DIFFERENTIATION

from first principles

Question 1 (**)

$$f(x) = x^2, \ x \in \mathbb{R}.$$

Use the formal definition of the derivative as a limit, to show that

$$f'(x) = 2x.$$

$\boxed{}$, $\boxed{\text{proof}}$

THE DEFINITION OF THE DERIVATIVE FOR $y = f(x)$ IS GIVEN BY

$$\frac{dy}{dx} = f'(x) = \lim_{h \to 0} \left[\frac{f(x+h) - f(x)}{h} \right]$$

$$f'(x) = \lim_{h \to 0} \left[\frac{(x+h)^2 - x^2}{h} \right]$$

$$f'(x) = \lim_{h \to 0} \left[\frac{\cancel{x^2} + 2xh + h^2 - \cancel{x^2}}{h} \right]$$

$$f'(x) = \lim_{h \to 0} \left[\frac{2xh + h^2}{h} \right]$$

DIVIDING h FROM ALL THE 3 TERMS

$$f'(x) = \lim_{h \to 0} \left[2x + h \right]$$

AS h TENDS TO ZERO THE ONLY TERM THAT SURVIVES IS $2x$

$$f'(x) = 2x$$

AS REQUIRED

Question 2 (**)

$$f(x) = x^4, \; x \in \mathbb{R}.$$

Use the formal definition of the derivative as a limit, to show that

$$f'(x) = 4x^3.$$

$\boxed{}$, $\boxed{\text{proof}}$

THE DERIVATIVE IS FORMALLY GIVEN BY

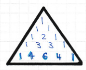
$$f'(x) = \lim_{h \to 0} \left[\frac{f(x+h) - f(x)}{h} \right]$$

IN THIS CASE WE HAVE

$$f(x) = x^4$$
$$f(x+h) = (x+h)^4$$

EXPANDING BINOMIALLY WE HAVE

$$(x+h)^4 = 1x^4h^0 + 4x^3h^1 + 6x^2h^2 + 4x^1h^3 + 1x^0h^4$$
$$(x+h)^4 = x^4 + 4x^3h + 6x^2h^2 + 4xh^3 + h^4$$



TIDYING UP NEXT

$$f(x+h) - f(x) = (x+h)^4 - x^4 = (\cancel{x^4} + 4x^3h + 6x^2h^2 + 4xh^3 + h^4) - \cancel{x^4}$$

FINALLY WE HAVE

$$f'(x) = \lim_{h \to 0} \left[\frac{f(x+h) - f(x)}{h} \right] = \lim_{h \to 0} \left[\frac{4x^3h + 6x^2h^2 + 4xh^3 + h^4}{h} \right]$$
$$= \lim_{h \to 0} \left[4x^3 + \cancel{6x^2h} + \cancel{4xh^2} + \cancel{h^3} \right]$$
$$= 4x^3$$

Question 3 (**)

$$f(x) = x^2 - 3x + 7\,,\ x \in \mathbb{R}\,.$$

Use the formal definition of the derivative as a limit, to show that

$$f'(x) = 2x - 3\,.$$

$\boxed{}$, $\boxed{\text{proof}}$

$f(x) = x^2 - 3x + 7\,,\ x \in \mathbb{R}$

- FIRST OBTAIN A SIMPLIFIED EXPRESSION FOR $f(x+h)$

$$f(x+h) = (x+h)^2 - 3(x+h) + 7$$
$$= x^2 + 2xh + h^2 - 3x - 3h + 7$$

- USING THE FORMAL DEFINITION OF THE DERIVATIVE

$$f'(x) = \lim_{h \to 0}\left[\frac{f(x+h) - f(x)}{h}\right]$$

$$f'(x) = \lim_{h \to 0}\left[\frac{(\cancel{x^2} + 2xh + h^2 \cancel{-3x} - 3h \cancel{+7}) - (\cancel{x^2} \cancel{-3x} \cancel{+7})}{h}\right]$$

$$f'(x) = \lim_{h \to 0}\left[\frac{2xh + h^2 - 3h}{h}\right]$$

$$f'(x) = \lim_{h \to 0}\left[2x + \cancel{h} - 3\right]$$

$$f'(x) = 2x - 3$$

AS REQUIRED

Question 4 (+)**

$$y = x^3 - 4x + 1, \ x \in \mathbb{R}.$$

Use the formal definition of the derivative as a limit, to show that

$$\frac{dy}{dx} = 3x^2 - 4.$$

$\boxed{}$, $\boxed{\text{proof}}$

LET $y = f(x) = x^3 - 4x + 1$

THEN
$$\begin{aligned} f(x+h) &= (x+h)^3 - 4(x+h) + 1 \\ &= (x+h)(x+h)^2 - 4x - 4h + 1 \\ &= (x+h)(x^2+2xh+h^2) - 4x - 4h + 1 \\ &= \begin{matrix} x^3 + 2x^2h + xh^2 \\ x^2h + 2xh^2 + h^3 \end{matrix} - 4x - 4h + 1 \\ &= x^3 + 3x^2h + 3xh^2 + h^3 - 4x - 4h + 1 \end{aligned}$$

BY THE FORMAL DEFINITION OF THE DERIVATIVE WE HAVE

$$\begin{aligned} \frac{dy}{dx} &= \lim_{h \to 0}\left[\frac{f(x+h) - f(x)}{h}\right] \\ &= \lim_{h \to 0}\left[\frac{(x^3+3x^2h+3xh^2+h^3-4x-4h+1) - (x^3-4x+1)}{h}\right] \\ &= \lim_{h \to 0}\left[\frac{\cancel{x^3}+3x^2h+3xh^2+h^3\cancel{-4x}-4h\cancel{+1}\cancel{-x^3}\cancel{+4x}\cancel{-1}}{h}\right] \\ &= \lim_{h \to 0}\left[\frac{3x^2h+3xh^2+h^3-4h}{h}\right] \\ &= \lim_{h \to 0}\left[3x^2+3xh+h^2-4\right] \\ &= 3x^2 - 4 \end{aligned}$$

// AS REQUIRED

Question 5 (**+)

$$f(x) = x^3 + 2, \; x \in \mathbb{R}.$$

a) State the value of $f(-1)$.

b) Find a simplified expression for $f(-1+h)$.

c) Use the formal definition of the derivative as a limit, to show that

$$f'(-1) = 3.$$

$\boxed{f(-1)=1}$, $\boxed{f(-1+h) = 1 + 3h - 3h^2 + h^3}$

$\boxed{f(x) = x^3 + 2}$

a) $f(-1) = (-1)^3 + 2 = -1 + 2 = 1$

b) $f(-1+h) = (-1+h)^3 + 2 = (h-1)^3 + 2 = (h-1)(h^2-2h+1) + 2$

$= h^3 - 2h^2 + h$
$\quad - h^2 + 2h - 1 + 2$

$= h^3 - 3h^2 + 3h + 1$

c) $f'(-1) = \lim_{h \to 0} \left[\frac{f(-1+h) - f(-1)}{h} \right]$

$= \lim_{h \to 0} \left[\frac{(h^3 - 3h^2 + 3h + 1) - (1)}{h} \right]$

$= \lim_{h \to 0} \left[\frac{h^3 - 3h^2 + 3h}{h} \right]$

$= \lim_{h \to 0} \left[h^2 - 3h + 3 \right]$

TAKING THE LIMIT NOW, AS $h \to 0$

$= 3$

AS REQUIRED

Question 6 (***)

$$f(x) = x^4 - 4x, \quad x \in \mathbb{R}.$$

a) Find a simplified expression for

$$f(2+h) - f(2).$$

b) Use the formal definition of the derivative as a limit, to show that

$$f'(2) = 28.$$

$$\boxed{f(2+h) - f(2) = 28h + 24h^2 + 8h^3 + h^4}$$

a) $f(x) = x^4 - 4x$

$$\begin{aligned}
f(2+h) - f(2) &= \left[(2+h)^4 - 4(2+h)\right] - \left[2^4 - 4\times 2\right] \\
&= (2+h)^4 - \cancel{8} - 4h - 16 + \cancel{8} \\
&= (2+h)^4 - 4h - 16 \\
&= (2+h)^2(2+h)^2 - 4h - 16 \\
&= (4 + 4h + h^2)(4 + 4h + h^2) - 4h - 16
\end{aligned}$$

$$\begin{array}{rl}
= & 16 + 16h + 4h^2 \\
 & \quad 16h + 16h^2 + 4h^3 \\
 & \quad\quad + 4h^2 + 4h^3 + h^4 \quad - 4h - 16 \\
\hline
 & \cancel{16} + 32h + 24h^2 + 8h^3 + h^4 - 4h - \cancel{16}
\end{array}$$

$$= h^4 + 8h^3 + 24h^2 + 28h$$

b) $\boxed{f'(x) = \lim_{h \to 0}\left[\dfrac{f(x+h) - f(x)}{h}\right]}$

$$f'(2) = \lim_{h \to 0}\left[\frac{f(2+h) - f(2)}{h}\right]$$

$$f'(2) = \lim_{h \to 0}\left[\frac{h^4 + 8h^3 + 24h^2 + 28h}{h}\right]$$

$$f'(2) = \lim_{h \to 0}\left[\cancel{h^3} + \cancel{8h^2} + \cancel{24h} + 28\right]$$

$$f'(2) = 28$$

Question 7 (***)

A reciprocal curve has equation

$$y = \frac{1}{x}, \quad x \neq 0.$$

Use the formal definition of the derivative as a limit, to show that

$$\frac{dy}{dx} = -\frac{1}{x^2}.$$

$\boxed{}$, $\boxed{\text{proof}}$

LET $y = f(x) = \frac{1}{x}$

$f(x+h) = \frac{1}{x+h}$

$f(x+h) - f(x) = \frac{1}{x+h} - \frac{1}{x} = \frac{x-(x+h)}{x(x+h)} = \frac{-h}{x(x+h)}$

USING THE FORMAL DEFINITION OF THE DERIVATIVE

$$\frac{dy}{dx} = f'(x) = \lim_{h \to 0}\left[\frac{f(x+h)-f(x)}{h}\right]$$

$$= \lim_{h \to 0}\left[\frac{\frac{-h}{x(x+h)}}{h}\right]$$

$$= \lim_{h \to 0}\left[\frac{-h}{x(x+h)} \div h\right]$$

$$= \lim_{h \to 0}\left[\frac{-\cancel{h}}{x^2+xh} \times \frac{1}{\cancel{h}}\right]$$

$$= \lim_{h \to 0}\left[-\frac{1}{x^2+xh}\right]$$

TAKING LIMITS, AS $h \to 0$

$$= -\frac{1}{x^2 + \cancel{xh}}$$

$$= -\frac{1}{x^2}$$ AS REQUIRED

Question 8 (***)

$$\frac{d}{dx}(\sin x) = \cos x .$$

Prove by first principles the validity of the above result by using the small angle approximations for $\sin x$ and $\cos x$.

$\boxed{}$, $\boxed{\text{proof}}$

STARTING WITH THE FORMAL DEFINITION OF THE DERIVATIVE

$$f'(x) = \lim_{h \to 0}\left[\frac{f(x+h)-f(x)}{h}\right] \quad \text{WITH } f(x) = \sin x$$

$$f'(x) = \lim_{h \to 0}\left[\frac{\sin(x+h) - \sin x}{h}\right]$$

USING COMPOUND ANGLE IDENTITIES

$$f'(x) = \lim_{h \to 0}\left[\frac{\sin x\cos h + \cos x\sin h - \sin x}{h}\right]$$

USING SMALL ANGLE APPROXIMATIONS

$$\sin h = h + O(h^3)$$
$$\cos h = 1 + O(h^2)$$

THUS WE OBTAIN:

$$f'(x) = \lim_{h \to 0}\left[\frac{\sin x\left[1+O(h^2)\right] + \cos x\left[h+O(h^3)\right] - \sin x}{h}\right]$$

$$= \lim_{h \to 0}\left[\frac{\cancel{\sin x} + O(h^2)\sin x + h\cos x + O(h^3)\cos x - \cancel{\sin x}}{h}\right]$$

$$= \lim_{h \to 0}\left[\frac{O(h^2)\sin x + h\cos x + O(h^3)\cos x}{h}\right]$$

$$= \lim_{h \to 0}\left[O(h)\sin x + \cos x + O(h^2)\cos x\right]$$

$$= \cos x$$

AS REQUIRED

Question 9 **(***)**

If x is in radians

$$\frac{d}{dx}(\sin x) = \cos x.$$

Prove the validity of the above result from first principles.

You may assume that if h is small and measured in radians, then as $h \to \infty$

$$\frac{\cos(h) - 1}{h} \to 0 \quad \text{and} \quad \frac{\sin(h)}{h} \to 1.$$

MP2 , proof

USING THE STANDARD FORMULA FOR THE DERIVATIVE

$$f'(x) = \lim_{h \to 0}\left[\frac{f(x+h) - f(x)}{h}\right] \quad \text{WITH} \quad f(x) = \sin x$$

$$\frac{d}{dx}(\sin x) = \lim\left[\frac{\sin(x+h) - \sin x}{h}\right]$$

USING THE IDENTITY $\sin(A+B) \equiv \sin A\cos B + \cos A\sin B$

$$\frac{d}{dx}(\sin x) = \lim_{h \to 0}\left[\frac{\sin x\cos h + \cos x\sin h - \sin x}{h}\right]$$

$$= \lim_{h \to 0}\left[\frac{\sin x\cos h - \sin x + \cos x\sin h}{h}\right]$$

$$= \lim_{h \to 0}\left[\frac{\sin x(\cos h - 1) + \cos x\sin h}{h}\right]$$

$$= \lim_{h \to 0}\left[\sin x\left(\frac{\cos h - 1}{h}\right) + \cos x\left(\frac{\sin h}{h}\right)\right]$$

$$= \sin x \times 0 + \cos x \times 1$$

$$= \cos x$$

Question 10 (***+)

$$\cos(A+B) \equiv \cos A\cos B - \sin A\sin B$$
$$\cos(A-B) \equiv \cos A\cos B + \sin A\sin B$$

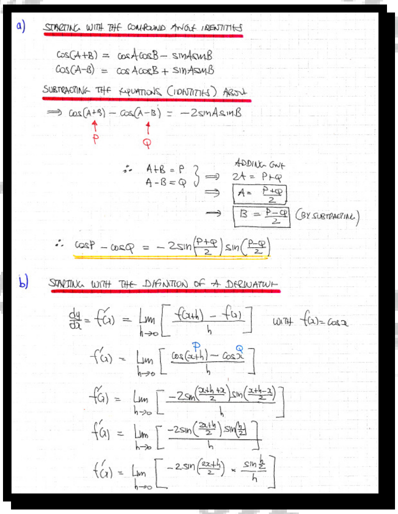
a) By using the above identities show that

$$\cos P - \cos Q \equiv -2\sin\left(\frac{P+Q}{2}\right)\sin\left(\frac{P-Q}{2}\right).$$

b) Hence, prove by first principles that

$$\frac{d}{dx}(\cos x) = -\sin x$$

$\boxed{}$, $\boxed{\text{proof}}$



a) STARTING WITH THE COMPOUND ANGLE IDENTITIES

$\cos(A+B) = \cos A\cos B - \sin A\sin B$
$\cos(A-B) = \cos A\cos B + \sin A\sin B$

SUBTRACTING THE EQUATIONS (IDENTITIES) ABOVE

$\Longrightarrow \cos(A+B) - \cos(A-B) = -2\sin A\sin B$

P Q

$\therefore \left.\begin{matrix} A+B=P \\ A-B=Q \end{matrix}\right\} \Longrightarrow$ ADDING GIVE $2A = P+Q$

$\Longrightarrow \boxed{A = \frac{P+Q}{2}}$

$\longrightarrow \boxed{B = \frac{P-Q}{2}}$ (BY SUBTRACTING)

$\therefore \cos P - \cos Q = -2\sin\left(\frac{P+Q}{2}\right)\sin\left(\frac{P-Q}{2}\right)$

b) STARTING WITH THE DEFINITION OF A DERIVATIVE

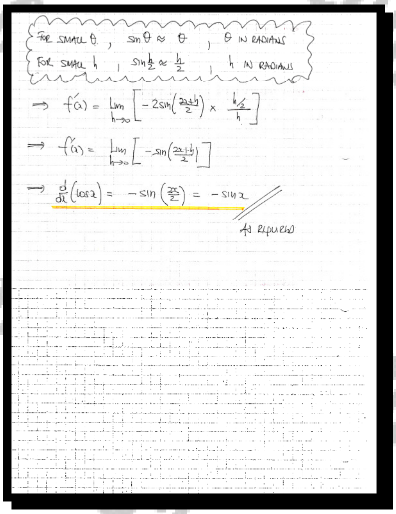
$\frac{dy}{dx} = f'(x) = \lim_{h\to 0}\left[\frac{f(x+h)-f(x)}{h}\right]$ WITH $f(x) = \cos x$

$f'(x) = \lim_{h\to 0}\left[\frac{\cos(x+h) - \cos x}{h}\right]$

$f'(x) = \lim_{h\to 0}\left[\frac{-2\sin\left(\frac{x+h+x}{2}\right)\sin\left(\frac{x+h-x}{2}\right)}{h}\right]$

$f'(x) = \lim_{h\to 0}\left[\frac{-2\sin\left(\frac{2x+h}{2}\right)\sin\left(\frac{h}{2}\right)}{h}\right]$

$f'(x) = \lim_{h\to 0}\left[-2\sin\left(\frac{2x+h}{2}\right) \times \frac{\sin\frac{h}{2}}{h}\right]$



FOR SMALL θ, $\sin\theta \approx \theta$, θ IN RADIANS
FOR SMALL h, $\sin\frac{h}{2} \approx \frac{h}{2}$, h IN RADIANS

$\Longrightarrow f'(x) = \lim_{h\to 0}\left[-2\sin\left(\frac{2x+h}{2}\right) \times \frac{h/2}{h}\right]$

$\Longrightarrow f'(x) = \lim_{h\to 0}\left[-\sin\left(\frac{2x+h}{2}\right)\right]$

$\longrightarrow \frac{d}{dx}(\cos x) = -\sin\left(\frac{2x}{2}\right) = -\sin x$

AS REQUIRED

Question 11 (***+)

Differentiate from first principles

$$\frac{1}{x^2}, \; x \neq 0.$$

$$\boxed{-\frac{2}{x^3}}$$

LET $f(x) = \frac{1}{x^2}$

$\Rightarrow f(x+h) = \frac{1}{(x+h)^2}$

WE REQUIRE FIRST

$$f(x+h) - f(x) = \frac{1}{(x+h)^2} - \frac{1}{x^2} = \frac{x^2-(x+h)^2}{x^2(x+h)^2}$$

$$= \frac{x^2-(x^2+2xh+h^2)}{x^2(x^2+2xh+h^2)} = \frac{-2xh-h^2}{x^4+2x^3h+x^2h^2}$$

USING THE FORMAL DEFINITION OF THE DERIVATIVE

$$\Rightarrow f'(x) = \lim_{h \to 0}\left[\frac{f(x+h)-f(x)}{h}\right]$$

$$\Rightarrow f'(x) = \lim_{h \to 0}\left[\frac{\frac{-2xh-h^2}{x^4+2x^3h+x^2h^2}}{h}\right]$$

$$\Rightarrow f'(x) = \lim_{h \to 0}\left[\frac{-2xh-h^2}{x^4-2x^3h+x^2h^2} \div h\right]$$

$$\Rightarrow f'(x) = \lim_{h \to 0}\left[\frac{-\cancel{h}(2x+h)}{x^4-2x^3h+x^2h^2} \times \frac{1}{\cancel{h}}\right]$$

$$\Rightarrow f'(x) = \lim_{h \to 0}\left[\frac{-(2x+h)}{x^4-2x^3h+x^2h^2}\right]$$

TAKING THE LIMIT AS $h \to 0$

$$\Rightarrow f'(x) = -\frac{2x+\cancel{h}}{x^4\cancel{-2x^3h+x^2h^2}} = -\frac{2x}{x^4} = -\frac{2}{x^3}$$

AS REQUIRED

Question 12 (***+)

Differentiate $\dfrac{1}{2-x}$ from first principles.

$$\boxed{\frac{d}{dx}\left(\frac{1}{2-x}\right)=\frac{1}{(2-x)^2}}$$

LET $f(x)=\dfrac{1}{2-x}$

$f(x+h)=\dfrac{1}{2-(x+h)}=\dfrac{1}{2-h-x}$

FROM THE DEFINITION OF THE DERIVATIVE

$$f'(x)=\lim_{h\to 0}\left[\frac{f(x+h)-f(x)}{h}\right]$$

$$f'(x)=\lim_{h\to 0}\left[\frac{\frac{1}{2-x-h}-\frac{1}{2-x}}{h}\right]$$

$$f'(x)=\lim_{h\to 0}\left[\frac{\frac{(\cancel{2-x})-(\cancel{2-x}-h)}{(2-x-h)(2-x)}}{h}\right]$$

$$f'(x)=\lim_{h\to 0}\left[\frac{\frac{h}{(2-x)(2-x-h)}}{h}\right]$$

$$f'(x)=\lim_{h\to 0}\left[\frac{h}{(2-x)(2-x-h)}\div h\right]$$

$$f'(x)=\lim_{h\to 0}\left[\frac{\cancel{h}}{(2-x)(2-x-h)}\times\frac{1}{\cancel{h}}\right]$$

$$f'(x)=\lim_{h\to 0}\left[\frac{1}{(2-x)(2-x-h)}\right]$$

$$f'(x)=\frac{1}{(2-x)^2}$$

Question 13 (***+)

$$f(x) = \frac{x-2}{x+2}, \; x \in \mathbb{R}.$$

Use the formal definition of the derivative as a limit, to show that

$$f'(x) = \frac{4}{(x+2)^2}.$$

$\boxed{}$, $\boxed{\text{proof}}$

$$\boxed{\text{IF } f(x) = \frac{x-2}{x+2} \text{ THEN } f(x+h) = \frac{(x+h)-2}{(x+h)+2}}$$

$$f(x+h) - f(x) = \frac{x+h-2}{x+h+2} - \frac{x-2}{x+2} = \frac{(x+2)(x-2+h) - (x-2)(x+2+h)}{(x+2)(x+h+2)}$$

$$= \frac{\cancel{x^2-4} + h(x+2) - [\cancel{x^2-4} + h(x-2)]}{(x+2)(x+h+2)}$$

$$= \frac{hx + 2h - hx + 2h}{(x+2)(x+h+2)} = \frac{4h}{(x+2)(x+h+2)}$$

FROM THE DEFINITION OF THE DERIVATIVE AS A LIMIT

$$f'(x) = \lim_{h \to 0} \left[\frac{f(x+h) - f(x)}{h} \right] = \lim_{h \to 0} \left[[f(x+h) - f(x)] \div h \right]$$

$$= \lim_{h \to 0} \left[\frac{4\cancel{h}}{(x+2)(x+h+2)} \times \frac{1}{\cancel{h}} \right]$$

$$= \lim_{h \to 0} \left[\frac{4}{(x+2)(x+h+2)} \right]$$

$$= \frac{4}{(x+2)^2}$$

Question 14 (***+)

Differentiate from first principles

$$\frac{x}{x+1}, \ x \neq -1 .$$

$\boxed{}$, $\boxed{\dfrac{1}{(x+1)^2}}$

LET $f(x) = \frac{x}{x+1}$ AND $f(x+h) = \frac{x+h}{x+h+1}$

$$f'(x) = \frac{dy}{dx} = \lim_{h \to 0}\left[\frac{f(x+h) - f(x)}{h}\right]$$

$$= \lim_{h \to 0}\left[\frac{\frac{x+h}{x+h+1} - \frac{x}{x+1}}{h}\right]$$

$$= \lim_{h \to 0}\left[\frac{\frac{(x+h)(x+1) - x(x+h+1)}{(x+h+1)(x+1)}}{h}\right]$$

$$= \lim_{h \to 0}\left[\frac{\frac{\cancel{x^2}+\cancel{x}+\cancel{xh}+h-\cancel{x^2}-\cancel{xh}-\cancel{x}}{(x+h+1)(x+1)}}{h}\right]$$

$$= \lim_{h \to 0}\left[\frac{\frac{h}{(x+h+1)(x+1)}}{h}\right]$$

$$= \lim_{h \to 0}\left[\frac{h}{(x+h+1)(x+1)} \div h\right]$$

$$= \lim_{h \to 0}\left[\frac{\cancel{h}}{(x+h+1)(x+1)} \times \frac{1}{\cancel{h}}\right]$$

$$= \lim_{h \to 0}\left[\frac{1}{(x+h+1)(x+1)}\right]$$

$$= \frac{1}{(x+1)(x+1)}$$

$$= \frac{1}{(x+1)^2}$$

Question 15 (***+)

Differentiate $\dfrac{1}{2+x^2}$ from first principles.

$\boxed{}$, $\boxed{\dfrac{d}{dx}\left(\dfrac{1}{2+x^2}\right) = -\dfrac{2x}{\left(2+x^2\right)^2}}$

LET $f(x) = \dfrac{1}{2+x^2}$

- $f(x+h) = \dfrac{1}{2+(x+h)^2}$
- $$\frac{f(x+h)-f(x)}{h} = \frac{\frac{1}{2+(x+h)^2} - \frac{1}{2+x^2}}{h} = \frac{\frac{(2+x^2)-[2+(x+h)^2]}{(2+x^2)[2+(x+h)^2]}}{h}$$
$$= \frac{(2+x^2)-[2+(x+h)^2]}{h(2+x^2)[2+(x+h)^2]}$$
$$= \frac{\cancel{2}+\cancel{x^2}-(\cancel{2}+\cancel{x^2}+2xh+h^2)}{h(2+x^2)[2+(x+h)^2]}$$
$$= \frac{-2xh-h^2}{h(2+x^2)[2+(x+h)^2]}$$
$$= \frac{-h(2x+h)}{h(2+x^2)[2+(x+h)^2]}$$
$$= \frac{-(2x+h)}{(2+x^2)[2+(x+h)^2]}$$

TAKING THE LIMIT AS $h \to 0$

$$\frac{d}{dx}\left(\frac{1}{2+x^2}\right) = \lim_{h\to 0}\left[\frac{f(x+h)-f(x)}{h}\right] = \lim_{h\to 0}\left[\frac{-(2x+h)}{(2+x^2)[2+(x+h)^2]}\right]$$
$$= \frac{-2x}{(2+x^2)(2+x^2)} = -\frac{2x}{(2+x^2)^2}$$

Question 16 (***+)

Differentiate $\dfrac{1}{x^2-2x}$ from first principles.

$\boxed{}$, $\boxed{\dfrac{d}{dx}\left(\dfrac{1}{x^2-2x}\right)=\dfrac{2(1-x)}{\left(x^2-2x\right)^2}}$

- WRITE THE EXPRESSION IN FUNCTION NOTATION FOR SIMPLICITY

$$f(x)=\frac{1}{x^2-2x}$$

$$f(x+h)-f(x)=\frac{1}{(x+h)^2-2(x+h)}-\frac{1}{x^2-2x}$$

$$=\frac{1}{x^2+2xh+h^2-2x-2h}-\frac{1}{x^2-2x}$$

$$=\frac{\cancel{x^2}-\cancel{2x}-(\cancel{x^2}+2xh+h^2-\cancel{2x}-2h)}{(x^2+2xh+h^2-2x-2h)(x^2-2x)}$$

$$=\frac{-2xh-h^2+2h}{(x^2+2xh+h^2-2x-2h)(x^2-2x)}$$

- HENCE THE DERIVATIVE NOW YIELDS

$$f'(x)=\lim_{h\to 0}\left[\frac{f(x+h)-f(x)}{h}\right]=\lim_{h\to 0}\left[\frac{1}{h}\left[f(x+h)-f(x)\right]\right]$$

$$=\lim_{h\to 0}\left[\frac{1}{\cancel{h}}\times\frac{\cancel{h}(-2x-h+2)}{(x^2+2xh+h^2-2x-2h)(x^2-2x)}\right]$$

$$=\lim_{h\to 0}\left[\frac{-h+2-2x}{(x^2+2xh+h^2-2x-2h)(x^2-2x)}\right]$$

$$=\frac{2-2x}{(x^2-2x)(x^2-2x)}=\frac{2(1-x)}{(x^2-2x)^2}$$

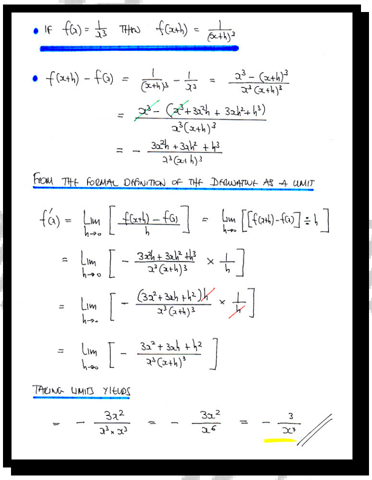
Question 17 (***+)

$$f(x) = \frac{1}{x^3}, \; x \in \mathbb{R}, \; x \neq 0 .$$

Use the formal definition of the derivative as a limit, to show that

$$f'(x) = -\frac{3}{x^4} .$$

$\boxed{}$, $\boxed{\text{proof}}$



- IF $f(x) = \frac{1}{x^3}$ THEN $f(x+h) = \frac{1}{(x+h)^3}$
- $f(x+h) - f(x) = \frac{1}{(x+h)^3} - \frac{1}{x^3} = \frac{x^3 - (x+h)^3}{x^3(x+h)^3}$

$$= \frac{\cancel{x^3} - (\cancel{x^3} + 3x^2h + 3xh^2 + h^3)}{x^3(x+h)^3}$$

$$= -\frac{3x^2h + 3xh^2 + h^3}{x^3(x+h)^3}$$

FROM THE FORMAL DEFINITION OF THE DERIVATIVE AS A LIMIT

$$f'(x) = \lim_{h \to 0}\left[\frac{f(x+h) - f(x)}{h}\right] = \lim_{h \to 0}\Big[\left[f(x+h) - f(x)\right] \div h\Big]$$

$$= \lim_{h \to 0}\left[-\frac{3x^2h + 3xh^2 + h^3}{x^3(x+h)^3} \times \frac{1}{h}\right]$$

$$= \lim_{h \to 0}\left[-\frac{(3x^2 + 3xh + h^2)\cancel{h}}{x^3(x+h)^3} \times \frac{1}{\cancel{h}}\right]$$

$$= \lim_{h \to 0}\left[-\frac{3x^2 + 3xh + h^2}{x^3(x+h)^3}\right]$$

TAKING LIMITS YIELDS

$$= -\frac{3x^2}{x^3 \times x^3} = -\frac{3x^2}{x^6} = -\frac{3}{x^4}$$

Question 18 (***+)

$$f(x)=\frac{1}{5x+3}, \; x\in\mathbb{R}, \; x\neq -\frac{3}{5}.$$

Use the formal definition of the derivative as a limit, to show that

$$f'(x)=-\frac{5}{(5x+3)^2}.$$

proof

LET $f(x)=\frac{1}{5x+3}$

$$f'(x)=\lim_{h\to 0}\left[\frac{f(x+h)-f(x)}{h}\right]=\lim_{h\to 0}\left[\frac{\frac{1}{5(x+h)+3}-\frac{1}{5x+3}}{h}\right]$$

$$=\lim_{h\to 0}\left[\frac{5x+3-5(x+h)-3}{(5x+3)[5(x+h)+3]}\times\frac{1}{h}\right]$$

$$=\lim_{h\to 0}\left[\frac{\cancel{5x}+\cancel{3}-\cancel{5x}-5\cancel{h}-\cancel{3}}{(5x+3)(5x+5h+3)\cancel{h}}\right]=\lim_{h\to 0}\left[\frac{-5}{(5x+3)(5x+5h+3)}\right]$$

$$=\frac{-5}{(5x+3)^2}$$

Question 19 (***+)

$$f(x) = \frac{3}{4x-1}, \quad x \in \mathbb{R}, \quad x \neq -\frac{1}{4}.$$

Use the formal definition of the derivative as a limit, to show that

$$f'(x) = -\frac{12}{(4x-1)^2}.$$

$\boxed{\text{proof}}$

$$y = f(x) = \frac{3}{4x-1}$$

$$f'(x) = \lim_{h \to 0}\left[\frac{f(x+h)-f(x)}{h}\right] = \lim_{h \to 0}\left[\frac{\frac{3}{4(x+h)-1} - \frac{3}{4x-1}}{h}\right]$$

$$= \lim_{h \to 0}\left[\frac{\frac{3(4x-1) - 3[4(x+h)-1]}{[4(x+h)-1][4x-1]}}{h}\right]$$

$$= \lim_{h \to 0}\left[\frac{\frac{12x-3-3[4x+4h-1]}{(4x+4h-1)(4x-1)}}{h}\right] = \lim_{h \to 0}\left[\frac{\frac{\cancel{12x}-\cancel{3}-\cancel{12x}-12h+\cancel{3}}{(4x+4h-1)(4x-1)}}{h}\right]$$

$$= \lim_{h \to 0}\left[\frac{\frac{-12\cancel{h}}{(4x+4h-1)(4x-1)}}{\frac{\cancel{h}}{1}}\right] = \lim_{h \to 0}\left[\frac{-12}{(4x+4h-1)(4x-1)}\right]$$

$$= \frac{-12}{(4x-1)(4x-1)} = -\frac{12}{(4x-1)^2}$$

Question 20 **(***+)**

$$\frac{d}{dx}(\sin x) = \cos x\,.$$

Prove the validity of the above result by …

a) … using $\displaystyle\lim_{x\to 0}\left(\frac{\sin x}{x}\right) = 1$

and the trigonometric identity

$$\sin A - \sin B \equiv 2\cos\frac{A+B}{2}\sin\frac{A-B}{2}\,.$$

b) … using small angle approximations for $\sin x$ and $\cos x$.

proof

(a)

$$\frac{d}{dx}[\sin x] = \lim_{h\to 0}\left[\frac{\sin(x+h)-\sin x}{h}\right]$$

$$= \lim_{h\to 0}\left[\frac{1}{h}\left[2\cos\left(\frac{x+h+x}{2}\right)\sin\left(\frac{x+h-x}{2}\right)\right]\right]$$

$$= \lim_{h\to 0}\left[\frac{2}{h}\cos\left(x+\frac{h}{2}\right)\sin\left(\frac{h}{2}\right)\right]$$

$$= \lim_{h\to 0}\left[\cos\left(x+\frac{h}{2}\right)\times\frac{\sin\frac{h}{2}}{\frac{h}{2}}\right] \leftarrow \text{LIMIT IS 1}$$

$$= \cos x$$

(b)

$$\frac{d}{dx}[\sin x] = \lim_{h\to 0}\left[\frac{\sin(x+h)-\sin x}{h}\right] =$$

$$= \lim_{h\to 0}\left[\frac{\sin x\cos(h)+\cos x\sin(h)-\sin x}{h}\right]$$

$$\cos(h) = 1+O(h^2)$$
$$\sin(h) = h+O(h^3)$$

$$= \lim_{h\to 0}\left[\frac{1}{h}\left[\sin x\,(1+O(h^2)) + \cos x\times(h+O(h^3)) - \sin x\right]\right]$$

$$= \lim_{h\to 0}\left[\frac{1}{h}\left[\cancel{\sin x} + O(h^2)\sin x + h\cos x + O(h^3)\cos x \cancel{-\sin x}\right]\right]$$

$$= \lim_{h\to 0}\left[O(h)\sin x + \cos x + O(h^2)\cos x\right]$$

$$= \cos x$$

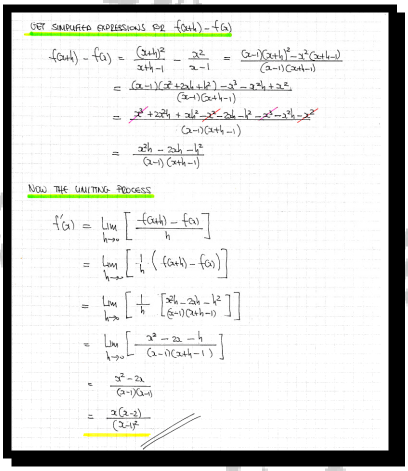
Question 21 (***+)

$$f(x) \equiv \frac{x^2}{x-1}, \; x \in \mathbb{R}, \; x \neq -\frac{1}{4}.$$

Use the formal definition of the derivative as a limit, to show that

$$f'(x) = \frac{x(x-2)}{(x-1)^2}.$$

$\boxed{}$, $\boxed{\text{proof}}$



GET SIMPLIFIED EXPRESSIONS FOR $f(x+h) - f(x)$

$$f(x+h) - f(x) = \frac{(x+h)^2}{x+h-1} - \frac{x^2}{x-1} = \frac{(x-1)(x+h)^2 - x^2(x+h-1)}{(x-1)(x+h-1)}$$

$$= \frac{(x-1)(x^2+2xh+h^2) - x^3 - x^2h + x^2}{(x-1)(x+h-1)}$$

$$= \frac{\cancel{x^3} + 2x^2h + xh^2 \cancel{- x^2} - 2xh - h^2 \cancel{- x^3} - x^2h \cancel{- x^2}}{(x-1)(x+h-1)}$$

$$= \frac{x^2h - 2xh - h^2}{(x-1)(x+h-1)}$$

NOW THE LIMITING PROCESS

$$f'(x) = \lim_{h \to 0}\left[\frac{f(x+h) - f(x)}{h}\right]$$

$$= \lim_{h \to 0}\left[\frac{1}{h}\left(f(x+h) - f(x)\right)\right]$$

$$= \lim_{h \to 0}\left[\frac{1}{h}\left[\frac{x^2h - 2xh - h^2}{(x-1)(x+h-1)}\right]\right]$$

$$= \lim_{h \to 0}\left[\frac{x^2 - 2x - h}{(x-1)(x+h-1)}\right]$$

$$= \frac{x^2 - 2x}{(x-1)(x-1)}$$

$$= \frac{x(x-2)}{(x-1)^2}$$

Question 22 (****)

Prove by first principles, and by using the small angle approximations for $\sin x$ and $\cos x$, that

$$\frac{d}{dx}\left(\tan x\right) = \sec^2 x \, .$$

[ ], [proof]

USING THE FORMAL DEFINITION OF THE DERIVATIVE

$$f'(x) = \lim_{h \to 0}\left[\frac{f(x+h) - f(x)}{h}\right]$$

$$\frac{d}{dx}(\tan x) = \lim_{h \to 0}\left[\frac{\tan(x+h) - \tan x}{h}\right]$$

WRITE AS SINES AND COSINES

$$\frac{d}{dx}(\tan x) = \lim_{h \to 0}\left[\frac{\frac{\sin(x+h)}{\cos(x+h)} - \frac{\sin x}{\cos x}}{h}\right]$$

$$= \lim_{h \to 0}\left[\frac{\frac{\sin(x+h)\cos x - \cos(x+h)\sin x}{\cos x \cos(x+h)}}{h}\right]$$

$$= \lim_{h \to 0}\left[\frac{\frac{\sin[(x+h)-x]}{\cos x \cos(x+h)}}{h}\right]$$

COMPOUND ANGLE IDENTITY FOR $\sin(A-B)$

$$= \lim_{h \to 0}\left[\frac{\frac{\sin(h)}{\cos x \cos(x+h)}}{h}\right]$$

$$= \lim_{h \to 0}\left[\frac{\sin(h)}{\cos x \cos(x+h)} \div h\right]$$

$$= \lim_{h \to 0}\left[\frac{\sin(h)}{\cos x \cos(x+h)} \times \frac{1}{h}\right]$$

AS $h \to 0$ $\quad \frac{\sin(h)}{h} \to 1 \quad$ $\sin x \approx x$ FOR SMALL x

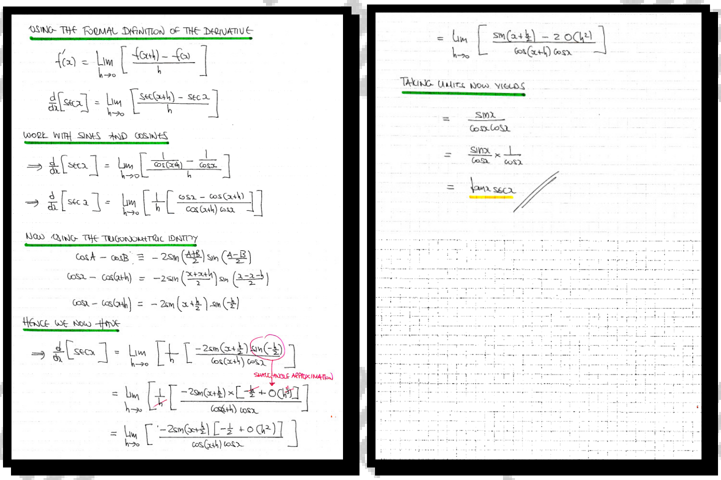
$$= \lim_{h \to 0}\left[\frac{1}{\cos x \cos(x+h)} \times \cancel{\frac{\sin h}{h}}\right] = \frac{1}{\cos^2 x} = \sec^2 x$$

Question 23 (****)

Prove by first principles, and by using the small angle approximations for $\sin x$ and $\cos x$, that

$$\frac{d}{dx}(\sec x) = \sec x \tan x .$$

$\boxed{}$, $\boxed{\text{proof}}$



USING THE FORMAL DEFINITION OF THE DERIVATIVE

$$f'(x) = \lim_{h \to 0}\left[\frac{f(x+h) - f(x)}{h}\right]$$

$$\frac{d}{dx}[\sec x] = \lim_{h \to 0}\left[\frac{\sec(x+h) - \sec x}{h}\right]$$

WORK WITH SINES AND COSINES

$$\Rightarrow \frac{d}{dx}[\sec x] = \lim_{h \to 0}\left[\frac{\frac{1}{\cos(x+h)} - \frac{1}{\cos x}}{h}\right]$$

$$\Rightarrow \frac{d}{dx}[\sec x] = \lim_{h \to 0}\left[\frac{1}{h}\left[\frac{\cos x - \cos(x+h)}{\cos(x+h)\cos x}\right]\right]$$

NOW USING THE TRIGONOMETRIC IDENTITY

$$\cos A - \cos B \equiv -2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$$

$$\cos x - \cos(x+h) = -2\sin\left(\frac{x+x+h}{2}\right)\sin\left(\frac{x-x-h}{2}\right)$$

$$\cos x - \cos(x+h) = -2\sin\left(x+\frac{h}{2}\right)\sin\left(-\frac{h}{2}\right)$$

HENCE WE NOW HAVE

$$\Rightarrow \frac{d}{dx}[\sec x] = \lim_{h \to 0}\left[\frac{1}{h}\left[\frac{-2\sin\left(x+\frac{h}{2}\right)\sin\left(-\frac{h}{2}\right)}{\cos(x+h)\cos x}\right]\right]$$

SMALL ANGLE APPROXIMATION

$$= \lim_{h \to 0}\left[\frac{1}{\cancel{h}}\left[\frac{-2\sin\left(x+\frac{h}{2}\right)\times\left[-\frac{\cancel{h}}{2} + O(h^{\cancel{3}\,2})\right]}{\cos(x+h)\cos x}\right]\right]$$

$$= \lim_{h \to 0}\left[\frac{-2\sin\left(x+\frac{h}{2}\right)\left[-\frac{1}{2} + O(h^2)\right]}{\cos(x+h)\cos x}\right]$$

$$= \lim_{h \to 0}\left[\frac{\sin\left(x+\frac{h}{2}\right) - 2\,O(h^2)}{\cos(x+h)\cos x}\right]$$

TAKING LIMITS NOW YIELDS

$$= \frac{\sin x}{\cos x \cos x}$$

$$= \frac{\sin x}{\cos x} \times \frac{1}{\cos x}$$

$$= \tan x \sec x$$

Question 24 (****+)

$$f(x) = \sqrt{1+x^2}\,,\ x \in \mathbb{R}\,.$$

Use the formal definition of the derivative as a limit, to show that

$$f'(x) = \frac{x}{\sqrt{1+x^2}}\,.$$

$\boxed{}$, $\boxed{\text{proof}}$

- IF $f(x) = \sqrt{1+x^2}$ THEN $f(x+h) = \sqrt{1+(x+h)^2}$
- BY THE FORMAL DEFINITION OF THE DERIVATIVE

$$f'(x) = \lim_{h\to 0}\left[\frac{f(x+h)-f(x)}{h}\right] = \lim_{h\to 0}\left[\frac{\sqrt{1+(x+h)^2}-\sqrt{1+x^2}}{h}\right]$$

$$= \lim_{h\to 0}\left[\frac{\left[\sqrt{1+(x+h)^2}-\sqrt{1+x^2}\right]\left[\sqrt{1+(x+h)^2}+\sqrt{1+x^2}\right]}{h\left[\sqrt{1+(x+h)^2}+\sqrt{1+x^2}\right]}\right]$$

$$= \lim_{h\to 0}\left[\frac{\left[\cancel{1}+(x+h)^2\right]-\left[\cancel{1}+x^2\right]}{h\left[\sqrt{1+(x+h)^2}+\sqrt{1+x^2}\right]}\right]$$

$$= \lim_{h\to 0}\left[\frac{\cancel{x^2}+2xh+h^2-\cancel{x^2}}{h\left[\sqrt{1+(x+h)^2}+\sqrt{1+x^2}\right]}\right]$$

$$= \lim_{h\to 0}\left[\frac{\cancel{h}(2x+h)}{\cancel{h}\left[\sqrt{1+(x+h)^2}+\sqrt{1+x^2}\right]}\right]$$

$$= \lim_{h\to 0}\left[\frac{2x+h}{\sqrt{1+(x+h)^2}+\sqrt{1+x^2}}\right]$$

$$= \frac{2x}{\sqrt{1+x^2}+\sqrt{1+x^2}}$$

$$= \frac{2x}{2\sqrt{1+x^2}}$$

$$= \frac{x}{\sqrt{1+x^2}} \quad //$$

Question 25 (****+)

$$f(x) = \frac{1}{\sqrt{x^2 - 1}}, \quad x \in \mathbb{R}, \ |x| > 1.$$

Use the formal definition of the derivative as a limit, to show that

$$f'(x) = -\frac{x}{\left(x^2 - 1\right)^{\frac{3}{2}}}.$$

$\boxed{\text{proof}}$

$$\lim_{h \to 0}\left[\frac{f(x+h) - f(x)}{h}\right] = \lim_{h \to 0}\left[\frac{\frac{1}{\sqrt{(x+h)^2-1}} - \frac{1}{\sqrt{x^2-1}}}{h}\right]$$

MULTIPLY TOP & BOTTOM BY $\sqrt{x^2-1}\sqrt{(x+h)^2-1}$

$$= \lim_{h \to 0}\left[\frac{\sqrt{x^2-1} - \sqrt{(x+h)^2-1}}{h\sqrt{x^2-1}\sqrt{(x+h)^2-1}}\right]$$

SETTING $h \to 0$ AT THIS STAGE PRODUCES $\frac{0}{0}$, SO WE NEED TO REMOVE THE ZERO FROM THE TOP BY SURD CONJUGATION

$$= \lim_{h \to 0}\left[\frac{\left[\sqrt{x^2-1} - \sqrt{(x+h)^2-1}\right]\left[\sqrt{x^2-1} + \sqrt{(x+h)^2-1}\right]}{h\sqrt{x^2-1}\sqrt{(x+h)^2-1}\left[\sqrt{x^2-1} + \sqrt{(x+h)^2-1}\right]}\right]$$

SIMPLIFY THE TOP (DIFFERENCE OF SQUARES)

$$= \lim_{h \to 0}\left[\frac{(x^2-1) - \left[(x+h)^2-1\right]}{h\sqrt{x^2-1}\sqrt{(x+h)^2-1}\left[\sqrt{x^2-1} + \sqrt{(x+h)^2-1}\right]}\right]$$

$$= \lim_{h \to 0}\left[\frac{(\cancel{x^2}\cancel{-1}) - (\cancel{x^2} + 2xh + h^2 \cancel{-1})}{h\sqrt{x^2-1}\sqrt{(x+h)^2-1}\left[\sqrt{x^2-1} + \sqrt{(x+h)^2-1}\right]}\right]$$

$$= \lim_{h \to 0}\left[\frac{-2xh + h^2}{h\sqrt{x^2-1}\sqrt{(x+h)^2-1}\left[\sqrt{x^2-1} + \sqrt{(x+h)^2-1}\right]}\right].$$

$$= \lim_{h \to 0}\left[\frac{-2x + h}{\sqrt{x^2-1}\sqrt{(x+h)^2-1}\left[\sqrt{x^2-1} + \sqrt{(x+h)^2-1}\right]}\right]$$

$$= \frac{-2x}{\sqrt{x^2-1}\sqrt{x^2-1}\left[\sqrt{x^2-1} + \sqrt{x^2-1}\right]}$$

$$= \frac{-2x}{(x^2-1) \times 2(x^2-1)^{\frac{1}{2}}}$$

$$= \frac{-x}{(x^2-1)^{3/2}}$$

Question 26 (****+)

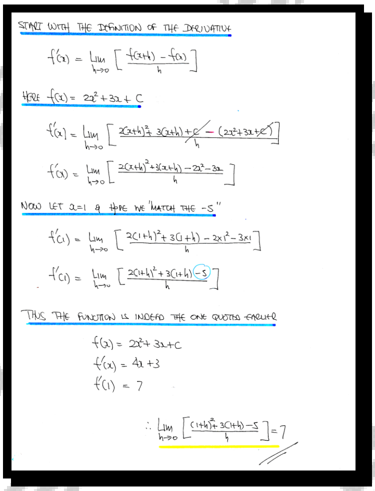
The limit expression shown below represents a student's evaluation for $f'(x)$, for a specific value of x.

$$\lim_{h\to 0}\left[\frac{2(1+h)^2+3(1+h)-5}{h}\right].$$

Determine an expression for $f(x)$ and once obtained, **differentiate it directly** to find the value of $f'(x)$, for the specific value of x the student was evaluating.

No credit will be given for evaluating the limit directly.

$\boxed{\quad}$, $\boxed{f(x)=2x^2+3x}$, $\boxed{f'(1)=7}$


START WITH THE DEFINITION OF THE DERIVATIVE

$$f'(x)=\lim_{h\to 0}\left[\frac{f(x+h)-f(x)}{h}\right]$$

HERE $f(x)=2x^2+3x+C$

$$f'(x)=\lim_{h\to 0}\left[\frac{2(x+h)^2+3(x+h)+\cancel{C}-(2x^2+3x+\cancel{C})}{h}\right]$$

$$f'(x)=\lim_{h\to 0}\left[\frac{2(x+h)^2+3(x+h)-2x^2-3x}{h}\right]$$

NOW LET $x=1$ & HOPE WE "MATCH THE -5"

$$f'(1)=\lim_{h\to 0}\left[\frac{2(1+h)^2+3(1+h)-2\times1^2-3\times1}{h}\right]$$

$$f'(1)=\lim_{h\to 0}\left[\frac{2(1+h)^2+3(1+h)-5}{h}\right]$$

THUS THE FUNCTION IS INDEED THE ONE QUOTED EARLIER

$f(x)=2x^2+3x+C$

$f'(x)=4x+3$

$f'(1)=7$

$$\therefore \lim_{h\to 0}\left[\frac{(1+h)^2+3(1+h)-5}{h}\right]=7$$

Question 27 (*****)

Use the formal definition of the derivative to prove that if

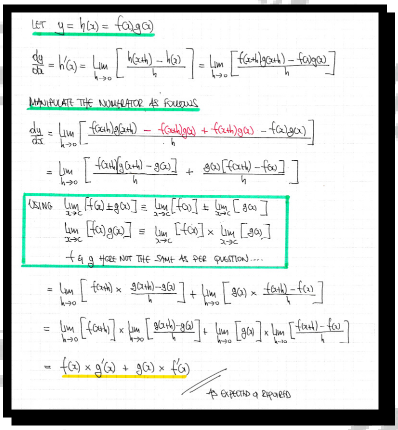
$$y = f(x)\, g(x),$$

then $\dfrac{dy}{dx} = f'(x)\, g(x) + f(x)\, g'(x)$

You may assume that

- $\lim\limits_{x \to c}\left[f(x) + g(x)\right] = \lim\limits_{x \to c}\left[f(x)\right] + \lim\limits_{x \to c}\left[g(x)\right]$
- $\lim\limits_{x \to c}\left[f(x) \times g(x)\right] = \lim\limits_{x \to c}\left[f(x)\right] \times \lim\limits_{x \to c}\left[g(x)\right]$

V, , proof



LET $y = h(x) = f(x)g(x)$

$$\frac{dy}{dx} = h'(x) = \lim_{h \to 0}\left[\frac{h(x+h) - h(x)}{h}\right] = \lim_{h \to 0}\left[\frac{f(x+h)g(x+h) - f(x)g(x)}{h}\right]$$

MANIPULATE THE NUMERATOR AS FOLLOWS

$$\frac{dy}{dx} = \lim_{h \to 0}\left[\frac{f(x+h)g(x+h) - f(x+h)g(x) + f(x+h)g(x) - f(x)g(x)}{h}\right]$$

$$= \lim_{h \to 0}\left[\frac{f(x+h)[g(x+h) - g(x)]}{h} + \frac{g(x)[f(x+h) - f(x)]}{h}\right]$$

USING $\lim\limits_{x \to c}[f(x) \pm g(x)] \equiv \lim\limits_{x \to c}[f(x)] \pm \lim\limits_{x \to c}[g(x)]$

$\lim\limits_{x \to c}[f(x)g(x)] \equiv \lim\limits_{x \to c}[f(x)] \times \lim\limits_{x \to c}[g(x)]$

f & g HERE NOT THE SAME AS PER QUESTION....

$$= \lim_{h \to 0}\left[f(x+h) \times \frac{g(x+h) - g(x)}{h}\right] + \lim_{h \to 0}\left[g(x) \times \frac{f(x+h) - f(x)}{h}\right]$$

$$= \lim_{h \to 0}\left[f(x+h)\right] \times \lim_{h \to 0}\left[\frac{g(x+h) - g(x)}{h}\right] + \lim_{h \to 0}\left[g(x)\right] \times \lim_{h \to 0}\left[\frac{f(x+h) - f(x)}{h}\right]$$

$$= f(x) \times g'(x) + g(x) \times f'(x)$$

AS EXPECTED & REQUIRED

Question 28 (*****)

$$f(x) = \sqrt{\frac{1-x}{1+x}}, \quad x \in \mathbb{R}, \quad |x| < 1.$$

Use the formal definition of the derivative as a limit, to show that

$$f'(x) = -\frac{1}{(1+x)\sqrt{1-x^2}}.$$

$\boxed{}$, $\boxed{\text{proof}}$

MANIPULATE STRAIGHT AWAY

$$f(x) = \sqrt{\frac{1-x}{1+x}} = \frac{\sqrt{1-x}}{\sqrt{1+x}} = \frac{\sqrt{1-x}\sqrt{1+x}}{\sqrt{1+x}\sqrt{1+x}} = \frac{\sqrt{1-x^2}}{1+x}$$

NOW BY THE FORMAL DEFINITION OF A LIMIT

$$f'(x) = \lim_{h \to 0}\left[\frac{f(x+h) - f(x)}{h}\right]$$

$$f'(x) = \lim_{h \to 0}\left[\frac{1}{h}\left[\frac{\sqrt{1-(x+h)^2}}{1+(x+h)} - \frac{\sqrt{1-x^2}}{1+x}\right]\right]$$

$$f'(x) = \lim_{h \to 0}\left[\frac{1}{h}\left[\frac{(1+x)\sqrt{1-(x+h)^2} - (1+x+h)\sqrt{1-x^2}}{(1+x+h)(1+x)}\right]\right]$$

"RATIONALIZE" THE NUMERATOR AS THE LIMIT IS OF THE FORM "ZERO OVER ZERO"

MULTIPLY "TOP & BOTTOM" BY $(1+x)\sqrt{1-(x+h)^2} + (1+x+h)\sqrt{1-x^2}$

$$f'(x) = \lim_{h \to 0}\left[\frac{1}{h}\left[\frac{(1+x)^2\left[1-(x+h)^2\right] - (1+x+h)^2(1-x^2)}{(1+x+h)(1+x)\left[(1+x)\sqrt{1-(x+h)^2} + (1+x+h)\sqrt{1-x^2}\right]}\right]\right]$$

DIFFERENCE OF SQUARES

$$f'(x) = \lim_{h \to 0}\left[\frac{1}{h}\left[\frac{(1+x)^2(1-x-h)(1+x+h) - (1+x+h)^2(1-x)(1+x)}{(1+x+h)(1+x)\left[(1+x)\sqrt{1-(x+h)^2} + (1+x+h)\sqrt{1-x^2}\right]}\right]\right]$$

FACTORIZE & TIDY IN THE NUMERATOR

$$f'(x) = \lim_{h \to 0}\left[\frac{1}{h}\left[\frac{(1+x)(1+x+h)\left[(1+x)(1-x-h) - (1-x)(1+x+h)\right]}{(1+x+h)(1+x)\left[(1+x)\sqrt{1-(x+h)^2} + (1+x+h)\sqrt{1-x^2}\right]}\right]\right]$$

$$(1+x)(1-x-h) - (1-x)(1+x+h)$$

$$= (1+x)(1-x-h) + (x-1)(1+x+h)$$

$$\begin{aligned} = \; & \cancel{1} - \cancel{x} - h \\ & \cancel{x} \quad - \cancel{x^2} - \cancel{xh} \\ & \cancel{x} \quad + \cancel{x^2} + \cancel{xh} \\ & -\cancel{1} - \cancel{x} - h \end{aligned}$$

$$= -2h$$

RETURNING TO THE LIMIT

$$f'(x) = \lim_{h \to 0}\left[\frac{1}{\cancel{h}}\left[\frac{-2\cancel{h}(1+x)(1+x+h)}{(1+x)(1+x+h)\left[(1+x)\sqrt{1-(x+h)^2} + (1+x+h)\sqrt{1-x^2}\right]}\right]\right]$$

NOW THE LIMIT IS NO LONGER ZERO OVER ZERO BECAUSE h CANCELS

$$f'(x) = \lim_{h \to 0}\left[\frac{-2(1+x)(1+x+h)}{(1+x)(1+x+h)\left[(1+x)\sqrt{1-(x+h)^2} + (1+x+h)\sqrt{1-x^2}\right]}\right]$$

$$f'(x) = \frac{-2\cancel{(1+x)}\cancel{(1+x)}}{\cancel{(1+x)}\cancel{(1+x)}\left[(1+x)\sqrt{1-x^2} + (1+x)\sqrt{1-x^2}\right]}$$

$$f'(x) = \frac{-2}{2(1+x)\sqrt{1-x^2}}$$

$$f'(x) = -\frac{1}{(1+x)\sqrt{1-x^2}}$$

// AS REQUIRED

Question 29 (*****)

Use the formal definition of the derivative of a suitable expression, to find the value for the following limit

$$\lim_{x\to 4}\left[\frac{\sqrt{x^3}+2\sqrt{x}-12}{x-4}\right].$$

No credit will be given for using L'Hospital's rule.

$\boxed{}$, $\boxed{\frac{7}{2}}$

START BY THE DEFINITION OF THE DERIVATIVE

$$f(x) = \lim_{h\to 0}\left[\frac{f(x+h)-f(x)}{h}\right]$$

NOW CONSIDER THE LIMIT GIVEN

$$\lim_{x\to 4}\left[\frac{x^{\frac{3}{2}}+2x^{\frac{1}{2}}-12}{x-4}\right]$$

$$= \lim_{h\to 0}\left[\frac{(4+h)^{\frac{3}{2}}+2(4+h)^{\frac{1}{2}}-12}{(4+h)-4}\right]$$

$h = x-4$
$x = 4+h$

$$= \lim_{h\to 0}\left[\frac{(4+h)^{\frac{3}{2}}+2(4+h)^{\frac{1}{2}}-12}{h}\right]$$

THIS COULD BE THE DERIVATIVE OF $f(x) = x^{3/2}+2x^{\frac{1}{2}}+C$ EVALUATED AT $x=4$, SO LONG AS THE "-12 MATCHES"

$$f'(x) = \lim_{h\to 0}\left[\frac{(x+h)^{\frac{3}{2}}+2(x+h)^{\frac{1}{2}}\cancel{+c}-\left[x^{3/2}+2x^{\frac{1}{2}}\cancel{+c}\right]}{h}\right]$$

$$f'(4) = \lim_{h\to 0}\left[\frac{(4+h)^{\frac{3}{2}}+2(4+h)^{\frac{1}{2}}-4^{3/2}-2\times 4^{1/2}}{h}\right]$$

$-4^{\frac{3}{2}}-2\times 4^{\frac{1}{2}}$
$= -8-2\times 2$
$= -12$

$$f'(4) = \lim_{h\to 0}\left[\frac{(4+h)^{3/2}+2(4+h)^{\frac{1}{2}}-12}{h}\right]$$

INDEED THIS IS $\frac{d}{dx}\left(x^{3/2}+2x^{1/2}+c\right)\Big|_{x=4}$

$$\therefore \lim_{x\to 4}\left[\frac{\sqrt{x^3}+2\sqrt{x}-12}{x-4}\right] = \left[\frac{3}{2}x^{\frac{1}{2}}+x^{-\frac{1}{2}}\right]_{x=4}$$

$$= \frac{3}{\cancel{2}}\times\cancel{2}+\frac{1}{\sqrt{4}} = \frac{7}{2}$$