CONIC SECTIONS

CIRCLE

Question 1 (***+)

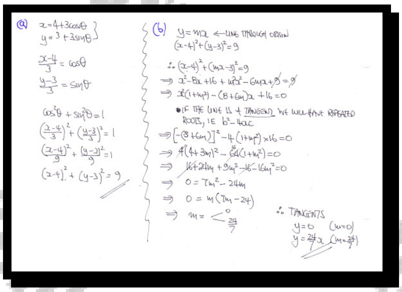
A circle is given parametrically by the equations

$$x = 4 + 3\cos\theta, \ y = 3 + 3\sin\theta, \ 0 \le \theta < 2\pi .$$

a) Find a Cartesian equation for the circle.

b) Find the equations of the two tangents to the circle, which pass through the origin O.

$\boxed{(x-4)^2 + (y-3)^2 = 9}$, $\boxed{y = 0 \quad \text{and} \quad y = \frac{24}{7}x}$



(a) $x = 4 + 3\cos\theta$, $y = 3 + 3\sin\theta$

$\frac{x-4}{3} = \cos\theta$

$\frac{y-3}{3} = \sin\theta$

$\cos^2\theta + \sin^2\theta = 1$

$\left(\frac{x-4}{3}\right)^2 + \left(\frac{y-3}{3}\right)^2 = 1$

$\frac{(x-4)^2}{9} + \frac{(y-3)^2}{9} = 1$

$(x-4)^2 + (y-3)^2 = 9$

(b) $y = mx \leftarrow$ LINE THROUGH ORIGIN

$(x-4)^2 + (y-3)^2 = 9$

$\therefore (x-4)^2 + (mx-3)^2 = 9$

$\Rightarrow x^2 - 8x + 16 + m^2x^2 - 6mx + \cancel{9} = \cancel{9}$

$\Rightarrow x^2(1+m^2) - (8+6m)x + 16 = 0$

• IF THE LINE IS A TANGENT WE WILL HAVE REPEATED ROOTS, IE $b^2 - 4ac$

$\Rightarrow [-(8+6m)]^2 - 4(1+m^2) \times 16 = 0$

$\Rightarrow 4(4+3m)^2 - 64(1+m^2) = 0$

$\Rightarrow \cancel{16} + 24m + 9m^2 - \cancel{16} - 16m^2 = 0$

$\Rightarrow 0 = 7m^2 - 24m$

$\Rightarrow 0 = m(7m - 24)$

$\Rightarrow m = 0, \ \frac{24}{7}$

$\therefore$ TANGENTS

$y = 0 \quad (m=0)$

$y = \frac{24}{7}x \quad (m = \frac{24}{7})$

Question 2 (****+)

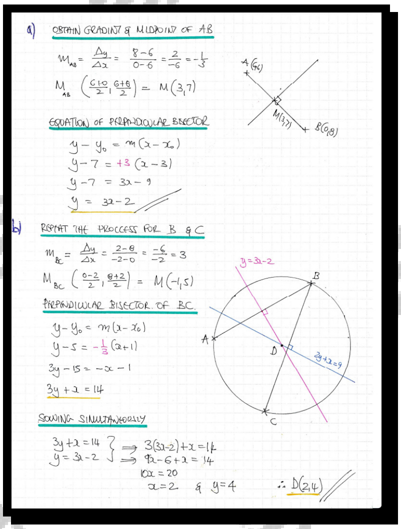
The points A, B and C have coordinates $(6,6)$, $(0,8)$ and $(-2,2)$, respectively.

a) Find an equation of the perpendicular bisector of AB.

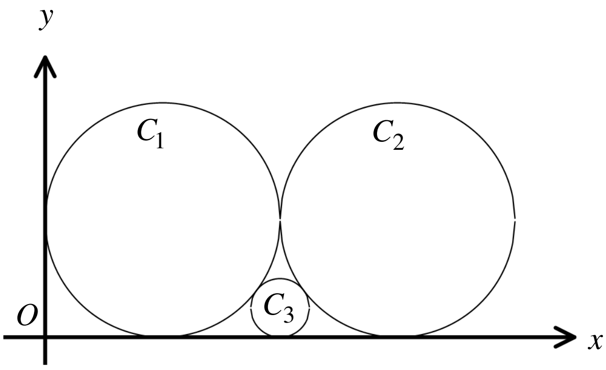
The points A, B and C lie on the circumference of a circle whose centre is located at the point D.

b) Determine the coordinates of D.

$\boxed{y = 3x - 2}$, $\boxed{D(2,4)}$



Question 3 (****+)



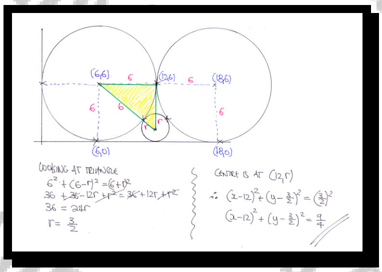
The figure above shows three circles C_1, C_2 and C_3.

The coordinates of the centres of all three circles are positive.

- The circle C_1 has centre at $(6,6)$ and **touches** both the x axis and the y axis.
- The circle C_2 has the same size radius as C_1 and **touches** the x axis.
- The circle C_3 **touches** the x axis and **both** C_1 and C_2.

Determine an equation of C_3.

$\boxed{}$, $\boxed{(x-12)^2+\left(y-\frac{3}{2}\right)^2=\frac{9}{4}}$



Question 4 (****+)

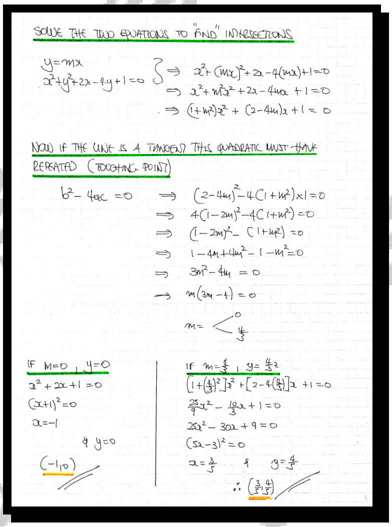
A circle C has equation

$$x^2 + y^2 + 2x - 4y + 1 = 0$$

The straight line L with equation $y = mx$ is a tangent to C.

Find the possible values of m and hence determine the possible coordinates at which L meets C.

$\boxed{}$, $\boxed{m = 0,\ m = \frac{4}{3}}$, $\boxed{(-1,0),\ \left(\frac{3}{5}, \frac{4}{5}\right)}$



SOLVE THE TWO EQUATIONS TO "FIND" INTERSECTIONS

$$\left.\begin{array}{l} y = mx \\ x^2+y^2+2x-4y+1=0 \end{array}\right\} \Rightarrow x^2+(mx)^2+2x-4(mx)+1=0$$

$\Rightarrow x^2+m^2x^2+2x-4mx+1=0$

$\Rightarrow (1+m^2)x^2+(2-4m)x+1=0$

NOW IF THE LINE IS A TANGENT THIS QUADRATIC MUST HAVE REPEATED (TOUCHING POINT)

$b^2-4ac=0 \Rightarrow (2-4m)^2-4(1+m^2)\times 1=0$

$\Rightarrow 4(1-2m)^2-4(1+m^2)=0$

$\Rightarrow (1-2m)^2-(1+m^2)=0$

$\Rightarrow 1-4m+4m^2-1-m^2=0$

$\Rightarrow 3m^2-4m=0$

$\Rightarrow m(3m-4)=0$

$m = \begin{cases} 0 \\ \frac{4}{3} \end{cases}$

IF $m=0$, $y=0$

$x^2+2x+1=0$

$(x+1)^2=0$

$x=-1$

& $y=0$

$(-1,0)$

IF $m=\frac{4}{3}$, $y=\frac{4}{3}x$

$\left[1+\left(\frac{4}{3}\right)^2\right]x^2+\left[2-4\left(\frac{4}{3}\right)\right]x+1=0$

$\frac{25}{9}x^2-\frac{10}{3}x+1=0$

$25x^2-30x+9=0$

$(5x-3)^2=0$

$x=\frac{3}{5}$ & $y=\frac{4}{5}$

$\therefore \left(\frac{3}{5},\frac{4}{5}\right)$

Question 5 (****+)

A circle C has equation

$$x^2 + y^2 + 4x - 10y + 9 = 0 .$$

a) Find the coordinates of the centre of C and the size of its radius.

A tangent to the circle T, passes through the point with coordinates $(0,-1)$ and has gradient m, where $m < 0$.

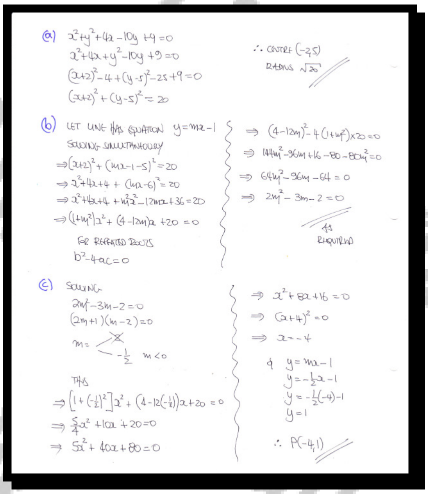
b) Show that m is a solution of the equation

$$2m^2 - 3m - 2 = 0 .$$

The tangent T meets C at the point P.

c) Determine the coordinates of P.

$\boxed{(-2,5),\ r = \sqrt{20}}$, $\boxed{P(-4,1)}$



(a) $x^2+y^2+4x-10y+9=0$

$x^2+4x+y^2-10y+9=0$

$(x+2)^2-4+(y-5)^2-25+9=0$

$(x+2)^2+(y-5)^2=20$

$\therefore$ CENTRE $(-2,5)$

RADIUS $\sqrt{20}$

(b) LET LINE HAS EQUATION $y=mx-1$

SOLVING SIMULTANEOUSLY

$\Rightarrow (x+2)^2+(mx-1-5)^2=20$

$\Rightarrow x^2+4x+4+(mx-6)^2=20$

$\Rightarrow x^2+4x+4+m^2x^2-12mx+36=20$

$\Rightarrow (1+m^2)x^2+(4-12m)x+20=0$

FOR REPEATED ROOTS

$b^2-4ac=0$

$\Rightarrow (4-12m)^2-4(1+m^2)\times 20=0$

$\Rightarrow 144m^2-96m+16-80-80m^2=0$

$\Rightarrow 64m^2-96m-64=0$

$\Rightarrow 2m^2-3m-2=0$

AS REQUIRED

(c) SOLVING

$2m^2-3m-2=0$

$(2m+1)(m-2)=0$

$m = \cancel{2}$, $-\frac{1}{2}$ $\quad m<0$

THUS

$\Rightarrow \left[1+\left(-\frac{1}{2}\right)^2\right]x^2+\left(4-12\left(-\frac{1}{2}\right)\right)x+20=0$

$\Rightarrow \frac{5}{4}x^2+10x+20=0$

$\Rightarrow 5x^2+40x+80=0$

$\Rightarrow x^2+8x+16=0$

$\Rightarrow (x+4)^2=0$

$\Rightarrow x=-4$

& $y=mx-1$

$y=-\frac{1}{2}x-1$

$y=-\frac{1}{2}(-4)-1$

$y=1$

$\therefore P(-4,1)$

Question 6 (****+)

A circle has equation

$$x^2 + y^2 - 4x - 2y = 13 .$$

a) Find the coordinates of the centre of the circle and the size of its radius.

The points A and B lie on the circle such that the length of AB is 6 units.

b) Show that $\measuredangle ACB = 90°$, where C is the centre of the circle.

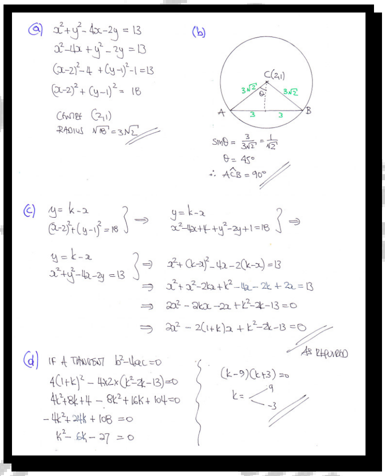
A tangent to the circle has equation $y = k - x$, where k is a constant.

c) Show clearly that

$$2x^2 + 2(1-k)x + k^2 - 2k - 13 = 0 .$$

d) Determine the possible values of k.

$\boxed{(2,1),\ r = \sqrt{18}}$, $\boxed{k = -3,\ 9}$



(a) $x^2+y^2-4x-2y=13$

$x^2-4x+y^2-2y=13$

$(x-2)^2-4+(y-1)^2-1=13$

$(x-2)^2+(y-1)^2=18$

CENTRE $(2,1)$

RADIUS $\sqrt{18}=3\sqrt{2}$

(b)

$\sin\theta = \frac{3}{3\sqrt{2}} = \frac{1}{\sqrt{2}}$

$\theta = 45°$

$\therefore A\hat{C}B = 90°$

(c) $\left.\begin{array}{l} y=k-x \\ (x-2)^2+(y-1)^2=18 \end{array}\right\} \Rightarrow \left.\begin{array}{l} y=k-x \\ x^2-4x+4+y^2-2y+1=18 \end{array}\right\} \Rightarrow$

$\left.\begin{array}{l} y=k-x \\ x^2+y^2-4x-2y=13 \end{array}\right\} \Rightarrow x^2+(k-x)^2-4x-2(k-x)=13$

$\Rightarrow x^2+x^2-2kx+k^2-4x-2k+2x=13$

$\Rightarrow 2x^2-2kx-2x+k^2-2k-13=0$

$\Rightarrow 2x^2-2(1+k)x+k^2-2k-13=0$

AS REQUIRED

(d) IF A TANGENT $b^2-4ac=0$

$4(1+k)^2-4\times2\times(k^2-2k-13)=0$

$4k^2+8k+4-8k^2+16k+104=0$

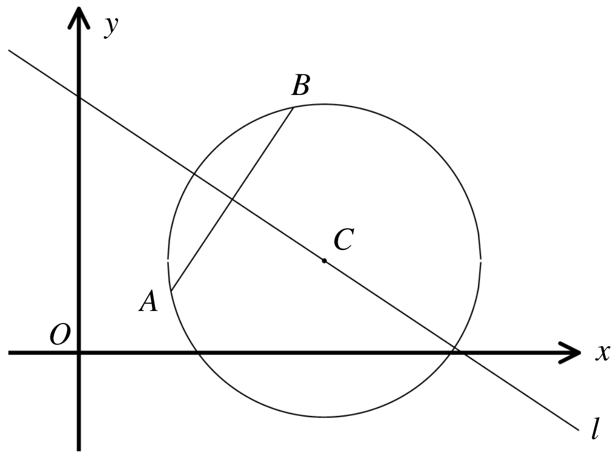
$-4k^2+24k+108=0$

$k^2-6k-27=0$

$(k-9)(k+3)=0$

$k = 9, -3$

Question 7 (****+)



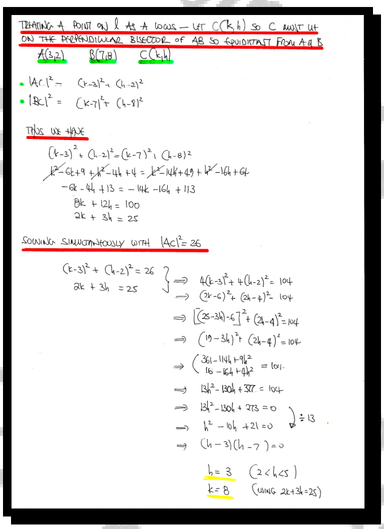
The figure above shows a circle whose centre is located at the point $C(k,h)$, where k and h are constants such that $2<h<5$.

The points $A(3,2)$ and $B(7,8)$ lie on this circle.

The straight line l passes through C and the midpoint of AB.

Given that the radius of the circle is $\sqrt{26}$, find an equation for l, the value of k and the value of h.

$\boxed{}$, $\boxed{2x+3y=25}$, $\boxed{k=8}$, $\boxed{h=3}$



TREATING A POINT ON l AS A LOCUS – LET $C(k,h)$ SO C MUST LIE ON THE PERPENDICULAR BISECTOR OF AB SO EQUIDISTANT FROM A & B

$A(3,2)$ $B(7,8)$ $C(k,h)$

- $|AC|^2 = (k-3)^2+(h-2)^2$
- $|BC|^2 = (k-7)^2+(h-8)^2$

THUS WE HAVE

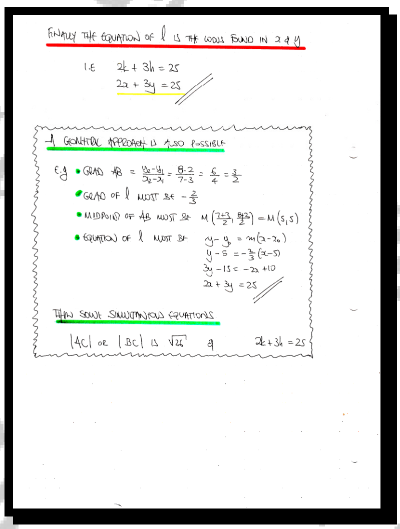
$$(k-3)^2+(h-2)^2=(k-7)^2+(h-8)^2$$
$$\cancel{k^2}-6k+9+\cancel{h^2}-4h+4=\cancel{k^2}-14k+49+\cancel{h^2}-16h+64$$
$$-6k-4h+13=-14k-16h+113$$
$$8k+12h=100$$
$$2k+3h=25$$

SOLVING SIMULTANEOUSLY WITH $|AC|^2=26$

$$\left.\begin{array}{l}(k-3)^2+(h-2)^2=26\\ 2k+3h=25\end{array}\right\} \Rightarrow 4(k-3)^2+4(h-2)^2=104$$
$$\Rightarrow (2k-6)^2+(2h-4)^2=104$$
$$\Rightarrow \left[(25-3h)-6\right]^2+(2h-4)^2=104$$
$$\Rightarrow (19-3h)^2+(2h-4)^2=104$$
$$\Rightarrow \left(\begin{array}{l}361-114h+9h^2\\16-16h+4h^2\end{array}\right.=104$$
$$\Rightarrow 13h^2-130h+377=104$$
$$\Rightarrow 13h^2-130h+273=0 \quad \div 13$$
$$\Rightarrow h^2-10h+21=0$$
$$\Rightarrow (h-3)(h-7)=0$$

$h=3$ $(2<h<5)$

$k=8$ (USING $2k+3h=25$)



FINALLY THE EQUATION OF l IS THE LOCUS FOUND IN x & y

I.E $2k+3h=25$

$2x+3y=25$

A GEOMETRIC APPROACH IS ALSO POSSIBLE

E.g
- GRAD $AB = \frac{y_2-y_1}{x_2-x_1}=\frac{8-2}{7-3}=\frac{6}{4}=\frac{3}{2}$
- GRAD OF l MUST BE $-\frac{2}{3}$
- MIDPOINT OF AB MUST BE $M\left(\frac{7+3}{2},\frac{8+2}{2}\right)=M(5,5)$
- EQUATION OF l MUST BE $y-y_0=m(x-x_0)$

$y-5=-\frac{2}{3}(x-5)$

$3y-15=-2x+10$

$2x+3y=25$

THEN SOLVE SIMULTANEOUS EQUATIONS

$|AC|$ OR $|BC|$ IS $\sqrt{26}$ & $2k+3h=25$

Question 8 (****+)

The straight line passing through the points $P(1,9)$ and $Q(5,5)$ is a tangent to a circle with centre at $C(6,8)$.

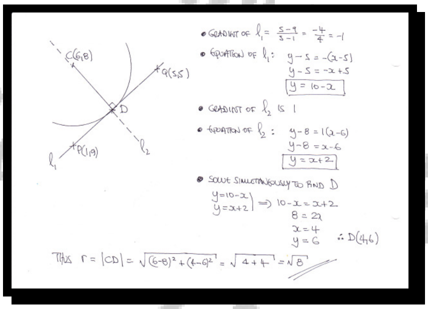
Determine, in exact surd form, the radius of the circle.

In this question you may **not** use …

… a standard formula which determines the shortest distance of a point from a straight line.

… any form of calculus.

$\boxed{r=\sqrt{8}}$



Question 9 (****+)

The straight line with equation $y = 2x - 3$ is a tangent to a circle with centre at the point $C(2,-3)$.

Determine, in exact surd form, the radius of the circle.

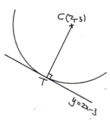
In this question you may **not** use …

… a standard formula which determines the shortest distance of a point from a straight line.

… any form of calculus.

$\boxed{r = \frac{4}{5}\sqrt{5}}$

METHOD A – BY CO-ORDINATE GEOMETRY



- GRADIENT OF THE TANGENT IS 2
- GRADIENT CT MUST BE $-\frac{1}{2}$
- EQUATION OF LINE THROUGH C & T

$$y-(-3) = -\tfrac{1}{2}(x-2)$$
$$y+3 = -\tfrac{1}{2}(x-2)$$
$$2y+6 = -x+2$$
$$2y+x+4=0$$

- SOLVING SIMULTANEOUSLY WITH $y=2x-3$

$$2(2x-3)+x+4=0$$
$$5x-2=0$$
$$x=\tfrac{2}{5}$$

AND $y = 2\left(\frac{2}{5}\right)-3 = \frac{4}{5}-3 = -\frac{11}{5}$ $\quad\therefore \left(\frac{2}{5}, -\frac{11}{5}\right)$

- DISTANCE CT FINALLY, $C(2,-3)$ & $T\left(\frac{2}{5},-\frac{11}{5}\right)$

$$d = \sqrt{(y_2-y_1)^2+(x_2-x_1)^2}$$
$$|CT| = \sqrt{\left(-3+\tfrac{11}{5}\right)^2+\left(\tfrac{2}{5}-2\right)^2}$$
$$r = \sqrt{\left(-\tfrac{4}{5}\right)^2+\left(-\tfrac{8}{5}\right)^2} = \sqrt{\tfrac{16}{25}+\tfrac{64}{25}} = \sqrt{\tfrac{80}{25}} = \tfrac{4}{5}\sqrt{5}$$

METHOD B – USING DISCRIMINANTS

- LET THE CIRCLE HAVE EQUATION

$$(x-2)^2+(y+3)^2=r^2$$

- SOLVING SIMULTANEOUSLY WITH $y=2x-3$ TO "FIND" T

$$\Rightarrow (x-2)^2+(2x-3+3)^2=r^2$$
$$\Rightarrow (x-2)^2+(2x)^2=r^2$$
$$\Rightarrow x^2-4x+4+4x^2=r^2$$
$$\Rightarrow 5x^2-4x+(4-r^2)=0$$

- THIS EQUATION MUST PRODUCE REPEATED ROOTS AS THE POINT T IS A POINT OF TANGENCY

$$b^2-4ac=0 \Rightarrow (-4)^2-4\times5\times(4-r^2)=0$$
$$\Rightarrow 16-20(4-r^2)=0$$
$$\Rightarrow 16-80+20r^2=0$$
$$\Rightarrow 20r^2=64$$
$$\Rightarrow r^2=\tfrac{64}{20}=\tfrac{64}{100}\times5$$
$$\Rightarrow r=\tfrac{4}{5}\sqrt{5}$$

AS BEFORE

METHOD C – BY MINIMIZATION (COMPLETING THE SQUARE)

- CONSIDER A POINT ON THE LINE $y=2x-3$, IE $(x, 2x-3)$
- THE DISTANCE FROM $(x,2x-3)$ TO THE CENTRE $(2,-3)$ IS GIVEN BY

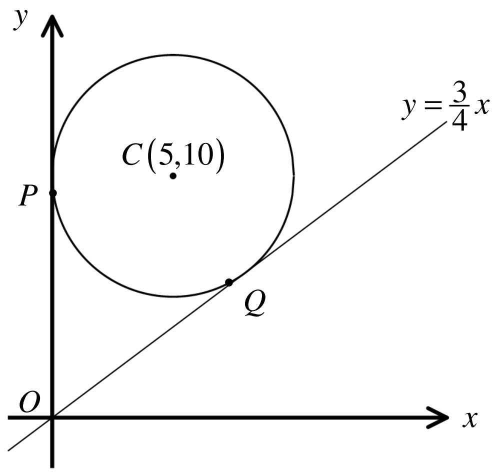
$$\Rightarrow d = \sqrt{(x-2)^2+[2x-3-(-3)]^2}$$
$$\Rightarrow d = \sqrt{(x-2)^2+4x^2}$$
$$\Rightarrow d^2 = x^2-4x+4+4x^2$$
$$\Rightarrow d^2 = 5x^2-4x+4$$
$$\Rightarrow d^2 = 5\left[x^2-\tfrac{4}{5}x+\tfrac{4}{5}\right]$$
$$\Rightarrow d^2 = 5\left[\left(x-\tfrac{2}{5}\right)^2-\tfrac{4}{25}+\tfrac{4}{5}\right]$$
$$\Rightarrow d^2 = 5\left(x-\tfrac{2}{5}\right)^2-\tfrac{4}{5}+4$$
$$\Rightarrow d^2 = 5\left(x-\tfrac{2}{5}\right)^2+\tfrac{16}{5}$$

$\therefore$ MINIMUM VALUE OF d^2 IS $\frac{16}{5}$ (OCCURS AT $x=\frac{2}{5}$)

$$\therefore d_{MIN} = r = \sqrt{\tfrac{16}{5}} = \sqrt{\tfrac{16\times5}{25}} = \tfrac{4}{5}\sqrt{5}$$

Question 10 (*****)

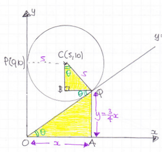
The figure below shows the circle with centre at $C(5,10)$ and radius 5.



The straight lines with equations, $x=0$ and $y=\frac{3}{4}x$ are tangents to the circle at the points P and Q respectively.

Show that the area of the triangle PCQ is 10 square units.

, proof



● A GEOMETRIC APPROACH

ON $\hat{OAQ}$, $\tan\theta = \frac{\frac{3}{4}x}{x}$

$\tan\theta = \frac{3}{4}$

$\sin\theta = \frac{3}{5}$

$\cos\theta = \frac{4}{5}$

LOOKING AT THE SIMILAR TRIANGLE $\hat{CBQ}$

$|BC| = 5\cos\theta = 5\times\frac{4}{5} = 4$

$|BQ| = 5\sin\theta = 5\times\frac{3}{5} = 3$

$\therefore Q(5+3, 10-4)$

$Q(8,6)$

● AN ALGEBRAIC APPROACH

$(x-5)^2+(y-10)^2 = 25$

$y = \frac{3}{4}x$

$\Rightarrow (x-5)^2 + (\frac{3}{4}x-10)^2 = 25$

$\Rightarrow \left\{ \begin{array}{l} x^2-10x+25 \\ \frac{9}{16}x^2-15x+100 \end{array} \right\} = 25$

$\Rightarrow \frac{25}{16}x^2 - 25x + 100 = 0$

$\Rightarrow \frac{1}{16}x^2 - x + 4 = 0$

$\Rightarrow x^2 - 16x + 64 = 0$

$\Rightarrow (x-8)^2 = 0$

$\therefore x = 8$ & $y = \frac{3}{4}\times 8 = 6$

$\therefore Q(8,6)$

● A THIRD APPROACH

$|OP| = |OQ| = 10$ (CIRCLE THEOREM)

BY PYTHAGORAS ON $\hat{OAQ}$

$|OA|^2 + |AQ|^2 = |OQ|^2$

$x^2 + (\frac{3}{4}x)^2 = 100$

$x^2 + \frac{9}{16}x^2 = 100$

$16x^2 + 9x^2 = 1600$

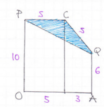
$25x^2 = 1600$

$(5x)^2 = (40)^2$

$5x = \pm 40$

$x = 8$ & $y = 6$

● FINALLY THE AREA OF THE REQUIRED TRIANGLE CAN BE FOUND



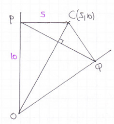
THE REQUIRED AREA IN BLUE IS GIVEN BY

$(5\times10) + \frac{1}{2}(10+6)\times3 - \frac{1}{2}(10+6)\times8$

$= 50 + 24 - 64$

$= 10$

ALTERNATIVE METHOD WITHOUT FINDING THE CO-ORDINATES OF Q



GRADIENT $OC = \frac{10}{5} = 2$

GRADIENT $PQ = -\frac{1}{2}$

LINE $OC \Rightarrow y = 2x$

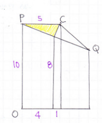
LINE $QP \Rightarrow y = 10 - \frac{1}{2}x$

$\Rightarrow 2x = 10 - \frac{1}{2}x$

$\Rightarrow 4x = 20 - x$

$\Rightarrow 5x = 20$

$\Rightarrow x = 4, y = 8$



AREA OF YELLOW TRIANGLE

$(10\times5) - \frac{1}{2}(10+8)\times4 - \frac{1}{2}(8+10)\times1$

$= 50 - 36 - 9$

$= 5$

$\therefore$ REQUIRED AREA OF $\hat{PCQ} = 10$

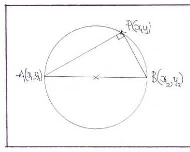
Question 11 (*****)

A circle passes through the points $A(x_1, y_1)$ and $A(x_2, y_2)$.

Given that AB is a diameter of the circle, show that the equation of the circle is

$$(x-x_1)(x-x_2)+(y-y_1)(y-y_2)=0.$$

$\boxed{}$, $\boxed{\text{proof}}$



- LET $P(x,y)$ BE AN ARBITRARY POINT ON THE CIRCLE
- GRAD $AP = \frac{y-y_1}{x-x_1}$
- GRAD $BP = \frac{y-y_2}{x-x_2}$
- BUT FOR ALL POSITIONS OF P

$AP \perp BP$

- HENCE THE GRADIENTS MULTIPLY TO -1

$\Rightarrow \frac{y-y_1}{x-x_1} \times \frac{y-y_2}{x-x_2} = -1$

$\Rightarrow \frac{(y-y_1)(y-y_2)}{(x-x_1)(x-x_2)} = -1$

$\Rightarrow (y-y_1)(y-y_2) = -(x-x_1)(x-x_2)$

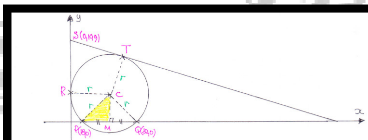
$\Rightarrow (y-y_1)(y-y_2) + (x-x_1)(x-x_2) = 0$

Question 12 (*****)

A circle passes through the points $P(18,0)$ and $Q(32,0)$. A tangent to this circle passes through the point $S(0,199)$ and touches the circle at the point T.

Given that the y axis is a tangent to this circle, determine the coordinates of T

$\boxed{}$, $\boxed{(49,31)}$



- START WITH A GOOD DIAGRAM & SOME BASIC GEOMETRY
  - THE MIDPOINT OF PQ IS $M(25,0)$
  - THIS IS THE x CO-ORDINATE OF THE CENTRE OF THE CIRCLE, AS THE CIRCLE TOUCHES THE y AXIS
  - LOOKING AT $P\hat{M}C$ (MARKED IN YELLOW)

$\Rightarrow |PM|^2 + |MC|^2 = |PC|^2$

$\Rightarrow 7^2 + |MC|^2 = 25^2$

$\Rightarrow |MC| = 24$ (7–24–25 PYTHAGOREAN TRIPLE)

  - CENTRE OF THE CIRCLE IS AT $C(25,24)$ AND ITS EQUATION IS

$\therefore \boxed{(x-25)^2 + (y-24)^2 = 25^2}$

- LET THE TANGENT TO THE ABOVE CIRCLE HAVE EQUATION $\boxed{y = mx + 199}$, AS IT PASSES THROUGH $S(0,199)$ — SOLVING SIMULTANEOUSLY WE OBTAIN

$\Rightarrow (x-25)^2 + (mx+199-24)^2 = 25^2$

$\Rightarrow (x-25)^2 + (mx+175)^2 = 25^2$

$\Rightarrow x^2 - 50x + \cancel{25^2} + m^2x^2 + 350mx + 175^2 = \cancel{25^2}$

$\Rightarrow m^2x^2 + x^2 + 350mx - 50x + 175^2 = 0$

$\Rightarrow \boxed{(m^2+1)x^2 + 50(7m-1)x + 175^2 = 0}$

- LOOKING FOR REPEATED ROOTS, AS THE LINE IS A TANGENT AT T

$\Rightarrow [50(7m-1)]^2 - 4(m^2+1) \times 175^2 = 0$

$\Rightarrow 50^2(7m-1)^2 - 4 \times 175^2(m^2+1) = 0$

$\Rightarrow 50^2(7m-1)^2 - 2^2 \times (25\times7)^2(m^2+1) = 0$ DIVIDE BY 50^2

$\Rightarrow \cancel{50^2}(7m-1)^2 - \cancel{2^2 \times 25^2} \times 7^2(m^2+1) = 0$

$\Rightarrow 49m^2 - 14m + 1 - 49(m^2+1) = 0$

$\Rightarrow \cancel{49m^2} - 14m + 1 - \cancel{49m^2} - 49 = 0$

$\Rightarrow -14m = 48$

$\Rightarrow \boxed{m = -\frac{24}{7}}$

& THE EQUATION OF THE TANGENT IS $\boxed{y = 199 - \frac{24}{7}x}$

- FINALLY USING $m = -\frac{24}{7}$ IN

$\Rightarrow (m^2+1)x^2 + 50x(7m-1) + 175^2 = 0$

$\Rightarrow \left[\left(-\frac{24}{7}\right)^2 + 1\right]x^2 + 50x\left[7\left(-\frac{24}{7}\right) - 1\right] + 175^2 = 0$

$\Rightarrow \left[\frac{24^2}{7^2} + 1\right]x^2 + 50x[-25] + 175^2 = 0$

$\Rightarrow \left[\frac{24^2+7^2}{7^2}\right]x^2 - 25 \times 50x + 175^2 = 0$

$\Rightarrow \frac{25^2}{7^2}x^2 - 25^2 \times 2x + (25\times7)^2 = 0$ DIVIDE BY 25^2

$\Rightarrow \frac{1}{7^2}x^2 - 2x + 7^2 = 0$ PERFECT SQUARE (EXPECTED)

$\Rightarrow \left(\frac{x}{7} - 7\right)^2 = 0$

$\Rightarrow \underline{x = 49}$

& $y = 199 - \frac{24}{7} \times 49 = 199 - 24 \times 7$

$y = 199 - 168 = 31$

$\therefore T(49,31)$

Question 13 (*****)

The circle C_1 has equation

$$x^2 + y^2 - 4x - 4y + 6 = 0 .$$

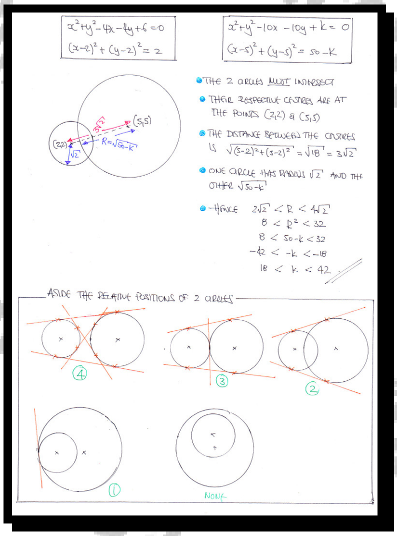
The circle C_2 has equation

$$x^2 + y^2 - 10x - 10y + k = 0 ,$$

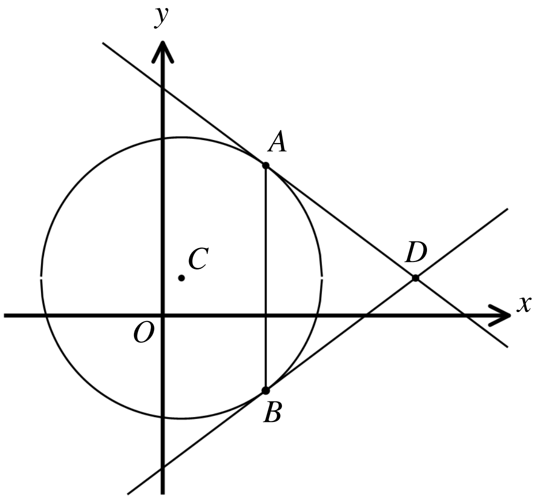
where k is a constant.

Given that C_1 and C_2 have exactly two common tangents, determine the range of possible values of k.

$\boxed{}$, $\boxed{18 < k < 42}$



Question 14 (*****) non calculator



The figure above shows the circle with equation

$$x^2 + y^2 - 4x - 8y = 205,$$

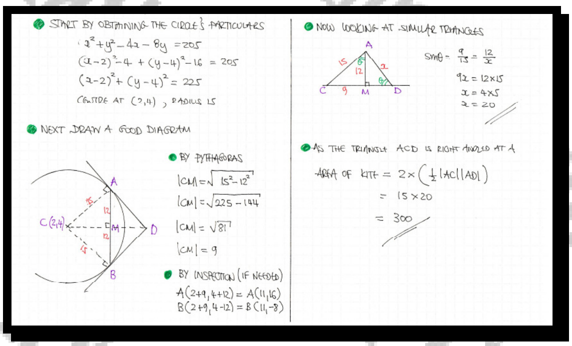
with centre at the point C and radius r.

The straight line AB is parallel to the y axis and has length 24 units.

The tangents to the circle at A and B meet at the point D.

Find the length of AD and hence deduce the area of the kite $CADB$.

$\boxed{}$, $\boxed{|AD| = 20}$, $\boxed{\text{area} = 300}$



START BY OBTAINING THE CIRCLE'S PARTICULARS

$x^2+y^2-4x-8y=205$

$(x-2)^2-4+(y-4)^2-16=205$

$(x-2)^2+(y-4)^2=225$

CENTRE AT $(2,4)$, RADIUS 15

NEXT DRAW A GOOD DIAGRAM

BY PYTHAGORAS

$|CM| = \sqrt{15^2-12^2}$

$|CM| = \sqrt{225-144}$

$|CM| = \sqrt{81}$

$|CM| = 9$

BY INSPECTION (IF NEEDED)

$A(2+9, 4+12) = A(11,16)$

$B(2+9, 4-12) = B(11,-8)$

NOW LOOKING AT SIMILAR TRIANGLES

$\sin\theta = \frac{9}{15} = \frac{12}{x}$

$9x = 12 \times 15$

$x = 4 \times 5$

$x = 20$

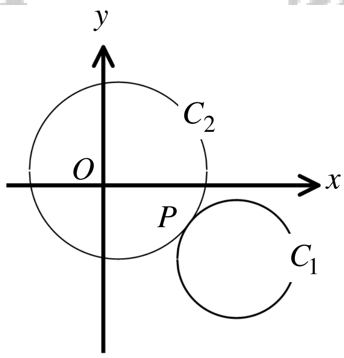
AS THE TRIANGLE ACD IS RIGHT ANGLED AT A

AREA OF KITE $= 2 \times \left(\frac{1}{2}|AC||AD|\right)$

$= 15 \times 20$

$= 300$

Question 15 (*****)



The figure above shows a circle C_1 with equation

$$x^2 + y^2 - 18x + ky + 90 = 0,$$

where k is a positive constant.

a) Determine, in terms of k, the coordinates of the centre of C_1 and the size of its radius.

Another circle C_2 has equation

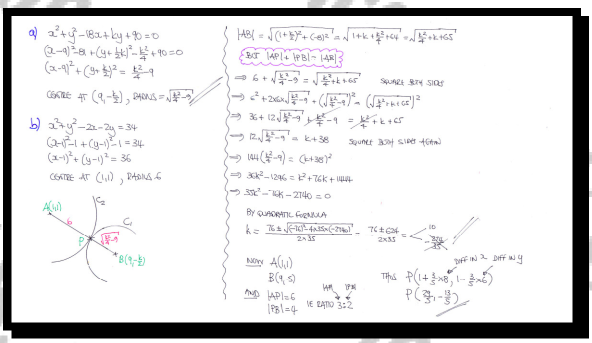
$$x^2 + y^2 - 2x - 2y = 34.$$

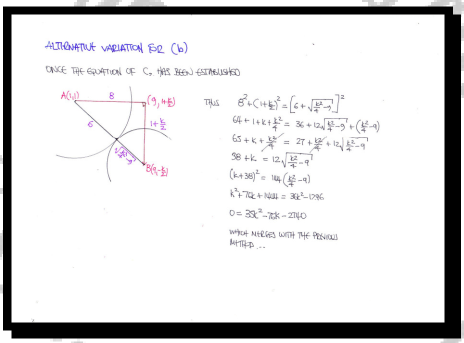
b) Given that C_1 and C_2 are **touching externally** at the point P, find …

i. … the value of k.

ii. … the coordinates of P.

$\boxed{}$, $\boxed{\left(9, -\frac{1}{2}k\right),\ r = \sqrt{\frac{k^2}{4} - 9}}$, $\boxed{k = 10}$, $\boxed{P\left(\frac{29}{5}, -\frac{13}{5}\right)}$





Question 16 (*****)

The curve C has equation

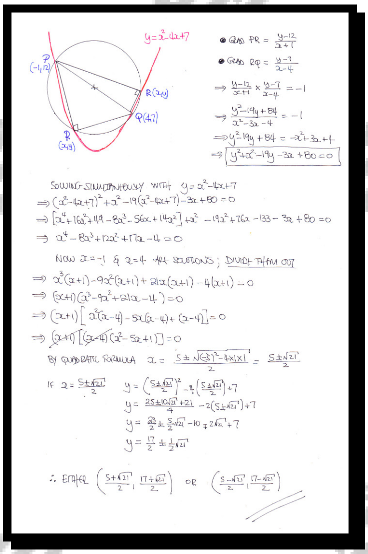
$$y = x^2 - 4x + 7 .$$

The points $P(-1,12)$ and $Q(4,7)$ lie on C.

The point R also lies on C so that $\measuredangle PRQ = 90°$.

Determine, as exact surds, the possible coordinates of R.

$$\boxed{\left(\frac{5+\sqrt{21}}{2},\frac{17+\sqrt{21}}{2}\right) \text{ or } \left(\frac{5-\sqrt{21}}{2},\frac{17-\sqrt{21}}{2}\right)}$$



• GRAD $PR = \frac{y-12}{x+1}$

• GRAD $RQ = \frac{y-7}{x-4}$

$\Rightarrow \frac{y-12}{x+1} \times \frac{y-7}{x-4} = -1$

$\Rightarrow \frac{y^2-19y+84}{x^2-3x-4} = -1$

$\Rightarrow y^2-19y+84 = -x^2+3x+4$

$\Rightarrow \boxed{y^2+x^2-19y-3x+80=0}$

SOLVING SIMULTANEOUSLY WITH $y=x^2-4x+7$

$\Rightarrow (x^2-4x+7)^2+x^2-19(x^2-4x+7)-3x+80=0$

$\Rightarrow [x^4+16x^2+49-8x^3-56x+14x^2]+x^2-19x^2+76x-133-3x+80=0$

$\Rightarrow x^4-8x^3+12x^2+17x-4=0$

NOW $x=-1$ & $x=4$ ARE SOLUTIONS; DIVIDE THEM OUT

$\Rightarrow x^3(x+1)-9x^2(x+1)+21x(x+1)-4(x+1)=0$

$\Rightarrow (x+1)(x^3-9x^2+21x-4)=0$

$\Rightarrow (x+1)[x^2(x-4)-5x(x-4)+(x-4)]=0$

$\Rightarrow \cancel{(x+1)}[\cancel{(x-4)}(x^2-5x+1)]=0$

BY QUADRATIC FORMULA $x = \frac{5 \pm \sqrt{(-5)^2-4\times1\times1}}{2} = \frac{5\pm\sqrt{21}}{2}$

IF $x=\frac{5\pm\sqrt{21}}{2}$ $\quad y=\left(\frac{5\pm\sqrt{21}}{2}\right)^2-4\left(\frac{5\pm\sqrt{21}}{2}\right)+7$

$y=\frac{25\pm10\sqrt{21}+21}{4}-2(5\pm\sqrt{21})+7$

$y=\frac{23}{2}\pm\frac{5}{2}\sqrt{21}-10\mp2\sqrt{21}+7$

$y=\frac{17}{2}\pm\frac{1}{2}\sqrt{21}$

$\therefore$ EITHER $\left(\frac{5+\sqrt{21}}{2},\frac{17+\sqrt{21}}{2}\right)$ OR $\left(\frac{5-\sqrt{21}}{2},\frac{17-\sqrt{21}}{2}\right)$

Question 17 (*****)

A circle C is centred at (a,a) and has radius a, where a is a positive constant.

The straight line L has equation

$$4x-3y+4=0.$$

Given that L is tangent to C at the point P, determine …

a) … an equation of C.

b) … the coordinates of P.

*You may **not** use a formula which determines the shortest distance of a point from a straight line in this question.*

$\boxed{}$, $\boxed{(x-1)^2+(y-1)^2=1}$, $\boxed{P\left(\frac{1}{5},\frac{8}{5}\right)}$

$\boxed{\begin{array}{l}\text{EQUATION OF } L: \; 4x-3y+4=0 \\ \text{EQUATION OF } C: \; (x-a)^2+(y-a)^2=a^2\end{array}}$

- SOLVING SIMULTANEOUSLY

$\left.\begin{array}{l}16(x-a)^2+16(y-a)^2=16a^2 \\ 4x=3y-4\end{array}\right\} \Rightarrow \left.\begin{array}{l}(4x-4a)^2+(4y-4a)^2=16a^2 \\ 4x=3y-4\end{array}\right\} \Rightarrow$

$\Rightarrow (3y-4-4a)^2+(4y-4a)^2=16a^2$

- $\boxed{(A+B+C)^2 \equiv A^2+B^2+C^2+2AB+2BC+2CA}$

$\Rightarrow \left\{\begin{array}{l}9y^2+16+16a^2-24y+32a-24ay \\ 16y^2 \quad \cancel{+16a^2} \quad -32ay\end{array}\right\} = \cancel{16a^2}$

$\Rightarrow 25y^2-24y-56ay+32a+16+16a^2=0$

$\Rightarrow 25y^2-(24+56a)y+(16a^2+32a+16)=0$

$\Rightarrow 25y^2-8(3+7a)y+16(a^2+2a+1)=0$

- IF L IS A TANGENT THE ABOVE EQUATION MUST HAVE REPEATED ROOTS, SO THE DISCRIMINANT MUST BE ZERO

$\Rightarrow [-8(3+7a)]^2-4\times25\times16(a^2+2a+1)=0$

$\Rightarrow 64(7a+3)^2-64\times25(a^2+2a+1)=0$

$\Rightarrow (7a+3)^2-25(a^2+2a+1)=0$

$\Rightarrow 49a^2-42a+9-25a^2-50a-25=0$

$\Rightarrow 24a^2-8a-16=0$

$\Rightarrow 3a^2-a-2=0$

$\Rightarrow (3a+2)(a-1)=0$

$\Rightarrow a = 1, \; \cancel{-\frac{2}{3}}$ ← RADIUS

$\therefore$ EQUATION OF THE CIRCLE

$(x-1)^2+(y-1)^2=1$ //

b) • IF $a=1$

$25y^2-8(3+7a)y+16(1+2a+a^2)=0$

$25y^2-80y+64=0$

- WE EXPECT A PERFECT SQUARE

$(5y-8)^2=0$

$\boxed{y=\frac{8}{5}}$

- $4x=3y-4$

$\Rightarrow 4x=3\left(\frac{8}{5}\right)-4$

$\Rightarrow 4x=\frac{24}{5}-4$

$\Rightarrow 20x=24-20$

$\Rightarrow 20x=4$

$\Rightarrow \boxed{x=\frac{1}{5}}$

$\therefore P\left(\frac{1}{5},\frac{8}{5}\right)$ //

Question 18 (*****)

A curve in the x-y plane has equation

$$x^2 + y^2 + 6x\cos\theta - 18y\sin\theta + 45 = 0,$$

where θ is a parameter such that $0 \le \theta < 2\pi$.

Given that curve represents a circle determine the range of possible values of θ.

$\boxed{}$, $\boxed{\left\{\frac{1}{4}\pi < \theta < \frac{3}{4}\pi\right\} \cup \left\{\frac{5}{4}\pi < \theta < \frac{7}{4}\pi\right\}}$

- START BY COMPLETING THE SQUARES

$\Rightarrow x^2 + y^2 + 6x\cos\theta - 18y\sin\theta + 45 = 0$

$\Rightarrow x^2 + 6x\cos\theta + y^2 - 18y\sin\theta + 45 = 0$

$\Rightarrow (x + 3\cos\theta)^2 - 9\cos^2\theta + (y - 9\sin\theta)^2 - 81\sin^2\theta + 45 = 0$

$\Rightarrow (x + 3\cos\theta)^2 + (y - 9\sin\theta)^2 = 9\cos^2\theta + 81\sin^2\theta - 45$

- NOW IF THIS REPRESENTS A CIRCLE, THEN

$\Rightarrow 9\cos^2\theta + 81\sin^2\theta - 45 > 0$

$\Rightarrow \cos^2\theta + 9\sin^2\theta - 5 > 0$

$\Rightarrow 1 + 8\sin^2\theta - 5 > 0$

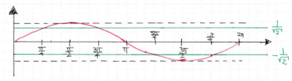
$\Rightarrow 8\sin^2\theta > 4$

$\Rightarrow \sin^2\theta > \frac{1}{2}$

- HENCE $\sin\theta > \frac{1}{\sqrt{2}}$ OR $\sin\theta < -\frac{1}{\sqrt{2}}$

$\left(\arcsin\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4}\right)$ $\left(\arcsin\left(-\frac{1}{\sqrt{2}}\right) = -\frac{\pi}{4}\right)$

- THUS IN THE REQUIRED RANGE



$\therefore \frac{\pi}{4} < \theta < \frac{3\pi}{4}$ OR $\frac{5\pi}{4} < \theta < \frac{7\pi}{4}$

Question 19 (*****)

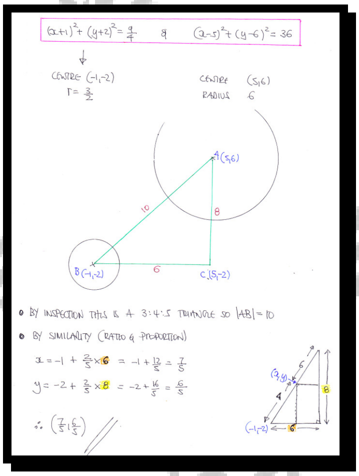
The circles C_1 and C_2 have respective equations

$$(x+1)^2+(y+2)^2=\frac{9}{4} \qquad \text{and} \qquad (x-5)^2+(y-6)^2=36.$$

The point P lies on C_2 so that the distance of P from C_1 is least.

Determine the exact coordinates of P.

$\boxed{}$, $\boxed{P\left(\frac{7}{5},\frac{6}{5}\right)}$



Question 20 (*****)

The straight line L and the circle C, have respective equations

$$L : y = \lambda(x-a) + a\sqrt{\lambda^2+1} \quad \text{and} \quad C : x^2 + y^2 = 2ax,$$

where a is a positive constant and λ is a parameter.

Show that for all values of λ, L is a tangent to C.

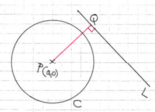
[ ] , [proof]

$L: \ y = \lambda(x-a) + a\sqrt{\lambda^2+1}$ $\qquad$ $C: \ x^2+y^2 = 2ax$

START BY REWRITING THE EQUATION OF THE CIRCLE

$x^2+y^2 = 2ax$
$x^2-2ax+y^2=0$
$(x-a)^2-a^2+y^2=0$
$(x-a)^2+y^2 = a^2$

CENTRE $(a,0)$, RADIUS a



THE GRADIENT OF L IS λ – PERPENDICULAR TO L PASSING THROUGH THE POINT $P(a,0)$ IS GIVEN BY

$y-0 = -\frac{1}{\lambda}(x-a)$
$-y\lambda = x-a$
$x = a-y\lambda$

SOLVING SIMULTANEOUSLY THE TWO LINES TO FIND THE COORDS OF Q

$y = \lambda(x-a) + a\sqrt{\lambda^2+1}$
$y = \lambda(a-y\lambda-a) + a\sqrt{\lambda^2+1}$
$y = -\lambda^2 y + a\sqrt{\lambda^2+1}$
$y+\lambda^2 y = a\sqrt{\lambda^2+1}$
$y(\lambda^2+1) = a\sqrt{\lambda^2+1}$
$y = \frac{a}{\sqrt{\lambda^2+1}}$

AND TO FIND THE x CO-ORDINATE

$x = a-y\lambda = a - \frac{a\lambda}{\sqrt{\lambda^2+1}}$

$\therefore Q\left(a-\frac{a\lambda}{\sqrt{\lambda^2+1}}, \frac{a}{\sqrt{\lambda^2+1}}\right)$

FINALLY THE DISTANCE PQ WHERE $P(a,0)$

$|PQ| = \sqrt{\left[\left(a-\frac{a\lambda}{\sqrt{\lambda^2+1}}\right)-a\right]^2 + \left[0-\frac{a}{\sqrt{\lambda^2+1}}\right]^2}$

$|PQ| = \sqrt{\left(-\frac{a\lambda}{\sqrt{\lambda^2+1}}\right)^2 + \left(-\frac{a}{\sqrt{\lambda^2+1}}\right)^2}$

$|PQ| = \sqrt{\frac{a^2\lambda^2}{\lambda^2+1} + \frac{a^2}{\lambda^2+1}}$

$|PQ| = \sqrt{\frac{a^2\lambda^2+a^2}{\lambda^2+1}} = \sqrt{\frac{a^2(\lambda^2+1)}{\lambda^2+1}} = \sqrt{a^2}$

$|PQ| = a$

$\therefore |PQ| = a =$ RADIUS OF THE CIRCLE AND IS INDEPENDENT OF λ

$\therefore$ THE LINE IS ALWAYS A TANGENT

NOTE THE STANDARD FORMULA WHICH FINDS THE DISTANCE OF A LINE FROM THE CENTRE OF THE CIRCLE CAN BE USED HERE TO MASSIVELY CUT OUT THE WORK

Question 21 (*****)

The straight line with equation

$$y = t(x-2),$$

where t is a parameter,

crosses the circle with equation

$$x^2 + y^2 = 1$$

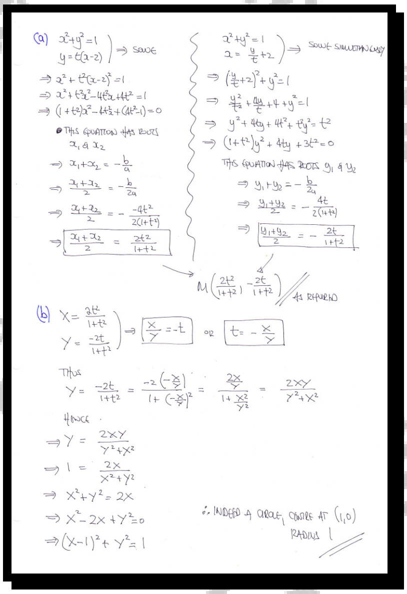
at two distinct points A and B.

a) Show that the coordinates of the midpoint of AB are given by

$$M\left(\frac{2t^2}{1+t^2}, -\frac{2t}{1+t^2}\right).$$

b) Hence show that the locus of M as t varies is a circle, stating its radius and the coordinates of its centre.

$$\boxed{(x-1)^2 + y^2 = 1}$$



(a) $\left.\begin{array}{l} x^2+y^2=1 \\ y=t(x-2) \end{array}\right\} \Rightarrow$ SOLVE

$\Rightarrow x^2 + t^2(x-2)^2 = 1$

$\Rightarrow x^2 + t^2x^2 - 4t^2x + 4t^2 = 1$

$\Rightarrow (1+t^2)x^2 - 4t^2x + (4t^2-1) = 0$

• THIS QUADRATIC HAS ROOTS x_1 & x_2

$\Rightarrow x_1 + x_2 = -\frac{b}{a}$

$\Rightarrow \frac{x_1+x_2}{2} = -\frac{b}{2a}$

$\Rightarrow \frac{x_1+x_2}{2} = -\frac{-4t^2}{2(1+t^2)}$

$\Rightarrow \boxed{\frac{x_1+x_2}{2} = \frac{2t^2}{1+t^2}}$

$\left.\begin{array}{l} x^2+y^2=1 \\ x=\frac{y}{t}+2 \end{array}\right\} \Rightarrow$ SOLVE SIMULTANEOUSLY

$\Rightarrow \left(\frac{y}{t}+2\right)^2 + y^2 = 1$

$\Rightarrow \frac{y^2}{t^2} + \frac{4y}{t} + 4 + y^2 = 1$

$\Rightarrow y^2 + 4ty + 4t^2 + t^2y^2 = t^2$

$\Rightarrow (1+t^2)y^2 + 4ty + 3t^2 = 0$

THIS QUADRATIC HAS ROOTS y_1 & y_2

$\Rightarrow y_1 + y_2 = -\frac{b}{2a}$

$\Rightarrow \frac{y_1+y_2}{2} = -\frac{4t}{2(1+t^2)}$

$\Rightarrow \boxed{\frac{y_1+y_2}{2} = -\frac{2t}{1+t^2}}$

$M\left(\frac{2t^2}{1+t^2}, -\frac{2t}{1+t^2}\right)$ // AS REQUIRED

(b) $\left.\begin{array}{l} X=\frac{2t^2}{1+t^2} \\ Y=\frac{-2t}{1+t^2} \end{array}\right\} \Rightarrow \boxed{\frac{X}{Y}=-t}$ OR $\boxed{t=-\frac{X}{Y}}$

THUS

$Y = \frac{-2t}{1+t^2} = \frac{-2\left(-\frac{X}{Y}\right)}{1+\left(-\frac{X}{Y}\right)^2} = \frac{\frac{2X}{Y}}{1+\frac{X^2}{Y^2}} = \frac{2XY}{Y^2+X^2}$

HENCE

$\Rightarrow Y = \frac{2XY}{Y^2+X^2}$

$\Rightarrow 1 = \frac{2X}{X^2+Y^2}$

$\Rightarrow X^2+Y^2 = 2X$

$\Rightarrow X^2 - 2X + Y^2 = 0$

$\Rightarrow (X-1)^2 + Y^2 = 1$

$\therefore$ INDEED A CIRCLE, CENTRE AT $(1,0)$ RADIUS 1 //

Question 22 (*****)

Two parallel straight lines, L_1 and L_2, have respective equations

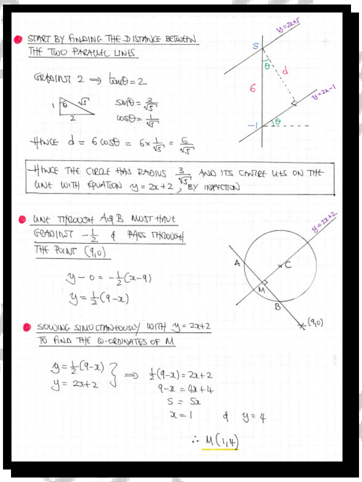
$$y = 2x + 5 \qquad \text{and} \qquad y = 2x - 1 .$$

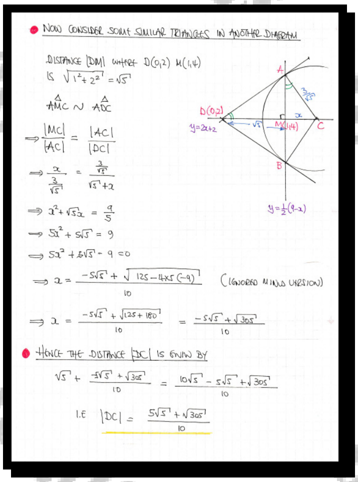
L_1 and L_2, are tangents to a circle centred at the point C .

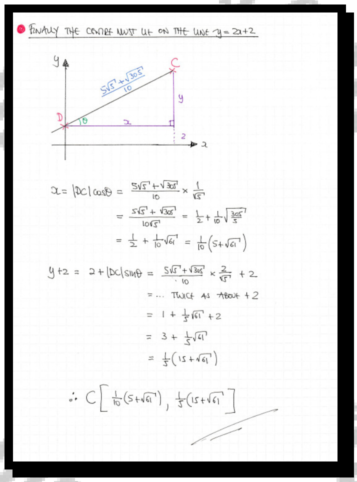
A third line L_3 is perpendicular to L_1 and L_2, and meets the circle in two distinct points, A and B.

Given that L_3 passes through the point $(9,0)$, find, in exact simplified surd form, the coordinates of C.

$$\boxed{}\,,\ \boxed{C\left[\tfrac{1}{10}\left(5+\sqrt{61}\right), \tfrac{1}{5}\left(15+\sqrt{61}\right)\right]}$$







Question 23 (*****)

Two circles, C_1 and C_2, have respective radii of 4 units and 1 unit and are touching each other externally at the point A.

The coordinates axes are tangents to C_1, whose centre P lies in the first quadrant.

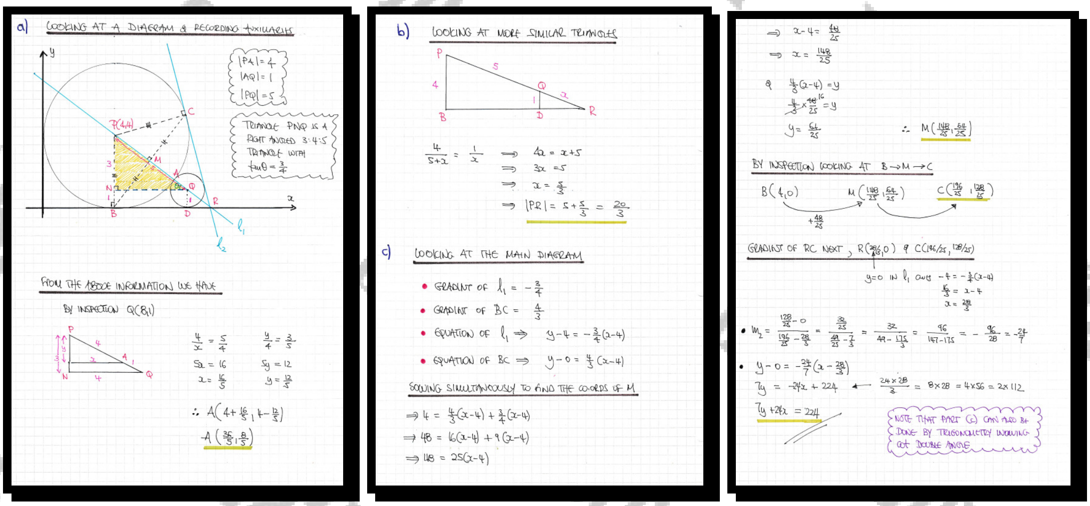
The x axis is a tangent to C_2, whose centre Q also lies in the first quadrant.

The straight line l_1, passes through P and Q, and meets the x axis at the point R.

The straight line l_2 has negative gradient, passes through R and is a common tangent to C_1 and C_2.

Determine, in any order and in exact form where appropriate, the coordinates of A, the length of PR and an equation of l_2.

$\boxed{}$, $\boxed{A\left(\frac{36}{5},\frac{8}{5}\right)}$, $\boxed{|PR|=\frac{20}{3}}$, $\boxed{24x+7y=224}$



Question 24 (*****)

A family of circles is passing through the points with coordinates $(2,1)$ and $(4,5)$

Show that the equation of every such circle has equation

$$x^2 + y^2 + 2x(2k-9) + 2ky = 6k - 41,$$

where k is a parameter.

$\boxed{}$, $\boxed{\text{proof}}$

- LET THE EQUATION OF THE CIRCLE BE

$$(x-A)^2 + (y-B)^2 = R^2$$

$$(2,1) \Rightarrow (2-A)^2 + (1-B)^2 = R^2$$
$$A^2 - 4A + 4 + B^2 - 2B + 1 = R^2$$
$$\underline{A^2 + B^2 - 4A - 2B = R^2 - 5}$$

$$(4,5) \Rightarrow (4-A)^2 + (5-B)^2 = R^2$$
$$A^2 - 8A + 16 + B^2 - 10B + 25 = R^2$$
$$\underline{A^2 + B^2 - 8A - 10B = R^2 - 41}$$

- SUBTRACTING $4A + 8B = 36$

$$A + 2B = 9$$
$$\boxed{A = 9 - 2B}$$

THUS WE HAVE

$$(9-2B)^2 + B^2 - 8(9-2B) - 10B = R^2 - 41$$

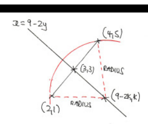
$$\begin{pmatrix} 4B^2 - 36B + 81 \\ B^2 + 16B - 72 \\ -10B \end{pmatrix} = R^2 - 41$$

$$\underline{5B^2 - 30B + 50 = R^2}$$

- HENCE THE EQUATION BECOMES

$$(x - 9 + 2B)^2 + (y-B)^2 = 5B^2 - 30B + 50$$
$$x^2 + \cancel{4B^2} + 81 - 18x + 4Bx - 36B + y^2 - 2By + \cancel{B^2} = \cancel{5B^2} - 30B + 50$$
$$x^2 + (4B-18)x + y^2 - 2By = 6B - 41$$
$$x^2 + 2(2k-9)x + y^2 + 2ky = 6k - 41$$

ALTERNATIVE



- GRADIENT $= \frac{5-1}{4-2} = 2$
- MIDPOINT $\left(\frac{1+5}{2}, \frac{2+4}{2}\right) = (3,3)$
- EQUATION OF THE CENTRE OF THE FAMILY OF CIRCLES IS GIVEN BY

$$y - 3 = -\tfrac{1}{2}(x-3)$$
$$2y - 6 = -x + 3$$
$$\underline{x = 9 - 2y}$$

- $\text{RADIUS}^2 = (9-2k-2)^2 + (k-1)^2 = (7-2k)^2 + (k-1)^2$
$$= 4k^2 - 28k + 49 + k^2 - 2k + 1 = 5k^2 - 30k + 50$$

- HENCE THE EQUATION OF THE CIRCLE WILL BE

$$(x - (9-2k))^2 + (y-k)^2 = (\text{RADIUS})^2$$
$$(x + 2k - 9)^2 + (y-k)^2 = 5k^2 - 30k + 50$$

⋮ WHICH AGREES WITH THE PREVIOUS SOLUTION TO GIVE THE DESIRED RESULT

$$x^2 + 2(2k-9)x + y^2 + 2ky = 6k - 41$$

Question 25 (*****)

Three circles, C_1, C_2 and C_3, have their centres at A, B and C, respectively, so that $|AB| = 5$, $|AC| = 4$ and $|BC| = 3$.

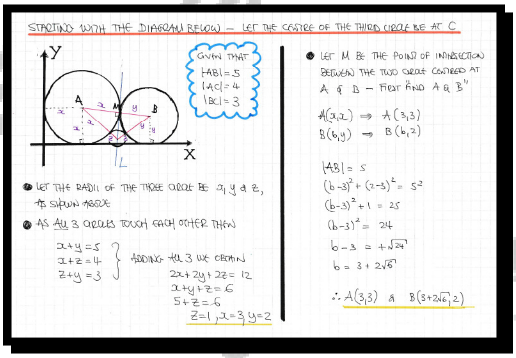
The positive x and y axis are tangents to C_1.

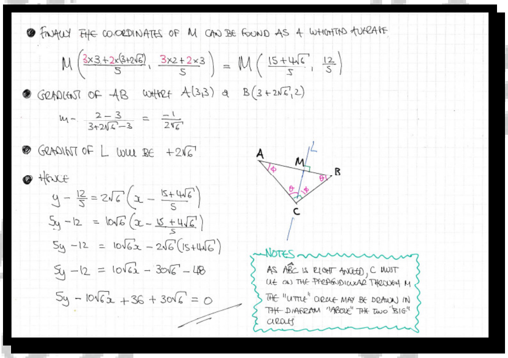
The positive x axis is a tangent to C_2.

C_1 and C_2 touch each other externally at the point M.

Given further that C_3 touches externally both C_1 and C_2, find, in exact simplified form, an equation of the straight line which passes through M and C.

$\boxed{}$, $\boxed{5y - 10\sqrt{6}\,x + 36 + 30\sqrt{6} = 0}$





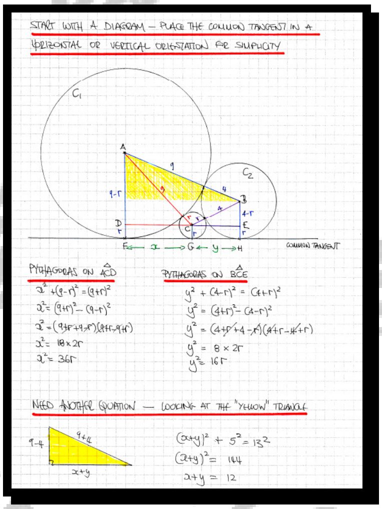
Question 26 (*****)

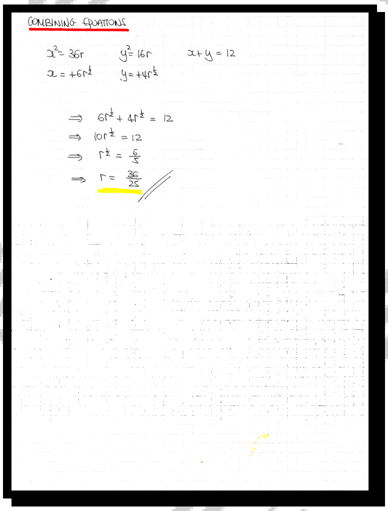
Two circles, C_1 and C_2, are touching each other **externally**, and have respective radii of 9 and 4 units.

A third circle C_3, of radius r, touches C_1 **and** C_2 **externally**.

Given further that all three circles have a common tangent, determine the value of r.

$\boxed{}$, $\boxed{r = \frac{36}{25} = 1.44}$





Question 27 (*****)

The point $A(6,-1)$ lies on the circle with equation

$$x^2 + y^2 - 4x + 6y = 7.$$

The tangent to the circle at A passes through the point P, so that the distance of P from the centre of the circle is $\sqrt{65}$.

Another tangent to the circle, at some point B, also passes through P.

Determine in any order the two sets of the possible coordinates of P and B.

$\boxed{}$, $\boxed{P(3,5) \cap B\left(-\frac{1}{13},-\frac{18}{13}\right) \ \cup \ P(9,-7) \cap B\left(\frac{30}{13},-\frac{97}{13}\right)}$

START WITH A DIAGRAM — THEN OBTAIN SOME STANDARD INFO

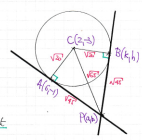
$\Rightarrow x^2+y^2-4x+6y=7$

$\Rightarrow x^2-4x+y^2+6y=7$

$\Rightarrow (x-2)^2-4+(y+3)^2-9=7$

$\Rightarrow (x-2)^2+(y+3)^2=20$

$C(2,-3)$, $r=\sqrt{20}$



TRIVIALLY BY PYTHAGORAS AND CIRCLE GEOMETRY $|AP|=|BP|=\sqrt{45}$

LET $P(a,b)$ & $B(k,h)$, AND MARK ALL KNOWN INFORMATION IN THE DIAGRAM

$\Rightarrow$ "GRAD AC" $\times$ "GRAD AP" $=-1$

$\Rightarrow \frac{-1-(-3)}{6-2} \times \frac{b-(-1)}{a-6} = -1$

$\Rightarrow \frac{2}{4} \times \frac{b+1}{a-6} = -1$

$\Rightarrow \frac{b+1}{a-6} = -2$

$\Rightarrow b+1 = -2a+12$

$\Rightarrow b = 11-2a$

NOW WE HAVE DISTANCE CONSTRAINTS, IE $|PA|=\sqrt{45}$ & $|PC|=\sqrt{65}$

$\left.\begin{array}{l}(a-6)^2+(b+1)^2=45\\(a-2)^2+(b+3)^2=65\end{array}\right\} \Rightarrow$ USING EITHER EQUATION WITH $b=11-2a$

$\Rightarrow (a-6)^2+\left[(11-2a)+1\right]^2=45$

$\Rightarrow a^2-12a+36+(12-2a)^2=45$

$\Rightarrow a^2-12a-9+(144-48a+4a^2)=0$

$\Rightarrow 5a^2-60a+135=0$

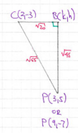
$\Rightarrow a^2-12a+27=0$

$\Rightarrow (a-9)(a-3)=0$

$\Rightarrow a = \left\langle\begin{array}{l}3\\9\end{array}\right. \qquad b=\left\langle\begin{array}{l}5\\-7\end{array}\right.$

$\therefore P(3,5)$ OR $P(9,-7)$

NOW LOOKING AT THE TRIANGLE $B\hat{C}P$



USE $P(3,5)$ FIRST

$|BP|=\sqrt{45}$ & $|BC|=\sqrt{20}$

$\Rightarrow \left.\begin{array}{l}(k-3)^2+(h-5)^2=45\\(k-2)^2+(h+3)^2=20\end{array}\right\}$

$\Rightarrow \begin{array}{r}k^2-6k+9+h^2-10h+25=45\\k^2-4k+4+h^2+6h+9=20\\\hline -2k+5 \quad -16h+16=25\end{array}$ SUBTRACT

$\Rightarrow -2k-16h-4=0$

$\Rightarrow -k-8h-2=0$

$\Rightarrow k=-2-8h$

SUBSTITUTE INTO $(k-2)^2+(h-3)^2=20$

$\Rightarrow \left[(-2-8h)-2\right]^2+(h+3)^2=20$

$\Rightarrow \left[-4-8h\right]^2+(h+3)^2=20$

$\Rightarrow (8h+4)^2+(h+3)^2=20$

$\Rightarrow 64h^2+64h+16+h^2+6h+9=20$

$\Rightarrow 65h^2+70h+5=0$

$\Rightarrow 13h+14h+1=0$

$\Rightarrow (13h+1)(h+1)=0$

$h=\left\langle\begin{array}{l}-1\\-\frac{1}{13}\end{array}\right. \qquad k=\left\langle\begin{array}{l}6\\-\frac{18}{13}\end{array}\right.$

$\therefore B\left(-\frac{1}{13},-\frac{18}{13}\right)$ & $A(-1,6)$ AS EXPECTED

USE $P(9,-7)$ NEXT

USING $|BP|=\sqrt{45}$ & $|BC|=\sqrt{20}$ AND SAME DIAGRAM

$\Rightarrow \left.\begin{array}{l}(k-9)^2+(h+7)^2=45\\(k-2)^2+(h+3)^2=20\end{array}\right\}$

$\Rightarrow \begin{array}{r}k^2-18k+81+h^2+14h+49=45\\k^2-4k+4+h^2+6h+9=20\\\hline -14k+77 \quad +8h+40=25\end{array}$ SUBTRACT

$\Rightarrow -14k+8h+92=0$

$\Rightarrow -7k+4h+46=0$

$\Rightarrow 4h=7k-46$

PROCEED BY THE SUBSTITUTION INTO $(k-2)^2+(h+3)^2=20$

$\Rightarrow (k-2)^2+(h+3)^2=20$

$\Rightarrow 16(k-2)^2+16(h+3)^2=320$

$\Rightarrow 16(k^2-4k+4)+(4h+12)^2=320$

$\Rightarrow 16k^2-64k+64+(7k-46+12)^2=320$

$\Rightarrow 16k^2-64k+64+(7k-34)^2=320$

$\Rightarrow 16k^2-64k+64+49k^2-476k+1156=320$

$\Rightarrow 65k^2-540k+900=0$

$\Rightarrow 13k^2-108k+180=0$

$\Rightarrow (13k-30)(k-6)=0$ FROM POINT A

$\Rightarrow k=\left\langle\begin{array}{l}6\\\frac{30}{13}\end{array}\right. \qquad h=\left\langle\begin{array}{l}-1\\-\frac{97}{13}\end{array}\right.$ $\leftarrow$ POINT A, $\leftarrow$ POINT B

HENCE THE REQUIRED ANSWERS ARE

EITHER $P(3,5)$ $B\left(-\frac{1}{13},-\frac{18}{13}\right)$

OR $P(9,-7)$ $B\left(\frac{30}{13},-\frac{97}{13}\right)$

PARABOLA

Question 1 **(**)**

The general point $P\left(9t^2, 18t\right)$, where t is a parameter, lies on the parabola with Cartesian equation

$$y^2 = 36x.$$

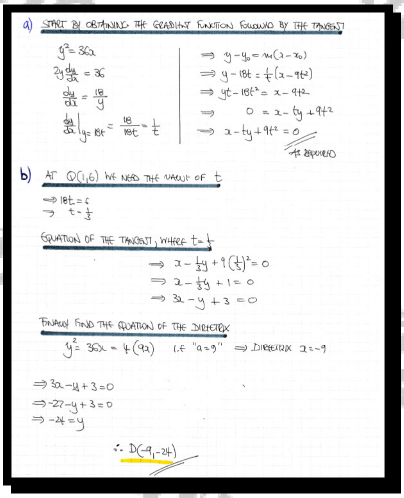
a) Show that the equation of a tangent at the point P is given by

$$x - ty + 9t^2 = 0.$$

The tangent to the parabola $y^2 = 36x$ at the point $Q\,(1,6)$ crosses the directrix of the parabola at the point D.

b) Find the coordinates of D.

$\boxed{}$, $\boxed{D(-9,-24)}$


a) START BY OBTAINING THE GRADIENT FUNCTION FOLLOWED BY THE TANGENT

$y^2 = 36x$

$2y\frac{dy}{dx} = 36$

$\frac{dy}{dx} = \frac{18}{y}$

$\left.\frac{dy}{dx}\right|_{y=18t} = \frac{18}{18t} = \frac{1}{t}$

$\Rightarrow y - y_0 = m(x - x_0)$

$\Rightarrow y - 18t = \frac{1}{t}(x - 9t^2)$

$\Rightarrow yt - 18t^2 = x - 9t^2$

$\Rightarrow 0 = x - ty + 9t^2$

$\Rightarrow x - ty + 9t^2 = 0$ AS REQUIRED

b) AT $Q(1,6)$ WE NEED THE VALUE OF t

$\Rightarrow 18t = 6$

$\Rightarrow t = \frac{1}{3}$

EQUATION OF THE TANGENT, WHERE $t = \frac{1}{3}$

$\Rightarrow x - \frac{1}{3}y + 9\left(\frac{1}{3}\right)^2 = 0$

$\Rightarrow x - \frac{1}{3}y + 1 = 0$

$\Rightarrow 3x - y + 3 = 0$

FINALLY FIND THE EQUATION OF THE DIRECTRIX

$y^2 = 36x = 4(9x)$ I.E "$a = 9$" $\Rightarrow$ DIRECTRIX $x = -9$

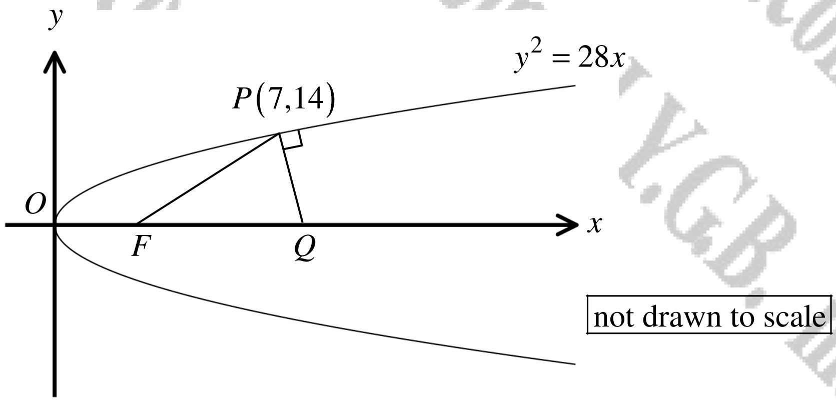
$\Rightarrow 3x - y + 3 = 0$

$\Rightarrow -27 - y + 3 = 0$

$\Rightarrow -24 = y$

$\therefore D(-9,-24)$

Question 2 **(**)**



The figure above shows the graph of the parabola with equation

$$y^2 = 28x,\ x \in \mathbb{R},\ x \geq 0.$$

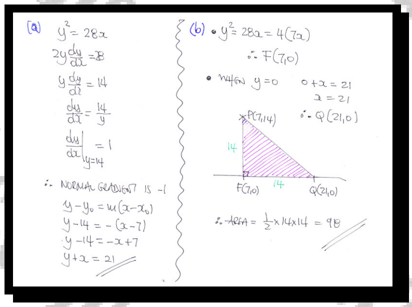
The point $P(7,14)$ lies on the parabola.

a) Find an equation of the normal to the parabola at P.

This normal meets the x axis at the point Q and F is the focus of the parabola.

b) Determine the area of the triangle PQF.

$\boxed{x + y = 21}$, $\boxed{\text{area} = 98}$


(a) $y^2 = 28x$

$2y\frac{dy}{dx} = 28$

$y\frac{dy}{dx} = 14$

$\frac{dy}{dx} = \frac{14}{y}$

$\left.\frac{dy}{dx}\right|_{y=14} = 1$

$\therefore$ NORMAL GRADIENT IS -1

$y - y_0 = m(x - x_0)$

$y - 14 = -(x - 7)$

$y - 14 = -x + 7$

$y + x = 21$

(b) • $y^2 = 28x = 4(7x)$

$\therefore F(7,0)$

• WHEN $y = 0$ $\quad 0 + x = 21$

$x = 21$

$\therefore Q(21,0)$

$\therefore$ AREA $= \frac{1}{2} \times 14 \times 14 = 98$

Question 3 (**+)

A parabola H has Cartesian equation

$$y^2 = 12x, \quad x \geq 0.$$

The point $P\left(3t^2, 6t\right)$, where t is a parameter, lies on H.

a) Show that the equation of a tangent to the parabola at P is given by

$$yt = x + 3t^2.$$

The tangent to the parabola at P meets the y axis at the point Q and the point S is the focus of the parabola.

b) Show further that …

i. … PQ is perpendicular to SQ.

ii. … the area of the triangle PQS is $\frac{9}{2}|t|\left(1+t^2\right)$.

proof

a) DIFFERENTIATING IMPLICITLY & FIND GRADIENT AT P

$$y^2 = 12x$$
$$2y\frac{dy}{dx} = 12$$
$$\frac{dy}{dx} = \frac{6}{y}$$
$$\left.\frac{dy}{dx}\right|_{y=6t} = \frac{6}{6t} = \frac{1}{t}$$

EQUATION OF THE TANGENT AT THE GENERAL POINT $P(3t^2, 6t)$

$$\Rightarrow y - y_0 = m(x - x_0)$$
$$\Rightarrow y - 6t = \frac{1}{t}(x - 3t^2)$$
$$\Rightarrow yt - 6t^2 = x - 3t^2$$
$$\Rightarrow yt = x + 3t^2$$

AS REQUIRED

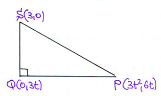
b) I) START BY OBTAINING THE COORDINATES OF Q & S

- WHEN $x = 0$
  $yt = 0 + 3t^2$
  $y = 3t$
  $\therefore Q(0, 3t)$
- $y^2 = 12x$
  $y^2 = 4 \times 3 \times x \quad (y^2 = 4ax)$
  $\therefore$ FOCUS $S(3, 0)$

- GRADIENT $PQ = \frac{6t - 3t}{3t^2 - 0} = \frac{3t}{3t^2} = \frac{1}{t}$ (INDEED AS EXPECTED!)
- GRADIENT $SQ = \frac{3t - 0}{0 - 3} = \frac{3t}{-3} = -t$

AS THEIR GRADIENTS ARE NEGATIVE RECIPROCALS OF ONE ANOTHER PQ IS PERPENDICULAR TO SQ

II) DRAWING A DIAGRAM TO RECORD THE INFO



- $|PQ| = \sqrt{(0-3t^2)^2 + (3t-6t)^2} = \sqrt{9t^4 + 9t^2}$
- $|SQ| = \sqrt{(3-0)^2 + (0-3t)^2} = \sqrt{9 + 9t^2}$
- $\text{AREA} = \frac{1}{2}|PQ||SQ| = \frac{1}{2}\sqrt{9t^4 + 9t^2}\sqrt{9 + 9t^2}$

$$= \frac{1}{2}|3t|\sqrt{t^2+1} \times 3\sqrt{1+t^2}$$
$$= \frac{9}{2}|t|(t^2+1)$$

AS REQUIRED

$\sqrt{x^2} \equiv |x|$

ALTERNATIVE FOR b II

AREA OF TRIANGLE WITH VERTICES AT $(x_1, y_1), (x_2, y_2), (x_3, y_3)$ IS GIVEN BY

$$\frac{1}{2}\left\|\begin{matrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{matrix}\right\|$$

$$\text{AREA} = \frac{1}{2}\left\|\begin{matrix} 1 & 1 & 1 \\ 3 & 0 & 3t^2 \\ 0 & 3t & 6t \end{matrix}\right\| = \frac{1}{2}\left|3 \times 3t \begin{vmatrix} 1 & 1 & 1 \\ 1 & 0 & t^2 \\ 0 & 1 & 2 \end{vmatrix}\right|$$

$$= \frac{9}{2}\left|t\left(\begin{vmatrix} 0 & t^2 \\ 1 & 2 \end{vmatrix} - \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix}\right)\right| \quad \leftarrow \text{EXPANDING BY THE FIRST COLUMN}$$

$$= \frac{9}{2}\left|t(-t^2-1)\right| = \frac{9}{2}\left|-t(t^2+1)\right| = \frac{9}{2}|t|(t^2+1)$$

Question 4 (***)

The general point $P\left(3t^2, 6t\right)$ lies on a parabola.

a) Show that the equation of a tangent at P is given by

$$ty = x + 3t^2.$$

The point $Q(-12, 9)$ does not lie on the parabola.

b) Find the equations of the two tangents to the parabola which pass through Q and deduce the coordinates of their corresponding points of tangency.

$\boxed{x + y + 3 = 0,\ \ (3, -6)}$, $\boxed{4y = x + 48,\ \ (48, 24)}$

(a) $\left.\begin{array}{l} x = 3t^2 \\ y = 6t \end{array}\right\} \Rightarrow \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{6}{6t} = \frac{1}{t}$ ← GRADIENT AT POINT WHERE $t = t$

HENCE AT $P(3t^2, 6t) \Rightarrow$ $y - 6t = \frac{1}{t}(x - 3t^2)$

$yt - 6t^2 = x - 3t^2$

$yt = x + 3t^2$ AS REQUIRED

(b) THE POINT $Q(-12, 9)$ LIES ON THE GENERAL TANGENT

THUS $9t = -12 + 3t^2$

$0 = 3t^2 - 9t + 12$

$0 = t^2 - 3t + 4$

$0 = (t + 1)(t - 4)$

$t = \begin{cases} -1 \\ 4 \end{cases} \Rightarrow x = \begin{cases} 3 \\ 48 \end{cases} \quad y = \begin{cases} -6 \\ 24 \end{cases}$

IF $t = -1 \Rightarrow -y = x + 3$

$x + y + 3 = 0$ TANGENT AT $(3, -6)$

IF $t = 4 \Rightarrow 4y = x + 48$ TANGENT AT $(48, 24)$

Question 5 (***)

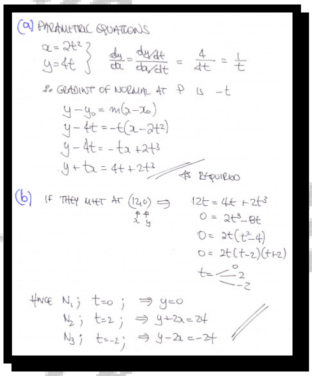
The general point $P\left(2t^2, 4t\right)$ lies on a parabola.

a) Show that the equation of a normal at P is given by

$$y + tx = 4t + 2t^3.$$

b) Find the equation of each of the three normals to the parabola that meet at the point with coordinates $(12,0)$.

$$\boxed{y = 0, \quad y + 2x = 24, \quad y - 2x = -24}$$



(a) PARAMETRIC EQUATIONS

$\left.\begin{array}{l} x = 2t^2 \\ y = 4t \end{array}\right\}$ $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{4}{4t} = \frac{1}{t}$

∴ GRADIENT OF NORMAL AT P IS $-t$

$y - y_0 = m(x - x_0)$

$y - 4t = -t(x - 2t^2)$

$y - 4t = -tx + 2t^3$

$y + tx = 4t + 2t^3$ // AS REQUIRED

(b) IF THEY MEET AT $(12,0) \Rightarrow$ $12t = 4t + 2t^3$

$0 = 2t^3 - 8t$

$0 = 2t(t^2 - 4)$

$0 = 2t(t-2)(t+2)$

$t = 0, 2, -2$

HENCE N_1; $t = 0$; $\Rightarrow y = 0$

N_2; $t = 2$; $\Rightarrow y + 2x = 24$

N_3; $t = -2$; $\Rightarrow y - 2x = -24$ //

Question 6 (***)

A parabola is defined parametrically by

$$x = at^2, \quad y = 2at, \quad t \in \mathbb{R},$$

where a is a positive constant and t is a parameter.

a) Show that an equation of a normal to the parabola at the point P, where $t = p$, $p \neq 0$, is given by

$$y + px = 2ap + ap^3.$$

The normal at P meets the x axis at the point Q.

The point R, lies on the x axis, so that PR is parallel to the y axis.

b) Show that the distance QR remains constant for all values of the parameter, and state this distance.

$$\boxed{|QR| = 2a}$$

(a) $\left.\begin{array}{l} x = at^2 \\ y = 2at \end{array}\right\} \Rightarrow \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2a}{2at} = \frac{1}{t}$

EQUATION OF NORMAL AT $P(ap^2, 2ap)$ IS

$y - 2ap = -p(x - ap^2)$

$y - 2ap = -px + ap^3$

$y + px = 2ap + ap^3$

AS REQUIRED

(b) WHEN $y = 0 \Rightarrow px = 2ap + ap^3$

$x = 2a + ap^2$

$x = a(2 + p^2)$

$\therefore Q\left[a(2+p^2), 0\right]$

AND

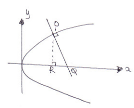
$R(ap^2, 0)$

$|QR| = a(2+p^2) - ap^2$

$= 2a + \cancel{ap^2} - \cancel{ap^2}$

$= 2a$

IT CONSTANT AT $2a$



Question 7 (***+)

The point $P\left(4p^2, 8p\right)$, $p \geq 0$, lies on the parabola with equation

$$y^2 = 16x\,,\ x \geq 0\,.$$

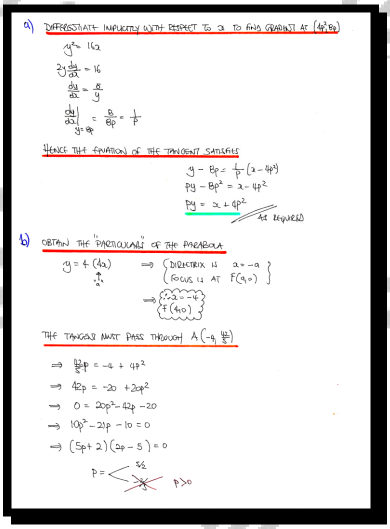
a) Show that the equation of the tangent to the parabola at P is given by

$$yp = x + 4p^2\,.$$

The tangent to the parabola at P meets the directrix of the parabola at the point A and the x axis at the point B. The point F is the focus of the parabola.

b) Given that the y coordinate of A is $\frac{42}{5}$, find the area of the triangle FBP.

$\boxed{}$, $\boxed{\text{area} = 290}$



a) DIFFERENTIATE IMPLICITLY WITH RESPECT TO x TO FIND GRADIENT AT $(4p^2, 8p)$

$$y^2 = 16x$$
$$2y\frac{dy}{dx} = 16$$
$$\frac{dy}{dx} = \frac{8}{y}$$
$$\left.\frac{dy}{dx}\right|_{y=8p} = \frac{8}{8p} = \frac{1}{p}$$

HENCE THE EQUATION OF THE TANGENT SATISFIES

$$y - 8p = \frac{1}{p}(x - 4p^2)$$
$$py - 8p^2 = x - 4p^2$$
$$py = x + 4p^2$$

AS REQUIRED

b) OBTAIN THE "PARTICULARS" OF THE PARABOLA

$$y = 4(4x) \implies \left\{\begin{array}{l}\text{DIRECTRIX IS } x = -a \\ \text{FOCUS IS AT } F(a, 0)\end{array}\right\}$$

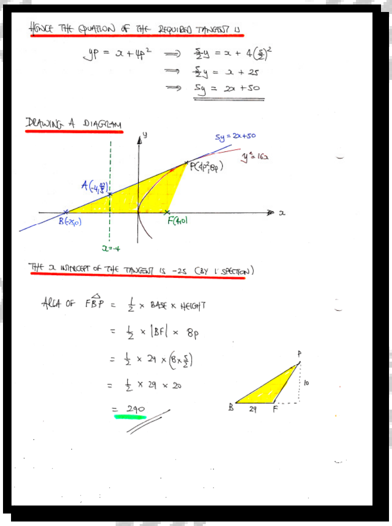
"a"

$$\implies \left\{\begin{array}{l} \therefore x = -4 \\ F(4, 0)\end{array}\right\}$$

THE TANGENT MUST PASS THROUGH $A\left(-4, \frac{42}{5}\right)$

$$\implies \frac{42}{5}p = -4 + 4p^2$$
$$\implies 42p = -20 + 20p^2$$
$$\implies 0 = 20p^2 - 42p - 20$$
$$\implies 10p^2 - 21p - 10 = 0$$
$$\implies (5p + 2)(2p - 5) = 0$$

$$p = \left\langle \begin{array}{l} 5/2 \\ \cancel{-2/5} \end{array}\right. \quad p > 0$$



HENCE THE EQUATION OF THE REQUIRED TANGENT IS

$$yp = x + 4p^2 \implies \frac{5}{2}y = x + 4\left(\frac{5}{2}\right)^2$$
$$\implies \frac{5}{2}y = x + 25$$
$$\implies 5y = 2x + 50$$

DRAWING A DIAGRAM

THE x INTERCEPT OF THE TANGENT IS -25 (BY INSPECTION)

$$\text{AREA OF } F\hat{B}P = \frac{1}{2} \times \text{BASE} \times \text{HEIGHT}$$
$$= \frac{1}{2} \times |BF| \times 8p$$
$$= \frac{1}{2} \times 29 \times \left(8 \times \frac{5}{2}\right)$$
$$= \frac{1}{2} \times 29 \times 20$$
$$= 290$$

Question 8 (***+)

The point $P\left(3p^2, 6p\right)$, $p > 0$, lies on the parabola with equation

$$y^2 = 12x \, , \; x \geq 0 \, .$$

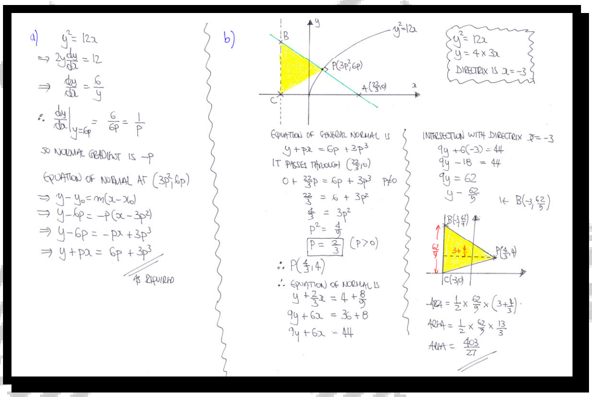
a) Show that the equation of the normal to the parabola at P is given by

$$y + px = 6p + 3p^3 \, .$$

The normal to the parabola at P meets the x axis at the point A and the directrix of the parabola at the point B. The point C is the point of intersection of the directrix of the parabola with the x axis.

b) Given that the coordinates of A are $\left(\frac{22}{3}, 0\right)$, find as an exact simplified fraction the area of the triangle BCP.

$$\boxed{\text{area} = \frac{403}{27}}$$



a) $y^2 = 12x$

$\Rightarrow 2y\frac{dy}{dx} = 12$

$\Rightarrow \frac{dy}{dx} = \frac{6}{y}$

$\therefore \left.\frac{dy}{dx}\right|_{y=6p} = \frac{6}{6p} = \frac{1}{p}$

SO NORMAL GRADIENT IS $-p$

EQUATION OF NORMAL AT $(3p^2, 6p)$

$\Rightarrow y - y_0 = m(x - x_0)$

$\Rightarrow y - 6p = -p(x - 3p^2)$

$\Rightarrow y - 6p = -px + 3p^3$

$\Rightarrow y + px = 6p + 3p^3$

AS REQUIRED

b)

$y^2 = 12x$

$y = 4 \times 3x$

DIRECTRIX IS $x = -3$

EQUATION OF GENERAL NORMAL IS

$y + px = 6p + 3p^3$

IT PASSES THROUGH $\left(\frac{22}{3}, 0\right)$

$0 + \frac{22}{3}p = 6p + 3p^3 \quad p \neq 0$

$\frac{22}{3} = 6 + 3p^2$

$\frac{4}{3} = 3p^2$

$p^2 = \frac{4}{9}$

$\boxed{p = \frac{2}{3}} \; (p > 0)$

$\therefore P\left(\frac{4}{3}, 4\right)$

$\therefore$ EQUATION OF NORMAL IS

$y + \frac{2}{3}x = 4 + \frac{8}{9}$

$9y + 6x = 36 + 8$

$9y + 6x = 44$

INTERSECTION WITH DIRECTRIX $x = -3$

$9y + 6(-3) = 44$

$9y - 18 = 44$

$9y = 62$

$y - \frac{62}{9}$ $\quad$ ie $B\left(-3, \frac{62}{9}\right)$

AREA $= \frac{1}{2} \times \frac{62}{9} \times \left(3 + \frac{4}{3}\right)$

AREA $= \frac{1}{2} \times \frac{62}{9} \times \frac{13}{3}$

AREA $= \frac{403}{27}$

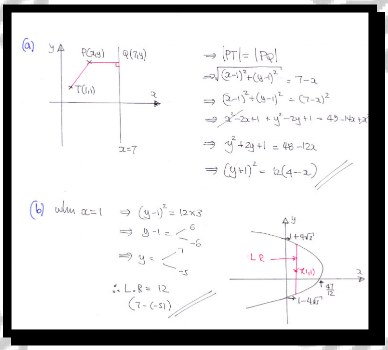
Question 9 **(***+)**

A parabola has its focus at $T(1,1)$ and its directrix has equation $x-7=0$.

a) Find an equation for the parabola.

b) Sketch the parabola and show that its latus rectum is 12 units.

$$\boxed{(y-1)^2 = 12(4-x)}$$



Question 10 (***+)

A parabola is given parametrically by the equations

$$x = 4 - t^2, \quad y = 1 - t, \quad t \in \mathbb{R}.$$

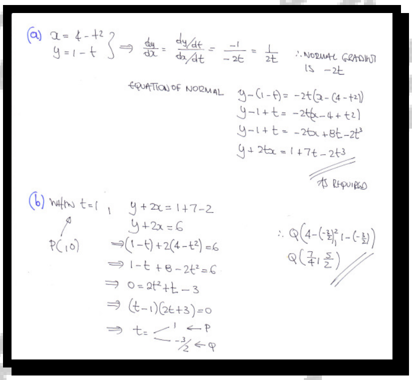
a) Show that an equation of the normal at the general point on the parabola is

$$y + 2tx = 1 + 7t - 2t^3.$$

The normal to parabola at $P(3,0)$ meets the parabola again at the point Q.

b) Find the coordinates of Q.

$$\boxed{Q\left(\frac{7}{4}, \frac{5}{2}\right)}$$



(a) $\left.\begin{array}{l} x = 4 - t^2 \\ y = 1 - t \end{array}\right\} \Rightarrow \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-1}{-2t} = \frac{1}{2t}$ ∴ NORMAL GRADIENT IS $-2t$

EQUATION OF NORMAL
$y - (1-t) = -2t(x - (4 - t^2))$
$y - 1 + t = -2t(x - 4 + t^2)$
$y - 1 + t = -2tx + 8t - 2t^3$
$y + 2tx = 1 + 7t - 2t^3$
AS REQUIRED

(b) WHEN $t = 1$, $y + 2x = 1 + 7 - 2$
$P(\ ,0)$ $\quad y + 2x = 6$
$\Rightarrow (1-t) + 2(4 - t^2) = 6$
$\Rightarrow 1 - t + 8 - 2t^2 = 6$
$\Rightarrow 0 = 2t^2 + t - 3$
$\Rightarrow (t-1)(2t+3) = 0$
$\Rightarrow t = 1 \leftarrow P, \ -\frac{3}{2} \leftarrow Q$

∴ $Q\left(4 - \left(-\frac{3}{2}\right)^2, 1 - \left(-\frac{3}{2}\right)\right)$
$Q\left(\frac{7}{4}, \frac{5}{2}\right)$

Question 11 (***+)

The points P and Q lie on the parabola with equation

$$y^2 = 2x,$$

so that OP is perpendicular to OQ, where O is the origin.

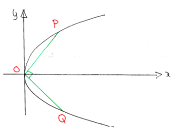
The point M is the midpoint of PQ.

Show that the Cartesian locus of M lies on the curve with equation

$$y^2 = x - 2.$$

FP3, proof

STARTING WITH A DIAGRAM



THE PARABOLA $y^2 = 4ax$ IS PARAMETERIZED AS $(at^2, 2at)$

LET $a = \frac{1}{2}$ & $y^2 = 2x$ IS PARAMETRIZED AS $\left(\frac{1}{2}t^2, t\right)$

$\Longrightarrow$ LET $P\left(\frac{1}{2}p^2, p\right)$ & $Q\left(\frac{1}{2}q^2, q\right)$

$\Longrightarrow$ GRAD OP $= \frac{p}{\frac{1}{2}p^2} = \frac{2}{p}$

$\Longrightarrow$ GRAD OQ $= \frac{q}{\frac{1}{2}q^2} = \frac{2}{q}$

$\Longrightarrow$ GRAD OP $\times$ GRAD OQ $= -1$

$\Longrightarrow \frac{2}{p} \times \frac{2}{q} = -1$

$\Longrightarrow pq = -4$

NOW GET AN EXPRESSION FOR THE MIDPOINT OF PQ

$$M\left(\frac{\frac{1}{2}p^2 + \frac{1}{2}q^2}{2}, \frac{p+q}{2}\right) = M\left(\frac{p^2+q^2}{4}, \frac{p+q}{2}\right)$$

ELIMINATE p & q, NOTING THAT $pq = -4$

$X = \frac{1}{4}(p^2+q^2)$

$4X = p^2 + q^2$

$Y = \frac{1}{2}(p+q)$

$2Y = p+q$

$4Y^2 = (p+q)^2$

$4Y^2 = p^2+q^2+2pq$

$4Y^2 = (p^2+q^2) + 2(pq)$

$4Y^2 = 4X + 2(-4)$

$4Y^2 = 4X - 8$

$Y^2 = X - 2$

Question 12 (***+)

The point P has coordinates

$$P\left(at^2, 2at\right),$$

where a is a positive constant and t is a real parameter.

The point P traces a parabola.

a) Show that the equation of a normal at the point P is given by

$$y + tx = 2at + at^3.$$

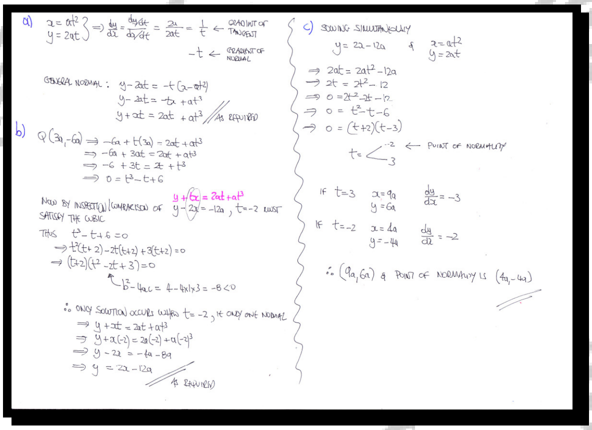
b) Show that the straight line with equation

$$y = 2x - 12a,$$

is the only normal to the parabola passing through the point $Q(3a, -6a)$.

c) Determine the coordinates of the two points of intersection between this normal and the parabola, indicating clearly which point of intersection represents the point of normality.

$\boxed{(9a, 6a), \text{ normal at } (4a, -4a)}$



a) $\left.\begin{array}{l} x = at^2 \\ y = 2at \end{array}\right\} \Rightarrow \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2a}{2at} = \frac{1}{t} \leftarrow$ GRADIENT OF TANGENT

$-t \leftarrow$ GRADIENT OF NORMAL

GENERAL NORMAL: $y - 2at = -t(x - at^2)$

$y - 2at = -tx + at^3$

$y + xt = 2at + at^3$ // AS REQUIRED

b) $Q(3a, -6a) \Rightarrow -6a + t(3a) = 2at + at^3$

$\Rightarrow -6a + 3at = 2at + at^3$

$\Rightarrow -6 + 3t = 2t + t^3$

$\Rightarrow 0 = t^3 - t + 6$

NOW BY INSPECTION/COMPARISON OF $y + tx = 2at + at^3$, $y - 2x = -12a$, $t = -2$ MUST SATISFY THE CUBIC

THUS $t^3 - t + 6 = 0$

$\Rightarrow t^2(t + 2) - 2t(t + 2) + 3(t + 2) = 0$

$\Rightarrow (t + 2)(t^2 - 2t + 3) = 0$

$b^2 - 4ac = 4 - 4 \times 1 \times 3 = -8 < 0$

$\therefore$ ONLY SOLUTION OCCURS WHEN $t = -2$, IE ONLY ONE NORMAL

$\Rightarrow y + xt = 2at + at^3$

$\Rightarrow y + x(-2) = 2a(-2) + a(-2)^3$

$\Rightarrow y - 2x = -4a - 8a$

$\Rightarrow y = 2x - 12a$ // AS REQUIRED

c) SOLVING SIMULTANEOUSLY

$y = 2x - 12a$ & $\begin{array}{l} x = at^2 \\ y = 2at \end{array}$

$\Rightarrow 2at = 2at^2 - 12a$

$\Rightarrow 2t = 2t^2 - 12$

$\Rightarrow 0 = 2t^2 - 2t - 12$

$\Rightarrow 0 = t^2 - t - 6$

$\Rightarrow 0 = (t + 2)(t - 3)$

$t = -2 \leftarrow$ POINT OF NORMALITY, $t = 3$

IF $t = 3$: $x = 9a$, $y = 6a$, $\frac{dy}{dx} = -3$

IF $t = -2$: $x = 4a$, $y = -4a$, $\frac{dy}{dx} = -2$

$\therefore (9a, 6a)$ & POINT OF NORMALITY IS $(4a, -4a)$

Question 13 (***+)

A straight line L is a tangent to the parabola with equation

$$y^2 = Ax$$

where A is a positive constant.

Given that L does not pass through the origin O, show that the product of the gradient and the y intercept of L equals the x coordinate of the focus of the parabola.

$\boxed{\text{proof}}$

$$\left.\begin{array}{l} y^2 = 4ax \\ y = mx + c \end{array}\right\} \text{SOLVING SIMULTANEOUSLY} \quad \left.\begin{array}{l} y^2 = 4ax \\ y^2 = (mx+c)^2 \end{array}\right\} \Rightarrow$$

$$\Rightarrow (mx+c)^2 = 4ax$$

$$\Rightarrow m^2x^2 + 2mcx + c^2 = 4ax$$

$$\Rightarrow m^2x^2 + 2mcx - 4ax + c^2 = 0$$

$$\Rightarrow m^2x^2 + (2mc - 4a)x + c^2 = 0$$

BUT $y = mx + c$ IS A TANGENT $\Rightarrow$ "$B^2 - 4AC = 0$"

$$\Rightarrow (2mc - 4a)^2 - 4m^2c^2 = 0$$

$$\Rightarrow \cancel{4m^2c^2} - 16mca + 16a^2 - \cancel{4m^2c^2} = 0$$

$$\Rightarrow -\cancel{16}mca + \cancel{16}a^2 = 0$$

$$\Rightarrow a^2 = mca$$

$$\Rightarrow mc = a$$

Question 14 (***+)

Sketch the parabola with equation

$$y^2 - 4y - 2x = 2.$$

The sketch must include the …

a) … coordinates of points of intersection with the coordinate axes.

b) … coordinates of the vertex of the parabola.

c) … coordinates of the focus of the parabola.

d) … equation of the directrix of the parabola.

graph

REWRITE IN THE "STANDARD" FORM

$\Rightarrow y^2 - 4y - 2x = 2$

$\Rightarrow (y-2)^2 - 4 = 2x + 2$

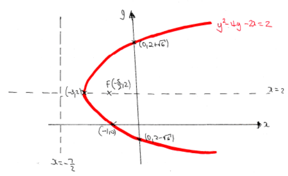
$\Rightarrow (y-2)^2 = 2x + 6$

$\Rightarrow (y-2)^2 = 2(x+3)$

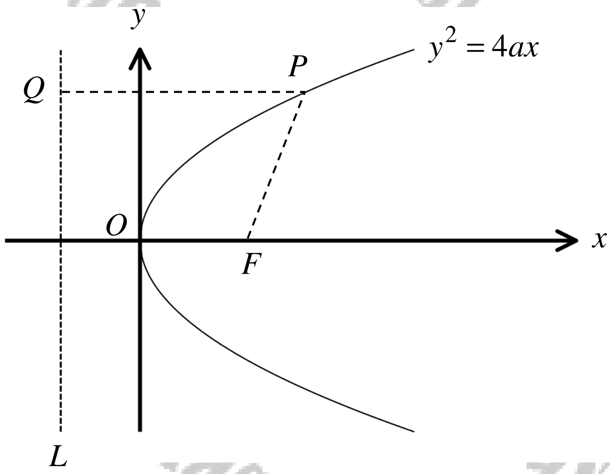
$Y^2 = 2X$

PARABOLA WITH $a = \frac{1}{2}$ WHEN COMPARED WITH $Y^2 = 4aX$, TRANSLATED BY THE VECTOR $\begin{pmatrix} -3 \\ 2 \end{pmatrix}$

- VERTEX $(0,0) \longmapsto (-3,2)$
- FOCUS $(\frac{1}{2},0) \longmapsto (-\frac{5}{2},2)$
- DIRECTRIX $x = -\frac{1}{2} \longmapsto x = -\frac{7}{2}$
- $x=0$ $(y-2)^2 = 6$, $y - 2 = \pm\sqrt{6}$, $y = 2 \pm \sqrt{6}$
- $y=0$ $x = -1$



Question 15 (***+)



The figure above shows the sketch of the parabola with equation

$$y^2 = 4ax,$$

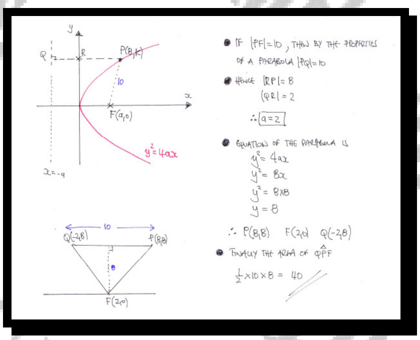
where a is a positive constant.

The straight line L and the point F are the directrix and the focus of the parabola, respectively.

The point $P(8, y)$, $y > 0$, lies on the parabola. The point Q lies on L, so that QP is parallel to the x axis.

Given further that $|PF| = 10$, determine the area of the triangle FPQ.

$\boxed{\text{area} = 40}$



Question 16 (***+)

The point $T\left(at^2, 2at\right)$, lies on the parabola with equation

$$y^2 = 4ax, \quad a > 0, \; x \geq 0.$$

a) Show clearly that an equation of a normal to the parabola at the point $P\left(ap^2, 2ap\right)$, $p \neq 0$, can be written as

$$y + px = 2ap + ap^3.$$

The normal at P meets the x axis at the point Q.

The midpoint of PQ is M.

b) Show that the locus of M as p varies is the parabola with equation

$$y^2 = a(x-a).$$

c) Find the coordinates of the focus of $y^2 = a(x-a)$.

$$\boxed{\left(\frac{5}{4}a, 0\right)}$$

(a) $\left.\begin{array}{l} x = at^2 \\ y = 2at \end{array}\right\} \Rightarrow \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2a}{2at} = \frac{1}{t}$

NORMAL AT $P(ap^2, 2ap)$, GRADIENT $-p$

$y - 2ap = -p(x - ap^2)$

$y - 2ap = -px + ap^3$

$y + px = 2ap + ap^3$

AS REQUIRED

(b) WHEN $y = 0$, $px = 2ap + ap^3$

$x = 2a + ap^2 \quad (p \neq 0)$

$\therefore Q(2a + ap^2, 0)$ & $P(ap^2, 2ap)$

$\therefore M\left[\frac{2a + ap^2 + ap^2}{2}, \frac{2ap + 0}{2}\right] = M\left[a + ap^2, ap\right]$

THUS $\left.\begin{array}{l} x = a(1+p^2) \\ y = ap \end{array}\right\} \Rightarrow \left.\begin{array}{l} x = a(1+p^2) \\ y^2 = a^2p^2 \end{array}\right\} \Rightarrow x = a\left(1 + \frac{y^2}{a^2}\right)$

$\Rightarrow x = a + \frac{y^2}{a}$

$\Rightarrow ax = a^2 + y^2$

$\Rightarrow y^2 = ax - a^2$

$\Rightarrow y^2 = a(x-a)$

AS REQUIRED

(c) $y^2 = 4ax$, HAS FOCUS AT $(a, 0)$

$y^2 = 4a\left(\frac{1}{4}x\right) = 4\left(\frac{1}{4}a\right)x = ax$, HAS FOCUS AT $\left(\frac{1}{4}a, 0\right)$

$y^2 = a(x-a)$ IS A TRANSLATION "RIGHT" BY a

$\therefore$ THE FOCUS IS AT $\left(\frac{5}{4}a, 0\right)$

Question 17 (***+)

The point $T\left(at^2, 2at\right)$, where a is a positive constant and t is a real parameter, lies on the parabola with equation

$$y^2 = 4ax.$$

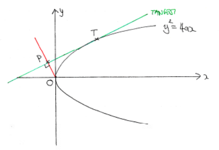
A straight line passing through the origin, intersects at right angles the tangent to the parabola at T, at the point P.

Show that as t varies, the Cartesian locus of P is

$$x^3 + xy^2 + ay^2 = 0.$$

FP3-[illegible], proof

STARTING WITH A DIAGRAM



$T(at^2, 2at)$

$y^2 = 4ax$

$2y\frac{dy}{dx} = 4a$

$\frac{dy}{dx} = \frac{2a}{y}$

$\left.\frac{dy}{dx}\right|_{y=2at} = \frac{1}{t}$

EQUATION OF THE TANGENT IS

$y - 2at = \frac{1}{t}(x - at^2)$

EQUATION OF THE LINE "OP" IS

$y = -tx$

SOLVING SIMULTANEOUSLY

$-tx - 2at = \frac{1}{t}(x - at^2)$

$-t^2x - 2at^2 = x - at^2$

$-at^2 = x + t^2x$

$-at^2 = x(1+t^2)$

$x = \frac{-at^2}{1+t^2}$ & $y = \frac{at^3}{1+t^2}$

i.e. $P\left(\frac{-at^2}{1+t^2}, \frac{at^3}{1+t^2}\right)$

ELIMINATE THE PARAMETER t

$\left.\begin{aligned} x &= -\frac{at^2}{1+t^2} \\ y &= \frac{at^3}{1+t^2} \end{aligned}\right\}$ DIVIDING $\frac{y}{x} = -t$

$t = -\frac{y}{x}$

SUBSTITUTE INTO EITHER EQUATION

$x = -\frac{a\left(-\frac{y}{x}\right)^2}{1+\left(-\frac{y}{x}\right)^2}$

$x = -\frac{\frac{ay^2}{x^2}}{1+\frac{y^2}{x^2}}$

$x = -\frac{ay^2}{x^2+y^2}$

$x^3 + xy^2 = -ay^2$

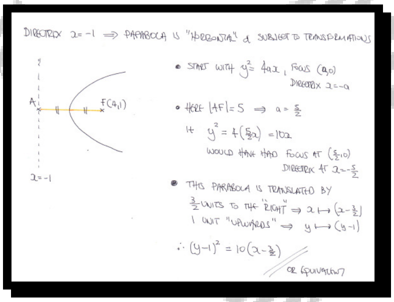
$x^3 + xy^2 + ay^2 = 0$ // AS REQUIRED

Question 18 (****)

A parabola has its focus at the point with coordinates $(4,1)$ and its directrix has equation $x=-1$.

Determine a Cartesian equation of the parabola.

$$\boxed{(y-1)^2 = 10\left(x-\frac{3}{2}\right)}$$



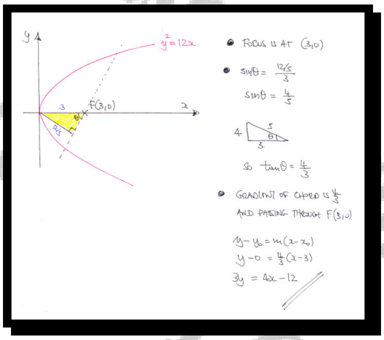
Question 19 (****)

A straight line L passes through the focus of the parabola with equation

$$y^2 = 12x.$$

Given further that the shortest distance of L from the origin O is $\frac{12}{5}$, determine an equation for L.

$$\boxed{3y = 4x - 12}$$



Question 20 (****)

A parabola P has Cartesian equation

$$y^2 - 4y - 8x + 28 = 0 .$$

a) Determine …

i. … the coordinates of the vertex of P.
ii. … the coordinates of the focus of P.
iii. … the equation of the directrix of P.

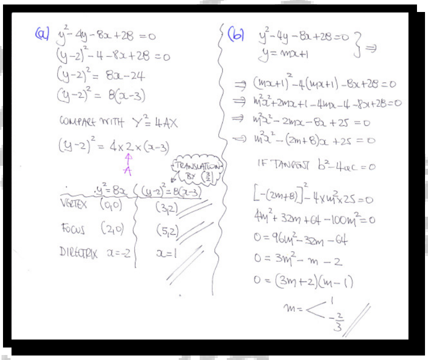
The line with equation

$$y = mx + 1, \text{ where } m \text{ is a constant,}$$

is a tangent at some point of P.

b) Find the possible values of m.

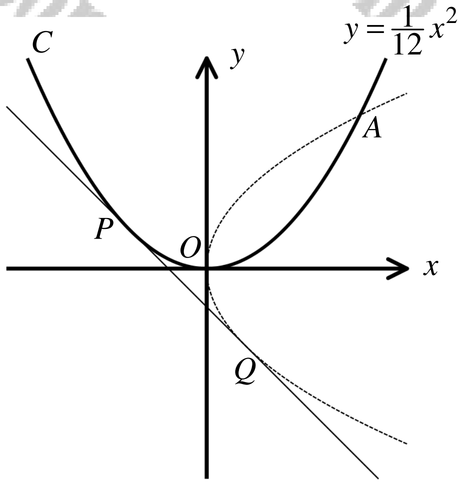
$\boxed{\text{vertex at } (3,2), \quad \text{focus at } (5,2), \quad \text{directrix} \quad x = 1}$, $\boxed{m = -\frac{2}{3}, 1}$


(a) $y^2 - 4y - 8x + 28 = 0$
$(y-2)^2 - 4 - 8x + 28 = 0$
$(y-2)^2 = 8x - 24$
$(y-2)^2 = 8(x-3)$
COMPARE WITH $Y^2 = 4AX$
$(y-2)^2 = 4 \times 2 \times (x-3)$
A
TRANSLATION BY $\binom{3}{2}$

	$y^2 = 8x$	$(y-2)^2 = 8(x-3)$
VERTEX	$(0,0)$	$(3,2)$
FOCUS	$(2,0)$	$(5,2)$
DIRECTRIX	$x = -2$	$x = 1$

(b) $\left.\begin{array}{l} y^2 - 4y - 8x + 28 = 0 \\ y = mx + 1 \end{array}\right\} \Rightarrow$
$\Rightarrow (mx+1)^2 - 4(mx+1) - 8x + 28 = 0$
$\Rightarrow m^2x^2 + 2mx + 1 - 4mx - 4 - 8x + 28 = 0$
$\Rightarrow m^2x^2 - 2mx - 8x + 25 = 0$
$\Rightarrow m^2x^2 - (2m+8)x + 25 = 0$
IF TANGENT $b^2 - 4ac = 0$
$[-(2m+8)]^2 - 4 \times m^2 \times 25 = 0$
$4m^2 + 32m + 64 - 100m^2 = 0$
$0 = 96m^2 - 32m - 64$
$0 = 3m^2 - m - 2$
$0 = (3m+2)(m-1)$
$m = \begin{cases} 1 \\ -\frac{2}{3} \end{cases}$

Question 21 (****)



The figure above shows the parabola C with equation $y = \dfrac{1}{12}x^2$.

The dotted line in the figure is the reflection of C in the line $y = x$.

a) Find the exact distance between the focus of C and the focus of its reflection.

The parabola intersects its reflection at the origin and at the point A.

b) Determine the coordinates of A.

The straight line L is a common tangent to both C and the reflection of C.

L touches C at the point P and the reflection of C at the point Q.

c) Determine the coordinates of P and Q.

$\boxed{}$, $\boxed{3\sqrt{2}}$, $\boxed{A(12,12)}$, $\boxed{P(-6,3)}$, $\boxed{Q(3,-6)}$

a) LOOKING AT THE REFLECTION OF THE PARABOLA

$y = \frac{1}{12}x^2 \longmapsto x = \frac{1}{12}y^2 \longmapsto y^2 = 12x \longmapsto y^2 = 4(3x)$

SWAP x AND y, FOR THE REFLECTION IN THE LINE $y=x$

FOCUS AT $(3,0)$

SO FOCUS OF $y = \frac{1}{12}x^2$ WILL BE AT $(0,3)$

$\therefore d = \sqrt{3^2 + 3^2} = \sqrt{18} = 3\sqrt{2}$

b) POINT A, MUST LIE ON THE LINE $y = x$, SO

$\left.\begin{array}{l} y = x \\ y = \frac{1}{12}x^2 \end{array}\right\} \Rightarrow x = \frac{1}{12}x^2 \Rightarrow 12x = x^2 \Rightarrow 12 = x$ AS $x \neq 0$ AT A

$\therefore A(12,12)$

c) THE COMMON TANGENT MUST BE AT RIGHT ANGLES TO $y = x$

$\therefore L: y = -x + c$

SOLVING SIMULTANEOUSLY WITH EITHER CURVE AND LOOK FOR REPEATED ROOTS

$\left.\begin{array}{l} y = -x + c \\ y = \frac{1}{12}x^2 \end{array}\right\} \Rightarrow \frac{1}{12}x^2 = -x + c \Rightarrow x^2 = -12x + 12c \Rightarrow x^2 + 12x - 12c = 0$

NOW $b^2 - 4ac = 0$

$\Rightarrow 12^2 - 4 \times 1 \times (-12c) = 0$

$\Rightarrow 144 + 48c = 0$

$\Rightarrow 48c = -144$

$\Rightarrow c = -3$

$\therefore$ THE QUADRATIC WHICH SHOULD YIELD REPEATED ROOTS IS

$x^2 + 12x - 12(-3) = 0$

$x^2 + 12x + 36 = 0$

$(x+6)^2 = 0$

$x = -6$ & $y = \frac{1}{12}(-6)^2 = 3$

$\therefore P(-6,3)$ & ITS REFLECTION ABOUT $y = x$ $(6,-3)$

Question 22 **(****)**

The point $T\left(at^2, 2at\right)$, lies on the parabola with equation

$$y^2 = 4ax, \quad a > 0, \; x \geq 0.$$

a) Show clearly that an equation of a normal to the parabola at the point $P\left(ap^2, 2ap\right)$, $p \neq 0$, can be written as

$$y + px = 2ap + ap^3.$$

The normal at P re-intersects the parabola at the point $Q\left(aq^2, 2aq\right)$.

b) Show that

$$q = -\frac{p^2+2}{p}.$$

c) Given that the midpoint of PQ has coordinates $(5a, -2a)$, find the value of p.

$\boxed{p = 1}$

(a) $y^2 = 4ax$

$2y\frac{dy}{dx} = 4a$

$\frac{dy}{dx} = \frac{2a}{y}$

$\left.\frac{dy}{dx}\right|_{y=2ap} = \frac{2a}{2ap} = \frac{1}{p}$

EQUATION OF NORMAL

$\Rightarrow y - 2ap = -p(x - ap^2)$

$\Rightarrow y - 2ap = -px + ap^3$

$\Rightarrow y + px = 2ap + ap^3$ // AS REQUIRED

(b) $y + px = 2ap + ap^3$ PASSES THROUGH $(aq^2, 2aq)$

$2aq + p(aq^2) = 2ap + ap^3$

$2q + pq^2 = 2p + p^3$

$2q - 2p = p^3 - pq^2$

$2(q-p) = p(p^2 - q^2)$

$2\cancel{(q-p)}_{-1} = p\cancel{(p-q)}(p+q)$ (AS $p \neq q$)

$-2 = p(p+q)$

$-\frac{2}{p} = p + q$

$-p - \frac{2}{p} = q$

$q = -\left(\frac{p^2+2}{p}\right)$ $\quad \therefore q = -\frac{p^2+2}{p}$ // AS REQUIRED

(c) $\left(\frac{ap^2+aq^2}{2}, \frac{2ap+2aq}{2}\right) \equiv (5a, -2a)$ FROM y CO-ORDINATE $\boxed{p+q=-2}$

SO $\left.\begin{array}{l} q = -2-p \\ q = -\frac{p^2+2}{p} \end{array}\right\} \Rightarrow -2-p = -\frac{p^2+2}{p}$

$\Rightarrow 2+p = \frac{p^2+2}{p}$

$\Rightarrow 2p + \cancel{p^2} = \cancel{p^2} + 2 \quad (p \neq \infty)$

$\Rightarrow p = 1$

Question 23 (****)

The point $P\left(2t^2,4t\right)$, lies on the parabola with equation

$$y^2 = 8x \, , \ x \geq 0 \, .$$

a) Show that an equation of a tangent to the parabola at P, can be written as

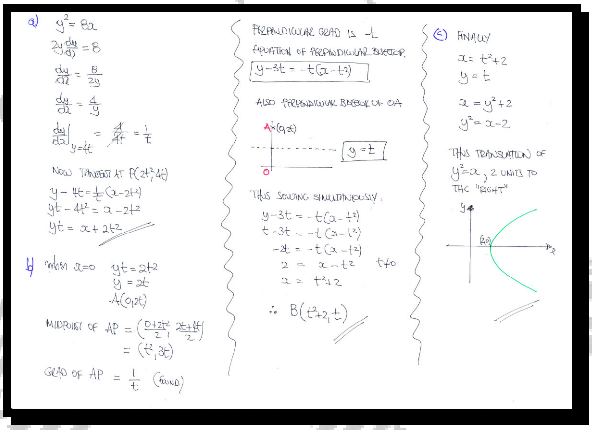
$$yt = x + 2t^2 \, , \ t \neq 0 \, .$$

The tangent to the parabola at P meets the y axis at the point A. The perpendicular bisectors of the straight line segments AP and OA, meets at the point B.

b) Find the coordinates of B, in terms of t.

c) Sketch the locus of B as t varies.

$$\boxed{B\left(t^2+2,t\right)}$$



a) $y^2 = 8x$

$2y\frac{dy}{dx} = 8$

$\frac{dy}{dx} = \frac{8}{2y}$

$\frac{dy}{dx} = \frac{4}{y}$

$\left.\frac{dy}{dx}\right|_{y=4t} = \frac{\cancel{4}}{\cancel{4}t} = \frac{1}{t}$

NOW TANGENT AT $P(2t^2, 4t)$

$y - 4t = \frac{1}{t}(x - 2t^2)$

$yt - 4t^2 = x - 2t^2$

$yt = x + 2t^2$

b) WHEN $x=0$ $\quad yt = 2t^2$

$y = 2t$

$A(0, 2t)$

MIDPOINT OF $AP = \left(\frac{0+2t^2}{2}, \frac{2t+4t}{2}\right) = (t^2, 3t)$

GRAD OF $AP = \frac{1}{t}$ (FOUND)

PERPENDICULAR GRAD IS $-t$

EQUATION OF PERPENDICULAR BISECTOR.

$\boxed{y - 3t = -t(x - t^2)}$

ALSO PERPENDICULAR BISECTOR OF OA

A $(0,2t)$

$\boxed{y = t}$

O

THUS SOLVING SIMULTANEOUSLY.

$y - 3t = -t(x - t^2)$

$t - 3t = -t(x - t^2)$

$-2t = -t(x - t^2)$

$2 = x - t^2 \quad t \neq 0$

$x = t^2 + 2$

$\therefore B(t^2+2, t)$

c) FINALLY

$x = t^2 + 2$

$y = t$

$x = y^2 + 2$

$y^2 = x - 2$

THUS TRANSLATION OF $y^2 = x$, 2 UNITS TO THE "RIGHT"

Question 24 (****)

A parabola C has Cartesian equation

$$y^2 + 4y - 16x + 36 = 0.$$

a) Describe the transformations that map the graph of the curve with equation $y^2 = 16x$ onto the graph of C.

b) Determine the coordinates of the focus of C.

c) Show that …

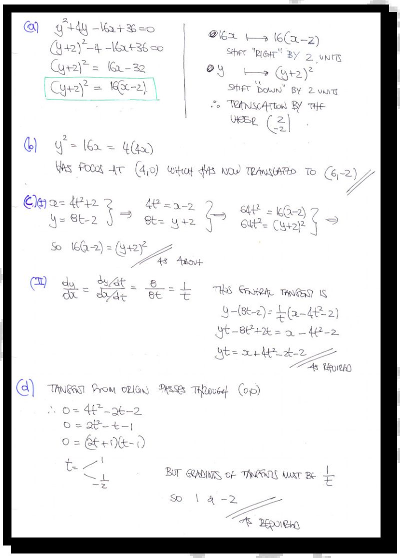
i. … the point $P\left(4t^2 + 2, 8t - 2\right)$, lies on the parabola.

ii. … the equation of a tangent to the parabola at the point P, is

$$yt = x + 4t^2 - 2t - 2.$$

d) Hence show that the gradients of the two tangents from the origin to the parabola have gradients -2 and 1.

$\boxed{\text{translation by vector } \begin{pmatrix} 2 \\ -2 \end{pmatrix}}$, $\boxed{(6,-2)}$



(a) $y^2+4y-16x+36=0$
$(y+2)^2-4-16x+36=0$
$(y+2)^2 = 16x-32$
$\boxed{(y+2)^2 = 16(x-2)}$

• $16x \longmapsto 16(x-2)$
SHIFT "RIGHT" BY 2 UNITS
• $y \longmapsto (y+2)^2$
SHIFT "DOWN" BY 2 UNITS
∴ TRANSLATION BY THE VECTOR $\begin{pmatrix} 2 \\ -2 \end{pmatrix}$

(b) $y^2 = 16x = 4(4x)$
HAS FOCUS AT $(4,0)$ WHICH HAS NOW TRANSLATED TO $(6,-2)$

(c)(I) $\left.\begin{array}{l} x = 4t^2+2 \\ y = 8t-2 \end{array}\right\} \Rightarrow \left.\begin{array}{l} 4t^2 = x-2 \\ 8t = y+2 \end{array}\right\} \Rightarrow \left.\begin{array}{l} 64t^2 = 16(x-2) \\ 64t^2 = (y+2)^2 \end{array}\right\} \Rightarrow$

SO $16(x-2) = (y+2)^2$
AS ABOVE

(II) $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{8}{8t} = \frac{1}{t}$

THUS GENERAL TANGENT IS
$y-(8t-2) = \frac{1}{t}(x-4t^2-2)$
$yt-8t^2+2t = x-4t^2-2$
$yt = x+4t^2-2t-2$
AS REQUIRED

(d) TANGENT FROM ORIGIN PASSES THROUGH $(0,0)$
$\therefore 0 = 4t^2-2t-2$
$0 = 2t^2-t-1$
$0 = (2t+1)(t-1)$
$t = \begin{cases} 1 \\ -\frac{1}{2} \end{cases}$

BUT GRADIENTS OF TANGENTS MUST BE $\frac{1}{t}$
SO 1 & -2
AS REQUIRED

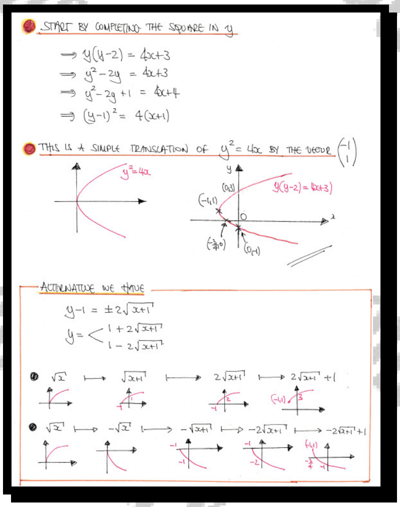
Question 25 (****)

Sketch the graph of the parabola with equation

$$y(y-2) = 4x+3.$$

The sketch must include the coordinates of any intersections with the axes and the coordinates of the vertex of the parabola.

graph



Question 26 (****)

The point $P\left(ap^2, 2ap\right)$, where p is a parameter, lies on the parabola, with Cartesian equation

$$y^2 = 4ax,$$

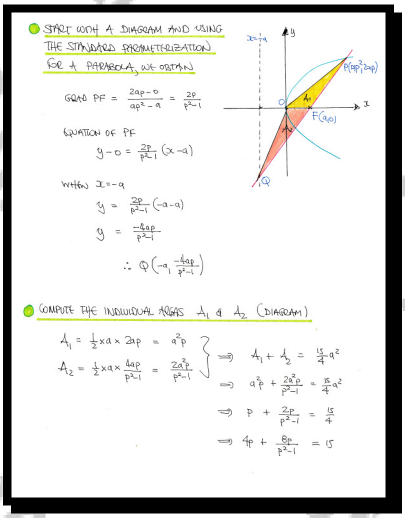
where a is a positive constant.

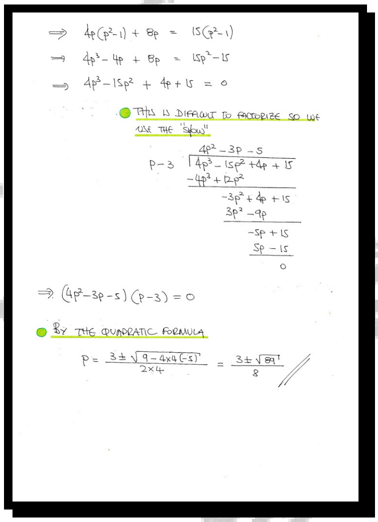
The point F is the focus of the parabola and O represents the origin.

The straight line which passes through P and F meets the directrix of the parabola at the point Q, so that the area of the triangle OPQ is $\frac{15}{4}a^2$.

Show that one of the possible values of p is 3 and find in exact surd form the other 2 possible values.

$\boxed{\quad\quad}$, $\boxed{p = \frac{1}{8}\left(3 \pm \sqrt{89}\right)}$





Question 27 (****+)

A parabola P has focus $S(6,0)$ and directrix the line $x=0$.

a) Show that a Cartesian equation for P is $y^2=12(x-3)$.

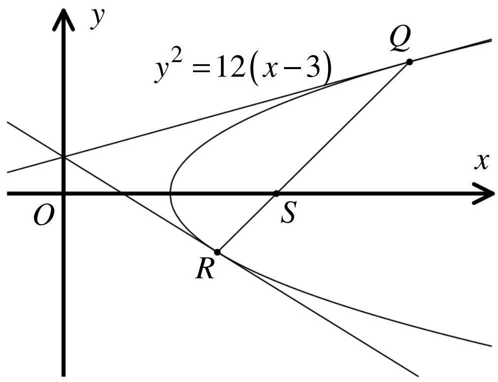
b) Verify that the parametric equations of P are

$$x=3t^2+3, \quad y=6t.$$

c) Show that the equation of the tangent at the point $Q\left(3q^2+3,6q\right)$ is

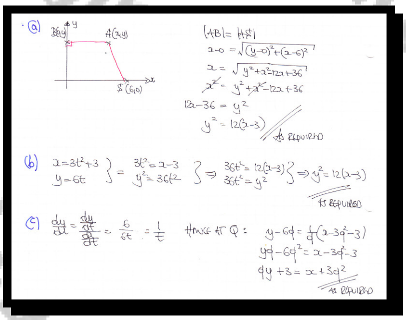
$$qy+3=x+3q^2$$

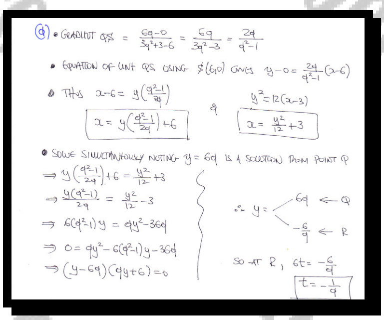
The diagram below shows the parabola and its tangents at the points Q and R. The point R lies on the parabola so that QSR is a straight line.

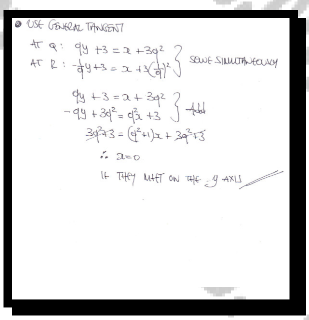


d) Show that the tangents to the parabola at Q and at R, meet on the y axis.

proof







Question 28 (****+)

A parabola C has parametric equations

$$x = -2t^2, \quad y = 4t$$

a) Determine the coordinates of the focus and the equation of directrix of C.

b) Show that an equation of the tangent to C, at the general point $T\left(-2t^2, 4t\right)$ is

$$yt + x = 2t^2$$

c) By considering the product of the roots of a suitable quadratic equation, show that any two tangents that meet on the directrix of C, are perpendicular.

$\boxed{F(-2,0)}$, $\boxed{x = 2}$

(a) $\left.\begin{array}{l} x = -2t^2 \\ y = 4t \end{array}\right\} \Rightarrow \left.\begin{array}{l} 8x = -16t^2 \\ y^2 = 16t^2 \end{array}\right\}$ Add $y^2 + 8x = 0$

$y^2 = -8x$

$y^2 = 4(-2x)$

∴ FOCUS AT $(-2,0)$

DIRECTRIX $x = 2$

(b) $y^2 = -8x$

$2y\frac{dy}{dx} = -8$

$\frac{dy}{dx} = -\frac{4}{y}$

$\left.\frac{dy}{dx}\right|_{y=4t} = -\frac{4}{4t} = -\frac{1}{t}$

TANGENT AT $(-2t^2, 4t)$, GRADIENT $-\frac{1}{t}$

$y - 4t = -\frac{1}{t}(x + 2t^2)$

$yt - 4t^2 = -x - 2t^2$

$yt + x = 2t^2$

AS REQUIRED

(c) IF TANGENTS CROSS AT THE DIRECTRIX SAY AT $P(2, \lambda)$, FOR SOME λ

$\lambda t + 2 = 2t^2$

$0 = 2t^2 - \lambda t - 2$

THE TWO ROOTS ARE t_1 & t_2

$t_1 t_2 = \frac{-2}{2}$

$\boxed{t_1 t_2 = -1}$

EQUATION OF TANGENTS ARE

- $yt_1 = -x + 2t_1^2$
  $y = -\frac{1}{t_1}x + 2t_1$ & SIMILARLY
- $y = -\frac{1}{t_2}x + 2t_2$

∴ $\frac{-1}{t_1} \times \frac{-1}{t_2} = \frac{1}{t_1 t_2} = -1$

∴ PERPENDICULAR

Question 29 (****+)

The point $P\left(at^2, 2at\right)$ lies on the parabola with equation

$$y^2 = 4ax,$$

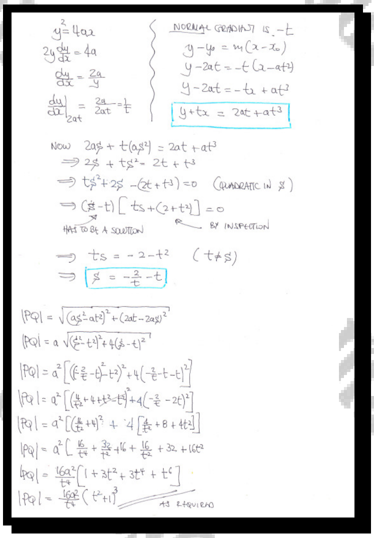
where a is a positive constant and t is a real parameter.

The normal to the parabola at P, meets the parabola again at the point $Q\left(as^2, 2as\right)$.

Show that

$$|PQ| = \frac{16a^2}{t^4}\left(t^2+1\right)^3.$$

$\boxed{\text{proof}}$



$y^2 = 4ax$

$2y\frac{dy}{dx} = 4a$

$\frac{dy}{dx} = \frac{2a}{y}$

$\left.\frac{dy}{dx}\right|_{2at} = \frac{2a}{2at} = \frac{1}{t}$

NORMAL GRADIENT IS $-t$

$y - y_0 = m(x - x_0)$

$y - 2at = -t(x - at^2)$

$y - 2at = -tx + at^3$

$\boxed{y + tx = 2at + at^3}$

NOW $2as + t(as^2) = 2at + at^3$

$\Rightarrow 2s + ts^2 = 2t + t^3$

$\Rightarrow ts^2 + 2s - (2t + t^3) = 0$ (QUADRATIC IN s)

$\Rightarrow (s - t)\left[ts + (2 + t^2)\right] = 0$

HAS TO BE A SOLUTION

BY INSPECTION

$\Rightarrow ts = -2 - t^2 \quad (t \neq s)$

$\Rightarrow \boxed{s = -\frac{2}{t} - t}$

$|PQ| = \sqrt{(as^2 - at^2)^2 + (2at - 2as)^2}$

$|PQ| = a\sqrt{(s^2 - t^2)^2 + 4(s - t)^2}$

$|PQ| = a^2\left[\left(\left(-\frac{2}{t} - t\right)^2 - t^2\right)^2 + 4\left(-\frac{2}{t} - t - t\right)^2\right]$

$|PQ| = a^2\left[\left(\frac{4}{t^2} + 4 + \cancel{t^2} - \cancel{t^2}\right)^2 + 4\left(-\frac{2}{t} - 2t\right)^2\right]$

$|PQ| = a^2\left[\left(\frac{4}{t^2} + 4\right)^2 + 4\left[\frac{4}{t^2} + 8 + 4t^2\right]\right]$

$|PQ| = a^2\left[\frac{16}{t^4} + \frac{32}{t^2} + 16 + \frac{16}{t^2} + 32 + 16t^2\right.$

$|PQ| = \frac{16a^2}{t^4}\left[1 + 3t^2 + 3t^4 + t^6\right]$

$|PQ| = \frac{16a^2}{t^4}(t^2 + 1)^3$

AS REQUIRED

Question 30 (****+)

A parabola C has Cartesian equation

$$y^2 = 4x \ , \ x \in \mathbb{R}, \ x \geq 0 .$$

The points $P\left(p^2, 2p\right)$ and $Q\left(q^2, 2q\right)$ are distinct and lie on C.

The tangent to C at P and the tangent to C at Q meet at $R\left(-1, \frac{15}{4}\right)$.

Calculate as an exact simplified fraction the area of the triangle PQR.

$$\boxed{\text{area} = \frac{4913}{128}}$$

$y^2 = 4x$

$2y\frac{dy}{dx} = 4$

$\frac{dy}{dx} = \frac{2}{y}$

$\left.\frac{dy}{dx}\right|_{y=2p} = \frac{2}{2p} = \frac{1}{p}$

EQUATION OF TANGENT AT P

$\Rightarrow y - y_0 = m(x - x_0)$

$\Rightarrow y - 2p = \frac{1}{p}(x - p^2)$

$\Rightarrow py - 2p^2 = x - p^2$

$\Rightarrow \boxed{py = x + p^2}$

& EQUATION OF TANGENT AT Q

$\boxed{qy = x + q^2}$

SOLVING SIMULTANEOUSLY

$\left.\begin{array}{l} py = x + p^2 \\ qy = x + q^2 \end{array}\right\} \Rightarrow$ SUBTRACT $(p-q)y = p^2 - q^2$

$(p-q)y = (p-q)(p+q)$

$\boxed{y = p + q}$ AS $p \neq q$

AND

$py = x + p^2$

$p(p+q) = x + p^2$

$p^2 + pq = x + p^2$

$\boxed{x = pq}$

$\left.\begin{array}{l} -1 = pq \\ p + q = \frac{15}{4} \end{array}\right\} \Rightarrow q = -\frac{1}{p}$

$\Rightarrow p - \frac{1}{p} = \frac{15}{4}$

$\Rightarrow p^2 - 1 = \frac{15}{4}p$

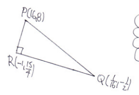
$\Rightarrow 4p^2 - 4 = 15p$

$\Rightarrow 4p^2 - 15p - 4 = 0$

$\Rightarrow (4p+1)(p-4) = 0$

$\Rightarrow p = \begin{cases} 4 \\ -\frac{1}{4} \end{cases} \quad \Rightarrow q = \begin{cases} -\frac{1}{4} \\ 4 \end{cases}$

$\therefore \boxed{p = 4, \ q = -\frac{1}{4}}$ OR THE OTHER ROUND



$P(p^2, 2p) \Rightarrow P(16, 8)$

$Q(q^2, 2q) \Rightarrow Q\left(\frac{1}{16}, -\frac{1}{2}\right)$

$|PR| = \sqrt{(16+1)^2 + \left(8 - \frac{15}{4}\right)^2} = \sqrt{17^2 + \left(\frac{17}{4}\right)^2} = \sqrt{289 + \frac{289}{16}} = 17\sqrt{1 + \frac{1}{16}} = 17\sqrt{\frac{17}{16}} = \frac{17}{4}\sqrt{17}$

$|QR| = \sqrt{\left(-1 - \frac{1}{16}\right)^2 + \left(-\frac{1}{2} - \frac{15}{4}\right)^2} = \sqrt{\left(-\frac{17}{16}\right)^2 + \left(-\frac{17}{4}\right)^2} = \sqrt{\frac{289}{256} + \frac{289}{16}} = \frac{17}{16}\sqrt{1 + 16} = \frac{17}{16}\sqrt{17}$

SO AREA $= \frac{1}{2}|PR||QR| = \frac{1}{2} \times \frac{17}{4}\sqrt{17} \times \frac{17}{16}\sqrt{17}$

$= \frac{289 \times 17}{8 \times 16}$

$= \frac{4913}{128}$

$\begin{array}{r} 289 \\ 17 \\ \hline 2023 \\ 289 \\ \hline 4913 \end{array}$

Question 31 (****+)

A parabola C has Cartesian equation

$$y^2 = 4ax \ ,$$

where a is a positive constant.

The points $P\left(ap^2, 2ap\right)$ and $Q\left(aq^2, 2aq\right)$ are distinct and lie on C.

The tangent to C at P and the tangent to C at Q meet at the point R.

Show that

$$|SR|^2 = |SP||SQ| ,$$

where S is the focus of the parabola.

$\boxed{}$, $\boxed{\text{proof}}$

START BY OBTAINING THE EQUATION OF THE TANGENT AT P & AT Q

$\Rightarrow y^2 = 4ax$

$\Rightarrow 2y\frac{dy}{dx} = 4a$

$\Rightarrow \frac{dy}{dx} = \frac{2a}{y}$

$\Rightarrow \left.\frac{dy}{dx}\right|_{y=2ap} = \frac{2a}{2ap} = \frac{1}{p}$

EQUATION OF TANGENT AT $P(ap^2, 2ap)$

$y - 2ap = \frac{1}{p}(x - ap^2)$

AND BY ANALOGY AT $Q(aq^2, 2aq)$

$y - 2aq = \frac{1}{q}(x - aq^2)$

SOLVING SIMULTANEOUSLY TO OBTAIN THE CO-ORDINATES OF R

$$\left.\begin{aligned} y - 2ap &= \tfrac{1}{p}(x - ap^2) \\ y - 2aq &= \tfrac{1}{q}(x - aq^2) \end{aligned}\right\} \text{SUBTRACT UPWARDS}$$

$\Rightarrow 2ap - 2aq = \frac{1}{q}(x - aq^2) - \frac{1}{p}(x - ap^2)$ $\times pq$

$\Rightarrow 2ap^2q - 2apq^2 = p(x - aq^2) - q(x - ap^2)$

$\Rightarrow 2apq(p-q) = px - apq^2 - qx + ap^2q$

$\Rightarrow 2apq(p-q) = x(p-q) + apq(p-q)$

$\Rightarrow apq(p-q) = x(p-q)$

$\Rightarrow x = apq$

($p \neq q$ AS POINTS ARE DISTINCT)

AND TO FIND y

$y = 2ap + \frac{1}{p}(x - ap^2)$

$y = 2ap + \frac{1}{p}(apq - ap^2)$

$y = 2ap + aq - ap$

$y = ap + aq$

$y = a(p+q)$

$\therefore R(apq, a(p+q))$

NEXT WE CALCULATE THE DISTANCES $|SP|$, $|SQ|$ AND $|SR|$

$S(a,0) \bullet P(ap^2, 2ap) \bullet Q(aq^2, 2aq)$

$|SP| = \sqrt{(ap^2-a)^2 + (2ap-0)^2} = \sqrt{a^2p^4 - 2a^2p^2 + a^2 + 4a^2p^2} = \sqrt{a^2p^4 + 2a^2p^2 + a^2}$

$= \sqrt{(ap^2+a)^2} = |ap^2 + a| = ap^2 + a = a(p^2+1)$

$|SQ| = a(q^2+1)$ BY ANALOGY

$|SR|^2 = \left[\sqrt{(apq-a)^2 + (ap+aq-0)^2}\right]^2 = (apq - a)^2 + (ap+aq)^2$

$= a^2(pq-1)^2 + a^2(p+q)^2 = a^2\left[(pq-1)^2 + (p+q)^2\right]$

$= a^2\left[p^2q^2 \cancel{- 2pq} + 1 + p^2 + q^2 \cancel{+ 2pq}\right]$

$= a^2\left[p^2q^2 + p^2 + q^2 + 1\right] = a^2\left[p^2(q^2+1) + (q^2+1)\right]$

$= a^2\left[(p^2+1)(q^2+1)\right] = a^2(p^2+1)(q^2+1)$

$= \left[a(p^2+1)\right]\left[a(q^2+1)\right]$

$= |SP||SQ|$

AS REQUIRED

Question 32 (****+)

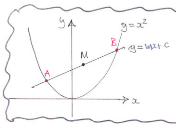
A parabola has Cartesian equation

$$y = x^2 \ , \ x \in \mathbb{R} \, .$$

A chord of the parabola is defined as the straight line segment joining any two distinct points on the parabola.

Find the equation of the locus of the midpoints of parallel chords of the parabola whose gradient is m .

$\boxed{\phantom{x = \frac{1}{2}m}}$, $\boxed{x = \frac{1}{2}m}$



- LET THE LINES HAVE EQUATION $y = mx + c$ WHERE m IS KNOWN, c VARYING
- $y = x^2$, $y = mx + c$

SOLVING SIMULTANEOUSLY

$x^2 = mx + c$
$x^2 - mx - c = 0$

WE DO NOT ACTUALLY NEED THE CO-ORDINATES OF A & B, BUT JUST THE MIDPOINT

LET THE ROOTS BE x_1 & x_2

$x_1 + x_2 = \frac{-(-m)}{1} = m$

$\frac{x_1 + x_2}{2} = \frac{m}{2}$

SOLVING SIMULTANEOUSLY AGAIN

$x^2 = y$
$mx = y - c$

$m^2x^2 = m^2y$
$m^2x^2 = (y-c)^2$

$\Rightarrow m^2y = y^2 - 2cy + c^2$

$\Rightarrow 0 = y^2 - m^2y - 2cy + c^2$

$\Rightarrow 0 = y^2 - y(m^2+2c) + c^2$

LET THE ROOTS BE y_1 & y_2

$y_1 + y_2 = -[-(m^2+c^2)]$

$\frac{y_1 + y_2}{2} = \frac{m^2+c^2}{2}$

$\therefore M(X, Y) = \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right) = \left(\frac{m}{2}, \frac{m^2+c^2}{2}\right)$

AS m IS CONSTANT & c VARIES THE LOCUS IS $x = \frac{m}{2}$

Question 33 (*****)

The points P and Q have respective coordinates $(-1,6)$ and $(-5,-1)$.

When the parabola with equation $y = 4ax$, where a is a constant, is translated by the vector $\begin{pmatrix} -3 \\ 2 \end{pmatrix}$ it passes through the point P.

Find the possible values of the gradient of the straight line which passes through Q and is a tangent to the **translated** parabola.

SPX-N, $\boxed{m = -\frac{1}{2} \cup m = 2}$

START BY TRANSLATING THE PARABOLA

$$y^2 = 4ax \xmapsto{\begin{pmatrix} -3 \\ 2 \end{pmatrix}} (y-2)^2 = 4a(x+3)$$

NOW THIS CURVE SATISFIES $P(-1,6)$

$$\Rightarrow (6-2)^2 = 4a(-1+3)$$
$$\Rightarrow 16 = 8a$$
$$\Rightarrow a = 2$$

NOW WORK IN PARAMETRIC

- $y^2 = 4ax$ IS PARAMETRIZED AS $(at^2, 2at)$
- $y^2 = 8x$ AS $(2t^2, 4t)$
- AND WHEN TRANSLATED $\begin{pmatrix} -3 \\ 2 \end{pmatrix}$ AS $(2t^2-3, 4t+2)$

FIND THE EQUATION OF A GENERAL TANGENT AT $t=p$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{4}{4t} = \frac{1}{t}$$

$\therefore \frac{dy}{dx} = \frac{1}{p}$ & POINT ON PARABOLA WILL BE $(2p^2-3, 4p+2)$

$$\Rightarrow y - 4p - 2 = \frac{1}{p}(x - 2p^2 + 3)$$

BUT $Q(-5,-1)$ LIES ON THIS TANGENT

$$\Rightarrow -1 - 4p - 2 = \frac{1}{p}(-5 - 2p^2 + 3)$$
$$\Rightarrow -3 - 4p = \frac{1}{p}(-2 - 2p^2)$$
$$\Rightarrow 4p + 3 = \frac{1}{p}(2p^2 + 2)$$
$$\Rightarrow 4p^2 + 3p = 2p^2 + 2$$
$$\Rightarrow 2p^2 + 3p - 2 = 0$$
$$\Rightarrow (2p-1)(p+2)$$
$$\Rightarrow p = -2 \text{ or } \frac{1}{2}$$

$\therefore$ GRADIENT OF THE TANGENT WILL BE $\frac{1}{p}$

EITHER $-\frac{1}{2}$ or 2

Question 34 (*****)

A parabola has Cartesian equation

$$y^2 = 12x, \quad x \geq 0.$$

The point P lies on the parabola and the point Q lies on the directrix of the parabola so that PQ is parallel to the x axis.

The area of the triangle PQF is $8\frac{2}{3}$ square units, where the point F represents the focus of the parabola.

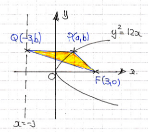
Determine the coordinates of P, given further that the y coordinate of P is a positive integer.

$\boxed{}$, $\boxed{P\left(\frac{4}{3}, 4\right)}$

START WITH A DIAGRAM

$y^2 = 12x = 4 \times 3x$

- FOCUS AT $(3,0)$
- DIRECTRIX $x = -3$



LET THE POINT P BE (a,b)

AREA OF $F\hat{P}Q = \frac{26}{3}$

$\frac{1}{2}b(a+3) = \frac{52}{3}$

$b(a+3) = \frac{52}{3}$

BUT $P(a,b)$ LIES ON $y^2 = 12x \implies b^2 = 12a$

$\implies 12b(a+3) = \frac{52}{3} \times 12$

$\implies b(12a+36) = 208$

$\implies b(b^2+36) = 208$

$\implies b^3 + 36b - 208 = 0$

GIVEN THE y CO-ORDINATE OF P IS AN INTEGER – ATTEMPT TO FACTORIZE AS FOLLOWS

$(b^3 - 1) + (36b - 207) = 0$ ×

$(b^3 - 8) + (36b - 200) = 0$ ×

$(b^3 - 27) + (36b - 181) = 0$ ×

$(b^3 - 64) + (36b - 144) = 0$ ✓

DIFFERENCE OF CUBES PRODUCES THE SAME FACTOR AS THAT IN THE BRACKET

$\implies (b^3 - 4^3) + 36(b-4) = 0$

$\implies (b-4)(b^2+8b+16) + 36(b-4) = 0$

$\implies (b-4)[b^2+8b+16+36] = 0$

$\implies (b-4)(b^2+8b+52) = 0$

IRREDUCIBLE

$\therefore b = 4$ & HENCE $b(a+3) = \frac{52}{3}$

$4(a+3) = \frac{52}{3}$

$a+3 = \frac{13}{3}$

$a = \frac{4}{3}$

$\therefore P\left(\frac{4}{3}, 4\right)$

Question 35 (*****)

The point $P\left(2p, p^2\right)$, where p is a parameter, lies on the parabola, with Cartesian equation

$$x^2 = 4y.$$

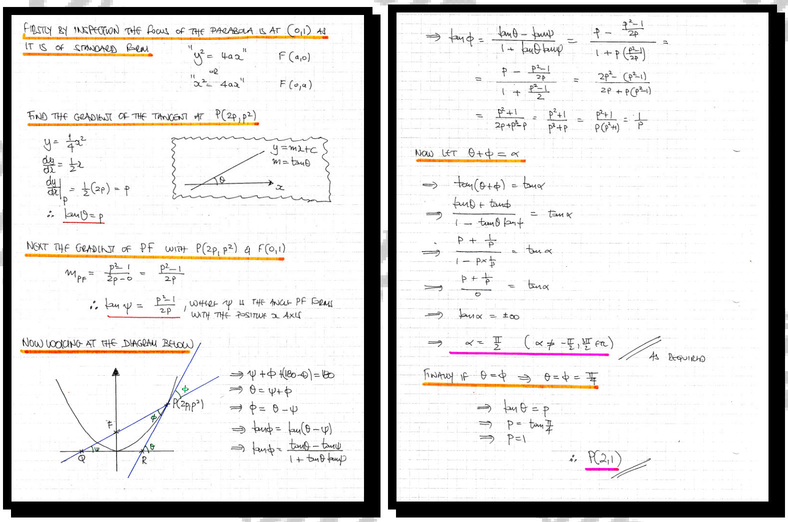
The point F is the focus of the parabola and O represents the origin.

The tangent to the parabola at P forms an angle θ with the positive x axis.

The straight line which passes through P and F forms an acute angle φ with the tangent to the parabola at P.

Show that $\theta + \varphi = \frac{1}{2}\pi$ and hence state the coordinates of P if $\theta = \varphi$.

$\boxed{}$, $\boxed{P(2,1)}$



Question 36 (*****)

A parabola has Cartesian equation

$$y = \frac{1}{2}x^2, \quad x \in \mathbb{R}.$$

The points P and Q both lie on the parabola so that POQ is a right angle, where O is the origin.

The point M represents the midpoint of PQ.

Show that as the position of P varies along the parabola, the locus of M is the curve with equation

$$y = x^2 - 2.$$

$\boxed{}$, $\boxed{\text{proof}}$

- IT BEST TO WORK IN PARAMETRIC

$y = \frac{1}{2}x^2$

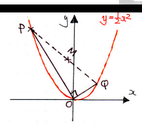
$2y = x^2$

LET $y = 2t^2$ (SO IT "SQUARE ROOTS")

$2(2t^2) = x^2$

$x^2 = 4t^2$

$x = 2t$



- LET THE GENERAL POINTS $P(2p, 2p^2)$ & $Q(2q, 2q^2)$, I.E WITH $t=p$ AT POINT P & $t=q$ AT POINT Q

$$\left.\begin{aligned} \text{GRADIENT OF } OP &= \frac{2p^2-0}{2p-0} = p \\ \text{GRADIENT OF } OQ &= \frac{2q^2-0}{2q-0} = q \end{aligned}\right\} \Rightarrow \text{PERPENDICULAR } OP \perp OQ$$

$\therefore pq = -1$

- NEXT WE CONSIDER THE MIDPOINT OF PQ

$$M\left(\frac{2p+2q}{2}, \frac{2p^2+2q^2}{2}\right) = M(p+q, p^2+q^2)$$

- IN PARAMETRIC WE HAVE

$X = p+q$

$Y = p^2+q^2$

WHERE p & q ARE PARAMETERS SATISFYING THE CONSTRAINT $pq = -1$

- ELIMINATING AS FOLLOWS

$\Rightarrow X = p+q$

$\Rightarrow X^2 = (p+q)^2$

$\Rightarrow X^2 = p^2+q^2-2pq$

$\Rightarrow X^2 = (p^2+q^2) - 2(pq)$

$\Rightarrow X^2 = Y - 2(-1)$

$\Rightarrow X^2 = Y + 2$

$\Rightarrow Y = X^2 + 2$

OR

$y = x^2 + 2$ //

Question 37 (*****)

The cubic equation

$$x^3 + px + q = 0,$$

has 2 distinct real roots.

a) Show that $27q^2 + 4p^3 < 0$.

A parabola has Cartesian equation

$$y = x^2, \quad x \in \mathbb{R}.$$

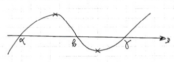
Three distinct normals to this parabola pass through the point, which does not lie on the parabola, whose coordinates are (a,b).

b) Show further that

$$b > \frac{1}{2} + 3\left(\frac{1}{4}a\right)^{\frac{2}{3}}.$$

proof

a) LOOKING AT THE GRAPH OF A CUBIC WITH 3 DISTINCT REAL ROOTS



- MUST HAVE 2 STATIONARY POINTS
- THE y CO-ORDINATES OF THE STATIONARY POINTS MUST HAVE OPPOSITE SIGNS

$f(x) = x^3 + px + q$

$f'(x) = 3x^2 + p$

SOLVING FOR ZERO, & NOTING THAT p HAS TO BE NEGATIVE

$3x^2 + p = 0$

$3x^2 = -p$

$x^2 = -\frac{p}{3}$

$x = \pm\sqrt{\frac{-p}{3}}$

FIND THE y CO-ORDINATES

$f\left(\sqrt{\frac{-p}{3}}\right) = \sqrt{\frac{-p}{3}}^{\,3} + p\sqrt{\frac{-p}{3}} + q = -\frac{p}{3}\sqrt{\frac{-p}{3}} + p\sqrt{\frac{-p}{3}} + q = q + \frac{2}{3}p\sqrt{\frac{-p}{3}}$

$f\left(-\sqrt{\frac{-p}{3}}\right) = \left(-\sqrt{\frac{-p}{3}}\right)^3 + p\left(-\sqrt{\frac{-p}{3}}\right) + q = +\frac{p}{3}\sqrt{\frac{-p}{3}} - p\sqrt{\frac{-p}{3}} + q = q - \frac{2}{3}p\sqrt{\frac{-p}{3}}$

FINALLY WE HAVE $f\left(\sqrt{\frac{-p}{3}}\right)f\left(-\sqrt{\frac{-p}{3}}\right) < 0$

$\Rightarrow \left[q + \frac{2}{3}p\sqrt{\frac{-p}{3}}\right]\left[q - \frac{2}{3}p\sqrt{\frac{-p}{3}}\right] < 0$

$\Rightarrow q^2 - \frac{4}{9}p^2\left(-\frac{p}{3}\right) < 0$

$\Rightarrow q^2 + \frac{4}{27}p^3 < 0$

$\Rightarrow 27q^2 + 4p^3 < 0$ // AS REQUIRED

b) LET THE GENERAL POINT ON THE PARABOLA HAVE CO-ORDINATES $P(p, p^2)$ SO THE GENERAL NORMAL CAN BE FOUND

$\frac{dy}{dx} = 2x$

$\left.\frac{dy}{dx}\right|_{x=p} = 2p$

$\therefore$ NORMAL GRADIENT $= -\frac{1}{2p}$

$y - p^2 = -\frac{1}{2p}(x - p)$

$2py - 2p^3 = -x + p$

$2py + x = 2p^3 + p$

NEXT WE NOW THAT ALL 3 NORMALS, AT 3 DISTINCT POINTS PASS THROUGH THE POINT (a,b)

$\Rightarrow 2pb + a = 2p^3 + p$

$\Rightarrow 2p^3 + (1 - 2b)p - a = 0$

$\Rightarrow p^3 + \left(\frac{1-2b}{2}\right)p + \left(\frac{-a}{2}\right) = 0$

FROM PART (a), THE INEQUALITY MUST HOLD

$\Rightarrow 4\left(\frac{1-2b}{2}\right)^3 + 27\left(\frac{-a}{2}\right)^2 < 0$

$\Rightarrow \frac{1}{2}(1-2b)^3 + \frac{27}{4}a^2 < 0$

$\Rightarrow (1-2b)^3 < -\frac{27}{2}a^2$

$\Rightarrow 1 - 2b < \left(-\frac{27}{2}a^2\right)^{\frac{1}{3}}$

$\Rightarrow 1 - 2b < -3\left(\frac{a}{4}\right)^{\frac{2}{3}}$

$\Rightarrow 2b - 1 > 3\left(\frac{a}{4}\right)^{\frac{2}{3}}$

$\Rightarrow 2b > 1 + 3\left(\frac{a}{4}\right)^{\frac{2}{3}}$

$\Rightarrow b > \frac{1}{2} + 3\left(\frac{a}{4}\right)^{2/3}$ // AS REQUIRED

Question 38 (*****)

A parabola is given parametrically by

$$x = \frac{1}{3}t^2, \quad y = \frac{2}{3}t, \quad t \in \mathbb{R}.$$

The normal to the parabola at the point P meets the parabola again at the point Q.

Show that the minimum value of $|PQ|$ is $\sqrt{12}$.

$\boxed{}$, $\boxed{\text{proof}}$

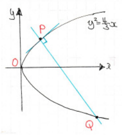
START COLLECTING INFORMATION

$$\left.\begin{array}{l} x = \frac{1}{3}t^2 \\ y = \frac{2}{3}t \end{array}\right\} \Rightarrow \left.\begin{array}{l} \frac{4}{3}x = \frac{4}{9}t^2 \\ y^2 = \frac{4}{9}t^2 \end{array}\right\} \Rightarrow y^2 = \frac{4}{3}x$$

$$\Rightarrow 2y\frac{dy}{dx} = \frac{4}{3}$$

$$\Rightarrow \frac{dy}{dx} = \frac{2}{3y}$$

LET THE POINT P LIE ON THE CURVE, AT THE POINT $t = p$, I.E $P\left(\frac{1}{3}p^2, \frac{2}{3}p\right)$



- $\left.\frac{dy}{dx}\right|_{y=\frac{2}{3}p} = \frac{2}{3\left(\frac{2}{3}p\right)} = \frac{1}{p}$
- NORMAL GRADIENT IS $-p$
- EQUATION OF THE NORMAL IS GIVEN BY

$$\Rightarrow y - \frac{2}{3}p = -p\left(x - \frac{1}{3}p^2\right)$$
$$\Rightarrow y - \frac{2}{3}p = -px + \frac{1}{3}p^3$$
$$\Rightarrow 3y - 2p = -3px + p^3$$
$$\Rightarrow 3y + 3px = 2p + p^3$$

- SOLVING SIMULTANEOUSLY WITH THE EQUATION OF THE CURVE

$$x = \frac{3}{4}y^2 \quad \& \quad 3y + 3px = 2p + p^3$$

$$\Rightarrow 3y + 3p\left(\frac{3}{4}y^2\right) = 2p + p^3$$
$$\Rightarrow 3y + \frac{9}{4}py^2 = 2p + p^3$$
$$\Rightarrow 12y + 9py^2 = 8p + 4p^3$$
$$\Rightarrow 9py^2 + 12y - 8p - 4p^3 = 0$$
$$\Rightarrow (3y - 2p)(3py - 4 + 2p^2) = 0$$

POINT P POINT Q BY INSPECTION

$$\Rightarrow y = \begin{cases} \frac{2}{3}p & \leftarrow \text{POINT } P \\ -\frac{4+2p^2}{3p} & \leftarrow \text{POINT } Q \end{cases}$$

WE REQUIRE THE VALUE OF t, AT POINT Q

$$\frac{2}{3}t = -\frac{4+2p^2}{3p}$$
$$t = -\frac{p^2+2}{p}$$

THUS WE CAN FIND THE x-COORDINATE OF Q

$$x = \frac{1}{3}t^2 = \frac{1}{3}\left[-\frac{p^2+2}{p}\right]^2 = \frac{(p^2+2)^2}{3p^2}$$

$P\left(\frac{1}{3}p^2, \frac{2}{3}p\right)$ & $Q\left(\frac{(p^2+2)^2}{3p^2}, -\frac{2p^2+4}{3p}\right)$

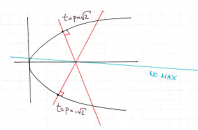
$$\Rightarrow |PQ|^2 = d^2 = \left[\frac{(p^2+2)^2}{3p^2} - \frac{1}{3}p^2\right]^2 + \left[\frac{2}{3}p + \frac{2p^2+4}{3p}\right]^2$$
$$\Rightarrow |PQ| = d^2 = \left[\frac{(p^2+2)^2 - p^4}{3p^2}\right]^2 + \left[\frac{2p^2 + 2p^2 + 4}{3p}\right]^2$$
$$\Rightarrow |PQ|^2 = d^2 = \left(\frac{\cancel{p^4} + 4p^2 + 4 \cancel{- p^4}}{3p^2}\right)^2 + \left(\frac{4p^2+4}{3p}\right)^2$$
$$\Rightarrow |PQ|^2 = d^2 = \left(\frac{4p^2+4}{3p^2}\right)^2 + \left(\frac{4p^2+4}{3p}\right)^2$$
$$\Rightarrow |PQ|^2 = d^2 = \frac{16(p^2+1)^2}{9p^4} + \frac{16(p^2+1)^2}{9p^2}$$
$$\Rightarrow |PQ|^2 = d^2 = \frac{16}{9}\left[\frac{(p^2+1)^2}{p^4} + \frac{(p^2+1)^2}{p^2}\right]$$
$$\Rightarrow |PQ|^2 = d^2 = \frac{16}{9}\left[\frac{(p^2+1)^2 + p^2(p^2+1)^2}{p^4}\right]$$
$$\Rightarrow |PQ|^2 = d^2 = \frac{16}{9}\left[\frac{(p^2+1)^2(1+p^2)}{p^4}\right] = \frac{16(p^2+1)^3}{9p^4}$$

LET $f(p) = \frac{(p^2+1)^3}{p^4}$

$$f'(p) = \frac{p^4 \times 3(p^2+1)^2 \times 2p - (p^2+1)^3 \times 4p^3}{p^8}$$
$$= \frac{6p^5(p^2+1)^2 - 4p^3(p^2+1)^3}{p^8}$$
$$= \frac{6p^2(p^2+1)^2 - 4(p^2+1)^3}{p^5}$$
$$= \frac{2(p^2+1)^2\left[3p^2 - 2(p^2+1)\right]}{p^5}$$
$$= \frac{2(p^2+1)^2(p^2-2)}{p^5}$$

SOLVING FOR ZERO, YIELDS $p = \pm\sqrt{2}$ (BY INSPECTION).

BOTH THESE VALUES SHOULD YIELD SYMMETRICAL MINIMUMS ON THE CURVE AS THERE IS NO MAX



when $p = \pm\sqrt{2}$, ie $t^2 = p^2 = 2$

$$|PQ|^2 = d^2 = \frac{16(p^2+1)^3}{9p^4} \quad \frac{16(2+1)^3}{9 \times 2^2} = \frac{\cancel{16}^{4} \times \cancel{27}^{3}}{\cancel{9} \times \cancel{4}} = 12$$

$\therefore$ MINIMUM DISTANCE IS $\sqrt{12} = 2\sqrt{3}$

ELLIPSE

Question 1 (**)

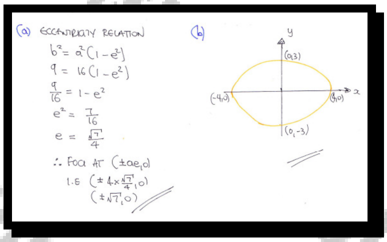
An ellipse has Cartesian equation

$$\frac{x^2}{16}+\frac{y^2}{9}=1.$$

a) Find the coordinates of its foci.

b) Sketch the ellipse.

$$\boxed{\left(\pm\sqrt{7},0\right)}$$



Question 2 (**)

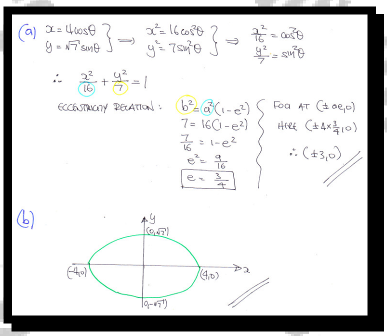
An ellipse has parametric equations

$$x=4\cos\theta,\quad y=\sqrt{7}\sin\theta,\quad 0\le\theta<2\pi.$$

a) Find the coordinates of its foci.

b) Sketch the ellipse.

$$\boxed{(\pm 3,0)}$$



Question 3 (**)

An ellipse has a focus at $(4,0)$ and the associated directrix has equation $x = \dfrac{25}{4}$.

Determine a Cartesian equation of the ellipse.

$$\boxed{\frac{x^2}{25} + \frac{y^2}{9} = 1}$$

FOCUS AT $(4,0)$

$ae = 4$

$\boxed{a = \frac{4}{e}}$

DIRECTRIX $x = \frac{25}{4}$

$\frac{a}{e} = \frac{25}{4}$

$\boxed{a = \frac{25}{4}e}$

$\frac{4}{e} = \frac{25}{4}e$

$\frac{16}{25} = e^2$

$\boxed{e = \frac{4}{5}}$ & $\boxed{a = 5}$

& BY ECCENTRICITY RELATION

$b^2 = a^2(1-e^2)$

$b^2 = 25\left(1 - \frac{16}{25}\right)$

$b^2 = 9$

$\boxed{b = 3}$

$\therefore \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \Rightarrow \frac{x^2}{25} + \frac{y^2}{9} = 1$

Question 4 (**)

$$\frac{x^2}{4} + y^2 = 1.$$

The ellipse with Cartesian equation above and a parabola with vertex at the origin share the same focal point.

Find the possible Cartesian equation for the parabola.

$\boxed{\text{YNE-P}}$, $\boxed{y^2 = \pm\sqrt{48}x}$

LOOK FOR THE ECCENTRICITY OF THE ELLIPSE

$\frac{x^2}{4} + \frac{y^2}{1} = 1$

$a^2 = 4$, $b^2 = 1$

$\Rightarrow b^2 = a^2(1-e^2)$

$\Rightarrow 1 = 4(1-e^2)$

$\Rightarrow \frac{1}{4} = 1 - e^2$

$\Rightarrow e^2 = \frac{3}{4}$

$\Rightarrow e = +\frac{\sqrt{3}}{2}$

THE ELLIPSE HAS FOCI AT $(\pm ae, 0)$, I.E. $\left(\pm\frac{\sqrt{3}}{2}\times 2, 0\right)$, $(\pm\sqrt{3}, 0)$

THE PARABOLA HAS EQUATION $y^2 = 4ax$

$a = \pm\sqrt{3}$

$\therefore y^2 = 4\sqrt{3}x$ OR $y^2 = -4\sqrt{3}x$

$y^2 = \sqrt{48}x$ OR $y^2 = -\sqrt{48}x$

Question 5 **(**+)**

An ellipse E is given parametrically by the equations

$$x = \cos t\,,\ \ y = 2\sin t\,,\ \ 0 \le t < 2\pi\,.$$

a) Show that an equation of the normal to E at the general point $P(\cos t, 2\sin t)$ can be written as

$$\frac{2y}{\sin t} - \frac{x}{\cos t} = 3\,.$$

The normal to E at P meets the x axis at the point Q. The midpoint of PQ is M.

b) Find the equation of the locus of M as t varies.

$$\boxed{x^2 + y^2 = 1}$$

(a) $\left.\begin{array}{l} x = \cos t \\ y = 2\sin t \end{array}\right\} \Rightarrow \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2\cos t}{-\sin t}$ $\therefore$ NORMAL GRADIENT IS $\frac{\sin t}{2\cos t}$

EQUATION OF THE NORMAL $\Rightarrow$ $y - 2\sin t = \frac{\sin t}{2\cos t}(x - \cos t)$

$2y\cos t - 4\sin t\cos t = x\sin t - \sin t\cos t$

$2y\cos t - x\sin t = 3\sin t\cos t$

$\frac{2y\cos t}{\sin t\cos t} - \frac{x\sin t}{\sin t\cos t} = \frac{3\sin t\cos t}{\sin t\cos t}$

$\frac{2y}{\sin t} - \frac{x}{\cos t} = 3$ // AS REQUIRED

(b) • $y = 0$ HENCE $\frac{-x}{\cos t} = 3$ IE $x = -3\cos t$ IE $Q(-3\cos t, 0)$

• MIDPOINT OF PQ WHERE $P(\cos t, 2\sin t)$ IS $M\left(\frac{-3\cos t + \cos t}{2}, \frac{0 + 2\sin t}{2}\right)$

I.E. $M(-\cos t, \sin t)$

IE $\left.\begin{array}{l} X = -\cos t \\ Y = \sin t \end{array}\right\} \Rightarrow X^2 + Y^2 = 1$

Question 6 (***)

$$\frac{x^2}{9}+\frac{y^2}{4}=1 .$$

The ellipse with the Cartesian equation given above, has foci S and S'.

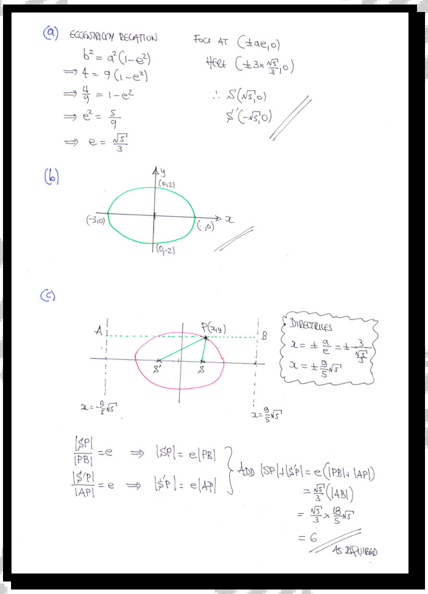
a) Find the coordinates S and S'.

b) Sketch the ellipse.

c) Show that for every point P on this ellipse,

$$|SP|+|S'P|=6 .$$

$$\boxed{\left(\pm\sqrt{5},0\right)}$$



Question 7 **(***)**

An ellipse E has Cartesian equation

$$\frac{x^2}{289}+\frac{y^2}{64}=1.$$

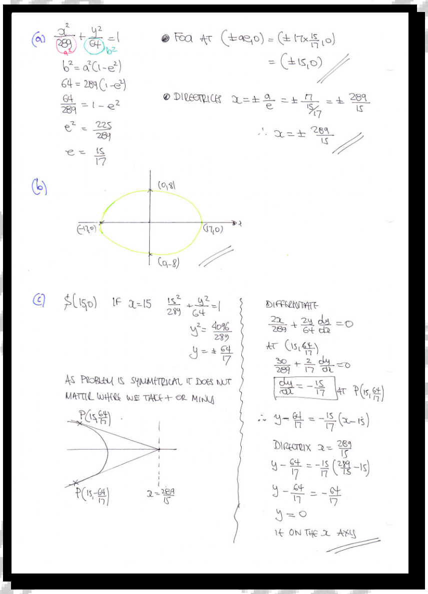
a) Find the coordinates of the foci of E, and the equations of its directrices.

b) Sketch the ellipse.

The point P lies on E so that PS is vertical, where S is the focus of the ellipse with positive x coordinate.

c) Show that the tangent to the ellipse at the point P meets one the directrices of the ellipse on the x axis.

$\boxed{(\pm 15,0)}$, $\boxed{x=\pm\frac{289}{15}}$


(a) $\frac{x^2}{289}+\frac{y^2}{64}=1$ a^2 b^2

$b^2=a^2(1-e^2)$

$64=289(1-e^2)$

$\frac{64}{289}=1-e^2$

$e^2=\frac{225}{289}$

$e=\frac{15}{17}$

• FOCI AT $(\pm ae,0)=(\pm 17\times\frac{15}{17},0)$

$=(\pm 15,0)$

• DIRECTRICES $x=\pm\frac{a}{e}=\pm\frac{17}{15/17}=\pm\frac{289}{15}$

$\therefore x=\pm\frac{289}{15}$

(b)

(c) $S(15,0)$ IF $x=15$ $\frac{15^2}{289}+\frac{y^2}{64}=1$

$y^2=\frac{4096}{289}$

$y=\pm\frac{64}{17}$

AS PROBLEM IS SYMMETRICAL IT DOES NOT MATTER WHERE WE TAKE + OR MINUS

DIFFERENTIATE

$\frac{2x}{289}+\frac{2y}{64}\frac{dy}{dx}=0$

AT $(15,\frac{64}{17})$

$\frac{30}{289}+\frac{2}{17}\frac{dy}{dx}=0$

$\boxed{\frac{dy}{dx}=-\frac{15}{17}}$ AT $P(15,\frac{64}{17})$

$\therefore y-\frac{64}{17}=-\frac{15}{17}(x-15)$

DIRECTRIX $x=\frac{289}{15}$

$y-\frac{64}{17}=-\frac{15}{17}\left(\frac{289}{15}-15\right)$

$y-\frac{64}{17}=-\frac{64}{17}$

$y=0$

IE ON THE x AXIS

Question 8 (***)

The point $P(5\cos\theta, 4\sin\theta)$ lies on the an ellipse E with Cartesian equation

$$16x^2 + 25y^2 = 400.$$

a) Find the coordinates of the foci of E.

b) Show that an equation of the normal to the ellipse at P is

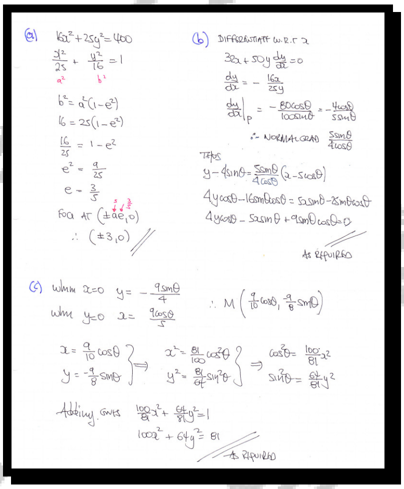
$$4y\cos\theta - 5x\sin\theta + 9\sin\theta\cos\theta = 0.$$

The normal to the ellipse intersects the coordinate axes at the points A and B, and the point M is the midpoint of AB.

c) Show that the locus of M, as θ varies, is the ellipse with equation

$$100x^2 + 64y^2 = 81.$$

$\boxed{(\pm 3, 0)}$


(a) $16x^2+25y^2=400$

$\frac{x^2}{25}+\frac{y^2}{16}=1$

a^2 b^2

$b^2=a^2(1-e^2)$

$16=25(1-e^2)$

$\frac{16}{25}=1-e^2$

$e^2=\frac{9}{25}$

$e=\frac{3}{5}$

FOCI AT $(\pm ae, 0)$

$\therefore (\pm 3, 0)$

(b) DIFFERENTIATE W.R.T x

$32x+50y\frac{dy}{dx}=0$

$\frac{dy}{dx}=-\frac{16x}{25y}$

$\left.\frac{dy}{dx}\right|_P=-\frac{80\cos\theta}{100\sin\theta}=-\frac{4\cos\theta}{5\sin\theta}$

$\therefore$ NORMAL GRAD $\frac{5\sin\theta}{4\cos\theta}$

THUS

$y-4\sin\theta=\frac{5\sin\theta}{4\cos\theta}(x-5\cos\theta)$

$4y\cos\theta-16\sin\theta\cos\theta=5x\sin\theta-25\sin\theta\cos\theta$

$4y\cos\theta-5x\sin\theta+9\sin\theta\cos\theta=0$

AS REQUIRED

(c) when $x=0$ $y=-\frac{9\sin\theta}{4}$

when $y=0$ $x=\frac{9\cos\theta}{5}$

$\therefore M\left(\frac{9}{10}\cos\theta, -\frac{9}{8}\sin\theta\right)$

$\left.\begin{array}{l} x=\frac{9}{10}\cos\theta \\ y=-\frac{9}{8}\sin\theta \end{array}\right\} \Rightarrow \left.\begin{array}{l} x^2=\frac{81}{100}\cos^2\theta \\ y^2=\frac{81}{64}\sin^2\theta \end{array}\right\} \Rightarrow \begin{array}{l} \cos^2\theta=\frac{100}{81}x^2 \\ \sin^2\theta=\frac{64}{81}y^2 \end{array}$

Adding gives $\frac{100}{81}x^2+\frac{64}{81}y^2=1$

$100x^2+64y^2=81$

AS REQUIRED

Question 9 (***)

The ellipse E has parametric equations

$$x = a\cos\theta \ , \ y = b\sin\theta , \ 0 \le \theta < 2\pi$$

where a and b are positive constants.

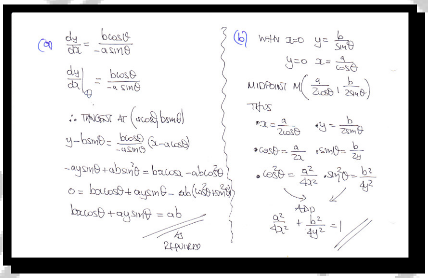
a) Show that an equation of the tangent at a general point on E is

$$bx\cos\theta + ay\sin\theta = ab .$$

This tangent to E intersects the coordinate axes at the points A and B, and the point M is the midpoint of AB.

b) Find a Cartesian locus of M, as θ varies.

$$\boxed{\frac{a^2}{ax^2} + \frac{b^2}{4y^2} = 1}$$



(a) $\frac{dy}{dx} = \frac{b\cos\theta}{-a\sin\theta}$

$\left.\frac{dy}{dx}\right|_{\theta} = \frac{b\cos\theta}{-a\sin\theta}$

$\therefore$ TANGENT AT $(a\cos\theta, b\sin\theta)$

$y - b\sin\theta = \frac{b\cos\theta}{-a\sin\theta}(x - a\cos\theta)$

$-ay\sin\theta + ab\sin^2\theta = bx\cos\theta - ab\cos^2\theta$

$0 = bx\cos\theta + ay\sin\theta - ab(\cos^2\theta + \sin^2\theta)$

$bx\cos\theta + ay\sin\theta = ab$

AS REQUIRED

(b) WHEN $x=0$ $y = \frac{b}{\sin\theta}$

$y=0$ $x = \frac{a}{\cos\theta}$

MIDPOINT $M\left(\frac{a}{2\cos\theta}, \frac{b}{2\sin\theta}\right)$

THUS

• $x = \frac{a}{2\cos\theta}$ • $y = \frac{b}{2\sin\theta}$

• $\cos\theta = \frac{a}{2x}$ • $\sin\theta = \frac{b}{2y}$

• $\cos^2\theta = \frac{a^2}{4x^2}$ • $\sin^2\theta = \frac{b^2}{4y^2}$

ADD

$\frac{a^2}{4x^2} + \frac{b^2}{4y^2} = 1$

Question 10 (***)

An ellipse has Cartesian equation

$$\frac{x^2}{16}+\frac{y^2}{4}=1.$$

The general point $P(4\cos\theta, 2\sin\theta)$ lies on the ellipse.

a) Show that the equation of the normal to the ellipse at P is

$$2x\sin\theta - y\cos\theta = 6\sin\theta\cos\theta.$$

The normal to the ellipse at P meets the x axis at the point Q and O is the origin.

b) Show clearly that as θ varies, the maximum area of the triangle OPQ is $4\frac{1}{2}$.

proof

(a) $\frac{x^2}{16}+\frac{y^2}{4}=1$

$\frac{2x}{16}+\frac{2y}{4}\frac{dy}{dx}=0$

$\frac{1}{2}y\frac{dy}{dx}=-\frac{1}{8}x$

$y\frac{dy}{dx}=-\frac{1}{4}x$

$\boxed{\frac{dy}{dx}=-\frac{x}{4y}}$

NOW $\left.\frac{dy}{dx}\right|_P=-\frac{4\cos\theta}{4(2\sin\theta)}=-\frac{\cos\theta}{2\sin\theta}$

$\therefore$ NORMAL GRADIENT IS $\frac{2\sin\theta}{\cos\theta}$

THUS $y-2\sin\theta=\frac{2\sin\theta}{\cos\theta}(x-4\cos\theta)$

$\Rightarrow y\cos\theta-2\sin\theta\cos\theta=2x\sin\theta-8\sin\theta\cos\theta$

$\Rightarrow 0=2x\sin\theta-y\cos\theta-6\sin\theta\cos\theta$

$\Rightarrow 2x\sin\theta-y\cos\theta=6\sin\theta\cos\theta$ // AS REQUIRED

(b) WHEN $x=0$ $y=-6\sin\theta$

WHEN $y=0$ $x=3\cos\theta$

AREA $=\frac{1}{2}|x||y|=\frac{1}{2}\times18\sin\theta\cos\theta=\frac{1}{2}\times9\sin2\theta=\frac{9}{2}\sin2\theta$

AS θ VARIES $|\sin2\theta|\le1$ $\therefore$ AREA$_{\text{MAX}}=\frac{9}{2}$ //

Question 11 (***+)

An ellipse with equation

$$\frac{x^2}{16}+\frac{y^2}{4}=1$$

is transformed by the enlargement matrix $\mathbf{E}$ into a circle of radius 3, with centre at the origin.

Determine the elements of $\mathbf{E}$.

SYN-[illegible], $\boxed{\mathbf{E}=\begin{pmatrix}\frac{3}{4} & 0\\ 0 & \frac{3}{2}\end{pmatrix}}$

MANIPULATE THE EQUATION AS FOLLOWS

$$\frac{x^2}{16}+\frac{y^2}{4}=1$$

$$\frac{9}{16}x^2+\frac{9}{4}y^2=9$$

$$\frac{9}{16}\left(\frac{4}{3}X\right)^2+\frac{9}{4}\left(\frac{2}{3}Y\right)^2=9$$

$$X^2+Y^2=9$$

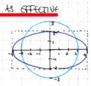
$x \longmapsto \frac{4}{3}X$ IE STRETCH PARALLEL TO x, SCALE FACTOR $\frac{3}{4}$

$y \longmapsto \frac{2}{3}Y$ IE STRETCH PARALLEL TO y, SCALE FACTOR $\frac{3}{2}$

$$\therefore \ \underline{E}=\begin{pmatrix}\frac{3}{4} & 0\\ 0 & \frac{3}{2}\end{pmatrix}$$

A GRAPHICAL APPROACH COULD BE AS EFFECTIVE

$$\frac{x^2}{16}+\frac{y^2}{4}=1$$



"SHRUNK IN x" BY $\frac{3}{4}$

"STRETCHED IN y" BY $\frac{3}{2}$

$$\therefore \ \underline{E}=\begin{pmatrix}\frac{3}{4} & 0\\ 0 & \frac{3}{2}\end{pmatrix}$$

Question 12 (***+)

An ellipse has Cartesian equation

$$2x^2 + 3y^2 - 4x + 12y + 8 = 0.$$

Determine …

a) … the coordinates of the centre of the ellipse.

b) … the eccentricity of the ellipse.

c) … the coordinates of the foci of the ellipse.

d) … the equations of the directrices of the ellipse.

$\boxed{(1,-2)}$, $\boxed{e = \frac{\sqrt{3}}{3}}$, $\boxed{(0,-2), (2,-2)}$, $\boxed{x=-2,\ x=4}$

- $2x^2 - 4x + 3y^2 + 12y + 8 = 0$

$\Rightarrow 2(x^2-2x) + 3(y^2+4y) + 8 = 0$

$\Rightarrow 2(x^2-2x+1-1) + 3(y^2+4y+4-4) + 8 = 0$

$\Rightarrow 2\left[(x-1)^2-1\right] + 3\left[(y+2)^2-4\right] + 8 = 0$

$\Rightarrow 2(x-1)^2 - 2 + 3(y+2)^2 - 12 + 8 = 0$

$\Rightarrow 2(x-1)^2 + 3(y+2)^2 = 6$

$\frac{(x-1)^2}{\underset{a^2}{3}} + \frac{(y+2)^2}{\underset{b^2}{2}} = 1$ $\leftarrow$ THIS IS A TRANSLATION OF A STANDARD ELLIPSE $\frac{x^2}{3}+\frac{y^2}{2}=1$ BY $\binom{1}{-2}$

(a) CENTRE $(0,0) \longmapsto (1,-2)$

(b) ECCENTRICITY IS UNAFFECTED BY TRANSLATIONS

$b^2 = a^2(1-e^2)$

$2 = 3(1-e^2)$

$\frac{2}{3} = 1-e^2$

$e^2 = \frac{1}{3}$

$e = \frac{\sqrt{3}}{3}$

(c) FOCI AT $(\pm ae, 0)$

I.E $\left(\pm\sqrt{3}\times\frac{\sqrt{3}}{3}, 0\right)$

$(\pm 1, 0)$

AFTER TRANSLATION FOCI IS AT $(0,-2)$

$(2,-2)$

(d) DIRECTRICES $x = \pm\frac{a}{e}$

$x = \pm\frac{\sqrt{3}}{\sqrt{3}/3}$

$x = \pm 3$

AFTER TRANSLATION $x=-2$

$x=4$

Question 13 (***+)

An ellipse has Cartesian equation

$$\frac{x^2}{a^2}+\frac{y^2}{12}=1,$$

where a is a positive constant.

The straight line with equation $x=8$ is a directrix for the ellipse.

Determine the possible set of coordinates for the foci of the ellipse.

$\boxed{(\pm 2,0) \text{ or } (\pm 6,0)}$

$\frac{x^2}{a^2}+\frac{y^2}{12}=1$

• DIRECTRIX $x=8$

$\Rightarrow \frac{a}{e}=8$

$\Rightarrow e=\frac{a}{8}$

$\Rightarrow \boxed{e^2=\frac{a^2}{64}}$

• ECCENTRICITY RELATION

$\Rightarrow b^2=a^2(1-e^2)$

$\Rightarrow \boxed{12=a^2(1-e^2)}$

$\Rightarrow 12=a^2\left(1-\frac{a^2}{64}\right)$

$\Rightarrow 12=a^2-\frac{a^4}{64}$

$\Rightarrow 768=64a^2-a^4$

$\Rightarrow a^4-64a^2+768=0$

$\Rightarrow (a^2-16)(a^2-48)=0$

$\Rightarrow a^2 = 16$ or 48

THUS $a = 4$ or $4\sqrt{3}$, $e = \frac{1}{2}$ or $\frac{\sqrt{3}}{2}$ $\Rightarrow$

$(\pm ae,0)=(\pm 2,0)$

$(\pm ae,0)=(\pm 6,0)$

Question 14 (***+)

An ellipse has equation

$$x^2 - 8x + 4y^2 + 12 = 0 .$$

a) Determine the coordinates of the foci and the equations of the directrices of the ellipse.

A straight line with positive gradient passes through the origin O and **touches** the ellipse at the point A.

b) Find the coordinates of A.

$\boxed{\left(4-\sqrt{3},0\right),\ \left(4+\sqrt{3},0\right)}$, $\boxed{x = 4-\frac{4}{3}\sqrt{3},\ x = 4+\frac{4}{3}\sqrt{3}}$, $\boxed{\left(3,\frac{1}{3}\sqrt{2}\right)}$

a) WRITE THE ELLIPSE IN "STANDARD" FORM

$x^2 - 8x + 4y^2 + 12 = 0$

$(x-4)^2 - 16 + 4y^2 + 12 = 0$

$(x-4)^2 + 4y^2 = 4$

$\frac{(x-4)^2}{4} + y^2 = 1$

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ $a>b$ HAS FOCI AT $(\pm ae, 0)$, DIRECTRICES AT $x = \pm\frac{a}{e}$

HERE $a=2$ $b=1$

$b^2 = a^2(1-e^2) \Rightarrow 1 = 4(1-e^2) \Rightarrow \frac{1}{4} = 1-e^2$, $e^2 = \frac{3}{4}$, $e = +\frac{\sqrt{3}}{2}$

HENCE THE ELLIPSE $\frac{x^2}{4} + \frac{y^2}{1} = 1$ HAS

FOCI AT $\left(\pm 2\frac{\sqrt{3}}{2}, 0\right) = (\pm\sqrt{3}, 0)$

DIRECTRICES AT $x = \pm\frac{2}{\sqrt{3}/2} = \pm\frac{4}{\sqrt{3}} = \pm\frac{4}{3}\sqrt{3}$

AS "OUR ELLIPSE" IS A HORIZONTAL TRANSLATION BY $+4$

- FOCI AT $(4\pm\sqrt{3})$
- DIRECTRICES $x = 4+\frac{4}{3}\sqrt{3}$, $4-\frac{4}{3}\sqrt{3}$

b) LET THE STRAIGHT LINE (SOON TO BE A TANGENT) HAVE EQUATION $y = mx$, $m>0$

$\left.\begin{array}{l} x^2 + 4y^2 - 8x + 12 = 0 \\ y = mx \end{array}\right\} \Rightarrow x^2 + 4m^2x^2 - 8x + 12 = 0$

$\Rightarrow (1+4m^2)x^2 - 8x + 12 = 0$

IF TANGENT $b^2 - 4ac = 0$

$\Rightarrow (-8)^2 - 4(1+4m^2)\times 12 = 0$

$\Rightarrow 64 - 48(1+4m^2) = 0$

$\Rightarrow 64 = 48(1+4m^2)$

$\Rightarrow \frac{4}{3} = 1+4m^2$

$\Rightarrow \left(m^2 = \frac{1}{12}\right)$

$\Rightarrow \left(m = +\frac{1}{\sqrt{12}}\right)$

SUBSTITUTE INTO THE QUADRATIC $4m^2+1 = \frac{4}{3}$

$\Rightarrow \frac{4}{3}x^2 - 8x + 12 = 0$ ← "EXPECT A PERFECT SQUARE"

$\Rightarrow 4x^2 - 24x + 36 = 0$

$\Rightarrow x^2 - 6x + 9 = 0$

$\Rightarrow (x-3)^2 = 0$

$\Rightarrow x = 3$

FINALLY $y = mx$ WITH $x=3$ & $m = \frac{1}{\sqrt{12}}$

$\Rightarrow y = \frac{1}{\sqrt{12}}\times 3 = \frac{1}{2\sqrt{3}}\times 3 = \frac{3}{2\sqrt{3}} = \frac{3\sqrt{3}}{6} = \frac{1}{2}\sqrt{3}$

$\therefore A\left(3, \frac{1}{2}\sqrt{3}\right)$

Question 15 (***+)

A point P lies on the ellipse with Cartesian equation

$$\frac{x^2}{64}+\frac{y^2}{16}=1 .$$

The point Q is the foot of the perpendicular from the point P to the straight line with equation $x=10$.

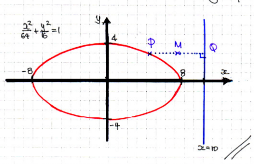
a) Sketch in the same diagram the ellipse, the straight line with equation $x=12$ and the straight line segment PQ.

The point M is the midpoint of PQ.

b) Determine a Cartesian equation for the locus of M as the position of P varies, further describing this locus geometrically.

$\boxed{}$, $\boxed{(x-5)^2+y^2=16}$

a) THIS IS A STANDARD ELLIPSE WITH $-8 \leq x \leq 8$, $-4 \leq y \leq 4$



b) PARAMETRIZE THE ELLIPSE

$$x = 8\cos\theta \ , \ y = 4\sin\theta \quad 0 \leq \theta < 2\pi$$

THEN THE CO-ORDINATES OF P, Q & R CAN BE FOUND

- $P(8\cos\theta, 4\sin\theta)$
- $Q(10, 4\sin\theta)$
- $M\left[\frac{8\cos\theta+10}{2}, \frac{4\sin\theta+4\sin\theta}{2}\right] = M(5+4\cos\theta, 4\sin\theta)$

ELIMINATE THE PARAMETER θ, OUT OF THE GENERAL CO-ORDINATES OF M (WRITTEN AS PARAMETRICS)

$$\left.\begin{array}{l} X = 5+4\cos\theta \\ Y = 4\sin\theta \end{array}\right\} \Longrightarrow \begin{array}{l} 4\cos\theta = X-5 \\ 4\sin\theta = Y \end{array}$$

$$\Longrightarrow \begin{array}{l} 16\cos^2\theta = (X-5)^2 \\ 16\sin^2\theta = Y^2 \end{array}$$

$$\Longrightarrow 16\cos^2\theta + 16\sin^2\theta = (X-5)^2+Y^2$$

$$\Longrightarrow 16(\cancel{\cos^2\theta+\sin^2\theta}) = (X-5)^2+Y^2$$

$$\Longrightarrow (X-5)^2+Y^2 = 16$$

$\therefore$ A CIRCLE, CENTRED AT $(5,0)$, RADIUS 4

Question 16 (***+)

An ellipse E has Cartesian equation

$$\frac{x^2}{16}+\frac{y^2}{4}=1.$$

a) Show that an equation of the tangent to E at the point $A(4\cos\theta, 2\sin\theta)$ is given by

$$2y\sin\theta + x\cos\theta = 4.$$

The point $B(4\cos\theta, 4\sin\theta)$ lies on the circle with Cartesian equation

$$x^2+y^2=16.$$

The tangent to the circle at the point B meets the tangent to the ellipse at the point A at the point P.

b) Determine the coordinates of P, in terms of θ.

c) Describe mathematically the locus of P as θ varies.

$\boxed{}$, $\boxed{P(4\sec\theta, 0)}$, $\boxed{\text{the } x \text{ axis, so that } x\in(-\infty,-4]\cup[4,\infty)}$

a) OBTAIN THE GRADIENT FUNCTION

$$\frac{d}{dx}\left(\frac{x^2}{16}+\frac{y^2}{4}\right)=\frac{d}{dx}(1)$$

$$\frac{1}{8}x+\frac{1}{2}y\frac{dy}{dx}=0$$

AT THE GENERAL POINT $A(4\cos\theta, 2\sin\theta)$

$$\frac{1}{8}(4\cos\theta)+\frac{1}{2}(2\sin\theta)\left.\frac{dy}{dx}\right|_A=0$$

$$\frac{1}{2}\cos\theta+\left.\frac{dy}{dx}\right|_A\sin\theta=0$$

$$\left.\frac{dy}{dx}\right|_A=-\frac{\cos\theta}{2\sin\theta}$$

OBTAIN THE EQUATION OF THE TANGENT

$$y-2\sin\theta=-\frac{\cos\theta}{2\sin\theta}(x-4\cos\theta)$$

$$2y\sin\theta-4\sin^2\theta=-x\cos\theta+4\cos^2\theta$$

$$2y\sin\theta+x\cos\theta=4(\cos^2\theta+\sin^2\theta)$$

$$2y\sin\theta+x\cos\theta=4$$ AS REQUIRED

b) CIRCLE HAS CENTRE AT $(0,0)$ – GRADIENT OF OB

$$m_{OB}=\frac{4\sin\theta-0}{4\cos\theta-0}=\frac{\sin\theta}{\cos\theta}$$

TANGENT GRADIENT AT B IS $-\frac{\cos\theta}{\sin\theta}$

EQUATION OF TANGENT AT B IS

$$y-4\sin\theta=-\frac{\cos\theta}{\sin\theta}(x-4\cos\theta)$$

$$y\sin\theta-4\sin^2\theta=-x\cos\theta+4\cos^2\theta$$

$$y\sin\theta+x\cos\theta=4(\cos^2\theta+\sin^2\theta)$$

$$y\sin\theta+x\cos\theta=4$$

SOLVING SIMULTANEOUSLY

$$\left.\begin{array}{l} y\sin\theta+x\cos\theta=4 \\ 2y\sin\theta+x\cos\theta=4\end{array}\right\}\Rightarrow y\sin\theta=0$$

$$\Rightarrow y=0$$

$$\Rightarrow x\cos\theta=4$$

$$\Rightarrow x=\frac{4}{\cos\theta}$$

$$\therefore P\left(\frac{4}{\cos\theta},0\right)$$

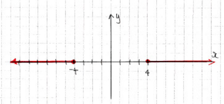
c) THE POINT $P\left(\frac{4}{\cos\theta},0\right)$ LIES ON THE x AXIS

$$-1\leqslant\cos\theta\leqslant 1$$

$$\frac{1}{\cos\theta}\leqslant -1 \;\cup\; \frac{1}{\cos\theta}\geqslant 1 \quad \text{(ESSENTIALLY } \sec\theta\text{)}$$

$$\frac{4}{\cos\theta}\leqslant -4 \;\cup\; \frac{4}{\cos\theta}\geqslant 1$$

HENCE THE REQUIRED LOCUS IS $\{y=0 : x\leqslant -4 \cup x\geqslant 4\}$



Question 17 (****)

An ellipse has Cartesian equation

$$\frac{x^2}{2} + y^2 = 1.$$

A straight line L has equation $y = mx + c$, where m and c are positive constants.

a) Show that the x coordinates of the points of intersection between L and the ellipse satisfy the equation

$$\left(2m^2+1\right)x^2 + 4mcx + 2\left(c^2-1\right) = 0.$$

b) Given that L is a tangent to the ellipse, show that $c^2 = 2m^2 + 1$.

The line L meets the negative x axis and the positive y axis at the points X and Y respectively. The point O is the origin.

c) Find the area of the triangle OXY, in terms of m

d) Show that as m varies, the minimum area of the triangle OXY is $\sqrt{2}$.

e) Find the x coordinate of the point of tangency between the line L and the ellipse when the area of the triangle is minimum.

$$\boxed{\text{area} = m + \frac{1}{2m}}, \quad \boxed{x = -1}$$

(a) $\left.\begin{array}{l}\frac{x^2}{2}+y^2=1\\ y=mx+c\end{array}\right\} \Rightarrow$

$\Rightarrow \frac{x^2}{2} + (mx+c)^2 = 1$

$\Rightarrow \frac{x^2}{2} + m^2x^2 + 2mcx + c^2 = 1$

$\Rightarrow x^2 + 2m^2x^2 + 4mcx + 2c^2 = 2$

$\Rightarrow (2m^2+1)x^2 + (4mc)x + 2(c^2-1) = 0$

AS REQUIRED

(b) $b^2 - 4ac = 0$

$\Rightarrow (4mc)^2 - 4(2m^2+1)\times 2(c^2-1) = 0$

$\Rightarrow 16m^2c^2 - 8(2m^2+1)(c^2-1) = 0$

$\Rightarrow 2m^2c^2 - (2m^2+1)(c^2-1) = 0$

$\Rightarrow 2m^2c^2 - (2m^2c^2 - 2m^2 + c^2 - 1) = 0$

$\Rightarrow \cancel{2m^2c^2} - \cancel{2m^2c^2} + 2m^2 - c^2 + 1 = 0$

$\Rightarrow c^2 = 2m^2+1$

AS REQUIRED

(c) $y = mx + c$

$x=0 \quad y=c$

$y=0 \quad x = -\frac{c}{m}$

AREA $= \frac{1}{2}\times c \times \frac{c}{m} = \frac{c^2}{2m}$

AREA $= \frac{2m^2+1}{2m}$

(d) $f(m) = m + \frac{1}{2m}$

$f'(m) = 1 - \frac{1}{2m^2}$

$f''(m) = \frac{1}{m^3} > 0$

AS $m>0$

$\therefore$ IT WILL BE MIN

$1 - \frac{1}{2m^2} = 0$

$2m^2 - 1 = 0$

$m^2 = \frac{1}{2}$

$m = +\frac{\sqrt{2}}{2} \quad m>0$

$\therefore f_{\text{MIN}}(m) = \frac{\sqrt{2}}{2} + \frac{1}{\sqrt{2}}$

$f_{\text{MIN}}(m) = \sqrt{2}$

AS REQUIRED

(e) NOW $\boxed{m = \frac{\sqrt{2}}{2}}$

$c^2 = 2m^2+1$

$c^2 = 2$

$\boxed{c = +\sqrt{2}}$

THUS

$(2m^2+1)x^2 + (4mc)x \; (c^2-1) = 0$

$c^2x^2 + (4mc)x + 2(c^2-1) = 0$

$2x^2 + 4x + 2 = 0$

$x^2 + 2x + 1 = 0$

$(x+1)^2 = 0$

$x = -1$

Question 18 (****)

The point $P(x,y)$ lies on an ellipse with foci at $A(2,0)$ and $B(6,0)$.

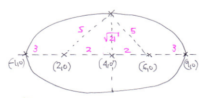
Given further that

$$|AP|+|BP|=10,$$

determine a simplified Cartesian equation for the ellipse, giving the final answer in the form

$$f(x,y)=1.$$

$$\boxed{\frac{(x-4)^2}{25}+\frac{y^2}{21}=1}$$



• BY SIMPLE GEOMETRY WE DRAW SKETCH

$(-1,0)$

$(9,0)$

$(4,\sqrt{21})$

$$\boxed{\frac{(x-A)^2}{B}+\frac{y^2}{C}=1}$$

• $(-1,0) \Rightarrow \frac{(-1-A)^2}{B}=1$

$\Rightarrow \frac{(A+1)^2}{B}=1$

$\Rightarrow \boxed{B=(A+1)^2}$

• $(9,0) \Rightarrow \frac{(9-A)^2}{B}=1$

$\boxed{B=(9-A)^2}$

• $(4,\sqrt{21}) \Rightarrow \boxed{\frac{(4-A)^2}{B}+\frac{21}{C}=1}$

THUS $(A+1)^2=(9-A)^2$

$\cancel{A^2}+2A+1=81-18A+\cancel{A^2}$

$20A=80$

$\boxed{A=4}$

& HENCE $\boxed{B=25}$

FINALLY

$\frac{4-4}{25}+\frac{21}{C}=1$

$\boxed{C=21}$

$\frac{(x-4)^2}{25}+\frac{y^2}{21}=1$

ALTERNATIVE BY LOCUS APPROACH

• LET A GENERAL POINT $P(x,y)$ LIE ON THE ELLIPSE

• THEN $\sqrt{(x-2)^2+(y-0)^2}+\sqrt{(x-6)^2+(y-0)^2}=10$

$\Rightarrow \sqrt{x^2-4x+4+y^2}=10-\sqrt{x^2-12x+36+y^2}$

$\Rightarrow \cancel{x^2}+\cancel{y^2}-4x+4=100-20\sqrt{x^2+y^2-12x+36}+(\cancel{x^2}+\cancel{y^2}-12x+36)$

$\Rightarrow 8x-132=-20\sqrt{x^2+y^2-12x+36}$

$\Rightarrow 33-2x=5\sqrt{x^2+y^2-12x+36}$

$\Rightarrow (33-2x)^2=25(x^2+y^2-12x+36)$

$\Rightarrow 4x^2-132x+1089=25x^2+25y^2-300x+900$

$\Rightarrow 0=21x^2+25y^2-168x-189$

$\Rightarrow 0=x^2+\frac{25}{21}y^2-8x-9$

$\Rightarrow (x-4)^2-16+\frac{25}{21}y^2-9=0$

$\Rightarrow (x-4)^2+\frac{25}{21}y^2=25$

$\Rightarrow \frac{(x-4)^2}{25}+\frac{y^2}{21}=1$

AS BEFORE

Question 19 (****)

An ellipse has Cartesian equation

$$\frac{x^2}{25}+\frac{y^2}{9}=1.$$

The general point $P(5\cos\theta, 3\sin\theta)$ lies on the ellipse.

a) Show that the equation of the normal to the ellipse at P is

$$3y\cos\theta - 5x\sin\theta + 16\sin\theta\cos\theta = 0.$$

The normal to the ellipse at P meets the x axis at the point Q and R is one of the foci of the ellipse.

b) Show clearly that

$$\frac{|QR|}{|PR|}=e,$$

where e is the eccentricity of the ellipse.

$\boxed{\text{proof}}$

(a) $\frac{x^2}{25}+\frac{y^2}{9}=1$

DIFF W.R.T x

$\frac{2}{25}x+\frac{2}{9}y\frac{dy}{dx}=0$

$\frac{1}{9}y\frac{dy}{dx}=-\frac{1}{25}x$

$\frac{dy}{dx}=-\frac{9x}{25y}$

$\left.\frac{dy}{dx}\right|_P=-\frac{9(5\cos\theta)}{25(3\sin\theta)}=-\frac{3\cos\theta}{5\sin\theta}$

EQUATION OF NORMAL THROUGH P

$\Rightarrow y-3\sin\theta=\frac{5\sin\theta}{3\cos\theta}(x-5\cos\theta)$

$\Rightarrow 3y\cos\theta-9\sin\theta\cos\theta=5x\sin\theta-25\sin\theta\cos\theta$

$\Rightarrow 3y\cos\theta-5x\sin\theta+16\sin\theta\cos\theta=0$

AS REQUIRED

(b) • WHEN $y=0$ $\quad -5x\sin\theta+16\sin\theta\cos\theta=0$

$x=\frac{16}{5}\cos\theta$ $\quad \therefore \boxed{Q\left(\frac{16}{5}\cos\theta, 0\right)}$

• ECCENTRICITY RELATION

$b^2=a^2(1-e^2)$

$9=25(1-e^2)$

$\frac{9}{25}=1-e^2$

$e^2=\frac{16}{25}$

$\boxed{e=\frac{4}{5}}$

FOCI ARE AT $(\pm ae, 0)$

SO $(\pm 5\times\frac{4}{5}, 0)$

$\therefore \boxed{R(4,0)}$

NOW $|QR|=4-\frac{16}{5}\cos\theta$

$|PR|=\sqrt{(4-5\cos\theta)^2+(3\sin\theta-0)^2}=\sqrt{16-40\cos\theta+25\cos^2\theta+9\sin^2\theta}$

$=\sqrt{16-40\cos\theta+9+16\cos^2\theta}=\sqrt{25-40\cos\theta+16\cos^2\theta}$

$=\sqrt{(5-4\cos\theta)^2}=5-4\cos\theta$

THUS $\frac{|QR|}{|PR|}=\frac{4-\frac{16}{5}\cos\theta}{5-4\cos\theta}=\frac{\frac{4}{5}\cancel{(5-4\cos\theta)}}{\cancel{(5-4\cos\theta)}}=\frac{4}{5}=e$

AS REQUIRED

Question 20 (****+)

An ellipse is given, in terms of a parameter θ, by the equations

$$x = 3\sqrt{2}\cos\theta\,, \quad y = 4\sin\theta\,, \quad 0 \le \theta < 2\pi\,.$$

a) Determine …

i. … the coordinates of the foci of the ellipse.

ii. … the equations of the directrices of the ellipse.

b) Show that an equation of the tangent at a general point on the ellipse is

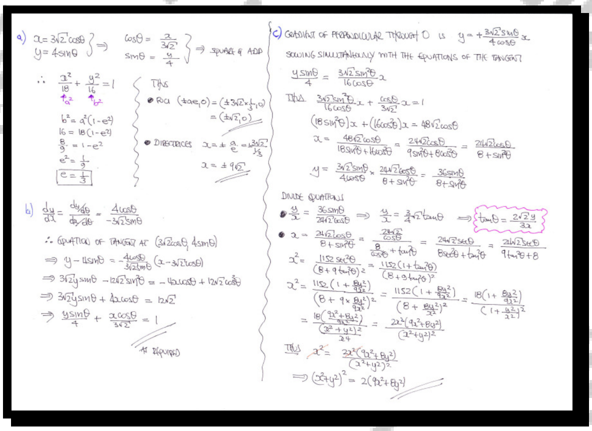
$$\frac{y\sin\theta}{4} + \frac{x\cos\theta}{3\sqrt{2}} = 1\,.$$

A straight line passes through the origin and meets the general tangent whose equation is given in part (b), at the point P.

c) Show that, as θ varies, P traces the curve with equation

$$\left(x^2 + y^2\right)^2 = 2\left(9x^2 + 8y^2\right).$$

$\boxed{F\left(\pm\sqrt{2},0\right)}$, $\boxed{x = \pm 9\sqrt{2}}$



Question 21 (****+)

The equation of an ellipse is given by

$$\frac{x^2}{4}+\frac{y^2}{3}=1.$$

a) Determine the coordinates of the foci of the ellipse, and the equation of each of its two directrices.

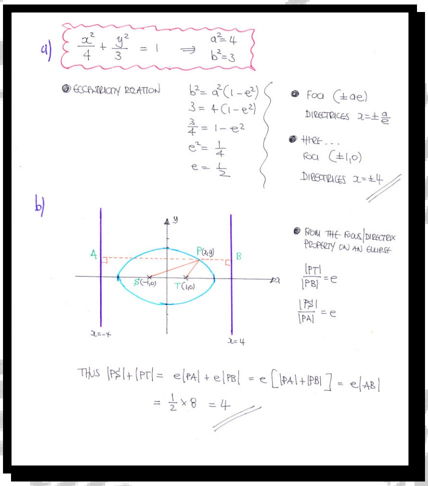
b) Show that

$$|SP|+|TP|=4.$$

A chord of the ellipse is defined as the straight line segment joining any two distinct points on the ellipse.

c) Find the equation of the locus of the midpoints of parallel chords of the ellipse whose gradient is 2.

$$\boxed{(\pm1,0)},\ \boxed{x=\pm4},\ \boxed{y=-\frac{1}{8}x}$$



a) $\frac{x^2}{4}+\frac{y^2}{3}=1 \Rightarrow \begin{matrix} a^2=4 \\ b^2=3 \end{matrix}$

• ECCENTRICITY RELATION

$b^2=a^2(1-e^2)$

$3=4(1-e^2)$

$\frac{3}{4}=1-e^2$

$e^2=\frac{1}{4}$

$e=\frac{1}{2}$

• FOCI $(\pm ae)$

DIRECTRICES $x=\pm\frac{a}{e}$

• HERE...

FOCI $(\pm1,0)$

DIRECTRICES $x=\pm4$

b)

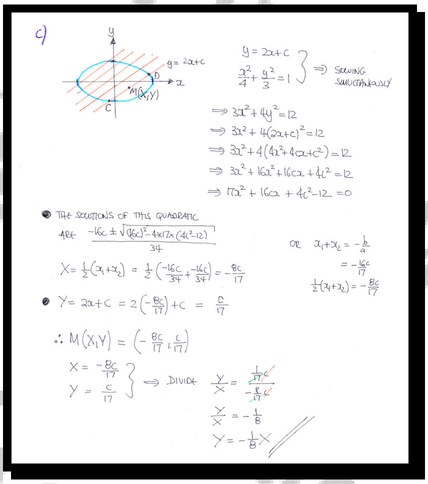
• FROM THE FOCUS/DIRECTRIX PROPERTY ON AN ELLIPSE

$\frac{|PT|}{|PB|}=e$

$\frac{|PS|}{|PA|}=e$

THUS $|PS|+|PT|=e|PA|+e|PB|=e\left[|PA|+|PB|\right]=e|AB|$

$=\frac{1}{2}\times8=4$



c)

$\left.\begin{matrix} y=2x+c \\ \frac{x^2}{4}+\frac{y^2}{3}=1 \end{matrix}\right\} \Rightarrow$ SOLVING SIMULTANEOUSLY

$\Rightarrow 3x^2+4y^2=12$

$\Rightarrow 3x^2+4(2x+c)^2=12$

$\Rightarrow 3x^2+4(4x^2+4cx+c^2)=12$

$\Rightarrow 3x^2+16x^2+16cx+4c^2=12$

$\Rightarrow 17x^2+16cx+4c^2-12=0$

• THE SOLUTIONS OF THIS QUADRATIC ARE $\frac{-16c\pm\sqrt{(16c)^2-4\times17\times(4c^2-12)}}{34}$

$X=\frac{1}{2}(x_1+x_2)=\frac{1}{2}\left(\frac{-16c}{34}+\frac{-16c}{34}\right)=-\frac{8c}{17}$

OR $x_1+x_2=-\frac{b}{a}$

$=-\frac{16c}{17}$

$\frac{1}{2}(x_1+x_2)=-\frac{8c}{17}$

• $Y=2x+c=2\left(-\frac{8c}{17}\right)+c=\frac{c}{17}$

$\therefore M(X,Y)=\left(-\frac{8c}{17},\frac{c}{17}\right)$

$\left.\begin{matrix} X=-\frac{8c}{17} \\ Y=\frac{c}{17} \end{matrix}\right\} \Rightarrow$ DIVIDE $\frac{Y}{X}=\frac{\cancel{\frac{1}{17}c}}{\cancel{-\frac{8}{17}c}}$

$\frac{Y}{X}=-\frac{1}{8}$

$Y=-\frac{1}{8}X$

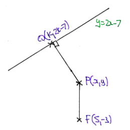
Question 22 (*****)

An ellipse has a focus at $(5,-3)$ and directrix with equation $y = 2x - 7$.

Given that the eccentricity of the ellipse is $\frac{\sqrt{5}}{10}$, find the coordinates of the points of intersection of the ellipse with the straight line with equation $y = -3$.

V, , $\boxed{\left(\frac{23}{4},-3\right), \left(\frac{9}{2},-3\right)}$

STARTING WITH A DIAGRAM AND SOME STANDARD RESULTS



- $\frac{|PF|}{|PQ|} = e = \frac{\sqrt{5}}{10} < 1$
- GRAD $PQ = -\frac{1}{2}$

$\frac{y-2k+7}{x-k} = -\frac{1}{2}$

$2y-4k+14 = -x+k$

$2y+x+14 = 5k$

$5k = 2y+x+14$

THUS WE NOW HAVE

$\Rightarrow |PF| = e\,|PQ|$

$\Rightarrow |PF|^2 = e^2 |PQ|^2$

$\Rightarrow (x-5)^2+(y+3)^2 = \frac{1}{20}\left[(x-k)^2+(y-2k+7)^2\right]$

MULTIPLY THE EQUATION BY 25 TO CREATE $5k= \dots$ IN THE R.H.S

$\Rightarrow 25\left[(x-5)^2+(y+3)^2\right] = \frac{1}{20}\left[(5x-5k)^2+(5y-2\times 5k+35)^2\right]$

$\Rightarrow 25\left[(x-5)^2+(y+3)^2\right] = \frac{1}{20}\left[(5x-2y-x-14)^2+(5y-4y-2x-28+35)^2\right]$

$\Rightarrow 25\left[(x-5)^2+(y+3)^2\right] = \frac{1}{20}\left[(4x-2y-14)^2+(y-2x+7)^2\right]$

NOW THIS IS THE EQUATION OF THE ELLIPSE – NO NEED TO SIMPLIFY AS WE ARE ONLY INTERESTED IN INTERSECTIONS WITH $y=-3$

$\Rightarrow 25\left[(x-5)^2+0^2\right] = \frac{1}{20}\left[(4x-8)^2+(-2x+4)^2\right]$

$\Rightarrow 500(x-5)^2 = (4x-8)^2+(2x-4)^2$

$\Rightarrow 500(x-5)^2 = 16(x-2)^2+4(x-2)^2$

$\Rightarrow 500(x-5)^2 = 20(x-2)^2$

$\Rightarrow 25(x-5)^2 = (x-2)^2$

$\Rightarrow \frac{(x-5)^2}{(x-2)^2} = \frac{1}{25}$

$\Rightarrow \frac{x-5}{x-2} = \frac{1}{5}$ OR $\frac{x-5}{x-2} = -\frac{1}{5}$

$5x-25 = x-2$

$4x = 23$

$x = \frac{23}{4}$

$5x-25 = -x+2$

$6x = 27$

$2x = 9$

$x = \frac{9}{2}$

AND FINALLY WE HAVE

$\left(\frac{23}{4},-3\right)$ & $\left(\frac{9}{2},-3\right)$

Question 23 (*****)

The point P lies on the ellipse with parametric equations

$$x = 3\cos\theta \qquad y = 2\sin\theta \qquad 0 \le \theta \le \frac{1}{2}\pi .$$

The point M is the midpoint of PY, where Y is the point where the normal to ellipse at P meets the y axis.

If O represents the origin, determine the maximum value of the area of the triangle OMP, as θ varies.

V, $\boxed{}$, $\boxed{\text{Area}_{\text{max}}\ \frac{15}{16}}$

START BY OBTAINING A GENERAL NORMAL AT $P(3\cos\theta, 2\sin\theta)$

$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{2\cos\theta}{-3\sin\theta}$ i.e. TANGENT GRADIENT AT P IS $-\frac{2}{3}\cot\theta$

$\therefore$ NORMAL GRADIENT AT P IS $+\frac{3}{2}\tan\theta$

EQUATION OF NORMAL

$y - 2\sin\theta = \frac{3}{2}\tan\theta(x - 3\cos\theta)$

MEETS THE y AXIS AT $x=0$

$y - 2\sin\theta = -\frac{9}{2}\tan\theta\cos\theta$

$y = 2\sin\theta - \frac{9}{2}\sin\theta$

$y = -\frac{5}{2}\sin\theta$

$\therefore Y(0, -\frac{5}{2}\sin\theta)$

COMPUTE THE CO-ORDINATES OF M

$M\left(\frac{3\cos\theta + 0}{2}, \frac{2\sin\theta - \frac{5}{2}\sin\theta}{2}\right) = M\left(\frac{3}{2}\cos\theta, -\frac{1}{4}\sin\theta\right)$

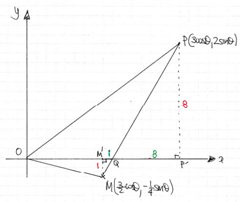
NEXT FIND THE AREA OF THE TRIANGLE $\hat{OMP}$

$$\text{AREA} = \frac{1}{2}\left\| \begin{matrix} 1 & 1 & 1 \\ 0 & 3\cos\theta & \frac{3}{2}\cos\theta \\ 0 & 2\sin\theta & -\frac{1}{4}\sin\theta \end{matrix} \right\| = \frac{1}{2}\left| -\frac{3}{4}\cos\theta\sin\theta - 3\cos\theta\sin\theta \right|$$

$$= \frac{1}{2} \times \frac{15}{4}\cos\theta\sin\theta = \frac{15}{8}\cos\theta\sin\theta = \frac{15}{16}\sin 2\theta$$

$\therefore \text{AREA}_{\text{MAX}} = \frac{15}{16}$ WHEN $\sin 2\theta = 1$

ALTERNATIVE FOR FINDING THE AREA OF OMP, BY GEOMETRY



BY SIMILAR TRIANGLES $MM'Q$ & $PP'Q$ (RATIO OFF THEIR HEIGHTS)

$|M'P'| = 3\cos\theta - \frac{3}{2}\cos\theta = \frac{3}{2}\cos\theta$

$|M'Q| = \frac{1}{9}|M'P'| = \frac{1}{9} \times \frac{3}{2}\cos\theta = \frac{1}{6}\cos\theta$

$|OQ| = |OM'| + |M'Q| = \frac{3}{2}\cos\theta + \frac{1}{6}\cos\theta = \frac{5}{3}\cos\theta$

THUS THE REQUIRED AREA IS

$$\frac{1}{2}|OQ||PP'| + \frac{1}{2}|OQ||MM'| = \frac{1}{2}|OQ|\left[|PP'| + |MM'|\right]$$
$$= \frac{1}{2} \times \frac{5}{3}\cos\theta \times \left[2\sin\theta + \frac{1}{4}\sin\theta\right]$$
$$= \frac{5}{6}\cos\theta \times \frac{9}{4}\sin\theta$$
$$= \frac{15}{8}\cos\theta\sin\theta$$

AS BEFORE

Question 24 (*****)

The straight line L with equation $y = mx + c$, where m and c are constants, passes through the point $(25, 25)$.

Given further that L is a tangent to the ellipse with equation

$$\frac{x^2}{25} + \frac{y^2}{9} = 1,$$

determine the possible equations of L.

$\boxed{y = \frac{4}{5}x + 5}$, $\boxed{y = \frac{77}{60}x - \frac{85}{12}}$

$\boxed{\frac{x^2}{25} + \frac{y^2}{9} = 1}$ & $\boxed{y = mx + c}$

- SOLVING SIMULTANEOUSLY AND LOOK FOR REPEATED ROOTS

$$\Rightarrow \frac{x^2}{25} + \frac{(mx+c)^2}{9} = 1$$
$$\Rightarrow 9x^2 + 25(mx+c)^2 = 225$$
$$\Rightarrow 9x^2 + 25m^2x^2 + 50mcx + 25c^2 = 225$$
$$\Rightarrow (9 + 25m^2)x^2 + (50mc)x + (25c^2 - 225) = 0$$

- LOOKING FOR REPEATED ROOTS $\Rightarrow b^2 - 4ac = 0$

$$\Rightarrow (50mc)^2 - 4(9 + 25m^2)(25c^2 - 225) = 0$$
$$\Rightarrow 2500m^2c^2 - 100(9 + 25m^2)(c^2 - 9) = 0$$
$$\Rightarrow 25m^2c^2 - (25m^2 + 9)(c^2 - 9) = 0$$
$$\Rightarrow 25m^2c^2 - (25m^2c^2 - 225m^2 + 9c^2 - 81) = 0$$
$$\Rightarrow \cancel{25m^2c^2} - \cancel{25m^2c^2} + 225m^2 - 9c^2 + 81 = 0$$
$$\Rightarrow 225m^2 - 9c^2 + 81 = 0$$
$$\Rightarrow 25m^2 - c^2 + 9 = 0$$
$$\Rightarrow c^2 - 25m^2 - 9 = 0$$
$$\Rightarrow \boxed{c^2 - 25m^2 = 9}$$

- NOW THE TANGENT PASSES THROUGH $(25, 25)$ $\Rightarrow y = mx + c$

$$\Rightarrow 25 = 25m + c$$
$$\Rightarrow c = 25 - 25m$$
$$\Rightarrow \boxed{c^2 = 625 - 1250m + 625m^2}$$

- SOLVING SIMULTANEOUSLY

$$\Rightarrow (625 - 1250m + 625m^2) - 25m^2 = 9$$
$$\Rightarrow 600m^2 - 1250m + 616 = 0$$
$$\Rightarrow 300m^2 - 625m + 308 = 0$$

- BY THE QUADRATIC FORMULA

$$m = \frac{625 \pm \sqrt{(-625)^2 - 4(300)(308)}}{2 \times 300}$$
$$m = \frac{625 \pm 145}{600} = \begin{cases} \frac{77}{60} \\ \frac{4}{5} \end{cases}$$
$$c = \begin{cases} -\frac{85}{12} \\ 5 \end{cases}$$

$$\therefore \ y = \frac{4}{5}x + 5$$

OR

$$y = \frac{77}{60}x - \frac{85}{12}$$

Question 25 (*****)

The point P lies on an ellipse whose foci are on the x axis at the points S and T.

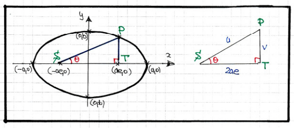
Given further that the triangle STP is right angled at T, show that

$$e = \frac{1-\tan\frac{1}{2}\theta}{1+\tan\frac{1}{2}\theta},$$

where e is the eccentricity of the ellipse, and θ is the angle PST.

[ ], proof

STARTING WITH A DIAGRAM



FROM THE RIGHT ANGLED TRIANGLE WE HAVE

$$\frac{v}{2ae} = \tan\theta \quad \& \quad \frac{2ae}{u} = \cos\theta$$

$$v = 2ae\tan\theta \qquad u = 2ae\sec\theta$$

FROM THE LOCUS PROPERTY OF THE ELLIPSE WE HAVE

$\Rightarrow |SP| + |TP| = \text{CONSTANT} = 2a$

$\Rightarrow u + v = 2a$

$\Rightarrow 2ae\sec\theta + 2ae\tan\theta = 2a$

$\Rightarrow e\sec\theta + e\tan\theta = 1$

$\Rightarrow e(\sec\theta + \tan\theta) = 1$

$\Rightarrow e = \dfrac{1}{\sec\theta + \tan\theta}$

MULTIPLYING "TOP & BOTTOM" OF THE FRACTION ON THE RHS BY $\cos\theta$ GIVES

$\Rightarrow e = \dfrac{\cos\theta}{\sec\theta\cos\theta + \tan\theta\cos\theta}$

$\Rightarrow e = \dfrac{\cos\theta}{1 + \frac{\sin\theta}{\cancel{\cos\theta}}\cancel{\cos\theta}} = \dfrac{\cos\theta}{1+\sin\theta}$

$\Rightarrow e = \dfrac{\overbrace{\cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2}}^{\cos\theta}}{\underbrace{\left(\cos^2\frac{\theta}{2} + \sin^2\frac{\theta}{2}\right)}_{1} + \underbrace{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}_{\sin\theta}}$

$\Rightarrow e = \dfrac{\left(\cos\frac{\theta}{2} - \sin\frac{\theta}{2}\right)\left(\cos\frac{\theta}{2} + \sin\frac{\theta}{2}\right)}{\left(\cos\frac{\theta}{2} + \sin\frac{\theta}{2}\right)^2}$

$\Rightarrow e = \dfrac{\cos\frac{\theta}{2} - \sin\frac{\theta}{2}}{\cos\frac{\theta}{2} + \sin\frac{\theta}{2}}$

DIVIDING "TOP & BOTTOM" OF THE FRACTION ON THE R.H.S BY $\cos\frac{\theta}{2}$

$\Rightarrow e = \dfrac{\frac{\cos\frac{\theta}{2}}{\cos\frac{\theta}{2}} - \frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}}{\frac{\cos\frac{\theta}{2}}{\cos\frac{\theta}{2}} + \frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}}$

$\Rightarrow e = \dfrac{1 - \tan\frac{\theta}{2}}{1 + \tan\frac{\theta}{2}}$ //

Question 26 (*****)

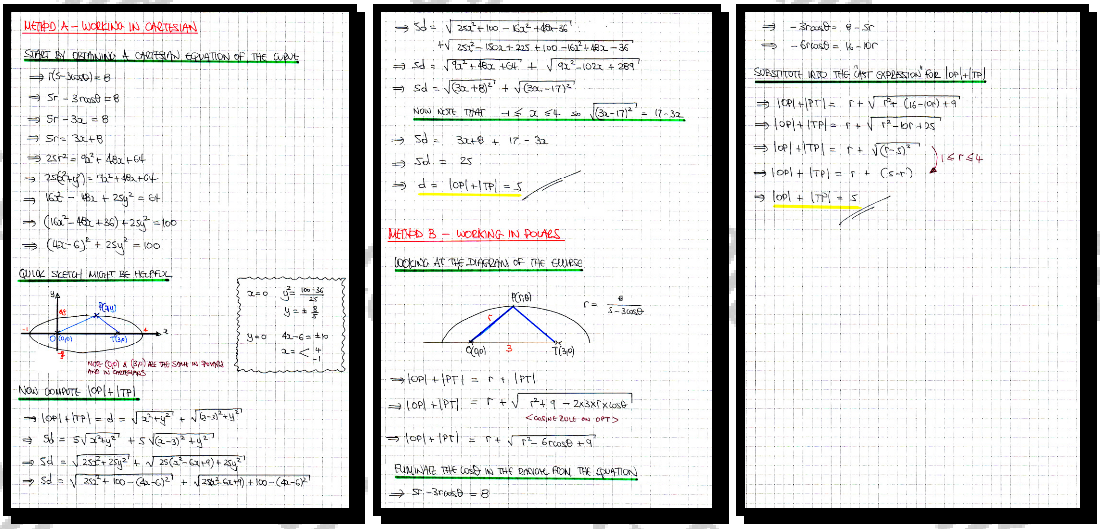
The point P lies on the ellipse with polar equation

$$r(5-3\cos\theta)=8, \ 0 \le \theta < 2\pi .$$

The ellipse has foci at $O(0,0)$ and at $T(3,0)$.

Show that $|OP|+|PT|$ is constant for all positions of P.

$\boxed{}$, $\boxed{|OP|+|PT|=5}$


METHOD A – WORKING IN CARTESIAN

START BY OBTAINING A CARTESIAN EQUATION OF THE CURVE

$\Rightarrow r(5-3\cos\theta)=8$

$\Rightarrow 5r-3r\cos\theta=8$

$\Rightarrow 5r-3x=8$

$\Rightarrow 5r=3x+8$

$\Rightarrow 25r^2=9x^2+48x+64$

$\Rightarrow 25(x^2+y^2)=9x^2+48x+64$

$\Rightarrow 16x^2-48x+25y^2=64$

$\Rightarrow (16x^2-48x+36)+25y^2=100$

$\Rightarrow (4x-6)^2+25y^2=100$

QUICK SKETCH MIGHT BE HELPFUL

$x=0 \quad y^2=\frac{100-36}{25}$, $y=\pm\frac{8}{5}$

$y=0 \quad 4x-6=\pm 10$, $x = \begin{cases}4\\-1\end{cases}$

NOW COMPUTE $|OP|+|TP|$

$\Rightarrow |OP|+|TP|=d=\sqrt{x^2+y^2}+\sqrt{(x-3)^2+y^2}$

$\Rightarrow 5d=5\sqrt{x^2+y^2}+5\sqrt{(x-3)^2+y^2}$

$\Rightarrow 5d=\sqrt{25x^2+25y^2}+\sqrt{25(x^2-6x+9)+25y^2}$

$\Rightarrow 5d=\sqrt{25x^2+100-(4x-6)^2}+\sqrt{25(x^2-6x+9)+100-(4x-6)^2}$

$\Rightarrow 5d=\sqrt{25x^2+100-16x^2+48x-36}$
$+\sqrt{25x^2-150x+225+100-16x^2+48x-36}$

$\Rightarrow 5d=\sqrt{9x^2+48x+64}+\sqrt{9x^2-102x+289}$

$\Rightarrow 5d=\sqrt{(3x+8)^2}+\sqrt{(3x-17)^2}$

NOW NOTE THAT $-1 \le x \le 4$ SO $\sqrt{(3x-17)^2}=17-3x$

$\Rightarrow 5d=3x+8+17-3x$

$\Rightarrow 5d=25$

$\Rightarrow d=|OP|+|TP|=5$

METHOD B – WORKING IN POLARS

LOOKING AT THE DIAGRAM OF THE ELLIPSE

$r=\frac{8}{5-3\cos\theta}$

$\Rightarrow |OP|+|PT|=r+|PT|$

$\Rightarrow |OP|+|PT|=r+\sqrt{r^2+9-2\times3\times r\times\cos\theta}$

<COSINE RULE ON OPT>

$\Rightarrow |OP|+|PT|=r+\sqrt{r^2-6r\cos\theta+9}$

ELIMINATE THE $\cos\theta$ IN THE RADICAL FROM THE EQUATION

$\Rightarrow 5r-3r\cos\theta=8$

$\Rightarrow -3r\cos\theta=8-5r$

$\Rightarrow -6r\cos\theta=16-10r$

SUBSTITUTE INTO THE "LAST EXPRESSION" FOR $|OP|+|TP|$

$\Rightarrow |OP|+|PT|=r+\sqrt{r^2+(16-10r)+9}$

$\Rightarrow |OP|+|TP|=r+\sqrt{r^2-10r+25}$

$\Rightarrow |OP|+|TP|=r+\sqrt{(r-5)^2}$ $\quad 1 \le r \le 4$

$\Rightarrow |OP|+|TP|=r+(5-r)$

$\Rightarrow |OP|+|TP|=5$

RECTANGULAR HYPERBOLA

Question 1 **(**)**

The rectangular hyperbola H has Cartesian equation

$$xy = 9, \quad x \neq 0, \quad y \neq 0.$$

The point $P\left(3t, \dfrac{3}{t}\right)$, $t \neq 0$, where t is a parameter, lies on H.

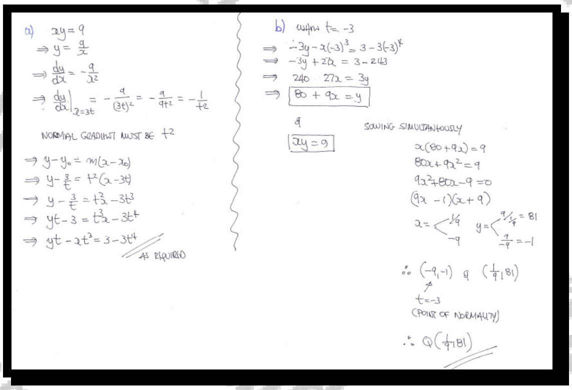
a) Show that the equation of a normal to H at P is given by

$$yt - xt^3 = 3 - 3t^4.$$

The normal to H at the point where $t = -3$ meets H again at the point Q.

b) Determine the coordinates of Q.

$$\boxed{Q\left(\frac{1}{9}, 81\right)}$$



a) $xy = 9$

$\Rightarrow y = \frac{9}{x}$

$\Rightarrow \frac{dy}{dx} = -\frac{9}{x^2}$

$\Rightarrow \left.\frac{dy}{dx}\right|_{x=3t} = -\frac{9}{(3t)^2} = -\frac{9}{9t^2} = -\frac{1}{t^2}$

NORMAL GRADIENT MUST BE t^2

$\Rightarrow y - y_0 = m(x - x_0)$

$\Rightarrow y - \frac{3}{t} = t^2(x - 3t)$

$\Rightarrow y - \frac{3}{t} = t^2x - 3t^3$

$\Rightarrow yt - 3 = t^3x - 3t^4$

$\Rightarrow yt - xt^3 = 3 - 3t^4$

AS REQUIRED

b) when $t = -3$

$\Rightarrow -3y - x(-3)^3 = 3 - 3(-3)^4$

$\Rightarrow -3y + 27x = 3 - 243$

$\Rightarrow 240 + 27x = 3y$

$\Rightarrow \boxed{80 + 9x = y}$

& $\boxed{xy = 9}$

SOLVING SIMULTANEOUSLY

$x(80 + 9x) = 9$

$80x + 9x^2 = 9$

$9x^2 + 80x - 9 = 0$

$(9x - 1)(x + 9)$

$x = \frac{1}{9}$ or -9, $\quad y = \frac{9}{1/9} = 81$ or $\frac{9}{-9} = -1$

$\therefore (-9, -1)$ & $\left(\frac{1}{9}, 81\right)$

$t = -3$ (POINT OF NORMALITY)

$\therefore Q\left(\frac{1}{9}, 81\right)$

Question 2 (**)

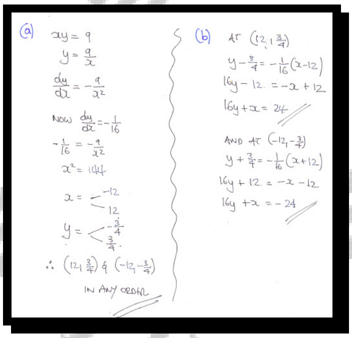
The tangents to the hyperbola with equation $xy = 9$, at two distinct points A and B, have gradient $-\dfrac{1}{16}$.

Determine in any order …

a) … the coordinates of A and B.

b) … the equation of each of the two tangents.

$\boxed{A\left(12,\frac{3}{4}\right),\ B\left(-12,-\frac{3}{4}\right)}$, $\boxed{x + 16y = 24,\ x + 16y = -24}$



(a) $xy = 9$

$y = \frac{9}{x}$

$\frac{dy}{dx} = -\frac{9}{x^2}$

NOW $\frac{dy}{dx} = -\frac{1}{16}$

$-\frac{1}{16} = -\frac{9}{x^2}$

$x^2 = 144$

$x = \begin{cases} -12 \\ 12 \end{cases}$

$y = \begin{cases} -\frac{3}{4} \\ \frac{3}{4} \end{cases}$

$\therefore \left(12, \frac{3}{4}\right)$ & $\left(-12, -\frac{3}{4}\right)$

IN ANY ORDER

(b) AT $\left(12, \frac{3}{4}\right)$

$y - \frac{3}{4} = -\frac{1}{16}(x - 12)$

$16y - 12 = -x + 12$

$16y + x = 24$

AND AT $\left(-12, -\frac{3}{4}\right)$

$y + \frac{3}{4} = -\frac{1}{16}(x + 12)$

$16y + 12 = -x - 12$

$16y + x = -24$

Question 3 **(**+)**

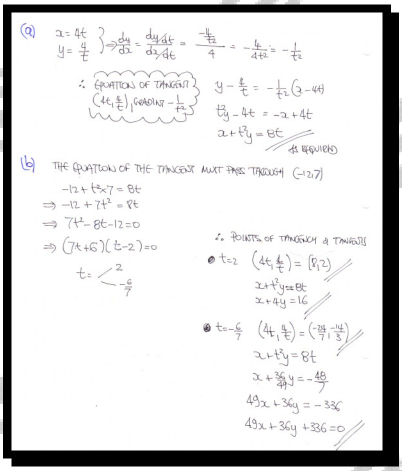
The general point $P\left(4t, \dfrac{4}{t}\right)$, $t \neq 0$, where t is a parameter, lies on a hyperbola H.

a) Show that the equation of a tangent at the point P is given by

$$x + t^2 y = 8t .$$

b) Find the equation of each of the two tangents to H which pass through the point $Q(-12,7)$, and further deduce the coordinates of their corresponding points of tangency.

$\boxed{x + 4y = 16, \ (8,2)}$, $\boxed{49x + 36y + 336 = 0, \ \left(-\frac{24}{7}, -\frac{14}{3}\right)}$


(a) $x = 4t$, $y = \frac{4}{t}$ $\Rightarrow \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-\frac{4}{t^2}}{4} = -\frac{4}{4t^2} = -\frac{1}{t^2}$

∴ EQUATION OF TANGENT $\left(4t, \frac{4}{t}\right)$, GRADIENT $-\frac{1}{t^2}$

$y - \frac{4}{t} = -\frac{1}{t^2}(x - 4t)$

$t^2y - 4t = -x + 4t$

$x + t^2y = 8t$ AS REQUIRED

(b) THE EQUATION OF THE TANGENT MUST PASS THROUGH $(-12,7)$

$-12 + t^2 \times 7 = 8t$

$\Rightarrow -12 + 7t^2 = 8t$

$\Rightarrow 7t^2 - 8t - 12 = 0$

$\Rightarrow (7t+6)(t-2) = 0$

$t = 2$ or $-\frac{6}{7}$

∴ POINTS OF TANGENCY & TANGENTS

- $t = 2$: $\left(4t, \frac{4}{t}\right) = (8,2)$; $x + t^2y = 8t$; $x + 4y = 16$
- $t = -\frac{6}{7}$: $\left(4t, \frac{4}{t}\right) = \left(-\frac{24}{7}, -\frac{14}{3}\right)$; $x + t^2y = 8t$; $x + \frac{36}{49}y = -\frac{48}{7}$; $49x + 36y = -336$; $49x + 36y + 336 = 0$

Question 4 (***)

The general point $P\left(ct, \frac{c}{t}\right)$, $c>0$, $t>0$, lies on a hyperbola H with Cartesian equation

$$xy = c^2 .$$

The tangent to H at P meets the coordinate axes at the points A and B.

Given the area of the triangle BOA is 72 square units, find the value of c.

$\boxed{c=6}$

$xy = c^2$

$y = \frac{c^2}{x}$

$\frac{dy}{dx} = -\frac{c^2}{x^2}$

$\left.\frac{dy}{dx}\right|_{x=ct} = -\frac{c^2}{c^2t^2} = -\frac{1}{t^2}$

GENERAL TANGENT

$y - \frac{c}{t} = -\frac{1}{t^2}(x - ct)$

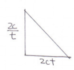
$yt^2 - ct = -x + ct$

$\boxed{yt^2 + x = 2ct}$

IF $x=0$ $y = \frac{2c}{t}$ ∴ $A\left(0, \frac{2c}{t}\right)$

IF $y=0$ $x = 2ct$ ∴ $B(2ct, 0)$

THUS



AREA $= 72$

$\frac{1}{2} \times 2ct \times \frac{2c}{t} = 72$

$2c^2 = 72$

$c^2 = 36$

$c = 6$ $(c>0)$

Question 5 **(***)**

The general point $P\left(3t, \frac{3}{t}\right)$, $t \neq 0$, where t is a parameter, lies on a hyperbola H.

a) Show that the equation of a tangent at the point P is given by

$$x + t^2 y = 6t.$$

The tangents to the hyperbola at points A and B intersect at the point $Q(-1, 7)$.

b) Determine in any order …

i. …the coordinates of A and B.

ii. … the equation of each of the two tangents.

$\boxed{A(3,3),\ \ B\left(-\frac{3}{7}, -21\right)}$, $\boxed{x + y = 6,\ \ 49x + y + 42 = 0}$

(a) $\left.\begin{array}{l} x = 3t \\ y = \frac{3}{t} \end{array}\right\}$ $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-\frac{3}{t^2}}{3} = -\frac{1}{t^2}$

TANGENT AT $\left(3t, \frac{3}{t}\right)$, GRADIENT $-\frac{1}{t^2}$

$y - \frac{3}{t} = -\frac{1}{t^2}(x - 3t)$

$t^2 y - 3t = -x + 3t$

$t^2 y + x = 6t$ // AS REQUIRED

(b) THE GENERAL TANGENT MUST PASS THROUGH $(-1, 7)$

$\Rightarrow 7t^2 - 1 = 6t$

$\Rightarrow 7t^2 - 6t - 1 = 0$

$\Rightarrow (7t + 1)(t - 1) = 0$

$\Rightarrow t = 1, -\frac{1}{7}$

• IF $t = 1$

$y + x = 6$ //

• IF $t = -\frac{1}{7}$

$\frac{1}{49} y + x = -\frac{6}{7}$

$y + 49x = -42$

$y + 49x + 42 = 0$ //

Question 6 (*+)**

The point $P\left(ap, \frac{a}{p}\right)$ lies on the rectangular hyperbola H, with Cartesian equation

$$xy = a^2,$$

where a is a positive constant and p is a parameter.

a) Show that the equation of a tangent at the point P is given by

$$x + p^2 y = 2ap.$$

The point $Q\left(aq, \frac{a}{q}\right)$ also lies on H, where q is a parameter, so that $q \neq p$.

The tangent at P and the tangent at Q intersect at the point R.

b) Find simplified expressions for the coordinates of R.

The values of p and q are such so that $p = 3q$.

c) Find a Cartesian locus of R as p varies.

$$\boxed{R\left(\frac{2apq}{p+q}, \frac{2a}{p+q}\right)}, \quad \boxed{xy = \frac{3}{4}a^2}$$

(a) $xy = a^2$

$y = \frac{a^2}{x}$

$\frac{dy}{dx} = -\frac{a^2}{x^2}$

$\left.\frac{dy}{dx}\right|_{x=ap} = -\frac{a^2}{(ap)^2} = -\frac{1}{p^2}$

EQUATION OF TANGENT AT $P\left(ap, \frac{a}{p}\right)$

$y - \frac{a}{p} = -\frac{1}{p^2}(x - ap)$

$p^2 y - ap = -x + ap$

$p^2 y + x = 2ap$

(b) TANGENT AT P: $p^2 y + x = 2ap$

TANGENT AT Q: $q^2 y + x = 2aq$

SUBTRACT $p^2 y - q^2 y = 2ap - 2aq$

$(p^2 - q^2)y = 2a(p-q)$

$y = \frac{2a(p-q)}{p^2-q^2}$

$y = \frac{2a(p-q)}{(p-q)(p+q)}$

AS $q \neq p$

$y = \frac{2a}{p+q}$

NOW $p^2 y + x = 2ap$

$p^2\left(\frac{2a}{p+q}\right) + x = 2ap$

$x = 2ap - \frac{2ap^2}{p+q}$

$x = \frac{2ap(p+q) - 2ap^2}{p+q}$

$x = \frac{2ap^2 + 2apq - 2ap^2}{p+q}$

$x = \frac{2apq}{p+q}$

$\therefore R\left(\frac{2apq}{p+q}, \frac{2a}{p+q}\right)$

(c) NOW $p = 3q$

$R\left(\frac{2a(3q)q}{3q+q}, \frac{2a}{3q+q}\right) = \left(\frac{6aq^2}{4q}, \frac{2a}{4q}\right) = \left(\frac{3}{2}aq, \frac{a}{2q}\right)$

$X = \frac{3}{2}aq$, $Y = \frac{a}{2q}$ $\Rightarrow$ $XY = \frac{3}{2}aq \times \frac{a}{2q}$

$XY = \frac{3}{4}a^2$

Question 7 (****)

The general point $P\left(cp, \dfrac{c}{p}\right)$, $p \neq 0$, where p is a parameter, lies on the rectangular hyperbola, with Cartesian equation

$$xy = c^2,$$

where c is a positive constant.

a) Show that an equation of the tangent to the hyperbola at P is given by

$$yp^2 + x = 2cp.$$

Another point $Q\left(cq, \dfrac{c}{q}\right)$, $p \neq \pm q$ also lies on the hyperbola.

The tangents to the hyperbola at P and Q meet at the point R.

b) Show that the coordinates of R are given by

$$\left(\frac{2cpq}{p+q}, \frac{2c}{p+q}\right).$$

c) Given that PQ is perpendicular to OR, show that

$$p^2q^2 = 1.$$

proof

(a) $y = \dfrac{c^2}{x}$

$\dfrac{dy}{dx} = -\dfrac{c^2}{x^2}$

$\left.\dfrac{dy}{dx}\right|_{x=cp} = \dfrac{-c^2}{c^2p^2} = -\dfrac{1}{p^2}$

EQUATION OF TANGENT AT $P\left(cp, \frac{c}{p}\right)$

$y - \frac{c}{p} = -\frac{1}{p^2}(x - cp)$

$p^2y - cp = -x + cp$

$x + p^2y = 2cp$ // AS REQUIRED

(b) SIMILARLY THE TANGENT AT Q MUST BE $x + q^2y = 2cq$

$\therefore \begin{cases} x + p^2y = 2cp \\ x + q^2y = 2cq \end{cases} \Rightarrow$ SUBTRACT $(p^2 - q^2)y = 2c(p-q)$

$\cancel{(p-q)}(p+q)y = 2c\cancel{(p-q)}$

$\boxed{y = \frac{2c}{p+q}}$ $p \neq \pm q$

& $\begin{cases} q^2x + p^2q^2y = 2cpq^2 \\ p^2x + q^2p^2y = 2cqp^2 \end{cases}$ SUBTRACT

$(q^2 - p^2)x = 2cpq(q-p)$

$\cancel{(q-p)}(q+p)x = 2cpq\cancel{(q-p)}$

$q \neq p$

$\boxed{x = \frac{2cpq}{p+q}}$

$\therefore \left(\frac{2cpq}{p+q}, \frac{2c}{p+q}\right)$ // AS REQUIRED

(c) GRADIENT $PQ = \dfrac{\frac{c}{p} - \frac{c}{q}}{cp - cq} = \dfrac{\frac{1}{p} - \frac{1}{q}}{p - q} = \dfrac{\frac{q-p}{pq}}{\frac{p-q}{1}}$

$= \dfrac{q-p}{(p-q)pq} = -\dfrac{\cancel{p-q}}{pq\cancel{(p-q)}} = \dfrac{-1}{pq}$

$(p \neq q)$

GRADIENT $OR = \dfrac{\frac{2c}{\cancel{p+q}}}{\frac{2cpq}{\cancel{p+q}}} = \dfrac{1}{pq}$

SO $-\dfrac{1}{pq} \times \dfrac{1}{pq} = -1$

$\dfrac{1}{p^2q^2} = 1$ $\therefore p^2q^2 = 1$ //

Question 8 (****)

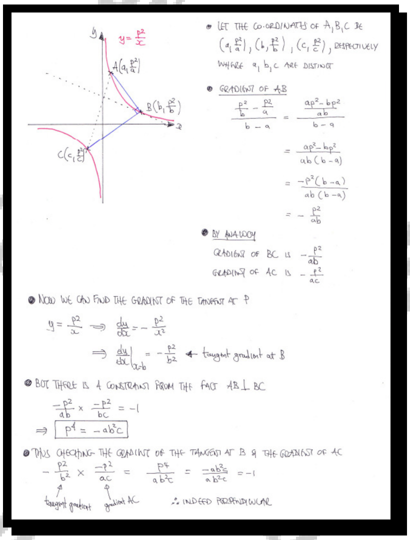
The distinct points A, B and C lie on the hyperbola with equation

$$xy = p^2,$$

where p is a positive constant.

Given that ABC is a right angle, show that the tangent to the hyperbola at B, is perpendicular to AC.

proof



- LET THE CO-ORDINATES OF A, B, C BE $\left(a, \frac{p^2}{a}\right), \left(b, \frac{p^2}{b}\right), \left(c, \frac{p^2}{c}\right)$, RESPECTIVELY WHERE a, b, c ARE DISTINCT

- GRADIENT OF AB

$$\frac{\frac{p^2}{b} - \frac{p^2}{a}}{b - a} = \frac{\frac{ap^2 - bp^2}{ab}}{b - a} = \frac{ap^2 - bp^2}{ab(b-a)} = \frac{-p^2(b-a)}{ab(b-a)} = -\frac{p^2}{ab}$$

- BY ANALOGY

GRADIENT OF BC IS $-\frac{p^2}{ab}$

GRADIENT OF AC IS $-\frac{p^2}{ac}$

- NOW WE CAN FIND THE GRADIENT OF THE TANGENT AT P

$$y = \frac{p^2}{x} \Rightarrow \frac{dy}{dx} = -\frac{p^2}{x^2}$$

$$\Rightarrow \left.\frac{dy}{dx}\right|_{x=b} = -\frac{p^2}{b^2} \leftarrow \text{tangent gradient at B}$$

- BUT THERE IS A CONSTRAINT FROM THE FACT $AB \perp BC$

$$\frac{-p^2}{ab} \times \frac{-p^2}{bc} = -1$$

$$\Rightarrow \boxed{p^4 = -ab^2c}$$

- THUS CHECKING THE GRADIENT OF THE TANGENT AT B & THE GRADIENT OF AC

$$\underbrace{-\frac{p^2}{b^2}}_{\text{tangent gradient}} \times \underbrace{\frac{-p^2}{ac}}_{\text{gradient AC}} = \frac{p^4}{ab^2c} = \frac{-ab^2c}{ab^2c} = -1$$

$\therefore$ INDEED PERPENDICULAR

Question 9 (****+)

The general point $P\left(5t, \dfrac{5}{t}\right)$, $t \neq 0$, where t is a parameter, lies on the hyperbola, with Cartesian equation

$$xy = 25.$$

a) Show that an equation of the normal to the hyperbola at the point P is

$$y = t^2 x + \frac{5}{t} - 5t^3$$

The normal to the hyperbola at P meets the hyperbola again at the point Q.

b) Show that the coordinates of Q are given by

$$\left(-\frac{5}{t^3}, -5t^3\right).$$

c) Show that the Cartesian form of the locus of the midpoint of PQ, as t varies, is given by

$$4xy + 25\left(\frac{y}{x} - \frac{x}{y}\right)^2 = 0.$$

$\boxed{\text{proof}}$

(a) $y = \frac{25}{x}$

$\frac{dy}{dx} = -\frac{25}{x^2}$

$\left.\frac{dy}{dx}\right|_{x=5t} = -\frac{25}{25t^2} = -\frac{1}{t^2}$

NORMAL GRADIENT $= t^2$, $P\left(5t, \frac{5}{t}\right)$

$y - \frac{5}{t} = t^2(x - 5t)$

$y = \frac{5}{t} + t^2 x - 5t^3$

$y = t^2 x + \frac{5}{t} - 5t^3$ // AS REQUIRED

(b) SOLVING SIMULTANEOUSLY $xy = 25$, $y = \frac{25}{x}$ & NORMAL

$\Rightarrow \frac{25}{x} = t^2 x + \frac{5}{t} - 5t^3$

$\Rightarrow \frac{25}{x} = t^2 x + \frac{5 - 5t^4}{t}$

$\Rightarrow \frac{25t}{x} = t^3 x + (5 - 5t^4)$

$\Rightarrow 25t = t^3 x^2 + (5 - 5t^4)x$

$\Rightarrow t^3 x^2 + (5 - 5t^4)x - 25t = 0$

• AS $x = 5t$ IS A SOLUTION (POINT P) FACTORIZE TO

$(x - 5t)(t^3 x + 5) = 0$

$x = 5t \leftarrow P$

$x = -\frac{5}{t^3} \leftarrow Q$

& $y = \frac{25}{x} = \frac{25}{-5/t^3} = \frac{-25t^3}{5} = -5t^3$

$\therefore Q\left(-\frac{5}{t^3}, -5t^3\right)$ //

(c) MIDPOINT OF PQ $M\left(\frac{5t - \frac{5}{t^3}}{2}, \frac{\frac{5}{t} - 5t^3}{2}\right) \Rightarrow M\left[\frac{5}{2}\left(t - \frac{1}{t^3}\right), \frac{5}{2}\left(\frac{1}{t} - t^3\right)\right]$

$\Rightarrow M\left[\frac{5}{2}\left(\frac{t^4 - 1}{t^3}\right), \frac{5}{2}\left(\frac{1 - t^4}{t}\right)\right] \Rightarrow M\left[\frac{5}{2}\left(\frac{t^4 - 1}{t^3}\right), -\frac{5}{2}\left(\frac{t^4 - 1}{t}\right)\right]$

PARAMETRIC LOCUS

$x = \frac{5}{2}\left(\frac{t^4 - 1}{t^3}\right)$, $y = -\frac{5}{2}\left(\frac{t^4 - 1}{t}\right)$ DIVIDE

$\frac{x}{y} = -\frac{\cancel{\frac{5}{2}}\left(\frac{\cancel{t^4-1}}{t^3}\right)}{\cancel{\frac{5}{2}}\left(\frac{\cancel{t^4-1}}{t}\right)} = -\frac{1}{t^2}$

$\boxed{t^2 = -\frac{y}{x}}$

THUS $y = -\frac{5}{2}\left(\frac{t^4 - 1}{t}\right)$

$\Rightarrow -\frac{2yt}{5} = t^4 - 1$

$\Rightarrow \frac{4y^2}{25} t^2 = (t^4 - 1)^2$

$\Rightarrow \frac{4y^2}{25}\left(\frac{-y}{x}\right) = \left(\frac{y^2}{x^2} - 1\right)^2$

$\Rightarrow -\frac{4y^3}{25x} = \left(\frac{y^2 - x^2}{x^2}\right)^2$

$\Rightarrow -\frac{4y^3}{25x} = \frac{(y^2 - x^2)^2}{x^4}$

$\Rightarrow -\frac{4}{25} x^3 y^3 = (y^2 - x^2)^2$

$\Rightarrow -\frac{4}{25} xy = \frac{(y^2 - x^2)^2}{x^2 y^2}$

$\Rightarrow -\frac{4}{25} xy = \left(\frac{y^2 - x^2}{xy}\right)^2$

$\Rightarrow -\frac{4xy}{25} = \left(\frac{y}{x} - \frac{x}{y}\right)^2$

$\Rightarrow -4xy = 25\left(\frac{y}{x} - \frac{x}{y}\right)^2$

$\Rightarrow 4xy + 25\left(\frac{y}{x} - \frac{x}{y}\right)^2$

// AS REQUIRED

Question 10 (****+)

The general point $P\left(cp, \frac{c}{p}\right)$, $p \neq 0$, where p is a parameter, lies on the rectangular hyperbola, with Cartesian equation

$$xy = c^2,$$

where c is a positive constant.

The normal to the hyperbola at P meets the hyperbola again at the point Q.

Show that the coordinates of Q are

$$\left(-\frac{c}{p^3}, -cp^3\right).$$

$\boxed{}$, $\boxed{\text{proof}}$

- START BY FINDING THE EQUATION OF THE NORMAL AT A GENERAL POINT ON THE HYPERBOLA

$$xy = c^2 \qquad P\left(cp, \frac{c}{p}\right)$$
$$y = \frac{c^2}{x}$$
$$\frac{dy}{dx} = -\frac{c^2}{x^2}$$
$$\left.\frac{dy}{dx}\right|_{x=cp} = -\frac{c^2}{c^2p^2} = -\frac{1}{p^2}$$

- NORMAL GRADIENT IS p^2
- EQUATION OF NORMAL

$$y - \frac{c}{p} = p^2(x - cp)$$

SOLVING SIMULTANEOUSLY WITH THE EQUATION OF THE CURVE

$$\Rightarrow \left(\frac{c^2}{x}\right) - \frac{c}{p} = p^2(x - cp)$$
$$\Rightarrow \frac{c^2}{x} - \frac{c}{p} = p^2x - cp^3$$
$$\Rightarrow c^2p - cx = p^3x^2 - cp^4x$$
$$\Rightarrow 0 = p^3x^2 + (c - cp^4)x - c^2p$$
$$\Rightarrow p^3x^2 + c(1-p^4)x - c^2p = 0$$

- AS $x = cp$ IS A SOLUTION (POINT OF NORMALITY), WE HAVE

$$\Rightarrow (x - cp)(p^3x + c) = 0$$

$$\Rightarrow x = \begin{cases} cp & \leftarrow P \\ -\frac{c}{p^3} & \leftarrow Q \end{cases}$$

$$\Rightarrow y = \frac{c^2}{-\frac{c}{p^3}} = -\frac{c^2p^3}{c} = -cp^3$$

$$\therefore Q\left(-\frac{c}{p^3}, -cp^3\right)$$

Question 11 (****+)

The general point $P\left(\frac{p}{2}, \frac{1}{2p}\right)$, $p \neq 0$, where p is a parameter, lies on the rectangular hyperbola, with Cartesian equation

$$4xy = 1.$$

The normal to the hyperbola at P meets the hyperbola again at the point Q.

Show that the Cartesian form of the locus of the midpoint of PQ, as p varies, is

$$\left(y^2 - x^2\right)^2 + 16x^3y^3 = 0.$$

FP3-V, proof

- START BY FINDING THE EQUATION OF THE NORMAL AT $P\left(\frac{p}{2}, \frac{1}{2p}\right)$

$$\Rightarrow 4xy = 1$$
$$\Rightarrow y = \frac{1}{4x} \quad \left(\text{OR } x = \frac{1}{4y}\right)$$
$$\Rightarrow \frac{dy}{dx} = -\frac{1}{4x^2}$$
$$\Rightarrow \left.\frac{dy}{dx}\right|_{x=\frac{p}{2}} = -\frac{1}{4\left(\frac{1}{4}p^2\right)} = -\frac{1}{p^2}$$

- THE EQUATION OF THE NORMAL IS GIVEN BY

$$y - \frac{1}{2p} = p^2\left(x - \frac{1}{2}p\right)$$

- SOLVING SIMULTANEOUSLY WITH $4xy=1$

$$\Rightarrow \frac{1}{4x} - \frac{1}{2p} = p^2x - \frac{1}{2}p^3 \qquad \times 4px$$
$$\Rightarrow p - 2x = 4p^3x^2 - 2p^4x$$
$$\Rightarrow 0 = 4p^3x^2 + (2 - 2p^4)x - p$$
$$\frac{\alpha+\beta}{2} = \frac{1}{2}\left(-\frac{b}{a}\right) = \frac{1}{2}\left[\frac{2p^4-2}{4p^3}\right] = \boxed{\frac{p^4-1}{4p^3}}$$

- REPEAT THE PROCESS FOR y

$$\Rightarrow y - \frac{1}{2p} = p^2\left(\frac{1}{4y} - \frac{1}{2}p\right)$$
$$\Rightarrow y - \frac{1}{2p} = \frac{p^2}{4y} - \frac{p^3}{2} \qquad \times 4py$$
$$\Rightarrow 4py^2 - 2y = p^3 - 2p^4y$$
$$\Rightarrow 4py^2 + (2p^4 - 2) - p^3 = 0$$
$$\frac{\alpha+\beta}{2} = \frac{1}{2}\left(-\frac{b}{a}\right) = \frac{1}{2}\left(\frac{2-2p^4}{4p}\right) = \boxed{\frac{1-p^4}{4p}}$$

$\therefore$ THE COORDINATES OF THE MIDPOINT OF PQ ARE

$$\left(\frac{p^4-1}{4p^3}, \frac{1-p^4}{4p}\right)$$

- ELIMINATING THE PARAMETER p

$$\bullet\ x = \frac{p^4-1}{4p^3} = \frac{1}{p^2}\left[\frac{p^4-1}{4p}\right]$$
$$\bullet\ y = -\frac{p^4-1}{4p}$$
$$\frac{1}{y} = -\frac{4p}{p^4-1}$$

$$\frac{x}{y} = \frac{1}{p^2}\left(\cancel{\frac{p^4-1}{4p}}\right)\left(\cancel{-\frac{4p}{p^4-1}}\right)$$
$$p^2 = -\frac{y}{x}$$

- FINALLY WE OBTAIN

$$y^2 = \frac{(p^4-1)^2}{16p^2} = \frac{\left[\left(-\frac{y}{x}\right)^2 - 1\right]^2}{16\left(-\frac{y}{x}\right)} = \frac{\left(\frac{y^2}{x^2} - 1\right)^2}{-\frac{16y}{x}} = \frac{\frac{(y^2-x^2)^2}{x^4}}{-\frac{16y}{x}}$$
$$\Rightarrow y^2 = \frac{x(y^2-x^2)^2}{-16yx^4}$$
$$\Rightarrow -16x^4y^3 = x(y^2-x^2)^2$$
$$\Rightarrow -16x^3y^3 = (y^2-x^2)^2$$
$$\Rightarrow (y^2-x^2)^2 + 16x^3y^3 = 0$$

Question 12 (****+)

Two distinct points $P\left(2p,\dfrac{2}{p}\right)$ and $Q\left(2q,\dfrac{2}{q}\right)$, lie on the hyperbola with Cartesian equation $xy=4$.

The tangents to the hyperbola at the points P and Q, meet at the point R.

a) Show that the coordinates of the point R are given by

$$x=\frac{4pq}{p+q}\ ,\quad y=\frac{4}{p+q}.$$

b) Given that the point R traces the rectangular hyperbola $xy=3$, find the two possible relationships between p and q, in the form $p=f(q)$

$\boxed{p=3q}$, $\boxed{p=\frac{1}{3}q}$

(a) $\left.\begin{array}{l} x=2t \\ y=\frac{2}{t}\end{array}\right\}$ $\dfrac{dy}{dx}=\dfrac{dy/dt}{dx/dt}=\dfrac{-\frac{2}{t^2}}{2}=-\dfrac{1}{t^2}$

• EQUATION OF TANGENT AT $P\left(2p,\frac{2}{p}\right)$, GRADIENT $-\frac{1}{p^2}$

$y-\frac{2}{p}=-\frac{1}{p^2}(x-2p)$

$y=\frac{2}{p}-\frac{1}{p^2}(x-2p)$

& SIMILARLY FOR TANGENT AT Q

$y=\frac{2}{q}-\frac{1}{q^2}(x-2q)$

So $\frac{2}{p}-\frac{1}{p^2}(x-2p)=\frac{2}{q}-\frac{1}{q^2}(x-2q)$

$2pq^2-q^2(x-2p)=2p^2q-p^2(x-2q)$

$2pq^2-q^2x+2pq^2=2p^2q-p^2x+2p^2q$

$p^2x-q^2x=4p^2q-4pq^2$

$(p^2-q^2)x=4pq(p-q)$

$\cancel{(p-q)}(p+q)x=4pq\cancel{(p-q)}$ BUT $p\neq q$ (DISTINCT)

$\boxed{x=\frac{4pq}{p+q}}$

& $y=\frac{2}{p}-\frac{1}{p^2}(x-2p)=\frac{2}{p}-\frac{1}{p^2}\times\frac{4pq}{p+q}+\frac{2}{p}=\frac{4}{p}-\frac{4q}{p(p+q)}$

$=\frac{4}{p}\left(1-\frac{q}{p+q}\right)=\frac{4}{p}\left(\frac{p+\cancel{q}-\cancel{q}}{p+q}\right)=\frac{4}{p+q}$

$\therefore R\left(\frac{4pq}{p+q},\frac{4}{p+q}\right)$ // AS REQUIRED

(b) Now $xy=3$

$\Rightarrow \frac{4pq}{p+q}\times\frac{4}{p+q}=3$

$\Rightarrow 16pq=3(p+q)^2$

$\Rightarrow 16pq=3p^2+6pq+3q^2$

$\Rightarrow 3p^2-10pq+3q^2$

$\Rightarrow (3p-q)(p-3q)=0$

$\Rightarrow p=\frac{1}{3}q$ or $3q$ //

Question 13 (****+)

The general point $P\left(2t, \dfrac{2}{t}\right)$, $t \neq 0$, where t is a parameter, lies on the rectangular hyperbola, with Cartesian equation

$$xy = 4.$$

a) Find an equation of the normal to the hyperbola at the point P.

The normal to the hyperbola at P meets the hyperbola again at the point Q.

The point M is the midpoint of PQ.

b) Find an equation of the locus of M, as t varies.
Give a simplified answer in the form $f(x,y) = 0$.

$\boxed{}$, $\boxed{ty - 2 = t^3x - 2t^4}$, $\boxed{\left(y^2 - x^2\right)^2 + x^3y^3 = 0}$

a) REARRANGE & DIFFERENTIATE

$\Rightarrow xy = 4$
$\Rightarrow y = \frac{4}{x}$
$\Rightarrow \frac{dy}{dx} = -\frac{4}{x^2}$
$\Rightarrow \left.\frac{dy}{dx}\right|_{x=2t} = -\frac{4}{(2t)^2} = -\frac{4}{4t^2} = -\frac{1}{t^2}$

TANGENT GRADIENT

FINALLY WE OBTAIN THE NORMAL

$y - \frac{2}{t} = t^2(x - 2t)$
$ty - 2 = t^3(x - 2t)$
$ty - 2 = t^3x - 2t^4$
$ty - t^3x = 2 - 2t^4$

b) PROCEED BY SOLVING SIMULTANEOUSLY THE CURVE & THE NORMAL — NOTE THAT THE POINT OF NORMALITY MUST BE A SOLUTION

$\Rightarrow ty - t^3x = 2 - 2t^4 \quad \times x$
$\Rightarrow txy - t^3x^2 = (2 - 2t^4)x$
$\Rightarrow 4t - t^3x^2 = (2 - 2t^4)x$
$\Rightarrow 0 = t^3x^2 + (2 - 2t^4)x - 4t$
$\Rightarrow (t^3x + 2)(x - 2t) = 0$

POINT Q (REINTERSECTION)

POINT OF NORMALITY $P\left(2t, \frac{2}{t}\right)$

FINDING THE CO-ORDINATES OF Q & M

WHEN $x = -\frac{2}{t^3}$ $\quad y = \frac{4}{-\frac{2}{t^3}} = -2t^3$ $\quad Q\left(-\frac{2}{t^3}, -2t^3\right)$

$M\left(\frac{2t - \frac{2}{t^3}}{2}, \frac{\frac{2}{t} - 2t^3}{2}\right) = M\left(t - \frac{1}{t^3}, \frac{1}{t} - t^3\right)$

FINALLY ELIMINATE t, TO OBTAIN A CARTESIAN EXPRESSION

$\left.\begin{array}{l} X = t - \frac{1}{t^3} \\ Y = \frac{1}{t} - t^3 \end{array}\right\} \Rightarrow \begin{array}{l} X = \frac{t^4 - 1}{t^3} \\ Y = \frac{1 - t^4}{t} = \frac{t^4 - 1}{-t} \end{array}$

DIVIDING THE EQUATIONS ABOVE

$\frac{Y}{X} = Y\left(\frac{1}{X}\right) = \frac{\cancel{t^4 - 1}}{-t}\left(\frac{t^3}{\cancel{t^4 - 1}}\right)$
$\frac{Y}{X} = -t^2$

SUB INTO EITHER PARAMETRIC

$\Rightarrow Y = \frac{t^4 - 1}{-t}$
$\Rightarrow Y^2 = \frac{(t^4 - 1)^2}{t^2}$
$\Rightarrow Y^2t^2 = (t^4 - 1)^2$

$\Rightarrow Y^2\left(-\frac{Y}{X}\right) = \left[\left(-\frac{Y}{X}\right)^2 - 1\right]^2$
$\Rightarrow -\frac{Y^3}{X} = \left[\frac{Y^2}{X^2} - 1\right]^2$
$\Rightarrow -\frac{Y^3}{X} = \left(\frac{Y^2 - X^2}{X^2}\right)^2$
$\Rightarrow -\frac{Y^3}{X} = \frac{(Y^2 - X^2)^2}{X^4}$
$\Rightarrow -Y^3X^3 = (Y^2 - X^2)^2$
$\Rightarrow (Y^2 - X^2)^2 + X^3Y^3 = 0$

Question 14 (*****)

The point $P\left(p+\frac{1}{p}, p-\frac{1}{p}\right)$, $p \neq 0$, lies on the rectangular hyperbola, with Cartesian equation

$$x^2 - y^2 = 4.$$

The normal to the hyperbola at P meets the y axis at the point $Q(0,-k)$, $k>0$.

The area of the triangle OPQ, where O is the origin, is $\frac{15}{4}$.

Determine the two possible sets of coordinates for P.

$\boxed{}$, $\boxed{\left(\frac{5}{2},-\frac{3}{2}\right), \left(\frac{5}{2},-\frac{3}{2}\right)}$

START BY OBTAINING THE EQUATION OF THE NORMAL AT $P\left(p+\frac{1}{p}, p-\frac{1}{p}\right)$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{1+\frac{1}{t^2}}{1-\frac{1}{t^2}} = \frac{t^2+1}{t^2-1}$$

$$\left.\frac{dy}{dx}\right|_{t=p} = \frac{p^2+1}{p^2-1}$$

HENCE THE EQUATION OF THE NORMAL AT P, WILL BE

$$y-\left(p-\frac{1}{p}\right) = -\frac{p^2-1}{p^2+1}\left[x-\left(p+\frac{1}{p}\right)\right]$$

NORMAL PASSES THROUGH $Q(0,-k)$, $k>0$

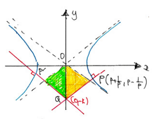
$$\Rightarrow -k-\left(p-\frac{1}{p}\right) = -\frac{p^2-1}{p^2+1}\left[0-\left(p+\frac{1}{p}\right)\right]$$

$$\Rightarrow -k-\frac{p^2-1}{p} = \frac{p^2-1}{p^2+1} \times \frac{p^2+1}{p}$$

$$\Rightarrow -k-\frac{p^2-1}{p} = \frac{p^2-1}{p}$$

$$\Rightarrow \boxed{-\frac{2(p^2-1)}{p} = k}$$

NEXT CONTINUE WITH A DRAWING



THE POSITION IS SYMMETRICAL SO THE AREA OF THE YELLOW (OR GREEN) TRIANGLE IS $\frac{15}{4}$

$$\Rightarrow \frac{1}{2}\times k \times\left(p+\frac{1}{p}\right) = \frac{15}{4}$$

$$\Rightarrow k\left(\frac{p^2+1}{p}\right) = \frac{15}{2}$$

$$\Rightarrow \boxed{k = \frac{15p}{2(p^2+1)}}$$

ELIMINATING k, BETWEEN THE LAST TWO EXPRESSIONS

$$\Rightarrow \frac{15p}{2(p^2+1)} = -\frac{2(p^2-1)}{p}$$

$$\Rightarrow 15p^2 = -4(p^2-1)(p^2+1)$$

$$\Rightarrow 15p^2 = -4p^4+4$$

$$\Rightarrow 4p^4+15p^2-4=0$$

$$\Rightarrow (4p^2-1)(p^2+4)=0$$

$$\Rightarrow p^2 = \begin{cases} \cancel{-4} \\ \frac{1}{4} \end{cases}$$

$$\Rightarrow p = +\frac{1}{2} \quad (\text{OTHERWISE } k<0)$$

FINALLY WE HAVE

$p=\frac{1}{2}$ $\quad x = \frac{1}{2}+\frac{1}{\frac{1}{2}} = \frac{5}{2}$

$\quad y = \frac{1}{2}-\frac{1}{\frac{1}{2}} = -\frac{3}{2}$

$\therefore \left(\frac{5}{2},-\frac{3}{2}\right), \left(-\frac{5}{2},-\frac{3}{2}\right)$ //

Question 15 (*****)

The points P and Q are two distinct points which lie on the curve with equation

$$y = \frac{1}{x}, \ x \in \mathbb{R}, \ x \neq 0.$$

P and Q are free to move on the curve so that the straight line segment PQ is a normal to the curve at P.

The tangents to the curve at P and Q meet at the point R.

Show that R is moving on the curve with Cartesian equation

$$\left(y^2 - x^2\right)^2 + 4xy = 0.$$

proof

- START BY FINDING THE GRADIENT FUNCTION ON THE CURVE

$$y = \frac{1}{x}$$
$$\frac{dy}{dx} = -\frac{1}{x^2}$$

- LET $P\left(p, \frac{1}{p}\right)$ $Q\left(q, \frac{1}{q}\right)$ $p \neq q$

  - GRADIENT OF CHORD $PQ = \dfrac{\frac{1}{q} - \frac{1}{p}}{q - p} = \dfrac{\frac{p-q}{pq}}{q-p}$

$$= \frac{\cancel{p-q}}{-pq(\cancel{p-q})} = -\frac{1}{pq}$$

  - CHORD $PQ \perp$ GRAD AT P (NORMAL)
  - GRAD AT P IS $-\frac{1}{p^2}$

(NORMAL GRAD AT P IS p^2)

$$\therefore \ -\frac{1}{p^2} \times \left(-\frac{1}{pq}\right) = -1$$
$$\frac{1}{p^3 q} = -1$$
$$p^3 q = -1$$

- NEXT WE FIND THE EQUATION OF THE TANGENT AT $P\left(p, \frac{1}{p}\right)$

$$y - \frac{1}{p} = -\frac{1}{p^2}(x - p)$$
$$y - \frac{1}{p} = -\frac{1}{p^2}x + \frac{1}{p}$$
$$y = \frac{2}{p} - \frac{1}{p^2}x$$

- SIMILARLY THE TANGENT AT $Q\left(q, \frac{1}{q}\right)$ WILL BE

$$y = \frac{2}{q} - \frac{1}{q^2}x$$

- SOLVING SIMULTANEOUSLY TO FIND THE POINT R

$$\frac{2}{p} - \frac{1}{p^2}x = \frac{2}{q} - \frac{1}{q^2}x$$
$$x\left(\frac{1}{q^2} - \frac{1}{p^2}\right) = \frac{2}{q} - \frac{2}{p}$$
$$\frac{p^2 - q^2}{p^2 q^2}x = 2\left(\frac{p-q}{pq}\right)$$
$$\frac{(p-q)(p+q)}{p^2 q^2}x = \frac{2(p-q)}{pq}$$

AS $p \neq q$ $\quad p \neq 0 \quad q \neq 0$

$$\frac{p+q}{pq}x = 2$$
$$x = \frac{2pq}{p+q}$$

AND $y = \dfrac{2}{q} - \dfrac{1}{q^2}\left(\dfrac{2pq}{p+q}\right) = \dfrac{2}{q} - \dfrac{2p}{q(p+q)}$

$$= \frac{2(p+q) - 2p}{q(p+q)} = \frac{2p + 2q - 2p}{q(p+q)}$$
$$= \frac{2q}{q(p+q)} = \frac{2}{p+q}$$

$$\therefore R\left(\frac{2pq}{p+q}, \frac{2}{p+q}\right)$$

- NOW WE CAN ELIMINATE THE "PARAMETERS" p & q FROM THE EQUATIONS

$$x = \frac{2pq}{p+q}$$
$$y = \frac{2}{p+q}$$

& THE CONSTRAINT

$$p^3 q = -1$$
$$q = -\frac{1}{p^3}$$

$$x = \frac{2p\left(-\frac{1}{p^3}\right)}{p - \frac{1}{p^3}} = \frac{-\frac{2}{p^2}}{\frac{p^4 - 1}{p^3}} = -\frac{2p}{p^4 - 1}$$

$$y = \frac{2}{p+q} = \frac{2}{p - \frac{1}{p^3}} = \frac{2}{\frac{p^4 - 1}{p^3}} = \frac{2p^3}{p^4 - 1}$$

- DIVIDE THE EQUATIONS

$$\frac{x}{y} = -\frac{2p}{2p^3} = -\frac{1}{p^2} \quad \text{IE} \quad p^2 = -\frac{y}{x}$$

- SUB INTO THE y EQUATION & TIDY UP

$$y = \frac{2p^3}{p^4 - 1} \implies y(p^4 - 1) = 2p^3$$
$$\implies y^2(p^4 - 1)^2 = 4p^6$$
$$\implies y^2\left[\frac{y^2}{x^2} - 1\right]^2 = 4\left(-\frac{y}{x}\right)^3$$
$$\implies y^2\left(\frac{y^2 - x^2}{x^2}\right)^2 = -\frac{4y^3}{x^3}$$
$$\implies y^2\frac{(y^2 - x^2)^2}{x^4} = -\frac{4y^3}{x^3}$$

$y \neq 0$

$$\implies \frac{(y^2 - x^2)^2}{x^4} = -\frac{4y}{x^3}$$

$x \neq 0$

$$\implies (y^2 - x^2)^2 = -4xy$$
$$\implies (y^2 - x^2)^2 + 4xy = 0$$

Question 16 (*****)

The variable point P lies on the rectangular hyperbola, with Cartesian equation

$$xy = a^2,$$

where a is a positive constant.

The normal to the hyperbola at P meets the hyperbola again at the point Q.

The point M is the midpoint of PQ.

Determine, in the form $f(x, y) = 0$, an equation of the locus of M, for all the possible positions of P.

$$\boxed{a^2\left(y^2 - x^2\right)^2 + 4x^3y^3 = 0}$$

LET THE POINT P HAVE CO-ORDINATES $\left(p, \frac{a^2}{p}\right)$, AS IT LIES ON THE CURVE $y = \frac{a^2}{x}$

- $\frac{dy}{dx} = -\frac{a^2}{x^2}$
- $\left.\frac{dy}{dx}\right|_p = -\frac{a^2}{p^2}$
- NORMAL GRADIENT $= +\frac{p^2}{a^2}$
- NORMAL EQUATION: $y - \frac{a^2}{p} = \frac{p^2}{a^2}(x - p)$

SOLVING SIMULTANEOUSLY WITH THE EQUATION OF THE CURVE

$$\Rightarrow \left(\frac{a^2}{x}\right) - \frac{a^2}{p} = \frac{p^2}{a^2}(x - p) \quad \times a^2px$$

$$\Rightarrow a^4p - a^4x = xp^3(x - p)$$

$$\Rightarrow a^4p - a^4x = x^2p^3 - xp^4$$

$$\Rightarrow 0 = p^3x^2 + (a^4 - p^4)x - a^4p$$

AS $x = p$ MUST ALSO BE A SOLUTION, FACTORIZE BY INSPECTION

$$\Rightarrow 0 = (x - p)(p^3x + a^4)$$

$$\Rightarrow x = \begin{cases} p & \leftarrow \text{POINT } P \\ -\frac{a^4}{p^3} & \leftarrow \text{POINT } Q \end{cases}$$

$$\Rightarrow y = \frac{a^2}{x} = a^2 \times \left(-\frac{p^3}{a^4}\right) = -\frac{p^3}{a^2} \qquad Q\left(-\frac{a^4}{p^3}, -\frac{p^3}{a^2}\right)$$

NEXT THE MIDPOINT M

$$M\left(\frac{p - \frac{a^4}{p^3}}{2}, \frac{\frac{a^2}{p} - \frac{p^3}{a^2}}{2}\right) = M\left(\frac{p^4 - a^4}{2p^3}, \frac{a^4 - p^4}{2pa^2}\right)$$

FINALLY ELIMINATE THE "PARAMETER" OUT OF THESE EQUATIONS

$$\left.\begin{aligned} X &= \frac{p^4 - a^4}{2p^3} \\ Y &= \frac{a^4 - p^4}{2a^2p} \end{aligned}\right\} \Rightarrow \frac{Y}{X} = \frac{a^4 - p^4}{2a^2p} \times \frac{2p^3}{p^4 - a^4}$$

$$\Rightarrow \frac{Y}{X} = \frac{-p^2}{a^2}$$

$$\Rightarrow p^2 = -\frac{a^2Y}{X}$$

$$Y^2 = \frac{(a^4 - p^4)^2}{4a^4p^2} \Rightarrow 4a^4p^2Y^2 = (a^4 - p^4)^2$$

$$\Rightarrow 4a^4\left(-\frac{a^2Y}{X}\right)Y^2 = \left(a^4 - \frac{a^4Y^2}{X^2}\right)^2$$

$$\Rightarrow -\frac{4a^6Y^3}{X} = \left(\frac{a^4X^2 - a^4Y^2}{X^2}\right)^2$$

$$\Rightarrow -\frac{4a^6Y^3}{X} = \frac{a^8}{X^4}(X^2 - Y^2)^2$$

$$\Rightarrow -4X^3Y^3 = a^2(X^2 - Y^2)^2$$

$$\Rightarrow a^2(X^2 - Y^2)^2 + 4X^3Y^3 = 0$$

OR

$$a^2(x^2 - y^2)^2 + 4x^3y^3 = 0$$

HYPERBOLA

Question 1 (**)

A hyperbola H has foci at the points with coordinates $(-10,0)$ and $(10,0)$, and its Cartesian equation is given by

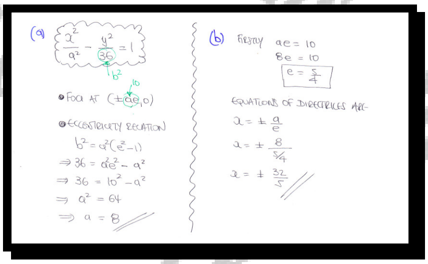
$$\frac{x^2}{a^2} - \frac{y^2}{36} = 1,$$

where a is a positive constant.

a) Find the value of a.

b) Deduce the equations of the directrices of H.

$\boxed{a = 8}$, $\boxed{x = \pm\frac{32}{5}}$



Question 2 **(**)**

A hyperbola H has foci at the points with coordinates $(\pm 13, 0)$ and the equations of its directrices are $x = \pm \dfrac{144}{13}$.

Determine a Cartesian equation for H.

$$\boxed{\frac{x^2}{144} - \frac{y^2}{25} = 1}$$

FOCUS

- $ae = 13$

  $\dfrac{a}{e} = \dfrac{144}{13}$

DIRECTRIX

$\Rightarrow$ MULTIPLY EQUATIONS

$a\cancel{e} \times \dfrac{a}{\cancel{e}} = \cancel{13} \times \dfrac{144}{\cancel{13}}$

$a^2 = 144$

$\boxed{a = 12}$

$\Rightarrow$ SO $ae = 13$

$12e = 13$

$\boxed{e = \dfrac{13}{12}}$

- ECCENTRICITY RELATION FOR HYPERBOLA IS

  $b^2 = a^2(e^2 - 1)$

  $b^2 = 144\left(\dfrac{169}{144} - 1\right)$

  $b^2 = 169 - 144$

  $b^2 = 25$

  $\boxed{b = 5}$

$\therefore \dfrac{x^2}{144} - \dfrac{y^2}{25} = 1$ //

Question 3 (***)

A hyperbola is given parametrically by

$$x=\frac{3}{2}\left(t+\frac{1}{t}\right), \quad y=\frac{5}{2}\left(t-\frac{1}{t}\right), \quad t\neq 0.$$

a) Show that the Cartesian equation of the hyperbola can be written as

$$\frac{x^2}{9}-\frac{y^2}{25}=1.$$

b) Find …

i. … the equations of its asymptotes.

ii. … the coordinates of its foci.

iii. … the equations of its directrices.

c) Sketch the hyperbola indicating any intersections with the coordinate axes, as well as the information stated in part **(b)**.

$\boxed{y=\pm\frac{5}{3}x}$, $\boxed{\left(\pm\sqrt{34},0\right)}$, $\boxed{x=\pm\frac{9}{\sqrt{34}}}$

a) $x=\frac{3}{2}\left(t+\frac{1}{t}\right)$ $\quad$ $y=\frac{5}{2}\left(t-\frac{1}{t}\right)$

$t+\frac{1}{t}=\frac{2x}{3}$ $\quad$ $t-\frac{1}{t}=\frac{2y}{5}$

ADDING THE EQUATIONS

$2t=\frac{2x}{3}+\frac{2y}{5}$

SUBTRACT THE EQUATIONS

$\frac{2}{t}=\frac{2x}{3}-\frac{2y}{5}$

MULTIPLY THE EXPRESSIONS

$$(2t)\left(\frac{2}{t}\right)=\left(\frac{2x}{3}+\frac{2y}{5}\right)\left(\frac{2x}{3}-\frac{2y}{5}\right)$$
$$4=\left(\frac{2x}{3}\right)^2-\left(\frac{2y}{5}\right)^2$$
$$4=\frac{4x^2}{9}-\frac{4y^2}{25}$$
$$\frac{x^2}{9}-\frac{y^2}{25}=1$$

AS REQUIRED

b) USING STANDARD RESULTS FOR $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$

$a=3$ $\quad$ $b=5$

I) ASYMPTOTES $y=\pm\frac{b}{a}x$

$y=\pm\frac{5}{3}x$

II) FIND THE ECCENTRICITY

$b^2=a^2(e^2-1)$

$25=9(e^2-1)$

$\frac{25}{9}=e^2-1$

$e^2=\frac{34}{9}$

$e=+\frac{\sqrt{34}}{3}$

FOCI AT $(\pm ae)$

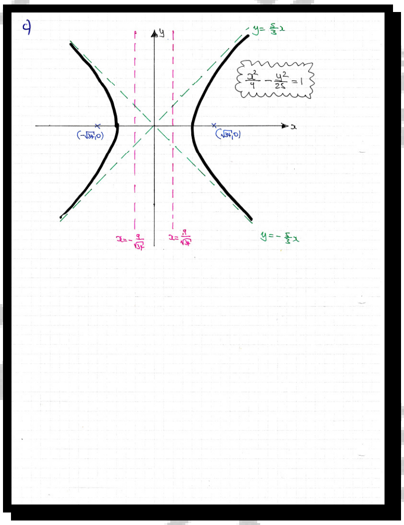
$\pm\left(3\times\frac{\sqrt{34}}{3},0\right)$ I.E. $\left(\pm\sqrt{34},0\right)$

DIRECTRICES AT $x=\pm\frac{a}{e}$

$x=\pm\frac{3}{\sqrt{34}/3}$

$x=\pm\frac{9}{\sqrt{34}}$

c)



Question 4 (***)

The point $P(a\sec\theta, b\tan\theta)$ lies on the hyperbola with equation

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1,$$

where a and b are positive constants.

a) Show that an equation of the normal at P is given by

$$by+ax\sin\theta=\left(a^2+b^2\right)\tan\theta .$$

The normal to the hyperbola meets the coordinate axes at the points A and B.

b) Show that, as θ varies, the Cartesian locus of the midpoint of AB is given by

$$4\left(a^2x^2-b^2y^2\right)=\left(a^2+b^2\right)^2 .$$

$\boxed{}$, $\boxed{\text{proof}}$

a) DETERMINE THE GRADIENT FUNCTION PARAMETRICALLY

$$\left.\begin{array}{l} x=a\sec\theta \\ y=b\tan\theta \end{array}\right\} \Rightarrow \frac{dy}{dx}=\frac{dy/d\theta}{dx/d\theta}=\frac{b\sec^2\theta}{a\sec\theta\tan\theta}=\frac{b\sec\theta}{a\tan\theta}$$

$$=\left(b\sec\theta\right)\left(\frac{1}{a}\cot\theta\right)=\frac{b}{\cos\theta}\times\frac{1}{a}\frac{\cos\theta}{\sin\theta}$$

$$=\frac{b}{a\sin\theta}$$

HENCE THE GRADIENT AT THE GENERAL POINT $(a\sec\theta, b\tan\theta)$ IS $\frac{b}{a\sin\theta}$

NORMAL $\Rightarrow$ $y-b\tan\theta=-\frac{a\sin\theta}{b}(x-a\sec\theta)$

$\Rightarrow by-b^2\tan\theta=-ax\sin\theta+a^2\sin\theta\sec\theta$

$\Rightarrow by+ax\sin\theta=b^2\tan\theta+a^2\sin\theta\times\frac{1}{\cos\theta}$

$\Rightarrow by+ax\sin\theta=b^2\tan\theta+a^2\tan\theta$

$\Rightarrow by+ax\sin\theta=(a^2+b^2)\tan\theta$ // AS REQUIRED

b) FIND THE CO-ORDS OF A & B

$x=0 \Rightarrow by=(a^2+b^2)\tan\theta$

$\Rightarrow y=\frac{a^2+b^2}{b}\tan\theta$ $\qquad A\left(0, \frac{a^2+b^2}{b}\tan\theta\right)$

$y=0 \Rightarrow ax\sin\theta=(a^2+b^2)\tan\theta$

$ax\sin\theta=(a^2+b^2)\frac{\sin\theta}{\cos\theta}$

$ax=\frac{a^2+b^2}{\cos\theta}$

$x=\frac{a^2+b^2}{a}\sec\theta$ $\qquad B\left(\frac{a^2+b^2}{a}\sec\theta, 0\right)$

THE MIDPOINT OF AB IS $M\left(\frac{a^2+b^2}{2a}\sec\theta, \frac{a^2+b^2}{2b}\tan\theta\right)$

$\uparrow x \qquad \uparrow y$

$$x=\frac{a^2+b^2}{2a}\sec\theta \qquad \frac{2ax}{a^2+b^2}=\sec\theta$$

$$y=\frac{a^2+b^2}{2b}\tan\theta \qquad \frac{2by}{a^2+b^2}=\tan\theta$$

$\Rightarrow 1+\tan^2\theta=\sec^2\theta$

$\Rightarrow 1+\left(\frac{2by}{a^2+b^2}\right)^2=\left(\frac{2ax}{a^2+b^2}\right)^2$

$\Rightarrow 1+\frac{4b^2y^2}{(a^2+b^2)^2}=\frac{4a^2x^2}{(a^2+b^2)^2}$

$\Rightarrow (a^2+b^2)^2+4b^2y^2=4a^2x^2$

$\Rightarrow (a^2+b^2)^2=4a^2x^2-4b^2y^2$

$\Rightarrow 4(a^2x^2-b^2y^2)=(a^2+b^2)^2$ // AS REQUIRED

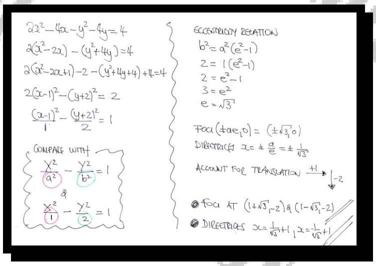
Question 5 **(***+)**

A hyperbola has Cartesian equation

$$2x^2 - 4x - y^2 - 4y = 4 .$$

Find the coordinates of its foci and the equations of its directrices.

$$\boxed{\left(1+\sqrt{3},-2\right),\ \left(1-\sqrt{3},-2\right)},\ \boxed{x = -\frac{1}{\sqrt{3}}+1,\ \ x = \frac{1}{\sqrt{3}}+1}$$



Question 6 (****)

The general point $P(a\sec\theta, b\tan\theta)$ lies on the hyperbola with Cartesian equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,$$

where a and b are positive constants.

a) Show that an equation of the normal at P is given by

$$by + ax\sin\theta = \left(a^2 + b^2\right)\tan\theta.$$

The normal to the hyperbola meets the x axis at the point X.

The eccentricity of the hyperbola is $\frac{3}{2}$ and its foci are denoted by S and S', where S has a positive x coordinate.

b) Given that $|OX| = 3|OS|$, find the possible values of θ for $0 \le \theta \le 2\pi$.

$$\boxed{\theta = \frac{\pi}{3}, \frac{5\pi}{3}}$$

a) DIFFERENTIATE W.R.T x

$$\frac{d}{dx}\left(\frac{x^2}{a^2} - \frac{y^2}{b^2}\right) = \frac{d}{dx}(1)$$

$$\frac{2x}{a^2} - \frac{2y}{b^2}\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{b^2 x}{a^2 y}$$

EVALUATE AT P

$$\left.\frac{dy}{dx}\right|_P = \frac{b^2(a\sec\theta)}{a^2(b\tan\theta)} = \frac{b}{a}\sec\theta\cot\theta = \frac{b}{a}\frac{1}{\cancel{\cos\theta}}\frac{\cancel{\cos\theta}}{\sin\theta}$$

$$= \frac{b}{a\sin\theta}$$

NORMAL EQUATION IS GIVEN BY

$$y - b\tan\theta = -\frac{a\sin\theta}{b}(x - a\sec\theta)$$

$$by - b^2\tan\theta = -ax\sin\theta + a^2\sin\theta\sec\theta$$

$$by + ax\sin\theta = b^2\tan\theta + a^2\sin\theta \times \frac{1}{\cos\theta}$$

$$by + ax\sin\theta = b^2\tan\theta + a^2\tan\theta$$

$$by + ax\sin\theta = (b^2 + a^2)\tan\theta$$

AS REQUIRED

b) FIRSTLY FIND THE CO-ORDINATES OF $X \Rightarrow y = 0$

$$ax\sin\theta = (a^2 + b^2)\tan\theta$$

$$x = \frac{a^2+b^2}{a}\frac{\tan\theta}{\sin\theta}$$

$$x = \frac{a^2+b^2}{a}\sec\theta$$

$$\therefore X\left(\frac{a^2+b^2}{a}\sec\theta, 0\right)$$

USING THE ECCENTRICITY RELATION

$$b^2 = a^2(e^2 - 1)$$

$$b^2 = a^2\left(\frac{9}{4} - 1\right)$$

$$b^2 = \frac{5}{4}a^2 \quad \Longrightarrow \quad X\left(\frac{a^2 + \frac{5}{4}a^2}{a}\sec\theta, 0\right)$$

$$X\left(\frac{9}{4}a\sec\theta, 0\right)$$

NEXT THE FOCI S WITH POSITIVE x CO-ORDINATE

$$S(ae, 0) \Rightarrow S\left(\frac{3}{2}a, 0\right)$$

FINALLY WE HAVE

$$\Rightarrow |OX| = 3|OS|$$

$$\Rightarrow \frac{9}{4}\cancel{a}\sec\theta = 3 \times \frac{3}{2}\cancel{a}$$

$$\Rightarrow \frac{9}{4}\sec\theta = \frac{9}{2}$$

$$\Rightarrow \sec\theta = 2$$

$$\Rightarrow \cos\theta = \frac{1}{2}$$

$$\therefore \theta = \frac{\pi}{3} \;\&\; \frac{5\pi}{3}$$

Question 7 (****)

The equation of a hyperbola H is given in terms of a parameter t by

$$x = \sinh t, \quad y = \cosh t, \quad t \in \mathbb{R}.$$

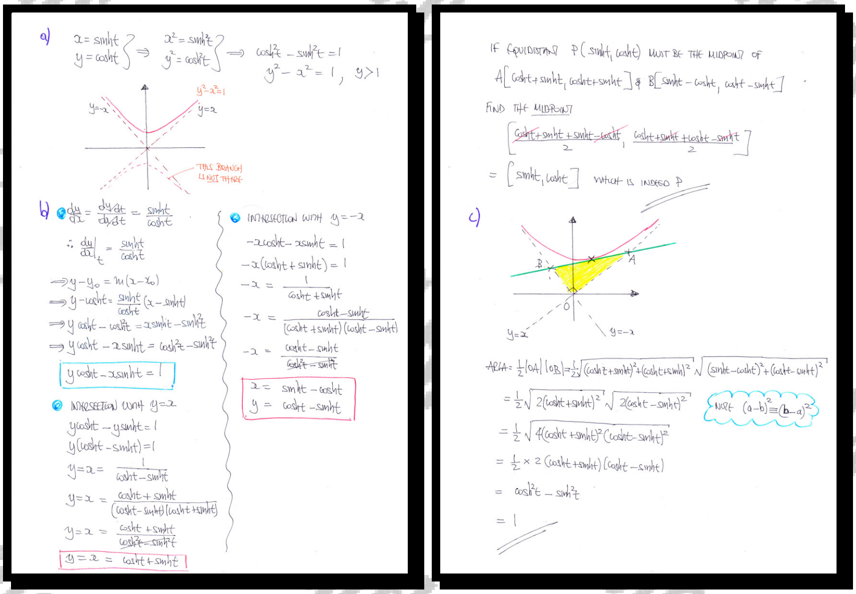
a) Sketch the graph of H, clearly marking the equation of each of its asymptotes.

The equation of the tangent to H at the point $P(\sinh t, \cosh t)$, meets each of the asymptotes at the points A and B.

b) Show that P is equidistant from A and B.

c) Show further that the area of the triangle OAB, where O is the origin, is exactly 1 square unit.

graph/proof



Question 8 (****+)

A hyperbola H and a line L have the following Cartesian equations

$$H: \ \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

$$L: \ y = mx + c,$$

where a, b, m and c are non zero constants.

a) Show that the x coordinates of the points of intersection between L and H satisfy the equation

$$\left(a^2m^2 - b^2\right)x^2 + \left(2a^2mc\right)x + a^2\left(b^2 + c^2\right) = 0.$$

b) Given the line is a tangent to the hyperbola show that

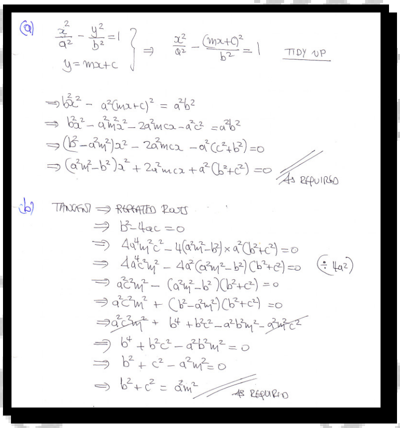
$$a^2m^2 = b^2 + c^2.$$

c) Find the equations of the two tangents to the hyperbola with Cartesian equation

$$\frac{x^2}{25} - \frac{y^2}{16} = 1$$

that pass through the point $(1,4)$, and for each tangent the coordinates of their point of tangency.

$\boxed{y = x + 3, \ \left(-\frac{25}{3}, -\frac{16}{3}\right)}$, $\boxed{y = -\frac{4}{3}x + \frac{16}{3}, \ \left(\frac{25}{4}, -3\right)}$



(a) $\left.\begin{array}{l}\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \\ y = mx + c\end{array}\right\} \Rightarrow \frac{x^2}{a^2} - \frac{(mx+c)^2}{b^2} = 1$ TIDY UP

$\Rightarrow b^2x^2 - a^2(mx+c)^2 = a^2b^2$

$\Rightarrow b^2x^2 - a^2m^2x^2 - 2a^2mcx - a^2c^2 = a^2b^2$

$\Rightarrow (b^2 - a^2m^2)x^2 - 2a^2mcx - a^2(c^2+b^2) = 0$

$\Rightarrow (a^2m^2 - b^2)x^2 + 2a^2mcx + a^2(b^2+c^2) = 0$ AS REQUIRED

(b) TANGENT $\Rightarrow$ REPEATED ROOTS

$\Rightarrow b^2 - 4ac = 0$

$\Rightarrow 4a^4m^2c^2 - 4(a^2m^2 - b^2) \times a^2(b^2+c^2) = 0$

$\Rightarrow 4a^4c^2m^2 - 4a^2(a^2m^2 - b^2)(b^2+c^2) = 0$ $(\div 4a^2)$

$\Rightarrow a^2c^2m^2 - (a^2m^2 - b^2)(b^2+c^2) = 0$

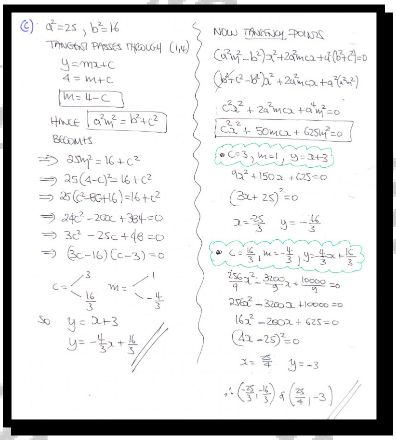
$\Rightarrow a^2c^2m^2 + (b^2 - a^2m^2)(b^2+c^2) = 0$

$\Rightarrow \cancel{a^2c^2m^2} + b^4 + b^2c^2 - a^2b^2m^2 - \cancel{a^2m^2c^2}$

$\Rightarrow b^4 + b^2c^2 - a^2b^2m^2 = 0$

$\Rightarrow b^2 + c^2 - a^2m^2 = 0$

$\Rightarrow b^2 + c^2 = a^2m^2$ AS REQUIRED



(c) $a^2 = 25$, $b^2 = 16$

TANGENT PASSES THROUGH $(1,4)$

$y = mx + c$

$4 = m + c$

$\boxed{m = 4 - c}$

HENCE $\boxed{a^2m^2 = b^2 + c^2}$ BECOMES

$\Rightarrow 25m^2 = 16 + c^2$

$\Rightarrow 25(4-c)^2 = 16 + c^2$

$\Rightarrow 25(c^2 - 8c + 16) = 16 + c^2$

$\Rightarrow 24c^2 - 200c + 384 = 0$

$\Rightarrow 3c^2 - 25c + 48 = 0$

$\Rightarrow (3c - 16)(c - 3) = 0$

$c = 3$ or $\frac{16}{3}$, $m = 1$ or $-\frac{4}{3}$

SO $y = x + 3$

$y = -\frac{4}{3}x + \frac{16}{3}$

NOW TANGENCY POINTS

$(a^2m^2 - b^2)x^2 + 2a^2mcx + a^2(b^2+c^2) = 0$

$(\cancel{b^2} + c^2 - \cancel{b^2})x^2 + 2a^2mcx + a^2(a^2m^2)$

$c^2x^2 + 2a^2mcx + a^4m^2 = 0$

$\boxed{c^2x^2 + 50mcx + 625m^2 = 0}$

- $c = 3$, $m = 1$, $y = x + 3$

$9x^2 + 150x + 625 = 0$

$(3x + 25)^2 = 0$

$x = -\frac{25}{3}$ $y = -\frac{16}{3}$

- $c = \frac{16}{3}$, $m = -\frac{4}{3}$, $y = -\frac{4}{3}x + \frac{16}{3}$

$\frac{256}{9}x^2 - \frac{3200}{9}x + \frac{10000}{9} = 0$

$256x^2 - 3200x + 10000 = 0$

$16x^2 - 200x + 625 = 0$

$(4x - 25)^2 = 0$

$x = \frac{25}{4}$ $y = -3$

$\therefore \left(-\frac{25}{3}, -\frac{16}{3}\right)$ & $\left(\frac{25}{4}, -3\right)$

Question 9 (****+)

A hyperbola H has Cartesian equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,$$

where a and b are positive constants.

The straight line T_1 is the tangent to H at the point $(a\cosh\theta, b\sinh\theta)$.

T_1 meets the x axis at the point P.

The straight line T_2 is a tangent to the hyperbola at the point $(a,0)$.

T_1 and T_2 meet each other at the point Q.

Given further that M is the midpoint of PQ, show that as θ varies, the locus of M traces the curve with equation

$$x\left(4y^2 + b^2\right) = ab^2.$$

proof

OBTAIN THE GRADIENT FUNCTION IN PARAMETRIC (OR CARTESIAN)

$$x = a\cosh t \qquad y = b\sinh t$$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{b\cosh t}{a\sinh t}$$

AT THE POINT GIVEN $t=\theta$ $\Rightarrow \left.\frac{dy}{dx}\right|_{t=\theta} = \frac{b\cosh\theta}{a\sinh\theta}$

EQUATION OF TANGENT T_1, AT $(a\cosh\theta, b\sinh\theta)$

$$\Rightarrow y - b\sinh\theta = \frac{b}{a}\frac{\cosh\theta}{\sinh\theta}(x - a\cosh\theta)$$

$$\Rightarrow ay\sinh\theta - ab\sinh^2\theta = bx\cosh\theta - ab\cosh^2\theta$$

$$\Rightarrow ay\sinh\theta - bx\cosh\theta = ab\sinh^2\theta - ab\cosh^2\theta$$

$$\Rightarrow bx\cosh\theta - ay\sinh\theta = ab\cosh^2\theta - ab\sinh^2\theta$$

$$\Rightarrow bx\cosh\theta - ay\sinh\theta = ab(\cosh^2\theta - \sinh^2\theta)$$

$$\Rightarrow bx\cosh\theta - ay\sinh\theta = ab$$

AT THE POINT $(a,0)$ $t=0$ (BY INSPECTION)

$$\left.\frac{dy}{dx}\right|_{t=0} = \infty \quad \text{, i.e. VERTICAL}$$

$$\therefore T_2 : \ x = a$$

ALTERNATIVE USE T_1 WITH $\theta = 0$

$$bx\cosh 0 - \cancel{ay\sinh 0} = ab$$

$$bx = ab$$

$$x = a$$

TO FIND P, SET $y=0$ IN T_1

$$bx\cosh\theta - \cancel{ay\sinh\theta} = ab$$

$$x\cosh\theta = a$$

$$x = \frac{a}{\cosh\theta} \qquad \therefore P\left(\frac{a}{\cosh\theta}, 0\right)$$

TO FIND Q, SOLVE SIMULTANEOUSLY T_1 & T_2

$$bx\cosh\theta - ay\sinh\theta = ab \quad \& \quad x = a$$

$$ab\cosh\theta - ay\sinh\theta = ab$$

$$b\cosh\theta - y\sinh\theta = b$$

$$b\cosh\theta - b = y\sinh\theta$$

$$y = \frac{b(\cosh\theta - 1)}{\sinh\theta} \qquad \therefore Q\left[a, \frac{b(\cosh\theta-1)}{\sinh\theta}\right]$$

MIDPOINT OF PQ IS $\left[\frac{1}{2}\left(\frac{a}{\cosh\theta} + a\right), \frac{1}{2}\frac{b(\cosh\theta-1)}{\sinh\theta}\right]$

- $x = \frac{1}{2}a\left(\frac{1}{\cosh\theta} + 1\right)$

$$\frac{2x}{a} = \frac{1}{\cosh\theta} + 1$$

$$\frac{2x}{a} - 1 = \frac{1}{\cosh\theta}$$

$$\frac{2x-a}{a} = \frac{1}{\cosh\theta}$$

$$\cosh\theta = \frac{a}{2x-a}$$

- $y = \frac{1}{2}b\,\frac{\cosh\theta - 1}{\sinh\theta}$

$$\frac{2y}{b} = \frac{\cosh\theta - 1}{\sinh\theta}$$

$$\frac{4y^2}{b^2} = \frac{(\cosh\theta-1)^2}{\sinh^2\theta}$$

$$\frac{4y^2}{b^2} = \frac{(\cosh\theta-1)^2}{\cosh^2\theta - 1}$$

$$\frac{4y^2}{b^2} = \frac{(\cosh\theta-1)^2}{(\cosh\theta-1)(\cosh\theta+1)}$$

$$\Rightarrow \frac{4y^2}{b^2} = \frac{\cosh\theta - 1}{\cosh\theta + 1}$$

$$\Rightarrow \frac{4y^2}{b^2} = \frac{\frac{a}{2x-a} - 1}{\frac{a}{2x-a} + 1}$$

MULTIPLY "TOP & BOTTOM" BY $(2x-a)$

$$\Rightarrow \frac{4y^2}{b^2} = \frac{a-(2x-a)}{a+(2x-a)}$$

$$\Rightarrow \frac{4y^2}{b^2} = \frac{2a-2x}{2x}$$

$$\Rightarrow \frac{4y^2}{b^2} = \frac{a-x}{x}$$

$$\Rightarrow 4xy^2 = ab^2 - b^2x$$

$$\Rightarrow 4xy^2 + b^2x = ab^2$$

$$\Rightarrow x(4y^2+b^2) = ab^2$$

AS REQUIRED

Question 10 (*****)

A hyperbola and an ellipse have respective equations

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \qquad \text{and} \qquad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

where $a > b > 0$.

The tangent to the hyperbola, at a point whose both coordinates are positive, passes through the focus of the ellipse with positive x coordinate.

Show that the gradient of the above described tangent is 1.

V , , proof

START BY DIFFERENTIATING THE HYPERBOLA

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \Rightarrow \frac{2x}{a^2} - \frac{2y}{b^2}\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{2x}{a^2} = \frac{2y}{b^2}\frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{xb^2}{ya^2}$$

THE GRADIENT AT A GENERAL POINT ON A STANDARD HYPERBOLA

$$P(a\sec\theta, b\tan\theta) \Rightarrow \left.\frac{dy}{dx}\right|_P = \frac{ab^2\sec\theta}{a^2 b\tan\theta} = \frac{b}{a}\frac{\sec\theta}{\tan\theta}$$

$$\left.\frac{dy}{dx}\right|_P = \frac{b}{a} \times \sec\theta \times \cot\theta = \frac{b}{a} \times \frac{1}{\cos\theta} \times \frac{\cos\theta}{\sin\theta}$$

$$\left.\frac{dy}{dx}\right|_P = \frac{b}{a\sin\theta}$$

THE EQUATION OF A GENERAL TANGENT WILL BE

$$y - b\tan\theta = \frac{b}{a\sin\theta}(x - a\sec\theta)$$

THIS TANGENT MUST PASS THROUGH THE FOCUS OF THE ELLIPSE $(ae, 0)$

$$\Rightarrow -\cancel{b}\tan\theta = \frac{\cancel{b}}{a\sin\theta}(ae - a\sec\theta)$$

$$\Rightarrow -\tan\theta = \frac{\cancel{a}}{\cancel{a}\sin\theta}(e - \sec\theta)$$

$$\Rightarrow -\tan\theta = \frac{1}{\sin\theta}(e - \sec\theta)$$

$$\Rightarrow -\tan\theta\sin\theta = e - \sec\theta$$

$$\Rightarrow \sec\theta - \tan\theta\sin\theta = e$$

$$\Rightarrow e = \frac{1}{\cos\theta} - \frac{\sin\theta}{\cos\theta} \times \sin\theta$$

$$\Rightarrow e = \frac{1 - \sin^2\theta}{\cos\theta} = \frac{\cos^2\theta}{\cos\theta}$$

$$\Rightarrow e = \cos\theta$$

BUT THE ECCENTRICITY OF THE ELLIPSE SATISFIES

$$b^2 = a^2(1 - e^2)$$

$$\frac{b^2}{a^2} = 1 \quad \cos^2\theta$$

$$\frac{b^2}{a^2} = \sin^2\theta$$

$$\frac{b}{a} = +\sin\theta$$

BUT THE GRADIENT OF TANGENT TO THE HYPERBOLA IS

$$\frac{b}{a\sin\theta} = \frac{b}{a} \times \frac{1}{\sin\theta} = \sin\theta \times \frac{1}{\sin\theta} = 1$$

INDEED THIS TANGENT MUST HAVE GRADIENT 1