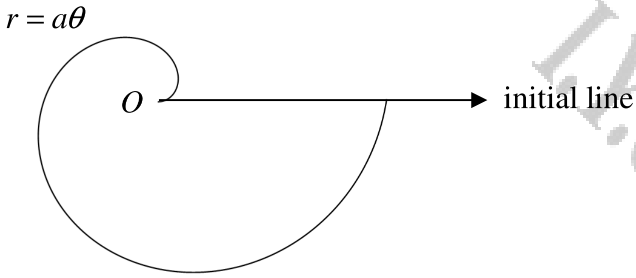
POLAR COORDINATES
54 EXAM QUESTIONS

8 BASIC QUESTIONS

Question 1 (**)



The figure above shows a spiral curve with polar equation

$$r = a\theta, \; 0 \le \theta \le 2\pi,$$

where a is a positive constant.

Find the area of the finite region bounded by the spiral and the initial line.

$\boxed{}$, $\boxed{\text{area} = \frac{4}{3}a^2\pi^3}$

USING THE STANDARD FORMULA FOR POLAR AREA

$$\text{AREA} = \int_{\theta_1}^{\theta_2} \frac{1}{2}r^2 \, d\theta = \int_{\theta=0}^{\theta=2\pi} \frac{1}{2}(a\theta)^2 \, d\theta = \frac{1}{2}a^2 \int_0^{2\pi} \theta^2 \, d\theta$$

$$= \frac{1}{2}a^2 \left[\frac{1}{3}\theta^3\right]_0^{2\pi} = \frac{1}{6}a^2\left[8\pi^3 - 0\right] = \frac{4}{3}\pi^3 a^2$$

Question 2 (**)

The polar curve C has equation

$$r = 2(\cos\theta - \sin\theta),\ 0 \le \theta < 2\pi .$$

Find a Cartesian equation for C and show it represents a circle, indicating its radius and the Cartesian coordinates of its centre.

$\boxed{}$, $\boxed{(x-1)^2 + (y+1)^2 = 2}$, $\boxed{r = \sqrt{2}}$, $\boxed{(1,-1)}$

USING THE POLAR TRANSFORMATION EQUATIONS

$$\left.\begin{array}{l} x = r\cos\theta \\ y = r\sin\theta \end{array}\right\} \Rightarrow \begin{array}{l} \cos\theta = \frac{x}{r} \\ \sin\theta = \frac{y}{r} \end{array}$$

SUBSTITUTE INTO THE EQUATION

$$\Rightarrow r = 2(\cos\theta - \sin\theta)$$
$$\Rightarrow r = 2\left(\frac{x}{r} - \frac{y}{r}\right)$$
$$\Rightarrow r^2 = 2x - 2y$$

BUT $r^2 = x^2 + y^2$

$$\Rightarrow x^2 + y^2 = 2x - 2y$$
$$\Rightarrow x^2 - 2x + y^2 + 2y = 0$$
$$\Rightarrow (x-1)^2 + (y+1)^2 = 2$$

INDEED A CIRCLE, CENTRE $(1,-1)$, RADIUS $\sqrt{2}$

Question 3 (**)

The polar curve C has equation

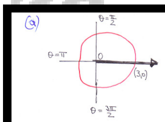
$$r = 2 + \cos\theta,\ 0 \le \theta < 2\pi .$$

a) Sketch the graph of C.

b) Show that the area enclosed by the curve is $\frac{9}{2}\pi$.

proof

(a)



(b)

$$A = \frac{1}{2}\int_{\theta_1}^{\theta_2} r^2\, d\theta$$
$$A = \frac{1}{2}\int_0^{2\pi} (2+\cos\theta)^2\, d\theta$$
$$A = \frac{1}{2}\int_0^{2\pi} 4 + 4\cos\theta + \cos^2\theta\, d\theta$$
$$A = \frac{1}{2}\int_0^{2\pi} 4 + 4\cos\theta + \frac{1}{2} + \frac{1}{2}\cos 2\theta\, d\theta$$
$$A = \frac{1}{2}\left[\frac{9}{2}\theta + 4\sin\theta + \frac{1}{4}\sin 2\theta\right]_0^{2\pi}$$
$$A = \frac{1}{2}\left[(9\pi + 0 + 0) - (0)\right]$$
$$A = \frac{9}{2}\pi$$

Question 4 (+)**

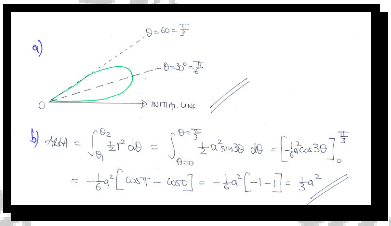
The curve C has polar equation

$$r^2 = a^2 \sin 3\theta, \ 0 \le \theta \le \frac{\pi}{3}.$$

a) Sketch the graph of C.

b) Find the exact value of area enclosed by the C.

$\boxed{\text{area} = \frac{1}{3}a^2}$



Question 5 (+)**

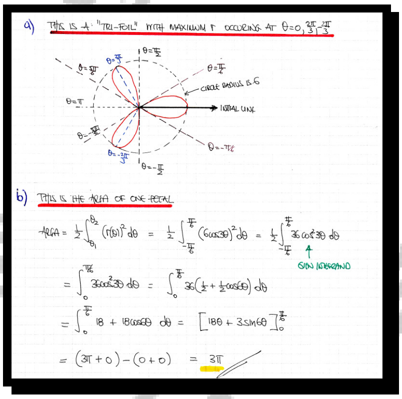
The curve C has polar equation

$$r = 6\cos 3\theta, \quad -\pi < \theta \le \pi.$$

a) Sketch the graph of C.

b) Find the exact value of area enclosed by the C, for $-\dfrac{\pi}{6} < \theta \le \dfrac{\pi}{6}$.

$\boxed{}$, $\boxed{\text{area} = 3\pi}$



a) THIS IS A "TRI-FOIL" WITH MAXIMUM r OCCURING AT $\theta = 0, \frac{2\pi}{3}, -\frac{2\pi}{3}$

b) THIS IS THE AREA OF ONE PETAL

$$\text{AREA} = \frac{1}{2}\int_{\theta_1}^{\theta_2} (r(\theta))^2 \, d\theta = \frac{1}{2}\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} (6\cos 3\theta)^2 \, d\theta = \frac{1}{2}\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} 36\cos^2 3\theta \, d\theta$$

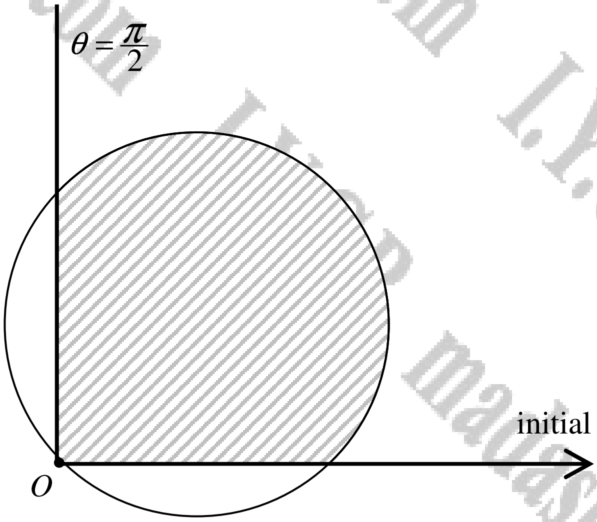
EVEN INTEGRAND

$$= \int_0^{\pi/6} 36\cos^2 3\theta \, d\theta = \int_0^{\frac{\pi}{6}} 36\left(\tfrac{1}{2} + \tfrac{1}{2}\cos 6\theta\right) d\theta$$

$$= \int_0^{\frac{\pi}{6}} 18 + 18\cos 6\theta \, d\theta = \Big[18\theta + 3\sin 6\theta\Big]_0^{\frac{\pi}{6}}$$

$$= (3\pi + 0) - (0 + 0) = 3\pi$$

Question 6 (**+)



The figure above shows a circle with polar equation

$$r = 4(\cos\theta + \sin\theta) \quad 0 \le \theta < 2\pi .$$

a) Find the exact area of the shaded region bounded by the circle, the initial line and the half line $\theta = \dfrac{\pi}{2}$.

b) Determine the Cartesian coordinates of the centre of the circle and the length of its radius.

$\boxed{\text{area} = 4\pi + 8}$, $\boxed{(2,2), \text{ radius} = \sqrt{8}}$

a) USING THE STANDARD FORMULA

$\Rightarrow \text{AREA} = \frac{1}{2}\int_{\theta_1}^{\theta_2} [r(\theta)]^2 \, d\theta$

$\Rightarrow \text{AREA} = \frac{1}{2}\int_0^{\pi/2} 16(\cos\theta + \sin\theta)^2 \, d\theta$

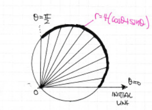
$\Rightarrow \text{AREA} = \frac{1}{2}\int_0^{\pi/2} 16(\cos^2\theta + 2\cos\theta\sin\theta + \sin^2\theta) \, d\theta$

$\Rightarrow \text{AREA} = \int_0^{\pi/2} 8(1 + 2\cos\theta\sin\theta) \, d\theta$

$\Rightarrow \text{AREA} = 8\left[\theta + \sin^2\theta\right]_0^{\pi/2}$

$\Rightarrow \text{AREA} = 8\left[\left(\frac{\pi}{2} + 1\right) - 0\right]$

$\Rightarrow \text{AREA} = 4\pi + 8$



b) PROBABLY EASIER TO WORK IN CARTESIAN

$\Rightarrow r = 4(\cos\theta + \sin\theta)$

$\Rightarrow r = 4\left(\frac{x}{r} + \frac{y}{r}\right)$

$\Rightarrow r^2 = 4x + 4y$

$\Rightarrow x^2 + y^2 = 4x + 4y$

$\Rightarrow x^2 - 4x + y^2 - 4y = 0$

$\Rightarrow (x-2)^2 - 4 + (y-2)^2 - 4 = 0$

$\Rightarrow (x-2)^2 + (y-2)^2 = 8$

$x = r\cos\theta$
$y = r\sin\theta$
$x^2 + y^2 = r$

$\therefore$ CENTRE AT $(2,2)$
RADIUS $2\sqrt{8}$

Question 7 (***)

Write the polar equation

$$r = \cos\theta + \sin\theta,\ 0 \le \theta < 2\pi$$

in Cartesian form, and hence show that it represents a circle, further determining the coordinates of its centre and the size of its radius.

$$\boxed{\left(x-\tfrac{1}{2}\right)^2 + \left(y-\tfrac{1}{2}\right)^2 = \tfrac{1}{2}}$$

USING THE "STANDARD TRANSFORMATION" EQUATIONS

$x = r\cos\theta$
$y = r\sin\theta$
$x^2+y^2=r$

$\Rightarrow r = \cos\theta + \sin\theta$
$\Rightarrow r = \frac{x}{r} + \frac{y}{r}$
$\Rightarrow r = \frac{x+y}{r}$
$\Rightarrow r^2 = x+y$
$\Rightarrow x^2+y^2-x-y=0$
$\Rightarrow \left(x-\frac{1}{2}\right)^2-\frac{1}{4}+\left(y-\frac{1}{2}\right)^2-\frac{1}{4}=0$
$\Rightarrow \left(x-\frac{1}{2}\right)^2+\left(y-\frac{1}{2}\right)^2=\frac{1}{2}$

$\therefore$ INDEED A CIRCLE

CENTRE AT $\left(\frac{1}{2},\frac{1}{2}\right)$
RADIUS $\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$

Question 8 (***)

A Cardioid has polar equation

$$r = 1 + 2\cos\theta, \ 0 \le \theta \le \frac{\pi}{2}.$$

The point P lies on the Cardioid so that the tangent to the Cardioid at P is parallel to the initial line.

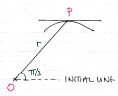
Determine the exact length of OP, where O is the pole.

$\boxed{}$, $\boxed{\frac{1}{4}\left(3+\sqrt{33}\right)}$

PARALLEL TO THE INITIAL LINE IMPLIES $\frac{dy}{dx} = 0$

$$\Rightarrow \frac{dy/d\theta}{dx/d\theta} = 0 \Rightarrow \frac{dy}{d\theta} = 0$$
$$\Rightarrow \frac{d}{d\theta}(y) = 0$$
$$\Rightarrow \frac{d}{d\theta}(r\sin\theta) = 0$$
$$\Rightarrow \frac{d}{d\theta}\left((1+2\cos\theta)\sin\theta\right) = 0$$
$$\Rightarrow \frac{d}{d\theta}(\sin\theta + \sin2\theta) = 0$$
$$\Rightarrow \cos\theta + 2\cos2\theta = 0$$
$$\Rightarrow \cos\theta + 2(2\cos^2\theta - 1) = 0$$
$$\Rightarrow 4\cos^2\theta + \cos\theta - 2 = 0$$
$$\therefore \cos\theta = \frac{-1 \pm \sqrt{33}}{8}$$
$$\cos\theta = \frac{-1+\sqrt{33}}{8} \quad \left(0 < \theta < \frac{\pi}{2}\right)$$

LOOKING AT THE DIAGRAM



$$\therefore |OP| = 1 + 2\cos\theta$$
$$= 1 + 2\left(\frac{-1+\sqrt{33}}{8}\right)$$
$$= 1 + \frac{-1+\sqrt{33}}{4}$$
$$= \frac{3+\sqrt{33}}{4}$$

26 STANDARD QUESTIONS

Question 1 (***+)

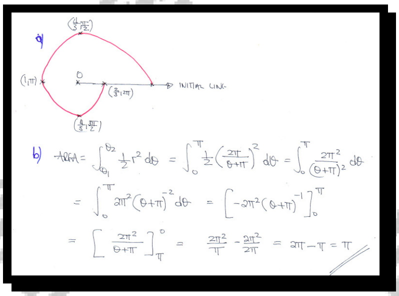
A curve has polar equation

$$r = \frac{2\pi}{\theta + \pi}, \; 0 \le \theta < 2\pi .$$

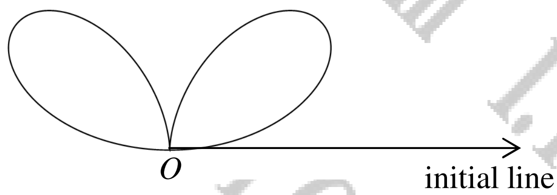
a) Sketch the curve.

b) Find the exact value of area enclosed by the curve, the initial line and the half line with equation $\theta = \pi$.

$\boxed{\text{area} = \pi}$



Question 2 (***+)

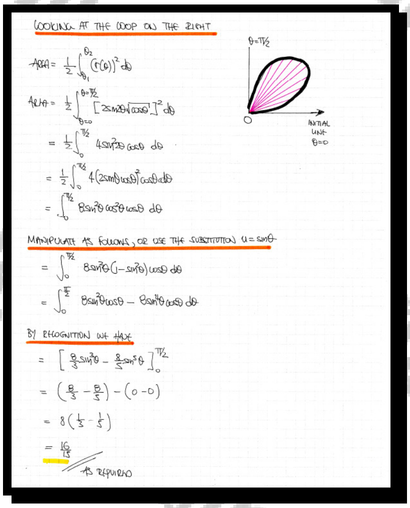


The figure above shows the polar curve C with equation

$$r = 2\sin 2\theta\sqrt{\cos\theta}\,, \quad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}.$$

Show that the area enclosed by one of the two identical loops of the curve is $\dfrac{16}{15}$.

$\boxed{}$, $\boxed{\text{proof}}$



LOOKING AT THE LOOP ON THE RIGHT

$$\text{AREA} = \frac{1}{2}\int_{\theta_1}^{\theta_2} (r(\theta))^2 \, d\theta$$

$$\text{AREA} = \frac{1}{2}\int_{\theta=0}^{\theta=\pi/2} \left[2\sin 2\theta\sqrt{\cos\theta}\right]^2 d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} 4\sin^2 2\theta\cos\theta \, d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} 4(2\sin\theta\cos\theta)^2\cos\theta \, d\theta$$

$$= \int_0^{\pi/2} 8\sin^2\theta\cos^2\theta\cos\theta \, d\theta$$

MANIPULATE AS FOLLOWS, OR USE THE SUBSTITUTION $u = \sin\theta$

$$= \int_0^{\pi/2} 8\sin^2\theta(1-\sin^2\theta)\cos\theta \, d\theta$$

$$= \int_0^{\frac{\pi}{2}} 8\sin^2\theta\cos\theta - 8\sin^4\theta\cos\theta \, d\theta$$

BY RECOGNITION WE HAVE

$$= \left[\frac{8}{3}\sin^3\theta - \frac{8}{5}\sin^5\theta\right]_0^{\pi/2}$$

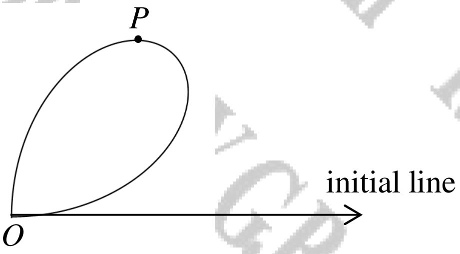
$$= \left(\frac{8}{3} - \frac{8}{5}\right) - (0-0)$$

$$= 8\left(\frac{1}{3} - \frac{1}{5}\right)$$

$$= \frac{16}{15}$$

AS REQUIRED

Question 3 (***+)



The figure above shows the polar curve with equation

$$r = \sin 2\theta \,, \; 0 \le \theta \le \frac{\pi}{2}.$$

a) Find the exact value of the area enclosed by the curve.

The point P lies on the curve so that the tangent at P is parallel to the initial line.

b) Find the **Cartesian** coordinates of P.

$\boxed{}$, $\boxed{\text{area} = \frac{\pi}{8}}$, $\boxed{\left(\frac{2}{9}\sqrt{6}, \frac{4}{9}\sqrt{3}\right)}$

a) USING THE STANDARD FORMULA FOR THE AREA IN POLARS

$$\text{AREA} = \int_{\theta_1}^{\theta_2} \frac{1}{2}r^2 \, d\theta = \int_0^{\frac{\pi}{2}} \frac{1}{2}(\sin 2\theta)^2 \, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \frac{1}{2}\sin^2 2\theta \, d\theta$$

NOW USING THE TRIGONOMETRIC IDENTITY FOR COSINE DOUBLE ANGLE

$$\cos 2A \equiv 1 - 2\sin^2 A$$

$$\cos 4A \equiv \cos[2(2A)] \equiv 1 - 2\sin^2 2A$$

$$\sin^2 2A \equiv \frac{1}{2} - \frac{1}{2}\cos 4A$$

$$\text{AREA} = \int_0^{\frac{\pi}{2}} \frac{1}{2}\left[\frac{1}{2} - \frac{1}{2}\cos 4\theta\right] d\theta$$

$$= \int_0^{\frac{\pi}{2}} \frac{1}{4} - \frac{1}{4}\cos 4\theta$$

$$= \left[\frac{1}{4}\theta - \frac{1}{16}\sin 4\theta\right]_0^{\frac{\pi}{2}}$$

$$= \left(\frac{\pi}{2} \times \frac{1}{4} - 0\right) - (0 - 0)$$

$$= \frac{\pi}{8}$$

b) FOR "HORIZONTAL TANGENT" $dy/dx = 0$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = 0$$

$$\frac{dy}{d\theta} = \frac{d}{d\theta}(r\sin\theta) = \frac{d}{d\theta}(\sin 2\theta \sin\theta) = 0$$

DIFFERENTIATE & SOLVE THE EQUATION

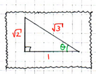
$$\Rightarrow 2\cos 2\theta \sin\theta + \sin 2\theta \cos\theta = 0$$

$$\Rightarrow 2\sin\theta(2\cos^2\theta - 1) + 2\sin\theta\cos^2\theta = 0$$

$$\Rightarrow 2\sin\theta\left[3\cos^2\theta - 1\right] = 0$$

$$\therefore \cancel{\sin\theta = 0} \qquad \cos\theta = \frac{1}{\sqrt{3}} \qquad \cancel{\cos\theta = -\frac{1}{\sqrt{3}}}$$

$$\therefore \theta = \arccos\left(\frac{1}{\sqrt{3}}\right)$$



$$\therefore r = \sin 2\theta = 2\sin\theta\cos\theta$$

$$= 2 \times \frac{\sqrt{2}}{\sqrt{3}} \times \frac{1}{\sqrt{3}} = \frac{2}{3}\sqrt{2}$$

POLAR CO-ORDINATES OF $P\left(\frac{2}{3}\sqrt{2}, \arccos\frac{1}{\sqrt{3}}\right)$

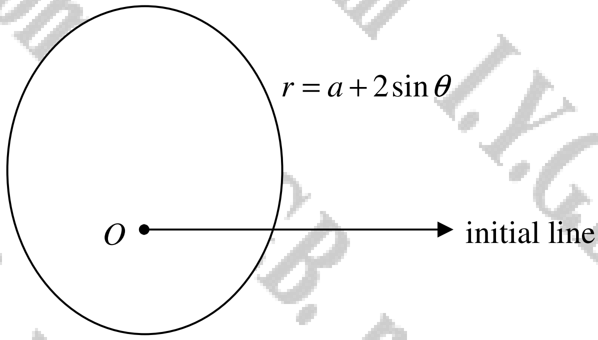
r θ

$$x = r\cos\theta = \frac{2}{3}\sqrt{2} \times \frac{1}{\sqrt{3}} = \frac{2\sqrt{2}}{3\sqrt{3}} = \frac{2}{9}\sqrt{6}$$

$$y = r\sin\theta = \frac{2}{3}\sqrt{2}\left(\frac{\sqrt{2}}{\sqrt{3}}\right) = \frac{4}{3\sqrt{3}} = \frac{4}{9}\sqrt{3}$$

CARTESIAN CO-ORDINATES $\left(\frac{2}{9}\sqrt{6}, \frac{4}{9}\sqrt{3}\right)$

Question 4 (***+)



The diagram above shows the curve with polar equation

$$r = a + 2\sin\theta,\ 0 \le \theta < 2\pi,$$

where a is a positive constant.

Determine the value of a given that the area bounded by the curve is 38π.

$\boxed{a = 6}$

$\boxed{A = \int_{\theta_1}^{\theta_2} \frac{1}{2}r^2 \, d\theta}$

$\Rightarrow 38\pi = \int_0^{2\pi} \frac{1}{2}(a + 2\sin\theta)^2 \, d\theta$

$\Rightarrow 38\pi = \frac{1}{2}\int_0^{2\pi} a^2 + 2a\sin\theta + 4\sin^2\theta \, d\theta$

$\Rightarrow 76\pi = \int_0^{2\pi} a^2 + 2a\sin\theta + 4\left(\frac{1}{2} - \frac{1}{2}\cos 2\theta\right) d\theta$

$\Rightarrow 76\pi = \int_0^{2\pi} a^2 + 2a\sin\theta + 2 - 2\cos 2\theta \, d\theta$

$\Rightarrow 76\pi = \left[a^2\theta - 2a\cos\theta + 2\theta - \sin 2\theta\right]_0^{2\pi}$

$\Rightarrow 76\pi = (2\pi a^2 - \cancel{2a} + 4\pi - 0) - (0 - \cancel{2a} + 0 - 0)$

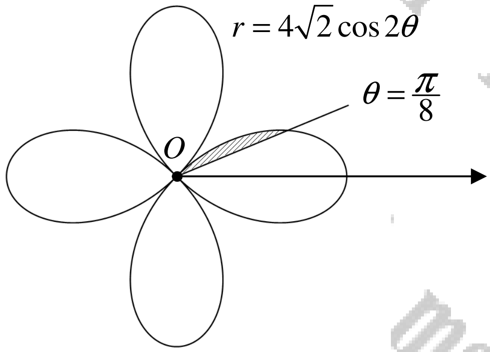
$\Rightarrow 76\pi = 2\pi a^2 + 4\pi$

$\Rightarrow 38 = a^2 + 2$

$\Rightarrow a^2 = 36$

$\Rightarrow a = 6$ // $a > 0$

Question 5 (***+)

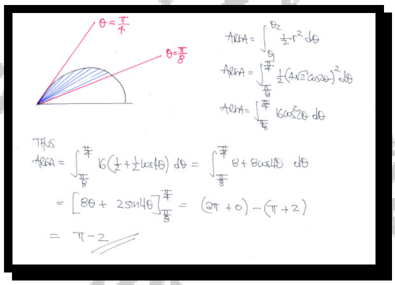


The figure above shows the curve with polar equation

$$r = 4\sqrt{2}\cos 2\theta \, , \; 0 \le \theta < 2\pi \, .$$

Find in exact form the area of the finite region bounded by the curve and the line with polar equation $\theta = \dfrac{\pi}{8}$, which is shown shaded in the above figure.

$\boxed{\text{area} = \pi - 2}$



Question 6 (***+)

A curve C_1 has polar equation

$$r = 2\sin\theta, \; 0 \le \theta < 2\pi .$$

a) Find a Cartesian equation for C_1, and describe it geometrically.

A different curve C_2 has Cartesian equation

$$y^2 = \frac{x^4}{1-x^2}, \; x \ne \pm 1 .$$

b) Find a polar equation for C_2, in the form $r = f(\theta)$.

$\boxed{x^2 + (y-1)^2 = 1}$, $\boxed{r = \tan\theta}$

(a) $r = 2\sin\theta$

$\Rightarrow r = 2\left(\frac{y}{r}\right)$

$\Rightarrow r^2 = 2y$

$\Rightarrow x^2 + y^2 - 2y = 0$

$\Rightarrow x^2 + (y-1)^2 - 1 = 0$

$\Rightarrow x^2 + (y-1)^2 = 1$

CIRCLE CENTRE $(0,1)$ RADIUS 1

(b) $y^2 = \frac{x^4}{1-x^2}$

$\Rightarrow y^2 - x^2y^2 = x^4$

$\Rightarrow y^2 = x^4 + x^2y^2$

$\Rightarrow y^2 = x^2(x^2+y^2)$

$\Rightarrow y^2 = x^2r^2$

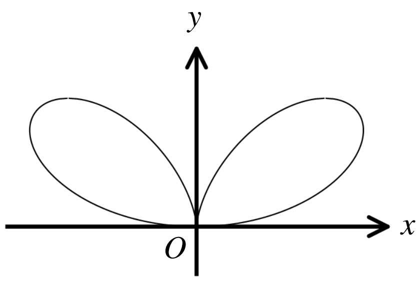
$\Rightarrow r^2 = \frac{y^2}{x^2}$

$\Rightarrow r^2 = \frac{\cancel{r^2}\sin^2\theta}{\cancel{r^2}\cos^2\theta}$

$\Rightarrow r^2 = \tan^2\theta$

$\Rightarrow r = \tan\theta$

Question 7 (***+)



The figure above shows the curve C with Cartesian equation

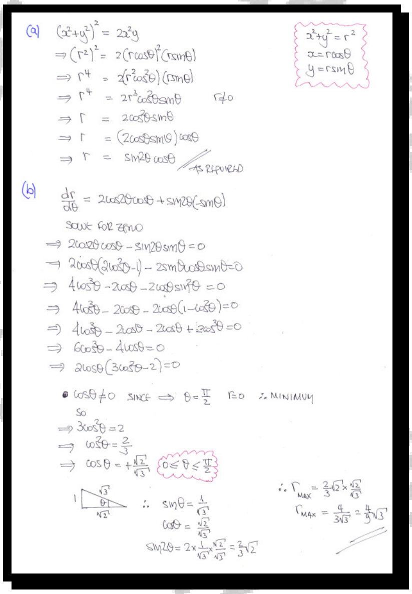
$$\left(x^2+y^2\right)^2 = 2x^2y.$$

a) Show that a polar equation for C can be written as

$$r = \sin 2\theta \cos\theta.$$

b) Determine in exact surd form the maximum value of r.

$$\boxed{r_{\text{max}} = \frac{4}{9}\sqrt{3}}$$



(a) $(x^2+y^2)^2 = 2x^2y$

$x^2+y^2=r^2$
$x=r\cos\theta$
$y=r\sin\theta$

$\Rightarrow (r^2)^2 = 2(r\cos\theta)^2(r\sin\theta)$

$\Rightarrow r^4 = 2(r^2\cos^2\theta)(r\sin\theta)$

$\Rightarrow r^4 = 2r^3\cos^2\theta\sin\theta \qquad r\neq 0$

$\Rightarrow r = 2\cos^2\theta\sin\theta$

$\Rightarrow r = (2\cos\theta\sin\theta)\cos\theta$

$\Rightarrow r = \sin 2\theta\cos\theta$ // AS REQUIRED

(b) $\frac{dr}{d\theta} = 2\cos 2\theta\cos\theta + \sin 2\theta(-\sin\theta)$

SOLVE FOR ZERO

$\Rightarrow 2\cos 2\theta\cos\theta - \sin 2\theta\sin\theta = 0$

$\Rightarrow 2\cos\theta(2\cos^2\theta-1) - 2\sin\theta\cos\theta\sin\theta = 0$

$\Rightarrow 4\cos^3\theta - 2\cos\theta - 2\cos\theta\sin^2\theta = 0$

$\Rightarrow 4\cos^3\theta - 2\cos\theta - 2\cos\theta(1-\cos^2\theta) = 0$

$\Rightarrow 4\cos^3\theta - 2\cos\theta - 2\cos\theta + 2\cos^3\theta = 0$

$\Rightarrow 6\cos^3\theta - 4\cos\theta = 0$

$\Rightarrow 2\cos\theta(3\cos^2\theta-2) = 0$

• $\cos\theta \neq 0$ SINCE $\Rightarrow \theta = \frac{\pi}{2} \quad r=0 \quad \therefore$ MINIMUM

SO

$\Rightarrow 3\cos^2\theta = 2$

$\Rightarrow \cos^2\theta = \frac{2}{3}$

$\Rightarrow \cos\theta = +\frac{\sqrt{2}}{\sqrt{3}} \quad 0 \leq \theta \leq \frac{\pi}{2}$

1, $\sqrt{3}$, θ, $\sqrt{2}$

$\therefore \sin\theta = \frac{1}{\sqrt{3}}$

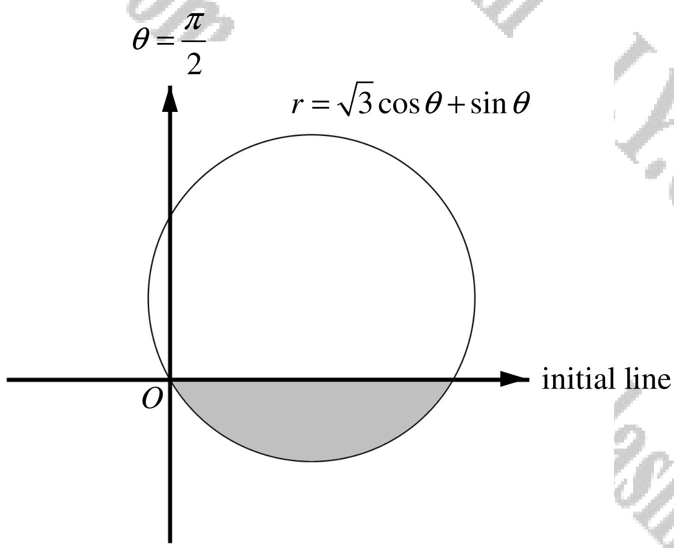
$\cos\theta = \frac{\sqrt{2}}{\sqrt{3}}$

$\sin 2\theta = 2\times\frac{1}{\sqrt{3}}\times\frac{\sqrt{2}}{\sqrt{3}} = \frac{2}{3}\sqrt{2}$

$\therefore r_{\text{MAX}} = \frac{2}{3}\sqrt{2}\times\frac{\sqrt{2}}{\sqrt{3}}$

$r_{\text{MAX}} = \frac{4}{3\sqrt{3}} = \frac{4}{9}\sqrt{3}$ //

Question 8 (***+)



The diagram above shows the curve with polar equation

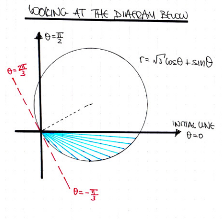
$$r = \sqrt{3}\cos\theta + \sin\theta, \ -\frac{\pi}{3} \le \theta < \frac{2\pi}{3}.$$

By using a method involving integration in polar coordinates, show that the area of the shaded region is

$$\frac{1}{12}\left(4\pi - 3\sqrt{3}\right).$$

$\boxed{}$, $\boxed{\text{proof}}$

LOOKING AT THE DIAGRAM BELOW



$$\text{AREA} = \int_{\theta_1}^{\theta_2} \frac{1}{2}r^2\,d\theta = \int_{\theta=-\frac{\pi}{3}}^{\theta=0} \frac{1}{2}\left(\sqrt{3}\cos\theta + \sin\theta\right)^2 d\theta$$

$$\text{AREA} = \int_{-\frac{\pi}{3}}^{0} \frac{1}{2}\left[3\cos^2\theta + \underbrace{2\sqrt{3}\cos\theta\sin\theta} + \sin^2\theta\right] d\theta$$

$$\text{AREA} = \frac{1}{2}\int_{-\frac{\pi}{3}}^{0} \underbrace{2\cos^2\theta + 1} + \sqrt{3}\sin 2\theta \ d\theta$$

$$\text{AREA} = \frac{1}{2}\int_{-\frac{\pi}{3}}^{0} \overbrace{(1+\cos 2\theta) + 1} + \sqrt{3}\sin 2\theta \ d\theta$$

$$[\cos 2\theta \equiv 2\cos^2\theta - 1]$$

$$\text{AREA} = \frac{1}{2}\int_{-\frac{\pi}{3}}^{0} 2 + \cos 2\theta + \sqrt{3}\sin 2\theta \ d\theta$$

$$\text{AREA} = \frac{1}{2}\left[2\theta + \frac{1}{2}\sin 2\theta - \frac{\sqrt{3}}{2}\cos 2\theta\right]_{-\frac{\pi}{3}}^{0}$$

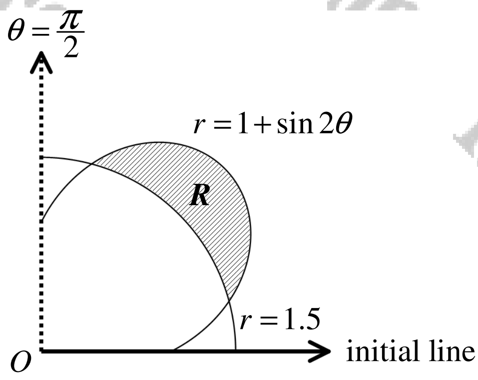
$$\text{AREA} = \frac{1}{2}\left[\left(0 + 0 - \frac{\sqrt{3}}{2}\right) - \left(-\frac{2\pi}{3} \cancel{- \frac{\sqrt{3}}{4}} \cancel{+ \frac{\sqrt{3}}{4}}\right)\right]$$

$$\text{AREA} = \frac{1}{2}\left[\frac{2\pi}{3} - \frac{\sqrt{3}}{2}\right]$$

$$\text{AREA} = \frac{1}{12}\left[4\pi - 3\sqrt{3}\right]$$

// AS REQUIRED

Question 9 (****)



The diagram above shows the curves with polar equations

$$r = 1 + \sin 2\theta, \ 0 \le \theta \le \tfrac{1}{2}\pi,$$

$$r = 1.5, \ 0 \le \theta \le \tfrac{1}{2}\pi.$$

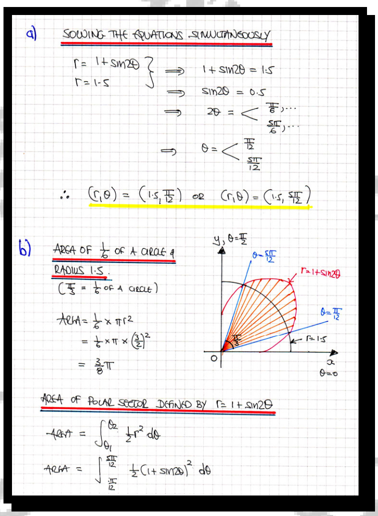
a) Find the polar coordinates of the points of intersection between the two curves.

The finite region R, is bounded by the two curves and is shown shaded in the figure.

b) Show that the area of R is

$$\frac{1}{16}\left(9\sqrt{3} - 2\pi\right).$$

FP2-J, $\boxed{\left(\frac{3}{2}, \frac{\pi}{12}\right), \ \left(\frac{3}{2}, \frac{5\pi}{12}\right)}$



a) SOLVING THE EQUATIONS SIMULTANEOUSLY

$\left.\begin{array}{l} r = 1 + \sin 2\theta \\ r = 1.5 \end{array}\right\} \Rightarrow 1 + \sin 2\theta = 1.5$

$\Rightarrow \sin 2\theta = 0.5$

$\Rightarrow 2\theta = \frac{\pi}{6}, \ldots \text{ or } \frac{5\pi}{6}, \ldots$

$\Rightarrow \theta = \frac{\pi}{12} \text{ or } \frac{5\pi}{12}$

$\therefore \ (r, \theta) = \left(1.5, \frac{\pi}{12}\right)$ OR $(r, \theta) = \left(1.5, \frac{5\pi}{12}\right)$

b) AREA OF $\frac{1}{6}$ OF A CIRCLE OF RADIUS 1.5.

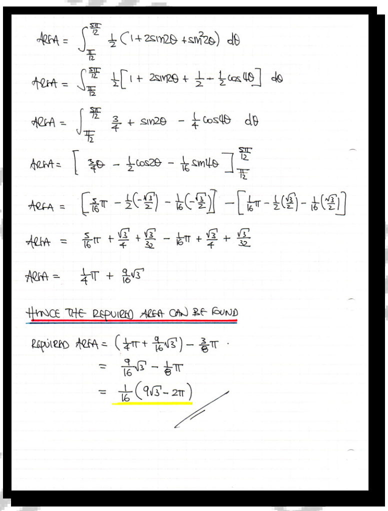
($\frac{\pi}{3} = \frac{1}{6}$ OF A CIRCLE)

$\text{AREA} = \frac{1}{6} \times \pi r^2 = \frac{1}{6} \times \pi \times \left(\frac{3}{2}\right)^2 = \frac{3}{8}\pi$

AREA OF POLAR SECTOR DEFINED BY $r = 1 + \sin 2\theta$

$\text{AREA} = \int_{\theta_1}^{\theta_2} \frac{1}{2} r^2 \, d\theta$

$\text{AREA} = \int_{\frac{\pi}{12}}^{\frac{5\pi}{12}} \frac{1}{2}(1 + \sin 2\theta)^2 \, d\theta$



$\text{AREA} = \int_{\frac{\pi}{12}}^{\frac{5\pi}{12}} \frac{1}{2}\left(1 + 2\sin 2\theta + \sin^2 2\theta\right) d\theta$

$\text{AREA} = \int_{\frac{\pi}{12}}^{\frac{5\pi}{12}} \frac{1}{2}\left[1 + 2\sin 2\theta + \frac{1}{2} - \frac{1}{2}\cos 4\theta\right] d\theta$

$\text{AREA} = \int_{\frac{\pi}{12}}^{\frac{5\pi}{12}} \frac{3}{4} + \sin 2\theta - \frac{1}{4}\cos 4\theta \, d\theta$

$\text{AREA} = \left[\frac{3}{4}\theta - \frac{1}{2}\cos 2\theta - \frac{1}{16}\sin 4\theta\right]_{\frac{\pi}{12}}^{\frac{5\pi}{12}}$

$\text{AREA} = \left[\frac{5}{16}\pi - \frac{1}{2}\left(-\frac{\sqrt{3}}{2}\right) - \frac{1}{16}\left(-\frac{\sqrt{3}}{2}\right)\right] - \left[\frac{1}{16}\pi - \frac{1}{2}\left(\frac{\sqrt{3}}{2}\right) - \frac{1}{16}\left(\frac{\sqrt{3}}{2}\right)\right]$

$\text{AREA} = \frac{5}{16}\pi + \frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{32} - \frac{1}{16}\pi + \frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{32}$

$\text{AREA} = \frac{1}{4}\pi + \frac{9}{16}\sqrt{3}$

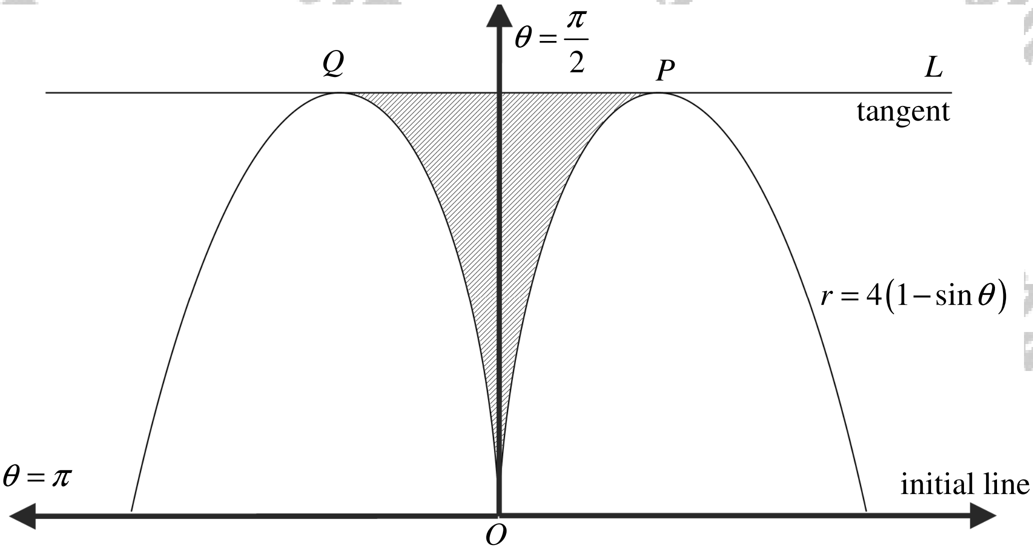
HENCE THE REQUIRED AREA CAN BE FOUND

$\text{REQUIRED AREA} = \left(\frac{1}{4}\pi + \frac{9}{16}\sqrt{3}\right) - \frac{3}{8}\pi$

$= \frac{9}{16}\sqrt{3} - \frac{1}{8}\pi$

$= \frac{1}{16}\left(9\sqrt{3} - 2\pi\right)$

Question 10 (****)



The figure above shows the graph of the curve with polar equation

$$r = 4(1 - \sin\theta),\ 0 \le \theta \le \pi .$$

The straight line L is a tangent to the curve parallel to the initial line, touching the curve at the points P and Q.

a) Find the polar coordinates of P and the polar coordinates of Q.

b) Show that the area of the shaded region is exactly

$$15\sqrt{3} - 8\pi .$$

$\boxed{}$, $\boxed{P\left(2, \tfrac{1}{6}\pi\right),\ Q\left(2, \tfrac{5}{6}\pi\right)}$

a) CONDITION FOR A "HORIZONTAL TANGENT"

$\frac{dy}{dx} = 0 \Rightarrow \frac{dy/d\theta}{dx/d\theta} = 0$

$\Rightarrow dy/d\theta = 0$

$\Rightarrow \frac{d}{d\theta}(y) = 0$

$\Rightarrow \frac{d}{d\theta}(r\sin\theta)$

$\Rightarrow \frac{d}{d\theta}\left[4(1-\sin\theta)\sin\theta\right] = 0$

$\Rightarrow \cancel{4}\frac{d}{d\theta}\left[\sin\theta - \sin^2\theta\right] = 0$

$\Rightarrow \cos\theta - 2\sin\theta\cos\theta = 0$

$\Rightarrow \cos\theta(1 - 2\sin\theta) = 0$

HENCE THE SOLUTIONS FOR $0 \le \theta \le \pi$

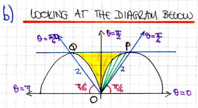
$\cos\theta = 0 \Rightarrow \cancel{\theta = \pi/2}$

$\sin\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{6}, \frac{5\pi}{6}$

$\Rightarrow r = 4(1-\frac{1}{2}) = 2$, $4(1-\frac{1}{2}) = 2$

$\therefore P(2, \frac{\pi}{6})$ & $Q(2, \frac{5\pi}{6})$

b) LOOKING AT THE DIAGRAM BELOW



AREA OF $\hat{OPQ} = \frac{1}{2}|OP||OQ|\sin(\pi - 2\times\frac{\pi}{6})$

$= \frac{1}{2}\times 2\times 2\times \sin(\frac{2\pi}{3})$

$= \sqrt{3}$

AREA OF THE "GREEN" POLAR SECTORS FROM $\theta = \frac{\pi}{6}$ TO $\theta = \frac{\pi}{2}$

$A = \int_{\theta_1}^{\theta_2} \frac{1}{2}r^2\,d\theta = \int_{\theta=\frac{\pi}{6}}^{\theta=\frac{\pi}{2}} \frac{1}{2}\left[4(1-\sin\theta)\right]^2 d\theta = \int_{\theta=\frac{\pi}{6}}^{\frac{\pi}{2}} 8(1-\sin\theta)^2\,d\theta$

$= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 8(1 - 2\sin\theta + \sin^2\theta)\,d\theta = \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 8 - 16\sin\theta + 8\sin^2\theta\,d\theta$

$= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 8 - 16\sin\theta + 8\left(\frac{1}{2} - \frac{1}{2}\cos 2\theta\right)d\theta$

$= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 12 - 16\sin\theta - 4\cos 2\theta\,d\theta = \left[12\theta + 16\cos\theta - 2\sin 2\theta\right]_{\frac{\pi}{6}}^{\frac{\pi}{2}}$

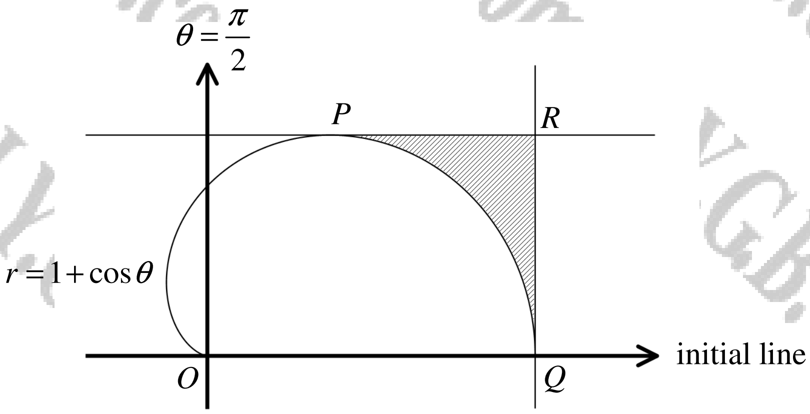
$= (6\pi + 0 - 0) - (2\pi + 8\sqrt{3} - \sqrt{3}) = 4\pi - 7\sqrt{3}$

THUS THE REQUIRED AREA IS GIVEN BY $\sqrt{3} - 2(4\pi - 7\sqrt{3})$

$= \sqrt{3} - 8\pi + 14\sqrt{3}$

$= 15\sqrt{3} - 8\pi$ AS REQUIRED

Question 11 (****)



The diagram above shows the curve with polar equation

$$r = 1 + \cos\theta, \quad 0 \le \theta \le \pi.$$

The curve meets the initial line at the origin O and at the point Q. The point P lies on the curve so that the tangent to the curve at P is parallel to the initial line.

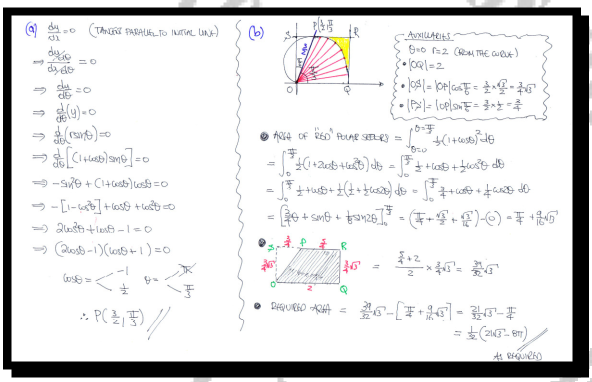
a) Determine the polar coordinates of P.

The tangent to the curve at Q is perpendicular to the initial line and meets the tangent to the curve at P, at the point R.

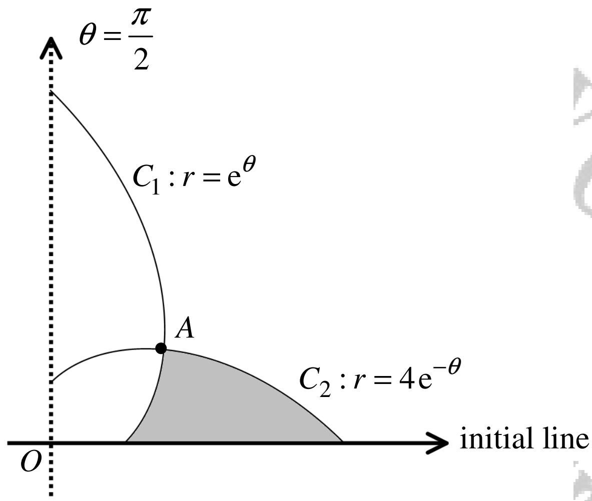
b) Show that the area of the finite region bounded by the line segments PR, QR and the arc PQ is

$$\frac{1}{32}\left(21\sqrt{3} - 8\pi\right).$$

$$\boxed{P\left(\frac{3}{2}, \frac{\pi}{3}\right)}$$



Question 12 (****)



The diagram below shows the curves with polar equations

$$C_1 : r = e^{\theta}, \quad 0 \le \theta \le \frac{\pi}{2}$$

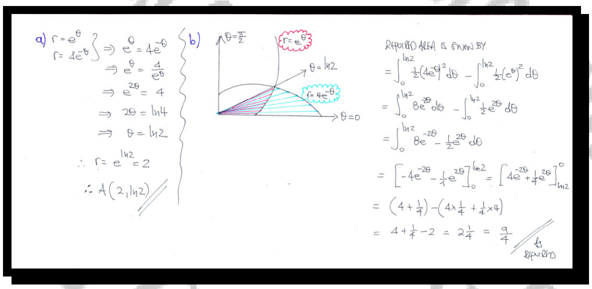
$$C_2 : r = 4e^{-\theta}, \quad 0 \le \theta \le \frac{\pi}{2}.$$

The curves intersect at the point A.

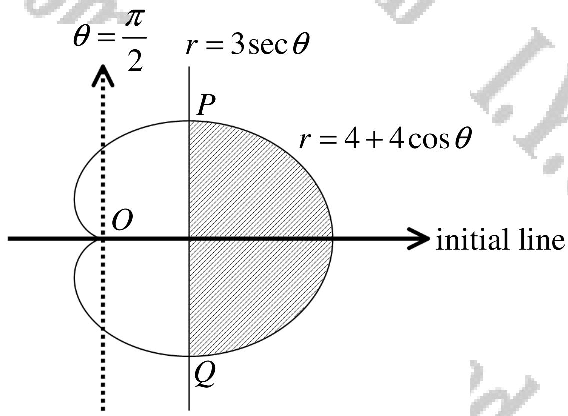
a) Find the exact polar coordinates of A.

b) Show that area of the shaded region is $\frac{9}{4}$.

$\boxed{A(2, \ln 2)}$



Question 13 (****)



The figure above shows a curve and a straight line with respective polar equations

$$r = 4 + 4\cos\theta,\ -\pi < \theta \le \pi \quad \text{and} \quad r = 3\sec\theta,\ -\frac{\pi}{2} < \theta \le \frac{\pi}{2}.$$

The straight line meets the curve at two points, P and Q.

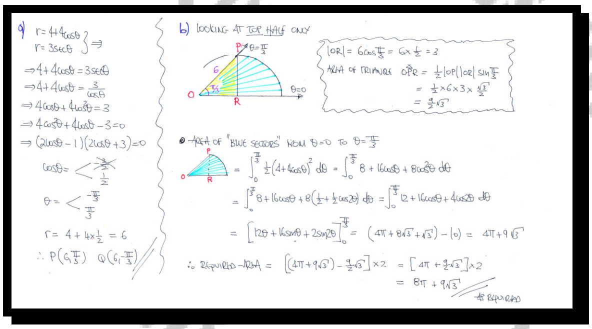
a) Determine the polar coordinates of P and Q.

The finite region, shown shaded in the figure, is bounded by the curve and the straight line.

b) Show that the area of this finite region is

$$8\pi + 9\sqrt{3}.$$

$$\boxed{P\left(6, \frac{\pi}{3}\right), Q\left(6, -\frac{\pi}{3}\right)}$$



a) $r = 4+4\cos\theta$, $r = 3\sec\theta$ $\Rightarrow$

$\Rightarrow 4+4\cos\theta = 3\sec\theta$

$\Rightarrow 4+4\cos\theta = \frac{3}{\cos\theta}$

$\Rightarrow 4\cos\theta + 4\cos^2\theta = 3$

$\Rightarrow 4\cos^2\theta + 4\cos\theta - 3 = 0$

$\Rightarrow (2\cos\theta - 1)(2\cos\theta + 3) = 0$

$\cos\theta = \cancel{-\frac{3}{2}},\ \frac{1}{2}$

$\theta = -\frac{\pi}{3},\ \frac{\pi}{3}$

$r = 4 + 4\times\frac{1}{2} = 6$

$\therefore P\left(6, \frac{\pi}{3}\right)\ Q\left(6, -\frac{\pi}{3}\right)$

b) LOOKING AT TOP HALF ONLY

P, $\theta = \frac{\pi}{3}$, 6, $\frac{\pi}{3}$, O, R, $\theta = 0$

$|OR| = 6\cos\frac{\pi}{3} = 6\times\frac{1}{2} = 3$

AREA OF TRIANGLE $O\hat{P}R = \frac{1}{2}|OP||OR|\sin\frac{\pi}{3}$

$= \frac{1}{2}\times 6\times 3\times\frac{\sqrt{3}}{2}$

$= \frac{9}{2}\sqrt{3}$

• AREA OF "BLUE SECTORS" FROM $\theta = 0$ TO $\theta = \frac{\pi}{3}$

$= \int_0^{\frac{\pi}{3}} \frac{1}{2}(4+4\cos\theta)^2\, d\theta = \int_0^{\frac{\pi}{3}} 8 + 16\cos\theta + 8\cos^2\theta\, d\theta$

$= \int_0^{\frac{\pi}{3}} 8 + 16\cos\theta + 8\left(\frac{1}{2} + \frac{1}{2}\cos 2\theta\right) d\theta = \int_0^{\frac{\pi}{3}} 12 + 16\cos\theta + 4\cos 2\theta\, d\theta$

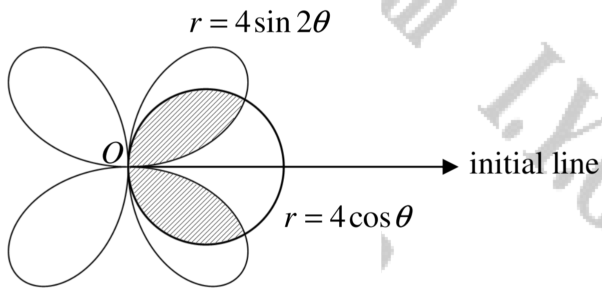
$= \left[12\theta + 16\sin\theta + 2\sin 2\theta\right]_0^{\frac{\pi}{3}} = \left(4\pi + 8\sqrt{3} + \sqrt{3}\right) - (0) = 4\pi + 9\sqrt{3}$

$\therefore$ REQUIRED AREA $= \left[\left(4\pi + 9\sqrt{3}\right) - \frac{9}{2}\sqrt{3}\right]\times 2 = \left[4\pi + \frac{9}{2}\sqrt{3}\right]\times 2$

$= 8\pi + 9\sqrt{3}$

AS REQUIRED

Question 14 (****)



The figure above shows the curves with polar equations

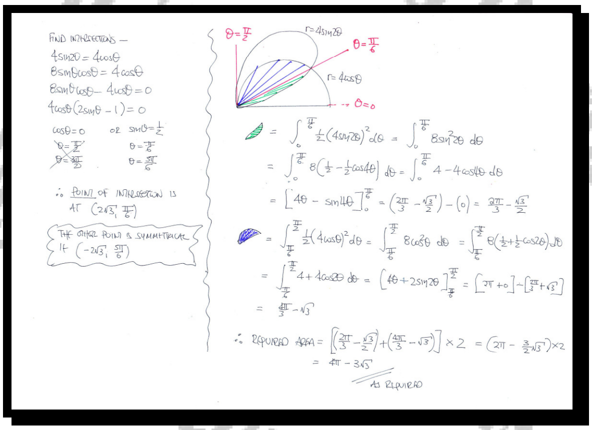
$$r = 4\cos\theta, \ 0 \le \theta \le 2\pi,$$

$$r = 4\sin 2\theta, \ 0 \le \theta \le 2\pi.$$

Show that the area of the shaded region which consists of all the points which are bounded by **both** curves is

$$4\pi - 3\sqrt{3}.$$

proof



FIND INTERSECTIONS –

$4\sin 2\theta = 4\cos\theta$
$8\sin\theta\cos\theta = 4\cos\theta$
$8\sin\theta\cos\theta - 4\cos\theta = 0$
$4\cos\theta(2\sin\theta - 1) = 0$

$\cos\theta = 0$ OR $\sin\theta = \frac{1}{2}$
$\cancel{\theta = \frac{\pi}{2}}$ $\quad \theta = \frac{\pi}{6}$
$\cancel{\theta = \frac{3\pi}{2}}$ $\quad \theta = \frac{5\pi}{6}$

$\therefore$ POINT OF INTERSECTION IS AT $(2\sqrt{3}, \frac{\pi}{6})$

THE OTHER POINT IS SYMMETRICAL AT $(-2\sqrt{3}, \frac{5\pi}{6})$

$= \int_0^{\frac{\pi}{6}} \frac{1}{2}(4\sin 2\theta)^2 \, d\theta = \int_0^{\frac{\pi}{6}} 8\sin^2 2\theta \, d\theta$

$= \int_0^{\frac{\pi}{6}} 8\left(\frac{1}{2} - \frac{1}{2}\cos 4\theta\right) d\theta = \int_0^{\frac{\pi}{6}} 4 - 4\cos 4\theta \, d\theta$

$= \left[4\theta - \sin 4\theta\right]_0^{\frac{\pi}{6}} = \left(\frac{2\pi}{3} - \frac{\sqrt{3}}{2}\right) - (0) = \frac{2\pi}{3} - \frac{\sqrt{3}}{2}$

$= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{1}{2}(4\cos\theta)^2 \, d\theta = \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 8\cos^2\theta \, d\theta = \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 8\left(\frac{1}{2} + \frac{1}{2}\cos 2\theta\right) d\theta$

$= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 4 + 4\cos 2\theta \, d\theta = \left[4\theta + 2\sin 2\theta\right]_{\frac{\pi}{6}}^{\frac{\pi}{2}} = \left[2\pi + 0\right] - \left[\frac{2\pi}{3} + \sqrt{3}\right]$

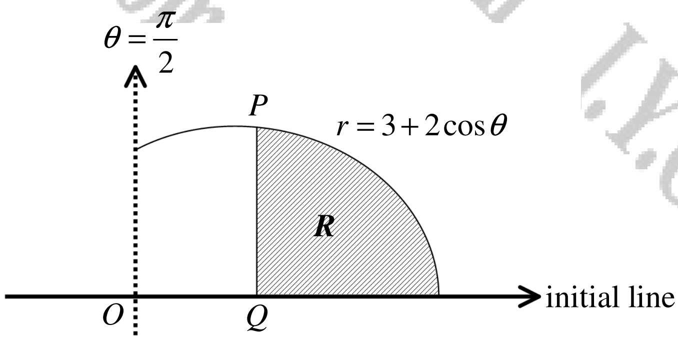
$= \frac{4\pi}{3} - \sqrt{3}$

$\therefore$ REQUIRED AREA $= \left[\left(\frac{2\pi}{3} - \frac{\sqrt{3}}{2}\right) + \left(\frac{4\pi}{3} - \sqrt{3}\right)\right] \times 2 = \left(2\pi - \frac{3}{2}\sqrt{3}\right) \times 2$

$= 4\pi - 3\sqrt{3}$

AS REQUIRED

Question 15 (****)



The figure above shows the cardioid with polar equation

$$r = 3 + 2\cos\theta, \ 0 < \theta \le \frac{\pi}{2}.$$

The point P lies on the cardioid and its distance from the pole O is 4 units.

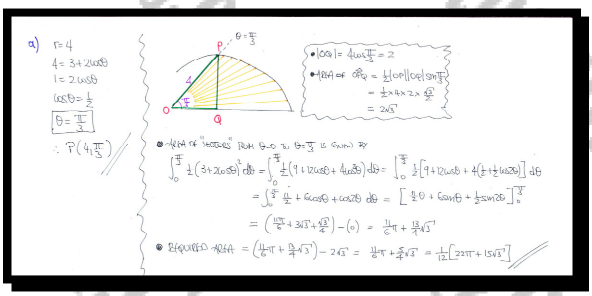
a) Determine the polar coordinates of P.

The point Q lies on the initial line so that the line segment PQ is perpendicular to the initial line. The finite region R, shown shaded in the figure, is bounded by the curve, the initial line and the line segment PQ.

b) Show that the area of R is

$$\frac{1}{12}\left(22\pi + 15\sqrt{3}\right).$$

$$\boxed{P\left(4, \frac{\pi}{3}\right)}$$



a) $r = 4$

$4 = 3 + 2\cos\theta$

$1 = 2\cos\theta$

$\cos\theta = \frac{1}{2}$

$\boxed{\theta = \frac{\pi}{3}}$

$\therefore P\left(4, \frac{\pi}{3}\right)$

$\theta = \frac{\pi}{3}$, P, 4, $\frac{\pi}{3}$, O, Q

• $|OQ| = 4\cos\frac{\pi}{3} = 2$

• AREA OF $O\hat{P}Q = \frac{1}{2}|OP||OQ|\sin\frac{\pi}{3}$
$= \frac{1}{2} \times 4 \times 2 \times \frac{\sqrt{3}}{2}$
$= 2\sqrt{3}$

• AREA OF "SECTORS" FROM $\theta = 0$ TO $\theta = \frac{\pi}{3}$ IS GIVEN BY

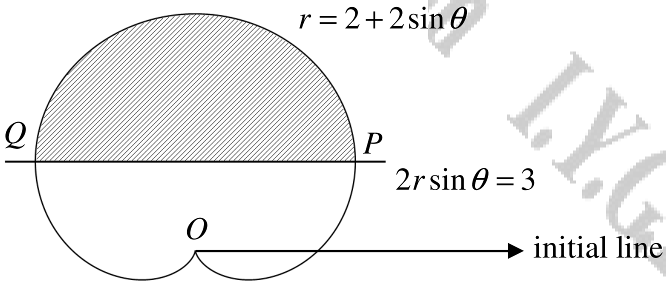
$\int_0^{\frac{\pi}{3}} \frac{1}{2}(3 + 2\cos\theta)^2 \, d\theta = \int_0^{\frac{\pi}{3}} \frac{1}{2}(9 + 12\cos\theta + 4\cos^2\theta) \, d\theta = \int_0^{\frac{\pi}{3}} \frac{1}{2}\left[9 + 12\cos\theta + 4\left(\frac{1}{2} + \frac{1}{2}\cos 2\theta\right)\right] d\theta$

$= \int_0^{\frac{\pi}{3}} \frac{11}{2} + 6\cos\theta + \cos 2\theta \, d\theta = \left[\frac{11}{2}\theta + 6\sin\theta + \frac{1}{2}\sin 2\theta\right]_0^{\frac{\pi}{3}}$

$= \left(\frac{11\pi}{6} + 3\sqrt{3} + \frac{\sqrt{3}}{4}\right) - (0) = \frac{11}{6}\pi + \frac{13}{4}\sqrt{3}$

• REQUIRED AREA $= \left(\frac{11}{6}\pi + \frac{13}{4}\sqrt{3}\right) - 2\sqrt{3} = \frac{11}{6}\pi + \frac{5}{4}\sqrt{3} = \frac{1}{12}\left[22\pi + 15\sqrt{3}\right]$

Question 16 (****)



The figure above shows the curve with polar equation

$$r = 2 + 2\sin\theta,\ 0 \le \theta \le 2\pi,$$

intersected by the straight line with polar equation

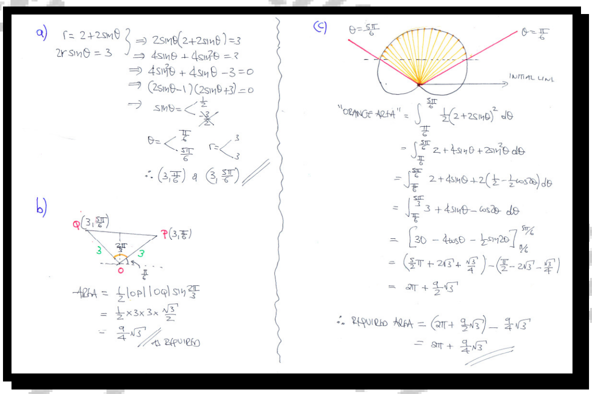
$$2r\sin\theta = 3,\ 0 < \theta < \pi.$$

a) Find the coordinates of the points P and Q, where the line meets the curve.

b) Show that the area of the triangle OPQ is $\frac{9}{4}\sqrt{3}$.

c) Hence find the exact area of the **shaded** region bounded by the curve and the straight line.

$\boxed{P\left(3, \frac{5\pi}{6}\right)}$, $\boxed{Q\left(3, \frac{\pi}{6}\right)}$, $\boxed{\text{area} = 2\pi + \frac{9}{4}\sqrt{3}}$



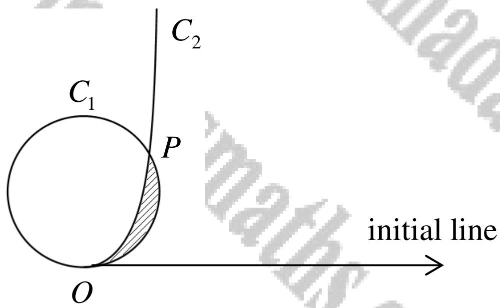
Question 17 (****)

The curves C_1 and C_2 have respective polar equations

$$C_1: \; r = 2\sin\theta, \; 0 \le \theta < 2\pi$$

$$C_2: \; r = \tan\theta, \; 0 \le \theta < \frac{\pi}{2}.$$

a) Find a Cartesian equation for C_1 and a Cartesian equation for C_2.



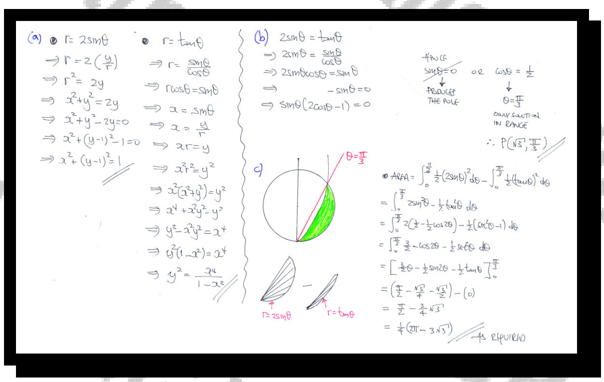
The figure above shows the two curves intersecting at the pole and at the point P.

The finite region, shown shaded in the figure, is bounded by the two curves.

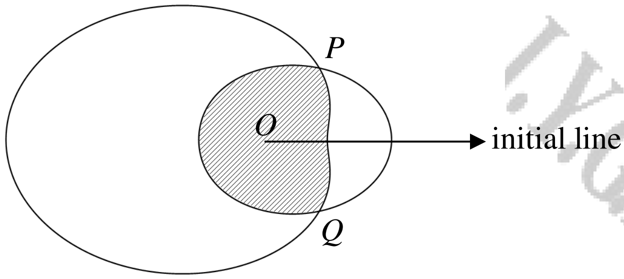
b) Determine the exact polar coordinates of P.

c) Show that the area of the shaded region is $\frac{1}{2}\left(2\pi - 3\sqrt{3}\right)$.

$\boxed{C_1 : x^2 + (y-1)^2 = 1}$, $\boxed{C_1 : x^2 + (y-1)^2 = 1}$, $\boxed{P\left(\sqrt{3}, \frac{\pi}{3}\right)}$



Question 18 (****)



The figure above shows two overlapping closed curves C_1 and C_2, with respective polar equations

$$C_1: \ r = 3 + \cos\theta, \ 0 \le \theta < 2\pi$$

$$C_2: \ r = 5 - 3\cos\theta, \ 0 \le \theta < 2\pi.$$

The curves meet at two points, P and Q.

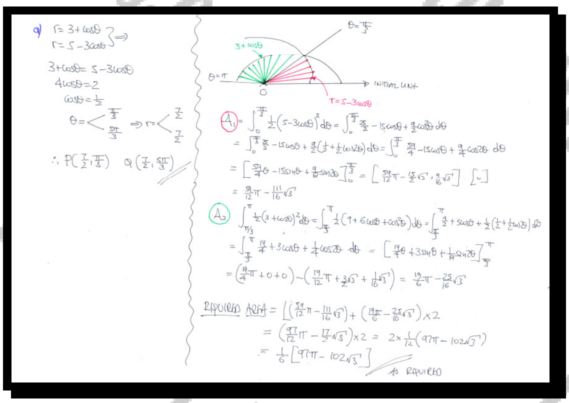
a) Determine the polar coordinates of P and Q.

The finite region R, shown shaded in the figure, consists of all the points which lie **inside both** C_1 and C_2.

b) Show that the area of R is

$$\frac{1}{6}\left(97\pi - 102\sqrt{3}\right).$$

$$\boxed{P\left(\frac{7}{2}, \frac{\pi}{3}\right), Q\left(\frac{7}{2}, \frac{5\pi}{3}\right)},$$



a) $r = 3 + \cos\theta$, $r = 5 - 3\cos\theta$ $\Rightarrow$

$3 + \cos\theta = 5 - 3\cos\theta$

$4\cos\theta = 2$

$\cos\theta = \frac{1}{2}$

$\theta = \frac{\pi}{3}, \frac{5\pi}{3} \Rightarrow r = \frac{7}{2}, \frac{7}{2}$

$\therefore P\left(\frac{7}{2}, \frac{\pi}{3}\right) \quad Q\left(\frac{7}{2}, \frac{5\pi}{3}\right)$

$\theta = \frac{\pi}{3}$, $3 + \cos\theta$, $\theta = \pi$, O, INITIAL LINE, $r = 5 - 3\cos\theta$

$A_1 = \int_0^{\frac{\pi}{3}} \frac{1}{2}(5 - 3\cos\theta)^2 \, d\theta = \int_0^{\frac{\pi}{3}} \frac{25}{2} - 15\cos\theta + \frac{9}{2}\cos^2\theta \, d\theta$

$= \int_0^{\frac{\pi}{3}} \frac{25}{2} - 15\cos\theta + \frac{9}{2}\left(\frac{1}{2} + \frac{1}{2}\cos 2\theta\right) d\theta = \int_0^{\frac{\pi}{3}} \frac{59}{4} - 15\cos\theta + \frac{9}{4}\cos 2\theta \, d\theta$

$= \left[\frac{59}{4}\theta - 15\sin\theta + \frac{9}{8}\sin 2\theta\right]_0^{\frac{\pi}{3}} = \left[\frac{59}{12}\pi - \frac{15}{2}\sqrt{3} + \frac{9}{16}\sqrt{3}\right] - [0]$

$= \frac{59}{12}\pi - \frac{111}{16}\sqrt{3}$

$A_2 = \int_{\pi/3}^{\pi} \frac{1}{2}(3 + \cos\theta)^2 \, d\theta = \int_{\frac{\pi}{3}}^{\pi} \frac{1}{2}(9 + 6\cos\theta + \cos^2\theta) \, d\theta = \int_{\frac{\pi}{3}}^{\pi} \frac{9}{2} + 3\cos\theta + \frac{1}{2}\left(\frac{1}{2} + \frac{1}{2}\cos 2\theta\right) d\theta$

$= \int_{\frac{\pi}{3}}^{\pi} \frac{19}{4} + 3\cos\theta + \frac{1}{4}\cos 2\theta \, d\theta = \left[\frac{19}{4}\theta + 3\sin\theta + \frac{1}{8}\sin 2\theta\right]_{\frac{\pi}{3}}^{\pi}$

$= \left(\frac{19}{4}\pi + 0 + 0\right) - \left(\frac{19}{12}\pi + \frac{3}{2}\sqrt{3} + \frac{1}{16}\sqrt{3}\right) = \frac{19}{6}\pi - \frac{25}{16}\sqrt{3}$

REQUIRED AREA $= \left[\left(\frac{59}{12}\pi - \frac{111}{16}\sqrt{3}\right) + \left(\frac{19\pi}{6} - \frac{25}{16}\sqrt{3}\right)\right] \times 2$

$= \left(\frac{97}{12}\pi - \frac{17}{2}\sqrt{3}\right) \times 2 = 2 \times \frac{1}{12}\left(97\pi - 102\sqrt{3}\right)$

$= \frac{1}{6}\left[97\pi - 102\sqrt{3}\right]$ AS REQUIRED

Question 19 (****)

The curve C with polar equation

$$r = \sqrt{6}\cos 2\theta, \ 0 \le \theta \le \frac{\pi}{4}.$$

The straight line l is parallel to the initial line and is a tangent to C.

Find an equation of l, giving the answer in the form $r = f(\theta)$.

$$\boxed{r = \frac{2}{3}\operatorname{cosec}\theta}$$

$r = \sqrt{6}\cos 2\theta$

"PARALLEL TO THE INITIAL LINE" $\Rightarrow \frac{d}{d\theta}(r\sin\theta) = 0$

THUS

$\frac{d}{d\theta}\left[\cancel{\sqrt{6}}\cos 2\theta \sin\theta\right] = 0$

$\Rightarrow -2\sin 2\theta \sin\theta + \cos 2\theta \cos\theta = 0$

$\Rightarrow -4\sin^2\theta\cos\theta + \cos\theta(1 - 2\sin^2\theta) = 0$

$\Rightarrow \cos\theta\left[-4\sin^2\theta + 1 - 2\sin^2\theta\right] = 0$

$\Rightarrow \cos\theta\left[1 - 6\sin^2\theta\right] = 0$

$\cancel{\cos\theta = 0}$ NO SOLUTIONS IN RANGE

$\sin^2\theta = \frac{1}{6}$

$\sin\theta = +\frac{1}{\sqrt{6}}$ ONLY SOLUTION IN RANGE

NOW

$r = \sqrt{6}\cos 2\theta$

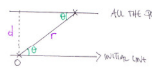
$r = \sqrt{6}\left[1 - 2\sin^2\theta\right]$

$r = \sqrt{6}\left[1 - 2 \times \frac{1}{6}\right]$

$r = \sqrt{6} \times \frac{2}{3}$

$r = \frac{2}{3}\sqrt{6}$

i.e. $\left[\frac{2}{3}\sqrt{6}, \arcsin\frac{1}{\sqrt{6}}\right]$



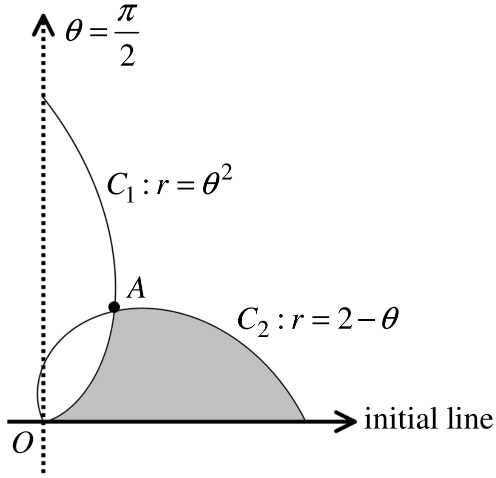
ALL THE POINTS MUST SATISFY

$r\sin\theta = d = \left(\frac{2}{3}\sqrt{6}\right)\left(\frac{1}{\sqrt{6}}\right)$

$r\sin\theta = \frac{2}{3}$

$r = \frac{2}{3}\operatorname{cosec}\theta$

Question 20 (****)



The diagram above shows the curves with polar equations

$$C_1 : r = \theta^2, \; 0 \le \theta \le \frac{\pi}{2}$$

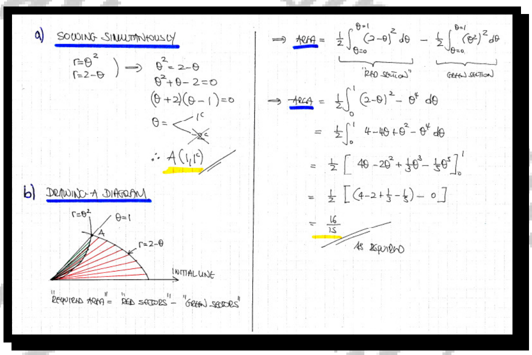
$$C_2 : r = 2 - \theta, \; 0 \le \theta \le 2.$$

The curves intersect at the point A.

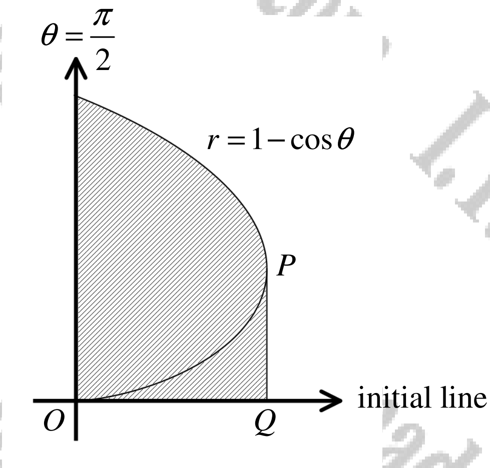
a) Find the polar coordinates of A.

b) Show that the area of the shaded region is $\frac{16}{15}$.

$\boxed{A(1,1)}$



Question 21 (****)



The figure above shows the curve C with polar equation

$$r = 1 - \cos\theta, \; 0 \le \theta < \frac{\pi}{2}.$$

The point P lies on C so that tangent to C is perpendicular to the initial line.

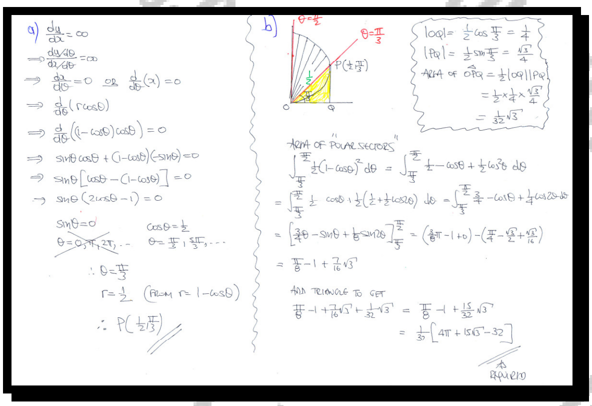
a) Determine the polar coordinates of P.

The finite region R consists of all the points which are bounded by C, the straight line segment PQ, the initial line and the line with equation $\theta = \frac{\pi}{2}$.

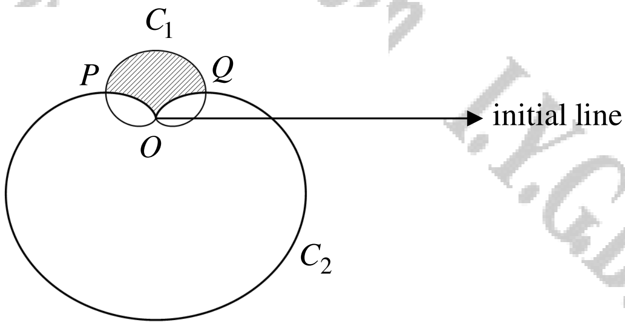
b) Show that the area of R, shown shaded in the figure above, is exactly

$$\frac{1}{32}\left(4\pi + 15\sqrt{3} - 32\right).$$

$$\boxed{P\left(\frac{1}{2}, \frac{\pi}{3}\right)}$$



Question 22 (****)



The figure above shows two closed curves with polar equations

$$C_1: \ r = a(1+\sin\theta), \ 0 \le \theta \le 2\pi \quad \text{and} \quad C_2: \ r = 3a(1-\sin\theta), \ 0 \le \theta \le 2\pi,$$

intersecting each other at the pole O and at the points P and Q.

a) Find the polar coordinates of the points P and Q.

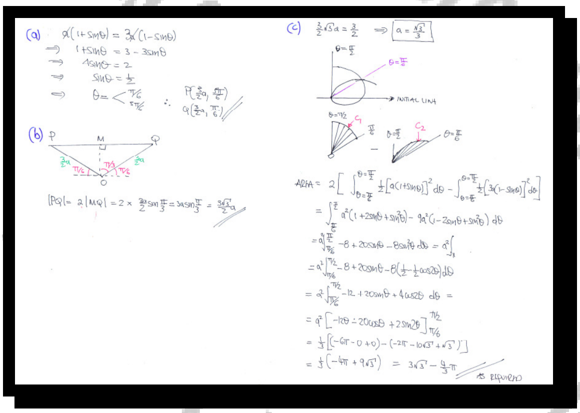
b) Show that the distance PQ is $\frac{3\sqrt{3}}{2}a$.

The finite region shown shaded in the above figure consists of all the points inside C_1 but outside C_2.

c) Given that the distance PQ is $\frac{3}{2}$, show that the area of the **shaded** region is

$$3\sqrt{3} - \frac{4}{3}\pi.$$

$\boxed{P\left(\frac{3}{2}a, \frac{5\pi}{6}\right)}$, $\boxed{Q\left(\frac{3}{2}a, \frac{\pi}{6}\right)}$



Question 23 (****)

The points A and B have respective coordinates $(-1,0)$ and $(1,0)$.

The locus of the point $P(x,y)$ traces a curve in such a way so that $|AP||BP| = 1$.

a) By forming a Cartesian equation of the locus of P, show that the polar equation of the curve is

$$r^2 = 2\cos 2\theta, \ 0 \le \theta < 2\pi .$$

b) Sketch the curve.

$\boxed{}$, $\boxed{\text{proof}}$

a) DETERMINE THE CARTESIAN LOCUS

$A(-1,0) \quad B(1,0) \quad P(x,y)$

- $|AP| = \sqrt{(x+1)^2 + y^2}$
- $|BP| = \sqrt{(x-1)^2 + y^2}$

$$\Rightarrow |AP||BP| = 1$$
$$\Rightarrow |AP|^2|BP|^2 = 1$$
$$\Rightarrow \left[(x+1)^2 + y^2\right]\left[(x-1)^2 + y^2\right] = 1$$
$$\Rightarrow \left\{\begin{matrix} y^2(x+1)^2 + (x+1)^2(x-1)^2 \\ y^2(x-1)^2 + y^4 \end{matrix}\right\} = 1$$
$$\Rightarrow y^2\left[(x+1)^2 + (x-1)^2\right] + y^4 + (x+1)^2(x-1)^2 = 1$$
$$\Rightarrow y^2\left[x^2+2x+1+x^2-2x+1\right] + y^4 + (x^2-1)^2 = 1$$
$$\Rightarrow y^2\left[2x^2+2\right] + y^4 + x^4 - 2x^2 + 1 = 1$$
$$\Rightarrow 2x^2y^2 + 2y^2 + y^4 + x^4 - 2x^2 = 0$$

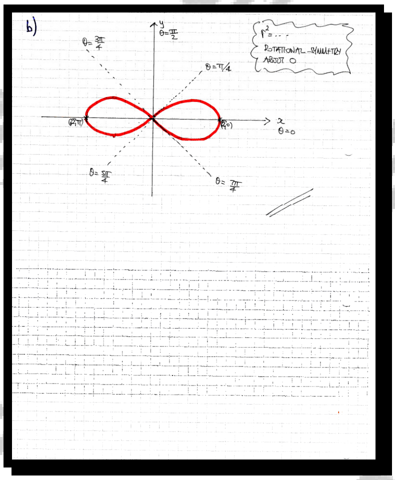
REGROUPING

$$\Rightarrow (y^4 + 2x^2y^2 + x^4) + 2(y^2 - x^2) = 0$$
$$\Rightarrow (x^2+y^2)^2 + 2(y^2 - x^2) = 0$$

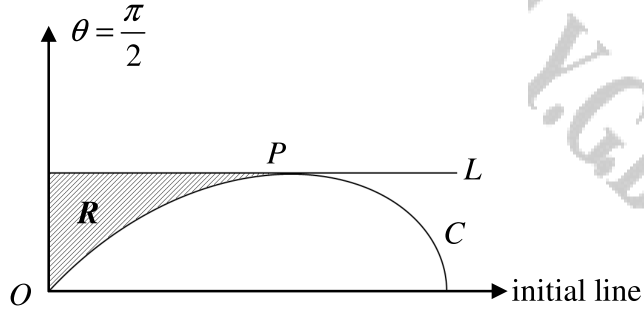
TRANSFORM INTO POLARS

$$\Rightarrow (r^2)^2 + 2(r^2\sin^2\theta - r^2\cos^2\theta) = 0$$
$$\Rightarrow r^4 + 2r^2(\sin^2\theta - \cos^2\theta) = 0$$
$$\Rightarrow r^2 + 2(\sin^2\theta - \cos^2\theta) = 0$$
$$\Rightarrow r^2 = 2(\cos^2\theta - \sin^2\theta)$$
$$\Rightarrow r^2 = 2\cos 2\theta$$

AS REQUIRED



Question 24 (****)



The figure above shows a curve C with polar equation

$$r^2 = 2\cos 2\theta,\ 0 \le \theta < \frac{\pi}{4}.$$

The straight line L is parallel to the initial line and is a tangent to C at the point P.

a) Show that the polar coordinates of P are $\left(1, \frac{\pi}{6}\right)$.

The finite region R, shown shaded in the figure above, is bounded by C, L and the half line with equation $\theta = \frac{\pi}{2}$.

b) Show that the area of R is

$$\tfrac{1}{8}\left(3\sqrt{3} - 4\right).$$

$\boxed{}$, $\boxed{\text{proof}}$

[solution overleaf]

a) PARALLEL TO THE INITIAL LINE IMPLIES $\frac{dy}{dx}=0$

$$\Rightarrow \frac{dy/d\theta}{dx/d\theta}=0$$

$$\Rightarrow dy/d\theta = 0$$

$$\Rightarrow \frac{d}{d\theta}(y)=0$$

$$\Rightarrow \frac{d}{d\theta}(r\sin\theta)=0$$

$$\Rightarrow \frac{d}{d\theta}(r^2\sin^2\theta)=0$$

$$\Rightarrow \frac{d}{d\theta}(2\cos2\theta\sin^2\theta)=0$$

$$\Rightarrow -4\sin2\theta\sin^2\theta + 4\cos2\theta\sin\theta\cos\theta=0$$

$\sin2\theta \equiv 2\sin\theta\cos\theta$

$$\Rightarrow -8\sin^3\theta\cos\theta + 4\cos2\theta\sin\theta\cos\theta=0$$

$$\Rightarrow \cancel{4\sin\theta\cos\theta}\,(\cos2\theta - 2\sin^2\theta)=0$$

$\sin\theta \neq 0$, $\cos\theta \neq 0$
AT THE REQUIRED POINT

$$\Rightarrow \cos2\theta - 2\left(\tfrac{1}{2}-\tfrac{1}{2}\cos2\theta\right)=0$$

$$\Rightarrow \cos2\theta - 1 + \cos2\theta = 0$$

$$\Rightarrow 2\cos2\theta = 1$$

$$\Rightarrow \cos2\theta = \tfrac{1}{2}$$

$$\Rightarrow 2\theta = \tfrac{\pi}{3}, \tfrac{5\pi}{3}, \tfrac{7\pi}{3}, \tfrac{11\pi}{3}, \ldots$$

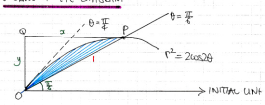
$$\Rightarrow \theta = \tfrac{\pi}{6} \qquad 0 \leqslant \theta < \tfrac{\pi}{4}$$

$$r^2 = 2\cos\left(\tfrac{\pi}{3}\right) = 2\times\tfrac{1}{2} = 1$$

$r=1 \qquad r \geqslant 0$

$\therefore P\left(1, \tfrac{\pi}{6}\right)$

LOOKING AT THE DIAGRAM



$$x = |OP|\cos\tfrac{\pi}{6} = 1\times\tfrac{\sqrt{3}}{2} = \tfrac{\sqrt{3}}{2}$$

$$y = |OP|\sin\tfrac{\pi}{6} = 1\times\tfrac{1}{2} = \tfrac{1}{2}$$

$$O\hat{Q}P \text{ AREA} = \tfrac{1}{2}\times\tfrac{1}{2}\times\tfrac{\sqrt{3}}{2} = \tfrac{\sqrt{3}}{8}$$

NOW THE "SHADED BLUE AREA" OF POLAR SECTOR IS GIVEN BY

$$= \tfrac{1}{2}\int_{\theta_1}^{\theta_2} r^2\,d\theta = \tfrac{1}{2}\int_{\pi/6}^{\pi/4} 2\cos2\theta\,d\theta$$

$$= \left[\tfrac{1}{2}\sin2\theta\right]_{\pi/6}^{\pi/4} = \tfrac{1}{2}\left(\sin\tfrac{\pi}{2} - \sin\tfrac{\pi}{3}\right)$$

$$= \tfrac{1}{2}\left(1-\tfrac{\sqrt{3}}{2}\right) = \tfrac{1}{2} - \tfrac{\sqrt{3}}{4}$$

FINALLY THE REQUIRED AREA IS GIVEN BY

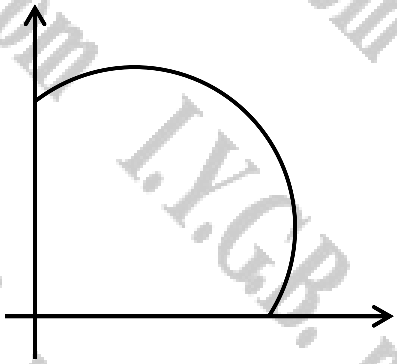
$$\tfrac{\sqrt{3}}{8} - \left(\tfrac{1}{2}-\tfrac{\sqrt{3}}{4}\right) = \tfrac{\sqrt{3}}{8} - \tfrac{1}{2} + \tfrac{\sqrt{3}}{4}$$

$$= \tfrac{3}{8}\sqrt{3} - \tfrac{1}{2}$$

$$= \tfrac{1}{8}\left(3\sqrt{3}-4\right)$$

AS REQUIRED

Question 25 (****)



The figure above shows the curve C, with Cartesian equation

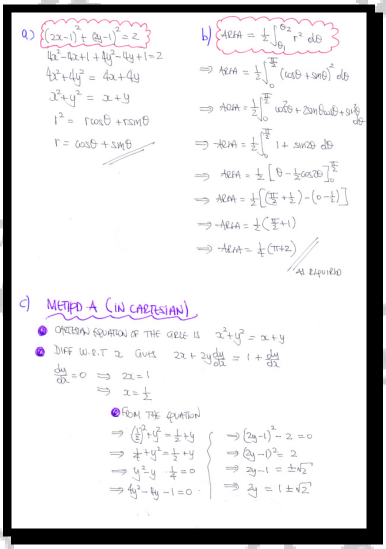
$$(2x-1)^2 + (2y-1)^2 = 2, \; x \ge 0, \; y \ge 0$$

a) Find a polar equation for C, in the form $r = f(\theta)$.

b) Show that the area bounded by C and the coordinate axes is $\frac{1}{4}(\pi + 2)$.

c) Determine, in exact simplified form, the polar coordinates of the point on C, where the tangent to C is parallel to the x axis.

$$\boxed{s = \frac{1}{4}\sqrt{5} + \ln\left[\frac{1}{4}\left(1+\sqrt{5}\right)\right]}$$



a) $(2x-1)^2 + (2y-1)^2 = 2$

$4x^2 - 4x + 1 + 4y^2 - 4y + 1 = 2$

$4x^2 + 4y^2 = 4x + 4y$

$x^2 + y^2 = x + y$

$r^2 = r\cos\theta + r\sin\theta$

$r = \cos\theta + \sin\theta$

b) $\text{AREA} = \frac{1}{2}\int_{\theta_1}^{\theta_2} r^2 \, d\theta$

$\Rightarrow \text{AREA} = \frac{1}{2}\int_0^{\frac{\pi}{2}} (\cos\theta + \sin\theta)^2 \, d\theta$

$\Rightarrow \text{AREA} = \frac{1}{2}\int_0^{\frac{\pi}{2}} \cos^2\theta + 2\sin\theta\cos\theta + \sin^2\theta \, d\theta$

$\Rightarrow \text{AREA} = \frac{1}{2}\int_0^{\frac{\pi}{2}} 1 + \sin 2\theta \, d\theta$

$\Rightarrow \text{AREA} = \frac{1}{2}\left[\theta - \frac{1}{2}\cos 2\theta\right]_0^{\frac{\pi}{2}}$

$\Rightarrow \text{AREA} = \frac{1}{2}\left[\left(\frac{\pi}{2} + \frac{1}{2}\right) - \left(0 - \frac{1}{2}\right)\right]$

$\Rightarrow \text{AREA} = \frac{1}{2}\left(\frac{\pi}{2} + 1\right)$

$\Rightarrow \text{AREA} = \frac{1}{4}(\pi + 2)$

AS REQUIRED

c) METHOD A (IN CARTESIAN)

- CARTESIAN EQUATION OF THE CIRCLE IS $x^2 + y^2 = x + y$
- DIFF W.R.T x GIVES $2x + 2y\frac{dy}{dx} = 1 + \frac{dy}{dx}$

$\frac{dy}{dx} = 0 \Rightarrow 2x = 1$

$\Rightarrow x = \frac{1}{2}$

- FROM THE EQUATION

$\Rightarrow \left(\frac{1}{2}\right)^2 + y^2 = \frac{1}{2} + y$

$\Rightarrow \frac{1}{4} + y^2 = \frac{1}{2} + y$

$\Rightarrow y^2 - y - \frac{1}{4} = 0$

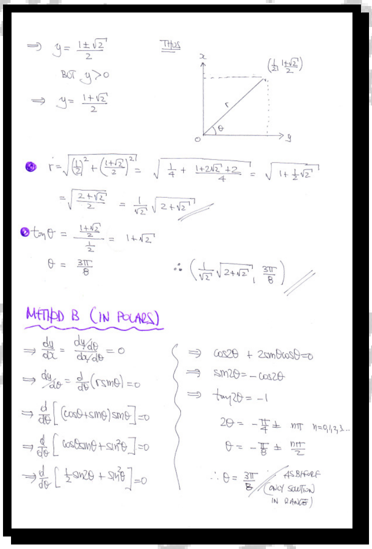
$\Rightarrow 4y^2 - 4y - 1 = 0$

$\Rightarrow (2y-1)^2 - 2 = 0$

$\Rightarrow (2y-1)^2 = 2$

$\Rightarrow 2y - 1 = \pm\sqrt{2}$

$\Rightarrow 2y = 1 \pm \sqrt{2}$



$\Rightarrow y = \frac{1 \pm \sqrt{2}}{2}$

BUT $y > 0$

$\Rightarrow y = \frac{1+\sqrt{2}}{2}$

THUS

$\left(\frac{1}{2}, \frac{1+\sqrt{2}}{2}\right)$

- $r = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{1+\sqrt{2}}{2}\right)^2} = \sqrt{\frac{1}{4} + \frac{1 + 2\sqrt{2} + 2}{4}} = \sqrt{1 + \frac{1}{2}\sqrt{2}}$

$= \sqrt{\frac{2+\sqrt{2}}{2}} = \frac{1}{\sqrt{2}}\sqrt{2+\sqrt{2}}$

- $\tan\theta = \frac{\frac{1+\sqrt{2}}{2}}{\frac{1}{2}} = 1 + \sqrt{2}$

$\theta = \frac{3\pi}{8}$

$\therefore \left(\frac{1}{\sqrt{2}}\sqrt{2+\sqrt{2}}, \frac{3\pi}{8}\right)$

METHOD B (IN POLARS)

$\Rightarrow \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = 0$

$\Rightarrow \frac{dy}{d\theta} = \frac{d}{d\theta}(r\sin\theta) = 0$

$\Rightarrow \frac{d}{d\theta}\left[(\cos\theta + \sin\theta)\sin\theta\right] = 0$

$\Rightarrow \frac{d}{d\theta}\left[\cos\theta\sin\theta + \sin^2\theta\right] = 0$

$\Rightarrow \frac{d}{d\theta}\left[\frac{1}{2}\sin 2\theta + \sin^2\theta\right] = 0$

$\Rightarrow \cos 2\theta + 2\sin\theta\cos\theta = 0$

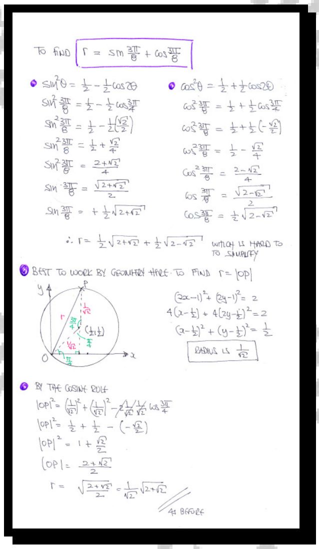
$\Rightarrow \sin 2\theta = -\cos 2\theta$

$\Rightarrow \tan 2\theta = -1$

$2\theta = -\frac{\pi}{4} \pm n\pi \quad n = 0, 1, 2, 3 \ldots$

$\theta = -\frac{\pi}{8} \pm \frac{n\pi}{2}$

$\therefore \theta = \frac{3\pi}{8}$ AS BEFORE (ONLY SOLUTION IN RANGE)



TO FIND $\boxed{r = \sin\frac{3\pi}{8} + \cos\frac{3\pi}{8}}$

- $\sin^2\theta = \frac{1}{2} - \frac{1}{2}\cos 2\theta$

$\sin^2\frac{3\pi}{8} = \frac{1}{2} - \frac{1}{2}\cos\frac{3\pi}{4}$

$\sin^2\frac{3\pi}{8} = \frac{1}{2} - \frac{1}{2}\left(\frac{\sqrt{2}}{2}\right)$

$\sin^2\frac{3\pi}{8} = \frac{1}{2} + \frac{\sqrt{2}}{4}$

$\sin^2\frac{3\pi}{8} = \frac{2+\sqrt{2}}{4}$

$\sin\frac{3\pi}{8} = \frac{\sqrt{2+\sqrt{2}}}{2}$

$\sin\frac{3\pi}{8} = +\frac{1}{2}\sqrt{2+\sqrt{2}}$

- $\cos^2\theta = \frac{1}{2} + \frac{1}{2}\cos 2\theta$

$\cos^2\frac{3\pi}{8} = \frac{1}{2} + \frac{1}{2}\cos\frac{3\pi}{4}$

$\cos^2\frac{3\pi}{8} = \frac{1}{2} + \frac{1}{2}\left(-\frac{\sqrt{2}}{2}\right)$

$\cos^2\frac{3\pi}{8} = \frac{1}{2} - \frac{\sqrt{2}}{4}$

$\cos^2\frac{3\pi}{8} = \frac{2-\sqrt{2}}{4}$

$\cos\frac{3\pi}{8} = \frac{\sqrt{2-\sqrt{2}}}{2}$

$\cos\frac{3\pi}{8} = \frac{1}{2}\sqrt{2-\sqrt{2}}$

$\therefore r = \frac{1}{2}\sqrt{2+\sqrt{2}} + \frac{1}{2}\sqrt{2-\sqrt{2}}$ WHICH IS HARD TO TO SIMPLIFY

- BEST TO WORK BY GEOMETRY HERE TO FIND $r = |OP|$

$(2x-1)^2 + (2y-1)^2 = 2$

$4\left(x-\frac{1}{2}\right) + 4\left(2y-\frac{1}{2}\right)^2 = 2$

$\left(x-\frac{1}{2}\right)^2 + \left(y-\frac{1}{2}\right)^2 = \frac{1}{2}$

$\boxed{\text{RADIUS IS } \frac{1}{\sqrt{2}}}$

- BY THE COSINE RULE

$|OP|^2 = \left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 - 2\frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}\cos\frac{3\pi}{4}$

$|OP|^2 = \frac{1}{2} + \frac{1}{2} - \left(-\frac{\sqrt{2}}{2}\right)$

$|OP|^2 = 1 + \frac{\sqrt{2}}{2}$

$|OP| = \frac{2+\sqrt{2}}{2}$

$r = \sqrt{\frac{2+\sqrt{2}}{2}} = \frac{1}{\sqrt{2}}\sqrt{2+\sqrt{2}}$

AS BEFORE

Question 26 (****)

A curve has polar equation

$$r = \frac{\cos\theta + \sin\theta}{\cos^2\theta + \sin 2\theta + 1}, \quad 0 \le \theta < 2\pi .$$

Find a Cartesian equation of the curve giving the answer in the form $f(x, y) = 0$.

V, $\boxed{}$, $\boxed{2x^2 + 2xy + y^2 - x - y = 0}$

START WITH SOME TRIGONOMETRIC MANIPULATIONS

$$r = \frac{\cos\theta + \sin\theta}{\cos^2\theta + \sin 2\theta + 1} = \frac{\cos\theta + \sin\theta}{\cos^2\theta + 2\sin\theta\cos\theta + (\sin^2\theta + \cos^2\theta)}$$

$$= \frac{\cos\theta + \sin\theta}{(\cos^2\theta + 2\cos\theta\sin\theta + \sin^2\theta) + \cos^2\theta} = \frac{\cos\theta + \sin\theta}{(\cos\theta + \sin\theta)^2 + \cos^2\theta}$$

NOW USING THE STANDARD TRANSFORMATIONS, $x = r\cos\theta$ & $y = r\sin\theta$

$$\Rightarrow r = \frac{\frac{x}{r} + \frac{y}{r}}{\left(\frac{x}{r} + \frac{y}{r}\right)^2 + \frac{x^2}{r^2}}$$

$$\Rightarrow r = \frac{\frac{x}{r} + \frac{y}{r}}{\frac{1}{r^2}(x+y)^2 + \frac{x^2}{r^2}}$$

MULTIPLY TOP & BOTTOM OF THE FRACTION BY r^2

$$\Rightarrow r = \frac{r(x+y)}{(x+y)^2 + x^2}$$

$$\Rightarrow 1 = \frac{x+y}{(x+y)^2 + x^2}$$

$$\Rightarrow (x+y)^2 + x^2 = x + y$$

$$\Rightarrow x^2 + 2xy + y^2 + x^2 = x + y$$

$$\Rightarrow 2x^2 + 2xy + y^2 - x - y = 0$$

10 HARD QUESTIONS

Question 1 (****+)

Show that the polar equation of the top half of the parabola with Cartesian equation

$$y = \sqrt{2x+1}\,,\ x \ge -\tfrac{1}{2}\,,$$

is given by the polar equation

$$r = \frac{1}{1-\cos\theta}\,,\ r \ge 0\,.$$

$\boxed{\text{proof}}$

$y = \sqrt{2x+1}$

$\Rightarrow y^2 = 2x+1$

$\Rightarrow y^2 + x^2 = x^2 + 2x + 1$

$\Rightarrow r^2 = (x+1)^2$

$\Rightarrow r = x+1$

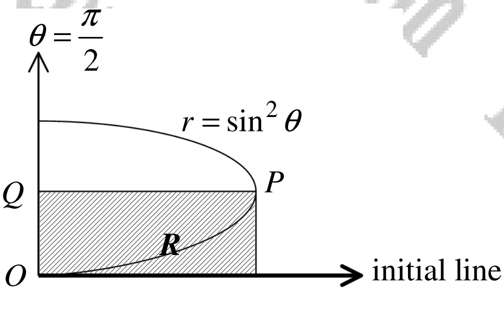
$\Rightarrow r - 1 = r\cos\theta$

$\Rightarrow r - r\cos\theta = 1$

$\Rightarrow r(1-\cos\theta) = 1$

$\Rightarrow r = \dfrac{1}{1-\cos\theta}$ //

Question 2 (**+)**



The figure above shows the curve with polar equation

$$r = \sin^2\theta, \quad 0 \le \theta \le \frac{\pi}{2}.$$

The point P lies on the curve so that the tangent to the curve at P is perpendicular to the initial line.

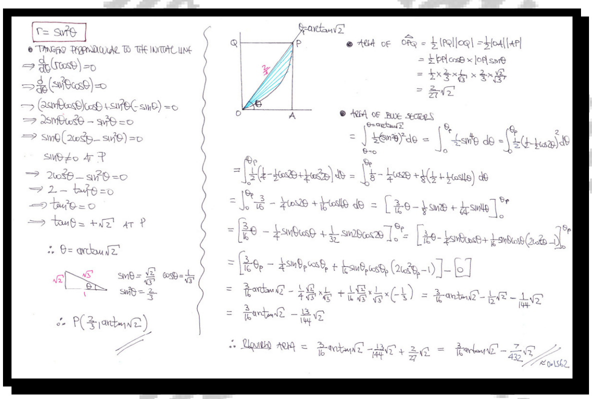
a) Find, in exact form, the polar coordinates of P

The point Q lies on the half line $\theta = \frac{\pi}{2}$, so that PQ is parallel to the initial line.

The finite region R, shown shaded in the above figure, is bounded by the curve and the straight line segments PQ and OQ, where O is the pole.

b) Determine the area of R, in exact simplified form.

$\boxed{P\left(\frac{2}{3}, \arctan\sqrt{2}\right)}$, $\boxed{\text{area} = \frac{1}{2}\arctan\sqrt{2} - \frac{7}{432}\sqrt{2} \approx 0.1562}$



Question 3 (****+)

A curve C has polar equation

$$r = \frac{2}{1+\cos\theta}, \quad 0 \le \theta < 2\pi .$$

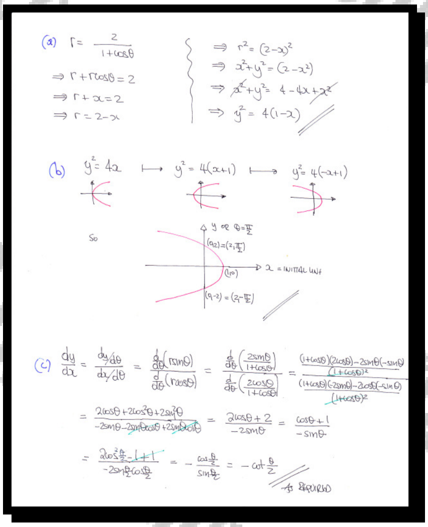
a) Find a Cartesian equation for C.

b) Sketch the graph of C.

c) Show that on any point on C with coordinates (r,θ)

$$\frac{dy}{dx} = -\cot\frac{\theta}{2}.$$

$$\boxed{y^2 = 4(1-x)}$$


(a) $r = \frac{2}{1+\cos\theta}$

$\Rightarrow r + r\cos\theta = 2$

$\Rightarrow r + x = 2$

$\Rightarrow r = 2 - x$

$\Rightarrow r^2 = (2-x)^2$

$\Rightarrow x^2 + y^2 = (2-x^2)$

$\Rightarrow \cancel{x^2} + y^2 = 4 - 4x + \cancel{x^2}$

$\Rightarrow y^2 = 4(1-x)$

(b) $y^2 = 4x \longmapsto y^2 = 4(x+1) \longmapsto y^2 = 4(-x+1)$

So

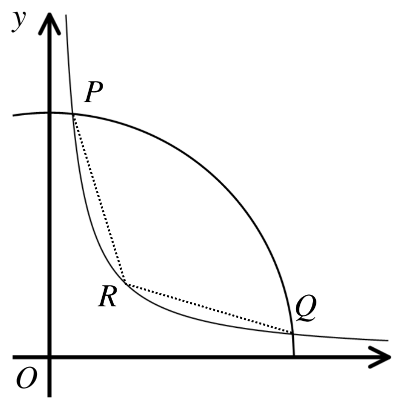
(c) $\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\frac{d}{d\theta}(r\sin\theta)}{\frac{d}{d\theta}(r\cos\theta)} = \frac{\frac{d}{d\theta}\left(\frac{2\sin\theta}{1+\cos\theta}\right)}{\frac{d}{d\theta}\left(\frac{2\cos\theta}{1+\cos\theta}\right)} = \frac{\frac{(1+\cos\theta)(2\cos\theta) - 2\sin\theta(-\sin\theta)}{\cancel{(1+\cos\theta)^2}}}{\frac{(1+\cos\theta)(-2\sin\theta) - 2\cos\theta(-\sin\theta)}{\cancel{(1+\cos\theta)^2}}}$

$= \frac{2\cos\theta + 2\cos^2\theta + 2\sin^2\theta}{-2\sin\theta - \cancel{2\sin\theta\cos\theta} + \cancel{2\sin\theta\cos\theta}} = \frac{2\cos\theta + 2}{-2\sin\theta} = \frac{\cos\theta + 1}{-\sin\theta}$

$= \frac{2\cos^2\frac{\theta}{2} \cancel{- 1 + 1}}{-2\sin\frac{\theta}{2}\cos\frac{\theta}{2}} = -\frac{\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}} = -\cot\frac{\theta}{2}$

AS REQUIRED

Question 4 (****+)



The figure above shows a hyperbola and a circle with respective Cartesian equations

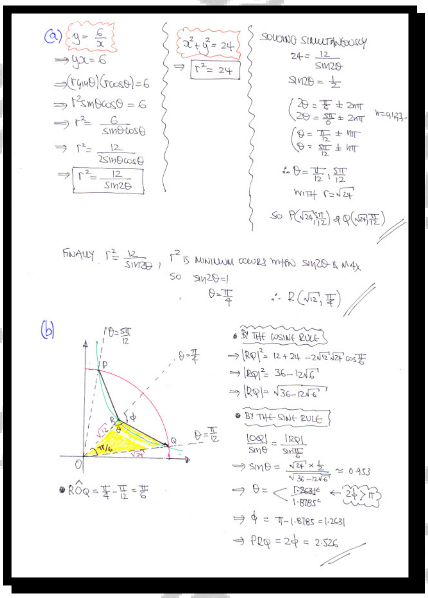
$$y = \frac{6}{x}, \; x > 0 \qquad \text{and} \qquad x^2 + y^2 = 8, \; x > 0, \; y > 0.$$

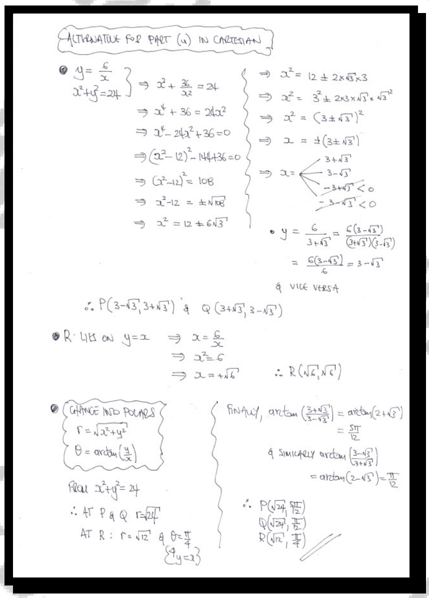
The points P and Q are the points of intersection between the hyperbola and the circle, and the point R lies on the hyperbola so that the distance OR is least.

a) Determine the **polar** coordinates of P, Q and R.

b) Calculate in radians the angle PRQ, correct to 3 decimal places.

$\boxed{P\left(\sqrt{24}, \frac{5\pi}{12}\right)}$, $\boxed{Q\left(\sqrt{24}, \frac{\pi}{12}\right)}$, $\boxed{R\left(\sqrt{12}, \frac{\pi}{4}\right)}$, $\boxed{\measuredangle ABC \approx 2.526^{c}}$





Question 5 (****+)

The curve C has Cartesian equation

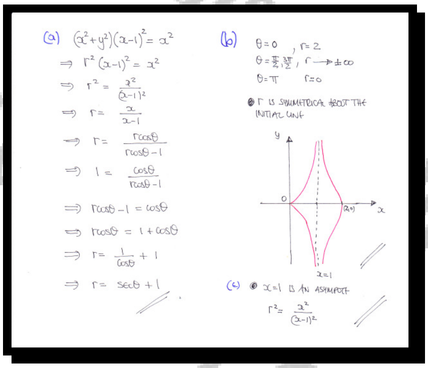
$$\left(x^2+y^2\right)(x-1)^2 = x^2 .$$

a) Find a polar equation of C in the form $r = f(\theta)$.

b) Sketch the curve in the Cartesian plane.

c) State the equation of the asymptote of the curve.

$\boxed{r = 1+\sec\theta}$, $\boxed{x=1}$



Question 6 (****+)

The following polar equations are given.

$$r_1 = \cos\theta , \quad 0 \le \theta \le \pi .$$

$$r_2 = \frac{1}{\cos\theta - \sin\theta} , \quad -\tfrac{1}{4}\pi \le \theta \le \tfrac{5}{4}\pi .$$

Find, in exact simplified form, the area of the **smaller** of the two finite regions, bounded by r_1 and r_2.

V , $\boxed{\text{area} = \dfrac{3\pi + 2}{16}}$

IT IS ACTUALLY BETTER TO WORK IN CARTESIAN, AT LEAST FOR THE SKETCH

$r = \dfrac{1}{\cos\theta - \sin\theta}$

$r\cos\theta - r\sin\theta = 1$

$x - y = 1$

$y = x - 1$

$r = \cos\theta$

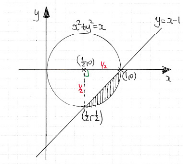
$r^2 = r\cos\theta$

$x^2 + y^2 = x$

$x^2 - x + y^2 = 0$

$(x - \frac{1}{2})^2 + y^2 = \frac{1}{4}$

DRAWING A SKETCH AND IDENTIFY THE REGION



$\left.\begin{array}{l} y^2 + x^2 = x \\ y = x - 1 \end{array}\right\} \Rightarrow$

$(x-1)^2 + x^2 = x$

$x^2 - 2x + 1 + x^2 = x$

$2x^2 - 3x + 1 = 0$

$(2x-1)(x-1) = 0$

$x = \begin{cases} 1 \\ \frac{1}{2} \end{cases} \quad y = \begin{cases} 0 \\ -\frac{1}{2} \end{cases}$

BY SIMPLE GEOMETRY THE AREA CAN BE FOUND

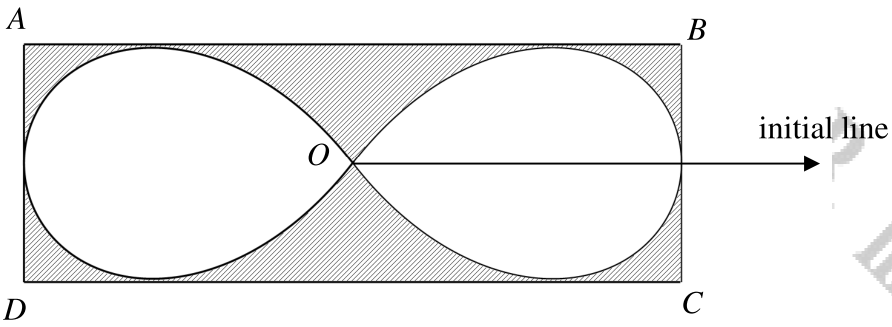
REQUIRED AREA = AREA OF QUARTER CIRCLE − AREA OF ISOSCELES, RIGHT ANGLE TRIANGLE

$= \frac{1}{4} \times \pi \times (\frac{1}{2})^2 - \frac{1}{2} \times (\frac{1}{2} \times \frac{1}{2})$

$= \frac{\pi}{16} - \frac{1}{8}$

$= \frac{\pi - 2}{16}$

Question 7 (****+)



The figure above shows the rectangle $ABCD$ enclosing the curve with polar equation

$$r^2 = \cos 2\theta, \quad \theta \in \left[0, \tfrac{1}{4}\pi\right] \cup \left[\tfrac{3}{4}\pi, \tfrac{5}{4}\pi\right] \cup \left[\tfrac{7}{4}\pi, 2\pi\right).$$

Each of the straight line segments AB and CD is a tangent to the curve parallel to the initial line, while each of the straight line segments AD and BC is a tangent to the curve perpendicular to the initial line.

Show with detailed calculations that the total area enclosed between the curve and the rectangle $ABCD$ is $\sqrt{2} - 1$.

$\boxed{}$, $\boxed{\text{proof}}$

- BY INSPECTION THE "VERTICAL" TANGENT HAS $r = 1$, AS $|\cos 2\theta| \leq 1$
- NEXT FIND THE "HORIZONTAL" TANGENT

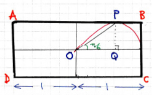
$$\frac{dy}{dx} = 0 \Rightarrow \frac{dy/d\theta}{dx/d\theta} = 0$$
$$\Rightarrow \frac{dy}{d\theta} = 0$$
$$\Rightarrow \frac{d}{d\theta}(r\sin\theta) = 0$$
$$\Rightarrow \frac{d}{d\theta}(r^2\sin^2\theta) = 0 \quad \text{(FOR SIMPLICITY)}$$
$$\Rightarrow \frac{d}{d\theta}(\cos 2\theta \sin^2\theta) = 0$$
$$\Rightarrow -2\sin 2\theta \sin^2\theta + \cos 2\theta(2\sin\theta\cos\theta) = 0$$
$$\Rightarrow -2\sin 2\theta \sin^2\theta + \sin 2\theta \cos 2\theta = 0$$
$$\Rightarrow \sin 2\theta\left[-2\sin^2\theta + \cos 2\theta\right] = 0$$

- IF $\sin 2\theta = 0$ $\quad 2\theta = 0, \pi, 2\pi, 3\pi, \dots$ $\quad \theta = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, \dots$ NONE ARE RELEVANT
- IF $-2\sin^2\theta + \cos 2\theta = 0$

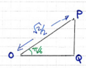
$$\Rightarrow -2\left(\tfrac{1}{2} - \tfrac{1}{2}\cos 2\theta\right) + \cos 2\theta = 0$$
$$\Rightarrow -1 + \cos 2\theta + \cos 2\theta = 0$$
$$\Rightarrow 2\cos 2\theta = 1$$
$$\Rightarrow \cos 2\theta = \tfrac{1}{2}$$
$$\Rightarrow 2\theta = \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}, \frac{11\pi}{3}, \dots$$
$$\Rightarrow \theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}, \dots$$

- WE JUST NEED ONE RELEVANT POINT TO CARRY OUT THE REQUIRED GEOMETRY

$r^2 = \cos 2\theta$
$r^2 = \cos\left(2 \times \frac{\pi}{6}\right)$
$r^2 = \cos\left(\frac{\pi}{3}\right)$
$r^2 = \frac{1}{2}$
$r = +\frac{\sqrt{2}}{2}$



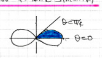
HENCE $|PQ| = \frac{\sqrt{2}}{2}\sin\frac{\pi}{6}$
$|PQ| = \frac{\sqrt{2}}{2} \times \frac{1}{2}$
$|PQ| = \frac{\sqrt{2}}{4}$



- THE AREA OF THE RECTANGLE ABCD IS

$$(2 \times 1) \times \left(2 \times \frac{\sqrt{2}}{4}\right) = \sqrt{2}$$

- NOW THE AREA ENCLOSED BY THE CURVE USING "QUARTER" SYMMETRY



$$\text{AREA} = 4\int_{\theta=0}^{\theta=\frac{\pi}{4}} \frac{1}{2}r^2 \, d\theta$$
$$\text{AREA} = 4\int_0^{\pi/4} \frac{1}{2}\cos 2\theta \, d\theta$$
$$\text{AREA} = 4\left[\frac{1}{4}\sin 2\theta\right]_0^{\frac{\pi}{4}}$$
$$\text{AREA} = \left[\sin 2\theta\right]_0^{\frac{\pi}{4}}$$
$$\text{AREA} = 1 - 0$$
$$\text{AREA} = 1$$

- HENCE BY SUBTRACTION THE REQUIRED AREA IS

$$\sqrt{2} - 1$$

// AS REQUIRED

Question 8 (****+)

The curves C_1 and C_2 have polar equations

$$C_1: \ r = 2\cos\theta - \sin\theta, \quad 0 < \theta \le \frac{\pi}{3}$$

$$C_2: \ r = \sqrt{2} + \sin\theta, \quad 0 \le \theta < 2\pi.$$

The point P lies on C_1 so that the tangent at P is parallel to the initial line.

a) Show clearly that at P

$$\tan 2\theta = 2$$

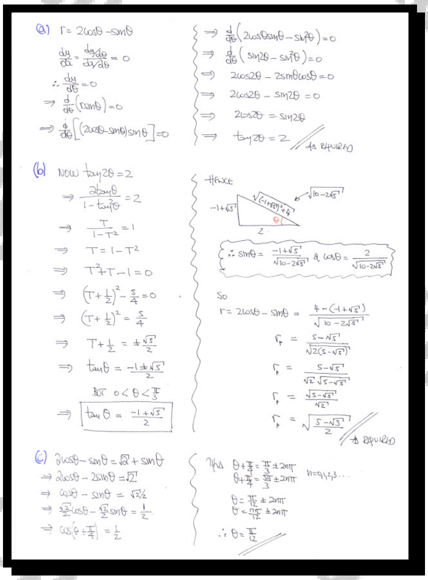
b) Hence show further that the exact distance of P from the origin O is

$$\sqrt{\frac{5-\sqrt{5}}{2}}.$$

The point Q is the point of intersection between C_1 and C_2.

c) Find the value of θ at Q.

$$\boxed{\theta = \frac{\pi}{12}}$$



(a) $r = 2\cos\theta - \sin\theta$

$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = 0$

$\therefore \frac{dy}{d\theta} = 0$

$\Rightarrow \frac{d}{d\theta}(r\sin\theta) = 0$

$\Rightarrow \frac{d}{d\theta}\left[(2\cos\theta - \sin\theta)\sin\theta\right] = 0$

$\Rightarrow \frac{d}{d\theta}(2\cos\theta\sin\theta - \sin^2\theta) = 0$

$\Rightarrow \frac{d}{d\theta}(\sin 2\theta - \sin^2\theta) = 0$

$\Rightarrow 2\cos 2\theta - 2\sin\theta\cos\theta = 0$

$\Rightarrow 2\cos 2\theta - \sin 2\theta = 0$

$\Rightarrow 2\cos 2\theta = \sin 2\theta$

$\Rightarrow \tan 2\theta = 2$ // AS REQUIRED

(b) NOW $\tan 2\theta = 2$

$\Rightarrow \frac{2\tan\theta}{1-\tan^2\theta} = 2$

$\Rightarrow \frac{T}{1-T^2} = 1$

$\Rightarrow T = 1 - T^2$

$\Rightarrow T^2 + T - 1 = 0$

$\Rightarrow \left(T+\frac{1}{2}\right)^2 - \frac{5}{4} = 0$

$\Rightarrow \left(T+\frac{1}{2}\right)^2 = \frac{5}{4}$

$\Rightarrow T + \frac{1}{2} = \pm\frac{\sqrt{5}}{2}$

$\Rightarrow \tan\theta = \frac{-1\pm\sqrt{5}}{2}$

BUT $0 < \theta < \frac{\pi}{3}$

$\Rightarrow \boxed{\tan\theta = \frac{-1+\sqrt{5}}{2}}$

HENCE

$\therefore \sin\theta = \frac{-1+\sqrt{5}}{\sqrt{10-2\sqrt{5}}}$ & $\cos\theta = \frac{2}{\sqrt{10-2\sqrt{5}}}$

SO

$r = 2\cos\theta - \sin\theta = \frac{4-(-1+\sqrt{5})}{\sqrt{10-2\sqrt{5}}}$

$r_P = \frac{5-\sqrt{5}}{\sqrt{2(5-\sqrt{5})}}$

$r_P = \frac{5-\sqrt{5}}{\sqrt{2}\sqrt{5-\sqrt{5}}}$

$r_P = \frac{\sqrt{5-\sqrt{5}}}{\sqrt{2}}$

$r_P = \sqrt{\frac{5-\sqrt{5}}{2}}$ // AS REQUIRED

(c) $2\cos\theta - \sin\theta = \sqrt{2} + \sin\theta$

$\Rightarrow 2\cos\theta - 2\sin\theta = \sqrt{2}$

$\Rightarrow \cos\theta - \sin\theta = \sqrt{2}/2$

$\Rightarrow \frac{\sqrt{2}}{2}\cos\theta - \frac{\sqrt{2}}{2}\sin\theta = \frac{1}{2}$

$\Rightarrow \cos\left(\theta + \frac{\pi}{4}\right) = \frac{1}{2}$

THUS $\theta + \frac{\pi}{4} = \frac{\pi}{3} \pm 2n\pi$

$\theta + \frac{\pi}{4} = \frac{5\pi}{3} \pm 2n\pi$ $\quad n = 0,1,2,3\ldots$

$\theta = \frac{\pi}{12} \pm 2n\pi$

$\theta = \frac{17\pi}{12} \pm 2n\pi$

$\therefore \theta = \frac{\pi}{12}$ //

Question 9 (****+)

The curve C has polar equation

$$r = \tan\theta, \ 0 \le \theta < \frac{\pi}{2}.$$

Find a Cartesian equation of C in the form $y = f(x)$.

$$\boxed{y = \frac{x^2}{\sqrt{1-x^2}}}$$

$r = \tan\theta$

$\Rightarrow r^2 = \tan^2\theta$

$\Rightarrow r^2 = \frac{\sin^2\theta}{\cos^2\theta}$

$\Rightarrow r^2\cos^2\theta = \sin^2\theta$

$\Rightarrow (r\cos\theta)^2 = \sin^2\theta$

$\Rightarrow x^2 = \sin^2\theta$

$\Rightarrow x^2 = 1-\cos^2\theta$

$\Rightarrow \cos^2\theta = 1-x^2$

$\Rightarrow \sec^2\theta = \frac{1}{1-x^2}$

$\Rightarrow 1+\tan^2\theta = \frac{1}{1-x^2}$

$\Rightarrow \tan^2\theta = \frac{1}{1-x^2} - 1$

$\Rightarrow r^2 = \frac{1-(1-x^2)}{1-x^2}$

$\Rightarrow x^2+y^2 = \frac{x^2}{1-x^2}$

$\Rightarrow y^2 = \frac{x^2}{1-x^2} - x^2$

$\Rightarrow y^2 = \frac{x^2-(x^2-x^4)}{1-x^2}$

$\Rightarrow y^2 = \frac{x^4}{1-x^2}$

$\Rightarrow y = +\sqrt{\frac{x^4}{1-x^2}}$

$\Rightarrow y = \frac{x^2}{\sqrt{1-x^2}}$

Question 10 (****+)

The curve C has polar equation

$$r = \frac{4}{4 - 3\cos\theta}, \quad 0 \le \theta < 2\pi .$$

a) Find a Cartesian equation of C in the form $y^2 = f(x)$.

b) Sketch the graph of C.

$$\boxed{y^2 = \frac{1}{16}\left(16 + 24x - 7x^2\right)}$$

$\boxed{r = \frac{4}{4 - 3\cos\theta}}$

a) MULTIPLY TOP & BOTTOM OF THE FRACTION ON THE R.H.S BY r

$\Rightarrow r = \frac{4r}{4r - 3r\cos\theta}$

$\Rightarrow r = \frac{4r}{4r - 3x}$

$\Rightarrow 1 = \frac{4}{4r - 3x}$

$\Rightarrow 4r - 3x = 4$

$\Rightarrow 4r = 3x + 4$

$\Rightarrow 16r^2 = (3x+4)^2$

$\Rightarrow 16(x^2+y^2) = (3x+4)^2$

$\Rightarrow 16x^2 + 16y^2 = (3x+4)^2$

$\Rightarrow 16y^2 = (3x+4)^2 - 16x^2$

$\Rightarrow 16y^2 = (3x+4+4x)(3x+4-4x)$

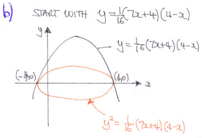
$\Rightarrow 16y^2 = (7x+4)(4-x)$

$\Rightarrow y^2 = \frac{1}{16}(7x+4)(4-x)$

OR EXPAND

$y^2 = \frac{1}{16}(16 + 24x - 7x^2)$

b) START WITH $y = \frac{1}{16}(7x+4)(4-x)$



10 ENRICHMENT QUESTIONS

Question 1 (*****)

Two curves, C_1 and C_2, have polar equations

$$C_1: \ r = 12\cos\theta, \ -\frac{\pi}{2} < \theta \le \frac{\pi}{2}$$

$$C_2: \ r = 4 + 4\cos\theta, \ -\pi < \theta \le \pi.$$

One of the points of intersection between the graphs of C_1 and C_2 is denoted by A. The area of the **smallest** of the two regions bounded by C_1 and the straight line segment OA is

$$6\pi - 9\sqrt{3}.$$

The finite region R represents points which lie inside C_1 but outside C_2.

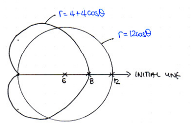
Show that the area of R is 16π.

$\boxed{}$, $\boxed{\text{proof}}$

START BY SKETCHING

$r_1 = 12\cos\theta$ IS A CIRCLE WITH DIAMETER AT $(0,0)$ AND $(12,0)$

$r_2 = 4 + 4\cos\theta$ IS A "STANDARD" CARDIOID

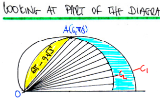


FINDING THE POINTS OF INTERSECTION

$$r_1 = r_2 \implies 12\cos\theta = 4 + 4\cos\theta \implies 8\cos\theta = 4 \implies \cos\theta = \tfrac{1}{2} \implies \theta = \pm\tfrac{\pi}{3}$$

$\therefore \ (6, \tfrac{\pi}{3})$ & $(6, -\tfrac{\pi}{3})$

LOOKING AT PART OF THE DIAGRAM – LET $A(6, \tfrac{\pi}{3})$



- WE REQUIRE THE "BLUE AREA TWICE"
- SUMMING ALL THE POLAR SECTORS, LEADING TO C_2 FROM $\theta = 0$ TO $\theta = \frac{\pi}{3}$

$$\int_0^{\frac{\pi}{3}} \tfrac{1}{2}(4 + 4\cos\theta)^2 \, d\theta$$

$$= \int_0^{\frac{\pi}{3}} 8 + 16\cos\theta + 8\cos^2\theta \, d\theta$$

$$= \int_0^{\pi/3} 8 + 16\cos\theta + 8\left(\tfrac{1}{2} + \tfrac{1}{2}\cos 2\theta\right) d\theta$$

$$= \int_0^{\frac{\pi}{3}} 12 + 16\cos\theta + 4\cos 2\theta \, d\theta$$

$$= \Big[12\theta + 16\sin\theta + 2\sin 2\theta\Big]_0^{\frac{\pi}{3}}$$

$$= \left(12 \times \tfrac{\pi}{3} + 16\sin\tfrac{\pi}{3} + 2\sin\tfrac{2\pi}{3}\right) - \left(0 + \cancel{16\sin 0} + \cancel{2\sin 0}\right)$$

$$= 4\pi + 16 \times \tfrac{\sqrt{3}}{2} + 2\left(\tfrac{\sqrt{3}}{2}\right)$$

$$= 4\pi + 9\sqrt{3}$$

NOW THE REQUIRED CAN BE FOUND

$$\Big[\text{AREA OF SEMICIRCLE} - (\underbrace{\text{AREA FOUND}}_{4\pi + 9\sqrt{3}} + \underbrace{\text{AREA GIVEN}}_{\text{"YELLOW IN DIAGRAM"}\ (6\pi - 9\sqrt{3})})\Big] \times 2$$

$$\therefore \ \Big[\tfrac{1}{2}\pi \times 6^2 - \left(4\pi + \cancel{9\sqrt{3}} + 6\pi - \cancel{9\sqrt{3}}\right)\Big] \times 2$$

$$= [18\pi - 10\pi] \times 2$$

$$= 16\pi$$

AS REQUIRED

Question 2 (*****)

A curve has polar equation

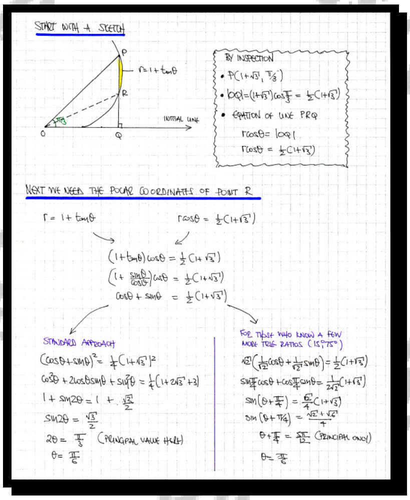
$$r = 1 + \tan\theta \;, \quad 0 \le \theta \le \tfrac{1}{2}\pi .$$

The point P lies on the curve where $\theta = \tfrac{1}{3}\pi$.

The point Q lies on the initial line so that the straight line L, which passes through P and Q meets the initial line at right angles.

Determine, in exact simplified form, the area of the finite region bounded by the curve and L.

$\boxed{}$, $\boxed{\tfrac{1}{2}[\ln 3 - 1]}$


START WITH A SKETCH

BY INSPECTION

- $P(1+\sqrt{3}, \tfrac{\pi}{3})$
- $|OQ| = (1+\sqrt{3})\cos\frac{\pi}{3} = \frac{1}{2}(1+\sqrt{3})$
- EQUATION OF LINE PRQ

$r\cos\theta = |OQ|$

$r\cos\theta = \frac{1}{2}(1+\sqrt{3})$

NEXT WE NEED THE POLAR COORDINATES OF POINT R

$r = 1+\tan\theta \qquad r\cos\theta = \frac{1}{2}(1+\sqrt{3})$

$(1+\tan\theta)\cos\theta = \frac{1}{2}(1+\sqrt{3})$

$\left(1+\frac{\sin\theta}{\cos\theta}\right)\cos\theta = \frac{1}{2}(1+\sqrt{3})$

$\cos\theta + \sin\theta = \frac{1}{2}(1+\sqrt{3})$

STANDARD APPROACH

$(\cos\theta+\sin\theta)^2 = \frac{1}{4}(1+\sqrt{3})^2$

$\cos^2\theta + 2\cos\theta\sin\theta + \sin^2\theta = \frac{1}{4}(1+2\sqrt{3}+3)$

$1+\sin 2\theta = 1 + \frac{\sqrt{3}}{2}$

$\sin 2\theta = \frac{\sqrt{3}}{2}$

$2\theta = \frac{\pi}{3}$ (PRINCIPAL VALUE HERE)

$\theta = \frac{\pi}{6}$

FOR THOSE WHO KNOW A FEW MORE TRIG RATIOS ($15°, 75°$)

$\sqrt{2}\left(\frac{1}{\sqrt{2}}\cos\theta + \frac{1}{\sqrt{2}}\sin\theta\right) = \frac{1}{2}(1+\sqrt{3})$

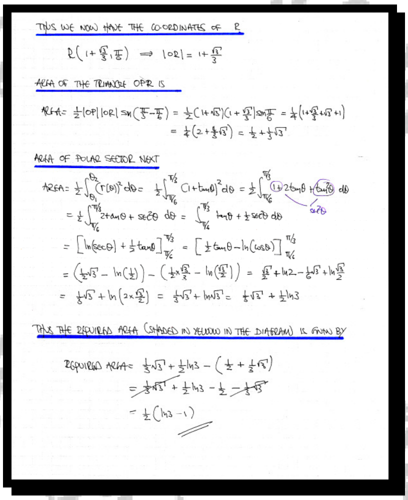
$\sin\frac{\pi}{4}\cos\theta + \cos\frac{\pi}{4}\sin\theta = \frac{1}{2\sqrt{2}}(1+\sqrt{3})$

$\sin\left(\theta+\frac{\pi}{4}\right) = \frac{\sqrt{2}}{4}(1+\sqrt{3})$

$\sin(\theta+\pi/4) = \frac{\sqrt{2}+\sqrt{6}}{4}$

$\theta+\frac{\pi}{4} = \frac{5\pi}{12}$ (PRINCIPAL ONLY)

$\theta = \frac{\pi}{6}$


THUS WE NOW HAVE THE CO-ORDINATES OF R

$R\left(1+\frac{\sqrt{3}}{3}, \frac{\pi}{6}\right) \implies |OR| = 1+\frac{\sqrt{3}}{3}$

AREA OF THE TRIANGLE OPR IS

$\text{AREA} = \frac{1}{2}|OP||OR|\sin\left(\frac{\pi}{3}-\frac{\pi}{6}\right) = \frac{1}{2}(1+\sqrt{3})\left(1+\frac{\sqrt{3}}{3}\right)\sin\frac{\pi}{6} = \frac{1}{4}\left(1+\frac{\sqrt{3}}{3}+\sqrt{3}+1\right)$

$= \frac{1}{4}\left(2+\frac{4}{3}\sqrt{3}\right) = \frac{1}{2}+\frac{1}{3}\sqrt{3}$

AREA OF POLAR SECTOR NEXT

$\text{AREA} = \frac{1}{2}\int_{\theta_1}^{\theta_2}(r(\theta))^2\,d\theta = \frac{1}{2}\int_{\pi/6}^{\pi/3}(1+\tan\theta)^2\,d\theta = \frac{1}{2}\int_{\pi/6}^{\pi/3} 1+2\tan\theta+\tan^2\theta\,d\theta$ $\sec^2\theta$

$= \frac{1}{2}\int_{\pi/6}^{\pi/3} 2\tan\theta + \sec^2\theta\,d\theta = \int_{\pi/6}^{\pi/3}\tan\theta + \frac{1}{2}\sec^2\theta\,d\theta$

$= \left[\ln|\sec\theta| + \frac{1}{2}\tan\theta\right]_{\pi/6}^{\pi/3} = \left[\frac{1}{2}\tan\theta - \ln(\cos\theta)\right]_{\pi/6}^{\pi/3}$

$= \left(\frac{1}{2}\sqrt{3} - \ln\left(\frac{1}{2}\right)\right) - \left(\frac{1}{2}\times\frac{\sqrt{3}}{3} - \ln\left(\frac{\sqrt{3}}{2}\right)\right) = \frac{\sqrt{3}}{2} + \ln 2 - \frac{1}{6}\sqrt{3} + \ln\frac{\sqrt{3}}{2}$

$= \frac{1}{3}\sqrt{3} + \ln\left(2\times\frac{\sqrt{3}}{2}\right) = \frac{1}{3}\sqrt{3} + \ln\sqrt{3} = \frac{1}{3}\sqrt{3} + \frac{1}{2}\ln 3$

THUS THE REQUIRED AREA (SHADED IN YELLOW IN THE DIAGRAM) IS GIVEN BY

$\text{REQUIRED AREA} = \frac{1}{3}\sqrt{3} + \frac{1}{2}\ln 3 - \left(\frac{1}{2}+\frac{1}{3}\sqrt{3}\right)$

$= \cancel{\frac{1}{3}\sqrt{3}} + \frac{1}{2}\ln 3 - \frac{1}{2} - \cancel{\frac{1}{3}\sqrt{3}}$

$= \frac{1}{2}(\ln 3 - 1)$

Question 3 **(*****)**

A set of cartesian axes is superimposed over a set of polar axes, so that both set of axes have a common origin O, and the positive x axis coincides with the initial line.

A parabola P has Cartesian equation

$$y^2 = 8(2-x), \quad x \le 2.$$

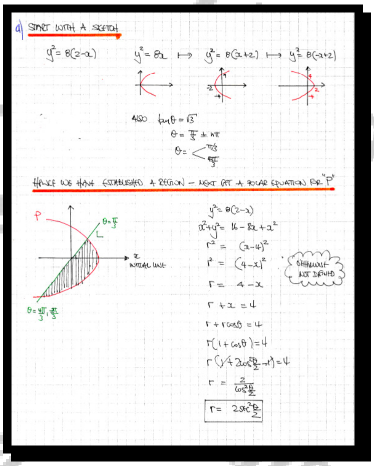
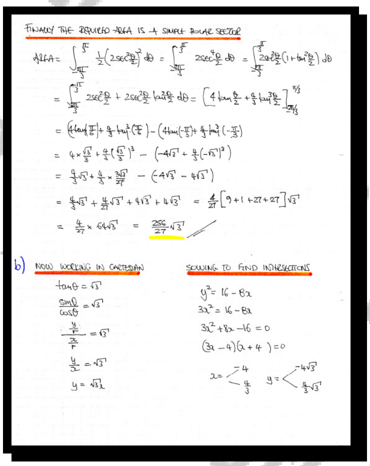
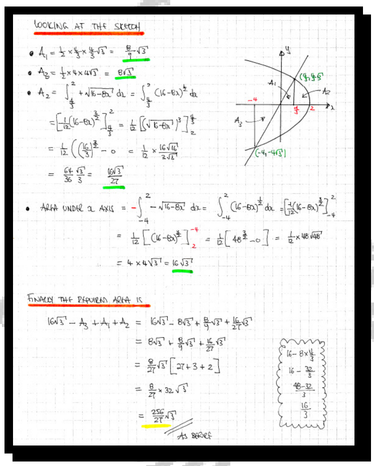
A straight line L has polar equation

$$\tan\theta = \sqrt{3}, \ -\pi < \theta \Leftrightarrow \pi.$$

a) Use polar coordinates to determine, in exact simplified form, the area of the finite region bounded by P and L.

b) Verify the answer of part **(a)** by using calculus in cartesian coordinates

$\boxed{}$, $\boxed{\frac{256}{27}\sqrt{3}}$

Question 4 (*****)

A curve has polar equation

$$r = 1 + \tan\theta \ , \quad 0 \le \theta \le \tfrac{1}{2}\pi \ ,$$

meets the initial line at the point P.

Another curve has polar equation

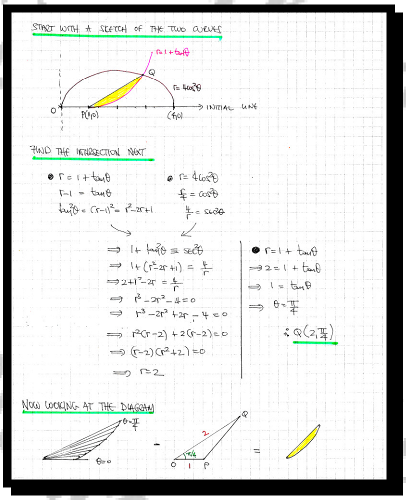
$$r = 4\cos^2\theta \ , \quad 0 \le \theta \le \tfrac{1}{2}\pi \ .$$

The two curves meet at the point Q.

Determine, in exact simplified form, the area of the finite region bounded by the straight line through P and Q, and the curve with equation $r = 1 + \tan\theta$.

Give the answer in the form $\frac{1}{k}\left[1 - \sqrt{k} + \ln k\right]$, where k is a positive integer.

$\boxed{}$, $\boxed{\frac{1}{2}\left[1 - \sqrt{2} + \ln 2\right]}$


START WITH A SKETCH OF THE TWO CURVES

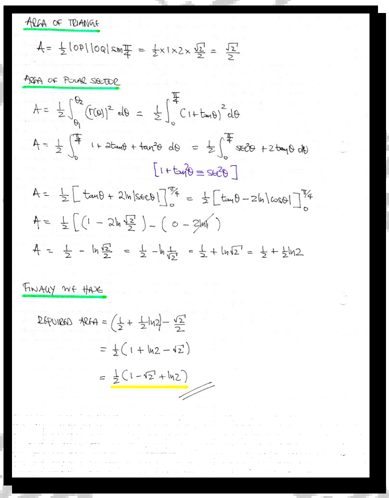
FIND THE INTERSECTION NEXT

- $r = 1 + \tan\theta$
  $r - 1 = \tan\theta$
  $\tan^2\theta = (r-1)^2 = r^2 - 2r + 1$
- $r = 4\cos^2\theta$
  $\frac{r}{4} = \cos^2\theta$
  $\frac{4}{r} = \sec^2\theta$

$\Rightarrow 1 + \tan^2\theta \equiv \sec^2\theta$
$\Rightarrow 1 + (r^2 - 2r + 1) = \frac{4}{r}$
$\Rightarrow 2 + r^2 - 2r = \frac{4}{r}$
$\Rightarrow r^3 - 2r^2 - 4 = 0$
$\Rightarrow r^3 - 2r^2 + 2r - 4 = 0$
$\Rightarrow r^2(r-2) + 2(r-2) = 0$
$\Rightarrow (r-2)(r^2+2) = 0$
$\Rightarrow r = 2$

- $r = 1 + \tan\theta$
$\Rightarrow 2 = 1 + \tan\theta$
$\Rightarrow 1 = \tan\theta$
$\Rightarrow \theta = \frac{\pi}{4}$
$\therefore Q\left(2, \frac{\pi}{4}\right)$

NOW LOOKING AT THE DIAGRAM


AREA OF TRIANGLE

$A = \frac{1}{2}|OP||OQ|\sin\frac{\pi}{4} = \frac{1}{2} \times 1 \times 2 \times \frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{2}$

AREA OF POLAR SECTOR

$A = \frac{1}{2}\int_{\theta_1}^{\theta_2} (r(\theta))^2 \, d\theta = \frac{1}{2}\int_0^{\frac{\pi}{4}} (1 + \tan\theta)^2 \, d\theta$

$A = \frac{1}{2}\int_0^{\frac{\pi}{4}} 1 + 2\tan\theta + \tan^2\theta \, d\theta = \frac{1}{2}\int_0^{\frac{\pi}{4}} \sec^2\theta + 2\tan\theta \, d\theta$

$[1 + \tan^2\theta \equiv \sec^2\theta]$

$A = \frac{1}{2}\left[\tan\theta + 2\ln|\sec\theta|\right]_0^{\pi/4} = \frac{1}{2}\left[\tan\theta - 2\ln|\cos\theta|\right]_0^{\pi/4}$

$A = \frac{1}{2}\left[\left(1 - 2\ln\frac{\sqrt{2}}{2}\right) - \left(0 - \cancel{2\ln 1}\right)\right]$

$A = \frac{1}{2} - \ln\frac{\sqrt{2}}{2} = \frac{1}{2} - \ln\frac{1}{\sqrt{2}} = \frac{1}{2} + \ln\sqrt{2} = \frac{1}{2} + \frac{1}{2}\ln 2$

FINALLY WE HAVE

REQUIRED AREA $= \left(\frac{1}{2} + \frac{1}{2}\ln 2\right) - \frac{\sqrt{2}}{2}$

$= \frac{1}{2}(1 + \ln 2 - \sqrt{2})$

$= \frac{1}{2}(1 - \sqrt{2} + \ln 2)$

Question 5 (*****)

A cardioid has polar equation

$$r = 4(1+\cos\theta), \quad 0 \le \theta \le \tfrac{1}{2}\pi .$$

A tangent to the curve at some point P has gradient -1.

Find, in the form $r = f(\theta)$, the polar equation of this tangent.

V, $\boxed{r = \dfrac{5+3\sqrt{3}}{\cos\theta+\sin\theta}}$

START BY OBTAINING THE GRADIENT FUNCTION

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\frac{d}{d\theta}(r\sin\theta)}{\frac{d}{d\theta}(r\cos\theta)} = \frac{\frac{d}{d\theta}(\cancel{4}(1+\cos\theta)\sin\theta)}{\frac{d}{d\theta}(\cancel{4}(1+\cos\theta)\cos\theta)}$$

$$= \frac{\frac{d}{d\theta}(\sin\theta+\cos\theta\sin\theta)}{\frac{d}{d\theta}(\cos\theta+\cos^2\theta)} = \frac{\cos\theta-\sin^2\theta+\cos^2\theta}{-\sin\theta-2\cos\theta\sin\theta}$$

$$= \frac{\cos\theta+\cos2\theta}{-\sin\theta-\sin2\theta}$$

SETTING $\frac{dy}{dx}=-1$ YIELDS THE FOLLOWING TRIGONOMETRIC EQUATION

$$\frac{\cos\theta+\cos2\theta}{\sin\theta-\sin2\theta} = -1$$

$$\cos\theta+\cos2\theta = \sin\theta+\sin2\theta$$

NOW NEED SOME IDENTITIES – IF NOT GIVEN OR KNOWN

$\cos(A+B) \equiv \cos A\cos B - \sin A\sin B$ | $\sin(A+B) \equiv \sin A\cos B + \cos A\sin B$

$\cos(A-B) \equiv \cos A\cos B + \sin A\sin B$ | $\sin(A-B) \equiv \sin A\cos B - \cos A\sin B$

$\therefore \cos(A+B)+\cos(A-B) \equiv 2\cos A\cos B$ | $\therefore \sin(A+B)+\sin(A-B) \equiv 2\sin A\cos B$

LET $A+B=P$ & $A-B=Q$

$\therefore \cos P+\cos Q \equiv 2\cos\frac{P+Q}{2}\cos\frac{P-Q}{2}$ | $\therefore \sin P+\sin Q \equiv 2\sin\frac{A+B}{2}\cos\frac{A-B}{2}$

RETURNING TO THE "MAIN LINE"

$$\Rightarrow \cos\theta+\cos2\theta = \sin\theta+\sin2\theta$$

$$\Rightarrow \cancel{2}\cos\frac{3\theta}{2}\cos\frac{\theta}{2} = \cancel{2}\sin\frac{3\theta}{2}\cos\frac{\theta}{2}$$

$$\Rightarrow \cos\frac{\theta}{2}\left(\cos\frac{3\theta}{2}-\sin\frac{3\theta}{2}\right)=0$$

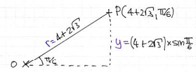
NOW $\cos\theta/2=0$, GIVES SOLUTIONS OUT OF RANGE.

$$\Rightarrow \cos\frac{3\theta}{2} = \sin\frac{3\theta}{2}$$

$$\Rightarrow 1 = \tan\frac{3\theta}{2}$$

$$\Rightarrow \theta = \pi/6 \quad \text{(ONLY SOLUTION IN RANGE)}$$

THUS $r = 4(1+\cos\pi/6) = 4(1+\sqrt{3}/2) = 4+2\sqrt{3}$



$y=(4+2\sqrt{3})\times\sin\frac{\pi}{6} = 2+\sqrt{3}$

$x=(4+2\sqrt{3})\cos\frac{\pi}{6} = (4+2\sqrt{3})\times\frac{\sqrt{3}}{2} = 2\sqrt{3}+3$

EQUATION OF TANGENT

$$y-(2+\sqrt{3}) = -1(x-(3+2\sqrt{3}))$$

$$y-2-\sqrt{3} = -x+3+2\sqrt{3}$$

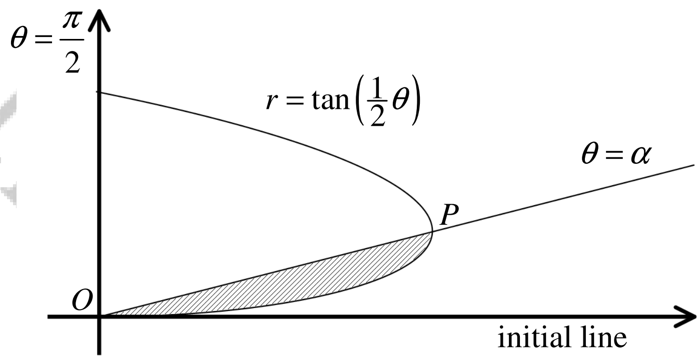
$$y+x = 5+3\sqrt{3}$$

$$r\sin\theta+r\cos\theta = 5+3\sqrt{3}$$

$$r(\cos\theta+\sin\theta) = 5+3\sqrt{3}$$

$$r = \frac{5+3\sqrt{3}}{\cos\theta+\sin\theta}$$

Question 6 (*****)



The figure above shows the curve C with polar equation

$$r = \tan\left(\tfrac{1}{2}\theta\right), \quad 0 \le \theta < \frac{\pi}{2}.$$

The point P lies on C so that tangent to C is perpendicular to the initial line.

The half line with equation $\theta = \alpha$ passes through P.

Find, in exact simplified form, the area of the finite region bounded by C and the above mentioned half line.

$$\boxed{\text{area} = \sqrt{-2+\sqrt{5}} - \arctan\sqrt{-2+\sqrt{5}}}$$

FIRSTLY WE NEED THE "θ CO-ORDINATE" OF P, WHICH IS WHERE THE TANGENT IS "VERTICAL"

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \infty \qquad \therefore \frac{dx}{d\theta} = 0$$

$$\Rightarrow \frac{d}{d\theta}(r\cos\theta) = 0$$

$$\Rightarrow \frac{d}{d\theta}\left[\tan\frac{\theta}{2}\cos\theta\right] = 0$$

$$\Rightarrow \frac{1}{2}\sec^2\frac{\theta}{2}\cos\theta - \tan\frac{\theta}{2}\sin\theta = 0$$

$$\Rightarrow \sec^2\frac{\theta}{2}\cos\theta - 2\tan\frac{\theta}{2}\sin\theta = 0$$

SOLVING THE ABOVE TRIGONOMETRIC EQUATION – START BY DIVIDING BY $\cos\theta$

$$\Rightarrow \sec^2\frac{\theta}{2} - 2\tan\frac{\theta}{2}\tan\theta = 0$$

$$\Rightarrow 1 + \tan^2\frac{\theta}{2} - 2\tan\frac{\theta}{2}\left[\frac{2\tan\frac{\theta}{2}}{1-\tan^2\frac{\theta}{2}}\right] = 0$$

$$\boxed{\tan 2A = \frac{2\tan A}{1-\tan^2 A} \qquad \tan A = \frac{2\tan A/2}{1-\tan^2 A/2}}$$

LET $\tan\frac{\theta}{2} = T$

$$\Rightarrow 1 + T^2 - 2T\left(\frac{2T}{1-T^2}\right) = 0$$

$$\Rightarrow 1 + T^2 - \frac{4T^2}{1-T^2} = 0$$

$$\Rightarrow (1+T^2)(1-T^2) - 4T^2 = 0$$

$$\Rightarrow 1 - T^4 - 4T^2 = 0$$

$$\Rightarrow 0 = T^4 + 4T^2 - 1$$

$$\Rightarrow (T^2+2)^2 - 4 - 1 = 0$$

$$\Rightarrow (T^2+2)^2 = 5$$

$$\Rightarrow T^2 + 2 = \pm\sqrt{5}$$

$$\Rightarrow \tan^2\frac{\theta}{2} = -2+\sqrt{5} \quad \text{or} \quad \cancel{-2-\sqrt{5}}$$

$$\Rightarrow \tan\frac{\theta}{2} = \sqrt{-2+\sqrt{5}} \quad \text{or} \quad \cancel{-\sqrt{-2+\sqrt{5}}} \qquad 0 < \theta < \frac{\pi}{2}$$

$$\Rightarrow \frac{\theta}{2} = \arctan\sqrt{-2+\sqrt{5}}$$

$$\Rightarrow \theta = 2\arctan\sqrt{-2+\sqrt{5}}$$

NOW FINDING THE REQUIRED AREA

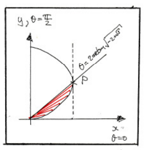
$$\Rightarrow \text{AREA} = \frac{1}{2}\int_{\theta_1}^{\theta_2} r^2\,d\theta = \frac{1}{2}\int_{\theta=0}^{2\arctan\sqrt{-2+\sqrt{5}}} \tan^2\frac{\theta}{2}\,d\theta$$

$$\Rightarrow \text{AREA} = \frac{1}{2}\int_0^{2\arctan\sqrt{-2+\sqrt{5}}} \sec^2\frac{\theta}{2} - 1\,d\theta$$

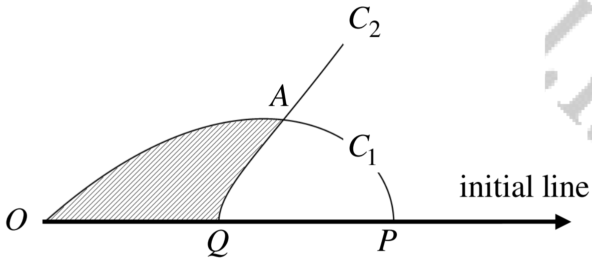
$$\Rightarrow \text{AREA} = \frac{1}{2}\left[2\tan\frac{\theta}{2} - \theta\right]_{\theta=0}^{\theta=2\arctan\sqrt{-2+\sqrt{5}}}$$

$$\Rightarrow \text{AREA} = \frac{1}{2}\left[2\sqrt{-2+\sqrt{5}} - 2\arctan\sqrt{-2+\sqrt{5}}\right]$$

$$\Rightarrow \text{AREA} = \sqrt{-2+\sqrt{5}} - \arctan\sqrt{-2+\sqrt{5}}$$



Question 7 (*****)



The figure above shows the curves C_1 and C_2 with respective polar equations

$$r_1 = \sec\theta\left(1-\tan^2\theta\right) \quad \text{and} \quad r_2 = \tfrac{1}{2}\sec^3\theta, \qquad 0 \le \theta < \tfrac{1}{4}\pi .$$

The points P and Q are the respective points where C_1 and C_2 meet the initial line, and the point A is the intersection of C_1 and C_2 .

a) Find the exact area of the curvilinear triangle OAQ, where O is the pole.

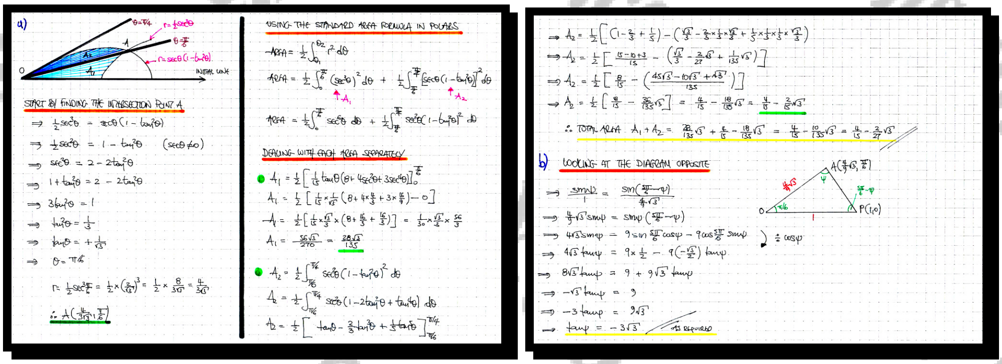
The angle OAP is denoted by ψ .

b) Show that $\tan\psi = -3\sqrt{3}$.

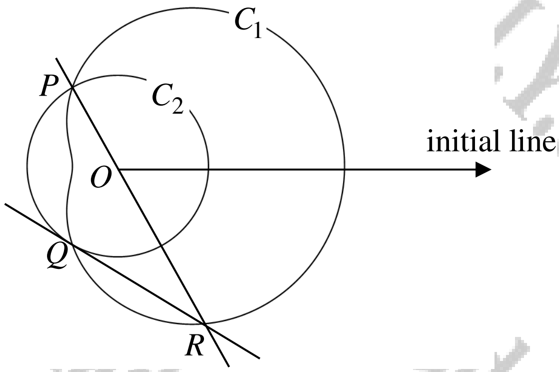
You may assume without proof

$$\int \sec^6 x \; dx = \frac{1}{15}\left(8+4\sec^2 x+3\sec^4 x\right)\tan x + C$$

$\boxed{}$, $\boxed{\dfrac{2\left(18-5\sqrt{3}\right)}{135}}$



Question 8 (*****)



The figure above shows the curves C_1 and C_2 with respective polar equations

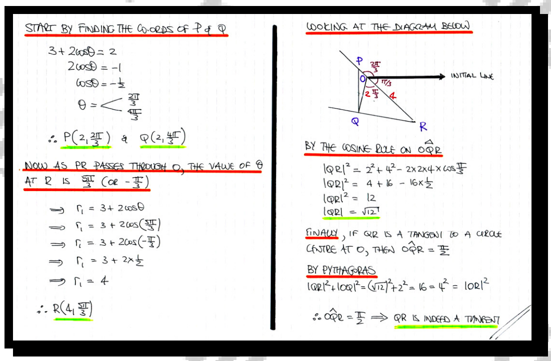
$$r_1 = 3 + 2\cos\theta,\ 0 \le \theta < 2\pi \qquad \text{and} \qquad r_2 = 2.$$

The two curves intersect at the points P and Q.

A straight line passing through P and the pole O intersects C_1 again at the point R.

Show that RQ is a tangent of C_1 at Q.

$\boxed{}$, $\boxed{\text{proof}}$



START BY FINDING THE CO-ORDS OF P & Q

$3 + 2\cos\theta = 2$

$2\cos\theta = -1$

$\cos\theta = -\frac{1}{2}$

$\theta = \begin{cases} \frac{2\pi}{3} \\ \frac{4\pi}{3} \end{cases}$

$\therefore P\left(2, \frac{2\pi}{3}\right)$ & $Q\left(2, \frac{4\pi}{3}\right)$

NOW AS PR PASSES THROUGH O, THE VALUE OF θ AT R IS $\frac{5\pi}{3}$ (OR $-\frac{\pi}{3}$)

$\Rightarrow r_1 = 3 + 2\cos\theta$

$\Rightarrow r_1 = 3 + 2\cos\left(\frac{5\pi}{3}\right)$

$\Rightarrow r_1 = 3 + 2\cos\left(-\frac{\pi}{3}\right)$

$\Rightarrow r_1 = 3 + 2 \times \frac{1}{2}$

$\Rightarrow r_1 = 4$

$\therefore R\left(4, \frac{5\pi}{3}\right)$

LOOKING AT THE DIAGRAM BELOW

BY THE COSINE RULE ON $O\hat{Q}R$

$|QR|^2 = 2^2 + 4^2 - 2 \times 2 \times 4 \times \cos\frac{\pi}{3}$

$|QR|^2 = 4 + 16 - 16 \times \frac{1}{2}$

$|QR|^2 = 12$

$|QR| = \sqrt{12}$

FINALLY, IF QR IS A TANGENT TO A CIRCLE CENTRE AT O, THEN $O\hat{Q}R = \frac{\pi}{2}$

BY PYTHAGORAS

$|QR|^2 + |OQ|^2 = (\sqrt{12})^2 + 2^2 = 16 = 4^2 = |OR|^2$

$\therefore O\hat{Q}R = \frac{\pi}{2} \Rightarrow$ QR IS INDEED A TANGENT

Question 9 (*****)

The curves C_1 and C_2 have respective polar equations

$$r = 1 + \sin\theta, \; 0 < \theta < \tfrac{1}{2}\pi \qquad \text{and} \qquad r = 1 + \cos 2\theta, \; 0 < \theta < \tfrac{1}{2}\pi.$$

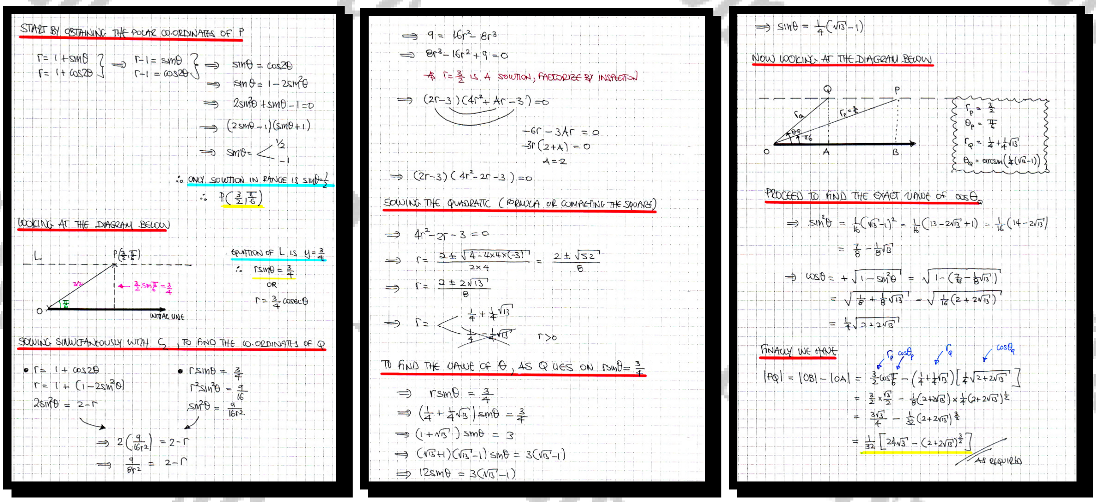
The point P is the point of intersection of C_1 and C_2.

A straight line, which is parallel to the initial line, passes through P and intersects C_2 at the point Q.

Show that

$$|PQ| = \frac{1}{32}\left[24\sqrt{3} - \left(2 + 2\sqrt{13}\right)^{\frac{3}{2}}\right].$$

$\boxed{}$, $\boxed{\text{proof}}$



START BY OBTAINING THE POLAR CO-ORDINATES OF P

$r = 1 + \sin\theta$, $r = 1 + \cos 2\theta$ $\Rightarrow$ $r - 1 = \sin\theta$, $r - 1 = \cos 2\theta$ $\Rightarrow \sin\theta = \cos 2\theta$

$\Rightarrow \sin\theta = 1 - 2\sin^2\theta$

$\Rightarrow 2\sin^2\theta + \sin\theta - 1 = 0$

$\Rightarrow (2\sin\theta - 1)(\sin\theta + 1)$

$\Rightarrow \sin\theta = \frac{1}{2}, -1$

$\therefore$ ONLY SOLUTION IN RANGE IS $\sin\theta = \frac{1}{2}$

$\therefore P\left(\frac{3}{2}, \frac{\pi}{6}\right)$

LOOKING AT THE DIAGRAM BELOW

EQUATION OF L IS $y = \frac{3}{4}$

$\therefore r\sin\theta = \frac{3}{4}$ OR $r = \frac{3}{4}\operatorname{cosec}\theta$

SOLVING SIMULTANEOUSLY WITH C_2, TO FIND THE CO-ORDINATES OF Q

- $r = 1 + \cos 2\theta$, $r = 1 + (1 - 2\sin^2\theta)$, $2\sin^2\theta = 2 - r$
- $r\sin\theta = \frac{3}{4}$, $r^2\sin^2\theta = \frac{9}{16}$, $\sin^2\theta = \frac{9}{16r^2}$

$\Rightarrow 2\left(\frac{9}{16r^2}\right) = 2 - r$

$\Rightarrow \frac{9}{8r^2} = 2 - r$

$\Rightarrow 9 = 16r^2 - 8r^3$

$\Rightarrow 8r^3 - 16r^2 + 9 = 0$

AS $r = \frac{3}{2}$ IS A SOLUTION, FACTORIZE BY INSPECTION

$\Rightarrow (2r - 3)(4r^2 + Ar - 3) = 0$

$-6r - 3Ar = 0$

$-3r(2 + A) = 0$

$A = -2$

$\Rightarrow (2r - 3)(4r^2 - 2r - 3) = 0$

SOLVING THE QUADRATIC (FORMULA OR COMPLETING THE SQUARE)

$\Rightarrow 4r^2 - 2r - 3 = 0$

$\Rightarrow r = \frac{2 \pm \sqrt{4 - 4 \times 4 \times (-3)}}{2 \times 4} = \frac{2 \pm \sqrt{52}}{8}$

$\Rightarrow r = \frac{2 \pm 2\sqrt{13}}{8}$

$\Rightarrow r = \frac{1}{4} + \frac{1}{4}\sqrt{13}$, $\cancel{\frac{1}{4} - \frac{1}{4}\sqrt{13}}$ $\quad r > 0$

TO FIND THE VALUE OF θ, AS Q LIES ON $r\sin\theta = \frac{3}{4}$

$\Rightarrow r\sin\theta = \frac{3}{4}$

$\Rightarrow \left(\frac{1}{4} + \frac{1}{4}\sqrt{13}\right)\sin\theta = \frac{3}{4}$

$\Rightarrow (1 + \sqrt{13})\sin\theta = 3$

$\Rightarrow (\sqrt{13} + 1)(\sqrt{13} - 1)\sin\theta = 3(\sqrt{13} - 1)$

$\Rightarrow 12\sin\theta = 3(\sqrt{13} - 1)$

$\Rightarrow \sin\theta = \frac{1}{4}(\sqrt{13} - 1)$

NOW LOOKING AT THE DIAGRAM BELOW

PROCEED TO FIND THE EXACT VALUE OF $\cos\theta_Q$

$\Rightarrow \sin^2\theta = \frac{1}{16}(\sqrt{13} - 1)^2 = \frac{1}{16}(13 - 2\sqrt{13} + 1) = \frac{1}{16}(14 - 2\sqrt{13})$

$= \frac{7}{8} - \frac{1}{8}\sqrt{13}$

$\Rightarrow \cos\theta = +\sqrt{1 - \sin^2\theta} = \sqrt{1 - \left(\frac{7}{8} - \frac{1}{8}\sqrt{13}\right)}$

$= \sqrt{\frac{1}{8} + \frac{1}{8}\sqrt{13}} = \sqrt{\frac{1}{16}(2 + 2\sqrt{13})}$

$= \frac{1}{4}\sqrt{2 + 2\sqrt{13}}$

FINALLY WE HAVE

$|PQ| = |OB| - |OA| = \underbrace{\tfrac{3}{2}}_{r_P}\underbrace{\cos\tfrac{\pi}{6}}_{\cos\theta_P} - \underbrace{\left(\tfrac{1}{4} + \tfrac{1}{4}\sqrt{13}\right)}_{r_Q}\underbrace{\left[\tfrac{1}{4}\sqrt{2 + 2\sqrt{13}}\right]}_{\cos\theta_Q}$

$= \frac{3}{2} \times \frac{\sqrt{3}}{2} - \frac{1}{8}(2 + 2\sqrt{13}) \times \frac{1}{4}(2 + 2\sqrt{13})^{\frac{1}{2}}$

$= \frac{3\sqrt{3}}{4} - \frac{1}{32}(2 + 2\sqrt{13})^{\frac{3}{2}}$

$= \frac{1}{32}\left[24\sqrt{3} - (2 + 2\sqrt{13})^{\frac{3}{2}}\right]$

AS REQUIRED

Question 10 (*****)

A straight line L, whose gradient is $-\frac{3}{11}$, is a tangent to the curve with polar equation

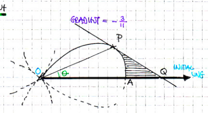
$$r = 25\cos 2\theta, \ 0 \le \theta \le \frac{1}{2}\pi$$

Show that the area of the finite region bounded by the curve, the straight line L and the initial line is

$$\frac{25}{12}\left[46 - 75\arctan\frac{1}{3}\right].$$

[ ], [proof]

START WITH A QUICK SKETCH OF $r = 25\cos 2\theta$, WHICH HERE IS JUST HALF OF THE 4 "LEAVES" OF THE CURVE



FIND AN EXPRESSION FOR THE GRADIENT FUNCTION

$$\Rightarrow \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\frac{d}{d\theta}(y)}{\frac{d}{d\theta}(x)}$$

$$\Rightarrow \frac{dy}{dx} = \frac{\frac{d}{d\theta}(r\sin\theta)}{\frac{d}{d\theta}(r\cos\theta)} = \frac{\frac{d}{d\theta}(\cancel{25}\cos 2\theta\sin\theta)}{\frac{d}{d\theta}(\cancel{25}\cos 2\theta\cos\theta)} = \frac{\cos 2\theta\cos\theta - 2\sin 2\theta\sin\theta}{-2\cos\theta\sin 2\theta - \cos 2\theta\sin\theta}$$

$$\Rightarrow \frac{dy}{dx} = \frac{2\sin\theta\sin 2\theta - \cos 2\theta\cos\theta}{2\sin 2\theta\cos\theta + \cos 2\theta\sin\theta}$$

SETTING THE GRADIENT TO $-\frac{3}{11}$

$$\Rightarrow \frac{2\sin\theta\sin 2\theta - \cos 2\theta\cos\theta}{2\sin 2\theta\cos\theta + \cos 2\theta\sin\theta} = -\frac{3}{11}$$

$$\Rightarrow 22\sin\theta\sin 2\theta - 11\cos 2\theta\cos\theta = -6\sin 2\theta\cos\theta - 3\cos 2\theta\sin\theta$$

$\div \cos\theta\cos 2\theta$

$$\Rightarrow 22\tan\theta\tan 2\theta - 11 = -6\tan 2\theta - 3\tan\theta$$

$$\Rightarrow 22\tan\theta\left(\frac{2\tan\theta}{1-\tan^2\theta}\right) - 11 = -6\left(\frac{2\tan\theta}{1-\tan^2\theta}\right) - 3\tan\theta$$

$$\Rightarrow 22T\left(\frac{2T}{1-T^2}\right) - 11 = -6\left(\frac{2T}{1-T^2}\right) - 3T$$

$$\Rightarrow 44T^2 - 11(1-T^2) = -12T - 3T(1-T^2)$$

$$\Rightarrow 44T^2 - 11 + 11T^2 = -12T - 3T + 3T^3$$

$$\Rightarrow 0 = 3T^3 - 55T^2 - 15T + 11$$

FACTORIZE BY LONG DIVISION/MANIPULATION

$$\Rightarrow T^2(3T-1) - 18T(3T-1) - 11(3T-1) = 0$$

AS THE FINAL ANSWER CONTAINS $\arctan\frac{1}{3}$

$$\Rightarrow (3T-1)(T^2 - 18T - 11) = 0$$

$$\Rightarrow \tan\theta = \frac{1}{3} \quad \text{OR} \quad \tan\theta = \frac{18 \pm \sqrt{368}}{2}$$

$$\tan\theta = \frac{18 \pm 2\sqrt{92}}{2}$$

$$\tan\theta = 9 \pm \sqrt{92}$$

LOOKING AT THE "STATIONARY POINT"

$$\frac{dy}{dx} = 0 \Longrightarrow 2\sin\theta\sin 2\theta - \cos 2\theta\cos\theta = 0$$

$$2\sin\theta(2\sin\theta\cos\theta) - \cos 2\theta\cos\theta = 0$$

$$4\sin^2\theta\cos\theta - \cos 2\theta\cos\theta = 0$$

$$\cos\theta(4\sin^2\theta - (1-2\sin^2\theta)) = 0$$

$$\cos\theta(6\sin^2\theta - 1) = 0$$

$$\cos\theta(\sqrt{6}\sin\theta - 1)(\sqrt{6}\sin\theta + 1) = 0$$

$\Rightarrow$ ONLY PLAUSIBLE SOLUTION $\sin\theta = \frac{1}{\sqrt{6}}$

$\theta \simeq 24.1°$

$$\Rightarrow \theta = \begin{cases} \arctan\frac{1}{3} \simeq 18.4° < 24.1° \\ \cancel{\arctan(9+\sqrt{92})} \simeq 86.9° > \\ \cancel{\arctan(9-\sqrt{92})} \simeq -30.6° \end{cases}$$

NEXT FIND THE CO-ORDS OF P

- $\tan\theta = \frac{1}{3} \Rightarrow \sin\theta = \frac{1}{\sqrt{10}}$

$\Rightarrow \cos\theta = \frac{3}{\sqrt{10}}$

$\Rightarrow \cos 2\theta = 2\cos^2\theta - 1 = 2\times\frac{9}{10} - 1 = \frac{4}{5}$

$\Rightarrow \sin 2\theta = \frac{3}{5}$

- $r = 25\cos 2\theta = 25 \times \frac{4}{5} = 20$
- IN CARTESIAN $x = r\cos\theta = 20 \times \frac{3}{\sqrt{10}} = \frac{60}{\sqrt{10}} = 6\sqrt{10}$

$y = r\sin\theta = 20 \times \frac{1}{\sqrt{10}} = \frac{20}{\sqrt{10}} = 2\sqrt{10}$

IE $P(6\sqrt{10}, 2\sqrt{10})$

EQUATION OF TANGENT

$$\Rightarrow y - 2\sqrt{10} = -\frac{3}{11}(x - 6\sqrt{10})$$

$$\Rightarrow 11y - 22\sqrt{10} = -3x + 18\sqrt{10}$$

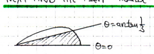
$$\Rightarrow 11y + 3x = 40\sqrt{10}$$

when $y = 0$

$x = \frac{40}{3}\sqrt{10}$

AREA OF OPQ $= \frac{1}{\cancel{2}} \times \frac{40}{3}\sqrt{10} \times \cancel{2}\sqrt{10} = \frac{400}{3}$

NEXT FIND THE AREA "INSIDE" THE LOOP



$$\text{AREA} = \frac{1}{2}\int_{\theta_1}^{\theta_2} (r(\theta))^2 \, d\theta$$

$$\Rightarrow \text{AREA} = \frac{1}{2}\int_{\theta=0}^{\arctan\frac{1}{3}} (25\cos 2\theta)^2 \, d\theta$$

$$\Rightarrow \text{AREA} = \frac{625}{2}\int_0^{\arctan\frac{1}{3}} \cos^2 2\theta \, d\theta = \frac{625}{2}\int_0^{\arctan\frac{1}{3}} \frac{1}{2} + \frac{1}{2}\cos 4\theta \, d\theta$$

$$\Rightarrow \text{AREA} = \frac{625}{2}\left[\frac{1}{2}\theta + \frac{1}{8}\sin 4\theta\right]_0^{\arctan\frac{1}{3}}$$

$$\Rightarrow \text{AREA} = \frac{625}{2}\left[\frac{1}{2}\theta + \frac{1}{4}\sin 2\theta\cos 2\theta\right]_0^{\arctan\frac{1}{3}}$$

$$\Rightarrow \text{AREA} = \frac{625}{2}\left[\left(\frac{1}{2}\arctan\frac{1}{3} + \frac{1}{4}\times\frac{3}{5}\times\frac{4}{5}\right) - 0\right]$$

$$\Rightarrow \text{AREA} = \frac{625}{2}\left[\frac{1}{2}\arctan\frac{1}{3} + \frac{3}{25}\right]$$

$$\Rightarrow \text{AREA} = \frac{625}{4}\arctan\frac{1}{3} + \frac{75}{2}$$

$\therefore$ REQUIRED AREA $= \frac{400}{3} - \left(\frac{625}{4}\arctan\frac{1}{3} + \frac{75}{2}\right)$

$$= \frac{575}{6} - \frac{625}{4}\arctan\frac{1}{3}$$

$$= \frac{25}{12}\left[46 - 75\arctan\frac{1}{3}\right]$$