NUMBER THEORY

Question 1 (**)

Use Euclid's algorithm to find the Highest common factor of 560 and 1169.

$\boxed{}$, $\boxed{7}$

SETTING UP EUCLID'S ALGORITHM FOR 560 & 1169

$$1169 = 2 \times 560 + 49$$
$$560 = 11 \times 49 + 21$$
$$49 = 2 \times 21 + 7 \quad \leftarrow \text{HCF}$$
$$21 = 3 \times 7 + 0$$

Question 2 (**)

$$f(n) = n^2 + n + 2, \; n \in \mathbb{N}.$$

Show that $f(n)$ is always even.

$\boxed{}$, $\boxed{\text{proof}}$

$n^2+n+2 = n(k+1)+2$

IF n IS EVEN $n(n+1)$ IS EVEN
$n(n+1)+2$ IS ALSO EVEN

IF n IS ODD
$n+1$ IS EVEN $n(n+1)$ IS EVEN
$n(n+1)+2$ IS EVEN

$\therefore$ n^2+n+2 IS EVEN FOR ALL $n \in \mathbb{N}$

Question 3 (**)

Prove that when the square of a positive odd integer is divided by 4 the remainder is always 1.

$\boxed{}$, $\boxed{\text{proof}}$

LET THE ODD NUMBER BE $2n+1$

$$(2n+1)^2 = n^2+4n+1$$
$$= 4(n^2+n)+1$$

I.E A MULTIPLE OF 4 WITH REMAINDER 1

Question 4 (**)

Use Euclid's algorithm to find the Highest common factor of 3059 and 7728.

$\boxed{}$, $\boxed{161}$

SETTING EUCLID'S ALGORITHM FOR 3059 & 7728

$7728 = 2 \times 3059 + 1610$

$3059 = 1 \times 1610 + 1449$

$1610 = 1 \times 1449 + 161$

$1449 = 9 \times 161 + 0$

THE H.C.F. OF 3059 & 7728 IS 161

Question 5 (**)

Prove that the square of a positive integer can never be of the form $3k+2$, $k \in \mathbb{N}$.

$\boxed{}$, $\boxed{\text{proof}}$

PROOF BY EXHAUSTION

"THE SQUARE OF ANY INTEGER CAN NEVER BE OF THE FORM $3k+2$, $k \in \mathbb{N}$"

THE NUMBER TO BE SQUARED, SAY a, CAN TAKE ONE OF THE FOLLOWING 3 FORMS

$a = 3m$, $a = 3m+1$, $a = 3m+2$, $m \in \mathbb{N}$

- IF $a = 3m \Rightarrow a^2 = 9m^2 = 3(3m^2) = 3k$, $k \in \mathbb{N}$
- IF $a = 3m+1 \Rightarrow a^2 = 9m^2+6m+1 = 3(3m^2+2m)+1 = 3k+1$, $k \in \mathbb{N}$
- IF $a = 3m+2 \Rightarrow a^2 = 9m^2+12m+4 = 3(3m^2+4m+1)+1 = 3k+1$, $k \in \mathbb{N}$

$\therefore$ SQUARING ANY INTEGER ONLY PRODUCES INTEGERS OF THE FORM $3k$ OR $3k+1$, $k \in \mathbb{N}$

$\therefore$ IT IS NOT POSSIBLE TO HAVE A SQUARE NUMBER OF THE FORM $3k+2$, $k \in \mathbb{N}$

Question 6 (**)

Show that $a^3 - a + 1$ is odd for all positive integer values of a.

$\boxed{}$, $\boxed{\text{proof}}$

METHOD A

- $a^3 - a + 1 = a(a^2-1) + 1 = a(a+1)(a-1) + 1$
- AS $a(a+1)(a-1)$ CONTAINS CONSECUTIVE INTEGERS, AT LEAST ONE OF THEM WILL BE EVEN, SO $a(a+1)(a-1)$ WILL BE EVEN FOR ALL $a \in \mathbb{N}$
- HENCE $a(a+1)(a-1)+1$ WILL BE ODD FOR ALL $a \in \mathbb{N}$

METHOD B (BY EXHAUSTION)

- LET a BE EVEN, $a = 2n$

$$(2n)^3 - 2n + 1 = 8n^3 - 2n + 1 = 2(4n^3 - n) + 1$$
$$= 2m + 1$$
$$\therefore \text{ODD}$$

- LET a BE ODD, $a = 2n+1$

$$(2n+1)^3 - (2n+1) + 1 = 8n^3 + 12n^2 + 6n + 1 - 2n - 1 + 1$$
$$= 8n^3 + 12n^2 + 4n + 1$$
$$= 2[4n^3 + 6n^2 + n] + 1$$
$$= 2m + 1$$
$$\therefore \text{ODD}$$

HENCE $a^3 - a + 1$ IS ODD FOR $a \in \mathbb{N}$

Question 7 **(**+)**

When a, $a \in \mathbb{N}$, is divided by b, $b \in \mathbb{N}$, the quotient is 20 and the remainder is 17.

a) Find the remainder when a is divided by 5.

Suppose that when a positive integer is divided by 8 the remainder is 6, and when the same positive integer is divided by 18 the remainder is 3.

b) Determine whether the positive integer of part **(b)** exists.

$\boxed{\text{FP4}}$, $\boxed{2}$, $\boxed{\text{no such integer}}$

a) • $\frac{a}{b} = 20 + \frac{17}{b}$ OR $a = 20b + 17$, $a, b \in \mathbb{N}$

THEN

• $\frac{a}{5} = \frac{20b}{5} + \frac{17}{5}$ OR $a = 20b + 15 + 2$

$\frac{a}{5} = 4b + 3 + \frac{2}{5}$ $\quad a = 5(4b) + (5 \times 3) + 2$

$\frac{a}{5} = (4b + 3) + \frac{2}{5}$ $\quad a = 5(4b + 3) + 2$

$\therefore$ THE REMAINDER IS 2

b) SUPPOSE THERE EXIST AN INTEGER a, SO THAT...

• ITS DIVISION BY 8, YIELDS REMAINDER 6

$\Rightarrow a = 8n + 6$, $n \in \mathbb{N}$

$\Rightarrow a = 2[4n + 3]$

$\therefore$ a IS EVEN

• ITS DIVISION BY 18, YIELDS REMAINDER 3

$\Rightarrow a = 18m + 3$, $m \in \mathbb{N}$

$\Rightarrow a = 18m + 2 + 1$

$\Rightarrow a = 2(9m + 1) + 1$

$\therefore$ a IS ODD

HENCE THERE IS NO SUCH INTEGER

Question 8 (**+)

$$f(n) \equiv n^2 + 4n + 3, \; n \in \mathbb{N}.$$

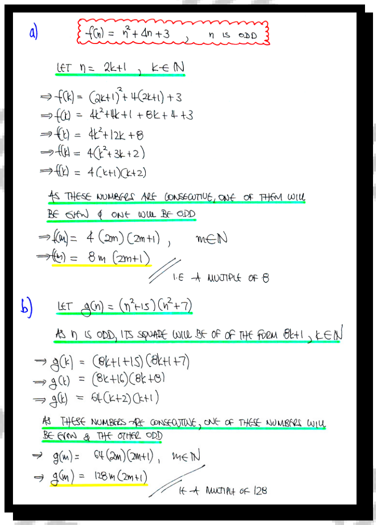
a) Given that n is odd show that $f(n)$ is a multiple of 8.

$$g(n) \equiv \left(n^2 + 15\right)\left(n^2 + 7\right), \; n \in \mathbb{N}.$$

b) Given that n is odd show that $g(n)$ is a multiple of 128.

You may assume that the square of an odd integer is of the form $8k + 1$, $k \in \mathbb{N}$.

$\boxed{\text{proof}}$



a) $f(n) = n^2 + 4n + 3$, n IS ODD

LET $n = 2k+1$, $k \in \mathbb{N}$

$\Rightarrow f(k) = (2k+1)^2 + 4(2k+1) + 3$
$\Rightarrow f(k) = 4k^2 + 4k + 1 + 8k + 4 + 3$
$\Rightarrow f(k) = 4k^2 + 12k + 8$
$\Rightarrow f(k) = 4(k^2 + 3k + 2)$
$\Rightarrow f(k) = 4(k+1)(k+2)$

AS THESE NUMBERS ARE CONSECUTIVE, ONE OF THEM WILL BE EVEN & ONE WILL BE ODD

$\Rightarrow f(m) = 4(2m)(2m+1)$, $m \in \mathbb{N}$
$\Rightarrow f(m) = 8m(2m+1)$

I.E A MULTIPLE OF 8

b) LET $g(n) = (n^2+15)(n^2+7)$

AS n IS ODD, ITS SQUARE WILL BE OF OF THE FORM $8k+1$, $k \in \mathbb{N}$

$\Rightarrow g(k) = (8k+1+15)(8k+1+7)$
$\Rightarrow g(k) = (8k+16)(8k+8)$
$\Rightarrow g(k) = 64(k+2)(k+1)$

AS THESE NUMBERS ARE CONSECUTIVE, ONE OF THESE NUMBERS WILL BE EVEN & THE OTHER ODD

$\Rightarrow g(m) = 64(2m)(2m+1)$, $m \in \mathbb{N}$
$\Rightarrow g(m) = 128m(2m+1)$

IE A MULTIPLE OF 128

Question 9 (***)

$$f(n) = 5^{2n} - 1, \; n \in \mathbb{N}.$$

Without using proof by induction, show that $f(n)$ is a multiple of 8.

proof

$5^{2n} - 1 = (5^n)^2 - 1 = (5^n - 1)(5^n + 1)$

- 5^n WILL BE ODD
- $5^n - 1$ & $5^n + 1$ ARE TWO CONSECUTIVE EVEN NUMBERS
- ONE OF THEM WILL A MULTIPLE OF 4
- WITHOUT LOSS OF GENERALITY SAY $5^n - 1 = 2m$ & $5^n + 1 = 4k$, $m \in \mathbb{N}$, $k \in \mathbb{N}$

$\therefore 5^{2n} - 1 = 2m \times 4k = 8mk$ IE MULTIPLE OF 8

Question 10 (***)

Bernoulli's inequality asserts that if $a \in \mathbb{R}$, $a > -1$ and $n \in \mathbb{N}$, $n \geq 2$, then

$$(1+a)^n > 1 + an.$$

Prove, by induction, the validity of Bernoulli's identity.

proof

BERNOULLI INEQUALITY

$(1+a)^n > 1 + an$ $\quad a \in \mathbb{R}, a > -1$, $n \in \mathbb{N}, n \geq 2$

PROOF BY INDUCTION

- IF $n = 2$ $\quad$ LHS $= (1+a)^2 = a^2 + 2a + 1$
  RHS $= 1 + 2a$
  $\therefore a^2 + 2a + 1 > 2a + 1$, SO THE RESULT HOLDS FOR $n = 2$
- SUPPOSE THAT THE INEQUALITY HOLDS FOR $n = k \in \mathbb{N}$, $k > 2$

$\Rightarrow (1+a)^k > 1 + ak$

$\Rightarrow (1+a)^k(1+a) > (1+a)(1+ak)$

$\Rightarrow (1+a)^{k+1} > 1 + ak + a + a^2k$

$\Rightarrow (1+a)^{k+1} > 1 + a(k+1) + \underbrace{a^2k}_{\text{(POSITIVE)}} > 1 + a(k+1)$

$\Rightarrow (1+a)^{k+1} > 1 + a(k+1)$

- IF THE INEQUALITY HOLDS FOR $n = k \in \mathbb{N}$, $k > 2$, THEN IT WILL ALSO HOLD FOR $n = k+1$.

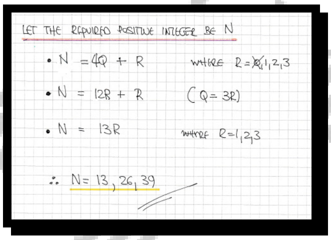
AS THE INEQUALITY HOLDS FOR $n = 2$, THEN IT MUST HOLD FOR ALL POSITIVE INTEGERS GREATER THAN 2

Question 11 (***)

When some positive integer N is divided by 4, the quotient is 3 times as large as the remainder.

Determine the possible values of N.

$\boxed{}$, $\boxed{N = \{13, 26, 39\}}$



LET THE REQUIRED POSITIVE INTEGER BE N

- $N = 4Q + R$ WHERE $R = \cancel{0}, 1, 2, 3$
- $N = 12R + R$ $(Q = 3R)$
- $N = 13R$ WHERE $R = 1, 2, 3$

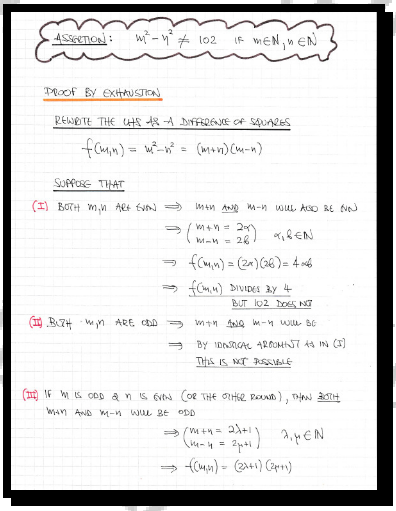
$\therefore N = 13, 26, 39$

Question 12 (***+)

Use **proof by exhaustion** to show that if $m \in \mathbb{N}$ and $n \in \mathbb{N}$, then

$$m^2 - n^2 \neq 102.$$

$\boxed{}$, $\boxed{\text{proof}}$



ASSERTION: $m^2 - n^2 \neq 102$ IF $m \in \mathbb{N}, n \in \mathbb{N}$

PROOF BY EXHAUSTION

REWRITE THE LHS AS A DIFFERENCE OF SQUARES

$$f(m,n) = m^2 - n^2 = (m+n)(m-n)$$

SUPPOSE THAT

(I) BOTH m, n ARE EVEN $\Rightarrow$ $m+n$ AND $m-n$ WILL ALSO BE EVEN

$\Rightarrow \begin{pmatrix} m+n = 2\alpha \\ m-n = 2\beta \end{pmatrix}$ $\alpha, \beta \in \mathbb{N}$

$\Rightarrow f(m,n) = (2\alpha)(2\beta) = 4\alpha\beta$

$\Rightarrow$ $f(m,n)$ DIVIDES BY 4 BUT 102 DOES NOT

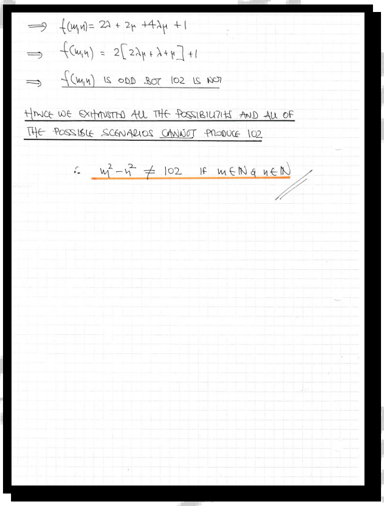
(II) BOTH m, n ARE ODD $\Rightarrow$ $m+n$ AND $m-n$ WILL BE

$\Rightarrow$ BY IDENTICAL ARGUMENT AS IN (I) THIS IS NOT POSSIBLE

(III) IF m IS ODD & n IS EVEN (OR THE OTHER ROUND), THEN BOTH $m+n$ AND $m-n$ WILL BE ODD

$\Rightarrow \begin{pmatrix} m+n = 2\lambda+1 \\ m-n = 2\mu+1 \end{pmatrix}$ $\lambda, \mu \in \mathbb{N}$

$\Rightarrow f(m,n) = (2\lambda+1)(2\mu+1)$



$\Rightarrow f(m,n) = 2\lambda + 2\mu + 4\lambda\mu + 1$

$\Rightarrow f(m,n) = 2[2\lambda\mu + \lambda + \mu] + 1$

$\Rightarrow$ $f(m,n)$ IS ODD BUT 102 IS NOT

HENCE WE EXHAUSTED ALL THE POSSIBILITIES AND ALL OF THE POSSIBLE SCENARIOS CANNOT PRODUCE 102

$\therefore m^2 - n^2 \neq 102$ IF $m \in \mathbb{N}$ & $n \in \mathbb{N}$

Question 13 (**+)

It is given that for $a \in \mathbb{N}$, $b \in \mathbb{N}$, $c \in \mathbb{N}$,

$$a^2 + b^2 + c^2 = 116 .$$

a) Prove that a, b and c are all even.

You may assume that the square of an odd integer is of the form $8k+1$, $k \in \mathbb{N}$.

b) Determine the values of a, b and c.

$\boxed{a = 8, \quad b = 6, \quad c = 4 \quad \text{in any order}}$

$a^2 + b^2 + c^2 = 116 \qquad a, b, c \in \mathbb{N}$

a) BREAK INTO CASES & PROVE BY EXHAUSTION

SUPPOSE ALL THREE ARE ODD

$\Rightarrow (2n+1)^2 + (2m+1)^2 + (2p+1)^2 = 116$

$\Rightarrow 4n^2 + 4n + 1 + 4m^2 + 4m + 1 + 4p^2 + 4p + 1 = 116$

$\Rightarrow 4n^2 + 4m^2 + 4p^2 + 4m + 4n + 4p + 3 = 116$

$\Rightarrow 2(2n^2 + 2m^2 + 2p^2 + 2m + 2n + 2p + 1) = 115$

WHICH IS A CONTRADICTION

SUPPOSE THAT TWO ARE ODD & ONE IS EVEN

$\Rightarrow (2n+1)^2 + (2m+1)^2 + (2p)^2 = 116$

$\Rightarrow 4n^2 + 4n + 1 + 4m^2 + 4m + 1 + 4p^2 = 116$

$\Rightarrow 4n^2 + 4m^2 + 4p^2 + 4n + 4m = 114$

$\Rightarrow 4(n^2 + m^2 + p^2 + n + m) = 114$

$\Rightarrow 2(n^2 + m^2 + p^2 + n + m) = 57$

WHICH IS A CONTRADICTION

SUPPOSE THAT ONE IS ODD & TWO ARE EVEN

$\Rightarrow (2n+1)^2 + (2m)^2 + (2p)^2 = 116$

$\Rightarrow 4n^2 + 4n + 1 + 4m^2 + 4p^2 = 116$

$\Rightarrow 4n^2 + 4m^2 + 4n + 4p^2 = 115$

$\Rightarrow 4(n^2 + m^2 + n + p^2) = 115$

WHICH IS A CONTRADICTION

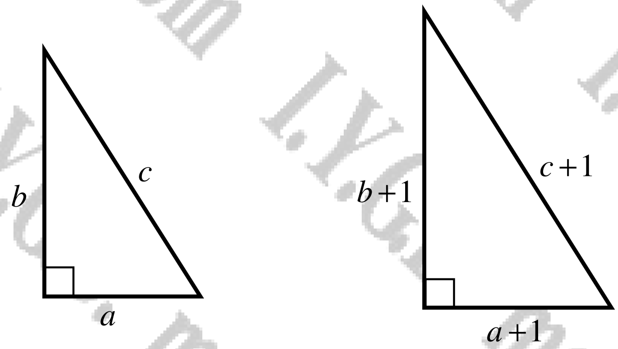
HENCE BY EXHAUSTION THE THREE NUMBERS (IT IS GIVEN THAT THEY EXIST) MUST ALL BE EVEN

b) BY QUICK INSPECTION NOTING THAT $\sqrt{121} = 11$ & $\sqrt{100} = 10$ THE LARGEST OF THE NUMBER CANNOT EXCEED 10

1ST	2ND	3RD	← ANY ORDER
10^2	4^2	2^2	too high (120)
10^2	2^2	2^2	too low (108)
8^2	8^2	2^2	too high (132)
8^2	6^2	6^2	too high (136)
8^2	6	4^2	"IT WORKS"

$\therefore$ $a = 8$, $b = 6$, $c = 4$ IN ANY ORDER

Question 14 (***+)



The figure above shows two right angled triangles.

- The triangle, on the left section of the figure, has side lengths of

  $$a, b \text{ and } c,$$

  where c is the length of its hypotenuse.

- The triangle, on the right section of the figure, has side lengths of

  $$a+1, \quad b+1 \quad \text{and} \quad c+1,$$

  where $c+1$ is the length of its hypotenuse.

Show that a, b and c cannot all be integers.

$\boxed{}$, $\boxed{\text{proof}}$

BY PYTHAGORAS ON THE TRIANGLE ON THE "LEFT"

$\Rightarrow a^2 + b^2 = c^2$

$\Rightarrow a^2 + b^2 - c^2 = 0$

BY PYTHAGORAS ON THE TRIANGLE ON THE "RIGHT"

$\Rightarrow (a+1)^2 + (b+1)^2 = (c+1)^2$

$\Rightarrow a^2 + 2a + 1 + b^2 + 2b \cancel{+1} = c^2 + 2c \cancel{+1}$

$\Rightarrow (a^2 + b^2 - c^2) + 2a + 2b + 1 = 2c$

$\Rightarrow 0 + 2(a+b) + 1 = 2c$

$\Rightarrow 2(a+b) + 1 = 2c$

L.H.S. WILL BE ODD IF a & b ARE BOTH INTEGERS

R.H.S WILL BE EVEN IF c IS AN INTEGER

HENCE NOT ALL OF a, b & c ARE INTEGERS

Question 15 (***+)

When 165 is divided by some integer the quotient is 7 and the remainder is R.

Determine the possible values of R.

$\boxed{}$, $\boxed{R = \{4, 11, 18\}}$

- LET THE REQUIRED POSITIVE INTEGER BE N

$\Rightarrow \quad 165 = \underbrace{7}_{\text{QUOTIENT}}N + R \qquad 0 \leq R < N$

$\Rightarrow \quad R = 165 - 7N$

$\Rightarrow \quad 0 \leq 165 - 7N < N$

$\Rightarrow \quad 7N \leq 165 < 8N$

- SOLVING EACH INEQUALITY

  - $7N \leq 165$, $N \leq 23\frac{4}{7}$
  - $8N > 165$, $N > 20\frac{5}{8}$

HENCE $N = 21, 22, 23$

- DETERMINE THE REMAINDER IN EACH CASE, USING $R = 165 - 7N$

  - $R_{21} = 165 - 7 \times 21 = 18$
  - $R_{22} = 165 - 7 \times 22 = 11$
  - $R_{23} = 165 - 7 \times 23 = 4$

Question 16 (***+)

It is given that a and b are positive integers, with $a > b$.

Use **proof by contradiction** to show that if $a+b$ is a multiple of 4, then $a-b$ **cannot** be a multiple of 4.

proof

IF a & b ARE ODD INTEGERS SUCH THAT 4 IS A FACTOR OF $a-b$, THEN 4 IS NOT A FACTOR OF $a+b$

- SUPPOSE THAT $a+b$ IS A MULTIPLE OF 4
- THEN $a+b = 4n$, FOR $n \in \mathbb{N}$
- AS $a-b$ IS ALSO A MULTIPLE OF 4, THEN $a-b = 4m$, FOR $m \in \mathbb{N}$
- ADDING THE TWO EXPRESSIONS WE OBTAIN

$$\left.\begin{array}{l} a+b = 4n \\ a-b = 4m \end{array}\right\} \Rightarrow \begin{array}{l} 2a = 4n+4m \\ a = 2n+2m \\ a = 2(n+m) \end{array}$$

- THIS IS A CONTRADICTION TO THE ASSERTION THAT a IS ODD

$\therefore$ THUS THE STATEMENT IS CORRECT

Question 17 (***+)

When a positive integer N is divided by 4 the remainder is 3.

When N is divided by 5 the remainder is 2.

Show that the remainder of the division of N by 20 is 5.

$\boxed{\text{proof}}$

PROCEED AS FOLLOWS

LET $N = 4n+3$ & $N = 5m+2$, $n \in \mathbb{N}$, $m \in \mathbb{N}$

$$\Rightarrow \begin{cases} 5N = 20n + 15 \\ 4N = 20m + 10 \end{cases}$$

SUBTRACTING WE OBTAIN

$$\Rightarrow N = 20(n-m) + 25$$

$$\Rightarrow N = 20(n-m) + 20 + 5$$

$$\Rightarrow N = 20(n-m+1) + 5$$

$\therefore$ IT LEAVES REMAINDER 5

Question 18 (***+)

a) Show that $9^{40} + 3^{40} + 6$ is a multiple of 8.

b) Show further that $3^{40} + 2$ divides $9^{40} + 3^{40} - 2$.

proof

a) REWRITE THE EXPRESSION AS FOLLOWS

$$9^{40} + 3^{40} + 6 = 9^{40} + (3^2)^{20} + 8 - 2$$
$$= (9^{40} - 1) + (9^{20} - 1) + 8$$
$$= (9-1)(9^{39} + 9^{38} + 9^{37} + \dots + 9 + 1) + (9-1)(9^{19} + 9^{18} + 9^{17} + \dots + 9 + 1) + 8$$
$$= 8(9^{39} + 9^{38} + 9^{37} + \dots + 9 + 1) + 8(9^{19} + 9^{18} + 9^{17} + \dots + 9 + 1) + 8$$
$$= 8\left[(9^{39} + 9^{38} + 9^{37} + \dots + 9 + 1) + (9^{19} + 9^{18} + 9^{17} + \dots + 9 + 1) + 1\right]$$
$$= 8m$$

AS REQUIRED

b) IF $3^{40} + 2$ DIVIDES $9^{40} + 3^{40} - 2$ THEN

$$\frac{9^{40} + 3^{40} - 2}{3^{40} + 2} = \frac{(3^{40})^2 + (3^{40}) - 2}{(3^{40}) + 2}$$
$$= \frac{(3^{40} - 1)(3^{40} + 2)}{3^{40} + 2}$$
$$= 3^{40} - 1$$

INDEED $3^{40} + 2$ DIVIDES $9^{40} + 3^{40} - 2$

Question 19 (***+)

Suppose that when a positive integer is divided by 6 the remainder is 4, and when the same positive integer is divided by 12 the remainder is 8.

a) Determine whether such positive exists.

Suppose next that when a positive integer is divided by 6, the quotient is q and the remainder is 1. When the square of the same positive integer is divided by q, the quotient is 984 and the remainder is 1.

b) Determine whether the positive integer of part **(b)** exists.

$\boxed{}$, $\boxed{\text{no such integer}}$, $\boxed{163}$

a) SUPPOSE THAT THERE EXIST POSITIVE INTEGER a SUCH THAT

- ITS DIVISION BY 6 YIELDS A REMAINDER OF 4

$a = 6n + 4, \; n \in \mathbb{N}$

$2a = 12n + 8$

- ITS DIVISION BY 12 GIVES REMAINDER 8

$a = 12m + 8, \; m \in \mathbb{N}$

$2a = 12n + 8$

$a = 12m + 8$

$a = 12n - 12m$ ← SUBTRACTING

$a = 12(n - m)$

I.E a IS DIVISIBLE BY 12, WHICH IS A CONTRADICTION TO THE SECOND STATEMENT

$\therefore$ THERE IS NO SUCH INTEGER

b) SUPPOSE THAT THERE EXISTS A POSITIVE INTEGER a, SUCH THAT

- ITS DIVISION BY 6, YIELDS A REMAINDER OF 1, AND QUOTIENT q

$\Rightarrow a = 6q + 1, \; q \in \mathbb{N}$

$\Rightarrow a - 1 = 6q$

- THE DIVISION OF a^2 BY q GIVES REMAINDER OF 1 AND QUOTIENT 984

$\Rightarrow a^2 = q \times 984 + 1, \; q \in \mathbb{N}$

$\Rightarrow a^2 - 1 = 984q$

$\Rightarrow (a+1)(a-1) = 984q$

$\Rightarrow (a+1) \times 6q = 984q$

$\Rightarrow 6\cancel{q}(a+1) = 984\cancel{q} \qquad q \neq 0$

THIS CAN BE SATISFIED IF 984 IS DIVISIBLE BY 6 WHICH IT IS, AS $984 \div 6 = 164$

$\therefore$ THERE EXISTS SUCH POSITIVE INTEGER a TO FIND IT SIMPLY $a + 1 = 164$ I.E $a = 163$

Question 20 (***+)

In the following question A, B and C are positive odd integers.

Show, using a clear method, that …

a) … $A^2 + B^2 + C^2 + 5$ is a multiple of 8.

b) … $A^2\left(A^2 + 6\right) - 7$ is divisible by 128.

c) … $A^4 - B^4$ is a multiple of 16.

, $\boxed{\text{proof}}$

a) BY DIRECT PROOF

$$A^2+B^2+C^2+5 = (8a+1)+(8b+1)+(8c+1)+5$$
$$= 8a+8b+8c+8$$
$$= 8(a+b+c+1)$$

INDEED A MULTIPLE OF 8

b)

$$A^2(A^2+6)-7 = A^4+6A^2-7$$
$$= (A^2-1)(A^2+7)$$
$$= [(8n+1)-1][(8n+1)+7]$$
$$= 8n(8n+8)$$
$$= 8 \times 8 \times n(n+1)$$
$$= 64 \times 2m$$
$$= 128m$$

INDEED DIVISIBLE BY 128

c)

$$A^4-B^4 = (A^2-B^2)(A^2+B^2)$$
$$= [(8m+1)-(8n+1)][(8m+1)+(8n+1)]$$
$$= [8m-8n][8m+8n+2]$$
$$= 8(m-n)\times 2(m+n+1)$$
$$= 16(m-n)(m+n+1)$$

INDEED A MULTIPLE OF 16

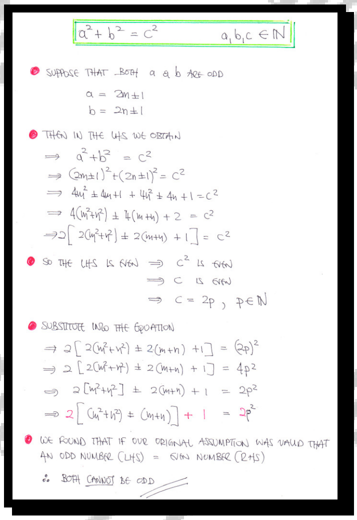
Question 21 (****)

It is given that

$$a^2 + b^2 = c^2, \quad a \in \mathbb{N}, \ b \in \mathbb{N}.$$

Show that a and b cannot both be odd.

$\boxed{}$, $\boxed{\text{proof}}$



$\boxed{a^2 + b^2 = c^2 \qquad a, b, c \in \mathbb{N}}$

- SUPPOSE THAT BOTH a & b ARE ODD

$$a = 2m \pm 1$$
$$b = 2n \pm 1$$

- THEN IN THE LHS WE OBTAIN

$$\Rightarrow a^2 + b^2 = c^2$$
$$\Rightarrow (2m \pm 1)^2 + (2n \pm 1)^2 = c^2$$
$$\Rightarrow 4m^2 \pm 4m + 1 + 4n^2 \pm 4n + 1 = c^2$$
$$\Rightarrow 4(m^2 + n^2) \pm 4(m + n) + 2 = c^2$$
$$\Rightarrow 2\left[2(m^2 + n^2) \pm 2(m + n) + 1\right] = c^2$$

- SO THE LHS IS EVEN $\Rightarrow c^2$ IS EVEN

$$\Rightarrow c \text{ IS EVEN}$$
$$\Rightarrow c = 2p, \ p \in \mathbb{N}$$

- SUBSTITUTE INTO THE EQUATION

$$\Rightarrow 2\left[2(m^2 + n^2) \pm 2(m + n) + 1\right] = (2p)^2$$
$$\Rightarrow 2\left[2(m^2 + n^2) \pm 2(m + n) + 1\right] = 4p^2$$
$$\Rightarrow 2\left[m^2 + n^2\right] \pm 2(m + n) + 1 = 2p^2$$
$$\Rightarrow 2\left[(m^2 + n^2) \pm (m + n)\right] + 1 = 2p^2$$

- WE FOUND THAT IF OUR ORIGINAL ASSUMPTION WAS VALID THAT AN ODD NUMBER (LHS) = EVEN NUMBER (RHS)

$\therefore$ BOTH CANNOT BE ODD

Question 22 (****)

Let $a \in \mathbb{N}$ with $\frac{1}{5}a \notin \mathbb{N}$.

a) Show that the remainder of the division of a^2 by 5 is either 1 or 4.

b) Given further that $b \in \mathbb{N}$ with $\frac{1}{5}b \notin \mathbb{N}$, deduce that $\frac{1}{5}\left(a^4 - b^4\right) \in \mathbb{N}$.

$\boxed{}$, $\boxed{\text{proof}}$

a) IF a IS NOT DIVISIBLE BY 5, THEN IT CAN ONLY BE OF THE FORMS

$a = 5n+1$, $a = 5n+2$, $a = 5n+3$, $a = 5n+4$ $\quad n \in \mathbb{N}$

HENCE WE HAVE BY EXHAUSTION

$$a^2 = (5n+1)^2 = 25n^2+10n+1 = 5(5n^2+2n)+1 = \text{"}5k+1\text{"}$$
$$a^2 = (5n+2)^2 = 25n^2+20n+4 = 5(5n^2+4n)+4 = \text{"}5k+4\text{"}$$
$$a^2 = (5n+3)^2 = 25n^2+30n+9 = 5(5n^2+6n+1)+4 = \text{"}5k+4\text{"}$$
$$a^2 = (5n+4)^2 = 25n^2+40n+16 = 5(5n^2+8n+3)+1 = \text{"}5k+1\text{"}$$

$\therefore$ THE ONLY POSSIBLE REMAINDERS ARE EITHER 1 OR 4

b) AGAIN BY EXHAUSTION WE HAVE

$$\left.\begin{array}{l} \bullet\ a^2 = 5k+1 \ \text{ OR } \ 5k+4 \\ \bullet\ b^2 = 5\ell+1 \ \text{ OR } \ 5\ell+4 \end{array}\right\} \begin{array}{l} k \in \mathbb{N},\ \ell \in \mathbb{N} \\ k > \ell \end{array}$$

- $a^4 - b^4 = (5k+1)^2 - (5\ell+1)^2 = 25k^2+10k+1-25\ell^2-10\ell-1$
  $= 25k^2-25\ell^2+10k-10\ell = 5(5k^2-5\ell^2+2k-2\ell)$
- $a^4 - b^4 = (5k+1)^2 - (5\ell+4)^2 = 25k^2+10k+1-25\ell^2-40\ell-16$
  $= 25k^2-25\ell^2+10k-40\ell-15 = 5(5k^2-5\ell^2+2k-8\ell-3)$
- $a^4 - b^4 = (5k+4)^2 - (5\ell+1)^2 = 25k^2+40k+16-25\ell^2-10\ell-1$
  $= 25k^2-25\ell^2+40k-10\ell+15 = 5(5k^2-5\ell^2+8k-2\ell+3)$
- $a^4 - b^4 = (5k+4)^2 - (5\ell+4)^2 = 25k^2+40k+16-25\ell^2-40\ell-16$
  $= 25k^2-25\ell^2+40k-40\ell = 5(5k^2-5\ell^2+8k+8\ell)$

HENCE IF a AND b ARE NOT DIVISIBLE BY 5, THEN $a^4 - b^4$ WILL BE DIVISIBLE BY 5

Question 23 (****)

It is given that k is a positive integer.

a) If $k-2$ divides k^2+4, determine the possible values of k.

It is further given that a and b are positive integers.

b) Show that $8a^2-b^2$ **cannot** equal 2017.

$\boxed{k = 3,\ 4,\ 6,\ 10}$

a) USING THE FOLLOWING METHODOLOGY

AS $k-2$ DIVIDES k^2+4 $\Rightarrow \dfrac{k^2+4}{k-2} \in \mathbb{N}$

$\Rightarrow \dfrac{k(k-2)+2(k-2)+8}{k-2} \in \mathbb{N}$

$\Rightarrow k+2+\dfrac{8}{k-2}$

HENCE $k-2$ MUST DIVIDE 8

$\Rightarrow k-2 = 1, 2, 4, 8$

$\Rightarrow k = 3, 4, 6, 10$

b) SUPPOSE THERE EXIST POSITIVE INTEGERS SO THAT $8a^2-b^2=2017$

- SUPPOSE THAT b IS EVEN, $b=2n$

$\Rightarrow 8a^2-(2n)^2 = 8a^2-4n^2 = \underbrace{2(4a^2-n^2)}_{\text{EVEN}} \neq \underbrace{2017}_{\text{ODD}}$

- SUPPOSE THAT b IS ODD, $b=2n+1$

$\Rightarrow 8a^2-(2n+1)^2 = 8a^2-(4n^2+4n+1) = 8a^2-4n^2-4n-1$

$= 4(2a^2-n^2-n)-1$

$\Rightarrow 4(2a^2-n^2-n)-1 = 2017$

$\Rightarrow 4(2a^2-n^2-n) = 2018$

$\Rightarrow \underbrace{2(2a^2-n^2-n)}_{\text{EVEN}} \neq \underbrace{1009}_{\text{ODD}}$

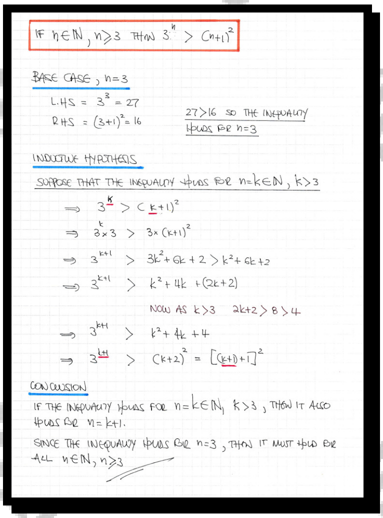
$\therefore$ NOT SUCH POSITIVE INTEGERS

Question 24 (****)

Prove by induction that if $n \in \mathbb{N}$, $n \geq 3$, then

$$3^n > (n+1)^2.$$

[ ] , [proof]


$\boxed{\text{IF } n \in \mathbb{N},\ n \geqslant 3 \text{ THEN } 3^n > (n+1)^2}$

BASE CASE, $n=3$

L.HS $= 3^3 = 27$

RHS $= (3+1)^2 = 16$

27>16 SO THE INEQUALITY HOLDS FOR $n=3$

INDUCTIVE HYPOTHESIS

SUPPOSE THAT THE INEQUALITY HOLDS FOR $n=k \in \mathbb{N}$, $k>3$

$\Rightarrow 3^k > (k+1)^2$

$\Rightarrow 3^k \times 3 > 3 \times (k+1)^2$

$\Rightarrow 3^{k+1} > 3k^2 + 6k + 2 > k^2 + 6k + 2$

$\Rightarrow 3^{k+1} > k^2 + 4k + (2k+2)$

NOW AS $k>3$ $\quad 2k+2 > 8 > 4$

$\Rightarrow 3^{k+1} > k^2 + 4k + 4$

$\Rightarrow 3^{k+1} > (k+2)^2 = [(k+1)+1]^2$

CONCLUSION

IF THE INEQUALITY HOLDS FOR $n=k \in \mathbb{N}$, $k>3$, THEN IT ALSO HOLDS FOR $n=k+1$.

SINCE THE INEQUALITY HOLDS FOR $n=3$, THEN IT MUST HOLD FOR ALL $n \in \mathbb{N}$, $n \geqslant 3$

Question 25 (****)

i. The function f is defined as

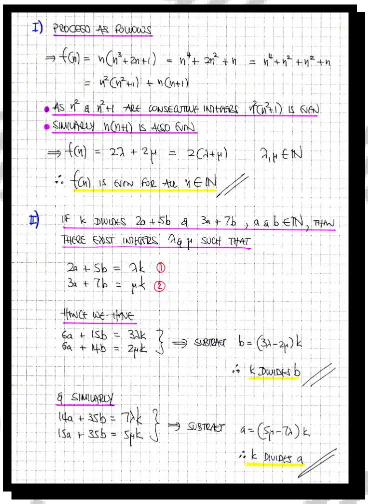
$$f(n) \equiv n\left(n^3 + 2n + 1\right), \quad n \in \mathbb{N}.$$

Show that f is even for all $n \in \mathbb{N}$.

ii. The positive integer k divides both $2a + 5b$ and $3a + 7b$, where $a \in \mathbb{N}$, $b \in \mathbb{N}$.

Show that k must then divide both a and b.

$\boxed{\text{SYN}}$, $\boxed{\text{proof}}$



I) PROCEED AS FOLLOWS

$\Rightarrow f(n) = n(n^3+2n+1) = n^4 + 2n^2 + n = n^4 + n^2 + n^2 + n$

$= n^2(n^2+1) + n(n+1)$

- AS n^2 & n^2+1 ARE CONSECUTIVE INTEGERS $n^2(n^2+1)$ IS EVEN
- SIMILARLY $n(n+1)$ IS ALSO EVEN

$\Rightarrow f(n) = 2\lambda + 2\mu = 2(\lambda+\mu) \qquad \lambda, \mu \in \mathbb{N}$

$\therefore f(n)$ IS EVEN FOR ALL $n \in \mathbb{N}$ //

II) IF k DIVIDES $2a+5b$ & $3a+7b$, a & $b \in \mathbb{N}$, THEN THERE EXIST INTEGERS λ & μ SUCH THAT

$2a + 5b = \lambda k$ ①

$3a + 7b = \mu k$ ②

HENCE WE HAVE

$\left.\begin{array}{l} 6a + 15b = 3\lambda k \\ 6a + 14b = 2\mu k \end{array}\right\} \Rightarrow$ SUBTRACT $b = (3\lambda - 2\mu)k$

$\therefore k$ DIVIDES b //

& SIMILARLY

$\left.\begin{array}{l} 14a + 35b = 7\lambda k \\ 15a + 35b = 5\mu k \end{array}\right\} \Rightarrow$ SUBTRACT $a = (5\mu - 7\lambda)k$

$\therefore k$ DIVIDES a //

Question 26 (****)

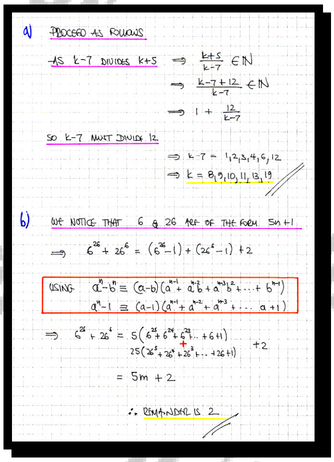
It is given that k is a positive integer.

a) If $k-7$ divides $k+5$, determine the possible values of k.

The second part of this question is unrelated to the first part.

b) By showing a detailed method, find the remainder of the division of $6^{26}+26^{6}$ by 5.

$\boxed{}$, $\boxed{k = 8,\ 9,\ 10,\ 11,\ 13,\ 19}$, $\boxed{2}$



a) PROCEED AS FOLLOWS

AS $k-7$ DIVIDES $k+5$ $\Rightarrow \frac{k+5}{k-7} \in \mathbb{N}$

$\Rightarrow \frac{k-7+12}{k-7} \in \mathbb{N}$

$\Rightarrow 1 + \frac{12}{k-7}$

SO $k-7$ MUST DIVIDE 12

$\Rightarrow k-7 = 1, 2, 3, 4, 6, 12$

$\Rightarrow k = 8, 9, 10, 11, 13, 19$

b) WE NOTICE THAT 6 & 26 ARE OF THE FORM $5n+1$

$\Rightarrow 6^{26} + 26^{6} = (6^{26}-1) + (26^{6}-1) + 2$

USING $a^n - b^n \equiv (a-b)(a^{n-1} + a^{n-2}b + a^{n-3}b^2 + \dots + b^{n-1})$

$a^n - 1 \equiv (a-1)(a^{n-1} + a^{n-2} + a^{n-3} + \dots a + 1)$

$\Rightarrow 6^{26} + 26^{6} = 5(6^{25} + 6^{24} + 6^{23} + \dots + 6 + 1) + 25(26^{5} + 26^{4} + 26^{3} + \dots + 26 + 1) + 2$

$= 5m + 2$

$\therefore$ REMAINDER IS 2

Question 27 (****)

i. The function f is defined as

$$f(n) \equiv \left(n^2+n\right)(n+5), \; n \in \mathbb{N}.$$

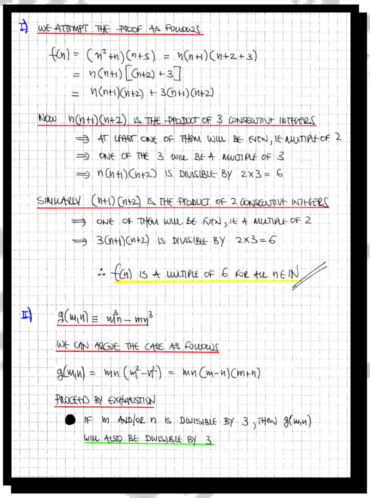
Show that f is multiple of 6 for all $n \in \mathbb{N}$.

ii. The function g is defined as

$$g(m,n) \equiv m^3 n - mn^3, \; m \in \mathbb{N}, \; n \in \mathbb{N}.$$

Show that g is divisible by 3 for all $m \in \mathbb{N}$, $n \in \mathbb{N}$.

, proof



I) WE ATTEMPT THE PROOF AS FOLLOWS

$f(n) = (n^2+n)(n+5) = n(n+1)(n+2+3)$
$= n(n+1)[(n+2)+3]$
$= n(n+1)(n+2) + 3(n+1)(n+2)$

NOW $n(n+1)(n+2)$ IS THE PRODUCT OF 3 CONSECUTIVE INTEGERS
$\Rightarrow$ AT LEAST ONE OF THEM WILL BE EVEN, IE MULTIPLE OF 2
$\Rightarrow$ ONE OF THE 3 WILL BE A MULTIPLE OF 3
$\Rightarrow$ $n(n+1)(n+2)$ IS DIVISIBLE BY $2 \times 3 = 6$

SIMILARLY $(n+1)(n+2)$ IS THE PRODUCT OF 2 CONSECUTIVE INTEGERS
$\Rightarrow$ ONE OF THEM WILL BE EVEN, IE A MULTIPLE OF 2
$\Rightarrow$ $3(n+1)(n+2)$ IS DIVISIBLE BY $2 \times 3 = 6$

$\therefore$ $f(n)$ IS A MULTIPLE OF 6 FOR ALL $n \in \mathbb{N}$

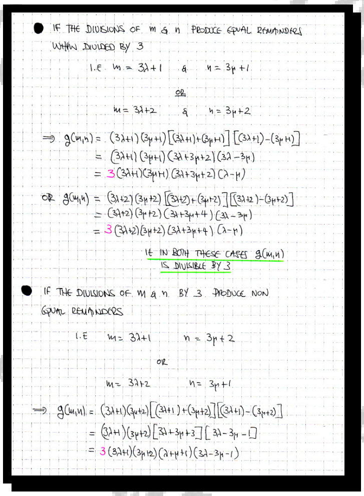
II) $g(m,n) \equiv m^3 n - mn^3$

WE CAN ARGUE THE CASE AS FOLLOWS

$g(m,n) = mn(m^2-n^2) = mn(m-n)(m+n)$

PROCEED BY EXHAUSTION

- IF m AND/OR n IS DIVISIBLE BY 3, THEN $g(m,n)$ WILL ALSO BE DIVISIBLE BY 3



- IF THE DIVISIONS OF m & n PRODUCE EQUAL REMAINDERS WHEN DIVIDED BY 3

I.E $m = 3\lambda+1$ & $n = 3\mu+1$

OR

$m = 3\lambda+2$ & $n = 3\mu+2$

$\Rightarrow g(m,n) = (3\lambda+1)(3\mu+1)[(3\lambda+1)+(3\mu+1)][(3\lambda+1)-(3\mu+1)]$
$= (3\lambda+1)(3\mu+1)(3\lambda+3\mu+2)(3\lambda-3\mu)$
$= 3(3\lambda+1)(3\mu+1)(3\lambda+3\mu+2)(\lambda-\mu)$

OR $g(m,n) = (3\lambda+2)(3\mu+2)[(3\lambda+2)+(3\mu+2)][(3\lambda+2)-(3\mu+2)]$
$= (3\lambda+2)(3\mu+2)(3\lambda+3\mu+4)(3\lambda-3\mu)$
$= 3(3\lambda+2)(3\mu+2)(3\lambda+3\mu+4)(\lambda-\mu)$

IE IN BOTH THESE CASES $g(m,n)$ IS DIVISIBLE BY 3

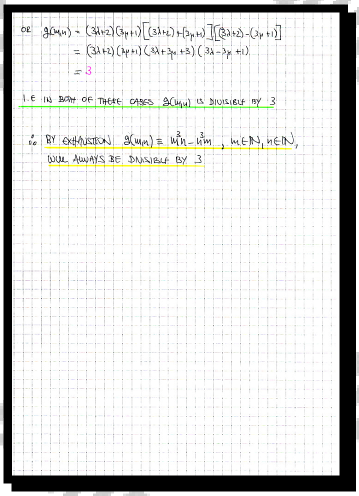
- IF THE DIVISIONS OF m & n BY 3 PRODUCE NON EQUAL REMAINDERS

I.E $m = 3\lambda+1$ $n = 3\mu+2$

OR

$m = 3\lambda+2$ $n = 3\mu+1$

$\Rightarrow g(m,n) = (3\lambda+1)(3\mu+2)[(3\lambda+1)+(3\mu+2)][(3\lambda+1)-(3\mu+2)]$
$= (3\lambda+1)(3\mu+2)[3\lambda+3\mu+3][3\lambda-3\mu-1]$
$= 3(3\lambda+1)(3\mu+2)(\lambda+\mu+1)(3\lambda-3\mu-1)$



OR $g(m,n) = (3\lambda+2)(3\mu+1)[(3\lambda+2)+(3\mu+1)][(3\lambda+2)-(3\mu+1)]$
$= (3\lambda+2)(3\mu+1)(3\lambda+3\mu+3)(3\lambda-3\mu+1)$
$= 3$

I.E IN BOTH OF THESE CASES $g(m,n)$ IS DIVISIBLE BY 3

$\therefore$ BY EXHAUSTION $g(m,n) \equiv m^3 n - n^3 m$, $m \in \mathbb{N}$, $n \in \mathbb{N}$, WILL ALWAYS BE DIVISIBLE BY 3

Question 28 (*****)

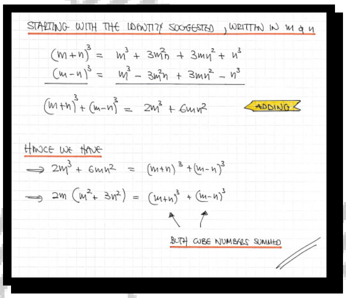
It is given that

$$f(m,n) \equiv 2m\left(m^2 + 3n^2\right),$$

where m and n are distinct positive integers, with $m > n$.

By using the expansion of $(A \pm B)^3$, prove that $f(m,n)$ can always be written as the sum of two cubes.

$\boxed{}$, $\boxed{\text{proof}}$



STARTING WITH THE IDENTITY SUGGESTED, WRITTEN IN m & n

$$(m+n)^3 = m^3 + 3m^2n + 3mn^2 + n^3$$
$$(m-n)^3 = m^3 - 3m^2n + 3mn^2 - n^3$$

$$(m+n)^3 + (m-n)^3 = 2m^3 + 6mn^2$$ ADDING

HENCE WE HAVE

$$\Rightarrow 2m^3 + 6mn^2 = (m+n)^3 + (m-n)^3$$

$$\Rightarrow 2m\left(m^2 + 3n^2\right) = (m+n)^3 + (m-n)^3$$

BOTH CUBE NUMBERS SUMMED

Question 29 (*****)

Prove that the sum of the squares of two distinct positive integers, when doubled, it can be written as the sum of two distinct square numbers

[ ] , [proof]

AS THIS MAY NOT BE OBVIOUS WHERE TO START WE LOOK FOR THE PROOF BY LOOKING DIRECTLY AT THE NUMBER PATTERNS

$2(1^2+2^2) = 10 = 1^2+3^2$
$2(1^2+3^2) = 20 = 2^2+4^2$
$2(1^2+4^2) = 34 = 3^2+5^2$
$2(1^2+5^2) = 52 = 4^2+6^2$

$2(2^2+3^2) = 26 = 1^2+5^2$
$2(2^2+4^2) = 40 = 2^2+6^2$
$2(2^2+5^2) = 58 = 3^2+7^2$
$2(2^2+6^2) = 80 = 4^2+8^2$

$2(3^2+4^2) = 50 = 1^2+7^2$
$2(3^2+5^2) = 68 = 2^2+8^2$
$2(3^2+6^2) = 90 = 3^2+9^2$
$2(3^2+7^2) = 116 = 4^2+10^2$

WE MAY CARRY ON A BIT MORE, IF NOT EVIDENT WHAT/WHY BUT HERE IS THE ALGEBRAIC PROOF, FOR $n \in \mathbb{N}, m \in \mathbb{N}, n \neq m$

$$\begin{aligned} 2(n^2+m^2) &= 2n^2+2m^2 = n^2+m^2+n^2+m^2 \\ &= (n^2+2nm+m^2)+(n^2-2nm+m^2) \\ &= (n+m)^2+(n-m)^2 \end{aligned}$$

Question 30 (*****)

Show that the square of an odd positive integer greater than 1 is of the form

$$8T+1,$$

where T is a triangular number.

proof

ASSERTION: THE SQUARE OF AN ODD POSITIVE INTEGER IS ALWAYS OF THE FORM $8T+1$, WHERE "T" IS A TRIANGLE NUMBER

PROOF BY EXHUSTION

- LET n BE ODD

$\Rightarrow n = 2m+1$

$\Rightarrow 2n+1 = 2(2m+1)+1$

$\Rightarrow 2n+1 = 4m+3$

e.g. $7 = (4\times1)+3$, $35 = (4\times8)+3$, $47 = (4\times11)+3$ ETC

- LET n BE EVEN

$\Rightarrow n = 2m$

$\Rightarrow 2n+1 = 2(2m)+1$

$\Rightarrow 2n+1 = 4m+1$

e.g. $5 = (4\times1)+1$, $21 = (4\times5)+1$, $33 = (4\times8)+1$ ETC

- SQUARING THE ODD NUMBER IN EACH CASE YIELDS

$$(2n+1)^2 = (4m+3)^2 = 16m^2+24m+9 = 8(2m^2+3m+1)+1 = 8(2m+1)(m+1)+1$$

$$(2n+1)^2 = (4m+1)^2 = 16m^2+8m+1 = 8m(2m+1)+1$$

I.E IN BOTH CASES THE OUTCOME IS OF THE FORM $8f(m)+1$

NOW TO PROVE THAT $f(m)$ IS A TRIANGLE NUMBER

- TRIANGLE NUMBERS ARE $1, 3, 6, 10, 15, 21, 28, 36, \dots$

- $1, 6, 15, 28$ (differences $5, 9, 13$; second differences $4, 4$)

$\underline{u_n = 2n^2+an+b}$

$2n^2$: $2, 8, 18, 32, \dots$

HERE: $1, 6, 15, 28$

$-1, -2, -3, -4$

$\therefore u_n = 2n^2-n$

$\Rightarrow u_n = n(2n-1)$

$n \longmapsto m+1$

$\Rightarrow u_m = (m+1)[2(m+1)-1]$

$\Rightarrow u_m = (m+1)(2m+1)$

WHICH WE OBTAINED

- $3, 10, 21, 36$ (differences $7, 11, 15$; second differences $4, 4$)

$u_n = 2n^2+\alpha n+\beta$

$2n^2$: $2, 8, 18, 28$

HERE: $3, 10, 21, 36$

$+1, +2, +3, +4$

$\therefore u_n = 2n^2+n$

$\Rightarrow u_n = n(2n+1)$

OR

$\Rightarrow u_m = m(2m+1)$

WHICH WE OBTAINED

EVERY SQUARE OF AN ODD NATURAL NUMBER GREATER THAN 3, IS OF THE FORM $8T+1$, WHERE T IS A TRIANGLE NUMBER

Question 31 (*****)

The product operator $\prod$, is defined as

$$\prod_{r=1}^{k}[u_r] = u_1 \times u_2 \times u_3 \times u_4 \times \ldots \times u_{k-1} \times u_k .$$

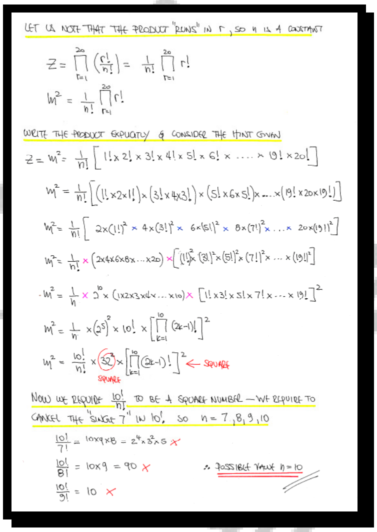
The integer Z is a **square number** and defined as

$$Z = \prod_{r=1}^{20}\left(\frac{r!}{n!}\right), \ \{n \in \mathbb{N} : 1 \le n \le 20\}.$$

By considering the terms inside the product operator in pairs, or otherwise, determine a possible value of n.

You must show a detailed method in this question.

$\boxed{}$, $\boxed{n = 10}$


LET US NOTE THAT THE PRODUCT "RUNS" IN r, SO n IS A CONSTANT

$$Z = \prod_{r=1}^{20}\left(\frac{r!}{n!}\right) = \frac{1}{n!}\prod_{r=1}^{20} r!$$

$$m^2 = \frac{1}{n!}\prod_{r=1}^{20} r!$$

WRITE THE PRODUCT EXPLICITLY & CONSIDER THE HINT GIVEN

$$Z = m^2 = \frac{1}{n!}\left[1! \times 2! \times 3! \times 4! \times 5! \times 6! \times \ldots \times 19! \times 20!\right]$$

$$m^2 = \frac{1}{n!}\left[(1! \times 2 \times 1!) \times (3! \times 4 \times 3!) \times (5! \times 6 \times 5!) \times \ldots \times (19! \times 20 \times 19!)\right]$$

$$m^2 = \frac{1}{n!}\left[2 \times (1!)^2 \times 4 \times (3!)^2 \times 6 \times (5!)^2 \times 8 \times (7!)^2 \times \ldots \times 20 \times (19!)^2\right]$$

$$m^2 = \frac{1}{n!} \times (2 \times 4 \times 6 \times 8 \times \ldots \times 20) \times \left[(1!)^2 \times (3!)^2 \times (5!)^2 \times (7!)^2 \times \ldots \times (19!)^2\right]$$

$$m^2 = \frac{1}{n} \times 2^{10} \times (1 \times 2 \times 3 \times 4 \times \ldots \times 10) \times \left[1! \times 3! \times 5! \times 7! \times \ldots \times 19!\right]^2$$

$$m^2 = \frac{1}{n} \times (2^5)^2 \times 10! \times \left[\prod_{k=1}^{10}(2k-1)!\right]^2$$

$$m^2 = \frac{10!}{n!} \times \underset{\text{SQUARE}}{32^2} \times \left[\prod_{k=1}^{10}(2k-1)!\right]^2 \leftarrow \text{SQUARE}$$

NOW WE REQUIRE $\frac{10!}{n!}$ TO BE A SQUARE NUMBER — WE REQUIRE TO CANCEL THE "SINGLE 7" IN $10!$, SO $n = 7, 8, 9, 10$

$\frac{10!}{7!} = 10 \times 9 \times 8 = 2^4 \times 3^2 \times 5$ ✗

$\frac{10!}{8!} = 10 \times 9 = 90$ ✗

$\frac{10!}{9!} = 10$ ✗

$\therefore$ POSSIBLE VALUE $n = 10$

Question 32 (*****)

Prove by induction that if $n \in \mathbb{N}$, $n \geq 3$, then

$$n^{n+1} > (n+1)^n,$$

and hence deduce that if $n \in \mathbb{N}$, $n \geq 3$, then

$$\sqrt[n]{n} > \sqrt[n+1]{n+1}$$

$\boxed{}$, $\boxed{\text{proof}}$

IF $n \in \mathbb{N}$, $n \geqslant 3$ THEN $n^{n+1} > (n+1)^n$

BASE CASE, $n=3$

L.H.S $= 3^4 = 81$

R.H.S. $= 4^3 = 64$

$81 > 64$, SO THE RESULT HOLDS FOR $n=3$

INDUCTIVE HYPOTHESIS

SUPPOSE THAT THE RESULT HOLDS FOR $n=k>3$, $k \in \mathbb{N}$

$\Rightarrow k^{k+1} > (k+1)^k$

$\Rightarrow k^{k+1}(k+1)^{k+2} > (k+1)^k(k+1)^{k+2}$

$\Rightarrow k^{k+1}(k+1)^{k+2} > (k+1)^{2k+2}$

$\Rightarrow (k+1)^{k+2} > \dfrac{(k+1)^{2k+2}}{k^{k+1}}$

NOW WE NEED TO SHOW THAT

$\dfrac{(k+1)^{2k+2}}{k^{k+1}} \geqslant (k+2)^{k+1} \Rightarrow (k+1)^{2k+2} \geqslant k^{k+1} \cdot (k+2)^{k+1}$

$\Rightarrow \left[(k+1)^2\right]^{k+1} \geqslant \left[k(k+2)\right]^{k+1}$

$\Rightarrow (k+1)^2 \geqslant k(k+2)$

$\Rightarrow k^2+2k+1 > k^2+2k$

WHICH HOLDS

RETURNING TO THE MAIN LINE OF THE INDUCTIVE HYPOTHESIS

- IF $k^{k+1} > (k+1)^k$

  . . . .

  . . . .

  . . . .

- THEN $(k+1)^{k+2} > \dfrac{(k+1)^{2k+2}}{k^{k+1}} > (k+2)^{k+1}$

I.E $(k+1)^{[k+1+1]} > \left[(k+1)+1\right]^{k+1}$

CONCLUSION

IF THE RESULT HOLDS FOR $n=k \in \mathbb{N}$, WITH $n \geqslant 3$ THEN IT MUST ALSO HOLD FOR $n=k+1$

AS THE RESULT HOLDS FOR $n=3$, THEN IT MUST HOLD FOR ALL $n \in \mathbb{N}$, WITH $n \geqslant 3$

FINALLY WE HAVE

$n^{n+1} > (n+1)^n \qquad n \in \mathbb{N}, n \geqslant 3$

$\Rightarrow \left(n^{\frac{1}{n}}\right)^{n(n+1)} > \left[(n+1)^{\frac{1}{n+1}}\right]^{(n+1)n}$

$\Rightarrow \left[n^{\frac{1}{n}}\right]^{n^2+2n} > \left[(n+1)^{\frac{1}{n+1}}\right]^{n^2+n}$

$\Rightarrow \sqrt[n]{n} > \sqrt[n+1]{n+1}$

Question 33 (*****)

It is given that $11a+13b$ is a multiple of $13-a$, where $a \in \mathbb{N}$, $b \in \mathbb{N}$.

It is then asserted that $(13+a)(11+b)$ is also a multiple of $13-a$.

Prove the validity of this assertion.

$\boxed{}$, $\boxed{\text{proof}}$

GIVEN

- $a \in \mathbb{N}$, $b \in \mathbb{N}$
- $11a+13b$ IS A MULTIPLE OF $13-a$

ASSERTION TO BE PROVEN

$(13+a)(11+b)$ IS ALSO A MULTIPLE OF $13-a$

IF $11a+13b$ IS A MULTIPLE OF $13-a$, THEN

$$11a+13b = (13-a)n, \quad n \in \mathbb{N}$$

NOW WE HAVE

$$\begin{aligned}(13+a)(11+b) &= 13\times 11 + 13b + 11a + ab \\ &= 13\times 11 + 2(13b+11a) - (13b+11a) + ab \\ &= 2(13b+11a) + 13\times 11 - 11a + ab - 13b \\ &= 2\left[(13-a)n\right] + 11(13-a) + b(a-13) \\ &= (13-a)\left[2n + 11 + b\right] \\ &= (13-a)m, \quad m \in \mathbb{N}\end{aligned}$$

INDEED THE ASSERTION IS TRUE