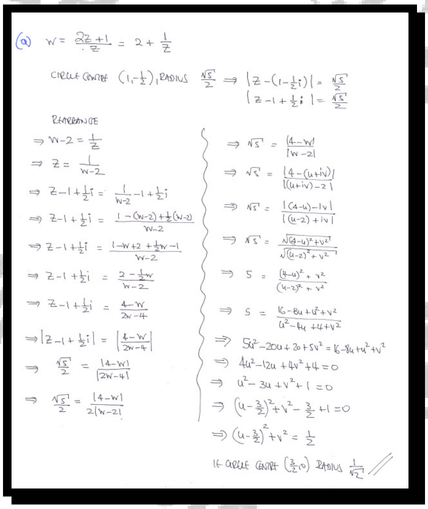
The circle C_1 with centre at $\left(1, -\frac{1}{2}\right)$ and radius $\frac{\sqrt{5}}{2}$ in the z plane is mapped in the w plane onto another curve C_2.

a) Show that C_2 is also a circle and determine the coordinates of its centre and the size of its radius.

The points inside C_1 in the z plane are mapped onto points of a region R in the w plane.

b) Sketch and shade R in a suitably labelled Argand diagram, fully justifying the choice of the region.

$$\boxed{\text{centre at } \left(\frac{3}{2}, 0\right), \text{ radius} = \frac{1}{\sqrt{2}}}$$



(a) $w = \frac{2z+1}{z} = 2 + \frac{1}{z}$

CIRCLE CENTRE $(1, -\frac{1}{2})$, RADIUS $\frac{\sqrt{5}}{2} \Rightarrow |z - (1 - \frac{1}{2}i)| = \frac{\sqrt{5}}{2}$

$|z - 1 + \frac{1}{2}i| = \frac{\sqrt{5}}{2}$

REARRANGE

$\Rightarrow w - 2 = \frac{1}{z}$

$\Rightarrow z = \frac{1}{w-2}$

$\Rightarrow z - 1 + \frac{1}{2}i = \frac{1}{w-2} - 1 + \frac{1}{2}i$

$\Rightarrow z - 1 + \frac{1}{2}i = \frac{1 - (w-2) + \frac{1}{2}(w-2)}{w-2}$

$\Rightarrow z - 1 + \frac{1}{2}i = \frac{1 - w + 2 + \frac{1}{2}w - 1}{w-2}$

$\Rightarrow z - 1 + \frac{1}{2}i = \frac{2 - \frac{1}{2}w}{w-2}$

$\Rightarrow z - 1 + \frac{1}{2}i = \frac{4-w}{2w-4}$

$\Rightarrow |z - 1 + \frac{1}{2}i| = \left|\frac{4-w}{2w-4}\right|$

$\Rightarrow \frac{\sqrt{5}}{2} = \frac{|4-w|}{|2w-4|}$

$\Rightarrow \frac{\sqrt{5}}{2} = \frac{|4-w|}{2|w-2|}$

$\Rightarrow \sqrt{5} = \frac{|4-w|}{|w-2|}$

$\Rightarrow \sqrt{5} = \frac{|4-(u+iv)|}{|(u+iv)-2|}$

$\Rightarrow \sqrt{5} = \frac{|(4-u)-iv|}{|(u-2)+iv|}$

$\Rightarrow \sqrt{5} = \frac{\sqrt{(4-u)^2+v^2}}{\sqrt{(u-2)^2+v^2}}$

$\Rightarrow 5 = \frac{(4-u)^2+v^2}{(u-2)^2+v^2}$

$\Rightarrow 5 = \frac{16-8u+u^2+v^2}{u^2-4u+4+v^2}$

$\Rightarrow 5u^2 - 20u + 20 + 5v^2 = 16 - 8u + u^2 + v^2$

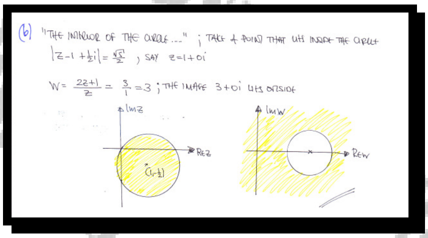
$\Rightarrow 4u^2 - 12u + 4v^2 + 4 = 0$

$\Rightarrow u^2 - 3u + v^2 + 1 = 0$

$\Rightarrow (u - \frac{3}{2})^2 + v^2 - \frac{3}{2} + 1 = 0$

$\Rightarrow (u - \frac{3}{2})^2 + v^2 = \frac{1}{2}$

IE CIRCLE CENTRE $(\frac{3}{2}, 0)$ RADIUS $\frac{1}{\sqrt{2}}$ //



(b) "THE INTERIOR OF THE CIRCLE..." ; TAKE A POINT THAT LIES INSIDE THE CIRCLE $|z - 1 + \frac{1}{2}i| = \frac{\sqrt{5}}{2}$, SAY $z = 1 + 0i$

$w = \frac{2z+1}{z} = \frac{3}{1} = 3$; THE IMAGE $3 + 0i$ LIES OUTSIDE

Question 96 (****)

The complex numbers z_1 and z_2 are given by

$$z_1 = 1 + \mathrm{i}\sqrt{3} \quad \text{and} \quad z_2 = \mathrm{i}z_1 .$$

a) Label accurately the points representing z_1 and z_2, in an Argand diagram.

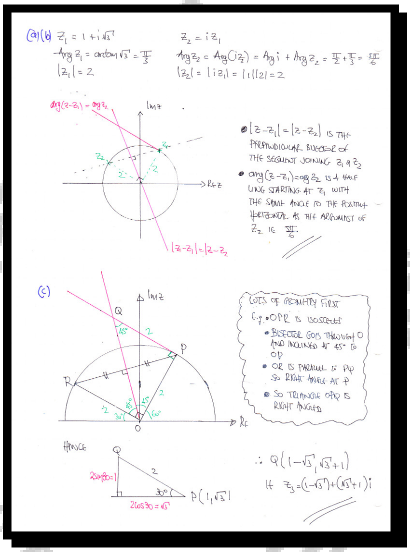
b) On the same Argand diagram, sketch the locus of the points z satisfying …

i. … $|z - z_1| = |z - z_2|$.

ii. … $\arg(z - z_1) = \arg z_2$.

c) Determine, in the form $x + \mathrm{i}y$, the complex number z_3 represented by the intersection of the two loci of part (**b**).

$$\boxed{z_3 = \left(1 - \sqrt{3}\right) + \mathrm{i}\left(1 + \sqrt{3}\right)}$$



(a)(b) $z_1 = 1 + i\sqrt{3}$ $\quad$ $z_2 = iz_1$

$\text{Arg}\, z_1 = \arctan\sqrt{3} = \frac{\pi}{3}$ $\quad$ $\text{Arg}\, z_2 = \text{Arg}(iz_2) = \text{Arg}\, i + \text{Arg}\, z_2 = \frac{\pi}{2} + \frac{\pi}{3} = \frac{5\pi}{6}$

$|z_1| = 2$ $\quad$ $|z_2| = |iz_1| = |i||2| = 2$

- $|z-z_1| = |z-z_2|$ IS THE PERPENDICULAR BISECTOR OF THE SEGMENT JOINING z_1 & z_2
- $\arg(z-z_1) = \arg z_2$ IS A HALF LINE STARTING AT z_1 WITH THE SAME ANGLE TO THE POSITIVE HORIZONTAL AS THE ARGUMENT OF z_2 IE $\frac{5\pi}{6}$

(c)

LOTS OF GEOMETRY FIRST

E.g. • OPR IS ISOSCELES

- BISECTOR GOES THROUGH O AND INCLINED AT 45° TO OP
- OR IS PARALLEL TO PQ SO RIGHT ANGLE AT P
- SO TRIANGLE OPQ IS RIGHT ANGLED

HENCE

$\therefore Q(1-\sqrt{3}, \sqrt{3}+1)$

IE $z_3 = (1-\sqrt{3}) + (\sqrt{3}+1)i$

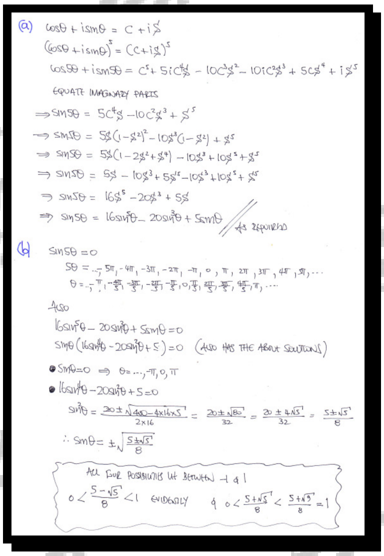
Question 97 **(****)**

a) Use De Moivre's theorem to show that

$$\sin 5\theta \equiv 16\sin^5\theta - 20\sin^3\theta + 5\sin\theta .$$

b) By considering the solutions of the equation $\sin 5\theta = 0$, find in exact surd form the values of $\sin\left(\frac{n\pi}{5}\right)$, for $n = 1, 2, 3, 4$.

$$\boxed{\sin\frac{\pi}{5} = \sin\frac{4\pi}{5} = \sqrt{\frac{5-\sqrt{5}}{8}}}\ , \ \boxed{\sin\frac{2\pi}{5} = \sin\frac{3\pi}{5} = \sqrt{\frac{5+\sqrt{5}}{8}}}$$



(a) $\cos\theta + i\sin\theta = c + is$

$(\cos\theta + i\sin\theta)^5 = (c+is)^5$

$\cos 5\theta + i\sin 5\theta = c^5 + 5ic^4s - 10c^3s^2 - 10ic^2s^3 + 5cs^4 + is^5$

EQUATE IMAGINARY PARTS

$\Rightarrow \sin 5\theta = 5c^4s - 10c^2s^3 + s^5$

$\Rightarrow \sin 5\theta = 5s(1-s^2)^2 - 10s^3(1-s^2) + s^5$

$\Rightarrow \sin 5\theta = 5s(1-2s^2+s^4) - 10s^3 + 10s^5 + s^5$

$\Rightarrow \sin 5\theta = 5s - 10s^3 + 5s^5 - 10s^3 + 10s^5 + s^5$

$\Rightarrow \sin 5\theta = 16s^5 - 20s^3 + 5s$

$\Rightarrow \sin 5\theta = 16\sin^5\theta - 20\sin^3\theta + 5\sin\theta$ // AS REQUIRED

(b) $\sin 5\theta = 0$

$5\theta = \dots, -5\pi, -4\pi, -3\pi, -2\pi, -\pi, 0, \pi, 2\pi, 3\pi, 4\pi, 5\pi, \dots$

$\theta = \dots, -\pi, -\frac{4\pi}{5}, -\frac{3\pi}{5}, -\frac{2\pi}{5}, -\frac{\pi}{5}, 0, \frac{\pi}{5}, \frac{2\pi}{5}, \frac{3\pi}{5}, \frac{4\pi}{5}, \pi, \dots$

ALSO

$16\sin^5\theta - 20\sin^3\theta + 5\sin\theta = 0$

$\sin\theta(16\sin^4\theta - 20\sin^2\theta + 5) = 0$ (ALSO HAS THE ABOVE SOLUTIONS)

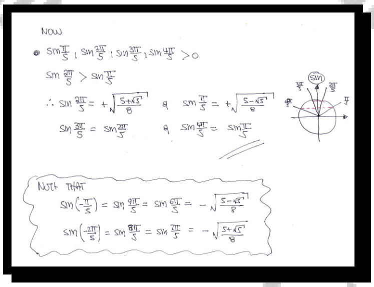
- $\sin\theta = 0 \Rightarrow \theta = \dots, -\pi, 0, \pi$
- $16\sin^4\theta - 20\sin^2\theta + 5 = 0$

$\sin^2\theta = \frac{20 \pm \sqrt{400 - 4\times16\times5}}{2\times16} = \frac{20\pm\sqrt{80}}{32} = \frac{20\pm4\sqrt{5}}{32} = \frac{5\pm\sqrt{5}}{8}$

$\therefore \sin\theta = \pm\sqrt{\frac{5\pm\sqrt{5}}{8}}$

ALL FOUR POSSIBILITIES LIE BETWEEN -1 & 1

$0 < \frac{5-\sqrt{5}}{8} < 1$ EVIDENTLY & $0 < \frac{5+\sqrt{5}}{8} < \frac{5+\sqrt{9}}{8} = 1$



NOW

- $\sin\frac{\pi}{5}, \sin\frac{2\pi}{5}, \sin\frac{3\pi}{5}, \sin\frac{4\pi}{5} > 0$

$\sin\frac{2\pi}{5} > \sin\frac{\pi}{5}$

$\therefore \sin\frac{2\pi}{5} = +\sqrt{\frac{5+\sqrt{5}}{8}}$ & $\sin\frac{\pi}{5} = +\sqrt{\frac{5-\sqrt{5}}{8}}$

$\sin\frac{3\pi}{5} = \sin\frac{2\pi}{5}$ & $\sin\frac{4\pi}{5} = \sin\frac{\pi}{5}$ //

NOTE THAT

$\sin\left(-\frac{\pi}{5}\right) = \sin\frac{9\pi}{5} = \sin\frac{6\pi}{5} = -\sqrt{\frac{5-\sqrt{5}}{8}}$

$\sin\left(-\frac{2\pi}{5}\right) = \sin\frac{8\pi}{5} = \sin\frac{7\pi}{5} = -\sqrt{\frac{5+\sqrt{5}}{8}}$

Question 98 **(****)**

A transformation of the z plane to the w plane is given by

$$w = z + \frac{1}{z}, \ z \in \mathbb{C}, \ z \neq 0,$$

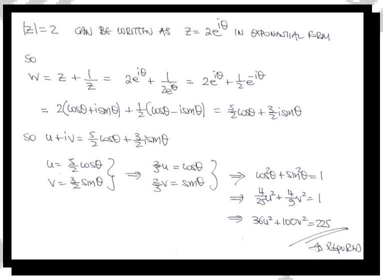
where $z = x + \mathrm{i}y$ and $w = u + \mathrm{i}v$.

The locus of the points in the z plane that satisfy the equation $|z| = 2$ are mapped in the w plane onto a curve C.

By considering the equation of the locus $|z| = 2$ in exponential form, or otherwise, show that a Cartesian equation of C is

$$36u^2 + 100v^2 = 225.$$

$\boxed{\text{proof}}$



$|z| = 2$ CAN BE WRITTEN AS $z = 2e^{i\theta}$ IN EXPONENTIAL FORM

SO

$$w = z + \frac{1}{z} = 2e^{i\theta} + \frac{1}{2e^{i\theta}} = 2e^{i\theta} + \frac{1}{2}e^{-i\theta}$$

$$= 2(\cos\theta + i\sin\theta) + \frac{1}{2}(\cos\theta - i\sin\theta) = \frac{5}{2}\cos\theta + \frac{3}{2}i\sin\theta$$

SO $u + iv = \frac{5}{2}\cos\theta + \frac{3}{2}i\sin\theta$

$$\left.\begin{array}{l} u = \frac{5}{2}\cos\theta \\ v = \frac{3}{2}\sin\theta \end{array}\right\} \Rightarrow \left.\begin{array}{l} \frac{2}{5}u = \cos\theta \\ \frac{2}{3}v = \sin\theta \end{array}\right\} \Rightarrow \cos^2\theta + \sin^2\theta = 1$$

$$\Rightarrow \frac{4}{25}u^2 + \frac{4}{9}v^2 = 1$$

$$\Rightarrow 36u^2 + 100v^2 = 225$$

AS REQUIRED

Question 99 (****)

a) Use De Moivre's theorem to show that

$$\sin 5\theta \equiv 16\sin^5\theta - 20\sin^3\theta + 5\sin\theta \, .$$

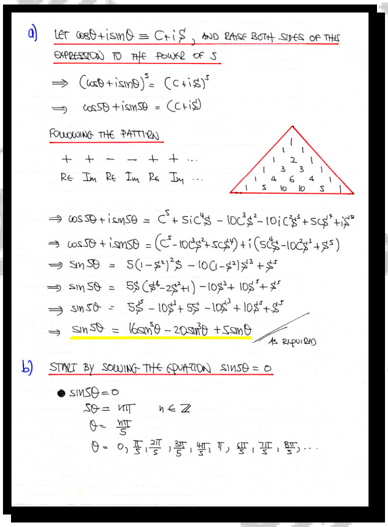
b) By considering the solutions of the equation $\sin 5\theta = 0$, find in trigonometric form the four solutions of the equation

$$16x^4 - 20x^2 + 5 = 0 \, .$$

c) Hence show, with full justification, that

$$\sin^2\left(\frac{\pi}{5}\right) = \frac{5-\sqrt{5}}{8} \, .$$

$\boxed{\text{FPT}}$, $\boxed{x = \sin\left(\tfrac{1}{5}k\pi\right), \quad k = 1,2,6,7}$



a) LET $\cos\theta + i\sin\theta \equiv C + iS$, AND RAISE BOTH SIDES OF THIS EXPRESSION TO THE POWER OF 5

$\Rightarrow (\cos\theta + i\sin\theta)^5 = (C+iS)^5$

$\Rightarrow \cos 5\theta + i\sin 5\theta = (C+iS)$

FOLLOWING THE PATTERN

$+ \quad + \quad - \quad - \quad + \quad + \quad \dots$

Re Im Re Im Re Im ...

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1

$\Rightarrow \cos 5\theta + i\sin 5\theta = C^5 + 5iC^4S - 10C^3S^2 - 10iC^2S^3 + 5CS^4 + iS^5$

$\Rightarrow \cos 5\theta + i\sin 5\theta = (C^5 - 10C^3S^2 + 5CS^4) + i(5C^4S - 10C^2S^3 + S^5)$

$\Rightarrow \sin 5\theta = 5(1-S^2)^2S - 10(1-S^2)S^3 + S^5$

$\Rightarrow \sin 5\theta = 5S(S^4 - 2S^2 + 1) - 10S^3 + 10S^5 + S^5$

$\Rightarrow \sin 5\theta = 5S^5 - 10S^3 + 5S - 10S^3 + 10S^5 + S^5$

$\Rightarrow \sin 5\theta = 16\sin^5\theta - 20\sin^3\theta + 5\sin\theta$ // AS REQUIRED

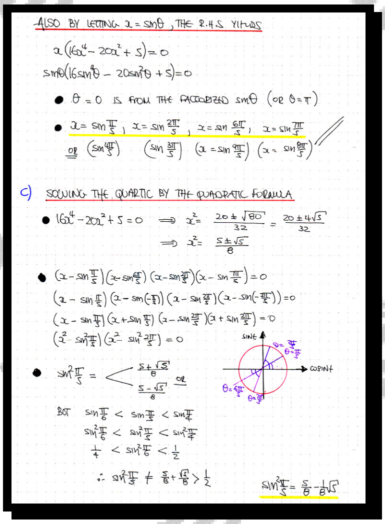
b) START BY SOLVING THE EQUATION $\sin 5\theta = 0$

- $\sin 5\theta = 0$

$5\theta = n\pi \quad n \in \mathbb{Z}$

$\theta = \frac{n\pi}{5}$

$\theta = 0, \frac{\pi}{5}, \frac{2\pi}{5}, \frac{3\pi}{5}, \frac{4\pi}{5}, \pi, \frac{6\pi}{5}, \frac{7\pi}{5}, \frac{8\pi}{5}, \dots$



ALSO BY LETTING $x = \sin\theta$, THE R.H.S YIELDS

$x(16x^4 - 20x^2 + 5) = 0$

$\sin\theta(16\sin^4\theta - 20\sin^2\theta + 5) = 0$

- $\theta = 0$ IS FROM THE FACTORIZED $\sin\theta$ (OR $\theta = \pi$)
- $x = \sin\frac{\pi}{5}, \; x = \sin\frac{2\pi}{5}, \; x = \sin\frac{6\pi}{5}, \; x = \sin\frac{7\pi}{5}$ //

OR $(\sin\frac{4\pi}{5}) \quad (\sin\frac{3\pi}{5}) \quad (x = \sin\frac{9\pi}{5}) \quad (x = \sin\frac{8\pi}{5})$

c) SOLVING THE QUARTIC BY THE QUADRATIC FORMULA

- $16x^4 - 20x^2 + 5 = 0 \Rightarrow x^2 = \frac{20 \pm \sqrt{80}}{32} = \frac{20 \pm 4\sqrt{5}}{32}$

$\Rightarrow x^2 = \frac{5 \pm \sqrt{5}}{8}$

- $(x - \sin\frac{\pi}{5})(x - \sin\frac{6\pi}{5})(x - \sin\frac{2\pi}{5})(x - \sin\frac{7\pi}{5}) = 0$

$(x - \sin\frac{\pi}{5})(x - \sin(-\frac{\pi}{5}))(x - \sin\frac{2\pi}{5})(x - \sin(-\frac{2\pi}{5})) = 0$

$(x - \sin\frac{\pi}{5})(x + \sin\frac{\pi}{5})(x - \sin\frac{2\pi}{5})(x + \sin\frac{2\pi}{5}) = 0$

$(x^2 - \sin^2\frac{\pi}{5})(x^2 - \sin^2\frac{2\pi}{5}) = 0$

- $\sin^2\frac{\pi}{5} = \left\langle \frac{5+\sqrt{5}}{8} \text{ OR } \frac{5-\sqrt{5}}{8} \right.$

SINE
COSINE
$\theta = \frac{2\pi}{5}$
$\theta = \frac{\pi}{5}$
$\theta = \frac{6\pi}{5}$
$\theta = \frac{7\pi}{5}$

BUT $\sin\frac{\pi}{6} < \sin\frac{\pi}{5} < \sin\frac{\pi}{4}$

$\sin^2\frac{\pi}{6} < \sin^2\frac{\pi}{5} < \sin^2\frac{\pi}{4}$

$\frac{1}{4} < \sin^2\frac{\pi}{6} < \frac{1}{2}$

$\therefore \sin^2\frac{\pi}{5} \neq \frac{5}{8} + \frac{\sqrt{5}}{8} > \frac{1}{2}$

$\sin^2\frac{\pi}{5} = \frac{5}{8} - \frac{1}{8}\sqrt{5}$

Question 100 (****)

The complex function $w = f(z)$ is given by

$$w = \frac{1}{1-z}\ ,\ z \neq 1 .$$

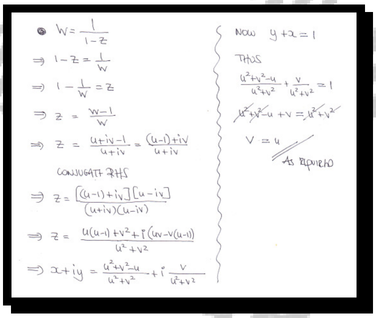
The point $P(x, y)$ in the z plane traces the line with Cartesian equation

$$y + x = 1 .$$

Show that the locus of the **image** of P in the w plane traces the line with equation

$$v = u .$$

proof



$$\bullet\ w = \frac{1}{1-z}$$

$$\Rightarrow 1 - z = \frac{1}{w}$$

$$\Rightarrow 1 - \frac{1}{w} = z$$

$$\Rightarrow z = \frac{w-1}{w}$$

$$\Rightarrow z = \frac{u+iv-1}{u+iv} = \frac{(u-1)+iv}{u+iv}$$

CONJUGATE RHS

$$\Rightarrow z = \frac{[(u-1)+iv][u-iv]}{(u+iv)(u-iv)}$$

$$\Rightarrow z = \frac{u(u-1)+v^2+i(uv-v(u-1))}{u^2+v^2}$$

$$\Rightarrow x+iy = \frac{u^2+v^2-u}{u^2+v^2} + i\frac{v}{u^2+v^2}$$

NOW $y + x = 1$

THUS

$$\frac{u^2+v^2-u}{u^2+v^2} + \frac{v}{u^2+v^2} = 1$$

$$\cancel{u^2}+\cancel{v^2}-u+v = \cancel{u^2}+\cancel{v^2}$$

$$v = u$$

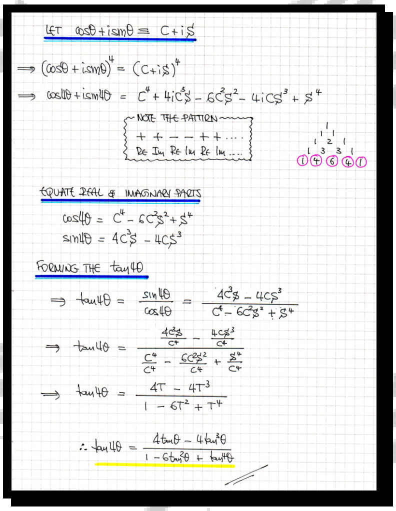
AS REQUIRED

Question 101 (****)

By considering the binomial expansion of $(\cos\theta + \mathrm{i}\sin\theta)^4$ show that

$$\tan 4\theta \equiv \frac{4\tan\theta - 4\tan^3\theta}{1 - 6\tan^2\theta + \tan^4\theta}.$$

$\boxed{\text{P2-M}}$, $\boxed{\text{proof}}$



LET $\cos\theta + i\sin\theta \equiv C + iS$

$\Longrightarrow (\cos\theta + i\sin\theta)^4 = (C+iS)^4$

$\Longrightarrow \cos 4\theta + i\sin 4\theta = C^4 + 4iC^3S - 6C^2S^2 - 4iCS^3 + S^4$

NOTE THE PATTERN
$+ + - - + + \dots$
Re Im Re Im Re Im ....

EQUATE REAL & IMAGINARY PARTS

$\cos 4\theta = C^4 - 6C^2S^2 + S^4$

$\sin 4\theta = 4C^3S - 4CS^3$

FORMING THE $\tan 4\theta$

$\Rightarrow \tan 4\theta = \frac{\sin 4\theta}{\cos 4\theta} = \frac{4C^3S - 4CS^3}{C^4 - 6C^2S^2 + S^4}$

$\Rightarrow \tan 4\theta = \frac{\frac{4C^3S}{C^4} - \frac{4CS^3}{C^4}}{\frac{C^4}{C^4} - \frac{6C^2S^2}{C^4} + \frac{S^4}{C^4}}$

$\Longrightarrow \tan 4\theta = \frac{4T - 4T^3}{1 - 6T^2 + T^4}$

$\therefore \tan 4\theta = \frac{4\tan\theta - 4\tan^3\theta}{1 - 6\tan^2\theta + \tan^4\theta}$

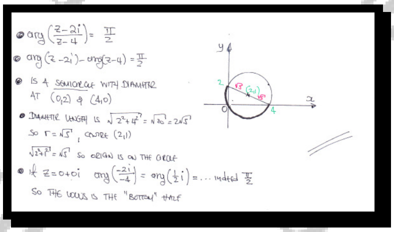
Question 102 (****)

In an Argand diagram which represents the z plane, the complex number $z = x + \mathrm{i}y$ satisfies the relationship

$$\arg\left(\frac{z-2\mathrm{i}}{z-4}\right) = \frac{\pi}{2}.$$

Sketch the curve that the locus of z traces.

$\boxed{\text{sketch}}$



- $\arg\left(\frac{z-2i}{z-4}\right) = \frac{\pi}{2}$
- $\arg(z-2i) - \arg(z-4) = \frac{\pi}{2}$
- IS A SEMICIRCLE WITH DIAMETER AT $(0,2)$ & $(4,0)$
- DIAMETER LENGTH IS $\sqrt{2^2+4^2} = \sqrt{20} = 2\sqrt{5}$
  SO $r = \sqrt{5}$, CENTRE $(2,1)$
  $\sqrt{2^2+1^2} = \sqrt{5}$ SO ORIGIN IS ON THE CIRCLE
- if $z = 0+0i$ $\arg\left(\frac{-2i}{-4}\right) = \arg\left(\frac{1}{2}i\right) = \dots$ indeed $\frac{\pi}{2}$
  SO THE LOCUS IS THE "BOTTOM" HALF

Question 103 **(****)**

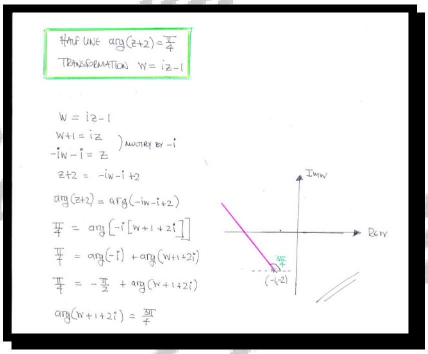
A transformation from the z plane to the w plane is defined by the equation

$$w = \mathrm{i}\,z - 1,\ z \in \mathbb{C}.$$

Sketch in the w plane, in Cartesian form, the equation of the image of the half line with equation

$$\arg(z+2) = \frac{\pi}{4},\ z \in \mathbb{C}.$$

$\boxed{\text{graph}}$



HALF LINE $\arg(z+2)=\frac{\pi}{4}$

TRANSFORMATION $w = iz-1$

$w = iz-1$

$w+1 = iz$) MULTIPLY BY $-i$

$-iw-i = z$

$z+2 = -iw-i+2$

$\arg(z+2) = \arg(-iw-i+2)$

$\frac{\pi}{4} = \arg[-i[w+1+2i]]$

$\frac{\pi}{4} = \arg(-i) + \arg(w+1+2i)$

$\frac{\pi}{4} = -\frac{\pi}{2} + \arg(w+1+2i)$

$\arg(w+1+2i) = \frac{3\pi}{4}$

Question 104 (****)

The complex function $w = f(z)$ maps points of the form $z = x + \mathrm{i}y$ from the z plane onto points of the form $w = u + \mathrm{i}v$ in the w plane.

It is given that

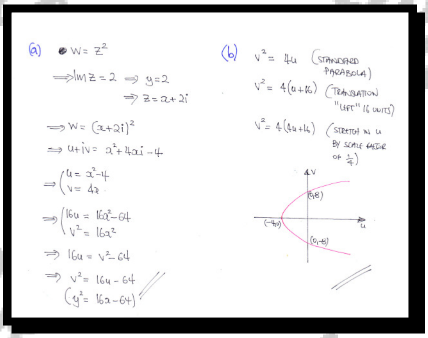
$$f(z) = z^2,\ z \in \mathbb{C}.$$

The line with equation $\operatorname{Im} z = 2$ in the z plane is mapped onto the curve C in the w plane.

a) Find a Cartesian equation for C.

b) Sketch the graph of C.

$$\boxed{v^2 = 16u - 64}$$



(a) ● $w = z^2$

$\Rightarrow \operatorname{Im} z = 2 \Rightarrow y = 2$

$\Rightarrow z = x + 2i$

$\Rightarrow w = (x+2i)^2$

$\Rightarrow u + iv = x^2 + 4xi - 4$

$\Rightarrow \begin{cases} u = x^2 - 4 \\ v = 4x \end{cases}$

$\Rightarrow \begin{cases} 16u = 16x^2 - 64 \\ v^2 = 16x^2 \end{cases}$

$\Rightarrow 16u = v^2 - 64$

$\Rightarrow v^2 = 16u - 64$ //

$(y^2 = 16x - 64)$

(b) $v^2 = 4u$ (STANDARD PARABOLA)

$v^2 = 4(u+16)$ (TRANSLATION "LEFT" 16 UNITS)

$v^2 = 4(4u+16)$ (STRETCH IN u BY SCALE FACTOR OF $\frac{1}{4}$)

Question 105 (****)

The complex function $w = f(z)$ maps points of the form $z = x + \mathrm{i}y$ from the z plane onto points of the form $w = u + \mathrm{i}v$ in the w plane.

It is given that

$$f(z) = \frac{4}{z}, \ z \in \mathbb{C}, \ z \neq 0.$$

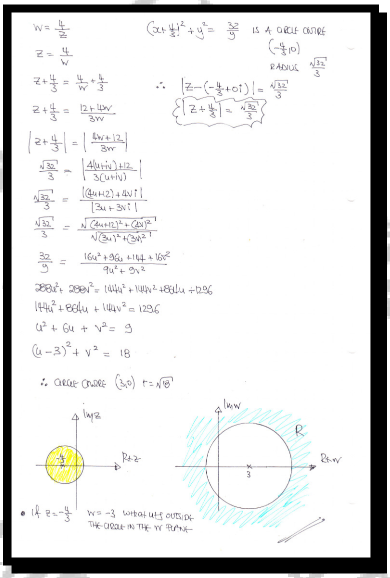
The points from the z plane, except the origin, which lie inside and on the boundary of the circle with equation

$$\left(x + \frac{4}{3}\right)^2 + y^2 = \frac{32}{9},$$

are mapped onto the region R in the w plane.

Shade the region R in a clearly labelled Argand diagram.

sketch



$w = \frac{4}{z}$

$z = \frac{4}{w}$

$z + \frac{4}{3} = \frac{4}{w} + \frac{4}{3}$

$z + \frac{4}{3} = \frac{12 + 4w}{3w}$

$\left|z + \frac{4}{3}\right| = \left|\frac{4w + 12}{3w}\right|$

$\frac{\sqrt{32}}{3} = \left|\frac{4(u+iv)+12}{3(u+iv)}\right|$

$\frac{\sqrt{32}}{3} = \frac{|(4u+12)+4vi|}{|3u+3vi|}$

$\frac{\sqrt{32}}{3} = \frac{\sqrt{(4u+12)^2+(4v)^2}}{\sqrt{(3u)^2+(3v)^2}}$

$\frac{32}{9} = \frac{16u^2+96u+144+16v^2}{9u^2+9v^2}$

$288u^2 + 288v^2 = 144u^2 + 144v^2 + 864u + 1296$

$144u^2 + 864u + 144v^2 = 1296$

$u^2 + 6u + v^2 = 9$

$(u-3)^2 + v^2 = 18$

$\therefore$ CIRCLE CENTRE $(3,0)$ $r = \sqrt{18}$

$(x+\frac{4}{3})^2 + y^2 = \frac{32}{9}$ IS A CIRCLE CENTRE $(-\frac{4}{3}, 0)$ RADIUS $\frac{\sqrt{32}}{3}$

$\therefore \ |z - (-\frac{4}{3} + 0i)| = \frac{\sqrt{32}}{3}$

$\left\{ \left|z + \frac{4}{3}\right| = \frac{\sqrt{32}}{3} \right\}$

- If $z = -\frac{4}{3}$ $w = -3$ WHICH LIES OUTSIDE THE CIRCLE IN THE w PLANE

Question 106 (****)

$$z = \mathrm{e}^{\mathrm{i}\theta}, \ -\pi < \theta \le \pi .$$

a) Show that …

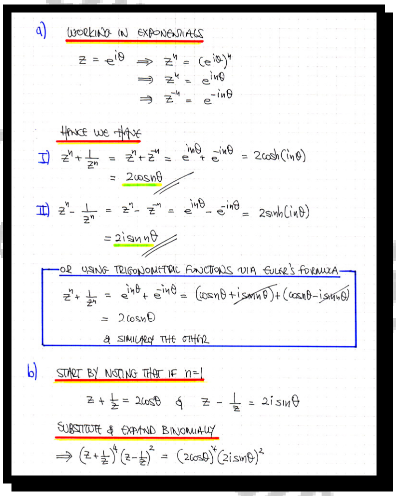
i. … $z^n + \dfrac{1}{z^n} = 2\cos n\theta$.

ii. ... $z^n - \dfrac{1}{z^n} = 2\mathrm{i}\sin n\theta$.

b) Hence show further that

$$\cos^4\theta\sin^2\theta = \frac{1}{16} + \frac{1}{32}\cos 2\theta - \frac{1}{16}\cos 4\theta - \frac{1}{32}\cos 6\theta .$$

☐ , proof



a) WORKING IN EXPONENTIALS

$z = e^{i\theta} \Rightarrow z^n = (e^{i\theta})^n$

$\Rightarrow z^n = e^{in\theta}$

$\Rightarrow z^{-n} = e^{-in\theta}$

HENCE WE HAVE

I) $z^n + \frac{1}{z^n} = z^n + z^{-n} = e^{in\theta} + e^{-in\theta} = 2\cosh(in\theta)$

$= 2\cos n\theta$

II) $z^n - \frac{1}{z^n} = z^n - z^{-n} = e^{in\theta} - e^{-in\theta} = 2\sinh(in\theta)$

$= 2i\sin n\theta$

OR USING TRIGONOMETRIC FUNCTIONS VIA EULER'S FORMULA

$z^n + \frac{1}{z^n} = e^{in\theta} + e^{-in\theta} = (\cos n\theta + \cancel{i\sin n\theta}) + (\cos n\theta - \cancel{i\sin n\theta})$

$= 2\cos n\theta$

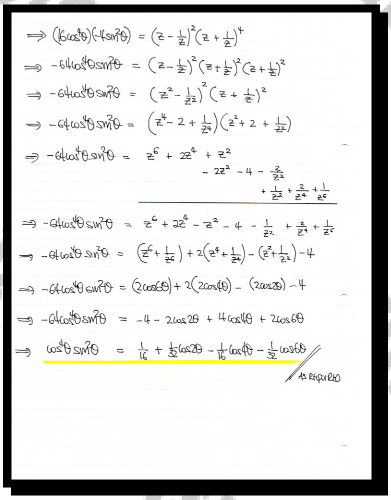
& SIMILARLY THE OTHER

b) START BY NOTING THAT IF n=1

$z + \frac{1}{z} = 2\cos\theta$ & $z - \frac{1}{z} = 2i\sin\theta$

SUBSTITUTE & EXPAND BINOMIALLY

$\Rightarrow \left(z+\frac{1}{z}\right)^4\left(z-\frac{1}{z}\right)^2 = (2\cos\theta)^4(2i\sin\theta)^2$



$\Rightarrow (16\cos^4\theta)(-4\sin^2\theta) = \left(z-\frac{1}{z}\right)^2\left(z+\frac{1}{z}\right)^4$

$\Rightarrow -64\cos^4\theta\sin^2\theta = \left(z-\frac{1}{z}\right)^2\left(z+\frac{1}{z}\right)^2\left(z+\frac{1}{z}\right)^2$

$\Rightarrow -64\cos^4\theta\sin^2\theta = \left(z^2-\frac{1}{z^2}\right)^2\left(z+\frac{1}{z}\right)^2$

$\Rightarrow -64\cos^4\theta\sin^2\theta = \left(z^4-2+\frac{1}{z^4}\right)\left(z^2+2+\frac{1}{z^2}\right)$

$\Rightarrow -64\cos^4\theta\sin^2\theta = z^6 + 2z^4 + z^2$

$- 2z^2 - 4 - \frac{2}{z^2}$

$+ \frac{1}{z^2} + \frac{2}{z^4} + \frac{1}{z^6}$

$\Rightarrow -64\cos^4\theta\sin^2\theta = z^6 + 2z^4 - z^2 - 4 - \frac{1}{z^2} + \frac{2}{z^4} + \frac{1}{z^6}$

$\Rightarrow -64\cos^4\theta\sin^2\theta = \left(z^6+\frac{1}{z^6}\right) + 2\left(z^4+\frac{1}{z^4}\right) - \left(z^2+\frac{1}{z^2}\right) - 4$

$\Rightarrow -64\cos^4\theta\sin^2\theta = (2\cos 6\theta) + 2(2\cos 4\theta) - (2\cos 2\theta) - 4$

$\Rightarrow -64\cos^4\theta\sin^2\theta = -4 - 2\cos 2\theta + 4\cos 4\theta + 2\cos 6\theta$

$\Rightarrow \cos^4\theta\sin^2\theta = \frac{1}{16} + \frac{1}{32}\cos 2\theta - \frac{1}{16}\cos 4\theta - \frac{1}{32}\cos 6\theta$

AS REQUIRED

Question 107 (****)

The locus of a point, represented by the complex number z, satisfies the relationship

$$|z+1+\mathrm{i}| = |z-1+2\mathrm{i}|.$$

When this locus is transformed by the complex function

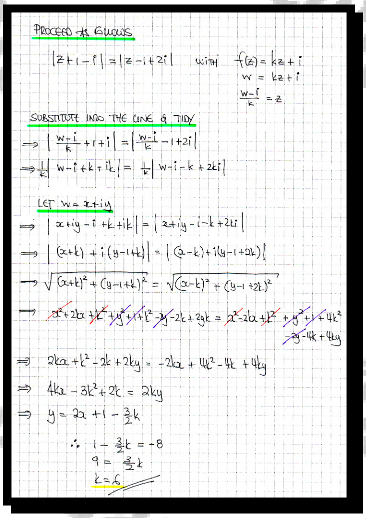
$$f(z) = kz + \mathrm{i}, \ k \in \mathbb{R},$$

the image of the locus traces the straight line with Cartesian equation

$$y = 2x - 8.$$

Determine the value of k.

$\boxed{k = 6}$



PROCEED AS FOLLOWS

$|z+1-i| = |z-1+2i|$ WITH $f(z) = kz+i$

$w = kz+i$

$\frac{w-i}{k} = z$

SUBSTITUTE INTO THE LINE & TIDY

$\Rightarrow \left|\frac{w-i}{k}+1+i\right| = \left|\frac{w-i}{k}-1+2i\right|$

$\Rightarrow \frac{1}{k}|w-i+k+ik| = \frac{1}{k}|w-i-k+2ki|$

LET $w = x+iy$

$\Rightarrow |x+iy-i+k+ik| = |x+iy-i-k+2ki|$

$\Rightarrow |(x+k)+i(y-1+k)| = |(x-k)+i(y-1+2k)|$

$\rightarrow \sqrt{(x+k)^2+(y-1+k)^2} = \sqrt{(x-k)^2+(y-1+2k)^2}$

$\Rightarrow \cancel{x^2}+2kx+\cancel{k^2}+\cancel{y^2}+\cancel{1}+k^2\cancel{-2y}-2k+2yk = \cancel{x^2}-2kx+\cancel{k^2}+\cancel{y^2}+\cancel{1}+4k^2\cancel{-2y}-4k+4ky$

$\Rightarrow 2kx+k^2-2k+2ky = -2kx+4k^2-4k+4ky$

$\Rightarrow 4kx-3k^2+2k = 2ky$

$\Rightarrow y = 2x+1-\frac{3}{2}k$

$\therefore 1-\frac{3}{2}k = -8$

$9 = \frac{3}{2}k$

$k = 6$

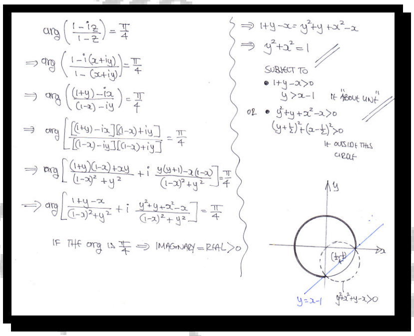
Question 108 **(****)**

The point P represents the number $z = x + \mathrm{i}y$ in an Argand diagram and further satisfies the equation

$$\arg\left(\frac{1-\mathrm{i}z}{1-z}\right) = \frac{\pi}{4}\ ,\ z \neq -\mathrm{i}\ .$$

Use an algebraic method to find an equation of the locus of P and sketch this locus accurately in an Argand diagram.

$$\boxed{u^2 + v^2 = 1,\ \ \text{such that}\ \ v > u - 1}$$



$\arg\left(\frac{1-iz}{1-z}\right) = \frac{\pi}{4}$

$\Rightarrow \arg\left(\frac{1-i(x+iy)}{1-(x+iy)}\right) = \frac{\pi}{4}$

$\Rightarrow \arg\left(\frac{(1+y)-ix}{(1-x)-iy}\right) = \frac{\pi}{4}$

$\Rightarrow \arg\left[\frac{[(1+y)-ix][(1-x)+iy]}{[(1-x)-iy][(1-x)+iy]}\right] = \frac{\pi}{4}$

$\Rightarrow \arg\left[\frac{(1+y)(1-x)+xy}{(1-x)^2+y^2} + i\,\frac{y(y+1)-x(1-x)}{(1-x)^2+y^2}\right] = \frac{\pi}{4}$

$\Rightarrow \arg\left[\frac{1+y-x}{(1-x)^2+y^2} + i\,\frac{y^2+y+x^2-x}{(1-x)^2+y^2}\right] = \frac{\pi}{4}$

IF THE arg is $\frac{\pi}{4} \Rightarrow$ IMAGINARY = REAL > 0

$\Rightarrow 1+y-x = y^2+y+x^2-x$

$\Rightarrow y^2+x^2 = 1$

SUBJECT TO

- $1+y-x>0$
  $y > x-1$ IE "ABOVE LINE"

OR

- $y^2+y+x^2-x>0$
  $(y+\frac{1}{2})^2+(x-\frac{1}{2})^2>0$
  IE OUTSIDE THIS CIRCLE

Question 109 **(****)**

The complex function $w = f(z)$ satisfies

$$w = \frac{1}{z}, \ z \in \mathbb{C}, \ z \neq 0.$$

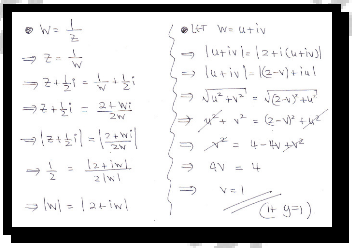
This function maps the point $P(x, y)$ in the z plane onto the point $Q(u, v)$ in the w plane.

It is further given that P traces the curve with equation

$$\left| z + \frac{1}{2}\mathrm{i} \right| = \frac{1}{2}.$$

Find, in Cartesian form, the equation of the locus of Q.

$\boxed{v = 1}$



$\bullet$ $w = \frac{1}{z}$

$\Rightarrow z = \frac{1}{w}$

$\Rightarrow z + \frac{1}{2}i = \frac{1}{w} + \frac{1}{2}i$

$\Rightarrow z + \frac{1}{2}i = \frac{2 + wi}{2w}$

$\Rightarrow \left| z + \frac{1}{2}i \right| = \left| \frac{2 + wi}{2w} \right|$

$\Rightarrow \frac{1}{2} = \frac{|2 + iw|}{2|w|}$

$\Rightarrow |w| = |2 + iw|$

$\bullet$ LET $w = u + iv$

$\Rightarrow |u + iv| = |2 + i(u + iv)|$

$\Rightarrow |u + iv| = |(2 - v) + iu|$

$\Rightarrow \sqrt{u^2 + v^2} = \sqrt{(2 - v)^2 + u^2}$

$\Rightarrow \cancel{u^2} + v^2 = (2 - v)^2 + \cancel{u^2}$

$\Rightarrow \cancel{v^2} = 4 - 4v \cancel{+ v^2}$

$\Rightarrow 4v = 4$

$\Rightarrow v = 1$

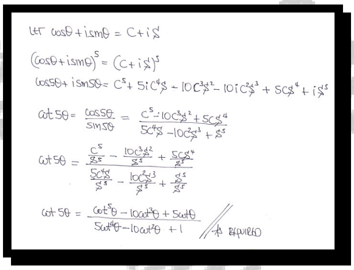
(i.e. $y = 1$)

Question 110 (****)

Use De Moivre's theorem to show that

$$\cot 5\theta \equiv \frac{\cot^5\theta - 10\cot^3\theta + 5\cot\theta}{5\cot^4\theta - 10\cot^2\theta + 1}.$$

$\boxed{\text{proof}}$



LET $\cos\theta + i\sin\theta = C + iS$

$(\cos\theta + i\sin\theta)^5 = (C + iS)^5$

$\cos 5\theta + i\sin 5\theta = C^5 + 5iC^4S - 10C^3S^2 - 10iC^2S^3 + 5CS^4 + iS^5$

$\cot 5\theta = \dfrac{\cos 5\theta}{\sin 5\theta} = \dfrac{C^5 - 10C^3S^2 + 5CS^4}{5C^4S - 10C^2S^3 + S^5}$

$\cot 5\theta = \dfrac{\frac{C^5}{S^5} - \frac{10C^3S^2}{S^5} + \frac{5CS^4}{S^5}}{\frac{5C^4S}{S^5} - \frac{10C^2S^3}{S^5} + \frac{S^5}{S^5}}$

$\cot 5\theta = \dfrac{\cot^5\theta - 10\cot^3\theta + 5\cot\theta}{5\cot^4\theta - 10\cot^2\theta + 1}$ // AS REQUIRED

Question 111 (****)

A transformation T from the z plane to the w plane is defined by

$$w = \frac{z-\mathrm{i}}{z+1}, \ z \in \mathbb{C}, \ z \neq -1.$$

T transforms the circle with equation $|z|=1$ in the z plane, into the straight line L in the w plane.

a) Find a Cartesian equation for L.

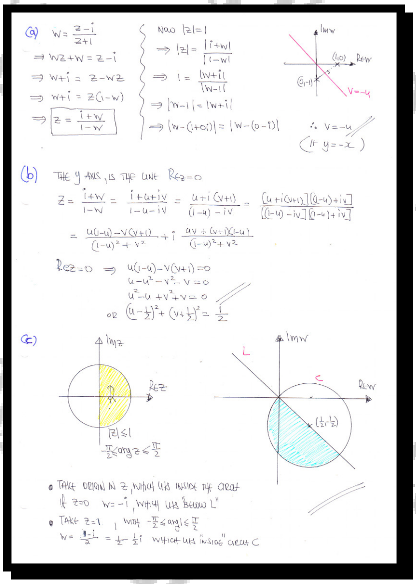
T transforms the y axis in the z plane, into the curve C in the w plane.

b) Find a Cartesian equation for C.

The region R in the z plane, satisfies $|z| \le 1$ such that $-\frac{\pi}{2} \le \arg z \le \frac{\pi}{2}$.

c) Shade the image of R under T in the w plane.

$\boxed{y=-x \text{ or } v=-u}$, $\boxed{u^2+v^2-u+v=0}$


(a) $w = \frac{z-i}{z+1}$

$\Rightarrow wz + w = z - i$

$\Rightarrow w + i = z - wz$

$\Rightarrow w + i = z(1-w)$

$\Rightarrow \boxed{z = \frac{i+w}{1-w}}$

Now $|z| = 1$

$\Rightarrow |z| = \frac{|i+w|}{|1-w|}$

$\Rightarrow 1 = \frac{|w+i|}{|w-1|}$

$\Rightarrow |w-1| = |w+i|$

$\Rightarrow |w-(1+0i)| = |w-(0-i)|$

$\therefore v=-u$ //

(i.e. $y=-x$)

(b) THE y AXIS, IS THE LINE $\text{Re}z=0$

$z = \frac{i+w}{1-w} = \frac{i+u+iv}{1-u-iv} = \frac{u+i(v+1)}{(1-u)-iv} = \frac{[u+i(v+1)][(1-u)+iv]}{[(1-u)-iv][(1-u)+iv]}$

$= \frac{u(1-u)-v(v+1)}{(1-u)^2+v^2} + i\frac{uv+(v+1)(1-u)}{(1-u)^2+v^2}$

$\text{Re}z=0 \Rightarrow u(1-u)-v(v+1)=0$

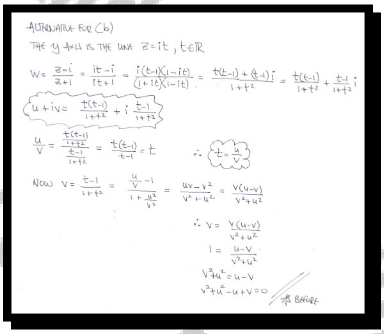
$u-u^2-v^2-v=0$

$u^2-u+v^2+v=0$ //

OR $\left(u-\frac{1}{2}\right)^2+\left(v+\frac{1}{2}\right)^2=\frac{1}{2}$

(c)

- TAKE ORIGIN IN Z, WHICH LIES INSIDE THE CIRCLE
  if $z=0$ $w=-i$, WHICH LIES "BELOW L"
- TAKE $z=1$, WITH $-\frac{\pi}{2} \le \arg 1 \le \frac{\pi}{2}$
  $w = \frac{1-i}{2} = \frac{1}{2} - \frac{1}{2}i$ WHICH LIES "INSIDE" CIRCLE C //


ALTERNATIVE FOR (b)

THE y AXIS IS THE LINE $z=it$, $t \in \mathbb{R}$

$w = \frac{z-i}{z+1} = \frac{it-i}{it+1} = \frac{i(t-1)(1-it)}{(1+it)(1-it)} = \frac{t(t-1)+(t-1)i}{1+t^2} = \frac{t(t-1)}{1+t^2} + \frac{t-1}{1+t^2}i$

$u+iv = \frac{t(t-1)}{1+t^2} + i\frac{t-1}{1+t^2}$

$\frac{u}{v} = \frac{\frac{t(t-1)}{1+t^2}}{\frac{t-1}{1+t^2}} = \frac{t(t-1)}{t-1} = t$ $\quad \therefore t = \frac{u}{v}$

NOW $v = \frac{t-1}{1+t^2} = \frac{\frac{u}{v}-1}{1+\frac{u^2}{v^2}} = \frac{uv-v^2}{v^2+u^2} = \frac{v(u-v)}{v^2+u^2}$

$\therefore v = \frac{v(u-v)}{v^2+u^2}$

$1 = \frac{u-v}{v^2+u^2}$

$v^2+u^2 = u-v$

$v^2+u^2-u+v=0$ // AS BEFORE

Question 112 (****)

A transformation T maps the point $x+\mathrm{i}y$ from the z plane to the point $u+\mathrm{i}v$ in the w plane, and is defined by

$$w = \frac{z+\mathrm{i}}{z}, \ z \in \mathbb{C}, \ z \neq 0 .$$

T transforms the line with equation $y = x$ in the z plane, except the origin, into the straight line L_1 in the w plane.

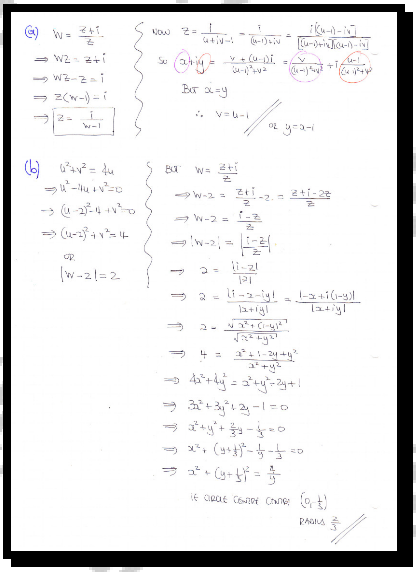
a) Find a Cartesian equation for L_1.

T transforms the circle C_1 in the z plane, into the circle C_2 in the w plane.

b) Find the coordinates of the centre of C_1 and the length of its radius, given the Cartesian equation of C_2 is

$$u^2 + v^2 = 4u .$$

$\boxed{y = x-1 \text{ or } v = u-1}$, $\boxed{\left(0,-\frac{1}{3}\right),\ r = \frac{2}{3}}$



(a) $w = \frac{z+i}{z}$

$\Rightarrow wz = z+i$

$\Rightarrow wz - z = i$

$\Rightarrow z(w-1) = i$

$\Rightarrow \boxed{z = \frac{i}{w-1}}$

NOW $z = \frac{i}{u+iv-1} = \frac{i}{(u-1)+iv} = \frac{i[(u-1)-iv]}{[(u-1)+iv][(u-1)-iv]}$

So $x+iy = \frac{v+(u-1)i}{(u-1)^2+v^2} = \frac{v}{(u-1)^2+v^2} + i\frac{u-1}{(u-1)^2+v^2}$

BUT $x=y$

$\therefore \ v = u-1$ // OR $y = x-1$

(b) $u^2+v^2 = 4u$

$\Rightarrow u^2-4u+v^2=0$

$\Rightarrow (u-2)^2-4+v^2=0$

$\Rightarrow (u-2)^2+v^2=4$

OR

$|w-2| = 2$

BUT $w = \frac{z+i}{z}$

$\Rightarrow w-2 = \frac{z+i}{z}-2 = \frac{z+i-2z}{z}$

$\Rightarrow w-2 = \frac{i-z}{z}$

$\Rightarrow |w-2| = \left|\frac{i-z}{z}\right|$

$\Rightarrow 2 = \frac{|i-z|}{|z|}$

$\Rightarrow 2 = \frac{|i-x-iy|}{|x+iy|} = \frac{|-x+i(1-y)|}{|x+iy|}$

$\Rightarrow 2 = \frac{\sqrt{x^2+(1-y)^2}}{\sqrt{x^2+y^2}}$

$\Rightarrow 4 = \frac{x^2+1-2y+y^2}{x^2+y^2}$

$\Rightarrow 4x^2+4y^2 = x^2+y^2-2y+1$

$\Rightarrow 3x^2+3y^2+2y-1=0$

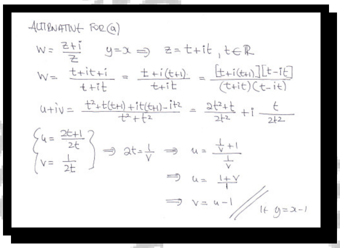
$\Rightarrow x^2+y^2+\frac{2}{3}y-\frac{1}{3}=0$

$\Rightarrow x^2+\left(y+\frac{1}{3}\right)^2-\frac{1}{9}-\frac{1}{3}=0$

$\Rightarrow x^2+\left(y+\frac{1}{3}\right)^2=\frac{4}{9}$

IE CIRCLE CENTRE CENTRE $\left(0,-\frac{1}{3}\right)$

RADIUS $\frac{2}{3}$ //



ALTERNATIVE FOR (a)

$w = \frac{z+i}{z}$ $\quad y=x \Rightarrow z = t+it, \ t \in \mathbb{R}$

$w = \frac{t+it+i}{t+it} = \frac{t+i(t+1)}{t+it} = \frac{[t+i(t+1)][t-it]}{(t+it)(t-it)}$

$u+iv = \frac{t^2+t(t+1)+it(t+1)-it^2}{t^2+t^2} = \frac{2t^2+t}{2t^2}+i\frac{t}{2t^2}$

$\left.\begin{cases} u = \frac{2t+1}{2t} \\ v = \frac{1}{2t} \end{cases}\right\} \Rightarrow 2t = \frac{1}{v} \Rightarrow u = \frac{\frac{1}{v}+1}{\frac{1}{v}}$

$\Rightarrow u = \frac{1+v}{1}$

$\Rightarrow v = u-1$ // IE $y = x-1$

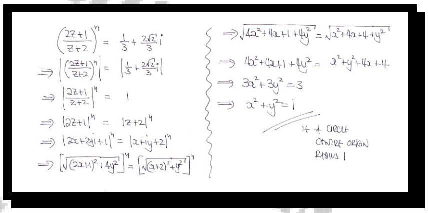
Question 113 (****)

The complex number z satisfies the relationship

$$\left(\frac{2z+1}{z+2}\right)^n = \frac{1}{3}+\frac{2\sqrt{2}}{3}\mathrm{i}, \quad z \neq -2, \quad n \in \mathbb{N}.$$

Show that the point represented by z in an Argand diagram represents a circle, stating the coordinates of its centre and the size of its radius.

$$\boxed{(0,0),\ r=1}$$



$$\left(\frac{2z+1}{z+2}\right)^n = \frac{1}{3}+\frac{2\sqrt{2}}{3}i$$

$$\Rightarrow \left|\left(\frac{2z+1}{z+2}\right)^n\right| = \left|\frac{1}{3}+\frac{2\sqrt{2}}{3}i\right|$$

$$\Rightarrow \left|\frac{2z+1}{z+2}\right|^n = 1$$

$$\Rightarrow |2z+1|^n = |z+2|^n$$

$$\Rightarrow |2x+2yi+1|^n = |x+iy+2|^n$$

$$\Rightarrow \left[\sqrt{(2x+1)^2+4y^2}\right]^n = \left[\sqrt{(x+2)^2+y^2}\right]^n$$

$$\Rightarrow \sqrt{4x^2+4x+1+4y^2} = \sqrt{x^2+4x+4+y^2}$$

$$\Rightarrow 4x^2+4x+1+4y^2 = x^2+y^2+4x+4$$

$$\Rightarrow 3x^2+3y^2=3$$

$$\Rightarrow x^2+y^2=1$$

IT A CIRCLE
CENTRE ORIGIN
RADIUS 1

Question 114 (****)

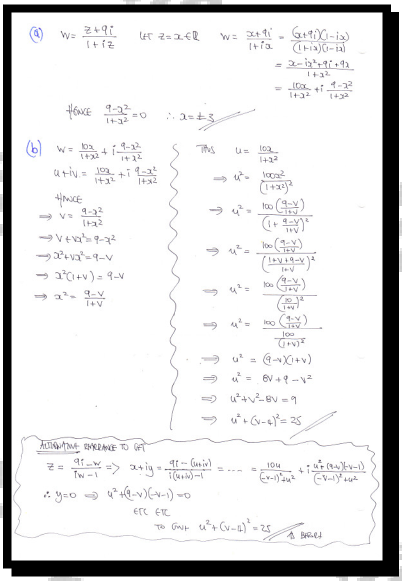
The numbers z and w satisfy the relationship

$$w = \frac{z+9\mathrm{i}}{1+\mathrm{i}z}, \quad z \neq \mathrm{i}.$$

a) Given that $w \in \mathbb{R}$, find the possible values of z.

b) Given instead that $z \in \mathbb{R}$, find a Cartesian equation of the locus of the point represented by w, in an Argand diagram.

$\boxed{z = \pm 3, \text{ or } x = \pm 3}$, $\boxed{u^2 + (v-4)^2 = 25}$



(a) $w = \frac{z+9i}{1+iz}$ LET $z = x \in \mathbb{R}$ $w = \frac{x+9i}{1+ix} = \frac{(x+9i)(1-ix)}{(1+ix)(1-ix)}$

$= \frac{x - ix^2 + 9i + 9x}{1+x^2}$

$= \frac{10x}{1+x^2} + i\frac{9-x^2}{1+x^2}$

HENCE $\frac{9-x^2}{1+x^2} = 0$ $\therefore x = \pm 3$

(b) $w = \frac{10x}{1+x^2} + i\frac{9-x^2}{1+x^2}$

$u + iv = \frac{10x}{1+x^2} + i\frac{9-x^2}{1+x^2}$

HENCE

$\Rightarrow v = \frac{9-x^2}{1+x^2}$

$\Rightarrow v + vx^2 = 9 - x^2$

$\Rightarrow x^2 + vx^2 = 9 - v$

$\Rightarrow x^2(1+v) = 9 - v$

$\Rightarrow x^2 = \frac{9-v}{1+v}$

THUS $u = \frac{10x}{1+x^2}$

$\Rightarrow u^2 = \frac{100x^2}{(1+x^2)^2}$

$\Rightarrow u^2 = \frac{100\left(\frac{9-v}{1+v}\right)}{\left(1+\frac{9-v}{1+v}\right)^2}$

$\Rightarrow u^2 = \frac{100\left(\frac{9-v}{1+v}\right)}{\left(\frac{1+v+9-v}{1+v}\right)^2}$

$\Rightarrow u^2 = \frac{100\left(\frac{9-v}{1+v}\right)}{\left(\frac{10}{1+v}\right)^2}$

$\Rightarrow u^2 = \frac{100\left(\frac{9-v}{1+v}\right)}{\frac{100}{(1+v)^2}}$

$\Rightarrow u^2 = (9-v)(1+v)$

$\Rightarrow u^2 = 8v + 9 - v^2$

$\Rightarrow u^2 + v^2 - 8v = 9$

$\Rightarrow u^2 + (v-4)^2 = 25$

ALTERNATIVE REARRANGE TO GET

$z = \frac{9i - w}{iw - 1} \Rightarrow x + iy = \frac{9i - (u+iv)}{i(u+iv) - 1} = \dots = \frac{10u}{(-v-1)^2 + u^2} + i\frac{u^2 + (9-v)(-v-1)}{(-v-1)^2 + u^2}$

$\therefore y = 0 \Rightarrow u^2 + (9-v)(-v-1) = 0$

ETC ETC

TO GIVE $u^2 + (v-4)^2 = 25$ AS BEFORE

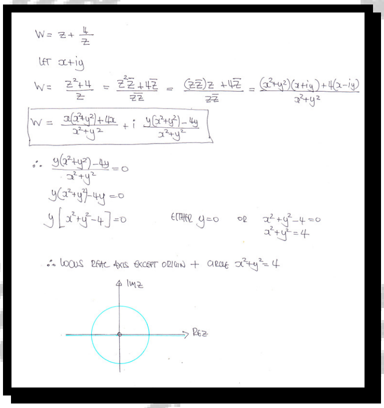
Question 115 (****)

The numbers z and w satisfy the relationship

$$w = z + \frac{4}{z}, \quad z \in \mathbb{C}, \quad z \neq 0 .$$

Given that w is always real sketch in a suitably labelled Argand diagram the locus of the possible positions of z.

graph



$$W = z + \frac{4}{z}$$

LET $x+iy$

$$W = \frac{z^2+4}{z} = \frac{z^2\bar{z}+4\bar{z}}{z\bar{z}} = \frac{(z\bar{z})z+4\bar{z}}{z\bar{z}} = \frac{(x^2+y^2)(x+iy)+4(x-iy)}{x^2+y^2}$$

$$\boxed{W = \frac{x(x^2+y^2)+4x}{x^2+y^2} + i\,\frac{y(x^2+y^2)-4y}{x^2+y^2}}$$

$$\therefore \frac{y(x^2+y^2)-4y}{x^2+y^2} = 0$$

$$y(x^2+y^2)-4y = 0$$

$$y\left[x^2+y^2-4\right] = 0$$

EITHER $y=0$ OR $x^2+y^2-4=0$

$x^2+y^2=4$

$\therefore$ LOCUS REAL AXIS EXCEPT ORIGIN + CIRCLE $x^2+y^2=4$

Question 116 (****)

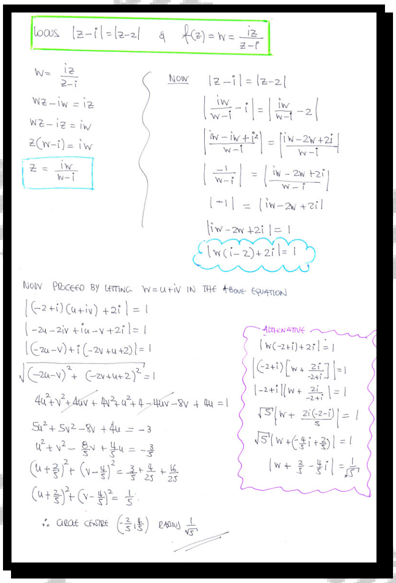
A transformation from the z plane to the w plane is defined by the equation

$$f(z) = \frac{\mathrm{i}\,z}{z-\mathrm{i}}, \quad z \in \mathbb{C}.$$

Find, in Cartesian form, the equation of the image of straight line with equation

$$|z-\mathrm{i}| = |z-2|, \quad z \in \mathbb{C}.$$

$$\boxed{\left(u+\tfrac{2}{5}\right)^2 + \left(v-\tfrac{4}{5}\right)^2 = \tfrac{1}{5}}$$



$$\boxed{\text{locus } |z-i| = |z-2| \ \ \& \ \ f(z) = w = \frac{iz}{z-i}}$$

$$w = \frac{iz}{z-i}$$

$$wz - iw = iz$$

$$wz - iz = iw$$

$$z(w-i) = iw$$

$$\boxed{z = \frac{iw}{w-i}}$$

NOW $|z-i| = |z-2|$

$$\left|\frac{iw}{w-i} - i\right| = \left|\frac{iw}{w-i} - 2\right|$$

$$\left|\frac{iw - iw + i^2}{w-i}\right| = \left|\frac{iw - 2w + 2i}{w-i}\right|$$

$$\left|\frac{-1}{w-i}\right| = \left|\frac{iw - 2w + 2i}{w-i}\right|$$

$$|-1| = |iw - 2w + 2i|$$

$$|iw - 2w + 2i| = 1$$

$$|w(i-2) + 2i| = 1$$

NOW PROCEED BY LETTING $w = u + iv$ IN THE ABOVE EQUATION

$$|(-2+i)(u+iv) + 2i| = 1$$

$$|-2u - 2iv + iu - v + 2i| = 1$$

$$|(-2u - v) + i(-2v + u + 2)| = 1$$

$$\sqrt{(-2u-v)^2 + (-2v+u+2)^2} = 1$$

$$4u^2 + v^2 + \cancel{4uv} + 4v^2 + u^2 + 4 - \cancel{4uv} - 8v + 4u = 1$$

$$5u^2 + 5v^2 - 8v + 4u = -3$$

$$u^2 + v^2 - \frac{8}{5}v + \frac{4}{5}u = -\frac{3}{5}$$

$$\left(u+\frac{2}{5}\right)^2 + \left(v-\frac{4}{5}\right)^2 = -\frac{3}{5} + \frac{4}{25} + \frac{16}{25}$$

$$\left(u+\frac{2}{5}\right)^2 + \left(v-\frac{4}{5}\right)^2 = \frac{1}{5}$$

$\therefore$ CIRCLE CENTRE $\left(-\frac{2}{5}, \frac{4}{5}\right)$ RADIUS $\frac{1}{\sqrt{5}}$

ALTERNATIVE

$$|w(-2+i) + 2i| = 1$$

$$\left|(-2+i)\left[w + \frac{2i}{-2+i}\right]\right| = 1$$

$$|-2+i|\left|w + \frac{2i}{-2+i}\right| = 1$$

$$\sqrt{5}\left|w + \frac{2i(-2-i)}{5}\right| = 1$$

$$\sqrt{5}\left|w + \left(-\frac{4}{5}i + \frac{2}{5}\right)\right| = 1$$

$$\left|w + \frac{2}{5} - \frac{4}{5}i\right| = \frac{1}{\sqrt{5}}$$

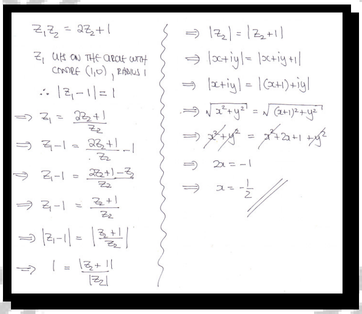
Question 117 (****)

The complex numbers z_1 and z_2, satisfy the relationship

$$z_1 z_2 = 2z_2 + 1, \; z_2 \neq 0.$$

Given that z_1 is tracing a circle with centre at $(1,0)$ and radius 1 in an Argand diagram, determine a Cartesian equation of the locus that z_2 is tracing.

$$\boxed{x = -\frac{1}{2}}$$



$z_1 z_2 = 2z_2 + 1$

z_1 LIES ON THE CIRCLE WITH CENTRE $(1,0)$, RADIUS 1

$\therefore |z_1 - 1| = 1$

$\Rightarrow z_1 = \frac{2z_2+1}{z_2}$

$\Rightarrow z_1 - 1 = \frac{2z_2+1}{z_2} - 1$

$\Rightarrow z_1 - 1 = \frac{2z_2+1-z_2}{z_2}$

$\Rightarrow z_1 - 1 = \frac{z_2+1}{z_2}$

$\Rightarrow |z_1 - 1| = \left|\frac{z_2+1}{z_2}\right|$

$\Rightarrow 1 = \frac{|z_2+1|}{|z_2|}$

$\Rightarrow |z_2| = |z_2+1|$

$\Rightarrow |x+iy| = |x+iy+1|$

$\Rightarrow |x+iy| = |(x+1)+iy|$

$\Rightarrow \sqrt{x^2+y^2} = \sqrt{(x+1)^2+y^2}$

$\Rightarrow \cancel{x^2} + \cancel{y^2} = \cancel{x^2} + 2x + 1 + \cancel{y^2}$

$\Rightarrow 2x = -1$

$\Rightarrow x = -\frac{1}{2}$

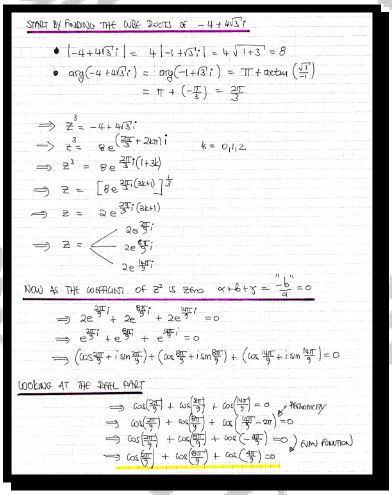
Question 118 **(****)**

$$z^3 + 4 = 4\sqrt{3}\,\mathrm{i}\,.$$

By considering the sum of the three roots of the above cubic equation show clearly that

$$\cos\frac{2\pi}{9} + \cos\frac{4\pi}{9} + \cos\frac{8\pi}{9} = 0\,.$$

$\boxed{}$, $\boxed{\text{proof}}$



START BY FINDING THE CUBE ROOTS OF $-4+4\sqrt{3}i$

- $|-4+4\sqrt{3}i| = 4|-1+\sqrt{3}i| = 4\sqrt{1+3} = 8$
- $\arg(-4+4\sqrt{3}i) = \arg(-1+\sqrt{3}i) = \pi + \arctan\left(\frac{\sqrt{3}}{-1}\right)$
$= \pi + \left(-\frac{\pi}{3}\right) = \frac{2\pi}{3}$

$$\Rightarrow z^3 = -4+4\sqrt{3}i$$
$$\Rightarrow z^3 = 8e^{\left(\frac{2\pi}{3}+2k\pi\right)i} \qquad k = 0,1,2$$
$$\Rightarrow z^3 = 8e^{\frac{2\pi}{3}i(1+3k)}$$
$$\Rightarrow z = \left[8e^{\frac{2\pi}{3}i(3k+1)}\right]^{\frac{1}{3}}$$
$$\Rightarrow z = 2e^{\frac{2\pi}{9}i(3k+1)}$$
$$\Rightarrow z = \begin{cases} 2e^{\frac{2\pi}{9}i} \\ 2e^{\frac{8\pi}{9}i} \\ 2e^{\frac{14\pi}{9}i} \end{cases}$$

NOW AS THE COEFFICIENT OF z^2 IS ZERO $\alpha+\beta+\gamma = \text{"}\frac{-b}{a}\text{"} = 0$

$$\Rightarrow 2e^{\frac{2\pi}{9}i} + 2e^{\frac{8\pi}{9}i} + 2e^{\frac{14\pi}{9}i} = 0$$
$$\Rightarrow e^{\frac{2\pi}{9}i} + e^{\frac{8\pi}{9}i} + e^{\frac{14\pi}{9}i} = 0$$
$$\Rightarrow \left(\cos\frac{2\pi}{9} + i\sin\frac{2\pi}{9}\right) + \left(\cos\frac{8\pi}{9} + i\sin\frac{8\pi}{9}\right) + \left(\cos\frac{14\pi}{9} + i\sin\frac{14\pi}{9}\right) = 0$$

LOOKING AT THE REAL PART

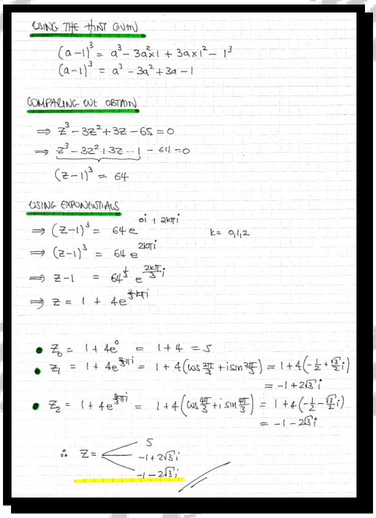
$$\Rightarrow \cos\left(\frac{2\pi}{9}\right) + \cos\left(\frac{8\pi}{9}\right) + \cos\left(\frac{14\pi}{9}\right) = 0 \quad \text{PERIODICITY}$$
$$\Rightarrow \cos\left(\frac{2\pi}{9}\right) + \cos\left(\frac{8\pi}{9}\right) + \cos\left(\frac{14\pi}{9} - 2\pi\right) = 0$$
$$\Rightarrow \cos\left(\frac{2\pi}{9}\right) + \cos\left(\frac{8\pi}{9}\right) + \cos\left(-\frac{4\pi}{9}\right) = 0 \quad \text{EVEN FUNCTION}$$
$$\Rightarrow \cos\left(\frac{2\pi}{9}\right) + \cos\left(\frac{8\pi}{9}\right) + \cos\left(\frac{4\pi}{9}\right) = 0$$

Question 119 **(****)**

$$z^3 - 3z^2 + 3z - 65 = 0, \; z \in \mathbb{C}.$$

By considering the binomial expansion of $(a-1)^3$, or otherwise, find in exact form where appropriate the three solutions of the above equation.

$\boxed{}$, $\boxed{z = 5, \; -1 \pm \mathrm{i}\frac{\sqrt{3}}{2}}$



USING THE HINT GIVEN

$$(a-1)^3 = a^3 - 3a^2 \times 1 + 3a \times 1^2 - 1^3$$
$$(a-1)^3 = a^3 - 3a^2 + 3a - 1$$

COMPARING WE OBTAIN

$$\Rightarrow z^3 - 3z^2 + 3z - 65 = 0$$
$$\Rightarrow \underbrace{z^3 - 3z^2 + 3z - 1}_{} - 64 = 0$$
$$(z-1)^3 = 64$$

USING EXPONENTIALS

$$\Rightarrow (z-1)^3 = 64e^{0i + 2k\pi i} \qquad k = 0,1,2$$
$$\Rightarrow (z-1)^3 = 64e^{2k\pi i}$$
$$\Rightarrow z - 1 = 64^{\frac{1}{3}} e^{\frac{2k\pi}{3}i}$$
$$\Rightarrow z = 1 + 4e^{\frac{2}{3}k\pi i}$$

- $z_0 = 1 + 4e^0 = 1 + 4 = 5$
- $z_1 = 1 + 4e^{\frac{2}{3}\pi i} = 1 + 4\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right) = 1 + 4\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)$
  $= -1 + 2\sqrt{3}i$
- $z_2 = 1 + 4e^{\frac{4}{3}\pi i} = 1 + 4\left(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}\right) = 1 + 4\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)$
  $= -1 - 2\sqrt{3}i$

$$\therefore z = \begin{cases} 5 \\ -1 + 2\sqrt{3}i \\ -1 - 2\sqrt{3}i \end{cases}$$

Question 120 (****+)

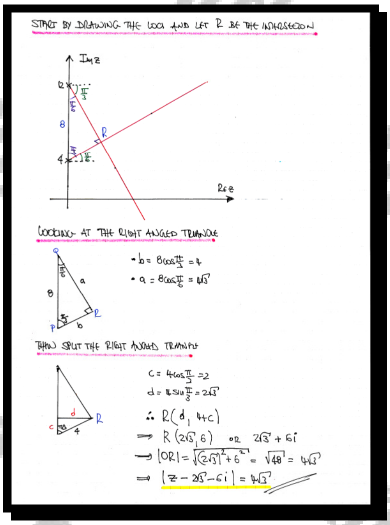
The complex number w is the point of intersection of the following two loci in a standard Argand diagram

$$\arg(z-4\mathrm{i})=\frac{\pi}{6} \quad \text{and} \quad \arg(z-12\mathrm{i})=-\frac{\pi}{3}.$$

Determine the equation of the circle which passes though w and the origin of the Argand diagram.

Give the answer in the form $|z-w|=r$, where w and r must be stated.

$\boxed{}$, $\boxed{|z-2\sqrt{3}-6\mathrm{i}|=4\sqrt{3}}$



START BY DRAWING THE LOCI AND LET R BE THE INTERSECTION

LOOKING AT THE RIGHT ANGLED TRIANGLE

- $b = 8\cos\frac{\pi}{3} = 4$
- $a = 8\cos\frac{\pi}{6} = 4\sqrt{3}$

THEN SPLIT THE RIGHT ANGLED TRIANGLE

$c = 4\cos\frac{\pi}{3} = 2$

$d = 4\sin\frac{\pi}{3} = 2\sqrt{3}$

$\therefore R(d, 4+c)$

$\Rightarrow R(2\sqrt{3}, 6)$ OR $2\sqrt{3}+6i$

$\Rightarrow |OR| = \sqrt{(2\sqrt{3})^2+6^2} = \sqrt{48} = 4\sqrt{3}$

$\Rightarrow |z-2\sqrt{3}-6i| = 4\sqrt{3}$

Question 121 (****+)

The complex number $17+k\mathrm{i}$, where k is a real constant, satisfies the locus

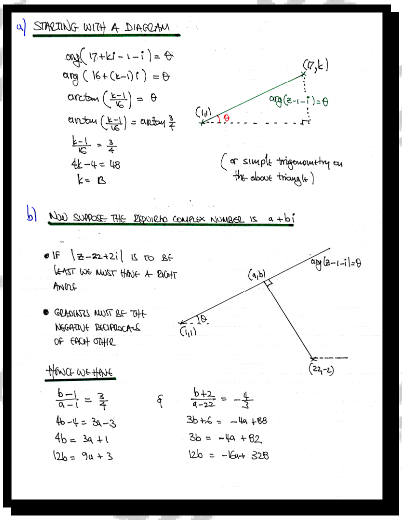
$$\arg\left(z-1-\mathrm{i}\right)=\theta,$$

where $\theta=\arctan\frac{3}{4}$.

a) Determine the value of k.

b) Find the complex number z which satisfies the locus $\arg\left(z-1-\mathrm{i}\right)=\theta$ so that $\left|z-22+2\mathrm{i}\right|$ is least.

$\boxed{}$, $\boxed{k=13}$, $\boxed{13+10\mathrm{i}}$



a) STARTING WITH A DIAGRAM

$\arg\left(17+k\mathrm{i}-1-\mathrm{i}\right)=\theta$

$\arg\left(16+(k-1)\mathrm{i}\right)=\theta$

$\arctan\left(\frac{k-1}{16}\right)=\theta$

$\arctan\left(\frac{k-1}{16}\right)=\arctan\frac{3}{4}$

$\frac{k-1}{16}=\frac{3}{4}$

$4k-4=48$

$k=13$

(or simple trigonometry on the above triangle)

b) NOW SUPPOSE THE REQUIRED COMPLEX NUMBER IS $a+b\mathrm{i}$

- IF $|z-22+2\mathrm{i}|$ IS TO BE LEAST WE MUST HAVE A RIGHT ANGLE
- GRADIENTS MUST BE THE NEGATIVE RECIPROCALS OF EACH OTHER

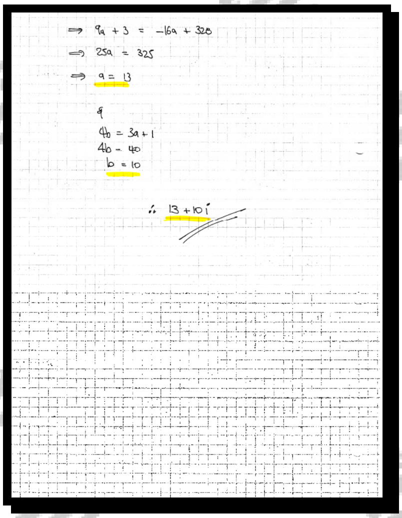
HENCE WE HAVE

$\frac{b-1}{a-1}=\frac{3}{4}$ & $\frac{b+2}{a-22}=-\frac{4}{3}$

$4b-4=3a-3$ $3b+6=-4a+88$

$4b=3a+1$ $3b=-4a+82$

$12b=9a+3$ $12b=-16a+328$



$\Rightarrow 9a+3=-16a+328$

$\Rightarrow 25a=325$

$\Rightarrow a=13$

&

$4b=3a+1$

$4b=40$

$b=10$

$\therefore 13+10\mathrm{i}$

Question 122 (****+)

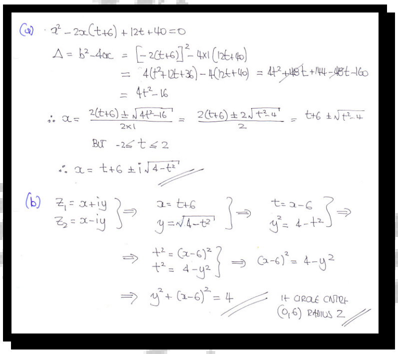
The quadratic equation

$$x^2 - 2x(t+6) + 12t + 40 = 0,$$

where t is a parameter such that $-2 \le t \le 2$, has complex roots.

Show that for all t such that $-2 \le t \le 2$, the roots of this quadratic equation lie on a circle in an Argand diagram.

$\boxed{\text{FP1-S}}$, $\boxed{x = t + 6 \pm \mathrm{i}\sqrt{4 - t^2}}$



(a) $x^2 - 2x(t+6) + 12t + 40 = 0$

$\Delta = b^2 - 4ac = \left[-2(t+6)\right]^2 - 4\times1(12t+40)$

$= 4(t^2+12t+36) - 4(12t+40) = 4t^2 \cancel{+48t} + 144 \cancel{-48t} - 160$

$= 4t^2 - 16$

$\therefore x = \dfrac{2(t+6) \pm \sqrt{4t^2-16}}{2\times1} = \dfrac{2(t+6) \pm 2\sqrt{t^2-4}}{2} = t+6 \pm \sqrt{t^2-4}$

BUT $-2 \le t \le 2$

$\therefore x = t+6 \pm i\sqrt{4-t^2}$

(b) $\left.\begin{array}{l} z_1 = x+iy \\ z_2 = x-iy \end{array}\right\} \Rightarrow \left.\begin{array}{l} x = t+6 \\ y = \sqrt{4-t^2} \end{array}\right\} \Rightarrow \left.\begin{array}{l} t = x-6 \\ y^2 = 4-t^2 \end{array}\right\} \Rightarrow$

$\Rightarrow \left.\begin{array}{l} t^2 = (x-6)^2 \\ t^2 = 4-y^2 \end{array}\right\} \Rightarrow (x-6)^2 = 4-y^2$

$\Rightarrow y^2 + (x-6)^2 = 4$

IT CIRCLE CENTRE (0,6) RADIUS 2

Question 123 (****+)

The complex function $w = f(z)$ is defined by

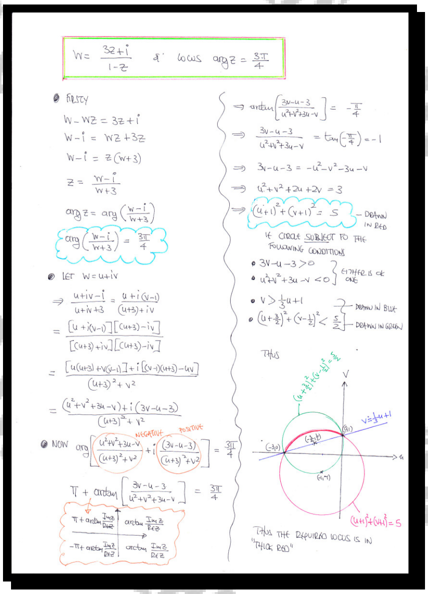
$$w = \frac{3z+\mathrm{i}}{1-z}, \quad z \in \mathbb{C}, \quad z \neq 1.$$

The half line with equation $\arg z = \frac{3\pi}{4}$ is transformed by this function.

a) Find a Cartesian equation of the locus of the **image** of the half line.

b) Sketch the **image** of the locus in an Argand diagram.

$$\boxed{(u+1)^2 + (v+1)^2 = 5, \quad v > \frac{1}{3}u + 1}$$



$$\boxed{w = \frac{3z+i}{1-z} \quad \& \quad \text{locus } \arg z = \frac{3\pi}{4}}$$

- FIRSTLY

$$w - wz = 3z + i$$
$$w - i = wz + 3z$$
$$w - i = z(w+3)$$
$$z = \frac{w-i}{w+3}$$
$$\arg z = \arg\left(\frac{w-i}{w+3}\right)$$
$$\arg\left(\frac{w-i}{w+3}\right) = \frac{3\pi}{4}$$

- LET $w = u + iv$

$$\Rightarrow \frac{u+iv-i}{u+iv+3} = \frac{u+i(v-1)}{(u+3)+iv}$$
$$= \frac{[u+i(v-1)][(u+3)-iv]}{[(u+3)+iv][(u+3)-iv]}$$
$$= \frac{[u(u+3)+v(v-1)] + i[(v-1)(u+3)-uv]}{(u+3)^2+v^2}$$
$$= \frac{(u^2+v^2+3u-v) + i(3v-u-3)}{(u+3)^2+v^2}$$

- NOW $\arg\left[\underbrace{\frac{u^2+v^2+3u-v}{(u+3)^2+v^2}}_{\text{NEGATIVE}} + i\underbrace{\frac{(3v-u-3)}{(u+3)^2+v^2}}_{\text{POSITIVE}}\right] = \frac{3\pi}{4}$

$$\pi + \arctan\left[\frac{3v-u-3}{u^2+v^2+3u-v}\right] = \frac{3\pi}{4}$$

$\pi + \arctan\frac{\mathrm{Im}\,z}{\mathrm{Re}\,z}$	$\arctan\frac{\mathrm{Im}\,z}{\mathrm{Re}\,z}$
$-\pi + \arctan\frac{\mathrm{Im}\,z}{\mathrm{Re}\,z}$	$\arctan\frac{\mathrm{Im}\,z}{\mathrm{Re}\,z}$

$$\Rightarrow \arctan\left[\frac{3v-u-3}{u^2+v^2+3u-v}\right] = -\frac{\pi}{4}$$
$$\Rightarrow \frac{3v-u-3}{u^2+v^2+3u-v} = \tan\left(-\frac{\pi}{4}\right) = -1$$
$$\Rightarrow 3v-u-3 = -u^2-v^2-3u-v$$
$$\Rightarrow u^2+v^2+2u+2v = 3$$
$$\Rightarrow (u+1)^2+(v+1)^2 = 5$$

– DRAWN IN RED

IE CIRCLE SUBJECT TO THE FOLLOWING CONDITIONS

- $3v-u-3>0$
- $u^2+v^2+3u-v<0$

} EITHER IS OK ONE

- $v > \frac{1}{3}u+1$ – DRAWN IN BLUE
- $\left(u+\frac{3}{2}\right)^2+\left(v-\frac{1}{2}\right)^2 < \frac{5}{2}$ – DRAWN IN GREEN

THUS

THUS THE REQUIRED LOCUS IS IN "THICK RED"

Question 124 (****+)

It is given that

$$\cot 4\theta = \frac{\cot^4\theta - 6\cot^2\theta + 1}{4\cot^3\theta - 4\cot\theta}.$$

a) Use de Moivre's theorem to prove the validity of the above trigonometric identity.

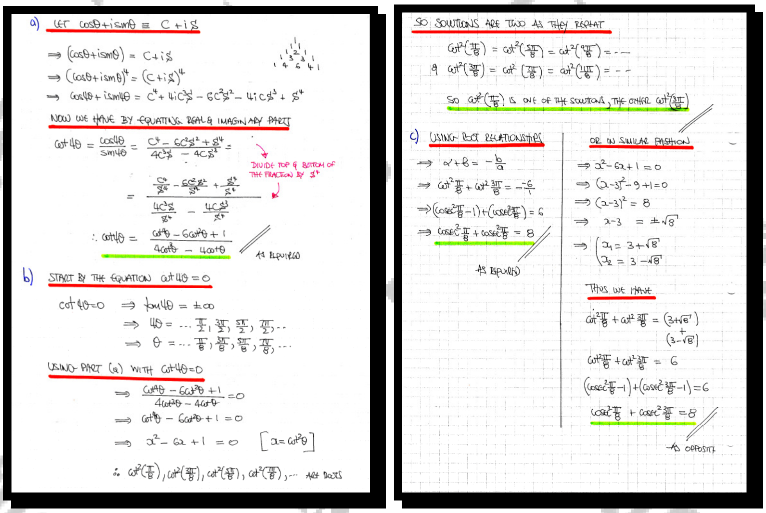
b) Deduce that $x = \cot^2\left(\frac{\pi}{8}\right)$ is one of the two solutions of the equation

$$x^2 - 6x + 1 = 0.$$

c) Show further that

$$\operatorname{cosec}^2\left(\frac{\pi}{8}\right) + \operatorname{cosec}^2\left(\frac{3\pi}{8}\right) = 8.$$

$\boxed{}$, $\boxed{\text{proof}}$



a) LET $\cos\theta + i\sin\theta \equiv C + iS$

$\Rightarrow (\cos\theta + i\sin\theta) = C + iS$

$\Rightarrow (\cos\theta + i\sin\theta)^4 = (C+iS)^4$

$\Rightarrow \cos 4\theta + i\sin 4\theta = C^4 + 4iC^3S^1 - 6C^2S^2 - 4iCS^3 + S^4$

NOW WE HAVE BY EQUATING REAL & IMAGINARY PARTS

$$\cot 4\theta = \frac{\cos 4\theta}{\sin 4\theta} = \frac{C^4 - 6C^2S^2 + S^4}{4C^3S - 4CS^3} =$$

DIVIDE TOP & BOTTOM OF THE FRACTION BY S^4

$$= \frac{\frac{C^4}{S^4} - \frac{6C^2S^2}{S^4} + \frac{S^4}{S^4}}{\frac{4C^3S}{S^4} - \frac{4CS^3}{S^4}}$$

$$\therefore \cot 4\theta = \frac{\cot^4\theta - 6\cot^2\theta + 1}{4\cot^3\theta - 4\cot\theta}$$

AS REQUIRED

b) START BY THE EQUATION $\cot 4\theta = 0$

$\cot 4\theta = 0 \Rightarrow \tan 4\theta = \pm\infty$

$\Rightarrow 4\theta = \dots \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}, \dots$

$\Rightarrow \theta = \dots \frac{\pi}{8}, \frac{3\pi}{8}, \frac{5\pi}{8}, \frac{7\pi}{8}, \dots$

USING PART (a) WITH $\cot 4\theta = 0$

$\Rightarrow \frac{\cot^4\theta - 6\cot^2\theta + 1}{4\cot^3\theta - 4\cot\theta} = 0$

$\Rightarrow \cot^4\theta - 6\cot^2\theta + 1 = 0$

$\Rightarrow x^2 - 6x + 1 = 0 \quad [x = \cot^2\theta]$

$\therefore \cot^2\left(\frac{\pi}{8}\right), \cot^2\left(\frac{3\pi}{8}\right), \cot^2\left(\frac{5\pi}{8}\right), \cot^2\left(\frac{7\pi}{8}\right), \dots$ ARE ROOTS

SO SOLUTIONS ARE TWO AS THEY REPEAT

$\cot^2\left(\frac{\pi}{8}\right) = \cot^2\left(\frac{5\pi}{8}\right) = \cot^2\left(\frac{9\pi}{8}\right) = \dots$

& $\cot^2\left(\frac{3\pi}{8}\right) = \cot^2\left(\frac{7\pi}{8}\right) = \cot^2\left(\frac{11\pi}{8}\right) = \dots$

SO $\cot^2\left(\frac{\pi}{8}\right)$ IS ONE OF THE SOLUTIONS, THE OTHER $\cot^2\left(\frac{3\pi}{8}\right)$

c) USING ROOT RELATIONSHIPS

$\Rightarrow \alpha + \beta = -\frac{b}{a}$

$\Rightarrow \cot^2\frac{\pi}{8} + \cot^2\frac{3\pi}{8} = -\frac{-6}{1}$

$\Rightarrow \left(\operatorname{cosec}^2\frac{\pi}{8} - 1\right) + \left(\operatorname{cosec}^2\frac{3\pi}{8}\right) = 6$

$\Rightarrow \operatorname{cosec}^2\frac{\pi}{8} + \operatorname{cosec}^2\frac{\pi}{8} = 8$

AS REQUIRED

OR IN SIMILAR FASHION

$\Rightarrow x^2 - 6x + 1 = 0$

$\Rightarrow (x-3)^2 - 9 + 1 = 0$

$\Rightarrow (x-3)^2 = 8$

$\Rightarrow x - 3 = \pm\sqrt{8}$

$\Rightarrow \begin{cases} x_1 = 3 + \sqrt{8} \\ x_2 = 3 - \sqrt{8} \end{cases}$

THUS WE HAVE

$\cot^2\frac{\pi}{8} + \cot^2\frac{3\pi}{8} = (3+\sqrt{8}) + (3-\sqrt{8})$

$\cot^2\frac{\pi}{8} + \cot^2\frac{3\pi}{8} = 6$

$\left(\operatorname{cosec}^2\frac{\pi}{8} - 1\right) + \left(\operatorname{cosec}^2\frac{3\pi}{8} - 1\right) = 6$

$\operatorname{cosec}^2\frac{\pi}{8} + \operatorname{cosec}^2\frac{3\pi}{8} = 8$

AS OPPOSITE

Question 125 (****+)

In an Argand diagram which represents the z plane, the complex number $z = x + \mathrm{i}y$ satisfies the relationship

$$\arg\left(\frac{z-2\mathrm{i}}{z-4}\right) = \frac{\pi}{2}.$$

a) Sketch the curve that the locus of z traces.

The complex function $w = f(z)$ maps points of the form $z = x + \mathrm{i}y$ from the z plane onto points of the form $w = u + \mathrm{i}v$ in the w plane.

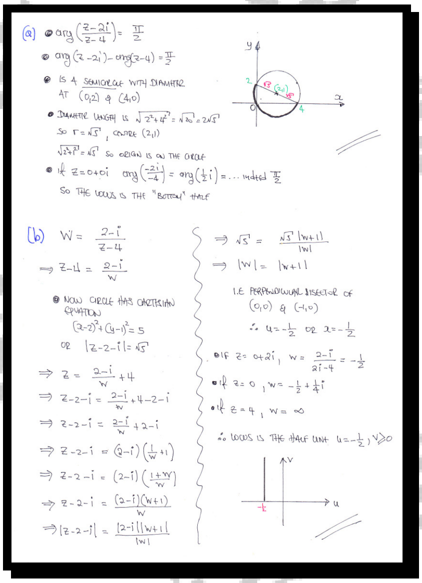
It is given that

$$f(z) = \frac{2-\mathrm{i}}{z-4}, \quad z \in \mathbb{C}, \quad z \neq 4.$$

The points in the z plane which lie on the locus described in part (a) are mapped onto a line in the w plane.

b) Sketch this line in an Argand diagram representing the w plane.

sketch



(a) • $\arg\left(\frac{z-2i}{z-4}\right) = \frac{\pi}{2}$

• $\arg(z-2i) - \arg(z-4) = \frac{\pi}{2}$

• IS A SEMICIRCLE WITH DIAMETER AT (0,2) & (4,0)

• DIAMETER LENGTH IS $\sqrt{2^2+4^2} = \sqrt{20} = 2\sqrt{5}$

SO $r = \sqrt{5}$, CENTRE (2,1)

$\sqrt{2^2+1^2} = \sqrt{5}$ SO ORIGIN IS ON THE CIRCLE

• if $z = 0+0i$ $\arg\left(\frac{-2i}{-4}\right) = \arg\left(\frac{1}{2}i\right) = \dots$ indeed $\frac{\pi}{2}$

SO THE LOCUS IS THE "BOTTOM" HALF

(b) $W = \frac{2-i}{z-4}$

$\Rightarrow z-4 = \frac{2-i}{w}$

• NOW CIRCLE HAS CARTESIAN EQUATION

$(x-2)^2 + (y-1)^2 = 5$

OR $|z-2-i| = \sqrt{5}$

$\Rightarrow z = \frac{2-i}{w} + 4$

$\Rightarrow z-2-i = \frac{2-i}{w} + 4 - 2 - i$

$\Rightarrow z-2-i = \frac{2-i}{w} + 2 - i$

$\Rightarrow z-2-i = (2-i)\left(\frac{1}{w}+1\right)$

$\Rightarrow z-2-i = (2-i)\left(\frac{1+w}{w}\right)$

$\Rightarrow z-2-i = \frac{(2-i)(w+1)}{w}$

$\Rightarrow |z-2-i| = \frac{|2-i||w+1|}{|w|}$

$\Rightarrow \sqrt{5} = \frac{\sqrt{5}|w+1|}{|w|}$

$\Rightarrow |w| = |w+1|$

I.E PERPENDICULAR BISECTOR OF (0,0) & (-1,0)

$\therefore u = -\frac{1}{2}$ OR $x = -\frac{1}{2}$

• IF $z = 0+2i$, $w = \frac{2-i}{2i-4} = -\frac{1}{2}$

• if $z = 0$, $w = -\frac{1}{2} + \frac{1}{4}i$

• if $z = 4$, $w = \infty$

$\therefore$ LOCUS IS THE HALF LINE $u = -\frac{1}{2}$, $v \geqslant 0$

Question 126 **(****+)**

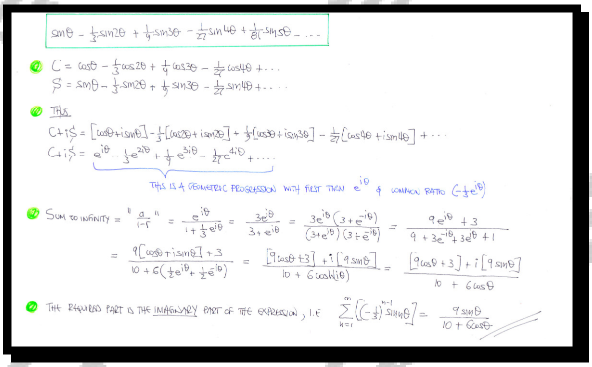
The following convergent series S is given below

$$S = \sin\theta - \frac{1}{3}\sin 2\theta + \frac{1}{9}\sin 3\theta - \frac{1}{27}\sin 4\theta \ ...$$

By considering the sum to infinity of a suitable geometric series involving the complex exponential function, show that

$$S = \frac{9\sin\theta}{10 + 6\cos\theta}.$$

$\boxed{\text{proof}}$



$$\boxed{\sin\theta - \tfrac{1}{3}\sin 2\theta + \tfrac{1}{9}\sin 3\theta - \tfrac{1}{27}\sin 4\theta + \tfrac{1}{81}\sin 5\theta - \dots}$$

$C = \cos\theta - \frac{1}{3}\cos 2\theta + \frac{1}{9}\cos 3\theta - \frac{1}{27}\cos 4\theta + \dots$

$S = \sin\theta - \frac{1}{3}\sin 2\theta + \frac{1}{9}\sin 3\theta - \frac{1}{27}\sin 4\theta + \dots$

THUS

$C + iS = [\cos\theta + i\sin\theta] - \frac{1}{3}[\cos 2\theta + i\sin 2\theta] + \frac{1}{9}[\cos 3\theta + i\sin 3\theta] - \frac{1}{27}[\cos 4\theta + i\sin 4\theta] + \dots$

$C + iS = \underbrace{e^{i\theta} - \frac{1}{3}e^{2i\theta} + \frac{1}{9}e^{3i\theta} - \frac{1}{27}e^{4i\theta} + \dots}$

THIS IS A GEOMETRIC PROGRESSION WITH FIRST TERM $e^{i\theta}$ & COMMON RATIO $(-\frac{1}{3}e^{i\theta})$

SUM TO INFINITY $= \text{"}\frac{a}{1-r}\text{"} = \frac{e^{i\theta}}{1 + \frac{1}{3}e^{i\theta}} = \frac{3e^{i\theta}}{3 + e^{i\theta}} = \frac{3e^{i\theta}(3 + e^{-i\theta})}{(3 + e^{i\theta})(3 + e^{-i\theta})} = \frac{9e^{i\theta} + 3}{9 + 3e^{-i\theta} + 3e^{i\theta} + 1}$

$= \frac{9[\cos\theta + i\sin\theta] + 3}{10 + 6(\frac{1}{2}e^{i\theta} + \frac{1}{2}e^{-i\theta})} = \frac{[9\cos\theta + 3] + i[9\sin\theta]}{10 + 6\cosh(i\theta)} = \frac{[9\cos\theta + 3] + i[9\sin\theta]}{10 + 6\cos\theta}$

THE REQUIRED PART IS THE IMAGINARY PART OF THE EXPRESSION, I.E $\sum_{n=1}^{\infty}\left[\left(-\frac{1}{3}\right)^{n-1}\sin n\theta\right] = \frac{9\sin\theta}{10 + 6\cos\theta}$

Question 127 (****+)

$$f(z) = z^6 + 8z^3 + 64, \quad z \in \mathbb{C}.$$

a) Given that $f(z) = 0$, show that

$$z^3 = -4 \pm 4\sqrt{3}\,\mathrm{i}.$$

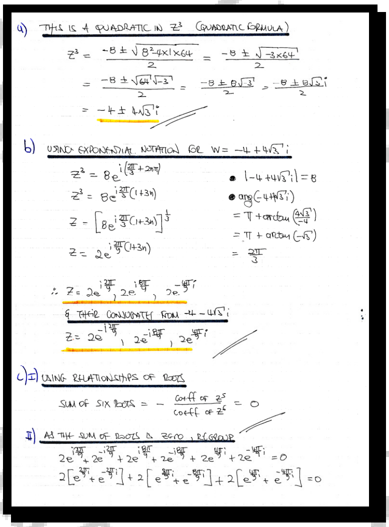
b) Find the six solutions of the equation $f(z) = 0$, giving the answers in the form $z = r\mathrm{e}^{\mathrm{i}\theta}$, where $r > 0$ and $-\pi < \theta \le \pi$.

c) Show further that …

i. … the sum of the six roots is zero.

ii. … $\cos\frac{2\pi}{9} + \cos\frac{4\pi}{9} + \cos\frac{6\pi}{9} + \cos\frac{8\pi}{9} = -\frac{1}{2}$.

$\boxed{}$, $\boxed{z = 2\mathrm{e}^{\mathrm{i}\varphi}, \varphi = \pm\frac{2\pi}{9}, \pm\frac{4\pi}{9}, \pm\frac{8\pi}{9}}$



a) THIS IS A QUADRATIC IN Z^3 (QUADRATIC FORMULA)

$$z^3 = \frac{-8 \pm \sqrt{8^2 - 4\times1\times64}}{2} = \frac{-8 \pm \sqrt{-3\times64}}{2}$$

$$= \frac{-8 \pm \sqrt{64}\sqrt{-3}}{2} = \frac{-8 \pm 8\sqrt{-3}}{2} = \frac{-8 \pm 8\sqrt{3}i}{2}$$

$$= -4 \pm 4\sqrt{3}i$$

b) USING EXPONENTIAL NOTATION FOR $W = -4 + 4\sqrt{3}i$

$$z^3 = 8e^{i\left(\frac{2\pi}{3} + 2n\pi\right)}$$

$$z^3 = 8e^{i\frac{2\pi}{3}(1+3n)}$$

$$z = \left[8e^{i\frac{2\pi}{3}(1+3n)}\right]^{\frac{1}{3}}$$

$$z = 2e^{i\frac{2\pi}{9}(1+3n)}$$

- $|-4+4\sqrt{3}i| = 8$
- $\arg(-4+4\sqrt{3}i)$

$= \pi + \arctan\left(\frac{4\sqrt{3}}{-4}\right)$

$= \pi + \arctan(-\sqrt{3})$

$= \frac{2\pi}{3}$

$$\therefore z = 2e^{i\frac{2\pi}{9}},\ 2e^{i\frac{8\pi}{9}},\ 2e^{-\frac{4\pi}{9}i}$$

& THEIR CONJUGATES FROM $-4 - 4\sqrt{3}i$

$$z = 2e^{-i\frac{2\pi}{9}},\ 2e^{-i\frac{8\pi}{9}},\ 2e^{\frac{4\pi}{9}i}$$

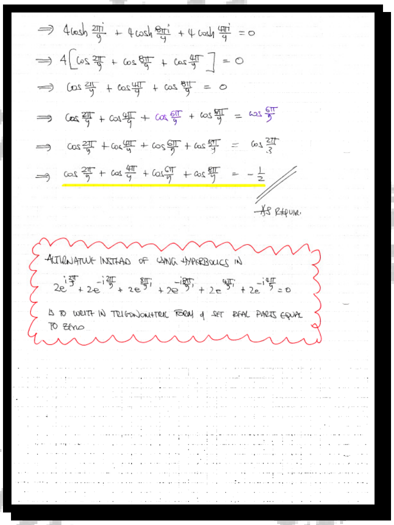
c)I) USING RELATIONSHIPS OF ROOTS

$$\text{SUM OF SIX ROOTS} = -\frac{\text{COEFF OF } z^5}{\text{COEFF OF } z^6} = 0$$

II) AS THE SUM OF ROOTS IS ZERO, REGROUP

$$2e^{i\frac{2\pi}{9}} + 2e^{-i\frac{2\pi}{9}} + 2e^{i\frac{8\pi}{9}} + 2e^{-i\frac{8\pi}{9}} + 2e^{\frac{4\pi}{9}i} + 2e^{-\frac{4\pi}{9}i} = 0$$

$$2\left[e^{\frac{2\pi}{9}i} + e^{-\frac{2\pi}{9}i}\right] + 2\left[e^{\frac{8\pi}{9}i} + e^{-\frac{8\pi}{9}i}\right] + 2\left[e^{\frac{4\pi}{9}i} + e^{-\frac{4\pi}{9}i}\right] = 0$$



$$\Rightarrow 4\cosh\frac{2\pi i}{9} + 4\cosh\frac{8\pi i}{9} + 4\cosh\frac{4\pi i}{9} = 0$$

$$\Rightarrow 4\left[\cos\frac{2\pi}{9} + \cos\frac{8\pi}{9} + \cos\frac{4\pi}{9}\right] = 0$$

$$\Rightarrow \cos\frac{2\pi}{9} + \cos\frac{4\pi}{9} + \cos\frac{8\pi}{9} = 0$$

$$\Rightarrow \cos\frac{2\pi}{9} + \cos\frac{4\pi}{9} + \cos\frac{6\pi}{9} + \cos\frac{8\pi}{9} = \cos\frac{6\pi}{9}$$

$$\Rightarrow \cos\frac{2\pi}{9} + \cos\frac{4\pi}{9} + \cos\frac{6\pi}{9} + \cos\frac{8\pi}{9} = \cos\frac{2\pi}{3}$$

$$\Rightarrow \cos\frac{2\pi}{9} + \cos\frac{4\pi}{9} + \cos\frac{6\pi}{9} + \cos\frac{8\pi}{9} = -\frac{1}{2}$$

AS REQUIRED

ALTERNATIVE INSTEAD OF USING HYPERBOLICS IN

$$2e^{i\frac{2\pi}{9}} + 2e^{-i\frac{2\pi}{9}} + 2e^{\frac{8\pi}{9}i} + 2e^{-i\frac{8\pi}{9}} + 2e^{\frac{4\pi}{9}i} + 2e^{-i\frac{4\pi}{9}} = 0$$

IS TO WRITE IN TRIGONOMETRIC FORM & SET REAL PARTS EQUAL TO ZERO

Question 128 **(****+)**

$$z = \cos\theta + \mathrm{i}\sin\theta\,,\ -\pi < \theta \le \pi\,.$$

a) Show clearly that

$$\frac{2}{1+z} = 1 - \mathrm{i}\tan\frac{\theta}{2}\,.$$

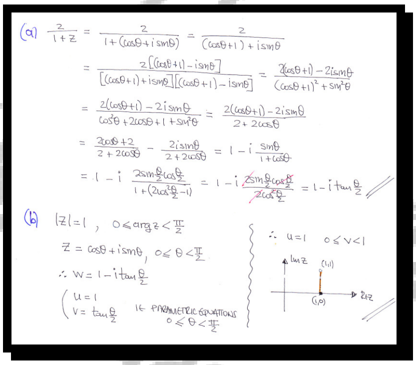
The complex function $w = f(z)$ is defined by

$$w = \frac{2}{1+z}\,,\ z \in \mathbb{C}\,,\ z \ne -1\,.$$

The circular arc $|z| = 1$, for which $0 \le \arg z < \frac{\pi}{2}$, is transformed by this function.

b) Sketch the image of this circular arc in a suitably labelled Argand diagram.

$\boxed{\text{proof/sketch}}$



(a) $\frac{2}{1+z} = \frac{2}{1+(\cos\theta+i\sin\theta)} = \frac{2}{(\cos\theta+1)+i\sin\theta}$

$= \frac{2[(\cos\theta+1)-i\sin\theta]}{[(\cos\theta+1)+i\sin\theta][(\cos\theta+1)-i\sin\theta]} = \frac{2(\cos\theta+1)-2i\sin\theta}{(\cos\theta+1)^2+\sin^2\theta}$

$= \frac{2(\cos\theta+1)-2i\sin\theta}{\cos^2\theta+2\cos\theta+1+\sin^2\theta} = \frac{2(\cos\theta+1)-2i\sin\theta}{2+2\cos\theta}$

$= \frac{2\cos\theta+2}{2+2\cos\theta} - \frac{2i\sin\theta}{2+2\cos\theta} = 1 - i\frac{\sin\theta}{1+\cos\theta}$

$= 1 - i\frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{1+(2\cos^2\frac{\theta}{2}-1)} = 1 - i\frac{\cancel{2}\sin\frac{\theta}{2}\cancel{\cos\frac{\theta}{2}}}{\cancel{2}\cos^{\cancel{2}}\frac{\theta}{2}} = 1 - i\tan\frac{\theta}{2}$ //

(b) $|z|=1\,,\quad 0 \le \arg z < \frac{\pi}{2}$

$z = \cos\theta + i\sin\theta,\ 0 \le \theta < \frac{\pi}{2}$

$\therefore w = 1 - i\tan\frac{\theta}{2}$

$\left(\begin{array}{l} u = 1 \\ v = \tan\frac{\theta}{2} \end{array}\right.$ IE PARAMETRIC EQUATIONS $0 \le \theta < \frac{\pi}{2}$

$\therefore u = 1 \quad 0 \le v < 1$

Question 129 (****+)

De Moivre's theorem states that

$$\left(\cos\theta + \mathrm{i}\sin\theta\right)^n = \cos n\theta + \mathrm{i}\sin n\theta, \ n \in \mathbb{Q}.$$

a) Use De Moivre's theorem to show that

$$\tan 5\theta \equiv \frac{5\tan\theta - 10\tan^3\theta + \tan^5\theta}{1 - 10\tan^2\theta + 5\tan^4\theta}.$$

b) Use part **(a)** to find the solutions of the equation

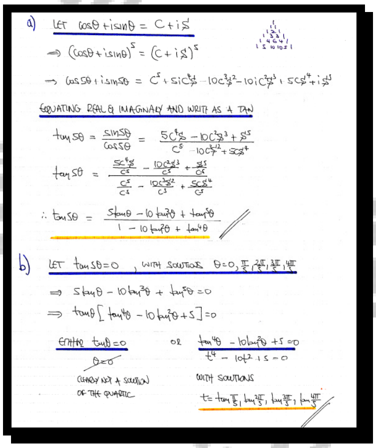
$$t^4 - 10t^2 + 5 = 0,$$

giving the answers in the form $t = \tan\varphi$, $0 < \varphi < \pi$.

c) Show further that

$$\tan\frac{\pi}{5}\tan\frac{2\pi}{5} = \sqrt{5}.$$

$\boxed{}$, $\boxed{t = \tan\frac{\pi}{5}, \tan\frac{2\pi}{5}, \tan\frac{3\pi}{5}, \tan\frac{4\pi}{5}}$



a) LET $\cos\theta + i\sin\theta = C + iS$

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1

$\Rightarrow (\cos\theta + i\sin\theta)^5 = (C + iS)^5$

$\rightarrow \cos 5\theta + i\sin 5\theta = C^5 + 5iC^4S - 10C^3S^2 - 10iC^2S^3 + 5CS^4 + iS^5$

EQUATING REAL & IMAGINARY AND WRITE AS A TAN

$\tan 5\theta = \frac{\sin 5\theta}{\cos 5\theta} = \frac{5C^4S - 10C^2S^3 + S^5}{C^5 - 10C^3S^2 + 5CS^4}$

$\tan 5\theta = \frac{\frac{5C^4S}{C^5} - \frac{10C^2S^3}{C^5} + \frac{S^5}{C^5}}{\frac{C^5}{C^5} - \frac{10C^3S^2}{C^5} + \frac{5CS^4}{C^5}}$

$\therefore \tan 5\theta = \frac{5\tan\theta - 10\tan^3\theta + \tan^5\theta}{1 - 10\tan^2\theta + \tan^4\theta}$ //

b) LET $\tan 5\theta = 0$, WITH SOLUTIONS $\theta = 0, \frac{\pi}{5}, \frac{2\pi}{5}, \frac{3\pi}{5}, \frac{4\pi}{5}$

$\Rightarrow 5\tan\theta - 10\tan^3\theta + \tan^5\theta = 0$

$\Rightarrow \tan\theta\left[\tan^4\theta - 10\tan^2\theta + 5\right] = 0$

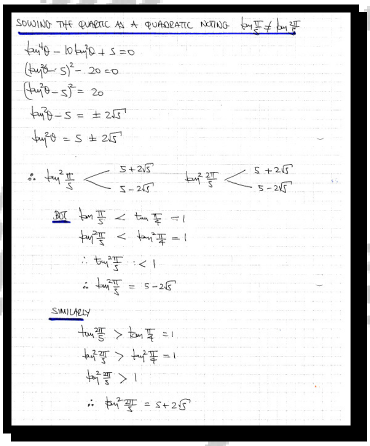
EITHER $\tan\theta = 0$ OR $\tan^4\theta - 10\tan^2\theta + 5 = 0$

$\cancel{\theta = 0}$ $\quad t^4 - 10t^2 + 5 = 0$

CLEARLY NOT A SOLUTION OF THE QUARTIC

WITH SOLUTIONS

$t = \tan\frac{\pi}{5}, \tan\frac{2\pi}{5}, \tan\frac{3\pi}{5}, \tan\frac{4\pi}{5}$ //



SOLVING THE QUARTIC AS A QUADRATIC NOTING $\tan\frac{\pi}{5} \neq \tan\frac{2\pi}{5}$

$\tan^4\theta - 10\tan^2\theta + 5 = 0$

$(\tan^2\theta - 5)^2 - 20 = 0$

$(\tan^2\theta - 5)^2 = 20$

$\tan^2\theta - 5 = \pm 2\sqrt{5}$

$\tan^2\theta = 5 \pm 2\sqrt{5}$

$\therefore \tan^2\frac{\pi}{5} \begin{cases} 5 + 2\sqrt{5} \\ 5 - 2\sqrt{5} \end{cases} \qquad \tan^2\frac{2\pi}{5} \begin{cases} 5 + 2\sqrt{5} \\ 5 - 2\sqrt{5} \end{cases}$

BUT $\tan\frac{\pi}{5} < \tan\frac{\pi}{4} = 1$

$\tan^2\frac{\pi}{5} < \tan^2\frac{\pi}{4} = 1$

$\therefore \tan^2\frac{\pi}{5} < 1$

$\therefore \tan^2\frac{\pi}{5} = 5 - 2\sqrt{5}$

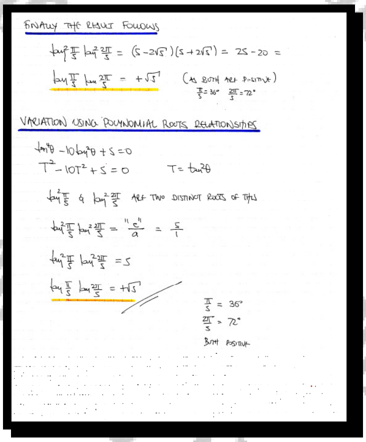
SIMILARLY

$\tan\frac{2\pi}{5} > \tan\frac{\pi}{4} = 1$

$\tan^2\frac{2\pi}{5} > \tan^2\frac{\pi}{4} = 1$

$\tan^2\frac{2\pi}{5} > 1$

$\therefore \tan^2\frac{2\pi}{5} = 5 + 2\sqrt{5}$



FINALLY THE RESULT FOLLOWS

$\tan^2\frac{\pi}{5}\tan^2\frac{2\pi}{5} = (5 - 2\sqrt{5})(5 + 2\sqrt{5}) = 25 - 20 =$

$\tan\frac{\pi}{5}\tan\frac{2\pi}{5} = +\sqrt{5}$ (AS BOTH ARE POSITIVE)

$\frac{\pi}{5} = 36°$ $\frac{2\pi}{5} = 72°$

VARIATION USING POLYNOMIAL ROOTS RELATIONSHIPS

$\tan^4\theta - 10\tan^2\theta + 5 = 0$

$T^2 - 10T^2 + 5 = 0 \qquad T = \tan^2\theta$

$\tan^2\frac{\pi}{5}$ & $\tan^2\frac{2\pi}{5}$ ARE TWO DISTINCT ROOTS OF THIS

$\tan^2\frac{\pi}{5}\tan^2\frac{2\pi}{5} = \frac{\text{"}c\text{"}}{a} = \frac{5}{1}$

$\tan^2\frac{\pi}{5}\tan^2\frac{2\pi}{5} = 5$

$\tan\frac{\pi}{5}\tan\frac{2\pi}{5} = +\sqrt{5}$ //

$\frac{\pi}{5} = 36°$

$\frac{2\pi}{5} = 72°$

BOTH POSITIVE

Question 130 (****+)

The complex function $w = f(z)$ maps points of the form $z = x + \mathrm{i}y$ from the z plane onto points of the form $w = u + \mathrm{i}v$ in the w plane.

It is given that

$$f(z) = \frac{z}{z - \mathrm{i}}, \; z \in \mathbb{C}, \; z \neq \mathrm{i}.$$

A circle C_1 with centre at $z = \mathrm{i}$ and radius 1 is mapped onto the circle C_2 in the w plane.

a) Find the coordinates of the centre of C_2, and the length of its radius.

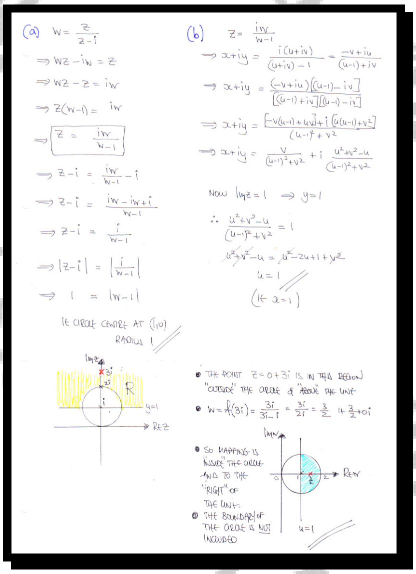
The straight line $z = \mathrm{i}$ is mapped onto another line L in the w plane.

b) Find an equation for this line.

The region R in the z plane lies outside C_1 such that $\operatorname{Im} z \geq 1$.

c) Shade in a clearly labelled diagram the image of R in the w plane.

$\boxed{(1,0), \; r = 1}$, $\boxed{u = 1 \text{ or } x = 1}$



(a) $w = \frac{z}{z-i}$

$\Rightarrow wz - iw = z$

$\Rightarrow wz - z = iw$

$\Rightarrow z(w-1) = iw$

$\Rightarrow \boxed{z = \frac{iw}{w-1}}$

$\Rightarrow z - i = \frac{iw}{w-1} - i$

$\Rightarrow z - i = \frac{iw - iw + i}{w-1}$

$\Rightarrow z - i = \frac{i}{w-1}$

$\Rightarrow |z-i| = \left|\frac{i}{w-1}\right|$

$\Rightarrow 1 = |w-1|$

IE CIRCLE CENTRE AT (1,0)
RADIUS 1

(b) $z = \frac{iw}{w-1}$

$\Rightarrow x+iy = \frac{i(u+iv)}{(u+iv)-1} = \frac{-v+iu}{(u-1)+iv}$

$\Rightarrow x+iy = \frac{(-v+iu)[(u-1)-iv]}{[(u-1)+iv][(u-1)-iv]}$

$\Rightarrow x+iy = \frac{[-v(u-1)+uv]+i[u(u-1)+v^2]}{(u-1)^2+v^2}$

$\Rightarrow x+iy = \frac{v}{(u-1)^2+v^2} + i\frac{u^2+v^2-u}{(u-1)^2+v^2}$

NOW $\operatorname{Im} z = 1 \Rightarrow y = 1$

$\therefore \frac{u^2+v^2-u}{(u-1)^2+v^2} = 1$

$\cancel{u^2}+\cancel{v^2}-u = \cancel{u^2}-2u+1+\cancel{v^2}$

$u = 1$

(IE $x = 1$)

- THE POINT $z = 0+3i$ IS IN THIS REGION "OUTSIDE" THE CIRCLE & "ABOVE" THE LINE
- $w = f(3i) = \frac{3i}{3i-i} = \frac{3i}{2i} = \frac{3}{2}$ IE $\frac{3}{2}+0i$
- SO MAPPING IS "INSIDE" THE CIRCLE AND TO THE "RIGHT" OF THE LINE.
- THE BOUNDARY OF THE CIRCLE IS NOT INCLUDED

Question 131 (****+)

$$z^5 - 1 = 0,\ z \in \mathbb{C},\ -\pi < \arg z \le \pi .$$

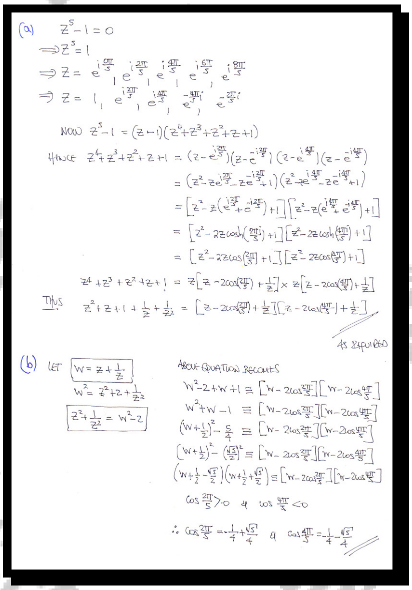
a) By considering the four complex roots of the above equation show clearly that

$$z^2 + z + 1 + \frac{1}{z} + \frac{1}{z^2} = \left[z + \frac{1}{z} - 2\cos\left(\frac{2\pi}{5}\right) \right]\left[z + \frac{1}{z} - 2\cos\left(\frac{4\pi}{5}\right) \right].$$

b) Use the substitution $w = z + \dfrac{1}{z}$ in the above equation, to find in exact surd form the values of

$$\cos\left(\frac{2\pi}{5}\right) \text{ and } \cos\left(\frac{4\pi}{5}\right).$$

$$\boxed{\cos\left(\frac{2\pi}{5}\right) = \frac{-1+\sqrt{5}}{4}},\ \boxed{\cos\left(\frac{4\pi}{5}\right) = \frac{-1-\sqrt{5}}{4}}$$



(a) $z^5 - 1 = 0$

$\Rightarrow z^5 = 1$

$\Rightarrow z = e^{i\frac{0\pi}{5}}, e^{i\frac{2\pi}{5}}, e^{i\frac{4\pi}{5}}, e^{i\frac{6\pi}{5}}, e^{i\frac{8\pi}{5}}$

$\Rightarrow z = 1, e^{i\frac{2\pi}{5}}, e^{i\frac{4\pi}{5}}, e^{-\frac{4\pi}{5}i}, e^{-\frac{2\pi}{5}i}$

NOW $z^5 - 1 = (z-1)(z^4+z^3+z^2+z+1)$

HENCE $z^4+z^3+z^2+z+1 = (z-e^{i\frac{2\pi}{5}})(z-e^{-i\frac{2\pi}{5}})(z-e^{i\frac{4\pi}{5}})(z-e^{-i\frac{4\pi}{5}})$

$= (z^2 - ze^{i\frac{2\pi}{5}} - ze^{-i\frac{2\pi}{5}} + 1)(z^2 - ze^{i\frac{4\pi}{5}} - ze^{-i\frac{4\pi}{5}} + 1)$

$= \left[z^2 - z(e^{i\frac{2\pi}{5}} + e^{-i\frac{2\pi}{5}}) + 1\right]\left[z^2 - z(e^{i\frac{4\pi}{5}} + e^{-i\frac{4\pi}{5}}) + 1\right]$

$= \left[z^2 - 2z\cosh\left(\frac{2\pi i}{5}\right) + 1\right]\left[z^2 - 2z\cosh\left(\frac{4\pi i}{5}\right) + 1\right]$

$= \left[z^2 - 2z\cos\left(\frac{2\pi}{5}\right) + 1\right]\left[z^2 - 2z\cos\left(\frac{4\pi}{5}\right) + 1\right]$

$z^4 + z^3 + z^2 + z + 1 = z\left[z - 2\cos\left(\frac{2\pi}{5}\right) + \frac{1}{z}\right] \times z\left[z - 2\cos\left(\frac{4\pi}{5}\right) + \frac{1}{z}\right]$

THUS $z^2 + z + 1 + \frac{1}{z} + \frac{1}{z^2} = \left[z - 2\cos\left(\frac{2\pi}{5}\right) + \frac{1}{z}\right]\left[z - 2\cos\left(\frac{4\pi}{5}\right) + \frac{1}{z}\right]$

AS REQUIRED

(b) LET $\boxed{w = z + \frac{1}{z}}$

$w^2 = z^2 + 2 + \frac{1}{z^2}$

$\boxed{z^2 + \frac{1}{z^2} = w^2 - 2}$

ABOVE EQUATION BECOMES

$w^2 - 2 + w + 1 \equiv \left[w - 2\cos\frac{2\pi}{5}\right]\left[w - 2\cos\frac{4\pi}{5}\right]$

$w^2 + w - 1 \equiv \left[w - 2\cos\frac{2\pi}{5}\right]\left[w - 2\cos\frac{4\pi}{5}\right]$

$(w+\frac{1}{2})^2 - \frac{5}{4} \equiv \left[w - 2\cos\frac{2\pi}{5}\right]\left[w - 2\cos\frac{4\pi}{5}\right]$

$(w+\frac{1}{2})^2 - (\frac{\sqrt{5}}{2})^2 \equiv \left[w - 2\cos\frac{2\pi}{5}\right]\left[w - 2\cos\frac{4\pi}{5}\right]$

$(w+\frac{1}{2}-\frac{\sqrt{5}}{2})(w+\frac{1}{2}+\frac{\sqrt{5}}{2}) \equiv \left[w - 2\cos\frac{2\pi}{5}\right]\left[w - 2\cos\frac{4\pi}{5}\right]$

$\cos\frac{2\pi}{5} > 0$ & $\cos\frac{4\pi}{5} < 0$

$\therefore \cos\frac{2\pi}{5} = -\frac{1}{4} + \frac{\sqrt{5}}{4}$ & $\cos\frac{4\pi}{5} = -\frac{1}{4} - \frac{\sqrt{5}}{4}$

Question 132 (****+)

The complex number $x+iy$ in the z plane of an Argand diagram satisfies the inequality

$$x^2+y^2+x>0 .$$

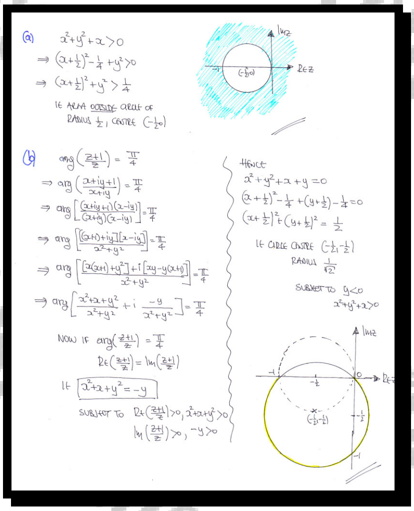
a) Sketch the region represented by this inequality.

A locus in the z plane of an Argand diagram is given by the equation

$$\arg\left(\frac{z+1}{z}\right)=\frac{\pi}{4} .$$

b) Sketch the locus represented by this equation.

sketch



(a) $x^2+y^2+x>0$

$\Rightarrow (x+\frac{1}{2})^2-\frac{1}{4}+y^2>0$

$\Rightarrow (x+\frac{1}{2})^2+y^2>\frac{1}{4}$

IE AREA OUTSIDE CIRCLE OF RADIUS $\frac{1}{2}$, CENTRE $(-\frac{1}{2},0)$

(b) $\arg\left(\frac{z+1}{z}\right)=\frac{\pi}{4}$

$\Rightarrow \arg\left(\frac{x+iy+1}{x+iy}\right)=\frac{\pi}{4}$

$\Rightarrow \arg\left[\frac{(x+iy+1)(x-iy)}{(x+iy)(x-iy)}\right]=\frac{\pi}{4}$

$\Rightarrow \arg\left[\frac{[(x+1)+iy][x-iy]}{x^2+y^2}\right]=\frac{\pi}{4}$

$\Rightarrow \arg\left[\frac{[x(x+1)+y^2]+i[xy-y(x+1)]}{x^2+y^2}\right]=\frac{\pi}{4}$

$\Rightarrow \arg\left[\frac{x^2+x+y^2}{x^2+y^2}+i\frac{-y}{x^2+y^2}\right]=\frac{\pi}{4}$

NOW IF $\arg\left(\frac{z+1}{z}\right)=\frac{\pi}{4}$

$\text{Re}\left(\frac{z+1}{z}\right)=\text{Im}\left(\frac{z+1}{z}\right)$

IE $\boxed{x^2+x+y^2=-y}$

SUBJECT TO $\text{Re}\left(\frac{z+1}{z}\right)>0,\ x^2+x+y^2>0$

$\text{Im}\left(\frac{z+1}{z}\right)>0,\ -y>0$

HENCE

$x^2+y^2+x+y=0$

$(x+\frac{1}{2})^2-\frac{1}{4}+(y+\frac{1}{2})-\frac{1}{4}=0$

$(x+\frac{1}{2})^2+(y+\frac{1}{2})^2=\frac{1}{2}$

IE CIRCLE CENTRE $(-\frac{1}{2},-\frac{1}{2})$

RADIUS $\frac{1}{\sqrt{2}}$

SUBJECT TO $y<0$

$x^2+y^2+x>0$

Question 133 **(****+)**

The following finite sums, C and S, are given by

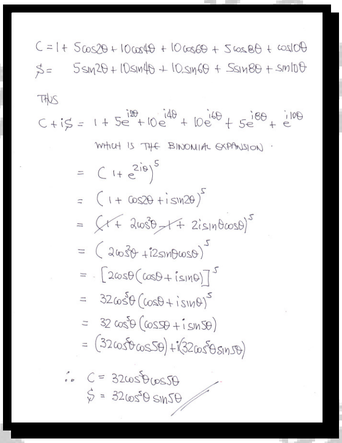
$$C = 1 + 5\cos 2\theta + 10\cos 4\theta + 10\cos 6\theta + 5\cos 8\theta + \cos 10\theta$$
$$S = 5\sin 2\theta + 10\sin 4\theta + 10\sin 6\theta + 5\sin 8\theta + \sin 10\theta$$

By considering the binomial expansion of $(1+A)^5$, show clearly that

$$C = 32\cos^5\theta\cos 5\theta,$$

and find a similar expression for S

$$\boxed{S = 32\cos^5\theta\sin 5\theta}$$



$C = 1 + 5\cos 2\theta + 10\cos 4\theta + 10\cos 6\theta + 5\cos 8\theta + \cos 10\theta$
$S = \quad 5\sin 2\theta + 10\sin 4\theta + 10\sin 6\theta + 5\sin 8\theta + \sin 10\theta$

THUS

$C + iS = 1 + 5e^{i2\theta} + 10e^{i4\theta} + 10e^{i6\theta} + 5e^{i8\theta} + e^{i10\theta}$

WHICH IS THE BINOMIAL EXPANSION

$= (1 + e^{2i\theta})^5$

$= (1 + \cos 2\theta + i\sin 2\theta)^5$

$= (\cancel{1} + 2\cos^2\theta \cancel{- 1} + 2i\sin\theta\cos\theta)^5$

$= (2\cos^2\theta + i2\sin\theta\cos\theta)^5$

$= [2\cos\theta(\cos\theta + i\sin\theta)]^5$

$= 32\cos^5\theta(\cos\theta + i\sin\theta)^5$

$= 32\cos^5\theta(\cos 5\theta + i\sin 5\theta)$

$= (32\cos^5\theta\cos 5\theta) + i(32\cos^5\theta\sin 5\theta)$

$\therefore \; C = 32\cos^5\theta\cos 5\theta$
$\quad\; S = 32\cos^5\theta\sin 5\theta$

Question 134 **(****+)**

The complex function with equation

$$f(z) = \frac{1}{z^2}, \; z \in \mathbb{C}, \; z \neq 0$$

maps the complex number $x + \mathrm{i}y$ from the z plane onto the complex number $u + \mathrm{i}v$ in the w plane.

The line with equation

$$y = mx, \; x \neq 0,$$

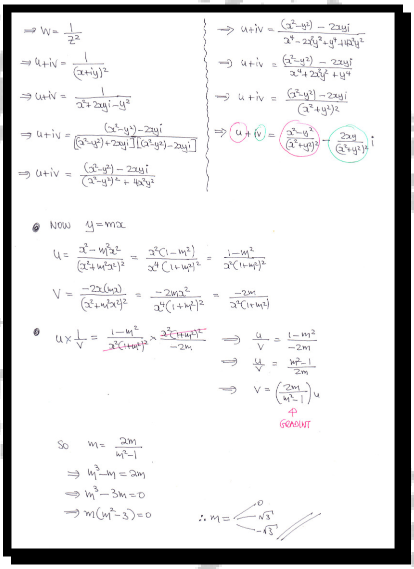
is mapped onto the line with equation

$$v = Mu,$$

where m and M are the respective gradients of the two lines.

Given that $m = M$, determine the three possible values of m.

$$\boxed{m = 0, \pm\sqrt{3}}$$



$$\Rightarrow w = \frac{1}{z^2}$$

$$\Rightarrow u + iv = \frac{1}{(x+iy)^2}$$

$$\Rightarrow u + iv = \frac{1}{x^2 + 2xyi - y^2}$$

$$\Rightarrow u + iv = \frac{(x^2-y^2) - 2xyi}{[(x^2-y^2)+2xyi][(x^2-y^2)-2xyi]}$$

$$\Rightarrow u + iv = \frac{(x^2-y^2) - 2xyi}{(x^2-y^2)^2 + 4x^2y^2}$$

$$\Rightarrow u + iv = \frac{(x^2-y^2) - 2xyi}{x^4 - 2x^2y^2 + y^4 + 4x^2y^2}$$

$$\Rightarrow u + iv = \frac{(x^2-y^2) - 2xyi}{x^4 + 2x^2y^2 + y^4}$$

$$\Rightarrow u + iv = \frac{(x^2-y^2) - 2xyi}{(x^2+y^2)^2}$$

$$\Rightarrow u + iv = \frac{x^2-y^2}{(x^2+y^2)^2} - \frac{2xy}{(x^2+y^2)^2}i$$

- NOW $y = mx$

$$u = \frac{x^2 - m^2x^2}{(x^2+m^2x^2)^2} = \frac{x^2(1-m^2)}{x^4(1+m^2)^2} = \frac{1-m^2}{x^2(1+m^2)^2}$$

$$v = \frac{-2x(mx)}{(x^2+m^2x^2)^2} = \frac{-2mx^2}{x^4(1+m^2)^2} = \frac{-2m}{x^2(1+m^2)}$$

- $$u \times \frac{1}{v} = \frac{1-m^2}{\cancel{x^2(1+m^2)^2}} \times \frac{\cancel{x^2(1+m^2)^2}}{-2m} \Rightarrow \frac{u}{v} = \frac{1-m^2}{-2m}$$

$$\Rightarrow \frac{u}{v} = \frac{m^2-1}{2m}$$

$$\Rightarrow v = \underset{\substack{\uparrow \\ \text{GRADIENT}}}{\left(\frac{2m}{m^2-1}\right)} u$$

So $m = \frac{2m}{m^2-1}$

$$\Rightarrow m^3 - m = 2m$$

$$\Rightarrow m^3 - 3m = 0$$

$$\Rightarrow m(m^2-3) = 0$$

$$\therefore m = \begin{cases} 0 \\ \sqrt{3} \\ -\sqrt{3} \end{cases} \; //$$

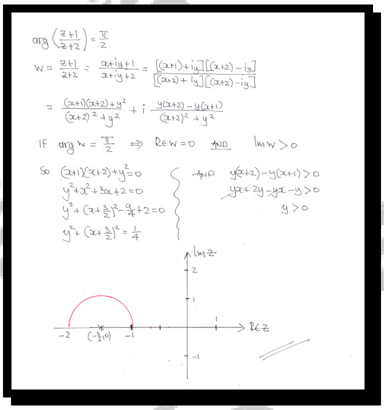
Question 135 (****+)

The point P represents the number $z = x + \mathrm{i}y$ in an Argand diagram and further satisfies the equation

$$\arg\left(\frac{z+1}{z+2}\right) = \frac{\pi}{2}\ ,\ z \neq -2\,.$$

Use an algebraic method to find an equation of the locus of P and sketch this locus accurately in an Argand diagram.

$$\boxed{\left(x+\frac{3}{2}\right)^2 + y^2 = \frac{1}{4},\quad \text{such that } \ y > 0}$$



$\arg\left(\frac{z+1}{z+2}\right) = \frac{\pi}{2}$

$w = \frac{z+1}{z+2} = \frac{x+iy+1}{x+iy+2} = \frac{[(x+1)+iy][(x+2)-iy]}{[(x+2)+iy][(x+2)-iy]}$

$= \frac{(x+1)(x+2)+y^2}{(x+2)^2+y^2} + i\,\frac{y(x+2)-y(x+1)}{(x+2)^2+y^2}$

IF $\arg w = \frac{\pi}{2} \Rightarrow \mathrm{Re}\,w = 0$ AND $\mathrm{Im}\,w > 0$

SO $(x+1)(x+2)+y^2=0$

$y^2+x^2+3x+2=0$

$y^2+(x+\frac{3}{2})^2-\frac{9}{4}+2=0$

$y^2+(x+\frac{3}{2})^2=\frac{1}{4}$

AND $y(x+2)-y(x+1)>0$

$\cancel{yx}+2y-\cancel{yx}-y>0$

$y>0$

Question 136 (****+)

$$z^n = 1,\ z \in \mathbb{C},\ n \in \mathbb{N}.$$

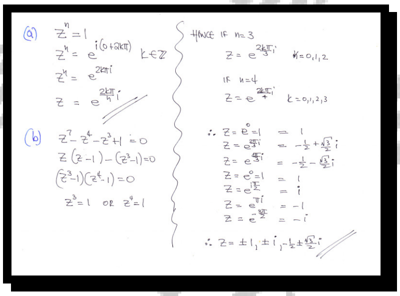
a) Solve the above equation, giving the general solution in terms of n and any suitably defined parameters.

b) Hence solve the equation

$$z^7 - z^4 - z^3 + 1 = 0,\ z \in \mathbb{C},$$

giving the answers in the form $x + \mathrm{i}y$, $x, y \in \mathbb{R}$, where appropriate.

$\boxed{z = \mathrm{e}^{\frac{2k\pi}{n}\mathrm{i}}, k \in \mathbb{Z}}$, $\boxed{z = \pm 1,\ \pm \mathrm{i},\ -\frac{1}{2} \pm \frac{\sqrt{3}}{2}\mathrm{i}}$



(a) $z^n = 1$

$z^n = e^{i(0+2k\pi)}$ $k \in \mathbb{Z}$

$z^n = e^{2k\pi i}$

$z = e^{\frac{2k\pi}{n}i}$

(b) $z^7 - z^4 - z^3 + 1 = 0$

$z(z-1) - (z^3-1) = 0$

$(z^3-1)(z^4-1) = 0$

$z^3 = 1$ OR $z^4 = 1$

HENCE IF $n=3$

$z = e^{\frac{2k\pi}{3}i}$ $k = 0,1,2$

IF $n=4$

$z = e^{\frac{2k\pi}{4}i}$ $k = 0,1,2,3$

$\therefore z = e^{0} = 1 = 1$

$z = e^{\frac{2\pi}{3}i} = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$

$z = e^{\frac{4\pi}{3}i} = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$

$z = e^{0} = 1 = 1$

$z = e^{i\frac{\pi}{2}} = i$

$z = e^{\pi i} = -1$

$z = e^{-\frac{3\pi}{2}} = -i$

$\therefore z = \pm 1, \pm i, -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$

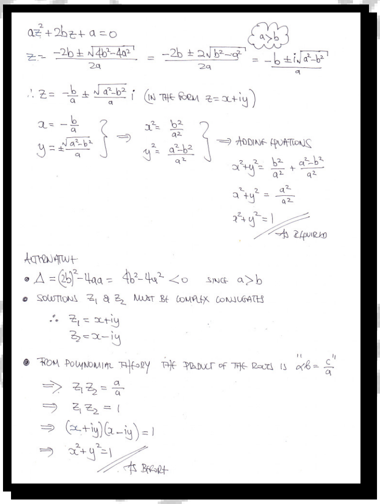
Question 137 (**+)**

Given that $a \in \mathbb{R}$, $b \in \mathbb{R}$, $a > b > 0$, show that in an Argand diagram, the roots of the quadratic equation

$$az^2 + 2bz + a = 0,$$

lie on the circle with equation $x^2 + y^2 = 1$.

proof



$az^2 + 2bz + a = 0$

a>b

$z = \frac{-2b \pm \sqrt{4b^2-4a^2}}{2a} = \frac{-2b \pm 2\sqrt{b^2-a^2}}{2a} = \frac{-b \pm i\sqrt{a^2-b^2}}{a}$

$\therefore z = -\frac{b}{a} \pm \frac{\sqrt{a^2-b^2}}{a} i$ (IN THE FORM $z = x+iy$)

$\left.\begin{array}{l} x = -\frac{b}{a} \\ y = \pm\frac{\sqrt{a^2-b^2}}{a} \end{array}\right\} \Rightarrow \left.\begin{array}{l} x^2 = \frac{b^2}{a^2} \\ y^2 = \frac{a^2-b^2}{a^2} \end{array}\right\} \Rightarrow$ ADDING EQUATIONS

$x^2+y^2 = \frac{b^2}{a^2} + \frac{a^2-b^2}{a^2}$

$x^2+y^2 = \frac{a^2}{a^2}$

$x^2+y^2 = 1$

AS REQUIRED

ALTERNATIVE

- $\Delta = (2b)^2 - 4aa = 4b^2 - 4a^2 < 0$ SINCE $a > b$
- SOLUTIONS z_1 & z_2 MUST BE COMPLEX CONJUGATES

$\therefore z_1 = x+iy$
$z_2 = x-iy$

- FROM POLYNOMIAL THEORY THE PRODUCT OF THE ROOTS IS "$\alpha\beta = \frac{c}{a}$"

$\Rightarrow z_1 z_2 = \frac{a}{a}$
$\Rightarrow z_1 z_2 = 1$
$\Rightarrow (x+iy)(x-iy) = 1$
$\Rightarrow x^2+y^2 = 1$

AS BEFORE

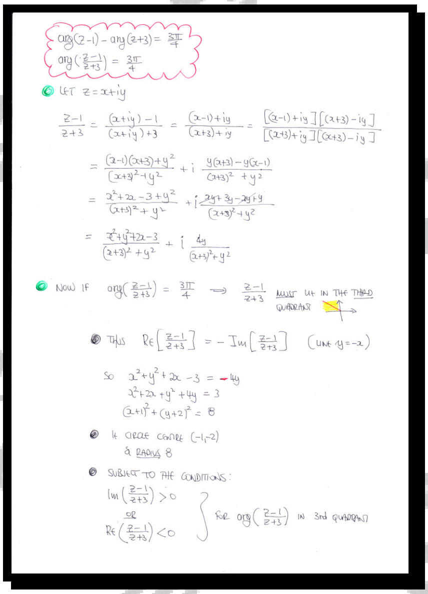
Question 138 (****+)

The point P represents the number $z = x + \mathrm{i}y$ in an Argand diagram and further satisfies the equation

$$\arg(z-1) - \arg(z+3) = \frac{3\pi}{4} \ , \ z \neq -3 .$$

Use an algebraic method to find an equation of the locus of P and sketch this locus accurately in an Argand diagram.

$$\boxed{(x+1)^2 + (y+2)^2 = 8, \quad \text{such that} \ \ y > 0}$$



$$\arg(z-1) - \arg(z+3) = \frac{3\pi}{4}$$
$$\arg\left(\frac{z-1}{z+3}\right) = \frac{3\pi}{4}$$

- LET $z = x + iy$

$$\frac{z-1}{z+3} = \frac{(x+iy)-1}{(x+iy)+3} = \frac{(x-1)+iy}{(x+3)+iy} = \frac{[(x-1)+iy][(x+3)-iy]}{[(x+3)+iy][(x+3)-iy]}$$
$$= \frac{(x-1)(x+3)+y^2}{(x+3)^2+y^2} + i\,\frac{y(x+3)-y(x-1)}{(x+3)^2+y^2}$$
$$= \frac{x^2+2x-3+y^2}{(x+3)^2+y^2} + i\,\frac{\cancel{xy}+3y-\cancel{xy}+y}{(x+3)^2+y^2}$$
$$= \frac{x^2+y^2+2x-3}{(x+3)^2+y^2} + i\,\frac{4y}{(x+3)^2+y^2}$$

- NOW IF $\arg\left(\frac{z-1}{z+3}\right) = \frac{3\pi}{4} \implies \frac{z-1}{z+3}$ MUST LIE IN THE THIRD QUADRANT

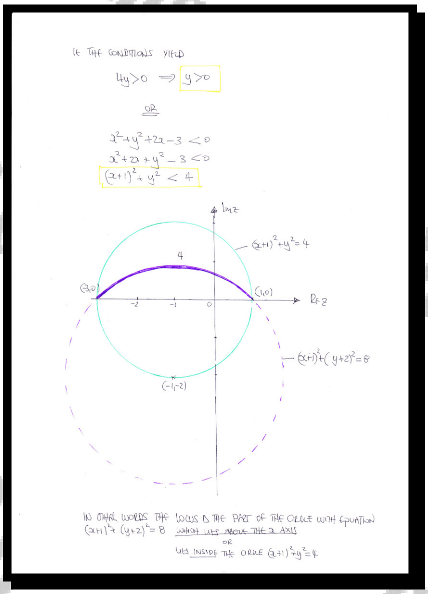
- THUS $\mathrm{Re}\left[\frac{z-1}{z+3}\right] = -\mathrm{Im}\left[\frac{z-1}{z+3}\right]$ (LINE $y=-x$)

SO $x^2+y^2+2x-3 = -4y$
$x^2+2x+y^2+4y = 3$
$(x+1)^2+(y+2)^2 = 8$

- ie CIRCLE CENTRE $(-1,-2)$
& RADIUS 8

- SUBJECT TO THE CONDITIONS:

$\mathrm{Im}\left(\frac{z-1}{z+3}\right) > 0$
OR
$\mathrm{Re}\left(\frac{z-1}{z+3}\right) < 0$
} FOR $\arg\left(\frac{z-1}{z+3}\right)$ IN 3rd QUADRANT



ie THE CONDITIONS YIELD

$4y > 0 \implies \boxed{y > 0}$

OR

$x^2+y^2+2x-3 < 0$
$x^2+2x+y^2-3 < 0$
$\boxed{(x+1)^2+y^2 < 4}$

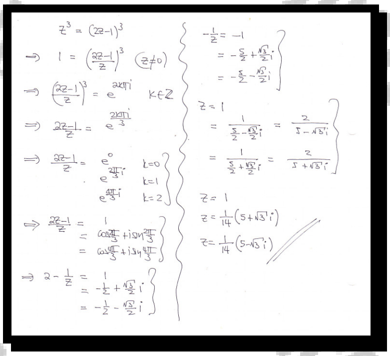
IN OTHER WORDS THE LOCUS IS THE PART OF THE CIRCLE WITH EQUATION $(x+1)^2+(y+2)^2=8$ WHICH LIES ABOVE THE x AXIS
OR
LIES INSIDE THE CIRCLE $(x+1)^2+y^2=4$

Question 139 (**+)**

$$z^3 = (2z-1)^3, \ z \in \mathbb{C}.$$

Find in the form $x + \mathrm{i}y$ the exact solutions of the above equation.

$$\boxed{z = 1, \ \frac{1}{14}\left(5 \pm \mathrm{i}\sqrt{3}\right)}$$



$$z^3 = (2z-1)^3$$

$$\Rightarrow 1 = \left(\frac{2z-1}{z}\right)^3 \quad (z \neq 0)$$

$$\Rightarrow \left(\frac{2z-1}{z}\right)^3 = e^{2k\pi i} \quad k \in \mathbb{Z}$$

$$\Rightarrow \frac{2z-1}{z} = e^{\frac{2k\pi i}{3}}$$

$$\Rightarrow \frac{2z-1}{z} = \left.\begin{array}{ll} e^0 & k=0 \\ e^{\frac{2\pi}{3}i} & k=1 \\ e^{\frac{4\pi}{3}i} & k=2 \end{array}\right\}$$

$$\Rightarrow \frac{2z-1}{z} = \left.\begin{array}{l} 1 \\ = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3} \\ = \cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3} \end{array}\right\}$$

$$\Rightarrow 2 - \frac{1}{z} = \left.\begin{array}{l} 1 \\ = -\frac{1}{2} + \frac{\sqrt{3}}{2}i \\ = -\frac{1}{2} - \frac{\sqrt{3}}{2}i \end{array}\right\}$$

$$-\frac{1}{z} = \left.\begin{array}{l} -1 \\ = -\frac{5}{2} + \frac{\sqrt{3}}{2}i \\ = -\frac{5}{2} - \frac{\sqrt{3}}{2}i \end{array}\right\}$$

$$z = \left.\begin{array}{l} 1 \\ = \frac{1}{\frac{5}{2} - \frac{\sqrt{3}}{2}i} = \frac{2}{5 - \sqrt{3}i} \\ = \frac{1}{\frac{5}{2} + \frac{\sqrt{3}}{2}i} = \frac{2}{5 + \sqrt{3}i} \end{array}\right\}$$

$$z = 1$$

$$z = \frac{1}{14}\left(5 + \sqrt{3}i\right)$$

$$z = \frac{1}{14}\left(5 - \sqrt{3}i\right)$$

Question 140 (****+)

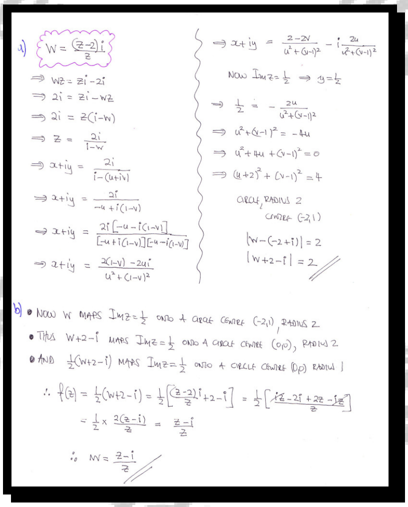
$$f(z) \equiv \frac{(z-2)\mathrm{i}}{z}, \ z = x + \mathrm{i}y, \ x \in \mathbb{R}, \ y \in \mathbb{R}.$$

The complex function f maps complex numbers onto complex numbers, which can be graphed in two separate Argand diagrams.

a) Given that $\operatorname{Im} z = \frac{1}{2}$, determine an equation of the locus of the image of the points under f.

b) Hence determine a complex function $g(z)$, which maps $\operatorname{Im} z = \frac{1}{2}$ onto a unit circle, centre at the origin O.

$\boxed{|w+2-\mathrm{i}|=2}$, $\boxed{g(z) = w = \frac{z-\mathrm{i}}{z}}$



a) $W = \frac{(z-2)i}{z}$

$\Rightarrow Wz = zi - 2i$

$\Rightarrow 2i = zi - wz$

$\Rightarrow 2i = z(i-w)$

$\Rightarrow z = \frac{2i}{i-w}$

$\Rightarrow x+iy = \frac{2i}{i-(u+iv)}$

$\Rightarrow x+iy = \frac{2i}{-u+i(1-v)}$

$\Rightarrow x+iy = \frac{2i[-u-i(1-v)]}{[-u+i(1-v)][-u-i(1-v)]}$

$\Rightarrow x+iy = \frac{2(1-v)-2ui}{u^2+(1-v)^2}$

$\Rightarrow x+iy = \frac{2-2v}{u^2+(v-1)^2} - i\frac{2u}{u^2+(v-1)^2}$

NOW $\operatorname{Im} z = \frac{1}{2} \Rightarrow y = \frac{1}{2}$

$\Rightarrow \frac{1}{2} = -\frac{2u}{u^2+(v-1)^2}$

$\Rightarrow u^2+(v-1)^2 = -4u$

$\Rightarrow u^2+4u+(v-1)^2 = 0$

$\Rightarrow (u+2)^2+(v-1)^2 = 4$

CIRCLE, RADIUS 2
CENTRE $(-2,1)$

$|w-(-2+i)| = 2$

$|w+2-i| = 2$

b)
- NOW W MAPS $\operatorname{Im} z = \frac{1}{2}$ ONTO A CIRCLE CENTRE $(-2,1)$, RADIUS 2
- THUS $W+2-i$ MAPS $\operatorname{Im} z = \frac{1}{2}$ ONTO A CIRCLE CENTRE $(0,0)$, RADIUS 2
- AND $\frac{1}{2}(W+2-i)$ MAPS $\operatorname{Im} z = \frac{1}{2}$ ONTO A CIRCLE CENTRE $(0,0)$ RADIUS 1

$\therefore f(z) = \frac{1}{2}(w+2-i) = \frac{1}{2}\left[\frac{(z-2)i}{z}+2-i\right] = \frac{1}{2}\left[\frac{\cancel{iz}-2i+2z-\cancel{iz}}{z}\right]$

$= \frac{1}{2} \times \frac{2(z-i)}{z} = \frac{z-i}{z}$

$\therefore w = \frac{z-i}{z}$

Question 141 (****+)

$$f(z) = (z-4)^3, \ z \in \mathbb{C}.$$

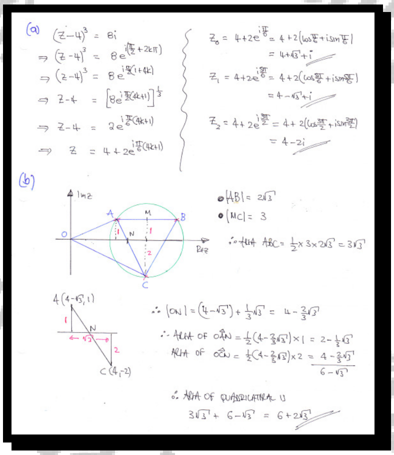
a) Solve the equation $f(z) = 8\text{i}$, giving the answers in the form $x + \text{i}y$.

The points A, B and C represent in an Argand diagram the roots of the equation $f(z) = 8\text{i}$. The points A and B represent the roots whose imaginary parts are positive and the point A represents the root with the smaller real part.

b) Show that the area of the quadrilateral $OABC$, where O is the origin, is

$$6 + 2\sqrt{3}.$$

$$\boxed{z = 4+\sqrt{3}+\text{i}, \ \ z = 4-\sqrt{3}+\text{i}, \ \ z = 4-2\text{i}}$$



(a)

$(z-4)^3 = 8i$

$\Rightarrow (z-4)^3 = 8e^{i(\frac{\pi}{2}+2k\pi)}$

$\Rightarrow (z-4)^3 = 8e^{i\frac{\pi}{2}(1+4k)}$

$\Rightarrow z-4 = \left[8e^{i\frac{\pi}{2}(4k+1)}\right]^{\frac{1}{3}}$

$\Rightarrow z-4 = 2e^{i\frac{\pi}{6}(4k+1)}$

$\Rightarrow z = 4+2e^{i\frac{\pi}{6}(4k+1)}$

$z_0 = 4+2e^{i\frac{\pi}{6}} = 4+2\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)$

$= 4+\sqrt{3}+i$

$z_1 = 4+2e^{i\frac{5\pi}{6}} = 4+2\left(\cos\frac{5\pi}{6}+i\sin\frac{5\pi}{6}\right)$

$= 4-\sqrt{3}+i$

$z_2 = 4+2e^{i\frac{3\pi}{2}} = 4+2\left(\cos\frac{3\pi}{2}+i\sin\frac{3\pi}{2}\right)$

$= 4-2i$

(b)

- $|AB| = 2\sqrt{3}$
- $|MC| = 3$

$\therefore$ AREA $ABC = \frac{1}{2} \times 3 \times 2\sqrt{3} = 3\sqrt{3}$

$\therefore |ON| = (4-\sqrt{3}) + \frac{1}{3}\sqrt{3} = 4-\frac{2}{3}\sqrt{3}$

$\therefore$ AREA OF $O\hat{A}N = \frac{1}{2}(4-\frac{2}{3}\sqrt{3}) \times 1 = 2-\frac{1}{3}\sqrt{3}$

AREA OF $O\hat{C}N = \frac{1}{2}(4-\frac{2}{3}\sqrt{3}) \times 2 = 4-\frac{2}{3}\sqrt{3}$

$6-\sqrt{3}$

$\therefore$ AREA OF QUADRILATERAL IS

$3\sqrt{3} + 6-\sqrt{3} = 6+2\sqrt{3}$

Question 142 (****+)

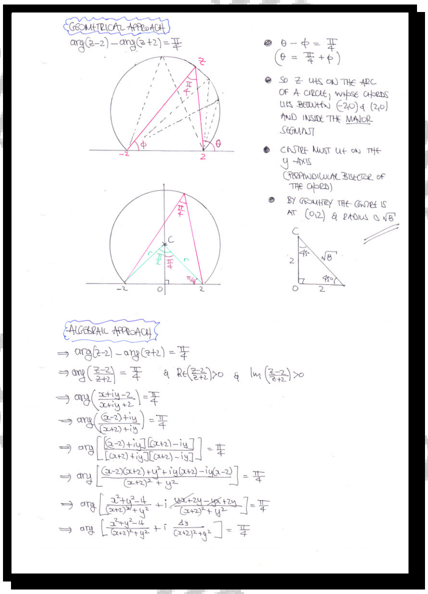
The complex number z satisfies the relationship

$$\arg(z-2)-\arg(z+2)=\frac{\pi}{4}.$$

Show that the locus of z is a circular arc, stating …

- … the coordinates of its endpoints.
- … the coordinates of its centre.
- … the length of its radius.

$\boxed{(-2,0),(2,0)}$, $\boxed{(0,2)}$, $\boxed{r=2\sqrt{2}}$



GEOMETRICAL APPROACH

$\arg(z-2)-\arg(z+2)=\frac{\pi}{4}$

- $\theta-\phi=\frac{\pi}{4}$
  $\left(\theta=\frac{\pi}{4}+\phi\right)$
- SO Z LIES ON THE ARC OF A CIRCLE, WHOSE CHORDS LIES BETWEEN (−2,0) & (2,0) AND INSIDE THE MAJOR SEGMENT
- CENTRE MUST LIE ON THE y-AXIS (PERPENDICULAR BISECTOR OF THE CHORD)
- BY GEOMETRY THE CENTRE IS AT (0,2) & RADIUS IS $\sqrt{8}$

ALGEBRAIC APPROACH

$\Rightarrow \arg(z-2)-\arg(z+2)=\frac{\pi}{4}$

$\Rightarrow \arg\left(\frac{z-2}{z+2}\right)=\frac{\pi}{4}$ & $\text{Re}\left(\frac{z-2}{z+2}\right)>0$ & $\text{Im}\left(\frac{z-2}{z+2}\right)>0$

$\Rightarrow \arg\left(\frac{x+iy-2}{x+iy+2}\right)=\frac{\pi}{4}$

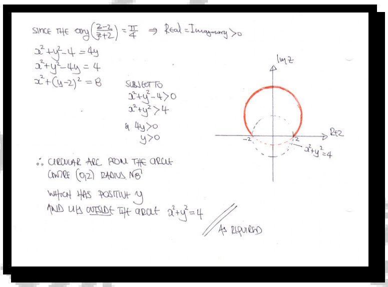
$\Rightarrow \arg\left(\frac{(x-2)+iy}{(x+2)+iy}\right)=\frac{\pi}{4}$

$\Rightarrow \arg\left[\frac{[(x-2)+iy][(x+2)-iy]}{[(x+2)+iy][(x+2)-iy]}\right]=\frac{\pi}{4}$

$\Rightarrow \arg\left[\frac{(x-2)(x+2)+y^2+iy(x+2)-iy(x-2)}{(x+2)^2+y^2}\right]=\frac{\pi}{4}$

$\Rightarrow \arg\left[\frac{x^2+y^2-4}{(x+2)^2+y^2}+i\,\frac{\cancel{yx}+2y-\cancel{yx}+2y}{(x+2)^2+y^2}\right]=\frac{\pi}{4}$

$\Rightarrow \arg\left[\frac{x^2+y^2-4}{(x+2)^2+y^2}+i\,\frac{4y}{(x+2)^2+y^2}\right]=\frac{\pi}{4}$



SINCE THE $\arg\left(\frac{z-2}{z+2}\right)=\frac{\pi}{4} \Rightarrow$ Real = Imaginary > 0

$x^2+y^2-4=4y$

$x^2+y^2-4y=4$

$x^2+(y-2)^2=8$

SUBJECT TO

$x^2+y^2-4>0$

$x^2+y^2>4$

& $4y>0$

$y>0$

∴ CIRCULAR ARC FROM THE CIRCLE CENTRE (0,2) RADIUS $\sqrt{8}$

WHICH HAS POSITIVE y

AND LIES OUTSIDE THE CIRCLE $x^2+y^2=4$

AS REQUIRED

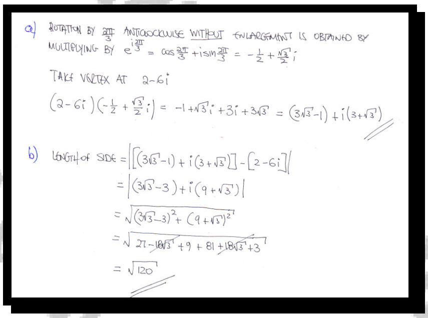
Question 143 (****+)

An equilateral triangle T is drawn in a standard Argand diagram. The origin O is located at the centre of T. One of the vertices of T is represented by the complex number $2-6\mathrm{i}$.

a) Find, in exact simplified form the complex number represented by another vertex of T.

b) Calculate, in exact surd form, the area of T.

$\boxed{\left(3\sqrt{3}-1\right)+\mathrm{i}\left(3+\sqrt{3}\right)}$, $\boxed{\text{area} = \sqrt{120}}$



a) ROTATION BY $\frac{2\pi}{3}$ ANTICLOCKWISE WITHOUT ENLARGEMENT IS OBTAINED BY MULTIPLYING BY $e^{i\frac{2\pi}{3}} = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3} = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$

TAKE VERTEX AT $2-6i$

$(2-6i)\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right) = -1+\sqrt{3}i+3i+3\sqrt{3} = (3\sqrt{3}-1)+i(3+\sqrt{3})$

b) LENGTH OF SIDE $= \left|\left[(3\sqrt{3}-1)+i(3+\sqrt{3})\right]-\left[2-6i\right]\right|$

$= \left|(3\sqrt{3}-3)+i(9+\sqrt{3})\right|$

$= \sqrt{(3\sqrt{3}-3)^2+(9+\sqrt{3})^2}$

$= \sqrt{27-\cancel{18\sqrt{3}}+9+81+\cancel{18\sqrt{3}}+3}$

$= \sqrt{120}$

Question 144 **(****+)**

The complex function $w = f(z)$ maps points of the form $z = x + \mathrm{i}y$ from the z plane onto points of the form $w = u + \mathrm{i}v$ in the w plane.

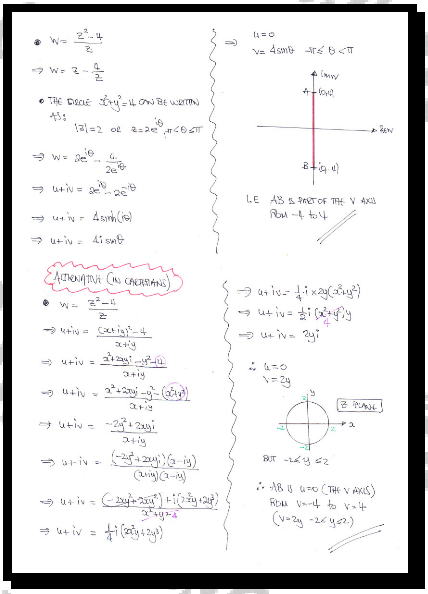
It is given that

$$f(z) = \frac{z^2 - 4}{z}, \quad z \in \mathbb{C}, \quad z \neq 0 .$$

The circle C with equation $x^2 + y^2 = 4$ in the z plane is mapped onto a line segment AB in the w plane.

Find a Cartesian equation for AB, stating the coordinates of its endpoints.

$\boxed{(1,0),\ r=1}$, $\boxed{u=1 \ \text{ or } \ x=1}$



• $w = \frac{z^2-4}{z}$

$\Rightarrow w = z - \frac{4}{z}$

• THE CIRCLE $x^2+y^2=4$ CAN BE WRITTEN AS:

$|z|=2$ OR $z=2e^{i\theta}, -\pi<\theta\leqslant\pi$

$\Rightarrow w = 2e^{i\theta} - \frac{4}{2e^{i\theta}}$

$\Rightarrow u+iv = 2e^{i\theta} - 2e^{-i\theta}$

$\Rightarrow u+iv = 4\sinh(i\theta)$

$\Rightarrow u+iv = 4i\sin\theta$

$\Rightarrow$ $u=0$
$v = 4\sin\theta \quad -\pi\leqslant\theta<\pi$

I.E AB IS PART OF THE V AXIS FROM -4 to 4

ALTERNATIVE (IN CARTESIANS)

• $w = \frac{z^2-4}{z}$

$\Rightarrow u+iv = \frac{(x+iy)^2-4}{x+iy}$

$\Rightarrow u+iv = \frac{x^2+2xyi-y^2-\textcircled{4}}{x+iy}$

$\Rightarrow u+iv = \frac{x^2+2xyi-y^2-\left(x^2+y^2\right)}{x+iy}$

$\Rightarrow u+iv = \frac{-2y^2+2xyi}{x+iy}$

$\Rightarrow u+iv = \frac{(-2y^2+2xyi)(x-iy)}{(x+iy)(x-iy)}$

$\Rightarrow u+iv = \frac{\cancel{(-2xy^2+2xy^2)}+i(2x^2y+2y^3)}{\cancel{x^2+y^2}\,4}$

$\Rightarrow u+iv = \frac{1}{4}i(2x^2y+2y^3)$

$\Rightarrow u+iv = \frac{1}{4}i\times2y(x^2+y^2)$

$\Rightarrow u+iv = \frac{1}{2}i\,(\underset{4}{\cancel{x^2+y^2}})y$

$\Rightarrow u+iv = 2yi$

$\therefore$ $u=0$
$v=2y$

BUT $-2\leqslant y\leqslant 2$

$\therefore$ AB IS $u=0$ (THE V AXIS) FROM $v=-4$ to $v=4$
$(v=2y \quad -2\leqslant y\leqslant 2)$

Question 145 (****+)

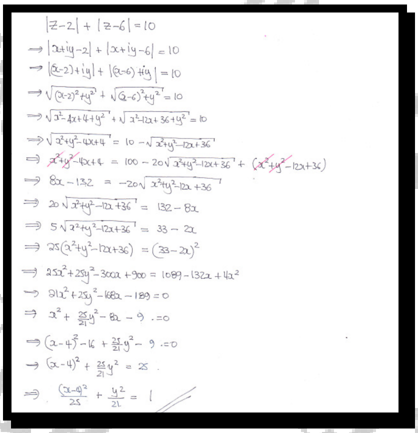
The complex number z satisfies the relationship

$$|z-2|+|z-6|=10.$$

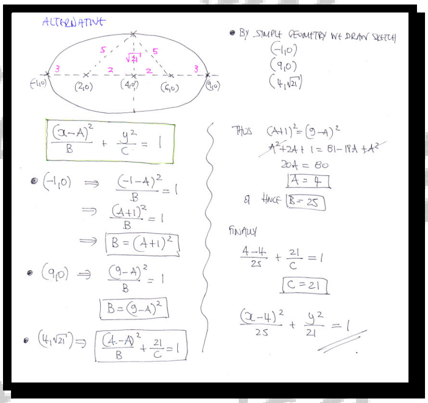
Determine a simplified Cartesian equation for the locus of z, giving the final answer in the form

$$f(x,y)=1.$$

$$\boxed{\frac{(x-4)^2}{25}+\frac{y^2}{21}=1}$$



$$|z-2|+|z-6|=10$$
$$\Rightarrow |x+iy-2|+|x+iy-6|=10$$
$$\Rightarrow |(x-2)+iy|+|(x-6)+iy|=10$$
$$\Rightarrow \sqrt{(x-2)^2+y^2}+\sqrt{(x-6)^2+y^2}=10$$
$$\Rightarrow \sqrt{x^2-4x+4+y^2}+\sqrt{x^2-12x+36+y^2}=10$$
$$\Rightarrow \sqrt{x^2+y^2-4x+4}=10-\sqrt{x^2+y^2-12x+36}$$
$$\Rightarrow \cancel{x^2}+\cancel{y^2}-4x+4=100-20\sqrt{x^2+y^2-12x+36}+(\cancel{x^2}+\cancel{y^2}-12x+36)$$
$$\Rightarrow 8x-132=-20\sqrt{x^2+y^2-12x+36}$$
$$\Rightarrow 20\sqrt{x^2+y^2-12x+36}=132-8x$$
$$\Rightarrow 5\sqrt{x^2+y^2-12x+36}=33-2x$$
$$\Rightarrow 25(x^2+y^2-12x+36)=(33-2x)^2$$
$$\Rightarrow 25x^2+25y^2-300x+900=1089-132x+4x^2$$
$$\Rightarrow 21x^2+25y^2-168x-189=0$$
$$\Rightarrow x^2+\frac{25}{21}y^2-8x-9=0$$
$$\Rightarrow (x-4)^2-16+\frac{25}{21}y^2-9=0$$
$$\Rightarrow (x-4)^2+\frac{25}{21}y^2=25$$
$$\Rightarrow \frac{(x-4)^2}{25}+\frac{y^2}{21}=1$$



ALTERNATIVE

- BY SIMPLE GEOMETRY WE DRAW SKETCH
  (-1,0)
  (9,0)
  $(4,\sqrt{21})$

$$\boxed{\frac{(x-A)^2}{B}+\frac{y^2}{C}=1}$$

- $(-1,0) \Rightarrow \frac{(-1-A)^2}{B}=1$
  $\Rightarrow \frac{(A+1)^2}{B}=1$
  $\Rightarrow \boxed{B=(A+1)^2}$
- $(9,0) \Rightarrow \frac{(9-A)^2}{B}=1$
  $\boxed{B=(9-A)^2}$
- $(4,\sqrt{21}) \Rightarrow \boxed{\frac{(4-A)^2}{B}+\frac{21}{C}=1}$

THUS $(A+1)^2=(9-A)^2$

$\cancel{A^2}+2A+1=81-18A+\cancel{A^2}$

$20A=80$

$\boxed{A=4}$

& HENCE $\boxed{B=25}$

FINALLY

$$\frac{4-4}{25}+\frac{21}{C}=1$$

$$\boxed{C=21}$$

$$\frac{(x-4)^2}{25}+\frac{y^2}{21}=1$$

Question 146 (****+)

$$f(z) \equiv (z+2\mathrm{i})^2, \; z \in \mathbb{C}.$$

The complex function f maps points, of the form $x+\mathrm{i}y$, from the z plane onto points, of the form $u+\mathrm{i}v$, in the w plane.

The straight line L lies in the z plane and has Cartesian equation

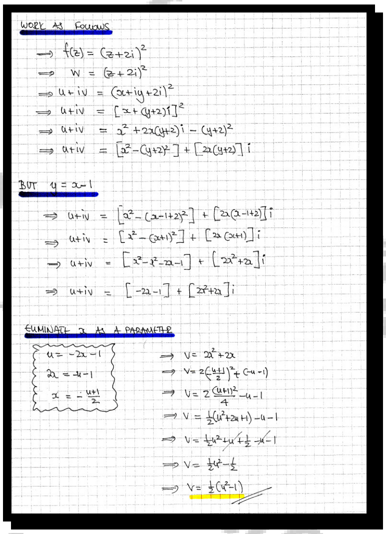
$$y = x-1.$$

Find an equation of the image of L in the w plane, giving the answer in the form

$$v = g(u),$$

where g, is a real function to be found.

$$\boxed{}, \; \boxed{v=\frac{1}{2}\left(u^2-1\right)}$$



WORK AS FOLLOWS

$\Longrightarrow f(z) = (z+2i)^2$

$\Longrightarrow w = (z+2i)^2$

$\Longrightarrow u+iv = (x+iy+2i)^2$

$\Longrightarrow u+iv = [x+(y+2)i]^2$

$\Longrightarrow u+iv = x^2+2x(y+2)i-(y+2)^2$

$\Longrightarrow u+iv = [x^2-(y+2)^2]+[2x(y+2)]i$

BUT $y = x-1$

$\Longrightarrow u+iv = [x^2-(x-1+2)^2]+[2x(x-1+2)]i$

$\Longrightarrow u+iv = [x^2-(x+1)^2]+[2x(x+1)]i$

$\Longrightarrow u+iv = [x^2-x^2-2x-1]+[2x^2+2x]i$

$\Longrightarrow u+iv = [-2x-1]+[2x^2+2x]i$

ELIMINATE x AS A PARAMETER

$u = -2x-1$

$2x = -u-1$

$x = -\frac{u+1}{2}$

$\Longrightarrow v = 2x^2+2x$

$\Longrightarrow v = 2\left(-\frac{u+1}{2}\right)^2+(-u-1)$

$\Longrightarrow v = 2\frac{(u+1)^2}{4}-u-1$

$\Longrightarrow v = \frac{1}{2}(u^2+2u+1)-u-1$

$\Longrightarrow v = \frac{1}{2}u^2+\cancel{u}+\frac{1}{2}\cancel{-u}-1$

$\Longrightarrow v = \frac{1}{2}u^2-\frac{1}{2}$

$\Longrightarrow v = \frac{1}{2}(u^2-1)$

Question 147 (****+)

Use de Moivre's theorem followed by a suitable trigonometric identity, to show that …

a) … $\cos 3\theta \equiv 4\cos^3\theta - 3\cos\theta$.

b) … $\cos 6\theta \equiv \left(2\cos^2\theta - 1\right)\left(16\cos^4\theta - 16\cos^2\theta + 1\right)$

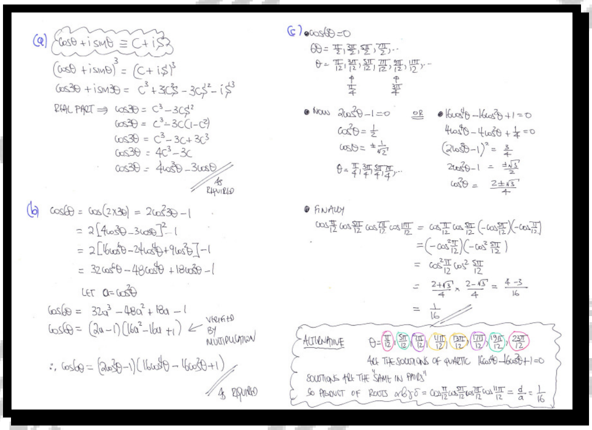
Consider the solutions of the equation.

$$\cos 6\theta = 0\,,\ 0 \le \theta \le \pi\,.$$

c) By fully justifying each step in the workings, find the exact value of

$$\cos\frac{\pi}{12}\cos\frac{5\pi}{12}\cos\frac{7\pi}{12}\cos\frac{11\pi}{12}\,.$$

$$\boxed{\frac{1}{16}}$$



(a) $\cos\theta + i\sin\theta \equiv C + iS$

$(\cos\theta + i\sin\theta)^3 = (C+iS)^3$

$\cos 3\theta + i\sin 3\theta = C^3 + 3iC^2S - 3CS^2 - iS^3$

REAL PART $\Rightarrow \cos 3\theta = C^3 - 3CS^2$

$\cos 3\theta = C^3 - 3C(1-C^2)$

$\cos 3\theta = C^3 - 3C + 3C^3$

$\cos 3\theta = 4C^3 - 3C$

$\cos 3\theta = 4\cos^3\theta - 3\cos\theta$ // AS REQUIRED

(b) $\cos 6\theta = \cos(2\times 3\theta) = 2\cos^2 3\theta - 1$

$= 2\left[4\cos^3\theta - 3\cos\theta\right]^2 - 1$

$= 2\left[16\cos^6\theta - 24\cos^4\theta + 9\cos^2\theta\right] - 1$

$= 32\cos^6\theta - 48\cos^4\theta + 18\cos^2\theta - 1$

LET $a = \cos^2\theta$

$\cos 6\theta = 32a^3 - 48a^2 + 18a - 1$

$\cos 6\theta = (2a-1)(16a^2 - 16a + 1) \leftarrow$ VERIFIED BY MULTIPLICATION

$\therefore \cos 6\theta = (2\cos^2\theta - 1)(16\cos^4\theta - 16\cos^2\theta + 1)$ // AS REQUIRED

(c) • $\cos 6\theta = 0$

$6\theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}, \ldots$

$\theta = \frac{\pi}{12}, \frac{3\pi}{12}, \frac{5\pi}{12}, \frac{7\pi}{12}, \frac{9\pi}{12}, \frac{11\pi}{12}, \ldots$

$\quad\ \ \overset{\uparrow}{\frac{\pi}{4}} \qquad \overset{\uparrow}{\frac{3\pi}{4}}$

• NOW $2\cos^2\theta - 1 = 0$

$\cos^2\theta = \frac{1}{2}$

$\cos\theta = \pm\frac{1}{\sqrt{2}}$

$\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}, \ldots$

OR

• $16\cos^4\theta - 16\cos^2\theta + 1 = 0$

$4\cos^4\theta - 4\cos^2\theta + \frac{1}{4} = 0$

$(2\cos^2\theta - 1)^2 = \frac{3}{4}$

$2\cos^2\theta - 1 = \frac{\pm\sqrt{3}}{2}$

$\cos^2\theta = \frac{2\pm\sqrt{3}}{4}$

• FINALLY

$\cos\frac{\pi}{12}\cos\frac{5\pi}{12}\cos\frac{7\pi}{12}\cos\frac{11\pi}{12} = \cos\frac{\pi}{12}\cos\frac{5\pi}{12}\left(-\cos\frac{5\pi}{12}\right)\left(-\cos\frac{\pi}{12}\right)$

$= \left(-\cos^2\frac{\pi}{12}\right)\left(-\cos^2\frac{5\pi}{12}\right)$

$= \cos^2\frac{\pi}{12}\cos^2\frac{5\pi}{12}$

$= \frac{2+\sqrt{3}}{4} \times \frac{2-\sqrt{3}}{4} = \frac{4-3}{16}$

$= \frac{1}{16}$ //

ALTERNATIVE $\theta = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{7\pi}{12}, \frac{11\pi}{12}, \frac{13\pi}{12}, \frac{17\pi}{12}, \frac{19\pi}{12}, \frac{23\pi}{12}$

ARE THE SOLUTIONS OF QUARTIC $16\cos^4\theta - 16\cos^2\theta + 1 = 0$

SOLUTIONS ARE THE "SAME IN PAIRS"

SO PRODUCT OF ROOTS $\alpha\beta\gamma\delta = \cos\frac{\pi}{12}\cos\frac{5\pi}{12}\cos\frac{7\pi}{12}\cos\frac{11\pi}{12} = \frac{d}{a} = \frac{1}{16}$

Question 148 **(****+)**

A transformation T maps points of the form $z = x + \mathrm{i}y$ from the z plane onto points of the form $v = u + \mathrm{i}v$ in the w plane, and is defined as

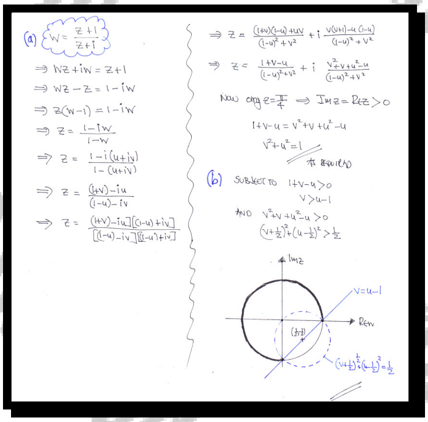
$$w = \frac{z+1}{z+\mathrm{i}}, \quad z \neq -\mathrm{i}.$$

The points that lie on the half line with equation $\arg z = \frac{\pi}{4}$ are mapped by T onto points which lie on a circle.

a) Determine a Cartesian equation for this circle.

b) Show that the image of the half line with equation $\arg z = \frac{\pi}{4}$ is not the entire circle found in part (b).

$$\boxed{u^2 + v^2 = 1}$$



(a) $w = \frac{z+1}{z+i}$

$\Rightarrow wz + iw = z + 1$

$\Rightarrow wz - z = 1 - iw$

$\Rightarrow z(w-1) = 1 - iw$

$\Rightarrow z = \frac{1-iw}{1-w}$

$\Rightarrow z = \frac{1-i(u+iv)}{1-(u+iv)}$

$\Rightarrow z = \frac{(1+v)-iu}{(1-u)-iv}$

$\Rightarrow z = \frac{(1+v)-iu][(1-u)+iv]}{[(1-u)-iv][(1-u)+iv]}$

$\Rightarrow z = \frac{(1+v)(1-u)+uv}{(1-u)^2+v^2} + i\frac{v(v+1)-u(1-u)}{(1-u)^2+v^2}$

$\Rightarrow z = \frac{1+v-u}{(1-u)^2+v^2} + i\frac{v^2+v+u^2-u}{(1-u)^2+v^2}$

Now $\arg z = \frac{\pi}{4} \Rightarrow \operatorname{Im} z = \operatorname{Re} z > 0$

$1+v-u = v^2+v+u^2-u$

$v^2+u^2=1$

AS REQUIRED

(b) SUBJECT TO $1+v-u>0$

$v > u-1$

AND $v^2+v+u^2-u>0$

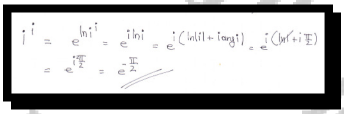
$(v+\frac{1}{2})^2+(u-\frac{1}{2})^2>\frac{1}{2}$

Question 149 **(****+)**

Show that if $z = \mathrm{i}$

$$z^z = \mathrm{e}^{-\frac{\pi}{2}}.$$

$$\boxed{\text{proof}}$$



$i^i = e^{\ln i^i} = e^{i \ln i} = e^{i(\ln|i| + i\arg i)} = e^{i(\cancel{\ln 1} + i\frac{\pi}{2})}$

$= e^{i\frac{i\pi}{2}} = e^{-\frac{\pi}{2}}$

Question 150 (****+)

The complex function $w = f(z)$ maps points of the form $z = x + \mathrm{i}y$ from the z plane onto points of the form $w = u + \mathrm{i}v$ in the w plane.

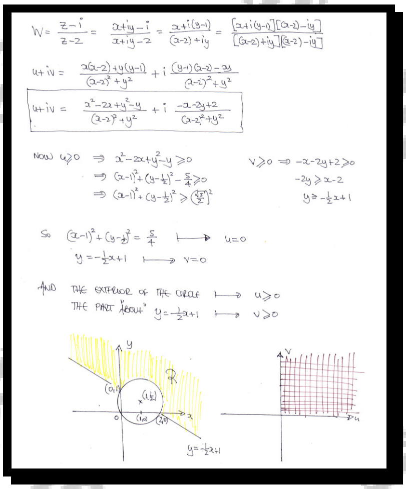
It is given that

$$f(z) = \frac{z - \mathrm{i}}{z - 2}, \ z \in \mathbb{C}, \ z \neq 2.$$

The points of a region R in the z plane are mapped onto points of a region R' in the w plane. The region R' consists of points such that $u \geq 0$ and $v \geq 0$.

Shade, with justification, in an accurate Argand diagram the region R.

sketch



$$W = \frac{z-i}{z-2} = \frac{x+iy-i}{x+iy-2} = \frac{x+i(y-1)}{(x-2)+iy} = \frac{[x+i(y-1)][(x-2)-iy]}{[(x-2)+iy][(x-2)-iy]}$$

$$u+iv = \frac{x(x-2)+y(y-1)}{(x-2)^2+y^2} + i\,\frac{(y-1)(x-2)-xy}{(x-2)^2+y^2}$$

$$\boxed{u+iv = \frac{x^2-2x+y^2-y}{(x-2)^2+y^2} + i\,\frac{-x-2y+2}{(x-2)^2+y^2}}$$

NOW $u \geqslant 0 \Rightarrow x^2-2x+y^2-y \geqslant 0$

$\Rightarrow (x-1)^2+(y-\frac{1}{2})^2-\frac{5}{4} \geqslant 0$

$\Rightarrow (x-1)^2+(y-\frac{1}{2})^2 \geqslant \left(\frac{\sqrt{5}}{2}\right)^2$

$v \geqslant 0 \Rightarrow -x-2y+2 \geqslant 0$

$-2y \geqslant x-2$

$y \geqslant -\frac{1}{2}x+1$

So $(x-1)^2+(y-\frac{1}{2})^2 = \frac{5}{4} \longmapsto u=0$

$y = -\frac{1}{2}x+1 \longmapsto v=0$

AND THE EXTERIOR OF THE CIRCLE $\longmapsto u \geqslant 0$

THE PART "ABOVE" $y = -\frac{1}{2}x+1 \longmapsto v \geqslant 0$

Question 151 **(****+)**

$$f(\theta)=(\cos\theta+\mathrm{i}\sin\theta)^4+(\cos\theta-\mathrm{i}\sin\theta)^4 .$$

a) By considering a simplified expression of $f(\theta)$, show that

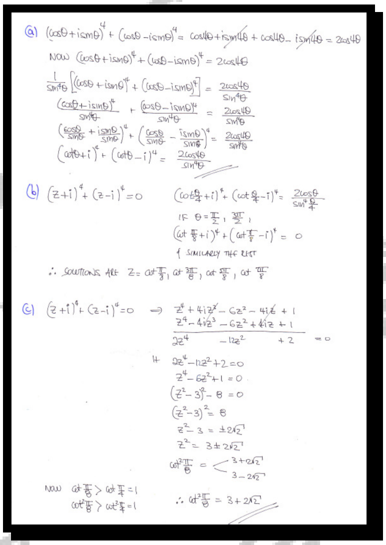
$$(\cot\theta+\mathrm{i})^4+(\cot\theta-\mathrm{i})^4=\frac{2\cos 4\theta}{\sin^4\theta} .$$

b) Find in the form $z=\cot\left(\frac{k\pi}{8}\right)$, the four solution of the equation

$$(z+\mathrm{i})^4+(z-\mathrm{i})^4=0 .$$

c) Hence, show clearly that $\cot^2\left(\frac{\pi}{8}\right)=3+2\sqrt{2}$.

$$\boxed{x=\cot\left(\frac{k\pi}{8}\right),\ k=1,3,5,7}$$



(a) $(\cos\theta+i\sin\theta)^4+(\cos\theta-i\sin\theta)^4=\cos4\theta+\cancel{i\sin4\theta}+\cos4\theta-\cancel{i\sin4\theta}=2\cos4\theta$

NOW $(\cos\theta+i\sin\theta)^4+(\cos\theta-i\sin\theta)^4=2\cos4\theta$

$\frac{1}{\sin^4\theta}\left[(\cos\theta+i\sin\theta)^4+(\cos\theta-i\sin\theta)^4\right]=\frac{2\cos4\theta}{\sin^4\theta}$

$\frac{(\cos\theta+i\sin\theta)^4}{\sin^4\theta}+\frac{(\cos\theta-i\sin\theta)^4}{\sin^4\theta}=\frac{2\cos4\theta}{\sin^4\theta}$

$\left(\frac{\cos\theta}{\sin\theta}+\frac{i\sin\theta}{\sin\theta}\right)^4+\left(\frac{\cos\theta}{\sin\theta}-\frac{i\sin\theta}{\sin\theta}\right)^4=\frac{2\cos4\theta}{\sin^4\theta}$

$(\cot\theta+i)^4+(\cot\theta-i)^4=\frac{2\cos4\theta}{\sin^4\theta}$

(b) $(z+i)^4+(z-i)^4=0$

$(\cot\frac{\theta}{4}+i)^4+(\cot\frac{\theta}{4}-i)^4=\frac{2\cos\theta}{\sin^4\frac{\theta}{4}}$

IF $\theta=\frac{\pi}{2},\frac{3\pi}{2},$

$(\cot\frac{\pi}{8}+i)^4+(\cot\frac{\pi}{8}-i)^4=0$

{ SIMILARLY THE REST

$\therefore$ SOLUTIONS ARE $z=\cot\frac{\pi}{8},\cot\frac{3\pi}{8},\cot\frac{5\pi}{8},\cot\frac{7\pi}{8}$

(c) $(z+i)^4+(z-i)^4=0 \implies$

$$\begin{array}{l} z^4+4i\cancel{z^3}-6z^2-4i\cancel{z}+1 \\ z^4-4iz^3-6z^2+4iz+1 \\ \hline 2z^4-12z^2+2 \end{array} =0$$

$\Rightarrow 2z^4-12z^2+2=0$

$z^4-6z^2+1=0$

$(z^2-3)^2-8=0$

$(z^2-3)^2=8$

$z^2-3=\pm2\sqrt{2}$

$z^2=3\pm2\sqrt{2}$

$\cot^2\frac{\pi}{8}=\begin{cases}3+2\sqrt{2}\\3-2\sqrt{2}\end{cases}$

NOW $\cot\frac{\pi}{8}>\cot\frac{\pi}{4}=1$

$\cot^2\frac{\pi}{8}>\cot^2\frac{\pi}{4}=1$

$\therefore \cot^2\frac{\pi}{8}=3+2\sqrt{2}$

Question 152 (****+)

The complex number z lies in the region R of an Argand diagram, defined by the inequalities

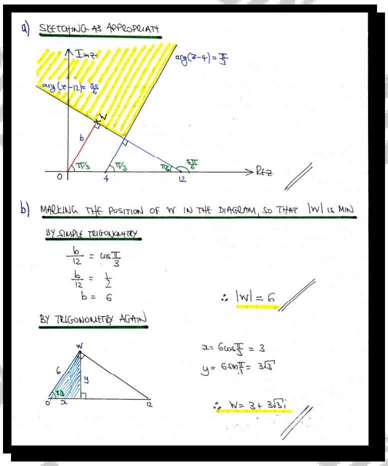
$$\frac{\pi}{3} \le \arg(z-4) \le \pi \quad \text{and} \quad 0 \le \arg(z-12) \le \frac{5\pi}{6}.$$

a) Sketch the region R, indicating clearly all the relevant details.

The complex number w lies in R, so that $|w|$ is minimum.

b) Find $|w|$, further giving w in the form $u + \mathrm{i}v$, where u and v are real numbers.

$\boxed{}$, $\boxed{|w| = 6}$, $\boxed{w = 3 + 3\sqrt{3}\mathrm{i}}$



a) SKETCHING AS APPROPRIATE

b) MARKING THE POSITION OF w IN THE DIAGRAM, SO THAT |w| IS MIN

BY SIMPLE TRIGONOMETRY

$$\frac{b}{12} = \cos\frac{\pi}{3}$$

$$\frac{b}{12} = \frac{1}{2}$$

$$b = 6$$

$\therefore |w| = 6$

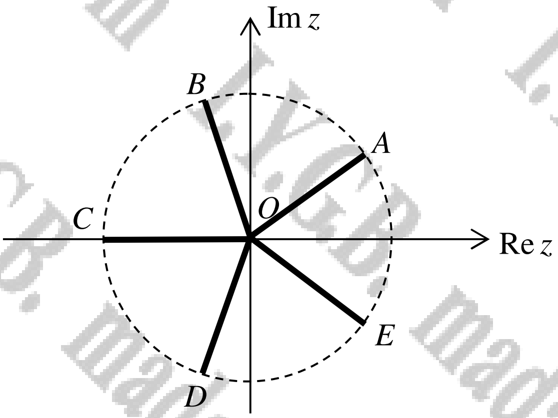
BY TRIGONOMETRY AGAIN

$x = 6\cos\frac{\pi}{3} = 3$

$y = 6\sin\frac{\pi}{3} = 3\sqrt{3}$

$\therefore w = 3 + 3\sqrt{3}i$

Question 153 (****+)



The figure above shows in a standard Argand diagram, the five roots of the equation $z^5+32=0$, indicated by the points A to E on a circle of radius r.

a) State the value of r.

b) State the five roots of the equation

$$z^5+32=0,$$

giving the answers in the form $z=r(\cos\theta+\mathrm{i}\sin\theta)$, $-\pi<\theta\le\pi$.

c) Show that a quadratic equation satisfied by the roots indicated by B and D is

$$z^2+4z\cos\left(\frac{2\pi}{5}\right)+4=0.$$

d) Find a similar quadratic satisfied by the roots indicated by A and E.

[continues overleaf]

[continued from overleaf]

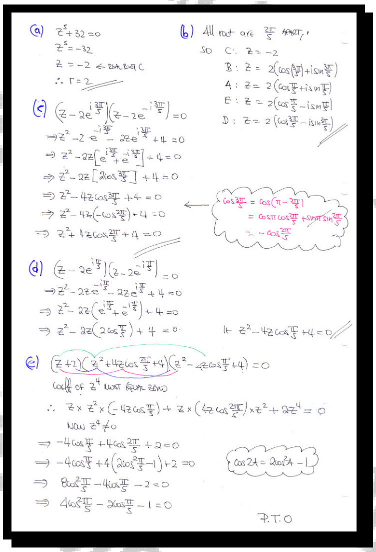
Consider the coefficients of z^4 in the following equations

$$z^5+32=0 \quad \text{and} \quad (z-z_C)\left[(z-z_B)(z-z_D)\right]\left[(z-z_A)(z-z_E)\right]=0.$$

e) Show that $\cos\left(\frac{\pi}{5}\right)=\frac{1}{4}+\frac{1}{4}\sqrt{5}$.

(you may find the cosine double angle formula useful)

$\boxed{r=2}$, $\boxed{z=2(\cos n\theta+\mathrm{i}\sin n\theta),n=-2,-1,0,1,2}$, $\boxed{z^2-4z\cos\left(\frac{\pi}{5}\right)+4=0}$



(a) $z^5+32=0$

$z^5=-32$

$z=-2$ ← REAL ROOT C

$\therefore r=2$

(b) All root are $\frac{2\pi}{5}$ APART, ...

SO C: $z=-2$

B: $z=2\left(\cos\left(\frac{3\pi}{5}\right)+i\sin\frac{3\pi}{5}\right)$

A: $z=2\left(\cos\frac{\pi}{5}+i\sin\frac{\pi}{5}\right)$

E: $z=2\left(\cos\frac{\pi}{5}-i\sin\frac{\pi}{5}\right)$

D: $z=2\left(\cos\frac{3\pi}{5}-i\sin\frac{3\pi}{5}\right)$

(c) $\left(z-2e^{i\frac{3\pi}{5}}\right)\left(z-2e^{-i\frac{3\pi}{5}}\right)=0$

$\Rightarrow z^2-2e^{-i\frac{3\pi}{5}}-2ze^{i\frac{3\pi}{5}}+4=0$

$\Rightarrow z^2-2z\left[e^{i\frac{3\pi}{5}}+e^{-i\frac{3\pi}{5}}\right]+4=0$

$\Rightarrow z^2-2z\left[2\cos\frac{3\pi}{5}\right]+4=0$

$\Rightarrow z^2-4z\cos\frac{3\pi}{5}+4=0$ ←

$\Rightarrow z^2-4z\left(-\cos\frac{2\pi}{5}\right)+4=0$

$\Rightarrow z^2+4z\cos\frac{2\pi}{5}+4=0$

$\cos\frac{3\pi}{5}=\cos\left(\pi-\frac{2\pi}{5}\right)$

$=\cos\pi\cos\frac{2\pi}{5}+\cancel{\sin\pi\sin\frac{2\pi}{5}}$

$=-\cos\frac{2\pi}{5}$

(d) $\left(z-2e^{i\frac{\pi}{5}}\right)\left(z-2e^{-i\frac{\pi}{5}}\right)=0$

$\Rightarrow z^2-2ze^{-i\frac{\pi}{5}}-2ze^{i\frac{\pi}{5}}+4=0$

$\Rightarrow z^2-2z\left(e^{i\frac{\pi}{5}}+e^{-i\frac{\pi}{5}}\right)+4=0$

$\Rightarrow z^2-2z\left(2\cos\frac{\pi}{5}\right)+4=0$. IE $z^2-4z\cos\frac{\pi}{5}+4=0$

(e) $(z+2)\left(z^2+4z\cos\frac{2\pi}{5}+4\right)\left(z^2-4z\cos\frac{\pi}{5}+4\right)=0$

COEFF OF z^4 MUST EQUAL ZERO

$\therefore z\times z^2\times\left(-4z\cos\frac{\pi}{5}\right)+z\times\left(4z\cos\frac{2\pi}{5}\right)\times z^2+2z^4=0$

NOW $z^4\neq0$

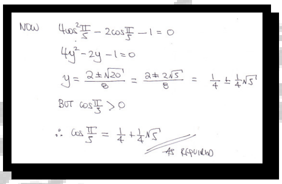
$\Rightarrow -4\cos\frac{\pi}{5}+4\cos\frac{2\pi}{5}+2=0$

$\Rightarrow -4\cos\frac{\pi}{5}+4\left(2\cos^2\frac{\pi}{5}-1\right)+2=0$ $\cos2A=2\cos^2A-1$

$\Rightarrow 8\cos^2\frac{\pi}{5}-4\cos\frac{\pi}{5}-2=0$

$\Rightarrow 4\cos^2\frac{\pi}{5}-2\cos\frac{\pi}{5}-1=0$

P.T.O



NOW $4\cos^2\frac{\pi}{5}-2\cos\frac{\pi}{5}-1=0$

$4y^2-2y-1=0$

$y=\frac{2\pm\sqrt{20}}{8}=\frac{2\pm2\sqrt{5}}{8}=\frac{1}{4}\pm\frac{1}{4}\sqrt{5}$

BUT $\cos\frac{\pi}{5}>0$

$\therefore \cos\frac{\pi}{5}=\frac{1}{4}+\frac{1}{4}\sqrt{5}$

AS REQUIRED

Question 154 (****+)

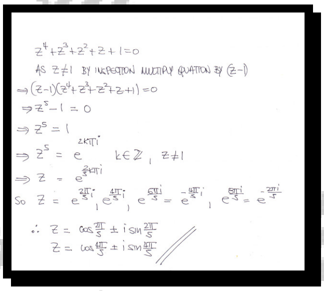
$$z^4 + z^3 + z^2 + z + 1 = 0, \ z \in \mathbb{C}.$$

By using the identity

$$a^n - 1 \equiv (a-1)\left(a^n + a^{n-1} + a^{n-2} + \ldots + a^2 + a + 1\right),$$

or otherwise, find in exact trigonometric form the four solutions of the above equation.

$$\boxed{z = \cos\frac{2\pi}{5} \pm \mathrm{i}\sin\frac{2\pi}{5}, \ \cos\frac{4\pi}{5} \pm \mathrm{i}\sin\frac{4\pi}{5}}$$



$z^4+z^3+z^2+z+1=0$

AS $z \neq 1$ BY INSPECTION MULTIPLY EQUATION BY $(z-1)$

$\Rightarrow (z-1)(z^4+z^3+z^2+z+1)=0$

$\Rightarrow z^5-1=0$

$\Rightarrow z^5=1$

$\Rightarrow z^5 = e^{2k\pi i} \quad k \in \mathbb{Z}, \ z \neq 1$

$\Rightarrow z = e^{\frac{2}{5}k\pi i}$

So $z = e^{\frac{2\pi}{5}i}, e^{\frac{4\pi}{5}i}, e^{\frac{6\pi i}{5}} = e^{-\frac{4\pi}{5}i}, e^{\frac{8\pi i}{5}} = e^{-\frac{2\pi i}{5}}$

$\therefore z = \cos\frac{2\pi}{5} \pm i\sin\frac{2\pi}{5}$

$z = \cos\frac{4\pi}{5} \pm i\sin\frac{4\pi}{5}$ //

Question 155 **(****+)**

$$f(z) \equiv z^2, \ z \in \mathbb{C}.$$

The complex function f maps points, of the form $x + \mathrm{i}y$, from the z plane onto points, of the form $u + \mathrm{i}v$, in the w plane.

The curve C lies in the z plane and has Cartesian equation

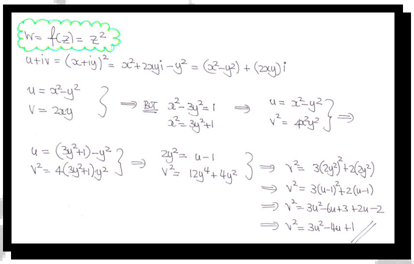
$$x^2 - 3y^2 = 1.$$

Find an equation of the image of C in the w plane, giving the answer in the form

$$v^2 = Au^2 + Bu + C,$$

where A, B and C are real constants to be found.

$$\boxed{v^2 = 3u^2 - 4u + 1}$$



$w = f(z) = z^2$

$u + iv = (x+iy)^2 = x^2 + 2xyi - y^2 = (x^2-y^2) + (2xy)i$

$\left.\begin{array}{l} u = x^2 - y^2 \\ v = 2xy \end{array}\right\} \Longrightarrow$ BUT $\begin{array}{l} x^2 - 3y^2 = 1 \\ x^2 = 3y^2 + 1 \end{array} \Longrightarrow \left.\begin{array}{l} u = x^2 - y^2 \\ v^2 = 4x^2y^2 \end{array}\right\} \Longrightarrow$

$\left.\begin{array}{l} u = (3y^2+1) - y^2 \\ v^2 = 4(3y^2+1)y^2 \end{array}\right\} \Longrightarrow \left.\begin{array}{l} 2y^2 = u - 1 \\ v^2 = 12y^4 + 4y^2 \end{array}\right\} \Longrightarrow v^2 = 3(2y^2)^2 + 2(2y^2)$

$\Longrightarrow v^2 = 3(u-1)^2 + 2(u-1)$

$\Longrightarrow v^2 = 3u^2 - 6u + 3 + 2u - 2$

$\Longrightarrow v^2 = 3u^2 - 4u + 1$ //

Question 156 (****+)

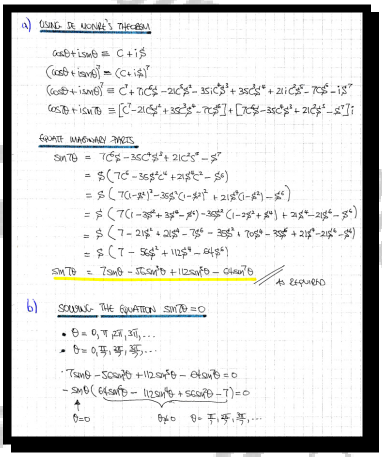
a) Show that

$$\sin 7\theta \equiv 7\sin\theta - 56\sin^3\theta + 112\sin^5\theta - 64\sin^7\theta$$

b) By considering a suitable polynomial equation based on the result of part **(a)** show further

$$\operatorname{cosec}^2\left(\tfrac{1}{7}\pi\right) + \operatorname{cosec}^2\left(\tfrac{2}{7}\pi\right) + \operatorname{cosec}^2\left(\tfrac{3}{7}\pi\right) = 8$$

☐ , $\boxed{\text{proof}}$



a) USING DE MOIVRE'S THEOREM

$\cos\theta + i\sin\theta \equiv C + iS$

$(\cos\theta + i\sin\theta)^7 \equiv (C + iS)^7$

$(\cos\theta + i\sin\theta)^7 \equiv C^7 + 7iC^6S - 21C^5S^2 - 35iC^4S^3 + 35C^3S^4 + 21iC^2S^5 - 7CS^6 - iS^7$

$\cos7\theta + i\sin7\theta \equiv [C^7 - 21C^5S^2 + 35C^3S^4 - 7CS^6] + [7C^6S - 35C^4S^3 + 21C^2S^5 - S^7]i$

EQUATE IMAGINARY PARTS

$\sin7\theta = 7C^6S - 35C^4S^3 + 21C^2S^5 - S^7$

$= S(7C^6 - 35S^2C^4 + 21S^4C^2 - S^6)$

$= S(7(1-S^2)^3 - 35S^2(1-S^2)^2 + 21S^4(1-S^2) - S^6)$

$= S(7(1-3S^2+3S^4-S^6) - 35S^2(1-2S^2+S^4) + 21S^4 - 21S^6 - S^6)$

$= S(7 - 21S^2 + 21S^4 - 7S^6 - 35S^2 + 70S^4 - 35S^6 + 21S^4 - 21S^6 - S^6)$

$= S(7 - 56S^2 + 112S^4 - 64S^6)$

$\sin7\theta = 7\sin\theta - 56\sin^3\theta + 112\sin^5\theta - 64\sin^7\theta$ // AS REQUIRED

b) SOLVING THE EQUATION $\sin7\theta = 0$

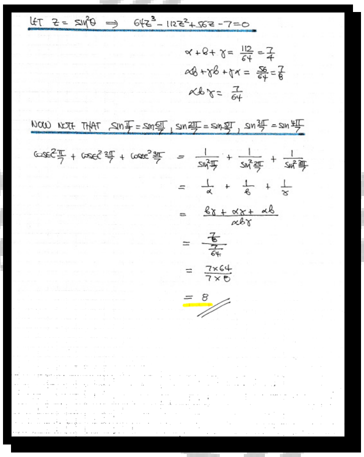
- $7\theta = 0, \pi, 2\pi, 3\pi, \dots$
- $\theta = 0, \frac{\pi}{7}, \frac{2\pi}{7}, \frac{3\pi}{7}, \dots$

$7\sin\theta - 56\sin^3\theta + 112\sin^5\theta - 64\sin^7\theta = 0$

$-\sin\theta(\underbrace{64\sin^6\theta - 112\sin^4\theta + 56\sin^2\theta - 7}) = 0$

$\uparrow$

$\theta = 0$ $\quad\theta \neq 0 \quad \theta = \frac{\pi}{7}, \frac{2\pi}{7}, \frac{3\pi}{7}, \dots$



LET $z = \sin^2\theta \Rightarrow 64z^3 - 112z^2 + 56z - 7 = 0$

$\alpha + \beta + \gamma = \frac{112}{64} = \frac{7}{4}$

$\alpha\beta + \gamma\beta + \gamma\alpha = \frac{56}{64} = \frac{7}{8}$

$\alpha\beta\gamma = \frac{7}{64}$

NOW NOTE THAT $\sin\frac{\pi}{7} = \sin\frac{6\pi}{7}$, $\sin\frac{2\pi}{7} = \sin\frac{5\pi}{7}$, $\sin\frac{3\pi}{7} = \sin\frac{4\pi}{7}$

$\operatorname{cosec}^2\frac{\pi}{7} + \operatorname{cosec}^2\frac{2\pi}{7} + \operatorname{cosec}^2\frac{3\pi}{7} = \frac{1}{\sin^2\frac{\pi}{7}} + \frac{1}{\sin^2\frac{2\pi}{7}} + \frac{1}{\sin^2\frac{3\pi}{7}}$

$= \frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$

$= \frac{\beta\gamma + \alpha\gamma + \alpha\beta}{\alpha\beta\gamma}$

$= \frac{\frac{7}{8}}{\frac{7}{64}}$

$= \frac{7\times64}{7\times8}$

$= 8$ //

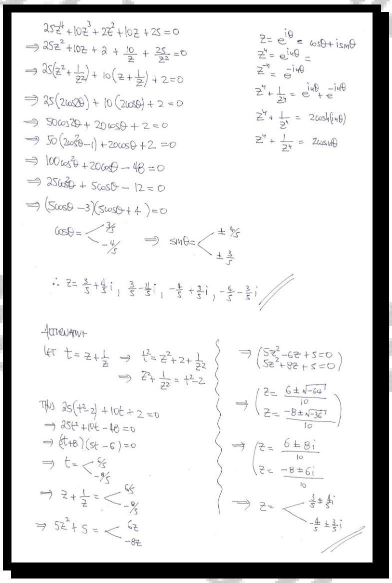
Question 157 **(****+)**

The following equation has no real solutions

$$25z^4 + 10z^3 + 2z^2 + 10z + 25 = 0 .$$

Find the four complex solution of the above equation, giving the answer in the form $a + b\mathrm{i}$, where $a \in \mathbb{C}$ and $b \in \mathbb{C}$.

$$\boxed{z = \frac{3}{5} + \frac{4}{5}\mathrm{i}, \quad z = \frac{3}{5} - \frac{4}{5}\mathrm{i}, \quad z = -\frac{4}{5} + \frac{3}{5}\mathrm{i}, \quad z = -\frac{4}{5} - \frac{3}{5}\mathrm{i}}$$



$25z^4 + 10z^3 + 2z^2 + 10z + 25 = 0$

$\Rightarrow 25z^2 + 10z + 2 + \frac{10}{z} + \frac{25}{z^2} = 0$

$\Rightarrow 25\left(z^2 + \frac{1}{z^2}\right) + 10\left(z + \frac{1}{z}\right) + 2 = 0$

$\Rightarrow 25(2\cos 2\theta) + 10(2\cos\theta) + 2 = 0$

$\Rightarrow 50\cos 2\theta + 20\cos\theta + 2 = 0$

$\Rightarrow 50(2\cos^2\theta - 1) + 20\cos\theta + 2 = 0$

$\Rightarrow 100\cos^2\theta + 20\cos\theta - 48 = 0$

$\Rightarrow 25\cos^2\theta + 5\cos\theta - 12 = 0$

$\Rightarrow (5\cos\theta - 3)(5\cos\theta + 4) = 0$

$\cos\theta = \begin{cases} 3/5 \\ -4/5 \end{cases} \Rightarrow \sin\theta = \begin{cases} \pm 4/5 \\ \pm \frac{3}{5} \end{cases}$

$\therefore z = \frac{3}{5} + \frac{4}{5}i, \ \frac{3}{5} - \frac{4}{5}i, \ -\frac{4}{5} + \frac{3}{5}i, \ -\frac{4}{5} - \frac{3}{5}i$ //

$z = e^{i\theta} = \cos\theta + i\sin\theta$

$z^4 = e^{i4\theta} =$

$z^{-4} = e^{-i4\theta}$

$z^4 + \frac{1}{z^4} = e^{i4\theta} + e^{-i4\theta}$

$z^4 + \frac{1}{z^4} = 2\cosh(i4\theta)$

$z^4 + \frac{1}{z^4} = 2\cos 4\theta$

ALTERNATIVE

LET $t = z + \frac{1}{z} \Rightarrow t^2 = z^2 + 2 + \frac{1}{z^2}$

$\Rightarrow z^2 + \frac{1}{z^2} = t^2 - 2$

THUS $25(t^2 - 2) + 10t + 2 = 0$

$\Rightarrow 25t^2 + 10t - 48 = 0$

$\Rightarrow (5t + 8)(5t - 6) = 0$

$\Rightarrow t = \begin{cases} 6/5 \\ -8/5 \end{cases}$

$\Rightarrow z + \frac{1}{z} = \begin{cases} 6/5 \\ -8/5 \end{cases}$

$\Rightarrow 5z^2 + 5 = \begin{cases} 6z \\ -8z \end{cases}$

$\Rightarrow \begin{pmatrix} 5z^2 - 6z + 5 = 0 \\ 5z^2 + 8z + 5 = 0 \end{pmatrix}$

$\Rightarrow \begin{pmatrix} z = \frac{6 \pm \sqrt{-64}}{10} \\ z = \frac{-8 \pm \sqrt{-36}}{10} \end{pmatrix}$

$\Rightarrow \begin{pmatrix} z = \frac{6 \pm 8i}{10} \\ z = \frac{-8 \pm 6i}{10} \end{pmatrix}$

$\Rightarrow z = \begin{cases} \frac{3}{5} \pm \frac{4}{5}i \\ -\frac{4}{5} \pm \frac{3}{5}i \end{cases}$ //

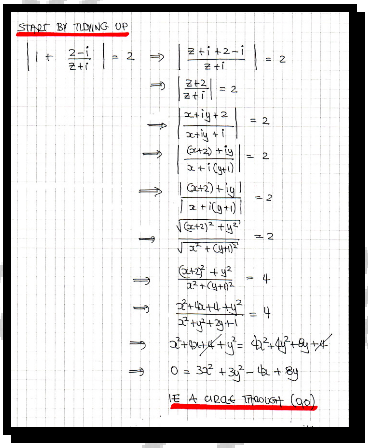
Question 158 (****+)

$$f(z)=\frac{2-\mathrm{i}}{z+\mathrm{i}},\ z\in\mathbb{C},\ z\neq-\mathrm{i}.$$

Find the greatest value of the modulus of z, given further that

$$\left|1+f(z)\right|=2.$$

$\boxed{}$, $\boxed{|z|_{\max}=\frac{4}{3}\sqrt{5}}$



START BY TIDYING UP

$$\left|1+\frac{2-i}{z+i}\right|=2 \Rightarrow \left|\frac{z+i+2-i}{z+i}\right|=2$$

$$\Rightarrow \left|\frac{z+2}{z+i}\right|=2$$

$$\Rightarrow \left|\frac{x+iy+2}{x+iy+i}\right|=2$$

$$\Rightarrow \left|\frac{(x+2)+iy}{x+i(y+1)}\right|=2$$

$$\Rightarrow \frac{|(x+2)+iy|}{|x+i(y+1)|}=2$$

$$\Rightarrow \frac{\sqrt{(x+2)^2+y^2}}{\sqrt{x^2+(y+1)^2}}=2$$

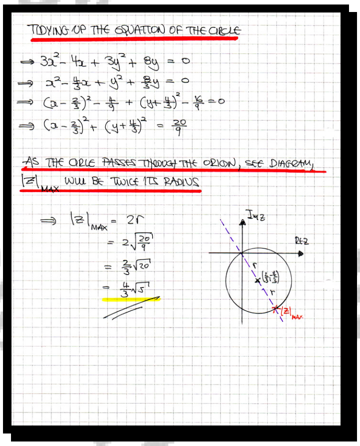
$$\Rightarrow \frac{(x+2)^2+y^2}{x^2+(y+1)^2}=4$$

$$\Rightarrow \frac{x^2+4x+4+y^2}{x^2+y^2+2y+1}=4$$

$$\Rightarrow x^2+4x+\cancel{4}+y^2=4x^2+4y^2+8y+\cancel{4}$$

$$\Rightarrow 0=3x^2+3y^2-4x+8y$$

IE A CIRCLE THROUGH (0,0)



TIDYING UP THE EQUATION OF THE CIRCLE

$$\Rightarrow 3x^2-4x+3y^2+8y=0$$

$$\Rightarrow x^2-\tfrac{4}{3}x+y^2+\tfrac{8}{3}y=0$$

$$\Rightarrow \left(x-\tfrac{2}{3}\right)^2-\tfrac{4}{9}+\left(y+\tfrac{4}{3}\right)^2-\tfrac{16}{9}=0$$

$$\Rightarrow \left(x-\tfrac{2}{3}\right)^2+\left(y+\tfrac{4}{3}\right)^2=\tfrac{20}{9}$$

AS THE CIRCLE PASSES THROUGH THE ORIGIN, SEE DIAGRAM, $|z|_{\text{MAX}}$ WILL BE TWICE ITS RADIUS

$$\Rightarrow |z|_{\text{MAX}}=2r$$

$$=2\sqrt{\tfrac{20}{9}}$$

$$=\tfrac{2}{3}\sqrt{20}$$

$$=\tfrac{4}{3}\sqrt{5}$$

Question 159 (****+)

The complex function $w = f(z)$ is defined by

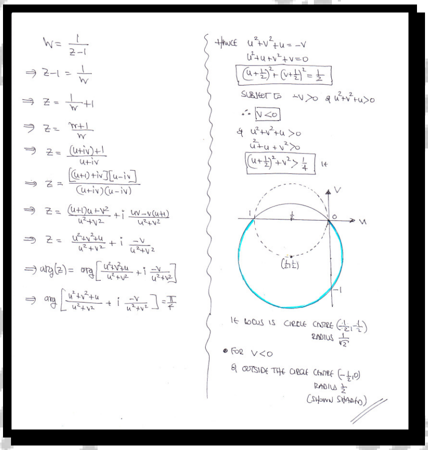
$$w = \frac{1}{z-1}, \quad z \in \mathbb{C}, \quad z \neq 1.$$

The half line with equation $\arg z = \frac{\pi}{4}$ is transformed by this function.

a) Find a Cartesian equation of the locus of the **image** of the half line.

b) Sketch the **image** of the locus in an Argand diagram.

$$\boxed{\left(u+\frac{1}{2}\right)^2 + \left(v+\frac{1}{2}\right)^2 = \frac{1}{2}, \; v<0, \; u^2+v^2+u>0}$$



$$W = \frac{1}{Z-1}$$

$$\Rightarrow Z-1 = \frac{1}{W}$$

$$\Rightarrow Z = \frac{1}{W}+1$$

$$\Rightarrow Z = \frac{W+1}{W}$$

$$\Rightarrow Z = \frac{(u+iv)+1}{u+iv}$$

$$\Rightarrow Z = \frac{[(u+1)+iv][u-iv]}{(u+iv)(u-iv)}$$

$$\Rightarrow Z = \frac{(u+1)u+v^2}{u^2+v^2} + i\,\frac{uv-v(u+1)}{u^2+v^2}$$

$$\Rightarrow Z = \frac{u^2+v^2+u}{u^2+v^2} + i\,\frac{-v}{u^2+v^2}$$

$$\Rightarrow \arg(z) = \arg\left[\frac{u^2+v^2+u}{u^2+v^2} + i\,\frac{-v}{u^2+v^2}\right]$$

$$\Rightarrow \arg\left[\frac{u^2+v^2+u}{u^2+v^2} + i\,\frac{-v}{u^2+v^2}\right] = \frac{\pi}{4}$$

HENCE $u^2+v^2+u = -v$

$u^2+u+v^2+v=0$

$$\boxed{\left(u+\tfrac{1}{2}\right)^2 + \left(v+\tfrac{1}{2}\right)^2 = \tfrac{1}{2}}$$

SUBJECT TO $-v>0$ & $u^2+v^2+u>0$

$\therefore \boxed{v<0}$

& $u^2+v^2+u>0$

$u^2+u+v^2>0$

$\boxed{\left(u+\tfrac{1}{2}\right)^2+v^2 > \tfrac{1}{4}}$ IE

IE LOCUS IS CIRCLE CENTRE $\left(-\frac{1}{2}, -\frac{1}{2}\right)$ RADIUS $\frac{1}{\sqrt{2}}$

- FOR $v<0$

& OUTSIDE THE CIRCLE CENTRE $\left(-\frac{1}{2}, 0\right)$ RADIUS $\frac{1}{2}$ (SHOWN SHADED)

Question 160 (****+)

$$\tan 3\theta \equiv \frac{3\tan\theta - \tan^3\theta}{1-3\tan^2\theta}.$$

a) Use De Moivre's theorem to prove the validity of the above trigonometric identity.

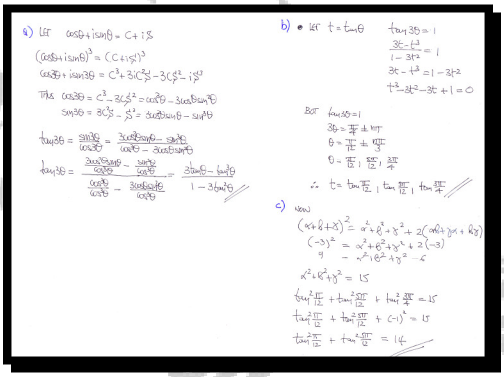
b) Hence find in exact trigonometric form the solutions of the equation

$$t^3 - 3t^2 - 3t + 1 = 0.$$

c) Use the answer of part (**b**) to show further that

$$\tan^2\frac{\pi}{12} + \tan^2\frac{5\pi}{12} = 14.$$

$$\boxed{t = \tan\frac{\pi}{12},\ \tan\frac{5\pi}{12},\ \tan\frac{3\pi}{4}}$$



a) LET $\cos\theta + i\sin\theta = C + iS$

$(\cos\theta + i\sin\theta)^3 = (C+iS)^3$

$\cos 3\theta + i\sin 3\theta = C^3 + 3iC^2S - 3CS^2 - iS^3$

THUS $\cos 3\theta = C^3 - 3CS^2 = \cos^3\theta - 3\cos\theta\sin^2\theta$

$\sin 3\theta = 3C^2S - S^3 = 3\cos^2\theta\sin\theta - \sin^3\theta$

$\tan 3\theta = \frac{\sin 3\theta}{\cos 3\theta} = \frac{3\cos^2\theta\sin\theta - \sin^3\theta}{\cos^3\theta - 3\cos\theta\sin^2\theta}$

$\tan 3\theta = \frac{\frac{3\cos^2\theta\sin\theta}{\cos^3\theta} - \frac{\sin^3\theta}{\cos^3\theta}}{\frac{\cos^3\theta}{\cos^3\theta} - \frac{3\cos\theta\sin^2\theta}{\cos^3\theta}} = \frac{3\tan\theta - \tan^3\theta}{1-3\tan^2\theta}$ //

b) • LET $t = \tan\theta$

$\tan 3\theta = 1$

$\frac{3t - t^3}{1-3t^2} = 1$

$3t - t^3 = 1 - 3t^2$

$t^3 - 3t^2 - 3t + 1 = 0$

BUT $\tan 3\theta = 1$

$3\theta = \frac{\pi}{4} \pm n\pi$

$\theta = \frac{\pi}{12} \pm \frac{n\pi}{3}$

$\theta = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{3\pi}{4}$

$\therefore\ t = \tan\frac{\pi}{12}, \tan\frac{5\pi}{12}, \tan\frac{3\pi}{4}$ //

c) NOW

$(\alpha+\beta+\gamma)^2 = \alpha^2+\beta^2+\gamma^2 + 2(\alpha\beta+\gamma\alpha+\beta\gamma)$

$(-3)^2 = \alpha^2+\beta^2+\gamma^2 + 2(-3)$

$9 = \alpha^2+\beta^2+\gamma^2 - 6$

$\alpha^2+\beta^2+\gamma^2 = 15$

$\tan^2\frac{\pi}{12} + \tan^2\frac{5\pi}{12} + \tan^2\frac{3\pi}{4} = 15$

$\tan^2\frac{\pi}{12} + \tan^2\frac{5\pi}{12} + (-1)^2 = 15$

$\tan^2\frac{\pi}{12} + \tan^2\frac{5\pi}{12} = 14$ //

Question 161 (****+)

The locus L_1 of a point in an Argand diagram satisfies

$$\arg(z-2)-\arg(z-2\mathrm{i})=\frac{3\pi}{4},\ z\in\mathbb{C}.$$

a) Find a Cartesian equation for L_1.

b) Show that all the points which lie on L_1 satisfy

$$\left|\frac{z-4}{z-1}\right|=k,$$

where k is an integer to be found.

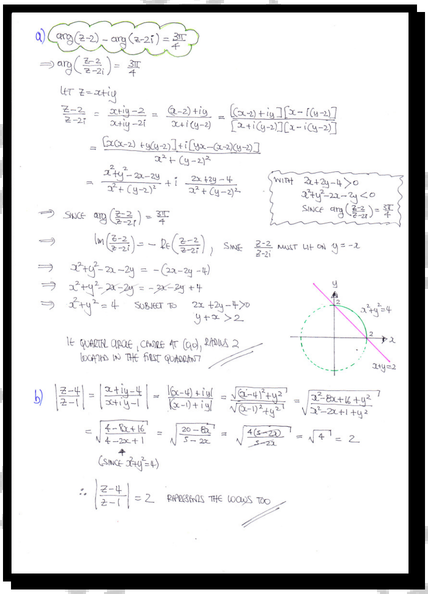
The locus L_2 of a different point in the same Argand diagram satisfies

$$|z-1|+|z-4|=6,\ z\in\mathbb{C}.$$

The point P lies on L_1 and L_1.

c) Find the complex number represented by P.

$$\boxed{L_1: x^2+y^2=4,\ x>0,\ y>0},\ \boxed{k=2},\ \boxed{P:\frac{1}{2}+\frac{1}{4}\mathrm{i}\sqrt{15}}$$



a) $\arg(z-2)-\arg(z-2i)=\frac{3\pi}{4}$

$\Rightarrow \arg\left(\frac{z-2}{z-2i}\right)=\frac{3\pi}{4}$

LET $z=x+iy$

$\frac{z-2}{z-2i}=\frac{x+iy-2}{x+iy-2i}=\frac{(x-2)+iy}{x+i(y-2)}=\frac{[(x-2)+iy][x-i(y-2)]}{[x+i(y-2)][x-i(y-2)]}$

$=\frac{[x(x-2)+y(y-2)]+i[yx-(x-2)(y-2)]}{x^2+(y-2)^2}$

$=\frac{x^2+y^2-2x-2y}{x^2+(y-2)^2}+i\,\frac{2x+2y-4}{x^2+(y-2)^2}$

WITH $2x+2y-4>0$
$x^2+y^2-2x-2y<0$
SINCE $\arg\left(\frac{z-2}{z-2i}\right)=\frac{3\pi}{4}$

$\Rightarrow$ SINCE $\arg\left(\frac{z-2}{z-2i}\right)=\frac{3\pi}{4}$

$\Rightarrow \operatorname{Im}\left(\frac{z-2}{z-2i}\right)=-\operatorname{Re}\left(\frac{z-2}{z-2i}\right)$, SINCE $\frac{z-2}{z-2i}$ MUST LIE ON $y=-x$

$\Rightarrow x^2+y^2-2x-2y=-(2x-2y-4)$

$\Rightarrow x^2+y^2\cancel{-2x}\cancel{-2y}=\cancel{-2x}\cancel{-2y}+4$

$\Rightarrow x^2+y^2=4$ SUBJECT TO $2x+2y-4>0$
$y+x>2$

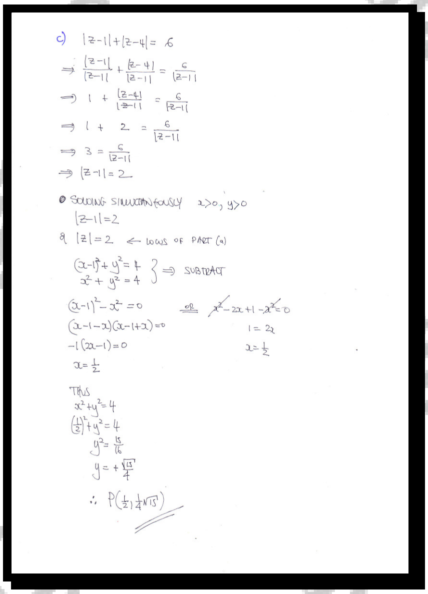
IE QUARTER CIRCLE, CENTRE AT (0,0), RADIUS 2 LOCATED IN THE FIRST QUADRANT

b) $\left|\frac{z-4}{z-1}\right|=\left|\frac{x+iy-4}{x+iy-1}\right|=\frac{|(x-4)+iy|}{|(x-1)+iy|}=\frac{\sqrt{(x-4)^2+y^2}}{\sqrt{(x-1)^2+y^2}}=\sqrt{\frac{x^2-8x+16+y^2}{x^2-2x+1+y^2}}$

$=\sqrt{\frac{4-8x+16}{4-2x+1}}=\sqrt{\frac{20-8x}{5-2x}}=\sqrt{\frac{4\cancel{(5-2x)}}{\cancel{5-2x}}}=\sqrt{4}=2$

(SINCE $x^2+y^2=4$)

$\therefore \left|\frac{z-4}{z-1}\right|=2$ REPRESENTS THE LOCUS TOO



c) $|z-1|+|z-4|=6$

$\Rightarrow \frac{|z-1|}{|z-1|}+\frac{|z-4|}{|z-1|}=\frac{6}{|z-1|}$

$\Rightarrow 1+\frac{|z-4|}{|z-1|}=\frac{6}{|z-1|}$

$\Rightarrow 1+2=\frac{6}{|z-1|}$

$\Rightarrow 3=\frac{6}{|z-1|}$

$\Rightarrow |z-1|=2$

- SOLVING SIMULTANEOUSLY $x>0,\ y>0$

$|z-1|=2$

& $|z|=2$ ← LOCUS OF PART (a)

$\left.\begin{matrix}(x-1)^2+y^2=4\\ x^2+y^2=4\end{matrix}\right\}\Rightarrow$ SUBTRACT

$(x-1)^2-x^2=0$ OR $\cancel{x^2}-2x+1-\cancel{x^2}=0$

$(x-1-x)(x-1+x)=0$ $1=2x$

$-1(2x-1)=0$ $x=\frac{1}{2}$

$x=\frac{1}{2}$

THUS

$x^2+y^2=4$

$\left(\frac{1}{2}\right)^2+y^2=4$

$y^2=\frac{15}{16}$

$y=+\frac{\sqrt{15}}{4}$

$\therefore P\left(\frac{1}{2},\frac{1}{4}\sqrt{15}\right)$

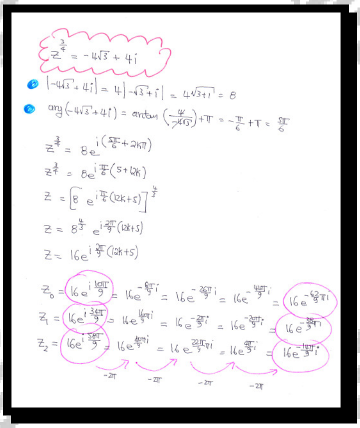
Question 162 (****+)

Solve the equation

$$z^{\frac{3}{4}} = -4\sqrt{3} + 4\mathrm{i}, \ z \in \mathbb{C}.$$

Give each of the roots in exponential form.

$$\boxed{z = 16\mathrm{e}^{\frac{8}{9}\pi \mathrm{i}} = 16\mathrm{e}^{-\frac{62}{9}\pi \mathrm{i}}}, \ \boxed{z = 16\mathrm{e}^{\frac{34}{9}\pi \mathrm{i}} = 16\mathrm{e}^{-\frac{38}{9}\pi \mathrm{i}}}, \ \boxed{z = 16\mathrm{e}^{\frac{58}{9}\pi \mathrm{i}} = 16\mathrm{e}^{-\frac{14}{9}\pi \mathrm{i}}}$$



$z^{\frac{3}{4}} = -4\sqrt{3} + 4i$

- $|-4\sqrt{3} + 4i| = 4|-\sqrt{3} + i| = 4\sqrt{3+1} = 8$
- $\arg(-4\sqrt{3} + 4i) = \arctan\left(\frac{4}{-4\sqrt{3}}\right) + \pi = -\frac{\pi}{6} + \pi = \frac{5\pi}{6}$

$z^{\frac{3}{4}} = 8e^{i\left(\frac{5\pi}{6} + 2k\pi\right)}$

$z^{\frac{3}{4}} = 8e^{i\frac{\pi}{6}(5+12k)}$

$z = \left[8 e^{i\frac{\pi}{6}(12k+5)}\right]^{\frac{4}{3}}$

$z = 8^{\frac{4}{3}} e^{i\frac{2\pi}{9}(12k+5)}$

$z = 16e^{i\frac{2\pi}{9}(12k+5)}$

$z_0 = 16e^{i\frac{10\pi}{9}} = 16e^{-\frac{8\pi}{9}i} = 16e^{-\frac{26\pi}{9}i} = 16e^{-\frac{44\pi}{9}i} = 16e^{-\frac{62}{9}\pi i}$

$z_1 = 16e^{i\frac{34\pi}{9}} = 16e^{\frac{16\pi}{9}i} = 16e^{-\frac{2\pi}{9}i} = 16e^{-\frac{20\pi}{9}i} = 16e^{-\frac{38}{9}\pi i}$

$z_2 = 16e^{i\frac{58\pi}{9}} = 16e^{\frac{40\pi}{9}i} = 16e^{\frac{22\pi}{9}\pi i} = 16e^{\frac{4\pi}{9}i} = 16e^{-\frac{14\pi}{9}i}$

-2π -2π -2π -2π

Question 163 **(****+)**

The complex number w is defined as $w = \mathrm{e}^{\frac{2}{5}\pi \mathrm{i}}$.

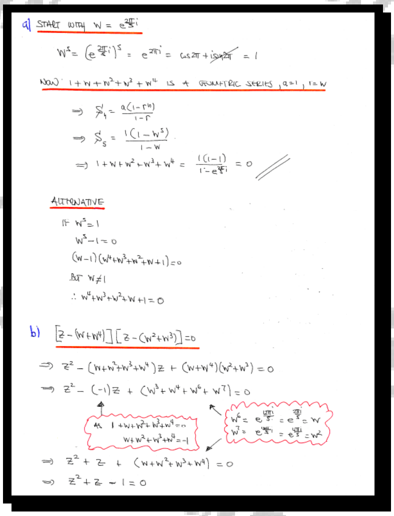
a) Prove that

$$1 + w + w^2 + w^3 + w^4 = 0 .$$

b) Derive a quadratic equation with integer coefficients whose roots are $\left(w + w^4\right)$ and $\left(w^2 + w^3\right)$, and hence show with full justification that

$$\cos\left(\tfrac{2}{5}\pi\right) = \frac{-1+\sqrt{5}}{4} \quad \text{and} \quad \cos\left(\tfrac{4}{5}\pi\right) = \frac{-1-\sqrt{5}}{4} .$$

$\boxed{}$, $\boxed{\text{proof}}$



a) START WITH $w = e^{\frac{2\pi}{5}i}$

$w^5 = \left(e^{\frac{2\pi}{5}i}\right)^5 = e^{2\pi i} = \cos 2\pi + \cancel{i\sin 2\pi} = 1$

NOW: $1 + w + w^2 + w^3 + w^4$ IS A GEOMETRIC SERIES, $a=1$, $r=w$

$\Rightarrow S_n = \frac{a(1-r^n)}{1-r}$

$\Rightarrow S_5 = \frac{1(1-w^5)}{1-w}$

$\Rightarrow 1 + w + w^2 + w^3 + w^4 = \frac{1(1-1)}{1-e^{\frac{2\pi}{5}i}} = 0$ //

ALTERNATIVE

IF $w^5 = 1$

$w^5 - 1 = 0$

$(w-1)(w^4+w^3+w^2+w+1) = 0$

BUT $w \neq 1$

$\therefore w^4+w^3+w^2+w+1 = 0$

b) $\left[z-(w+w^4)\right]\left[z-(w^2+w^3)\right] = 0$

$\Rightarrow z^2 - (w+w^2+w^3+w^4)z + (w+w^4)(w^2+w^3) = 0$

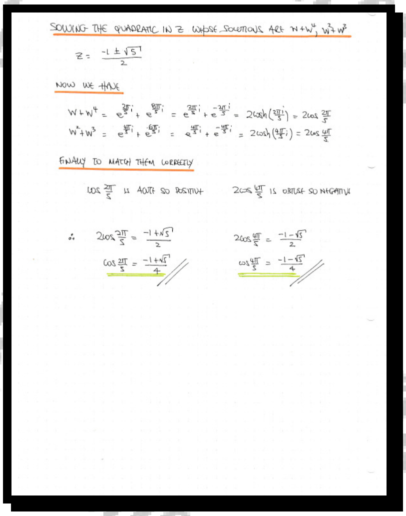
$\Rightarrow z^2 - (-1)z + (w^3+w^4+w^6+w^7) = 0$

AS $1+w+w^2+w^3+w^4 = 0$
$w+w^2+w^3+w^4 = -1$

$w^6 = e^{\frac{12\pi i}{5}} = e^{\frac{2\pi i}{5}} = w$
$w^7 = e^{\frac{14\pi i}{5}} = e^{\frac{4\pi i}{5}} = w^2$

$\Rightarrow z^2 + z + (w+w^2+w^3+w^4) = 0$

$\Rightarrow z^2 + z - 1 = 0$



SOLVING THE QUADRATIC IN z WHOSE SOLUTIONS ARE $w+w^4$, w^2+w^3

$z = \frac{-1 \pm \sqrt{5}}{2}$

NOW WE HAVE

$w+w^4 = e^{\frac{2\pi}{5}i} + e^{\frac{8\pi}{5}i} = e^{\frac{2\pi}{5}i} + e^{-\frac{2\pi}{5}i} = 2\cosh\left(\frac{2\pi i}{5}\right) = 2\cos\frac{2\pi}{5}$

$w^2+w^3 = e^{\frac{4\pi}{5}i} + e^{\frac{6\pi}{5}i} = e^{\frac{4\pi}{5}i} + e^{-\frac{4\pi}{5}i} = 2\cosh\left(\frac{4\pi}{5}i\right) = 2\cos\frac{4\pi}{5}$

FINALLY TO MATCH THEM CORRECTLY

$\cos\frac{2\pi}{5}$ IS ACUTE SO POSITIVE

$2\cos\frac{4\pi}{5}$ IS OBTUSE SO NEGATIVE

$\therefore 2\cos\frac{2\pi}{5} = \frac{-1+\sqrt{5}}{2}$

$\cos\frac{2\pi}{5} = \frac{-1+\sqrt{5}}{4}$ //

$2\cos\frac{4\pi}{5} = \frac{-1-\sqrt{5}}{2}$

$\cos\frac{4\pi}{5} = \frac{-1-\sqrt{5}}{4}$ //

Question 164 (****+)

A complex transformation of points from the z plane onto points in the w plane is defined by the equation

$$w = z^2, \; z \in \mathbb{C}.$$

The point represented by $z = x + \mathrm{i}y$ is mapped onto the point represented by $w = u + \mathrm{i}v$.

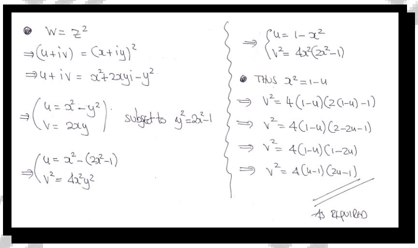
Show that if z traces the curve with Cartesian equation

$$y^2 = 2x^2 - 1,$$

the locus of w satisfies the equation

$$v^2 = 4(u-1)(2u-1).$$

$\boxed{\text{proof}}$



- $W = Z^2$

$\Rightarrow (u+iv) = (x+iy)^2$

$\Rightarrow u+iv = x^2+2xyi-y^2$

$\Rightarrow \begin{pmatrix} u = x^2-y^2 \\ v = 2xy \end{pmatrix}$ subject to $y^2=2x^2-1$

$\Rightarrow \begin{pmatrix} u = x^2-(2x^2-1) \\ v^2 = 4x^2y^2 \end{pmatrix}$

$\Rightarrow \begin{cases} u = 1-x^2 \\ v^2 = 4x^2(2x^2-1) \end{cases}$

- THUS $x^2 = 1-u$

$\Rightarrow v^2 = 4(1-u)(2(1-u)-1)$

$\Rightarrow v^2 = 4(1-u)(2-2u-1)$

$\Rightarrow v^2 = 4(1-u)(1-2u)$

$\Rightarrow v^2 = 4(u-1)(2u-1)$

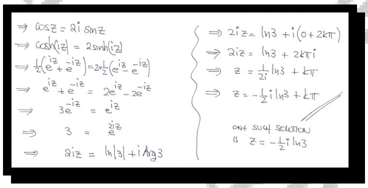
AS REQUIRED

Question 165 (*****)

Find a solution of the equation

$$\cos z = 2\mathrm{i}\sin z, \; z \in \mathbb{C}.$$

$\boxed{z = k\pi - \tfrac{1}{2}\mathrm{i}\ln 3, \; k \in \mathbb{Z}}$



$\Rightarrow \cos z = 2i\sin z$

$\Rightarrow \cosh(iz) = 2\sinh(iz)$

$\Rightarrow \frac{1}{2}(e^{iz}+e^{-iz}) = 2\times\frac{1}{2}(e^{iz}-e^{-iz})$

$\Rightarrow e^{iz}+e^{-iz} = 2e^{iz}-2e^{-iz}$

$\Rightarrow 3e^{-iz} = e^{iz}$

$\Rightarrow 3 = e^{2iz}$

$\Rightarrow 2iz = \ln|3| + i\,\mathrm{Arg}\,3$

$\Rightarrow 2iz = \ln 3 + i(0+2k\pi)$

$\Rightarrow 2iz = \ln 3 + 2k\pi i$

$\Rightarrow z = \frac{1}{2i}\ln 3 + k\pi$

$\Rightarrow z = -\frac{1}{2}i\ln 3 + k\pi$

ONE SUCH SOLUTION IS $z = -\frac{1}{2}i\ln 3$

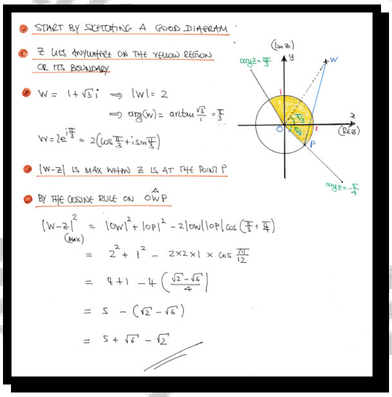
Question 166 (****+)

The complex number z lies in the region R of an Argand diagram, defined by the inequalities

$$-\frac{1}{4}\pi \le \arg z \le \frac{2}{3}\pi \quad \text{and} \quad |z| \le 1 .$$

Determine, in exact surd form, the maximum value of $|w-z|^2$, where $w = 1 + i\sqrt{3}$.

$\boxed{\text{P1}}$, $\boxed{|w-z|^2 = 5+\sqrt{6}-\sqrt{2}}$



- START BY SKETCHING A GOOD DIAGRAM
- Z LIES ANYWHERE ON THE YELLOW REGION OR ITS BOUNDARY
- $W = 1+\sqrt{3}i \Rightarrow |W| = 2$

$\Rightarrow \arg(W) = \arctan\frac{\sqrt{3}}{1} = \frac{\pi}{3}$

$W = 2e^{i\frac{\pi}{3}} = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$

- $|W-Z|$ IS MAX WHEN Z IS AT THE POINT P
- BY THE COSINE RULE ON $\overset{\triangle}{OWP}$

$$\underset{(\text{MAX})}{|W-Z|^2} = |OW|^2 + |OP|^2 - 2|OW||OP|\cos\left(\frac{\pi}{3}+\frac{\pi}{4}\right)$$
$$= 2^2 + 1^2 - 2\times2\times1\times\cos\frac{7\pi}{12}$$
$$= 4+1-4\left(\frac{\sqrt{2}-\sqrt{6}}{4}\right)$$
$$= 5-\left(\sqrt{2}-\sqrt{6}\right)$$
$$= 5+\sqrt{6}-\sqrt{2}$$

Question 167 (****+)

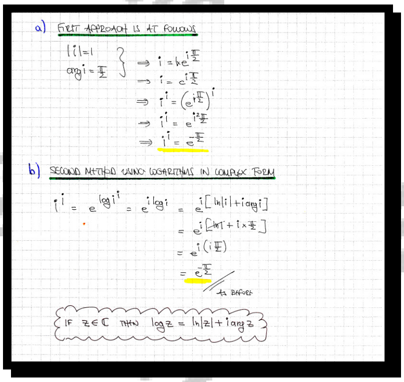
It is required to find the principal value of i^i, in exact simplified form, where i is the imaginary unit.

a) Show, with detailed workings, that

$$i^i = e^{-\frac{1}{2}\pi}.$$

b) Use a different method to that used in part **(a)**, to verify the exact answer given in part **(a)**.

$\boxed{}$, $\boxed{\text{proof}}$



a) FIRST APPROACH IS AS FOLLOWS

$$\left.\begin{array}{l} |i| = 1 \\ \arg i = \frac{\pi}{2} \end{array}\right\} \Rightarrow i = 1e^{i\frac{\pi}{2}}$$

$$\Rightarrow i = e^{i\frac{\pi}{2}}$$

$$\Rightarrow i^i = \left(e^{i\frac{\pi}{2}}\right)^i$$

$$\Rightarrow i^i = e^{i^2\frac{\pi}{2}}$$

$$\Rightarrow i^i = e^{-\frac{\pi}{2}}$$

b) SECOND METHOD USING LOGARITHMS IN COMPLEX FORM

$$i^i = e^{\log i^i} = e^{i\log i} = e^{i[\ln|i| + i\arg i]}$$

$$= e^{i[\cancel{\ln 1} + i \times \frac{\pi}{2}]}$$

$$= e^{i(i\frac{\pi}{2})}$$

$$= e^{-\frac{\pi}{2}}$$

AS BEFORE

IF $z \in \mathbb{C}$ THEN $\log z = \ln|z| + i\arg z$

Question 168 **(*****)**

The finite sum C is given below.

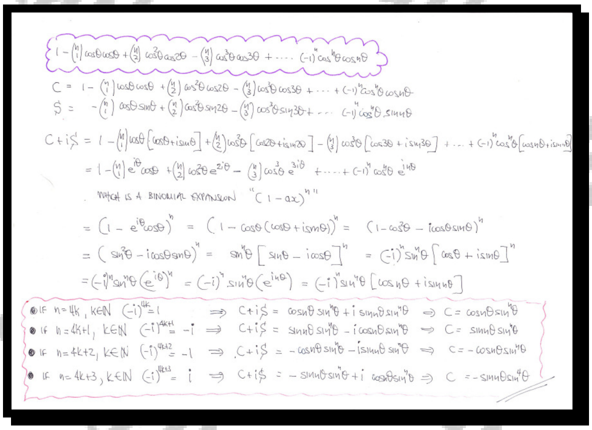
$$C = \sum_{r=0}^{n} \left[\binom{n}{r} (-1)^n \cos^n \theta \cos n\theta \right].$$

Given that $n \in \mathbb{N}$ determine the 4 possible expressions for C.

Give the answers in exact fully simplified form.

$\boxed{\text{SPX-M}}$, $\boxed{n = 4k,\ k \in \mathbb{N} : C = \cos n\theta \sin^n \theta}$, $\boxed{n = 4k+1,\ k \in \mathbb{N} : C = \sin n\theta \sin^n \theta}$,

$\boxed{n = 4k+2,\ k \in \mathbb{N} : C = -\cos n\theta \sin^n \theta}$, $\boxed{n = 4k+3,\ k \in \mathbb{N} : C = -\sin n\theta \sin^n \theta}$



$$1-\binom{n}{1}\cos\theta\cos\theta+\binom{n}{2}\cos^2\theta\cos2\theta-\binom{n}{3}\cos^3\theta\cos3\theta+\dots(-1)^n\cos^n\theta\cos n\theta$$

$$C = 1-\binom{n}{1}\cos\theta\cos\theta+\binom{n}{2}\cos^2\theta\cos2\theta-\binom{n}{3}\cos^3\theta\cos3\theta+\dots+(-1)^n\cos^n\theta\cos n\theta$$

$$S = -\binom{n}{1}\cos\theta\sin\theta+\binom{n}{2}\cos^2\theta\sin2\theta-\binom{n}{3}\cos^3\theta\sin3\theta+\dots(-1)^n\cos^n\theta\sin n\theta$$

$$C+iS = 1-\binom{n}{1}\cos\theta\left[\cos\theta+i\sin\theta\right]+\binom{n}{2}\cos^2\theta\left[\cos2\theta+i\sin2\theta\right]-\binom{n}{3}\cos^3\theta\left[\cos3\theta+i\sin3\theta\right]+\dots+(-1)^n\cos^n\theta\left[\cos n\theta+i\sin n\theta\right]$$

$$= 1-\binom{n}{1}e^{i\theta}\cos\theta+\binom{n}{2}\cos^2\theta\, e^{2i\theta}-\binom{n}{3}\cos^3\theta\, e^{3i\theta}+\dots+(-1)^n\cos^n\theta\, e^{in\theta}$$

WHICH IS A BINOMIAL EXPANSION "$(1-ax)^n$"

$$= \left(1-e^{i\theta}\cos\theta\right)^n = \left(1-\cos\theta(\cos\theta+i\sin\theta)\right)^n = \left(1-\cos^2\theta-i\cos\theta\sin\theta\right)^n$$

$$= \left(\sin^2\theta-i\cos\theta\sin\theta\right)^n = \sin^n\theta\left[\sin\theta-i\cos\theta\right]^n = (-i)^n\sin^n\theta\left[\cos\theta+i\sin\theta\right]^n$$

$$= (-i)^n\sin^n\theta\left(e^{i\theta}\right)^n = (-i)^n\sin^n\theta\left(e^{in\theta}\right) = (-i)^n\sin^n\theta\left[\cos n\theta+i\sin n\theta\right]$$

- IF $n=4k,\ k\in\mathbb{N}\quad (-i)^{4k}=1 \Rightarrow C+iS = \cos n\theta\sin^n\theta+i\sin n\theta\sin^n\theta \Rightarrow C=\cos n\theta\sin^n\theta$
- IF $n=4k+1,\ k\in\mathbb{N}\quad (-i)^{4k+1}=-i \Rightarrow C+iS = \sin n\theta\sin^n\theta-i\cos n\theta\sin^n\theta \Rightarrow C=\sin n\theta\sin^n\theta$
- IF $n=4k+2,\ k\in\mathbb{N}\quad (-i)^{4k+2}=-1 \Rightarrow C+iS = -\cos n\theta\sin^n\theta-i\sin n\theta\sin^n\theta \Rightarrow C=-\cos n\theta\sin^n\theta$
- IF $n=4k+3,\ k\in\mathbb{N}\quad (-i)^{4k+3}=i \Rightarrow C+iS = -\sin n\theta\sin^n\theta+i\cos n\theta\sin^n\theta \Rightarrow C=-\sin n\theta\sin^n\theta$

Question 169 (*****)

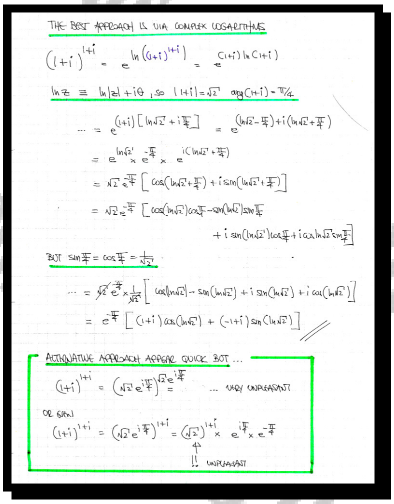
The complex number w is defined as $w = z^z$, where $z = 1 + \mathrm{i}$.

Show, with details workings, that

$$w = \mathrm{e}^{-\frac{1}{4}\pi}\left[(1+\mathrm{i})\cos(\ln k) + (-1+\mathrm{i})\sin(\ln k)\right],$$

where $(1+\mathrm{i})\cos(\ln k)+$ is an exact real constant to be found.

$\boxed{}$, $\boxed{k = \sqrt{2}}$



THE BEST APPROACH IS VIA COMPLEX LOGARITHMS

$$(1+i)^{1+i} = e^{\ln\left((1+i)^{1+i}\right)} = e^{(1+i)\ln(1+i)}$$

$\ln z \equiv \ln|z| + i\theta$, so $|1+i| = \sqrt{2}$ $\arg(1+i) = \pi/4$

$$\dots = e^{(1+i)\left[\ln\sqrt{2} + i\frac{\pi}{4}\right]} = e^{\left(\ln\sqrt{2} - \frac{\pi}{4}\right) + i\left(\ln\sqrt{2} + \frac{\pi}{4}\right)}$$

$$= e^{\ln\sqrt{2}} \times e^{-\frac{\pi}{4}} \times e^{i\left(\ln\sqrt{2} + \frac{\pi}{4}\right)}$$

$$= \sqrt{2}\,e^{-\frac{\pi}{4}}\left[\cos\left(\ln\sqrt{2} + \frac{\pi}{4}\right) + i\sin\left(\ln\sqrt{2} + \frac{\pi}{4}\right)\right]$$

$$= \sqrt{2}\,e^{-\frac{\pi}{4}}\left[\cos(\ln\sqrt{2})\cos\frac{\pi}{4} - \sin(\ln\sqrt{2})\sin\frac{\pi}{4} + i\sin(\ln\sqrt{2})\cos\frac{\pi}{4} + i\cos\ln\sqrt{2}\sin\frac{\pi}{4}\right]$$

BUT $\sin\frac{\pi}{4} = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$

$$\dots = \cancel{\sqrt{2}}\,e^{-\frac{\pi}{4}} \times \frac{1}{\cancel{\sqrt{2}}}\left[\cos(\ln\sqrt{2}) - \sin(\ln\sqrt{2}) + i\sin(\ln\sqrt{2}) + i\cos(\ln\sqrt{2})\right]$$

$$= e^{-\frac{\pi}{4}}\left[(1+i)\cos(\ln\sqrt{2}) + (-1+i)\sin(\ln\sqrt{2})\right] //$$

ALTERNATIVE APPROACH APPEAR QUICK BUT ...

$$(1+i)^{1+i} = \left(\sqrt{2}\,e^{i\frac{\pi}{4}}\right)^{\sqrt{2}e^{i\frac{\pi}{4}}} = \quad \dots \text{ VERY UNPLEASANT}$$

OR EVEN

$$(1+i)^{1+i} = \left(\sqrt{2}\,e^{i\frac{\pi}{4}}\right)^{1+i} = \underset{\substack{\uparrow \\ !!\ \text{UNPLEASANT}}}{\left(\sqrt{2}\right)^{1+i}} \times e^{i\frac{\pi}{4}} \times e^{-\frac{\pi}{4}}$$

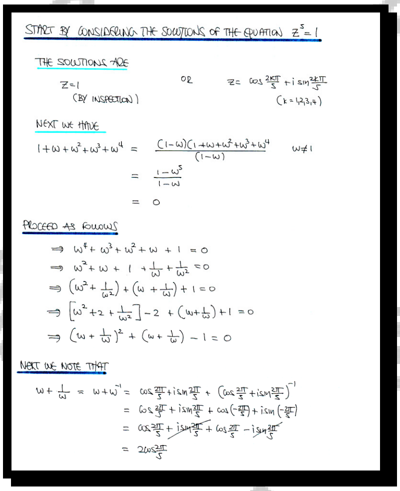
Question 170 (*****)

Use complex numbers to prove that

$$\cos\left(\tfrac{2}{5}\pi\right) = -\tfrac{1}{4} + \tfrac{1}{4}\sqrt{5}$$

A detailed method must support this proof.

☐ , $\boxed{\text{proof}}$



START BY CONSIDERING THE SOLUTIONS OF THE EQUATION $z^5=1$

THE SOLUTIONS ARE

$z=1$ (BY INSPECTION) OR $z=\cos\frac{2k\pi}{5}+i\sin\frac{2k\pi}{5}$ $(k=1,2,3,4)$

NEXT WE HAVE

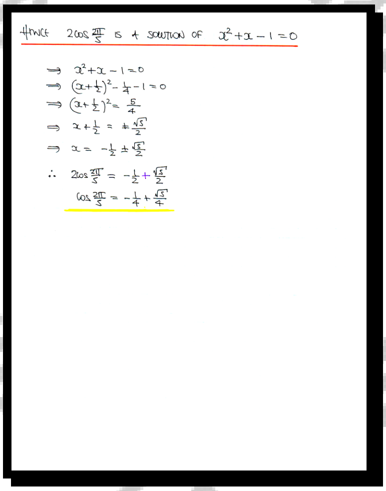
$$1+\omega+\omega^2+\omega^3+\omega^4 = \frac{(1-\omega)(1+\omega+\omega^2+\omega^3+\omega^4}{(1-\omega)} \quad \omega\neq 1$$
$$= \frac{1-\omega^5}{1-\omega}$$
$$= 0$$

PROCEED AS FOLLOWS

$$\Rightarrow \omega^4+\omega^3+\omega^2+\omega+1=0$$
$$\Rightarrow \omega^2+\omega+1+\frac{1}{\omega}+\frac{1}{\omega^2}=0$$
$$\Rightarrow \left(\omega^2+\frac{1}{\omega^2}\right)+\left(\omega+\frac{1}{\omega}\right)+1=0$$
$$\Rightarrow \left[\omega^2+2+\frac{1}{\omega^2}\right]-2+\left(\omega+\frac{1}{\omega}\right)+1=0$$
$$\Rightarrow \left(\omega+\frac{1}{\omega}\right)^2+\left(\omega+\frac{1}{\omega}\right)-1=0$$

NEXT WE NOTE THAT

$$\omega+\frac{1}{\omega} = \omega+\omega^{-1} = \cos\frac{2\pi}{5}+i\sin\frac{2\pi}{5}+\left(\cos\frac{2\pi}{5}+i\sin\frac{2\pi}{5}\right)^{-1}$$
$$= \cos\frac{2\pi}{5}+i\sin\frac{2\pi}{5}+\cos\left(-\frac{2\pi}{5}\right)+i\sin\left(-\frac{2\pi}{5}\right)$$
$$= \cos\frac{2\pi}{5}+\cancel{i\sin\frac{2\pi}{5}}+\cos\frac{2\pi}{5}-\cancel{i\sin\frac{2\pi}{5}}$$
$$= 2\cos\frac{2\pi}{5}$$



HENCE $2\cos\frac{2\pi}{5}$ IS A SOLUTION OF $x^2+x-1=0$

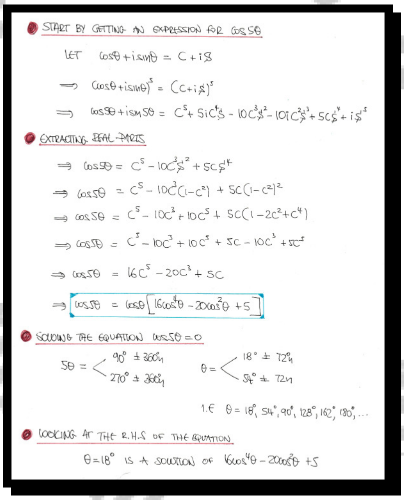
$$\Rightarrow x^2+x-1=0$$
$$\Rightarrow \left(x+\frac{1}{2}\right)^2-\frac{1}{4}-1=0$$
$$\Rightarrow \left(x+\frac{1}{2}\right)^2=\frac{5}{4}$$
$$\Rightarrow x+\frac{1}{2}=\pm\frac{\sqrt{5}}{2}$$
$$\Rightarrow x=-\frac{1}{2}\pm\frac{\sqrt{5}}{2}$$
$$\therefore 2\cos\frac{2\pi}{5}=-\frac{1}{2}+\frac{\sqrt{5}}{2}$$
$$\cos\frac{2\pi}{5}=-\frac{1}{4}+\frac{\sqrt{5}}{4}$$

Question 171 (*****)

Use De Moivre's theorem to find a multiple angle cosine expression and use this expression to show that

$$\cos 36° = \frac{1}{4}\left(1+\sqrt{5}\right).$$

SP-A, proof



START BY GETTING AN EXPRESSION FOR $\cos 5\theta$

LET $\cos\theta + i\sin\theta = C + iS$

$\Rightarrow (\cos\theta + i\sin\theta)^5 = (C+iS)^5$

$\Rightarrow \cos 5\theta + i\sin 5\theta = C^5 + 5iC^4S - 10C^3S^2 - 10iC^2S^3 + 5CS^4 + iS^5$

EXTRACTING REAL PARTS

$\Rightarrow \cos 5\theta = C^5 - 10C^3S^2 + 5CS^4$

$\Rightarrow \cos 5\theta = C^5 - 10C^3(1-C^2) + 5C(1-C^2)^2$

$\Rightarrow \cos 5\theta = C^5 - 10C^3 + 10C^5 + 5C(1-2C^2+C^4)$

$\Rightarrow \cos 5\theta = C^5 - 10C^3 + 10C^5 + 5C - 10C^3 + 5C^5$

$\Rightarrow \cos 5\theta = 16C^5 - 20C^3 + 5C$

$\Rightarrow \boxed{\cos 5\theta = \cos\theta\left[16\cos^4\theta - 20\cos^2\theta + 5\right]}$

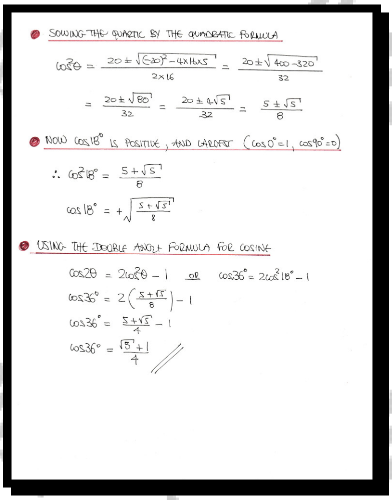
SOLVING THE EQUATION $\cos 5\theta = 0$

$5\theta = \begin{cases} 90° \pm 360n \\ 270° \pm 360n \end{cases} \qquad \theta = \begin{cases} 18° \pm 72n \\ 54° \pm 72n \end{cases}$

I.E $\theta = 18°, 54°, 90°, 128°, 162°, 180°, \ldots$

LOOKING AT THE R.H.S OF THE EQUATION

$\theta = 18°$ IS A SOLUTION OF $16\cos^4\theta - 20\cos^2\theta + 5$



SOLVING THE QUARTIC BY THE QUADRATIC FORMULA

$\cos^2\theta = \frac{20 \pm \sqrt{(-20)^2 - 4\times16\times5}}{2\times16} = \frac{20 \pm \sqrt{400-320}}{32}$

$= \frac{20 \pm \sqrt{80}}{32} = \frac{20 \pm 4\sqrt{5}}{32} = \frac{5 \pm \sqrt{5}}{8}$

NOW $\cos 18°$ IS POSITIVE, AND LARGEST ($\cos 0° = 1$, $\cos 90° = 0$)

$\therefore \cos^2 18° = \frac{5+\sqrt{5}}{8}$

$\cos 18° = +\sqrt{\frac{5+\sqrt{5}}{8}}$

USING THE DOUBLE ANGLE FORMULA FOR COSINE

$\cos 2\theta = 2\cos^2\theta - 1$ OR $\cos 36° = 2\cos^2 18° - 1$

$\cos 36° = 2\left(\frac{5+\sqrt{5}}{8}\right) - 1$

$\cos 36° = \frac{5+\sqrt{5}}{4} - 1$

$\cos 36° = \frac{\sqrt{5}+1}{4}$ //

Question 172 (*****)

$$w = \frac{2 - \mathrm{i}z}{z}, \quad z \in \mathbb{C}, \quad z \neq 0 .$$

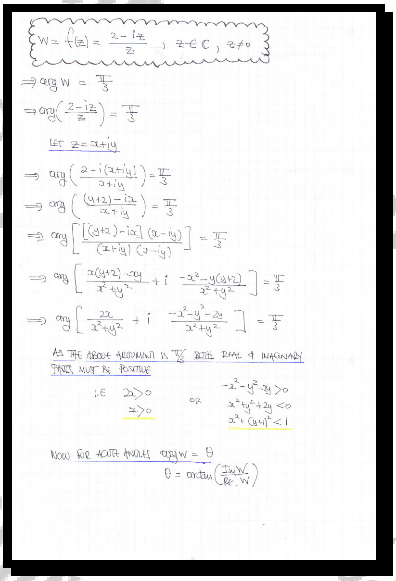
The complex function $w = f(z)$, maps the point $P(x, y)$ from the z complex plane onto the point $Q(u, v)$ on the w complex plane.

The curve C in the z complex plane is mapped in the w complex plane onto the curve with equation

$$\arg w = \frac{1}{3}\pi .$$

Determine a Cartesian equation of C, and hence find an exact simplified value for the area of the finite region bounded by C, and the y axis.

$$\boxed{}, \boxed{\left(x+\sqrt{3}\right)^2 + (y+1)^2 = 4 \quad \cup \quad x > 0}, \boxed{\frac{2}{3}\pi - \sqrt{3}}$$



$$\boxed{w = f(z) = \frac{2-iz}{z}, \quad z \in \mathbb{C}, \quad z \neq 0}$$

$\Rightarrow \arg w = \frac{\pi}{3}$

$\Rightarrow \arg\left(\frac{2-iz}{z}\right) = \frac{\pi}{3}$

LET $z = x + iy$

$\Rightarrow \arg\left(\frac{2-i(x+iy)}{x+iy}\right) = \frac{\pi}{3}$

$\Rightarrow \arg\left(\frac{(y+2)-ix}{x+iy}\right) = \frac{\pi}{3}$

$\Rightarrow \arg\left[\frac{[(y+2)-ix](x-iy)}{(x+iy)(x-iy)}\right] = \frac{\pi}{3}$

$\Rightarrow \arg\left[\frac{x(y+2)-xy}{x^2+y^2} + i\,\frac{-x^2-y(y+2)}{x^2+y^2}\right] = \frac{\pi}{3}$

$\Rightarrow \arg\left[\frac{2x}{x^2+y^2} + i\,\frac{-x^2-y^2-2y}{x^2+y^2}\right] = \frac{\pi}{3}$

AS THE ABOVE ARGUMENT IS $\pi/3$ BOTH REAL & IMAGINARY PARTS MUST BE POSITIVE

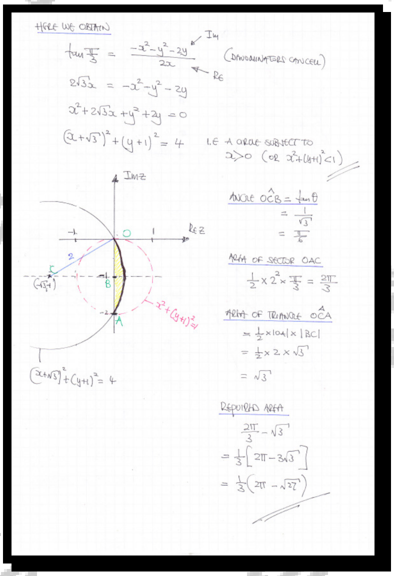
I.E $2x > 0$ OR $-x^2-y^2-2y > 0$

$x > 0$ $\qquad$ $x^2+y^2+2y < 0$

$x^2+(y+1)^2 < 1$

NOW FOR ACUTE ANGLES $\arg w = \theta$

$\theta = \arctan\left(\frac{\text{Im } w}{\text{Re. } w}\right)$



HERE WE OBTAIN

$\tan\frac{\pi}{3} = \frac{-x^2-y^2-2y}{2x}$ (DENOMINATORS CANCEL)

Im

Re

$2\sqrt{3}x = -x^2-y^2-2y$

$x^2+2\sqrt{3}x+y^2+2y = 0$

$(x+\sqrt{3})^2+(y+1)^2 = 4$ I.E A CIRCLE SUBJECT TO $x > 0$ (OR $x^2+(y+1)^2<1$)

ANGLE $O\hat{C}B = \tan\theta$

$= \frac{1}{\sqrt{3}}$

$= \frac{\pi}{6}$

AREA OF SECTOR OAC

$\frac{1}{2} \times 2^2 \times \frac{\pi}{3} = \frac{2\pi}{3}$

AREA OF TRIANGLE $O\hat{C}A$

$= \frac{1}{2} \times |OA| \times |BC|$

$= \frac{1}{2} \times 2 \times \sqrt{3}$

$= \sqrt{3}$

REQUIRED AREA

$\frac{2\pi}{3} - \sqrt{3}$

$= \frac{1}{3}\left[2\pi - 3\sqrt{3}\right]$

$= \frac{1}{3}\left(2\pi - \sqrt{27}\right)$

Question 173 (*****)

a) Show that

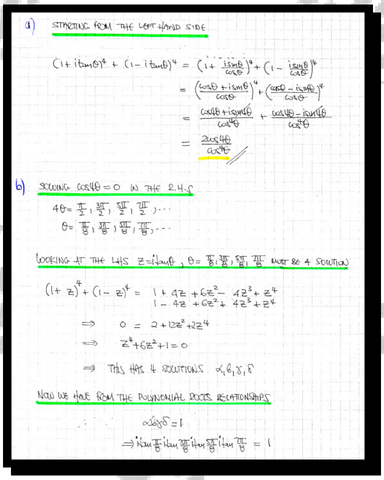
$$\left(1+\mathrm{i}\tan\theta\right)^4+\left(1-\mathrm{i}\tan\theta\right)^4 \equiv \frac{2\cos 4\theta}{\cos^4\theta}$$

b) By considering a suitable polynomial equation based on the result of part **(a)** show further

i. $\tan^2\left(\frac{1}{8}\pi\right)\tan^2\left(\frac{3}{8}\pi\right)=1$

ii. $\tan^2\left(\frac{1}{8}\pi\right)+\tan^2\left(\frac{3}{8}\pi\right)=6$

[illegible], proof



a) STARTING FROM THE LEFT HAND SIDE

$$(1+\mathrm{i}\tan\theta)^4+(1-\mathrm{i}\tan\theta)^4=\left(1+\frac{\mathrm{i}\sin\theta}{\cos\theta}\right)^4+\left(1-\frac{\mathrm{i}\sin\theta}{\cos\theta}\right)^4$$
$$=\left(\frac{\cos\theta+\mathrm{i}\sin\theta}{\cos\theta}\right)^4+\left(\frac{\cos\theta-\mathrm{i}\sin\theta}{\cos\theta}\right)^4$$
$$=\frac{\cos4\theta+\mathrm{i}\sin4\theta}{\cos^4\theta}+\frac{\cos4\theta-\mathrm{i}\sin4\theta}{\cos^4\theta}$$
$$=\frac{2\cos4\theta}{\cos^4\theta}$$

b) SOLVING $\cos4\theta=0$ IN THE R.H.S

$$4\theta=\frac{\pi}{2},\frac{3\pi}{2},\frac{5\pi}{2},\frac{7\pi}{2},\dots$$
$$\theta=\frac{\pi}{8},\frac{3\pi}{8},\frac{5\pi}{8},\frac{7\pi}{8},\dots$$

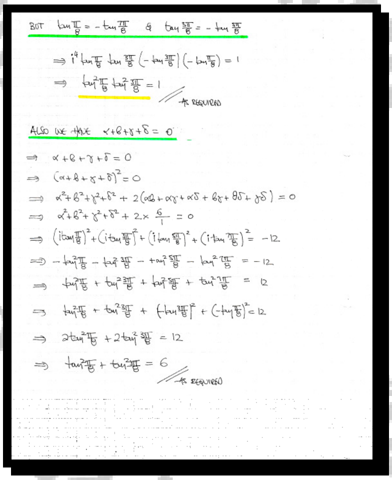
LOOKING AT THE LHS $z=\mathrm{i}\tan\theta$, $\theta=\frac{\pi}{8},\frac{3\pi}{8},\frac{5\pi}{8},\frac{7\pi}{8}$ MUST BE 4 SOLUTION

$$(1+z)^4+(1-z)^4=\begin{matrix}1+4z+6z^2-4z^3+z^4\\1-4z+6z^2+4z^3+z^4\end{matrix}$$
$$\Rightarrow\quad 0=2+12z^2+2z^4$$
$$\Rightarrow\quad z^4+6z^2+1=0$$

$\Rightarrow$ THIS HAS 4 SOLUTIONS $\alpha,\beta,\gamma,\delta$

NOW WE HAVE FROM THE POLYNOMIAL ROOTS RELATIONSHIPS

$$\alpha\beta\gamma\delta=1$$
$$\Rightarrow \mathrm{i}\tan\frac{\pi}{8}\,\mathrm{i}\tan\frac{3\pi}{8}\,\mathrm{i}\tan\frac{5\pi}{8}\,\mathrm{i}\tan\frac{7\pi}{8}=1$$



BUT $\tan\frac{\pi}{8}=-\tan\frac{7\pi}{8}$ & $\tan\frac{3\pi}{8}=-\tan\frac{5\pi}{8}$

$$\Rightarrow \mathrm{i}^4\tan\frac{\pi}{8}\tan\frac{3\pi}{8}\left(-\tan\frac{3\pi}{8}\right)\left(-\tan\frac{\pi}{8}\right)=1$$
$$\Rightarrow \tan^2\frac{\pi}{8}\tan^2\frac{3\pi}{8}=1$$

AS REQUIRED

ALSO WE HAVE $\alpha+\beta+\gamma+\delta=0$

$$\Rightarrow \alpha+\beta+\gamma+\delta=0$$
$$\Rightarrow (\alpha+\beta+\gamma+\delta)^2=0$$
$$\Rightarrow \alpha^2+\beta^2+\gamma^2+\delta^2+2(\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\theta\delta+\gamma\delta)=0$$
$$\Rightarrow \alpha^2+\beta^2+\gamma^2+\delta^2+2\times\frac{6}{1}=0$$
$$\Rightarrow \left(\mathrm{i}\tan\frac{\pi}{8}\right)^2+\left(\mathrm{i}\tan\frac{3\pi}{8}\right)^2+\left(\mathrm{i}\tan\frac{5\pi}{8}\right)^2+\left(\mathrm{i}\tan\frac{7\pi}{8}\right)^2=-12$$
$$\Rightarrow -\tan^2\frac{\pi}{8}-\tan^2\frac{3\pi}{8}-\tan^2\frac{5\pi}{8}-\tan^2\frac{7\pi}{8}=-12$$
$$\Rightarrow \tan^2\frac{\pi}{8}+\tan^2\frac{3\pi}{8}+\tan^2\frac{5\pi}{8}+\tan^2\frac{7\pi}{8}=12$$
$$\Rightarrow \tan^2\frac{\pi}{8}+\tan^2\frac{3\pi}{8}+\left(-\tan\frac{3\pi}{8}\right)^2+\left(-\tan\frac{\pi}{8}\right)^2=12$$
$$\Rightarrow 2\tan^2\frac{\pi}{8}+2\tan^2\frac{3\pi}{8}=12$$
$$\Rightarrow \tan^2\frac{\pi}{8}+\tan^2\frac{3\pi}{8}=6$$

AS REQUIRED

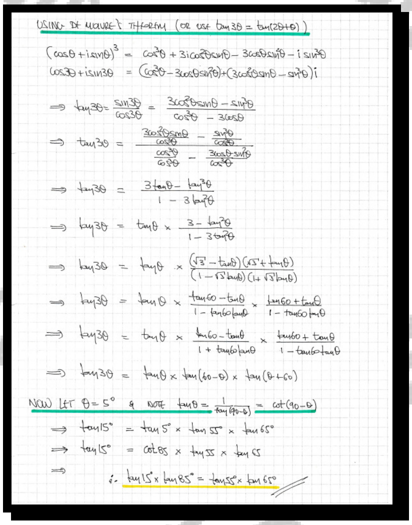
Question 174 (*****)

$$\tan(3\theta°) \equiv \tan(\theta°)\times\tan(60°-\theta°)\times\tan(60°+\theta°)$$

Prove the validity of the above trigonometric identity and hence show that

$$\tan 15°\times\tan 85° \;=\; \tan 55°\times\tan 65° .$$

$\boxed{\quad\quad}$, $\boxed{\text{proof}}$



USING DE MOIVRE'S THEOREM (OR USE $\tan 3\theta = \tan(2\theta+\theta)$)

$$(\cos\theta + i\sin\theta)^3 = \cos^3\theta + 3i\cos^2\theta\sin\theta - 3\cos\theta\sin^2\theta - i\sin^3\theta$$

$$\cos 3\theta + i\sin 3\theta = (\cos^3\theta - 3\cos\theta\sin^2\theta) + (3\cos^2\theta\sin\theta - \sin^3\theta)i$$

$$\Rightarrow \tan 3\theta = \frac{\sin 3\theta}{\cos 3\theta} = \frac{3\cos^2\theta\sin\theta - \sin^3\theta}{\cos^3\theta - 3\cos\theta}$$

$$\Rightarrow \tan 3\theta = \frac{\frac{3\cos^2\theta\sin\theta}{\cos^3\theta} - \frac{\sin^3\theta}{\cos^3\theta}}{\frac{\cos^3\theta}{\cos^3\theta} - \frac{3\cos\theta\sin^2\theta}{\cos^3\theta}}$$

$$\Rightarrow \tan 3\theta = \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$$

$$\Rightarrow \tan 3\theta = \tan\theta \times \frac{3 - \tan^2\theta}{1 - 3\tan^2\theta}$$

$$\Rightarrow \tan 3\theta = \tan\theta \times \frac{(\sqrt{3} - \tan\theta)(\sqrt{3} + \tan\theta)}{(1 - \sqrt{3}\tan\theta)(1 + \sqrt{3}\tan\theta)}$$

$$\Rightarrow \tan 3\theta = \tan\theta \times \frac{\tan 60 - \tan\theta}{1 - \tan 60\tan\theta} \times \frac{\tan 60 + \tan\theta}{1 - \tan 60\tan\theta}$$

$$\Rightarrow \tan 3\theta = \tan\theta \times \frac{\tan 60 - \tan\theta}{1 + \tan 60\tan\theta} \times \frac{\tan 60 + \tan\theta}{1 - \tan 60\tan\theta}$$

$$\Rightarrow \tan 3\theta = \tan\theta \times \tan(60 - \theta) \times \tan(\theta + 60)$$

NOW LET $\theta = 5°$ & NOTE $\tan\theta = \frac{1}{\tan(90-\theta)} = \cot(90-\theta)$

$$\Rightarrow \tan 15° = \tan 5° \times \tan 55° \times \tan 65°$$

$$\Rightarrow \tan 15° = \cot 85 \times \tan 55 \times \tan 65$$

$$\Rightarrow$$

$$\therefore \tan 15° \times \tan 85° = \tan 55° \times \tan 65°$$

Question 175 (*****)

$$I = \int \cos(\ln x)\,dx \quad \text{and} \quad J = \int \sin(\ln x)\,dx$$

a) Use an appropriate method to find expressions for I and J.

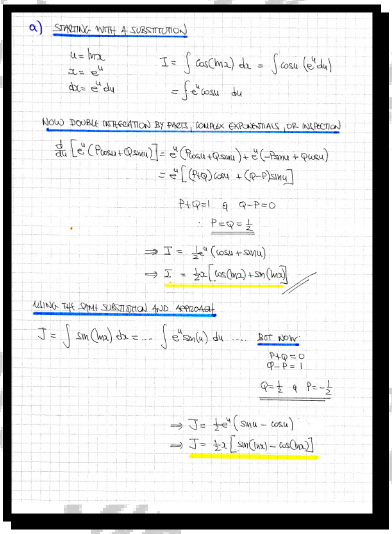
b) Use the integral $\int x^{\mathrm{i}}\,dx$, where i is the imaginary unit, to verify the answers given in part (**a**).

c) Find an exact simplified value for

$$\int_1^{e^{\frac{\pi}{2}}} 2x^{\mathrm{i}}\,dx.$$

$\boxed{}$, $\boxed{I = \frac{1}{2}x\left[\sin(\ln x) + \cos(\ln x)\right]}$, $\boxed{J = \frac{1}{2}x\left[\sin(\ln x) - \cos(\ln x)\right]}$,

$$\boxed{\int_1^{e^{\frac{\pi}{2}}} 2x^{\mathrm{i}}\,dx = \left(e^{\frac{1}{2}\pi} - 1\right) + \left(e^{\frac{1}{2}\pi} + 1\right)\mathrm{i}}$$



a) STARTING WITH A SUBSTITUTION

$u = \ln x$

$x = e^u$

$dx = e^u\,du$

$I = \int \cos(\ln x)\,dx = \int \cos u\,(e^u du)$

$= \int e^u \cos u \; du$

NOW DOUBLE INTEGRATION BY PARTS, COMPLEX EXPONENTIALS, OR INSPECTION

$\frac{d}{du}\left[e^u(P\cos u + Q\sin u)\right] = e^u(P\cos u + Q\sin u) + e^u(-P\sin u + Q\cos u)$

$= e^u\left[(P+Q)\cos u + (Q-P)\sin u\right]$

$P+Q=1$ & $Q-P=0$

$\therefore \; P = Q = \frac{1}{2}$

$\Rightarrow I = \frac{1}{2}e^u(\cos u + \sin u)$

$\Rightarrow I = \frac{1}{2}x\left[\cos(\ln x) + \sin(\ln x)\right]$

USING THE SAME SUBSTITUTION AND APPROACH

$J = \int \sin(\ln x)\,dx = \dots \int e^u \sin(u)\,du \; \dots$ BUT NOW

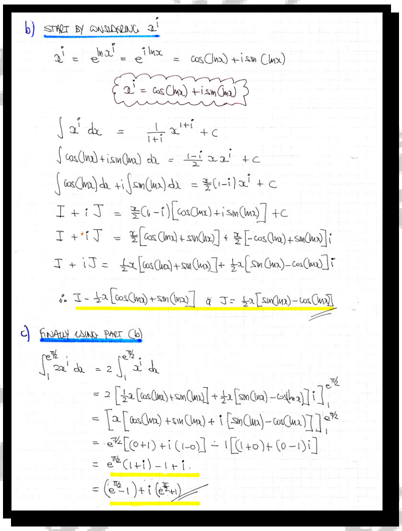
$P+Q=0$

$Q-P=1$

$Q = \frac{1}{2}$ & $P = -\frac{1}{2}$

$\Rightarrow J = \frac{1}{2}e^u(\sin u - \cos u)$

$\Rightarrow J = \frac{1}{2}x\left[\sin(\ln x) - \cos(\ln x)\right]$



b) START BY CONSIDERING x^i

$x^i = e^{\ln x^i} = e^{i\ln x} = \cos(\ln x) + i\sin(\ln x)$

$x^i = \cos(\ln x) + i\sin(\ln x)$

$\int x^i\,dx = \frac{1}{1+i}x^{1+i} + C$

$\int \cos(\ln x) + i\sin(\ln x)\,dx = \frac{1-i}{2}\,x\,x^i + C$

$\int \cos(\ln x)\,dx + i\int \sin(\ln x)\,dx = \frac{x}{2}(1-i)x^i + C$

$I + iJ = \frac{x}{2}(1-i)\left[\cos(\ln x) + i\sin(\ln x)\right] + C$

$I + iJ = \frac{x}{2}\left[\cos(\ln x) + \sin(\ln x)\right] + \frac{x}{2}\left[-\cos(\ln x) + \sin(\ln x)\right]i$

$I + iJ = \frac{1}{2}x\left[\cos(\ln x) + \sin(\ln x)\right] + \frac{1}{2}x\left[\sin(\ln x) - \cos(\ln x)\right]i$

$\therefore \; I - \frac{1}{2}x\left[\cos(\ln x) + \sin(\ln x)\right]$ & $J = \frac{1}{2}x\left[\sin(\ln x) - \cos(\ln x)\right]$

c) FINALLY USING PART (b)

$\int_1^{e^{\pi/2}} 2x^i\,dx = 2\int_1^{e^{\pi/2}} x^i\,dx$

$= 2\left[\frac{1}{2}x\left[\cos(\ln x) + \sin(\ln x)\right] + \frac{1}{2}x\left[\sin(\ln x) - \cos(\ln x)\right]i\right]_1^{e^{\pi/2}}$

$= \left[x\left[\cos(\ln x) + \sin(\ln x) + i\left[\sin(\ln x) - \cos(\ln x)\right]\right]\right]_1^{e^{\pi/2}}$

$= e^{\pi/2}\left[(0+1) + i(1-0)\right] - 1\left[(1+0) + (0-1)i\right]$

$= e^{\pi/2}(1+i) - 1 + i.$

$= \left(e^{\pi/2} - 1\right) + i\left(e^{\frac{\pi}{2}} + 1\right)$

Question 176 (*****)

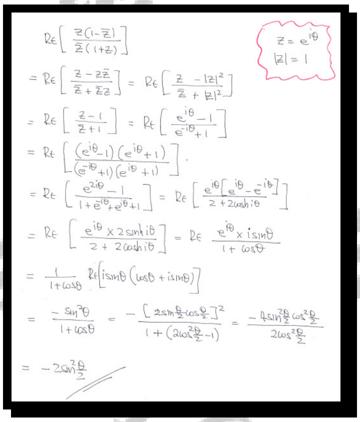
The complex number z has unit modulus and $\arg z = \theta$, $-\pi < \theta \le \pi$.

The complex conjugate of z is denoted by $\bar{z}$.

Using a detailed method, show that

$$\mathrm{Re}\left[\frac{z(1-\bar{z})}{\bar{z}(1+z)}\right] = -2\sin\left(\tfrac{1}{2}\theta\right).$$

$\boxed{}$, $\boxed{\text{proof}}$



$$\mathrm{Re}\left[\frac{z(1-\bar{z})}{\bar{z}(1+z)}\right]$$

$$z = e^{i\theta}$$
$$|z| = 1$$

$$= \mathrm{Re}\left[\frac{z - z\bar{z}}{\bar{z} + \bar{z}z}\right] = \mathrm{Re}\left[\frac{z \;- |z|^2}{\bar{z} + |z|^2}\right]$$

$$= \mathrm{Re}\left[\frac{z-1}{\bar{z}+1}\right] = \mathrm{Re}\left[\frac{e^{i\theta}-1}{e^{-i\theta}+1}\right]$$

$$= \mathrm{Re}\left[\frac{(e^{i\theta}-1)(e^{i\theta}+1)}{(e^{-i\theta}+1)(e^{i\theta}+1)}\right].$$

$$= \mathrm{Re}\left[\frac{e^{2i\theta}-1}{1+e^{-i\theta}+e^{i\theta}+1}\right] = \mathrm{Re}\left[\frac{e^{i\theta}\left[e^{i\theta}-e^{-i\theta}\right]}{2+2\cosh i\theta}\right]$$

$$= \mathrm{Re}\left[\frac{e^{i\theta} \times 2\sinh i\theta}{2+2\cosh i\theta}\right] = \mathrm{Re}\ \frac{e^{i\theta} \times i\sin\theta}{1+\cos\theta}$$

$$= \frac{1}{1+\cos\theta}\,\mathrm{Re}\left[i\sin\theta\,(\cos\theta + i\sin\theta)\right]$$

$$= \frac{-\sin^2\theta}{1+\cos\theta} = -\frac{\left[2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right]^2}{1+\left(2\cos^2\frac{\theta}{2}-1\right)} = \frac{-4\sin^2\frac{\theta}{2}\cos^2\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}}$$

$$= -2\sin^2\frac{\theta}{2}$$

Question 177 (*****)

The complex number $z = z_1 + z_2$ where

$$z_1 = 3 + 4\mathrm{i} \quad \text{and} \quad z_2 = 4\mathrm{e}^{\mathrm{i}\theta}, \ -\pi < \theta \le \pi$$

a) Sketch in an Argand diagram the locus of z.

The complex number z_3 lies on the locus of z such that the argument of z_3 takes its maximum value.

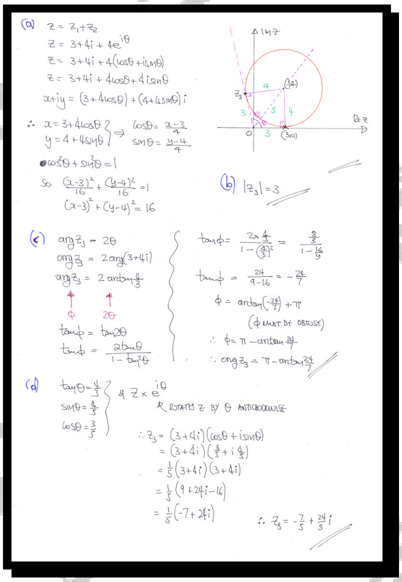
b) State the value of $|z_3|$.

c) Show clearly that

$$\arg z_3 = \pi - \arctan\frac{24}{7}.$$

d) Find z_3 in the form $x + \mathrm{i}y$.

$\boxed{}$, $\boxed{|z_3| = 3}$, $\boxed{|z|_{\text{max}} = 3}$, $\boxed{z_3 = -\frac{7}{5} + \frac{24}{5}\mathrm{i}}$



(a) $z = z_1 + z_2$

$z = 3 + 4i + 4e^{i\theta}$

$z = 3 + 4i + 4(\cos\theta + i\sin\theta)$

$z = 3 + 4i + 4\cos\theta + 4i\sin\theta$

$x + iy = (3 + 4\cos\theta) + (4 + 4\sin\theta)i$

$\therefore \left.\begin{array}{l} x = 3 + 4\cos\theta \\ y = 4 + 4\sin\theta \end{array}\right\} \Rightarrow \begin{array}{l} \cos\theta = \frac{x-3}{4} \\ \sin\theta = \frac{y-4}{4} \end{array}$

$\cos^2\theta + \sin^2\theta = 1$

So $\frac{(x-3)^2}{16} + \frac{(y-4)^2}{16} = 1$

$(x-3)^2 + (y-4)^2 = 16$

(b) $|z_3| = 3$

(c) $\arg z_3 = 2\theta$

$\arg z_3 = 2\arg(3+4i)$

$\arg z_3 = 2\arctan\frac{4}{3}$

$\uparrow \phi \qquad \uparrow 2\theta$

$\tan\phi = \tan 2\theta$

$\tan\phi = \frac{2\tan\theta}{1 - \tan^2\theta}$

$\tan\phi = \frac{2 \times \frac{4}{3}}{1 - \left(\frac{4}{3}\right)^2} = \frac{\frac{8}{3}}{1 - \frac{16}{9}}$

$\tan\phi = \frac{24}{9-16} = -\frac{24}{7}$

$\phi = \arctan\left(-\frac{24}{7}\right) + \pi$

(ϕ MUST BE OBTUSE)

$\therefore \phi = \pi - \arctan\frac{24}{7}$

$\therefore \arg z_3 = \pi - \arctan\frac{24}{7}$

(d) $\left.\begin{array}{l} \tan\theta = \frac{4}{3} \\ \sin\theta = \frac{4}{5} \\ \cos\theta = \frac{3}{5} \end{array}\right\}$ a $z \times e^{i\theta}$

ROTATES z BY θ ANTICLOCKWISE

$\therefore z_3 = (3+4i)(\cos\theta + i\sin\theta)$

$= (3+4i)\left(\frac{3}{5} + i\frac{4}{5}\right)$

$= \frac{1}{5}(3+4i)(3+4i)$

$= \frac{1}{5}(9 + 24i - 16)$

$= \frac{1}{5}(-7 + 24i)$

$\therefore z_3 = -\frac{7}{5} + \frac{24}{5}i$

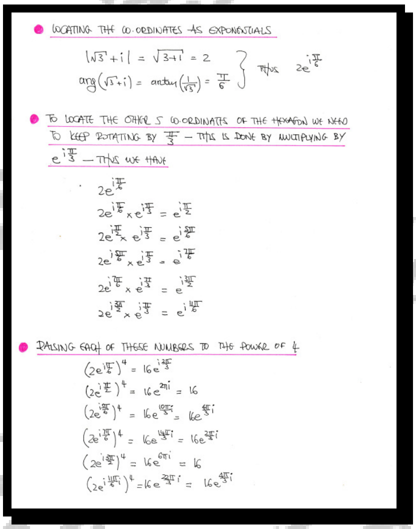
Question 178 (*****)

In a standard Argand diagram the complex number $\sqrt{3}+\mathrm{i}$, represents one of the vertices of a regular hexagon, with centre at the origin O.

The complex numbers that represent these 6 vertices are all raised to the power of 4, creating a closed shape S, whose sides are straight line segments.

Determine the area of S.

$\boxed{\phantom{192\sqrt{3}}}$, $\boxed{\text{proof}}$



- LOCATING THE CO-ORDINATES AS EXPONENTIALS

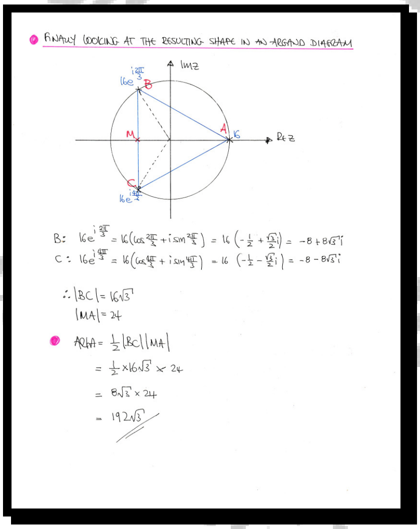
$$\left.\begin{array}{l} |\sqrt{3}+i| = \sqrt{3+1} = 2 \\ \arg(\sqrt{3}+i) = \arctan\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6} \end{array}\right\} \text{ THUS } 2e^{i\frac{\pi}{6}}$$

- TO LOCATE THE OTHER 5 CO-ORDINATES OF THE HEXAGON WE NEED TO KEEP ROTATING BY $\frac{\pi}{3}$ – THIS IS DONE BY MULTIPLYING BY $e^{i\frac{\pi}{3}}$ – THUS WE HAVE

$$2e^{i\frac{\pi}{6}}$$
$$2e^{i\frac{\pi}{6}} \times e^{i\frac{\pi}{3}} = e^{i\frac{\pi}{2}}$$
$$2e^{i\frac{\pi}{2}} \times e^{i\frac{\pi}{3}} = e^{i\frac{5\pi}{6}}$$
$$2e^{i\frac{5\pi}{6}} \times e^{i\frac{\pi}{3}} = e^{i\frac{7\pi}{6}}$$
$$2e^{i\frac{7\pi}{6}} \times e^{i\frac{\pi}{3}} = e^{i\frac{3\pi}{2}}$$
$$2e^{i\frac{3\pi}{2}} \times e^{i\frac{\pi}{3}} = e^{i\frac{11\pi}{6}}$$

- RAISING EACH OF THESE NUMBERS TO THE POWER OF 4

$$\left(2e^{i\frac{\pi}{6}}\right)^4 = 16e^{i\frac{2\pi}{3}}$$
$$\left(2e^{i\frac{\pi}{2}}\right)^4 = 16e^{2\pi i} = 16$$
$$\left(2e^{i\frac{5\pi}{6}}\right)^4 = 16e^{\frac{10\pi}{3}i} = 16e^{\frac{4\pi}{3}i}$$
$$\left(2e^{i\frac{7\pi}{6}}\right)^4 = 16e^{\frac{14\pi}{3}i} = 16e^{\frac{2\pi}{3}i}$$
$$\left(2e^{i\frac{3\pi}{2}}\right)^4 = 16e^{6\pi i} = 16$$
$$\left(2e^{i\frac{11\pi}{6}i}\right)^4 = 16e^{\frac{22\pi}{3}i} = 16e^{\frac{4\pi}{3}i}$$



- FINALLY LOOKING AT THE RESULTING SHAPE IN AN ARGAND DIAGRAM

$$B: \ 16e^{i\frac{2\pi}{3}} = 16\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right) = 16\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right) = -8+8\sqrt{3}i$$
$$C: \ 16e^{i\frac{4\pi}{3}} = 16\left(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}\right) = 16\left(-\frac{1}{2}-\frac{\sqrt{3}}{2}i\right) = -8-8\sqrt{3}i$$

$$\therefore |BC| = 16\sqrt{3}$$
$$|MA| = 24$$

- $\text{AREA} = \frac{1}{2}|BC||MA|$

$$= \frac{1}{2} \times 16\sqrt{3} \times 24$$
$$= 8\sqrt{3} \times 24$$
$$= 192\sqrt{3}$$

Question 179 (*****)

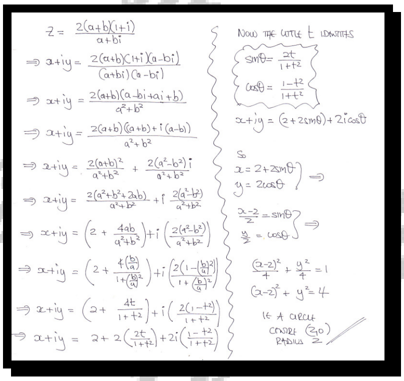
The complex number z is given by

$$z = \frac{2(a+b)(1+\mathrm{i})}{a+b\mathrm{i}}, \ a+b \neq 0,$$

where a and b are real parameters.

Show, that for all allowable values of a and b, the point represented by z is tracing a circle, determining the coordinates of its centre and the size of its radius.

$\boxed{\text{SP}}$, $\boxed{(2,0)}$, $\boxed{r=2}$



$z = \frac{2(a+b)(1+i)}{a+bi}$

$\Rightarrow x+iy = \frac{2(a+b)(1+i)(a-bi)}{(a+bi)(a-bi)}$

$\Rightarrow x+iy = \frac{2(a+b)(a-bi+ai+b)}{a^2+b^2}$

$\Rightarrow x+iy = \frac{2(a+b)((a+b)+i(a-b))}{a^2+b^2}$

$\Rightarrow x+iy = \frac{2(a+b)^2}{a^2+b^2} + \frac{2(a^2-b^2)i}{a^2+b^2}$

$\Rightarrow x+iy = \frac{2(a^2+b^2+2ab)}{a^2+b^2} + i\,\frac{2(a^2-b^2)}{a^2+b^2}$

$\Rightarrow x+iy = \left(2 + \frac{4ab}{a^2+b^2}\right) + i\left(\frac{2(a^2-b^2)}{a^2+b^2}\right)$

$\Rightarrow x+iy = \left(2 + \frac{4\left(\frac{b}{a}\right)}{1+\left(\frac{b}{a}\right)^2}\right) + i\left(\frac{2\left(1-\left(\frac{b}{a}\right)^2\right)}{1+\left(\frac{b}{a}\right)^2}\right)$

$\Rightarrow x+iy = \left(2 + \frac{4t}{1+t^2}\right) + i\left(\frac{2(1-t^2)}{1+t^2}\right)$

$\Rightarrow x+iy = 2 + 2\left(\frac{2t}{1+t^2}\right) + 2i\left(\frac{1-t^2}{1+t^2}\right)$

NOW THE LITTLE t IDENTITIES

$\sin\theta = \frac{2t}{1+t^2}$

$\cos\theta = \frac{1-t^2}{1+t^2}$

$x+iy = (2+2\sin\theta) + 2i\cos\theta$

So

$\left.\begin{array}{l} x = 2+2\sin\theta \\ y = 2\cos\theta \end{array}\right\} \Rightarrow$

$\left.\begin{array}{l} \frac{x-2}{2} = \sin\theta \\ \frac{y}{2} = \cos\theta \end{array}\right\} \Rightarrow$

$\frac{(x-2)^2}{4} + \frac{y^2}{4} = 1$

$(x-2)^2 + y^2 = 4$

IE A CIRCLE
CENTRE $(2,0)$
RADIUS 2

Question 180 (*****)

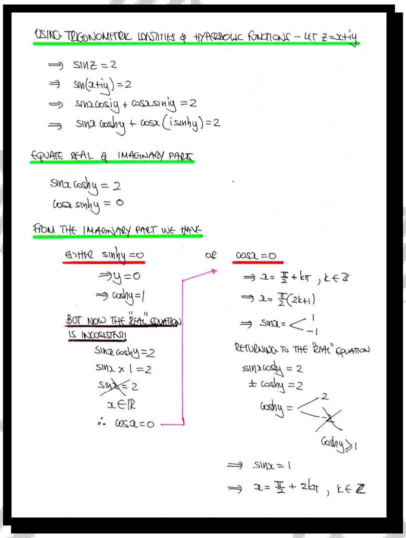
Show clearly that the general solution of the equation

$$\sin z = 2 ,\ z \in \mathbb{C} ,$$

can be written in the form

$$z = \frac{\pi}{2}(4k+1) \pm \mathrm{i}\,\text{arcosh}\,2 ,\ k \in \mathbb{Z} .$$

☐ , $\boxed{\text{proof}}$



USING TRIGONOMETRIC IDENTITIES & HYPERBOLIC FUNCTIONS – LET $z = x + iy$

$\Rightarrow \sin z = 2$

$\Rightarrow \sin(x+iy) = 2$

$\Rightarrow \sin x \cos iy + \cos x \sin iy = 2$

$\Rightarrow \sin x \cosh y + \cos x (i \sinh y) = 2$

EQUATE REAL & IMAGINARY PARTS

$\sin x \cosh y = 2$

$\cos x \sinh y = 0$

FROM THE IMAGINARY PART WE HAVE

EITHER $\sinh y = 0$

$\Rightarrow y = 0$

$\Rightarrow \cosh y = 1$

BUT NOW THE "REAL" EQUATION IS INCONSISTENT!

$\sin x \cosh y = 2$

$\sin x \times 1 = 2$

$\cancel{\sin x = 2}$

$x \in \mathbb{R}$

$\therefore \cos x = 0$

OR $\cos x = 0$

$\Rightarrow x = \frac{\pi}{2} + k\pi ,\ k \in \mathbb{Z}$

$\Rightarrow x = \frac{\pi}{2}(2k+1)$

$\Rightarrow \sin x = <^{\,1}_{\,-1}$

RETURNING TO THE "REAL" EQUATION

$\sin x \cosh y = 2$

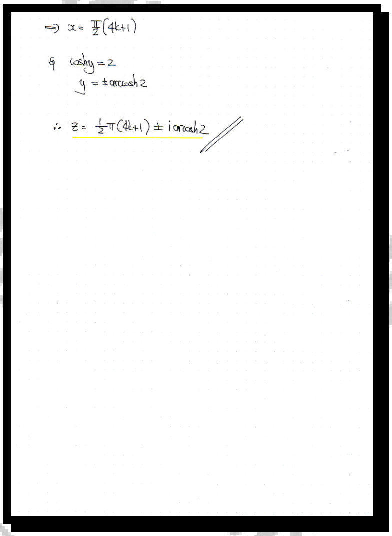
$\pm \cosh y = 2$

$\cosh y = <^{\,2}_{\,\cancel{-2}}$

$\cosh y \geqslant 1$

$\Rightarrow \sin x = 1$

$\Rightarrow x = \frac{\pi}{2} + 2k\pi ,\ k \in \mathbb{Z}$



$\Rightarrow x = \frac{\pi}{2}(4k+1)$

& $\cosh y = 2$

$y = \pm \text{arcosh}\,2$

$\therefore z = \frac{1}{2}\pi(4k+1) \pm i\,\text{arcosh}\,2$ //

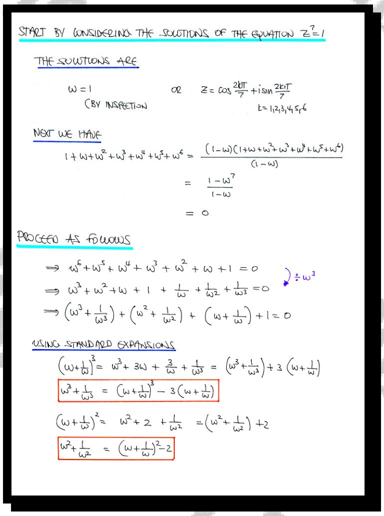
Question 181 (*****)

Use complex numbers to prove that $\cos\left(\frac{2}{7}\pi\right)$ is a solution of the cubic equation

$$x^3 + x^2 - 2x - 1 = 0 .$$

You may **not** use verification in this proof.

proof



START BY CONSIDERING THE SOLUTIONS OF THE EQUATION $z^7=1$

THE SOLUTIONS ARE

$\omega = 1$ (BY INSPECTION OR $z = \cos\frac{2k\pi}{7} + i\sin\frac{2k\pi}{7}$, $k = 1,2,3,4,5,6$

NEXT WE HAVE

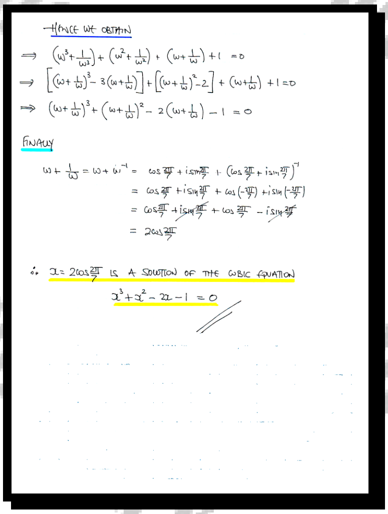
$$1+\omega+\omega^2+\omega^3+\omega^4+\omega^5+\omega^6 = \frac{(1-\omega)(1+\omega+\omega^2+\omega^3+\omega^4+\omega^5+\omega^6)}{(1-\omega)}$$
$$= \frac{1-\omega^7}{1-\omega}$$
$$= 0$$

PROCEED AS FOLLOWS

$$\Rightarrow \omega^6+\omega^5+\omega^4+\omega^3+\omega^2+\omega+1 = 0 \quad \Big) \div \omega^3$$
$$\Rightarrow \omega^3+\omega^2+\omega+1+\frac{1}{\omega}+\frac{1}{\omega^2}+\frac{1}{\omega^3} = 0$$
$$\Rightarrow \left(\omega^3+\frac{1}{\omega^3}\right)+\left(\omega^2+\frac{1}{\omega^2}\right)+\left(\omega+\frac{1}{\omega}\right)+1 = 0$$

USING STANDARD EXPANSIONS

$$\left(\omega+\frac{1}{\omega}\right)^3 = \omega^3+3\omega+\frac{3}{\omega}+\frac{1}{\omega^3} = \left(\omega^3+\frac{1}{\omega^3}\right)+3\left(\omega+\frac{1}{\omega}\right)$$
$$\boxed{\omega^3+\frac{1}{\omega^3} = \left(\omega+\frac{1}{\omega}\right)^3-3\left(\omega+\frac{1}{\omega}\right)}$$
$$\left(\omega+\frac{1}{\omega}\right)^2 = \omega^2+2+\frac{1}{\omega^2} = \left(\omega^2+\frac{1}{\omega^2}\right)+2$$
$$\boxed{\omega^2+\frac{1}{\omega^2} = \left(\omega+\frac{1}{\omega}\right)^2-2}$$



HENCE WE OBTAIN

$$\Rightarrow \left(\omega^3+\frac{1}{\omega^3}\right)+\left(\omega^2+\frac{1}{\omega^2}\right)+\left(\omega+\frac{1}{\omega}\right)+1 = 0$$
$$\Rightarrow \left[\left(\omega+\frac{1}{\omega}\right)^3-3\left(\omega+\frac{1}{\omega}\right)\right]+\left[\left(\omega+\frac{1}{\omega}\right)^2-2\right]+\left(\omega+\frac{1}{\omega}\right)+1 = 0$$
$$\Rightarrow \left(\omega+\frac{1}{\omega}\right)^3+\left(\omega+\frac{1}{\omega}\right)^2-2\left(\omega+\frac{1}{\omega}\right)-1 = 0$$

FINALLY

$$\omega+\frac{1}{\omega} = \omega+\omega^{-1} = \cos\frac{2\pi}{7}+i\sin\frac{2\pi}{7}+\left(\cos\frac{2\pi}{7}+i\sin\frac{2\pi}{7}\right)^{-1}$$
$$= \cos\frac{2\pi}{7}+i\sin\frac{2\pi}{7}+\cos\left(-\frac{2\pi}{7}\right)+i\sin\left(-\frac{2\pi}{7}\right)$$
$$= \cos\frac{2\pi}{7}+\cancel{i\sin\frac{2\pi}{7}}+\cos\frac{2\pi}{7}-\cancel{i\sin\frac{2\pi}{7}}$$
$$= 2\cos\frac{2\pi}{7}$$

$\therefore$ $x = 2\cos\frac{2\pi}{7}$ IS A SOLUTION OF THE CUBIC EQUATION

$$x^3+x^2-2x-1 = 0$$

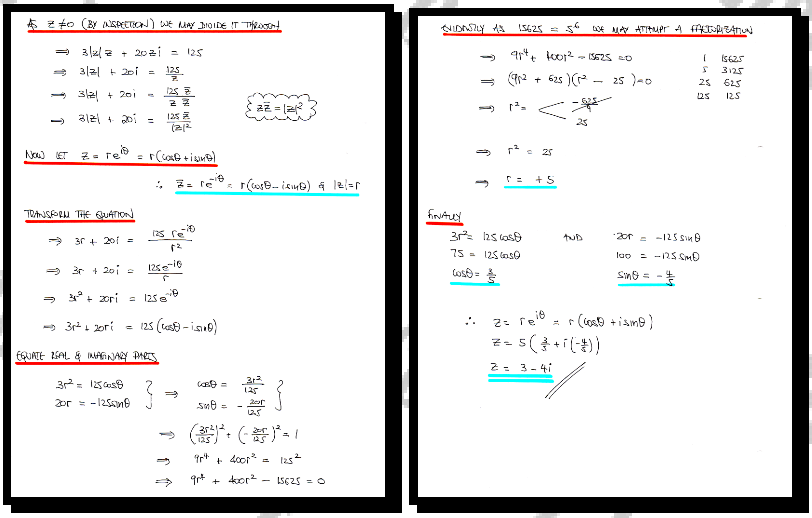
Question 182 (*****)

Solve the following equation

$$3|z|z + 20z\mathrm{i} = 125, \quad z \in \mathbb{C}.$$

Give the answer in the form $x + \mathrm{i}y$, where x and y are real.

$\boxed{}$, $\boxed{z = 3 - 4\mathrm{i}}$



AS $z \neq 0$ (BY INSPECTION) WE MAY DIVIDE IT THROUGH

$$\Rightarrow 3|z|z + 20z\mathrm{i} = 125$$
$$\Rightarrow 3|z| + 20\mathrm{i} = \frac{125}{z}$$
$$\Rightarrow 3|z| + 20\mathrm{i} = \frac{125\,\bar{z}}{z\,\bar{z}}$$
$$\Rightarrow 3|z| + 20\mathrm{i} = \frac{125\,\bar{z}}{|z|^2}$$

$z\bar{z} = |z|^2$

NOW LET $z = re^{i\theta} = r(\cos\theta + i\sin\theta)$

$\therefore \bar{z} = re^{-i\theta} = r(\cos\theta - i\sin\theta)$ & $|z| = r$

TRANSFORM THE EQUATION

$$\Rightarrow 3r + 20i = \frac{125\,re^{-i\theta}}{r^2}$$
$$\Rightarrow 3r + 20i = \frac{125e^{-i\theta}}{r}$$
$$\Rightarrow 3r^2 + 20ri = 125e^{-i\theta}$$
$$\Rightarrow 3r^2 + 20ri = 125(\cos\theta - i\sin\theta)$$

EQUATE REAL & IMAGINARY PARTS

$$\left.\begin{array}{l} 3r^2 = 125\cos\theta \\ 20r = -125\sin\theta \end{array}\right\} \Rightarrow \left.\begin{array}{l} \cos\theta = \frac{3r^2}{125} \\ \sin\theta = -\frac{20r}{125} \end{array}\right\}$$
$$\Rightarrow \left(\frac{3r^2}{125}\right)^2 + \left(-\frac{20r}{125}\right)^2 = 1$$
$$\Rightarrow 9r^4 + 400r^2 = 125^2$$
$$\Rightarrow 9r^4 + 400r^2 - 15625 = 0$$

EVIDENTLY AS $15625 = 5^6$ WE MAY ATTEMPT A FACTORIZATION

$$\Rightarrow 9r^4 + 400r^2 - 15625 = 0$$
$$\Rightarrow (9r^2 + 625)(r^2 - 25) = 0$$
$$\Rightarrow r^2 = \begin{cases} \cancel{-\frac{625}{9}} \\ 25 \end{cases}$$

1	15625
5	3125
25	625
125	125

$$\Rightarrow r^2 = 25$$
$$\Rightarrow r = +5$$

FINALLY

$3r^2 = 125\cos\theta$ AND $20r = -125\sin\theta$

$75 = 125\cos\theta$ $\quad$ $100 = -125\sin\theta$

$\cos\theta = \frac{3}{5}$ $\quad$ $\sin\theta = -\frac{4}{5}$

$$\therefore z = re^{i\theta} = r(\cos\theta + i\sin\theta)$$
$$z = 5\left(\frac{3}{5} + i\left(-\frac{4}{5}\right)\right)$$
$$z = 3 - 4i$$

Question 183 (*****)

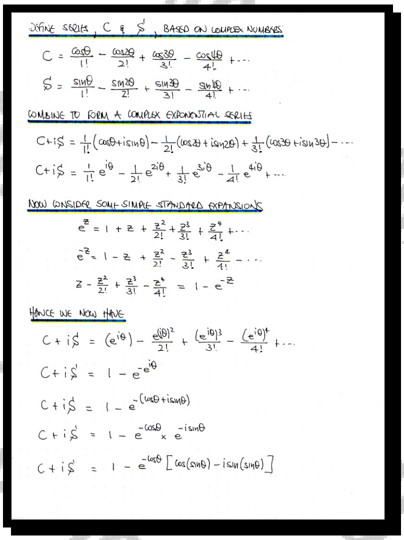
The following convergent series S is given below

$$S = \frac{\sin\theta}{1!} - \frac{\sin 2\theta}{2!} + \frac{\sin 3\theta}{3!} - \frac{\sin 4\theta}{4!} + \dots$$

By considering the sum to infinity of a suitable series involving the complex exponential function, show that

$$S = \mathrm{e}^{-\cos\theta} \sin(\sin\theta).$$

☐ , $\boxed{\text{proof}}$



DEFINE SERIES, C & S, BASED ON COMPLEX NUMBERS

$$C = \frac{\cos\theta}{1!} - \frac{\cos 2\theta}{2!} + \frac{\cos 3\theta}{3!} - \frac{\cos 4\theta}{4!} + \dots$$

$$S = \frac{\sin\theta}{1!} - \frac{\sin 2\theta}{2!} + \frac{\sin 3\theta}{3!} - \frac{\sin 4\theta}{4!} + \dots$$

COMBINE TO FORM A COMPLEX EXPONENTIAL SERIES

$$C + iS = \frac{1}{1!}(\cos\theta + i\sin\theta) - \frac{1}{2!}(\cos 2\theta + i\sin 2\theta) + \frac{1}{3!}(\cos 3\theta + i\sin 3\theta) - \dots$$

$$C + iS = \frac{1}{1!}e^{i\theta} - \frac{1}{2!}e^{2i\theta} + \frac{1}{3!}e^{3i\theta} - \frac{1}{4!}e^{4i\theta} + \dots$$

NOW CONSIDER SOME SIMPLE STANDARD EXPANSIONS

$$e^{z} = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \dots$$

$$e^{-z} = 1 - z + \frac{z^2}{2!} - \frac{z^3}{3!} + \frac{z^4}{4!} - \dots$$

$$z - \frac{z^2}{2!} + \frac{z^3}{3!} - \frac{z^4}{4!} = 1 - e^{-z}$$

HENCE WE NOW HAVE

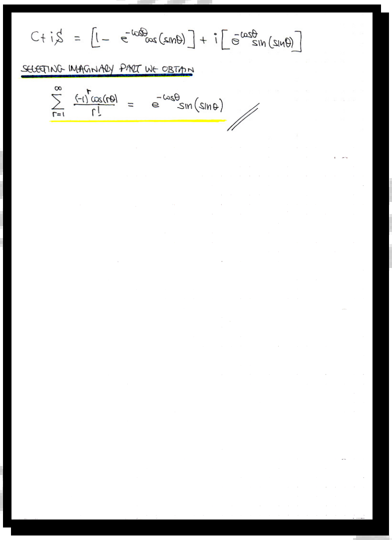
$$C + iS = (e^{i\theta}) - \frac{e^{(i\theta)^2}}{2!} + \frac{(e^{i\theta})^3}{3!} - \frac{(e^{i\theta})^4}{4!} + \dots$$

$$C + iS = 1 - e^{-e^{i\theta}}$$

$$C + iS = 1 - e^{-(\cos\theta + i\sin\theta)}$$

$$C + iS = 1 - e^{-\cos\theta} \times e^{-i\sin\theta}$$

$$C + iS = 1 - e^{-\cos\theta}\left[\cos(\sin\theta) - i\sin(\sin\theta)\right]$$



$$C + iS = \left[1 - e^{-\cos\theta}\cos(\sin\theta)\right] + i\left[e^{-\cos\theta}\sin(\sin\theta)\right]$$

SELECTING IMAGINARY PART WE OBTAIN

$$\sum_{r=1}^{\infty} \frac{(-1)^r \cos(r\theta)}{r!} = e^{-\cos\theta}\sin(\sin\theta) \quad //$$

Question 184 (*****)

The point P in an Argand diagram represents the complex number z, which satisfies

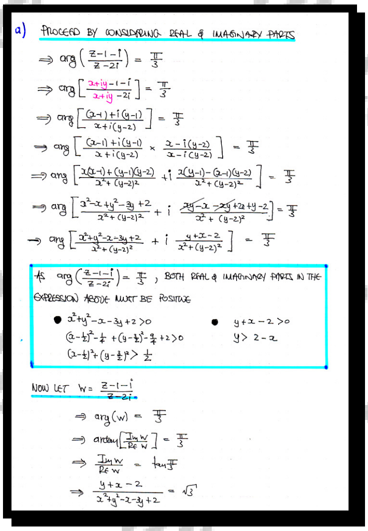
$$\arg\left[\frac{z-1-\mathrm{i}}{z-2\mathrm{i}}\right]=\frac{\pi}{3},\quad z\neq 2\mathrm{i}.$$

It further given that P lies on the arc AB of a circle centred at C and of radius r.

a) Sketch in an Argand diagram the circular arc AB, stating the coordinates of C and the value of r.

b) Given further that $|PA|=|PB|$, find the complex number represented by P.

$\boxed{}$, $\boxed{C\left[\frac{1}{2}\left(1+\frac{1}{3}\sqrt{3}\right),\frac{1}{2}\left(3+\frac{1}{3}\sqrt{3}\right)\right]}$, $\boxed{r=\sqrt{\frac{2}{3}}}$, $\boxed{\frac{1}{2}\left(1+\sqrt{3}\right)+\frac{1}{2}\left(3+\sqrt{3}\right)\mathrm{i}}$



a) PROCEED BY CONSIDERING REAL & IMAGINARY PARTS

$$\Rightarrow \arg\left(\frac{z-1-i}{z-2i}\right)=\frac{\pi}{3}$$

$$\Rightarrow \arg\left[\frac{x+iy-1-i}{x+iy-2i}\right]=\frac{\pi}{3}$$

$$\Rightarrow \arg\left[\frac{(x-1)+i(y-1)}{x+i(y-2)}\right]=\frac{\pi}{3}$$

$$\Rightarrow \arg\left[\frac{(x-1)+i(y-1)}{x+i(y-2)}\times\frac{x-i(y-2)}{x-i(y-2)}\right]=\frac{\pi}{3}$$

$$\Rightarrow \arg\left[\frac{x(x-1)+(y-1)(y-2)}{x^2+(y-2)^2}+i\,\frac{x(y-1)-(x-1)(y-2)}{x^2+(y-2)^2}\right]=\frac{\pi}{3}$$

$$\Rightarrow \arg\left[\frac{x^2-x+y^2-3y+2}{x^2+(y-2)^2}+i\,\frac{\cancel{xy}-x-\cancel{xy}+2x+y-2}{x^2+(y-2)^2}\right]=\frac{\pi}{3}$$

$$\Rightarrow \arg\left[\frac{x^2+y^2-x-3y+2}{x^2+(y-2)^2}+i\,\frac{y+x-2}{x^2+(y-2)^2}\right]=\frac{\pi}{3}$$

AS $\arg\left(\frac{z-1-i}{z-2i}\right)=\frac{\pi}{3}$, BOTH REAL & IMAGINARY PARTS IN THE EXPRESSION ABOVE MUST BE POSITIVE

- $x^2+y^2-x-3y+2>0$
  $(x-\frac{1}{2})^2-\frac{1}{4}+(y-\frac{3}{2})^2-\frac{9}{4}+2>0$
  $(x-\frac{1}{2})^2+(y-\frac{3}{2})^2>\frac{1}{2}$
- $y+x-2>0$
  $y>2-x$

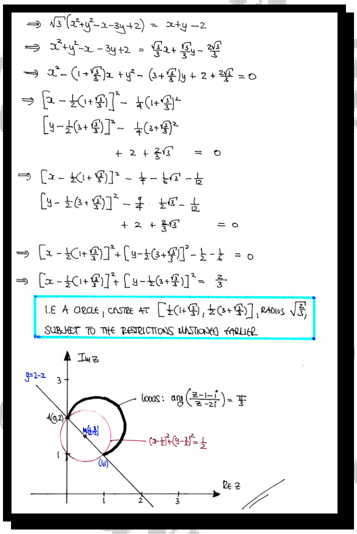
NOW LET $w=\frac{z-1-i}{z-2i}$

$$\Rightarrow \arg(w)=\frac{\pi}{3}$$

$$\Rightarrow \arctan\left[\frac{\operatorname{Im}w}{\operatorname{Re}w}\right]=\frac{\pi}{3}$$

$$\Rightarrow \frac{\operatorname{Im}w}{\operatorname{Re}w}=\tan\frac{\pi}{3}$$

$$\Rightarrow \frac{y+x-2}{x^2+y^2-x-3y+2}=\sqrt{3}$$



$$\Rightarrow \sqrt{3}(x^2+y^2-x-3y+2)=x+y-2$$

$$\Rightarrow x^2+y^2-x-3y+2=\frac{\sqrt{3}}{3}x+\frac{\sqrt{3}}{3}y-\frac{2\sqrt{3}}{3}$$

$$\Rightarrow x^2-\left(1+\frac{\sqrt{3}}{3}\right)x+y^2-\left(3+\frac{\sqrt{3}}{3}\right)y+2+\frac{2\sqrt{3}}{3}=0$$

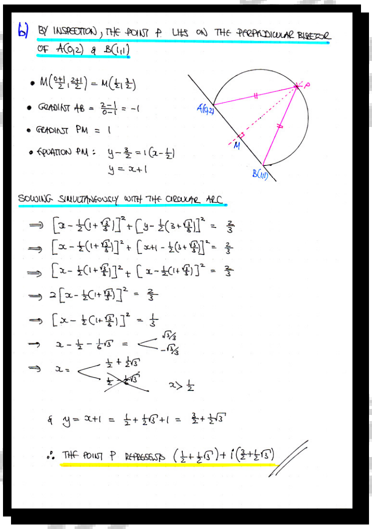
$$\Rightarrow \left[x-\frac{1}{2}\left(1+\frac{\sqrt{3}}{3}\right)\right]^2-\frac{1}{4}\left(1+\frac{\sqrt{3}}{3}\right)^2$$
$$\left[y-\frac{1}{2}\left(3+\frac{\sqrt{3}}{3}\right)\right]^2-\frac{1}{4}\left(3+\frac{\sqrt{3}}{3}\right)^2$$
$$+2+\frac{2}{3}\sqrt{3}=0$$

$$\Rightarrow \left[x-\frac{1}{2}\left(1+\frac{\sqrt{3}}{3}\right)\right]^2-\frac{1}{4}-\frac{1}{6}\sqrt{3}-\frac{1}{12}$$
$$\left[y-\frac{1}{2}\left(3+\frac{\sqrt{3}}{3}\right)\right]^2-\frac{9}{4}\quad\frac{1}{2}\sqrt{3}-\frac{1}{12}$$
$$+2+\frac{2}{3}\sqrt{3}=0$$

$$\Rightarrow \left[x-\frac{1}{2}\left(1+\frac{\sqrt{3}}{3}\right)\right]^2+\left[y-\frac{1}{2}\left(3+\frac{\sqrt{3}}{3}\right)\right]^2-\frac{1}{2}-\frac{1}{6}=0$$

$$\Rightarrow \left[x-\frac{1}{2}\left(1+\frac{\sqrt{3}}{3}\right)\right]^2+\left[y-\frac{1}{2}\left(3+\frac{\sqrt{3}}{3}\right)\right]^2=\frac{2}{3}$$

I.E A CIRCLE, CENTRE AT $\left[\frac{1}{2}\left(1+\frac{\sqrt{3}}{3}\right),\frac{1}{2}\left(3+\frac{\sqrt{3}}{3}\right)\right]$, RADIUS $\sqrt{\frac{2}{3}}$, SUBJECT TO THE RESTRICTIONS MENTIONED EARLIER



b) BY INSPECTION, THE POINT P LIES ON THE PERPENDICULAR BISECTOR OF $A(0,2)$ & $B(1,1)$

- $M\left(\frac{0+1}{2},\frac{2+1}{2}\right)=M\left(\frac{1}{2},\frac{3}{2}\right)$
- GRADIENT AB $=\frac{2-1}{0-1}=-1$
- GRADIENT PM $=1$
- EQUATION PM: $y-\frac{3}{2}=1(x-\frac{1}{2})$
  $y=x+1$

SOLVING SIMULTANEOUSLY WITH THE CIRCULAR ARC

$$\Rightarrow \left[x-\frac{1}{2}\left(1+\frac{\sqrt{3}}{3}\right)\right]^2+\left[y-\frac{1}{2}\left(3+\frac{\sqrt{3}}{3}\right)\right]^2=\frac{2}{3}$$

$$\Rightarrow \left[x-\frac{1}{2}\left(1+\frac{\sqrt{3}}{3}\right)\right]^2+\left[x+1-\frac{1}{2}\left(3+\frac{\sqrt{3}}{3}\right)\right]^2=\frac{2}{3}$$

$$\Rightarrow \left[x-\frac{1}{2}\left(1+\frac{\sqrt{3}}{3}\right)\right]^2+\left[x-\frac{1}{2}\left(1+\frac{\sqrt{3}}{3}\right)\right]^2=\frac{2}{3}$$

$$\Rightarrow 2\left[x-\frac{1}{2}\left(1+\frac{\sqrt{3}}{3}\right)\right]^2=\frac{2}{3}$$

$$\Rightarrow \left[x-\frac{1}{2}\left(1+\frac{\sqrt{3}}{3}\right)\right]^2=\frac{1}{3}$$

$$\Rightarrow x-\frac{1}{2}-\frac{1}{6}\sqrt{3}=\begin{cases}\sqrt{3}/3\\-\sqrt{3}/3\end{cases}$$

$$\Rightarrow x=\begin{cases}\frac{1}{2}+\frac{1}{2}\sqrt{3}\\\cancel{\frac{1}{2}-\frac{1}{6}\sqrt{3}}\end{cases}\qquad x>\frac{1}{2}$$

& $y=x+1=\frac{1}{2}+\frac{1}{2}\sqrt{3}+1=\frac{3}{2}+\frac{1}{2}\sqrt{3}$

∴ THE POINT P REPRESENTS $\left(\frac{1}{2}+\frac{1}{2}\sqrt{3}\right)+i\left(\frac{3}{2}+\frac{1}{2}\sqrt{3}\right)$

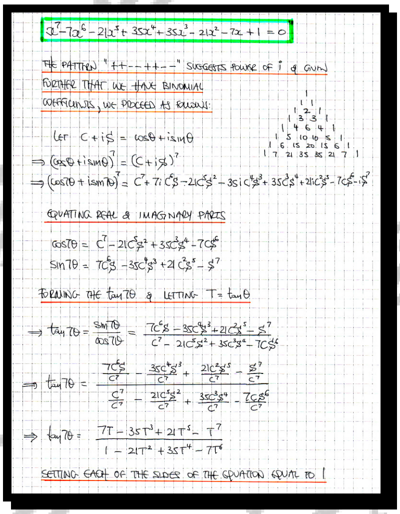
Question 185 (*****)

Find, in exact trigonometric form where appropriate, the real solutions of the following polynomial equation

$$x^7 - 7x^6 - 21x^5 + 35x^4 + 35x^3 - 21x^2 - 7x + 1 = 0 .$$

$\boxed{x = \tan\left(\frac{\pi}{28}\right)}$, $\boxed{x = \tan\left(\frac{5\pi}{28}\right)}$, $\boxed{x = \tan\left(\frac{9\pi}{28}\right)}$, $\boxed{x = \tan\left(\frac{13\pi}{28}\right)}$,

$\boxed{x = \tan\left(\frac{17\pi}{28}\right)}$, $\boxed{x = \tan\left(\frac{3\pi}{4}\right) = -1}$, $\boxed{x = \tan\left(\frac{25\pi}{28}\right)}$



$\boxed{x^7 - 7x^6 - 21x^5 + 35x^4 + 35x^3 - 21x^2 - 7x + 1 = 0}$

THE PATTERN "++−−++−−" SUGGESTS POWER OF i & GIVEN FURTHER THAT WE HAVE BINOMIAL COEFFICIENTS, WE PROCEED AS FOLLOWS:

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1

LET $C + iS = \cos\theta + i\sin\theta$

$\Rightarrow (\cos\theta + i\sin\theta)^7 = (C + iS)^7$

$\Rightarrow (\cos 7\theta + i\sin 7\theta) = C^7 + 7iC^6S - 21C^5S^2 - 35iC^4S^3 + 35C^3S^4 + 21iC^2S^5 - 7CS^6 - iS^7$

EQUATING REAL & IMAGINARY PARTS

$\cos 7\theta = C^7 - 21C^5S^2 + 35C^3S^4 - 7CS^6$

$\sin 7\theta = 7C^6S - 35C^4S^3 + 21C^2S^5 - S^7$

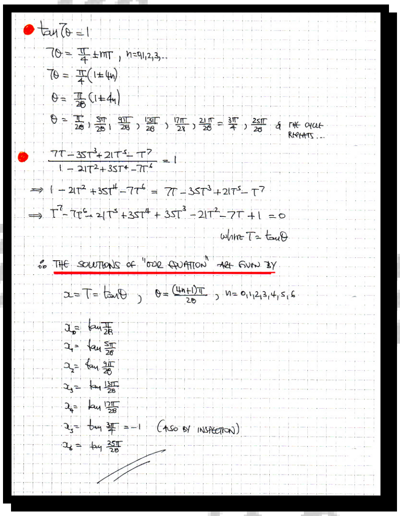
FORMING THE $\tan 7\theta$ & LETTING $T = \tan\theta$

$\Rightarrow \tan 7\theta = \frac{\sin 7\theta}{\cos 7\theta} = \frac{7C^6S - 35C^4S^3 + 21C^2S^5 - S^7}{C^7 - 21C^5S^2 + 35C^3S^4 - 7CS^6}$

$\Rightarrow \tan 7\theta = \frac{\frac{7C^6S}{C^7} - \frac{35C^4S^3}{C^7} + \frac{21C^2S^5}{C^7} - \frac{S^7}{C^7}}{\frac{C^7}{C^7} - \frac{21C^5S^2}{C^7} + \frac{35C^3S^4}{C^7} - \frac{7CS^6}{C^7}}$

$\Rightarrow \tan 7\theta = \frac{7T - 35T^3 + 21T^5 - T^7}{1 - 21T^2 + 35T^4 - 7T^6}$

SETTING EACH OF THE SIDES OF THE EQUATION EQUAL TO 1



- $\tan 7\theta = 1$

$7\theta = \frac{\pi}{4} \pm n\pi$, $n = 0,1,2,3,\ldots$

$7\theta = \frac{\pi}{4}(1 \pm 4n)$

$\theta = \frac{\pi}{28}(1 \pm 4n)$

$\theta = \frac{\pi}{28}, \frac{5\pi}{28}, \frac{9\pi}{28}, \frac{13\pi}{28}, \frac{17\pi}{28}, \frac{21\pi}{28} = \frac{3\pi}{4}, \frac{25\pi}{28}$ & THE CYCLE REPEATS...

- $\frac{7T - 35T^3 + 21T^5 - T^7}{1 - 21T^2 + 35T^4 - 7T^6} = 1$

$\Rightarrow 1 - 21T^2 + 35T^4 - 7T^6 = 7T - 35T^3 + 21T^5 - T^7$

$\Rightarrow T^7 - 7T^6 - 21T^5 + 35T^4 + 35T^3 - 21T^2 - 7T + 1 = 0$

WHERE $T = \tan\theta$

$\therefore$ THE SOLUTIONS OF "OUR EQUATION" ARE GIVEN BY

$x = T = \tan\theta$, $\theta = \frac{(4n+1)\pi}{28}$, $n = 0,1,2,3,4,5,6$

$x_0 = \tan\frac{\pi}{28}$

$x_1 = \tan\frac{5\pi}{28}$

$x_2 = \tan\frac{9\pi}{28}$

$x_3 = \tan\frac{13\pi}{28}$

$x_4 = \tan\frac{17\pi}{28}$

$x_5 = \tan\frac{3\pi}{4} = -1$ (ALSO BY INSPECTION)

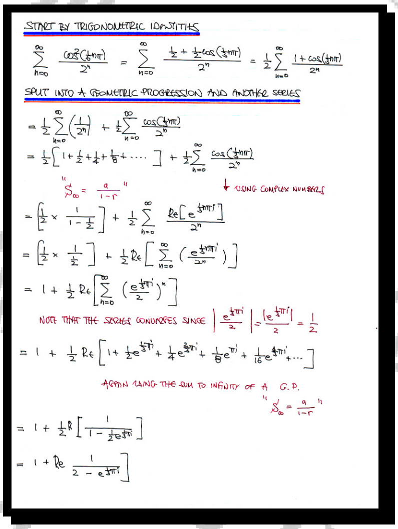
$x_6 = \tan\frac{25\pi}{28}$

Question 186 (*****)

By showing a detailed method involving complex numbers, sum the following series.

$$\sum_{n=0}^{\infty}\left[\frac{\cos^2\left(\frac{1}{6}n\pi\right)}{2^n}\right].$$

$\boxed{\quad}$, $\boxed{\frac{3}{2}}$



START BY TRIGONOMETRIC IDENTITIES

$$\sum_{n=0}^{\infty}\frac{\cos^2\left(\frac{1}{6}n\pi\right)}{2^n} = \sum_{n=0}^{\infty}\frac{\frac{1}{2}+\frac{1}{2}\cos\left(\frac{1}{3}n\pi\right)}{2^n} = \frac{1}{2}\sum_{n=0}^{\infty}\frac{1+\cos\left(\frac{1}{3}n\pi\right)}{2^n}$$

SPLIT INTO A GEOMETRIC PROGRESSION AND ANOTHER SERIES

$$= \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{1}{2^n}\right) + \frac{1}{2}\sum_{n=0}^{\infty}\frac{\cos\left(\frac{1}{3}n\pi\right)}{2^n}$$

$$= \frac{1}{2}\left[1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\dots\right] + \frac{1}{2}\sum_{n=0}^{\infty}\frac{\cos\left(\frac{1}{3}n\pi\right)}{2^n}$$

"$S_\infty = \frac{a}{1-r}$"

$\downarrow$ USING COMPLEX NUMBERS

$$= \left[\frac{1}{2}\times\frac{1}{1-\frac{1}{2}}\right] + \frac{1}{2}\sum_{n=0}^{\infty}\frac{\text{Re}\left[e^{\frac{1}{3}n\pi i}\right]}{2^n}$$

$$= \left[\frac{1}{2}\times\frac{1}{\frac{1}{2}}\right] + \frac{1}{2}\text{Re}\left[\sum_{n=0}^{\infty}\left(\frac{e^{\frac{1}{3}n\pi i}}{2^n}\right)\right]$$

$$= 1 + \frac{1}{2}\text{Re}\left[\sum_{n=0}^{\infty}\left(\frac{e^{\frac{1}{3}\pi i}}{2}\right)^n\right]$$

NOTE THAT THE SERIES CONVERGES SINCE $\left|\frac{e^{\frac{1}{3}\pi i}}{2}\right| = \frac{\left|e^{\frac{1}{3}\pi i}\right|}{2} = \frac{1}{2}$

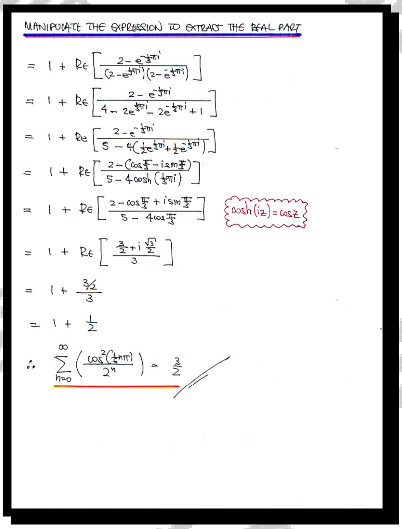
$$= 1 + \frac{1}{2}\text{Re}\left[1+\frac{1}{2}e^{\frac{1}{3}\pi i}+\frac{1}{4}e^{\frac{2}{3}\pi i}+\frac{1}{8}e^{\pi i}+\frac{1}{16}e^{\frac{4}{3}\pi i}+\dots\right]$$

AGAIN USING THE SUM TO INFINITY OF A G.P.

"$S_\infty = \frac{a}{1-r}$"

$$= 1 + \frac{1}{2}R\left[\frac{1}{1-\frac{1}{2}e^{\frac{1}{3}\pi i}}\right]$$

$$= 1 + \text{Re}\ \frac{1}{2-e^{\frac{1}{3}\pi i}}]$$



MANIPULATE THE EXPRESSION TO EXTRACT THE REAL PART

$$= 1 + \text{Re}\left[\frac{2-e^{-\frac{1}{3}\pi i}}{\left(2-e^{\frac{1}{3}\pi i}\right)\left(2-e^{-\frac{1}{3}\pi i}\right)}\right]$$

$$= 1 + \text{Re}\left[\frac{2-e^{-\frac{1}{3}\pi i}}{4-2e^{\frac{1}{3}\pi i}-2e^{-\frac{1}{3}\pi i}+1}\right]$$

$$= 1 + \text{Re}\left[\frac{2-e^{-\frac{1}{3}\pi i}}{5-4\left(\frac{1}{2}e^{\frac{1}{3}\pi i}+\frac{1}{2}e^{-\frac{1}{3}\pi i}\right)}\right]$$

$$= 1 + \text{Re}\left[\frac{2-\left(\cos\frac{\pi}{3}-i\sin\frac{\pi}{3}\right)}{5-4\cosh\left(\frac{1}{3}\pi i\right)}\right]$$

$$= 1 + \text{Re}\left[\frac{2-\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}}{5-4\cos\frac{\pi}{3}}\right]$$

$\cosh(iz) = \cos z$

$$= 1 + \text{Re}\left[\frac{\frac{3}{2}+i\frac{\sqrt{3}}{2}}{3}\right]$$

$$= 1 + \frac{3/2}{3}$$

$$= 1 + \frac{1}{2}$$

$$\therefore \sum_{n=0}^{\infty}\left(\frac{\cos^2\left(\frac{1}{6}n\pi\right)}{2^n}\right) = \frac{3}{2}$$