SERIES EXPANSIONS

59 QUESTIONS

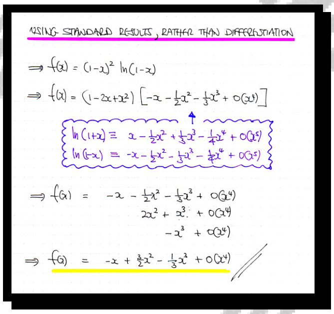
MACLAURIN EXPANSIONS 6 BASIC QUESTIONS

Question 1 (**)

$$f(x) = (1-x)^2 \ln(1-x), \; -1 \le x < 1.$$

Find the Maclaurin expansion of $f(x)$ up and including the term in x^3.

$$f(x) = -x + \frac{3}{2}x^2 - \frac{1}{3}x^3 + O(x^4)$$



USING STANDARD RESULTS, RATHER THAN DIFFERENTIATION

$\Rightarrow f(x) = (1-x)^2 \ln(1-x)$

$\Rightarrow f(x) = (1-2x+x^2)\left[-x - \frac{1}{2}x^2 - \frac{1}{3}x^3 + O(x^4)\right]$

$\ln(1+x) \equiv x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + O(x^5)$

$\ln(1-x) \equiv -x - \frac{1}{2}x^2 - \frac{1}{3}x^3 - \frac{1}{4}x^4 + O(x^5)$

$\Rightarrow f(x) = -x - \frac{1}{2}x^2 - \frac{1}{3}x^3 + O(x^4)$

$\qquad 2x^2 + x^3 + O(x^4)$

$\qquad -x^3 + O(x^4)$

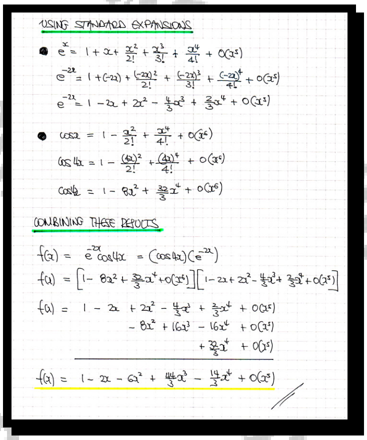
$\Rightarrow f(x) = -x + \frac{3}{2}x^2 - \frac{1}{3}x^3 + O(x^4)$

Question 2 (**+)

$$f(x) = e^{-2x}\cos 4x.$$

Find the Maclaurin expansion of $f(x)$ up and including the term in x^4.

$$e^{-2x}\cos 4x = 1 - 2x - 6x^2 + \frac{44}{3}x^3 - \frac{14}{3}x^4 + O(x^5)$$



USING STANDARD EXPANSIONS

- $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + O(x^5)$

$e^{-2x} = 1 + (-2x) + \frac{(-2x)^2}{2!} + \frac{(-2x)^3}{3!} + \frac{(-2x)^4}{4!} + O(x^5)$

$e^{-2x} = 1 - 2x + 2x^2 - \frac{4}{3}x^3 + \frac{2}{3}x^4 + O(x^5)$

- $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + O(x^6)$

$\cos 4x = 1 - \frac{(4x)^2}{2!} + \frac{(4x)^4}{4!} + O(x^6)$

$\cos 4x = 1 - 8x^2 + \frac{32}{3}x^4 + O(x^6)$

COMBINING THESE RESULTS

$f(x) = e^{-2x}\cos 4x = (\cos 4x)(e^{-2x})$

$f(x) = \left[1 - 8x^2 + \frac{32}{3}x^4 + O(x^6)\right]\left[1 - 2x + 2x^2 - \frac{4}{3}x^3 + \frac{2}{3}x^4 + O(x^5)\right]$

$f(x) = 1 - 2x + 2x^2 - \frac{4}{3}x^3 + \frac{2}{3}x^4 + O(x^5)$

$\qquad - 8x^2 + 16x^3 - 16x^4 + O(x^5)$

$\qquad + \frac{32}{3}x^4 + O(x^5)$

$f(x) = 1 - 2x - 6x^2 + \frac{44}{3}x^3 - \frac{14}{3}x^4 + O(x^5)$

Question 3 (**+)

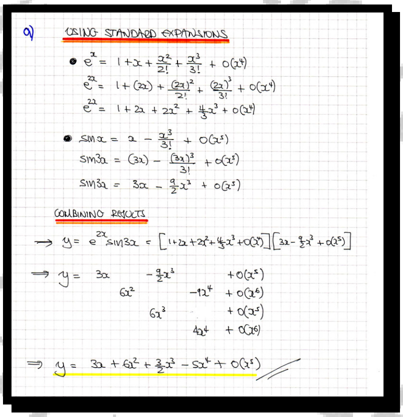
$$y = e^{2x} \sin 3x .$$

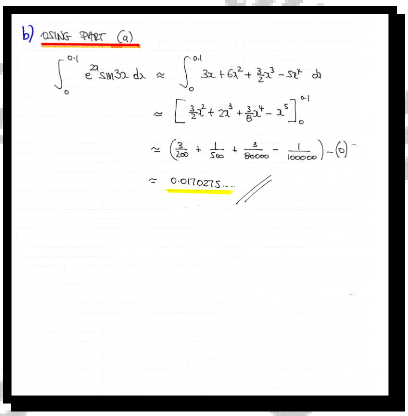
a) Use standard results to find the series expansion of y, up and including the term in x^4.

b) Hence find an approximate value for

$$\int_0^{0.1} e^{2x} \sin 3x \; dx .$$

$\boxed{}$, $\boxed{e^{2x} \sin 3x = 3x + 6x^2 + \frac{3}{2}x^3 - 5x^4 + O\left(x^5\right)}$, $\boxed{\approx 0.0170275}$

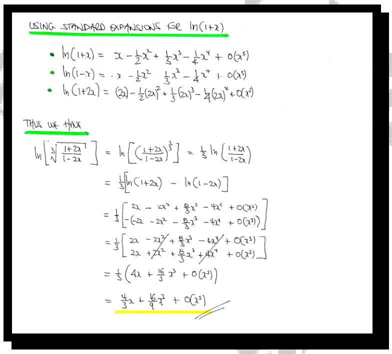




Question 4 (**+)

Find the Maclaurin's expansion of $\ln\left[\sqrt[3]{\dfrac{1+2x}{1-2x}}\right]$, up and including the term in x^3.

$$\ln\left[\sqrt[3]{\frac{1+2x}{1-2x}}\right] = \frac{4}{3}x + \frac{16}{9}x^3 + O\left(x^5\right)$$



Question 5 (***)

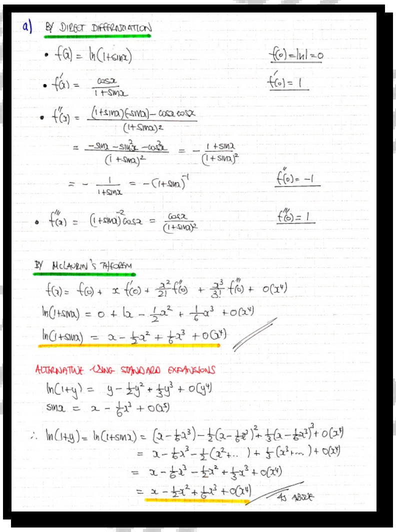
$$f(x) = \ln(1+\sin x), \ \sin x \neq -1.$$

a) Find the Maclaurin expansion of $f(x)$ up and including the term in x^3.

b) Hence show that

$$\int_0^{\frac{1}{4}} \ln(1+\sin x) \ dx \ \approx \ 0.028809.$$

FP2, $\boxed{\ln(1+\sin x) = x - \frac{1}{2}x^2 + \frac{1}{6}x^3 + O(x^4)}$



a) BY DIRECT DIFFERENTIATION

- $f(x) = \ln(1+\sin x)$ $\quad f(0) = \ln 1 = 0$
- $f'(x) = \frac{\cos x}{1+\sin x}$ $\quad f'(0) = 1$
- $f''(x) = \frac{(1+\sin x)(-\sin x) - \cos x \cos x}{(1+\sin x)^2}$

$= \frac{-\sin x - \sin^2 x - \cos^2 x}{(1+\sin x)^2} = -\frac{1+\sin x}{(1+\sin x)^2}$

$= -\frac{1}{1+\sin x} = -(1+\sin x)^{-1}$ $\quad f''(0) = -1$

- $f'''(x) = (1+\sin x)^{-2}\cos x = \frac{\cos x}{(1+\sin x)^2}$ $\quad f'''(0) = 1$

BY MCLAURIN'S THEOREM

$f(x) = f(0) + x f'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + O(x^4)$

$\ln(1+\sin x) = 0 + 1x - \frac{1}{2}x^2 + \frac{1}{6}x^3 + O(x^4)$

$\ln(1+\sin x) = x - \frac{1}{2}x^2 + \frac{1}{6}x^3 + O(x^4)$

ALTERNATIVE USING STANDARD EXPANSIONS

$\ln(1+y) = y - \frac{1}{2}y^2 + \frac{1}{3}y^3 + O(y^4)$

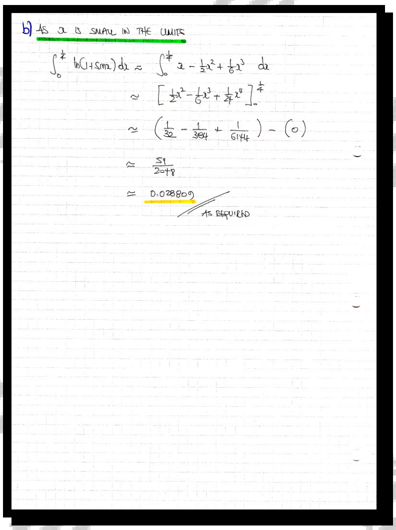
$\sin x = x - \frac{1}{6}x^3 + O(x^5)$

$\therefore \ln(1+y) = \ln(1+\sin x) = (x - \frac{1}{6}x^3) - \frac{1}{2}(x - \frac{1}{6}x^2)^2 + \frac{1}{3}(x - \frac{1}{6}x^3)^3 + O(x^4)$

$= x - \frac{1}{6}x^3 - \frac{1}{2}(x^2 + \dots) + \frac{1}{3}(x^3 + \dots) + O(x^4)$

$= x - \frac{1}{6}x^3 - \frac{1}{2}x^2 + \frac{1}{3}x^3 + O(x^4)$

$= x - \frac{1}{2}x^2 + \frac{1}{6}x^3 + O(x^4)$ AS ABOVE



b) AS x IS SMALL IN THE LIMITS

$\int_0^{\frac{1}{4}} \ln(1+\sin x)\,dx \approx \int_0^{\frac{1}{4}} x - \frac{1}{2}x^2 + \frac{1}{6}x^3 \, dx$

$\approx \left[\frac{1}{2}x^2 - \frac{1}{6}x^3 + \frac{1}{24}x^4\right]_0^{\frac{1}{4}}$

$\approx \left(\frac{1}{32} - \frac{1}{384} + \frac{1}{6144}\right) - (0)$

$\approx \frac{59}{2048}$

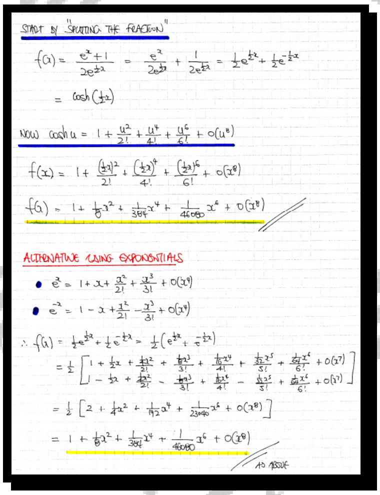
≈ 0.028809 AS REQUIRED

Question 6 (***)

$$f(x) \equiv \frac{e^{x}+1}{2e^{\frac{1}{2}x}}, \quad x \in \mathbb{R}.$$

Use standard results to determine the Maclaurin series expansion of $f(x)$, up and including the term in x^6.

$$f(x) = 1 + \frac{1}{8}x^2 + \frac{1}{384}x^4 + \frac{1}{7680}x^6 + O(x^8)$$



START BY "SPLITTING THE FRACTION"

$$f(x) = \frac{e^{x}+1}{2e^{\frac{1}{2}x}} = \frac{e^{x}}{2e^{\frac{1}{2}x}} + \frac{1}{2e^{\frac{1}{2}x}} = \frac{1}{2}e^{\frac{1}{2}x} + \frac{1}{2}e^{-\frac{1}{2}x}$$

$$= \cosh\left(\tfrac{1}{2}x\right)$$

NOW $\cosh u = 1 + \frac{u^2}{2!} + \frac{u^4}{4!} + \frac{u^6}{6!} + O(u^8)$

$$f(x) = 1 + \frac{(\frac{1}{2}x)^2}{2!} + \frac{(\frac{1}{2}x)^4}{4!} + \frac{(\frac{1}{2}x)^6}{6!} + O(x^8)$$

$$f(x) = 1 + \frac{1}{8}x^2 + \frac{1}{384}x^4 + \frac{1}{46080}x^6 + O(x^8)$$

ALTERNATIVE USING EXPONENTIALS

- $e^{x} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + O(x^4)$
- $e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + O(x^4)$

$$\therefore f(x) = \tfrac{1}{2}e^{\frac{1}{2}x} + \tfrac{1}{2}e^{-\frac{1}{2}x} = \tfrac{1}{2}\left(e^{\frac{1}{2}x} + e^{-\frac{1}{2}x}\right)$$

$$= \frac{1}{2}\left[\begin{matrix} 1 + \frac{1}{2}x + \frac{\frac{1}{4}x^2}{2!} + \frac{\frac{1}{8}x^3}{3!} + \frac{\frac{1}{16}x^4}{4!} + \frac{\frac{1}{32}x^5}{5!} + \frac{\frac{1}{64}x^6}{6!} + O(x^7) \\ 1 - \frac{1}{2}x + \frac{\frac{1}{4}x^2}{2!} - \frac{\frac{1}{8}x^3}{3!} + \frac{\frac{1}{16}x^4}{4!} - \frac{\frac{1}{32}x^5}{5!} + \frac{\frac{1}{64}x^6}{6!} + O(x^7) \end{matrix}\right]$$

$$= \frac{1}{2}\left[2 + \frac{1}{4}x^2 + \frac{1}{192}x^4 + \frac{1}{23040}x^6 + O(x^8)\right]$$

$$= 1 + \frac{1}{8}x^2 + \frac{1}{384}x^4 + \frac{1}{46080}x^6 + O(x^8)$$

AS ABOVE

MACLAURIN EXPANSIONS 20 STANDARD QUESTIONS

Question 1 (***+)

$$y = (1+x)^2 \cos x .$$

Show clearly that …

a) … $\dfrac{d^3y}{dx^3} = \left(x^2+2x-5\right)\sin x - 6(x+1)\cos x$.

b) … $y \approx 1 + Ax + Bx^2 + Cx^3$, where A , B and C are constants to be found.

, proof

a) DIFFERENTIATE & MANIPULATE BY THE PRODUCT RULE

- $y = (1+x)^2 \cos x$
- $\frac{dy}{dx} = 2(1+x)\cos x - (1+x)^2 \sin x$
- $\frac{d^2y}{dx^2} = 2\cos x - 2(1+x)\sin x - 2(1+x)\sin x - (1+x)^2\cos x$
  $= \left[2 - (1+x)^2\right]\cos x - 4(1+x)\sin x$
  $= (2 - 1 - 2x - x^2)\cos x - 4(1+x)\sin x$
  $= (1 - 2x - x^2)\cos x - 4(1+x)\sin x$
- $\frac{d^3y}{dx^3} = (-2-2x)\cos x - (1-2x-x^2)\sin x - 4\sin x - 4(1+x)\cos x$

$\frac{d^3y}{dx^3} = (-2-2x-4-4x)\cos x + (-1+2x+x^2-4)\sin x$

$\frac{d^3y}{dx^3} = (-6x-6)\cos x + (x^2+2x-5)\sin x$

$\frac{d^3y}{dx^3} = (x^2+2x-5)\sin x - 6(x+1)\cos x$ // AS REQUIRED

b) OBTAIN ALL THE DERIVATIVES AT $x=0$

$y\big|_{x=0} = 1$ $\quad \frac{dy}{dx}\big|_{x=0} = 2$

$\frac{d^2y}{dx^2}\big|_{x=0} = 1$

$\frac{d^3y}{dx^3}\big|_{x=0} = -6$

BY THE MACLAURIN THEOREM

$$y = y_0 + xy_0' + \frac{x^2}{2!}y_0'' + \frac{x^3}{3!}y_0''' + O(x^4)$$

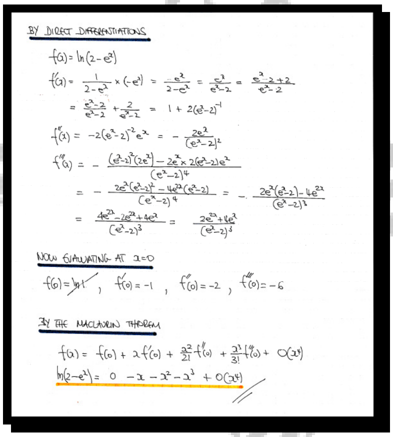
$$(1+x)^2\cos x = 1 + x\times 2 + \frac{x^2}{2}\times 1 + \frac{x^3}{6}\times(-6) + O(x^4)$$

$$(1+x)^2\cos x = 1 + 2x + \frac{1}{2}x^2 - x^3 + O(x^4)$$ //

Question 2 (***+)

Find the Maclaurin expansion of $\ln(2-\mathrm{e}^{x})$, up and including the term in x^3.

$$\ln\left(2-\mathrm{e}^{x}\right) = -x - x^2 - x^3 + O\left(x^4\right)$$



BY DIRECT DIFFERENTIATIONS

$f(x) = \ln(2-e^x)$

$f'(x) = \frac{1}{2-e^x} \times (-e^x) = \frac{-e^x}{2-e^x} = \frac{e^x}{e^x-2} = \frac{e^x-2+2}{e^x-2}$

$= \frac{e^x-2}{e^x-2} + \frac{2}{e^x-2} = 1 + 2(e^x-2)^{-1}$

$f''(x) = -2(e^x-2)^{-2}e^x = -\frac{2e^x}{(e^x-2)^2}$

$f'''(x) = -\frac{(e^x-2)^2(2e^x) - 2e^x \times 2(e^x-2)e^x}{(e^x-2)^4}$

$= -\frac{2e^x(e^x-2)^2 - 4e^{2x}(e^x-2)}{(e^x-2)^4} = -\frac{2e^x(e^x-2) - 4e^{2x}}{(e^x-2)^3}$

$= \frac{4e^{2x} - 2e^{2x} + 4e^x}{(e^x-2)^3} = \frac{2e^{2x}+4e^x}{(e^x-2)^3}$

NOW EVALUATING AT $x=0$

$f(0) = \ln 1$, $f'(0) = -1$, $f''(0) = -2$, $f'''(0) = -6$

BY THE MACLAURIN THEOREM

$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + O(x^4)$

$\ln(2-e^x) = 0 - x - x^2 - x^3 + O(x^4)$

Question 3 (***+)

$$f(x) = \ln(1+\cos 2x), \quad 0 \le x < \frac{\pi}{2}.$$

a) Find an expression for $f'(x)$.

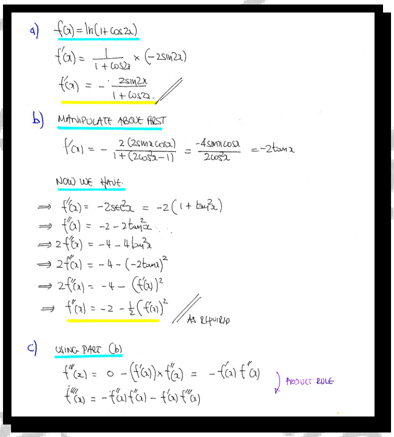
b) Show clearly that

$$f''(x) = -2 - \frac{1}{2}\big(f'(x)\big)^2.$$

c) Show further that the series expansion of the first three non zero terms of $f(x)$ is given by

$$\ln 2 - x^2 - \frac{1}{6}x^4.$$

$$\boxed{f'(x) = -\frac{2\sin 2x}{1+\cos 2x}}$$



a) $f(x) = \ln(1+\cos 2x)$

$f'(x) = \frac{1}{1+\cos 2x} \times (-2\sin 2x)$

$f'(x) = -\frac{2\sin 2x}{1+\cos 2x}$

b) MANIPULATE ABOVE FIRST

$f'(x) = -\frac{2(2\sin x\cos x)}{1+(2\cos^2 x-1)} = \frac{-4\sin x\cos x}{2\cos^2 x} = -2\tan x$

NOW WE HAVE

$\Rightarrow f''(x) = -2\sec^2 x = -2(1+\tan^2 x)$

$\Rightarrow f''(x) = -2 - 2\tan^2 x$

$\Rightarrow 2f''(x) = -4 - 4\tan^2 x$

$\Rightarrow 2f''(x) = -4 - (-2\tan x)^2$

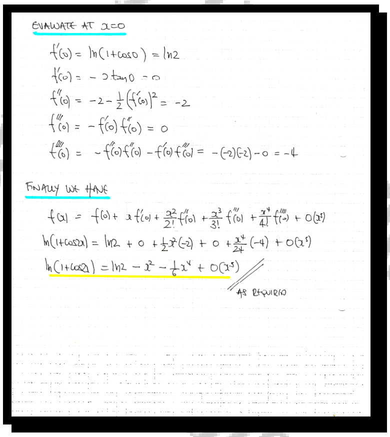
$\Rightarrow 2f''(x) = -4 - (f'(x))^2$

$\Rightarrow f''(x) = -2 - \frac{1}{2}(f'(x))^2$ // AS REQUIRED

c) USING PART (b)

$f'''(x) = 0 - (f'(x)) \times f''(x) = -f'(x)f''(x)$) PRODUCT RULE

$f''''(x) = -f''(x)f''(x) - f'(x)f'''(x)$



EVALUATE AT $x=0$

$f(0) = \ln(1+\cos 0) = \ln 2$

$f'(0) = -2\tan 0 = 0$

$f''(0) = -2 - \frac{1}{2}(f'(0))^2 = -2$

$f'''(0) = -f'(0)f''(0) = 0$

$f''''(0) = -f''(0)f''(0) - f'(0)f'''(0) = -(-2)(-2) - 0 = -4$

FINALLY WE HAVE

$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \frac{x^4}{4!}f''''(0) + O(x^5)$

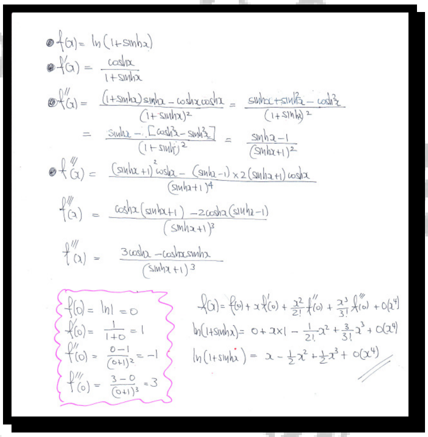
$\ln(1+\cos 2x) = \ln 2 + 0 + \frac{1}{2}x^2(-2) + 0 + \frac{x^4}{24}(-4) + O(x^5)$

$\ln(1+\cos 2x) = \ln 2 - x^2 - \frac{1}{6}x^4 + O(x^5)$ // AS REQUIRED

Question 4 (***+)

Find the Maclaurin expansion of $\ln(1+\sinh x)$ up and including the term in x^3.

$$\boxed{\ln\left(1+\sinh x\right)=x-\tfrac{1}{2}x^2+\tfrac{1}{2}x^3+O\left(x^4\right)}$$



$f(x)=\ln(1+\sinh x)$

$f'(x)=\dfrac{\cosh x}{1+\sinh x}$

$f''(x)=\dfrac{(1+\sinh x)\sinh x-\cosh x\cosh x}{(1+\sinh x)^2}=\dfrac{\sinh x+\sinh^2 x-\cosh^2 x}{(1+\sinh x)^2}$

$=\dfrac{\sinh x-\left[\cosh^2 x-\sinh^2 x\right]}{(1+\sinh x)^2}=\dfrac{\sinh x-1}{(\sinh x+1)^2}$

$f'''(x)=\dfrac{(\sinh x+1)^2\cosh x-(\sinh x-1)\times 2(\sinh x+1)\cosh x}{(\sinh x+1)^4}$

$f'''(x)=\dfrac{\cosh x(\sinh x+1)-2\cosh x(\sinh x-1)}{(\sinh x+1)^3}$

$f'''(x)=\dfrac{3\cosh x-\cosh x\sinh x}{(\sinh x+1)^3}$

$f(0)=\ln 1=0$, $f'(0)=\dfrac{1}{1+0}=1$, $f''(0)=\dfrac{0-1}{(0+1)^2}=-1$, $f'''(0)=\dfrac{3-0}{(0+1)^3}=+3$

$f(x)=f(0)+xf'(0)+\dfrac{x^2}{2!}f''(0)+\dfrac{x^3}{3!}f'''(0)+O(x^4)$

$\ln(1+\sinh x)=0+x\times 1-\dfrac{1}{2!}x^2+\dfrac{3}{3!}x^3+O(x^4)$

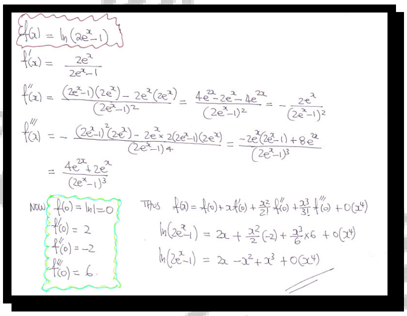
$\ln(1+\sinh x)=x-\tfrac{1}{2}x^2+\tfrac{1}{2}x^3+O(x^4)$

Question 5 (***+)

$$f(x)\equiv\ln\left(2e^x-1\right),\ x\in\mathbb{R}.$$

Find the Maclaurin expansion of $f(x)$, up and including the term in x^3.

$$\boxed{f(x)\equiv 2x-x^2+x^3+O\left(x^4\right)}$$



$f(x)=\ln(2e^x-1)$

$f'(x)=\dfrac{2e^x}{2e^x-1}$

$f''(x)=\dfrac{(2e^x-1)(2e^x)-2e^x(2e^x)}{(2e^x-1)^2}=\dfrac{4e^{2x}-2e^x-4e^{2x}}{(2e^x-1)^2}=-\dfrac{2e^x}{(2e^x-1)^2}$

$f'''(x)=-\dfrac{(2e^x-1)^2(2e^x)-2e^x\times 2(2e^x-1)(2e^x)}{(2e^x-1)^4}=\dfrac{-2e^x(2e^x-1)+8e^{2x}}{(2e^x-1)^3}$

$=\dfrac{4e^{2x}+2e^x}{(2e^x-1)^3}$

Now $f(0)=\ln 1=0$, $f'(0)=2$, $f''(0)=-2$, $f'''(0)=6$

Thus $f(x)=f(0)+xf'(0)+\dfrac{x^2}{2!}f''(0)+\dfrac{x^3}{3!}f'''(0)+O(x^4)$

$\ln(2e^x-1)=2x+\dfrac{x^2}{2}(-2)+\dfrac{x^3}{6}\times 6+O(x^4)$

$\ln(2e^x-1)=2x-x^2+x^3+O(x^4)$

Question 6 **(***+)**

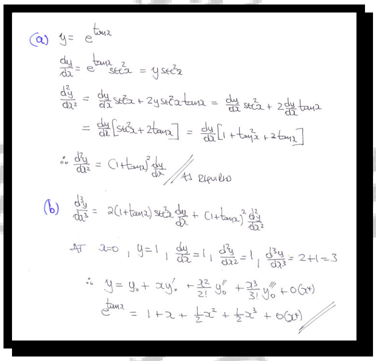
$$y = \mathrm{e}^{\tan x}, \ x \in \mathbb{R}.$$

a) Show clearly that

$$\frac{d^2y}{dx^2} = (1 + \tan x)^2 \frac{dy}{dx}.$$

b) Find a series expansion for $\mathrm{e}^{\tan x}$, up and including the term in x^3.

$$\boxed{\mathrm{e}^{\tan x} = 1 + x + \frac{1}{2}x^2 + \frac{1}{2}x^3 + O(x^4)}$$



(a) $y = e^{\tan x}$

$\frac{dy}{dx} = e^{\tan x}\sec^2 x = y\sec^2 x$

$\frac{d^2y}{dx^2} = \frac{dy}{dx}\sec^2 x + 2y\sec^2 x\tan x = \frac{dy}{dx}\sec^2 x + 2\frac{dy}{dx}\tan x$

$= \frac{dy}{dx}\left[\sec^2 x + 2\tan x\right] = \frac{dy}{dx}\left[1 + \tan^2 x + 2\tan x\right]$

$\therefore \frac{d^2y}{dx^2} = (1+\tan x)^2\frac{dy}{dx}$ // AS REQUIRED

(b) $\frac{d^3y}{dx^3} = 2(1+\tan x)\sec^2 x\frac{dy}{dx} + (1+\tan x)^2\frac{d^2y}{dx^2}$

AT $x=0$, $y=1$, $\frac{dy}{dx}=1$, $\frac{d^2y}{dx^2}=1$, $\frac{d^3y}{dx^3} = 2+1 = 3$

$\therefore y = y_0 + xy_0' + \frac{x^2}{2!}y_0'' + \frac{x^3}{3!}y_0''' + O(x^4)$

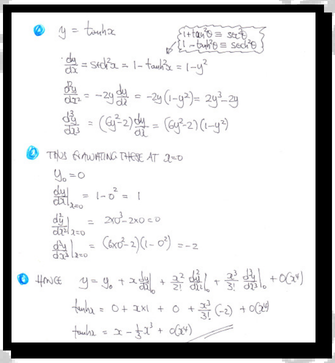
$e^{\tan x} = 1 + x + \frac{1}{2}x^2 + \frac{1}{2}x^3 + O(x^4)$ //

Question 7 (***+)

$$y = \tanh x\,,\ \ x \in \mathbb{R}\,.$$

By expressing the derivatives of $\tanh x$ in terms of y, or otherwise find the first 2 non zero terms of a series expansion for $\tanh x$.

$$\boxed{y \approx x - \frac{1}{3}x^3 + O\left(x^5\right)}$$


$y = \tanh x$

$\begin{cases} 1+\tan^2\theta \equiv \sec^2\theta \\ 1-\tanh^2\theta \equiv \operatorname{sech}^2\theta \end{cases}$

$\frac{dy}{dx} = \operatorname{sech}^2 x = 1 - \tanh^2 x = 1 - y^2$

$\frac{d^2y}{dx^2} = -2y\frac{dy}{dx} = -2y(1-y^2) = 2y^3 - 2y$

$\frac{d^3y}{dx^3} = (6y^2-2)\frac{dy}{dx} = (6y^2-2)(1-y^2)$

THUS EVALUATING THESE AT $x=0$

$y_0 = 0$

$\left.\frac{dy}{dx}\right|_{x=0} = 1 - 0^2 = 1$

$\left.\frac{d^2y}{dx^2}\right|_{x=0} = 2\times0^3 - 2\times0 = 0$

$\left.\frac{d^3y}{dx^3}\right|_{x=0} = (6\times0^2-2)(1-0^2) = -2$

HENCE $y = y_0 + x\left.\frac{dy}{dx}\right|_0 + \frac{x^2}{2!}\left.\frac{d^2y}{dx^2}\right|_0 + \frac{x^3}{3!}\left.\frac{d^3y}{dx^3}\right|_0 + O(x^4)$

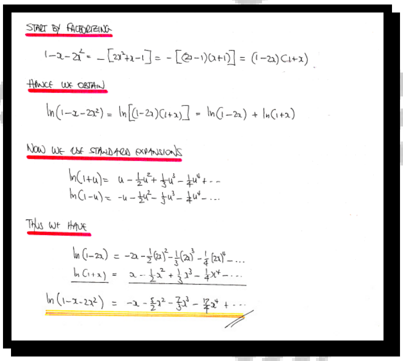
$\tanh x = 0 + x\times1 + 0 + \frac{x^3}{3!}(-2) + O(x^4)$

$\tanh x = x - \frac{1}{3}x^3 + O(x^4)$

Question 8 (***+)

By using results for series expansions of standard functions, find the series expansion of $\ln\left(1-x-2x^2\right)$ up and including the term in x^4.

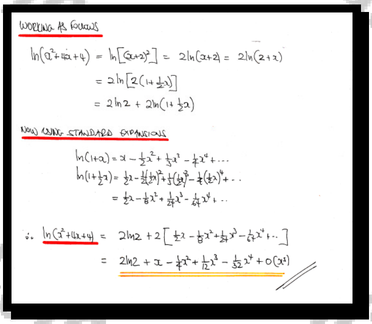
$$\ln\left(1-x-2x^2\right) = -x - \frac{5}{2}x^2 - \frac{7}{3}x^3 - \frac{17}{4}x^4 + O\left(x^5\right)$$



Question 9 (***+)

By using results for series expansions of standard functions, or otherwise, find the series expansion of $\ln\left(x^2+4x+4\right)$ up and including the term in x^4.

V, $\ln\left(x^2+4x+4\right) = 2\ln 2 + x - \frac{1}{4}x^2 + \frac{1}{12}x^3 - \frac{1}{32}x^4 + O\left(x^5\right)$



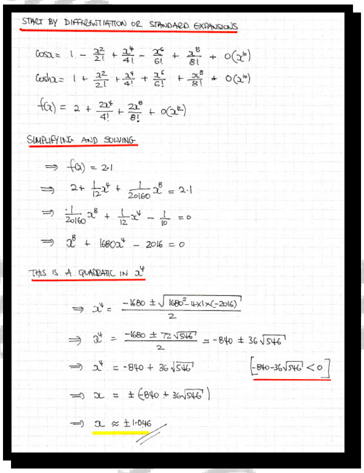
Question 10 **(***+)**

$$f(x) \equiv \cos x + \cosh x \, , \;\; x \in \mathbb{R} \, .$$

Use the first 3 non zero terms of the Maclaurin expansion of $f(x)$ to approximate the solutions of the equation

$$f(x) = 2.1 \, .$$

$\boxed{}$, $\boxed{x \approx \pm 1.046}$


START BY DIFFERENTIATION OR STANDARD EXPANSIONS

$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} + O(x^{10})$

$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \frac{x^8}{8!} + O(x^{10})$

$f(x) = 2 + \frac{2x^4}{4!} + \frac{2x^8}{8!} + O(x^{12})$

SIMPLIFYING AND SOLVING

$\Rightarrow f(x) = 2.1$

$\Rightarrow 2 + \frac{1}{12}x^4 + \frac{1}{20160}x^8 = 2.1$

$\Rightarrow \frac{1}{20160}x^8 + \frac{1}{12}x^4 - \frac{1}{10} = 0$

$\Rightarrow x^8 + 1680x^4 - 2016 = 0$

THIS IS A QUADRATIC IN x^4

$\Rightarrow x^4 = \frac{-1680 \pm \sqrt{1680^2 - 4 \times 1 \times (-2016)}}{2}$

$\Rightarrow x^4 = \frac{-1680 \pm 72\sqrt{546}}{2} = -840 \pm 36\sqrt{546}$

$\Rightarrow x^4 = -840 + 36\sqrt{546}$ $\quad [-840 - 36\sqrt{546} < 0]$

$\Rightarrow x = \pm(-840 + 36\sqrt{546})$

$\Rightarrow x \approx \pm 1.046$

Question 11 (****)

$$f(x) \equiv \sin\left[\ln(1+x)\right], \quad x \in \mathbb{R}, \quad x > -1.$$

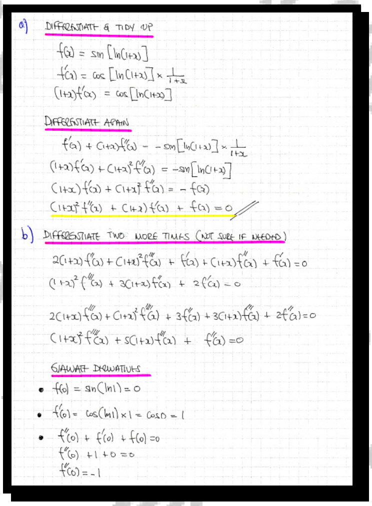
a) Show that

$$(1+x)^2 f''(x) + (1+x)f'(x) + f(x) = 0$$

b) Hence find first 3 non zero terms of the Maclaurin expansion of $f(x)$.

c) Use the result of part **(b)** to find first 2 non zero terms of the Maclaurin expansion of $\sin\left[\ln(1+x)\right]$.

FP, $\sin\left[\ln(1+x)\right] \approx x - \frac{1}{2}x^2 + \frac{1}{6}x^3$, $\cos\left[\ln(1+x)\right] \approx 1 - \frac{1}{2}x^2$



a) DIFFERENTIATE & TIDY UP

$f(x) = \sin[\ln(1+x)]$

$f'(x) = \cos[\ln(1+x)] \times \frac{1}{1+x}$

$(1+x)f'(x) = \cos[\ln(1+x)]$

DIFFERENTIATE AGAIN

$f'(x) + (1+x)f''(x) = -\sin[\ln(1+x)] \times \frac{1}{1+x}$

$(1+x)f'(x) + (1+x)^2 f''(x) = -\sin[\ln(1+x)]$

$(1+x)f'(x) + (1+x)^2 f''(x) = -f(x)$

$(1+x)^2 f''(x) + (1+x)f'(x) + f(x) = 0$

b) DIFFERENTIATE TWO MORE TIMES (NOT SURE IF NEEDED)

$2(1+x)f''(x) + (1+x)^2 f'''(x) + f'(x) + (1+x)f''(x) + f'(x) = 0$

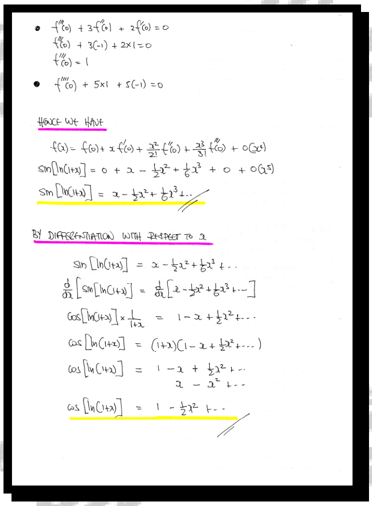
$(1+x)^2 f'''(x) + 3(1+x)f''(x) + 2f'(x) = 0$

$2(1+x)f'''(x) + (1+x)^2 f''''(x) + 3f''(x) + 3(1+x)f'''(x) + 2f''(x) = 0$

$(1+x)^2 f''''(x) + 5(1+x)f'''(x) + f''(x) = 0$

EVALUATE DERIVATIVES

- $f(0) = \sin(\ln 1) = 0$
- $f'(0) = \cos(\ln 1) \times 1 = \cos 0 = 1$
- $f''(0) + f'(0) + f(0) = 0$
  $f''(0) + 1 + 0 = 0$
  $f''(0) = -1$



- $f'''(0) + 3f''(0) + 2f'(0) = 0$
  $f'''(0) + 3(-1) + 2 \times 1 = 0$
  $f'''(0) = 1$
- $f''''(0) + 5 \times 1 + 5(-1) = 0$

HENCE WE HAVE

$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + O(x^4)$

$\sin[\ln(1+x)] = 0 + x - \frac{1}{2}x^2 + \frac{1}{6}x^3 + 0 + O(x^5)$

$\sin[\ln(1+x)] = x - \frac{1}{2}x^2 + \frac{1}{6}x^3 + \dots$

BY DIFFERENTIATION WITH RESPECT TO x

$\sin[\ln(1+x)] = x - \frac{1}{2}x^2 + \frac{1}{6}x^3 + \dots$

$\frac{d}{dx}\left[\sin[\ln(1+x)]\right] = \frac{d}{dx}\left[x - \frac{1}{2}x^2 + \frac{1}{6}x^3 + \dots\right]$

$\cos[\ln(1+x)] \times \frac{1}{1+x} = 1 - x + \frac{1}{2}x^2 + \dots$

$\cos[\ln(1+x)] = (1+x)(1 - x + \frac{1}{2}x^2 + \dots)$

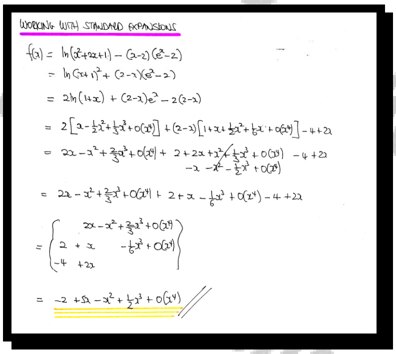
$\cos[\ln(1+x)] = 1 - x + \frac{1}{2}x^2 + \dots$
$\qquad\qquad\qquad\quad x - x^2 + \dots$

$\cos[\ln(1+x)] = 1 - \frac{1}{2}x^2 + \dots$

Question 12 (****)

By using results for series expansions of standard functions, or otherwise, find the series expansion of $\ln\left(x^2+2x+1\right)-(x-2)\left(\mathrm{e}^x-2\right)$ up and including the term in x^3.

$$\ln\left(x^2+2x+1\right)-(x-2)\left(\mathrm{e}^x-2\right)=-2+5x-x^2+\frac{1}{2}x^3+O\left(x^4\right)$$



Question 13 (****)

$$f(x) = \mathrm{e}^{x} \cos x \,, \; x \in \mathbb{R} .$$

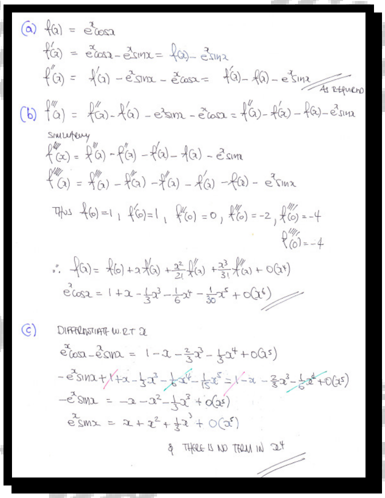
a) Show clearly that

$$f''(x) = f'(x) - f(x) - \mathrm{e}^{x} \sin x .$$

b) Find a series expansion for $f(x)$, up and including the term in x^5.

c) Hence find a series expansion for $\mathrm{e}^{x} \sin x$, up and including the term in x^4, showing further that the coefficient of x^4 is zero.

$f(x) = 1 + x + \frac{1}{3}x^3 - \frac{1}{6}x^4 - \frac{1}{30}x^5 + O(x^6)$, $\mathrm{e}^{x} \sin x = x + x^2 + \frac{1}{3}x^3 + O(x^5)$


(a) $f(x) = e^{x}\cos x$

$f'(x) = e^{x}\cos x - e^{x}\sin x = f(x) - e^{x}\sin x$

$f''(x) = f'(x) - e^{x}\sin x - e^{x}\cos x = f'(x) - f(x) - e^{x}\sin x$ AS REQUIRED

(b) $f'''(x) = f''(x) - f'(x) - e^{x}\sin x - e^{x}\cos x = f''(x) - f'(x) - f(x) - e^{x}\sin x$

SIMILARLY

$f''''(x) = f'''(x) - f''(x) - f'(x) - f(x) - e^{x}\sin x$

$f'''''(x) = f''''(x) - f'''(x) - f''(x) - f'(x) - f(x) - e^{x}\sin x$

THUS $f(0)=1$, $f'(0)=1$, $f''(0)=0$, $f'''(0)=-2$, $f''''(0)=-4$, $f'''''(0)=-4$

$\therefore f(x) = f(0) + x f'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + O(x^4)$

$e^{x}\cos x = 1 + x - \frac{1}{3}x^3 - \frac{1}{6}x^4 - \frac{1}{30}x^5 + O(x^6)$

(c) DIFFERENTIATE W.R.T x

$e^{x}\cos x - e^{x}\sin x = 1 - x - \frac{2}{3}x^3 - \frac{1}{3}x^4 + O(x^5)$

$-e^{x}\sin x + 1 + x - \frac{1}{3}x^3 - \frac{1}{6}x^4 - \frac{1}{15}x^5 = 1 - x - \frac{2}{3}x^3 - \frac{1}{6}x^4 + O(x^5)$

$-e^{x}\sin x = -x - x^2 - \frac{1}{3}x^3 + O(x^5)$

$e^{x}\sin x = x + x^2 + \frac{1}{3}x^3 + O(x^5)$

& THERE IS NO TERM IN x^4

Question 14 (****)

The functions f and g are given below.

$$f(x) = \arctan\left(\tfrac{2}{3}x\right), \quad x \in \mathbb{R}.$$

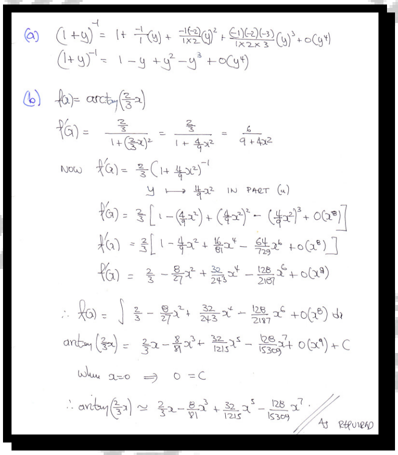
$$g(y) = \frac{1}{1+y}, \quad y \in \mathbb{R}, \ -1 < y < 1.$$

a) Expand $g(y)$ as a binomial series , up and including the term in y^3.

b) Use $f'(x)$ and the answer to part (a) to show clearly that

$$\arctan\left(\tfrac{2}{3}x\right) \approx \frac{2}{3}x - \frac{8}{81}x^3 + \frac{32}{1215}x^5 - \frac{128}{15309}x^7.$$

$$\boxed{g(y) = 1 - y + y^2 - y^3 + O(y^4)}$$



(a) $(1+y)^{-1} = 1 + \frac{-1}{1}(y) + \frac{-1(-2)}{1\times 2}(y)^2 + \frac{(-1)(-2)(-3)}{1\times 2\times 3}(y)^3 + O(y^4)$

$(1+y)^{-1} = 1 - y + y^2 - y^3 + O(y^4)$

(b) $f(x) = \arctan\left(\frac{2}{3}x\right)$

$f'(x) = \dfrac{\frac{2}{3}}{1+\left(\frac{2}{3}x\right)^2} = \dfrac{\frac{2}{3}}{1+\frac{4}{9}x^2} = \dfrac{6}{9+4x^2}$

NOW $f'(x) = \frac{2}{3}\left(1+\frac{4}{9}x^2\right)^{-1}$

$y \longmapsto \frac{4}{9}x^2$ IN PART (a)

$f'(x) = \frac{2}{3}\left[1 - \left(\frac{4}{9}x^2\right) + \left(\frac{4}{9}x^2\right)^2 - \left(\frac{4}{9}x^2\right)^3 + O(x^8)\right]$

$f'(x) = \frac{2}{3}\left[1 - \frac{4}{9}x^2 + \frac{16}{81}x^4 - \frac{64}{729}x^6 + O(x^8)\right]$

$f'(x) = \frac{2}{3} - \frac{8}{27}x^2 + \frac{32}{243}x^4 - \frac{128}{2187}x^6 + O(x^8)$

$\therefore f(x) = \int \frac{2}{3} - \frac{8}{27}x^2 + \frac{32}{243}x^4 - \frac{128}{2187}x^6 + O(x^8)\, dx$

$\arctan\left(\frac{2}{3}x\right) = \frac{2}{3}x - \frac{8}{81}x^3 + \frac{32}{1215}x^5 - \frac{128}{15309}x^7 + O(x^9) + C$

When $x = 0 \Rightarrow 0 = C$

$\therefore \arctan\left(\frac{2}{3}x\right) \simeq \frac{2}{3}x - \frac{8}{81}x^3 + \frac{32}{1215}x^5 - \frac{128}{15309}x^7$ // AS REQUIRED

Question 15 (****)

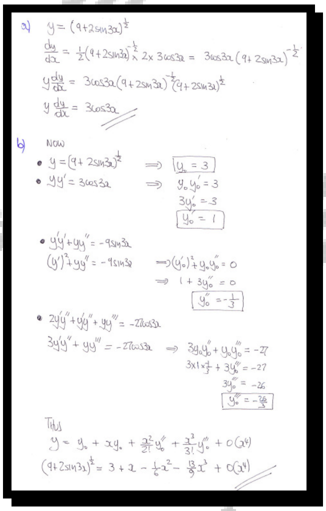
$$y = \sqrt{9 + 2\sin 3x} \,.$$

a) Find a simplified expression for $y\dfrac{dy}{dx}$.

b) Hence show that if x is numerically small

$$y \approx 3 + x - \frac{1}{6}x^2 - \frac{13}{9}x^3 \,.$$

$\boxed{3\cos 3x}$


a) $y = (9+2\sin 3x)^{\frac{1}{2}}$

$\frac{dy}{dx} = \frac{1}{2}(9+2\sin 3x)^{-\frac{1}{2}} \times 2 \times 3\cos 3x = 3\cos 3x\,(9+2\sin 3x)^{-\frac{1}{2}}$

$y\frac{dy}{dx} = 3\cos 3x(9+2\sin 3x)^{-\frac{1}{2}}(9+2\sin 3x)^{\frac{1}{2}}$

$y\frac{dy}{dx} = 3\cos 3x$

b) Now

- $y = (9+2\sin 3x)^{\frac{1}{2}} \implies \boxed{y_0 = 3}$
- $yy' = 3\cos 3x \implies y_0 y_0' = 3$, $3y_0' = 3$, $\boxed{y_0' = 1}$
- $y'y' + yy'' = -9\sin 3x$, $(y')^2 + yy'' = -9\sin 3x \implies (y_0')^2 + y_0 y_0'' = 0 \implies 1 + 3y_0'' = 0$, $\boxed{y_0'' = -\frac{1}{3}}$
- $2y'y'' + y'y'' + yy''' = -27\cos 3x$, $3y'y'' + yy''' = -27\cos 3x \implies 3y_0 y_0'' + y_0 y_0''' = -27$, $3 \times 1 \times \frac{1}{3} + 3y_0''' = -27$, $3y_0''' = -26$, $\boxed{y_0''' = -\frac{26}{3}}$

Thus

$y = y_0 + xy_0 + \frac{x^2}{2!}y_0'' + \frac{x^3}{3!}y_0''' + O(x^4)$

$(9+2\sin 3x)^{\frac{1}{2}} = 3 + x - \frac{1}{6}x^2 - \frac{13}{9}x^3 + O(x^4)$

Question 16 (****)

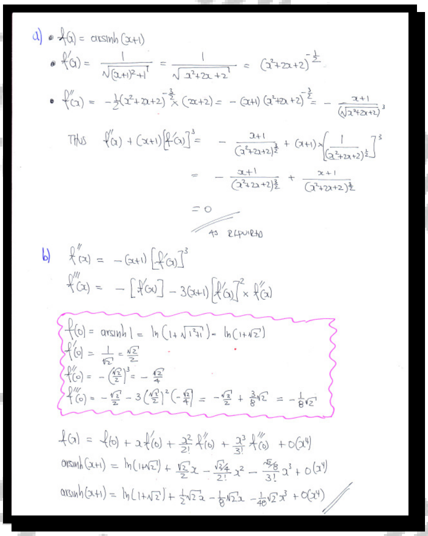
$$f(x) = \text{arsinh}(x+1), \ x \in \mathbb{R}.$$

Show clearly that …

a) … $f''(x) + (x+1)\left[f'(x)\right]^3 = 0$.

b) … $\text{arsinh}(x+1) \approx \ln\left(1+\sqrt{2}\right) + \frac{\sqrt{2}}{2}x - \frac{\sqrt{2}}{8}x^2 + \frac{\sqrt{2}}{48}x^3$.

proof



a) • $f(x) = \text{arsinh}(x+1)$

• $f'(x) = \frac{1}{\sqrt{(x+1)^2+1}} = \frac{1}{\sqrt{x^2+2x+2}} = (x^2+2x+2)^{-\frac{1}{2}}$

• $f''(x) = -\frac{1}{2}(x^2+2x+2)^{-\frac{3}{2}} \times (2x+2) = -(x+1)(x^2+2x+2)^{-\frac{3}{2}} = -\frac{x+1}{\left(\sqrt{x^2+2x+2}\right)^3}$

THUS $f''(x) + (x+1)\left[f'(x)\right]^3 = -\frac{x+1}{(x^2+2x+2)^{\frac{3}{2}}} + (x+1)\times\left[\frac{1}{(x^2+2x+2)^{\frac{1}{2}}}\right]^3$

$= -\frac{x+1}{(x^2+2x+2)^{\frac{3}{2}}} + \frac{x+1}{(x^2+2x+2)^{\frac{3}{2}}}$

$= 0$

AS REQUIRED

b) $f''(x) = -(x+1)\left[f'(x)\right]^3$

$f'''(x) = -\left[f'(x)\right] - 3(x+1)\left[f'(x)\right]^2 \times f''(x)$

$f(0) = \text{arsinh}\,1 = \ln\left(1+\sqrt{1^2+1}\right) = \ln\left(1+\sqrt{2}\right)$

$f'(0) = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$

$f''(0) = -\left(\frac{\sqrt{2}}{2}\right)^3 = -\frac{\sqrt{2}}{4}$

$f'''(0) = -\frac{\sqrt{2}}{2} - 3\left(\frac{\sqrt{2}}{2}\right)^2\left(-\frac{\sqrt{2}}{4}\right) = -\frac{\sqrt{2}}{2} + \frac{3}{8}\sqrt{2} = -\frac{1}{8}\sqrt{2}$

$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + O(x^4)$

$\text{arsinh}(x+1) = \ln\left(1+\sqrt{2}\right) + \frac{\sqrt{2}}{2}x - \frac{\sqrt{2}/4}{2!}x^2 - \frac{\sqrt{2}/8}{3!}x^3 + O(x^4)$

$\text{arsinh}(x+1) = \ln\left(1+\sqrt{2}\right) + \frac{1}{2}\sqrt{2}x - \frac{1}{8}\sqrt{2}x - \frac{1}{48}\sqrt{2}x^3 + O(x^4)$

Question 17 (****)

$$y = \tan x\,,\ 0 \le x < \frac{\pi}{2}\,.$$

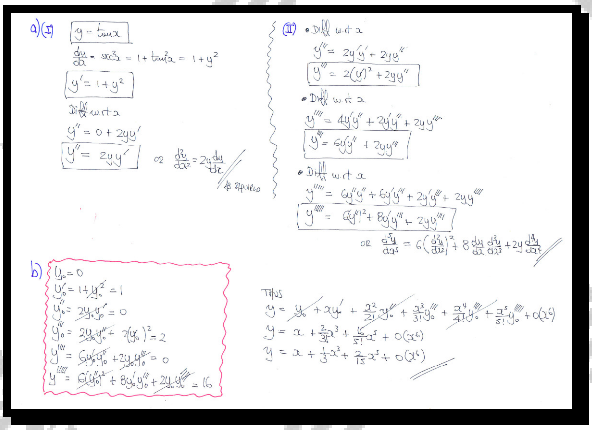
a) Show clearly that …

i. … $\dfrac{d^2y}{dx^2} = 2y\dfrac{dy}{dx}$.

ii. … $\dfrac{d^5y}{dx^5} = 6\left(\dfrac{d^2y}{dx^2}\right)^2 + 8\dfrac{dy}{dx}\dfrac{d^3y}{dx^3} + 2y\dfrac{d^4y}{dx^4}$.

b) Use these results to find the first 3 non zero terms of a series expansion for y.

$$\boxed{y \approx x + \frac{1}{3}x^3 + \frac{2}{15}x^5}$$



a)(I) $y = \tan x$

$\frac{dy}{dx} = \sec^2 x = 1 + \tan^2 x = 1 + y^2$

$y' = 1 + y^2$

Diff w.r.t x

$y'' = 0 + 2yy'$

$y'' = 2yy'$ OR $\frac{d^2y}{dx^2} = 2y\frac{dy}{dx}$ // AS REQUIRED

(II) • Diff w.r.t x

$y''' = 2y'y' + 2yy''$

$y''' = 2(y')^2 + 2yy''$

• Diff w.r.t x

$y'''' = 4y'y'' + 2y'y'' + 2yy'''$

$y'''' = 6y'y'' + 2yy'''$

• Diff w.r.t x

$y''''' = 6y''y'' + 6y'y''' + 2y'y''' + 2yy''''$

$y''''' = 6(y'')^2 + 8y'y''' + 2yy''''$

OR $\frac{d^5y}{dx^5} = 6\left(\frac{d^2y}{dx^2}\right)^2 + 8\frac{dy}{dx}\frac{d^3y}{dx^3} + 2y\frac{d^4y}{dx^4}$ //

b) $y_0 = 0$

$y_0' = 1 + y_0^2 = 1$

$y_0'' = 2y_0y_0' = 0$

$y_0''' = 2y_0y_0'' + 2(y_0')^2 = 2$

$y'''' = 6y_0'y_0'' + 2y_0y_0''' = 0$

$y''''' = 6(y_0'')^2 + 8y_0'y_0''' + 2y_0y_0'''' = 16$

THUS

$y = y_0 + xy_0' + \frac{x^2}{2!}y_0'' + \frac{x^3}{3!}y_0''' + \frac{x^4}{4!}y_0'''' + \frac{x^5}{5!}y_0''''' + O(x^6)$

$y = x + \frac{2}{3!}x^3 + \frac{16}{5!}x^5 + O(x^6)$

$y = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + O(x^6)$ //

Question 18 (****)

$$y = \ln(4+3x), \ x > -\frac{4}{3}.$$

a) Find simplified expressions for $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$ and $\frac{d^3y}{dx^3}$.

b) Hence, find the first 4 terms in the Maclaurin expansion of $y = \ln(4+3x)$.

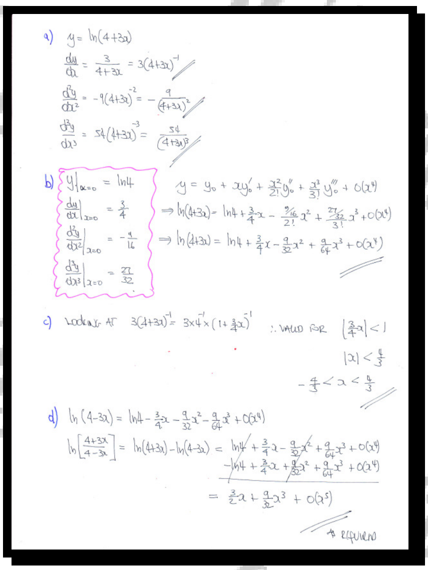
c) State the range of values of x for which the expansion is valid.

d) Show that for small values of x,

$$\ln\left(\frac{4+3x}{4-3x}\right) \approx \frac{3}{2}x + \frac{9}{32}x^3.$$

$\boxed{\frac{dy}{dx} = \frac{3}{3x+4}}$, $\boxed{\frac{d^2y}{dx^2} = -\frac{9}{(3x+4)^2}}$, $\boxed{\frac{d^3y}{dx^3} = \frac{54}{(3x+4)^2}}$, $\boxed{-\frac{4}{3} < x \le \frac{4}{3}}$,

$\boxed{\ln(4+3x) = \ln 4 + \frac{3}{4}x - \frac{9}{32}x^2 + \frac{9}{64}x^3 + O(x^4)}$



a) $y = \ln(4+3x)$

$\frac{dy}{dx} = \frac{3}{4+3x} = 3(4+3x)^{-1}$

$\frac{d^2y}{dx^2} = -9(4+3x)^{-2} = -\frac{9}{(4+3x)^2}$

$\frac{d^3y}{dx^3} = 54(4+3x)^{-3} = \frac{54}{(4+3x)^3}$

b) $y|_{x=0} = \ln 4$, $\frac{dy}{dx}\big|_{x=0} = \frac{3}{4}$, $\frac{d^2y}{dx^2}\big|_{x=0} = -\frac{9}{16}$, $\frac{d^3y}{dx^3}\big|_{x=0} = \frac{27}{32}$

$y = y_0 + xy_0' + \frac{x^2}{2!}y_0'' + \frac{x^3}{3!}y_0''' + O(x^4)$

$\Rightarrow \ln(4+3x) = \ln 4 + \frac{3}{4}x - \frac{9/16}{2!}x^2 + \frac{27/32}{3!}x^3 + O(x^4)$

$\Rightarrow \ln(4+3x) = \ln 4 + \frac{3}{4}x - \frac{9}{32}x^2 + \frac{9}{64}x^3 + O(x^4)$

c) LOOKING AT $3(4+3x)^{-1} = 3 \times 4^{-1} \times (1 + \frac{3}{4}x)^{-1}$ ∴ VALID FOR $|\frac{3}{4}x| < 1$

$|x| < \frac{4}{3}$

$-\frac{4}{3} < x < \frac{4}{3}$

d) $\ln(4-3x) = \ln 4 - \frac{3}{4}x - \frac{9}{32}x^2 - \frac{9}{64}x^3 + O(x^4)$

$\ln\left[\frac{4+3x}{4-3x}\right] = \ln(4+3x) - \ln(4-3x) = \ln 4 + \frac{3}{4}x - \frac{9}{32}x^2 + \frac{9}{64}x^3 + O(x^4)$
$\quad - \ln 4 + \frac{3}{4}x + \frac{9}{32}x^2 + \frac{9}{64}x^3 + O(x^4)$

$= \frac{3}{2}x + \frac{9}{32}x^3 + O(x^5)$

AS REQUIRED

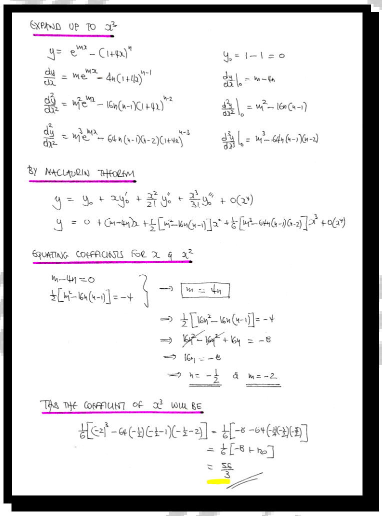
Question 19 (****)

If m and n are non zero constants, then the first non zero term in the Maclaurin expansion of $e^{mx} - (1+4x)^n$ is $-4x^2$.

Find the coefficient of x^3 in this expansion.

*You may **NOT** use standard series expansions in this question.*

$\boxed{}$, $\boxed{[x^3] = \frac{56}{3}}$


EXPAND UP TO x^3

$y = e^{mx} - (1+4x)^n \qquad y_0 = 1 - 1 = 0$

$\frac{dy}{dx} = me^{mx} - 4n(1+4x)^{n-1} \qquad \left.\frac{dy}{dx}\right|_0 = m - 4n$

$\frac{d^2y}{dx^2} = m^2e^{mx} - 16n(n-1)(1+4x)^{n-2} \qquad \left.\frac{d^2y}{dx^2}\right|_0 = m^2 - 16n(n-1)$

$\frac{d^3y}{dx^3} = m^3e^{mx} - 64n(n-1)(n-2)(1+4x)^{n-3} \qquad \left.\frac{d^3y}{dx^3}\right|_0 = m^3 - 64n(n-1)(n-2)$

BY MACLAURIN THEOREM

$y = y_0 + xy'_0 + \frac{x^2}{2!}y''_0 + \frac{x^3}{3!}y'''_0 + O(x^4)$

$y = 0 + (m-4n)x + \frac{1}{2}[m^2 - 16n(n-1)]x^2 + \frac{1}{6}[m^3 - 64n(n-1)(n-2)]x^3 + O(x^4)$

EQUATING COEFFICIENTS FOR x & x^2

$m - 4n = 0$, $\frac{1}{2}[m^2 - 16n(n-1)] = -4 \implies m = 4n$

$\implies \frac{1}{2}[16n^2 - 16n(n-1)] = -4$

$\implies 16n^2 - 16n^2 + 16n = -8$

$\implies 16n = -8$

$\implies n = -\frac{1}{2}$ & $m = -2$

THUS THE COEFFICIENT OF x^3 WILL BE

$\frac{1}{6}\left[(-2)^3 - 64\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)\left(-\frac{1}{2}-2\right)\right] = \frac{1}{6}\left[-8 - 64\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\right]$

$= \frac{1}{6}[-8 + 120]$

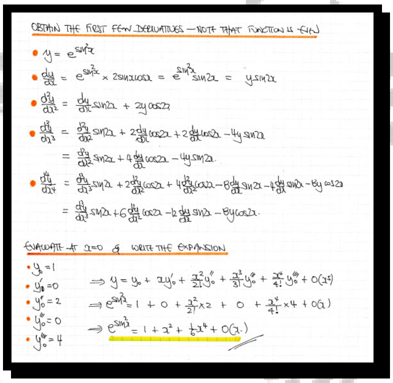
$= \frac{56}{3}$

Question 20 (****)

Determine the first 3 no zero terms in the Maclaurin expansion of

$$y = \mathrm{e}^{\sin^2 x}.$$

$\boxed{}$, $\boxed{y = 1 + x^2 + \frac{1}{2}x^4 + O\left(x^6\right)}$



OBTAIN THE FIRST FEW DERIVATIVES — NOTE THAT FUNCTION IS EVEN

- $y = e^{\sin^2 x}$
- $\frac{dy}{dx} = e^{\sin^2 x} \times 2\sin x\cos x = e^{\sin^2 x}\sin 2x = y\sin 2x$
- $\frac{d^2y}{dx^2} = \frac{dy}{dx}\sin 2x + 2y\cos 2x$
- $\frac{d^3y}{dx^3} = \frac{d^2y}{dx^2}\sin 2x + 2\frac{dy}{dx}\cos 2x + 2\frac{dy}{dx}\cos 2x - 4y\sin 2x$
  $= \frac{d^2y}{dx^2}\sin 2x + 4\frac{dy}{dx}\cos 2x - 4y\sin 2x.$
- $\frac{d^4y}{dx^4} = \frac{d^3y}{dx^3}\sin 2x + 2\frac{d^2y}{dx^2}\cos 2x + 4\frac{d^2y}{dx^2}\cos 2x - 8\frac{dy}{dx}\sin 2x - 4\frac{dy}{dx}\sin 2x - 8y\cos 2x$
  $= \frac{d^3y}{dx^3}\sin 2x + 6\frac{d^2y}{dx^2}\cos 2x - 12\frac{dy}{dx}\sin 2x - 8y\cos 2x.$

EVALUATE AT $x=0$ & WRITE THE EXPANSION

- $y_0 = 1$
- $y_0' = 0$
- $y_0'' = 2$
- $y_0''' = 0$
- $y_0'''' = 4$

$\Rightarrow y = y_0 + xy_0' + \frac{x^2}{2!}y_0'' + \frac{x^3}{3!}y_0''' + \frac{x^4}{4!}y_0'''' + O(x^5)$

$\Rightarrow e^{\sin^2 x} = 1 + 0 + \frac{x^2}{2!}\times 2 + 0 + \frac{x^4}{4!}\times 4 + O(x)$

$\Rightarrow e^{\sin^2 x} = 1 + x^2 + \frac{1}{6}x^4 + O(x)$

MACLAURIN EXPANSIONS 7 HARD QUESTIONS

Question 1 (****+)

$$y = \ln(1+\sin x), \ \sin x \neq -1.$$

a) Show clearly that

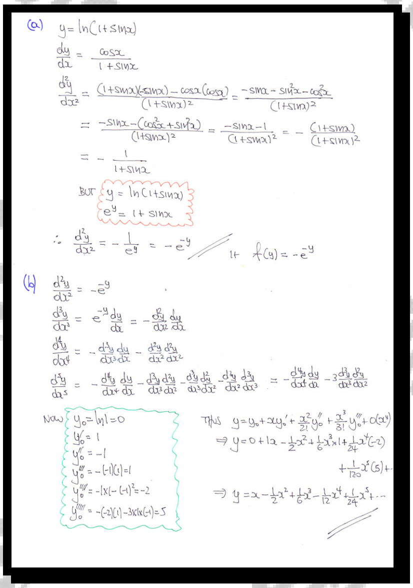
$$\frac{dy}{dx} = f(y),$$

where $f(y)$ is a function to be found.

b) Hence show further that

$$\ln(1+\sin x) \approx x - \frac{1}{2}x^2 + \frac{1}{6}x^3 - \frac{1}{12}x^4 + \frac{1}{24}x^5.$$

$\boxed{y = -\mathrm{e}^{-y}}$



(a) $y = \ln(1+\sin x)$

$\frac{dy}{dx} = \frac{\cos x}{1+\sin x}$

$\frac{d^2y}{dx^2} = \frac{(1+\sin x)(-\sin x) - \cos x(\cos x)}{(1+\sin x)^2} = \frac{-\sin x - \sin^2 x - \cos^2 x}{(1+\sin x)^2}$

$= \frac{-\sin x - (\cos^2 x + \sin^2 x)}{(1+\sin x)^2} = \frac{-\sin x - 1}{(1+\sin x)^2} = -\frac{(1+\sin x)}{(1+\sin x)^2}$

$= -\frac{1}{1+\sin x}$

BUT $y = \ln(1+\sin x)$, $e^y = 1+\sin x$

$\therefore \frac{d^2y}{dx^2} = -\frac{1}{e^y} = -e^{-y}$ // ie $f(y) = -e^{-y}$

(b) $\frac{d^2y}{dx^2} = -e^{-y}$

$\frac{d^3y}{dx^3} = e^{-y}\frac{dy}{dx} = -\frac{d^2y}{dx^2}\frac{dy}{dx}$

$\frac{d^4y}{dx^4} = -\frac{d^3y}{dx^3}\frac{dy}{dx} - \frac{d^2y}{dx^2}\frac{d^2y}{dx^2}$

$\frac{d^5y}{dx^5} = -\frac{d^4y}{dx^4}\frac{dy}{dx} - \frac{d^3y}{dx^3}\frac{d^2y}{dx^2} - \frac{d^3y}{dx^3}\frac{d^2y}{dx^2} - \frac{d^2y}{dx^2}\frac{d^3y}{dx^3} = -\frac{d^4y}{dx^4}\frac{dy}{dx} - 3\frac{d^3y}{dx^3}\frac{d^2y}{dx^2}$

Now $y_0 = \ln 1 = 0$
$y_0' = 1$
$y_0'' = -1$
$y_0''' = -(-1)(1) = 1$
$y_0'''' = -1\times(-(-1)^2 = -2$
$y_0''''' = -(-2)(1) - 3\times1\times(-1) = 5$

Thus $y = y_0 + xy_0' + \frac{x^2}{2!}y_0'' + \frac{x^3}{3!}y_0''' + O(x^4)$

$\Rightarrow y = 0 + 1x - \frac{1}{2}x^2 + \frac{1}{6}x^3\times1 + \frac{1}{24}x^4(-2) + \frac{1}{120}x^5(5) + \dots$

$\Rightarrow y = x - \frac{1}{2}x^2 + \frac{1}{6}x^3 - \frac{1}{12}x^4 + \frac{1}{24}x^5 + \dots$ //

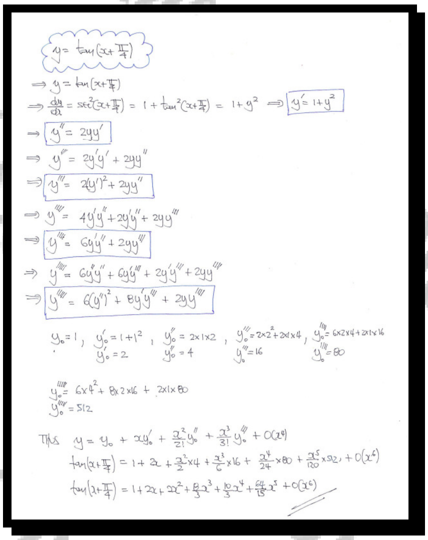
Question 2 (****+)

$$y=\tan\left(x+\frac{\pi}{4}\right),\quad -\frac{3\pi}{4}<x<\frac{\pi}{4}.$$

Use the Maclaurin theorem to show that

$$y=\tan\left(x+\frac{\pi}{4}\right)\approx 1+2x+2x^2+\frac{8}{3}x^3+\frac{10}{3}x^4+\frac{64}{15}x^5.$$

proof



$y=\tan\left(x+\frac{\pi}{4}\right)$

$\Rightarrow y=\tan\left(x+\frac{\pi}{4}\right)$

$\Rightarrow \frac{dy}{dx}=\sec^2\left(x+\frac{\pi}{4}\right)=1+\tan^2\left(x+\frac{\pi}{4}\right)=1+y^2 \Rightarrow y'=1+y^2$

$\Rightarrow y''=2yy'$

$\Rightarrow y'''=2y'y'+2yy''$

$\Rightarrow y'''=2(y')^2+2yy''$

$\Rightarrow y''''=4y'y''+2y'y''+2yy'''$

$\Rightarrow y''''=6y'y''+2yy'''$

$\Rightarrow y^{v}=6y'y''+6y'y'''+2y'y'''+2yy''''$

$\Rightarrow y^{v}=6(y'')^2+8y'y'''+2yy''''$

$y_0=1,\quad y_0'=1+1^2,\ y_0'=2,\quad y_0''=2\times1\times2,\ y_0''=4,\quad y_0'''=2\times2^2+2\times1\times4,\ y_0'''=16,\quad y_0''''=6\times2\times4+2\times1\times16,\ y_0''''=80$

$y_0^{v}=6\times4^2+8\times2\times16+2\times1\times80$

$y_0^{v}=512$

Thus $y=y_0+xy_0'+\frac{x^2}{2!}y_0''+\frac{x^3}{3!}y_0'''+\frac{x^4}{4!}y_0''''+O(x^5)$

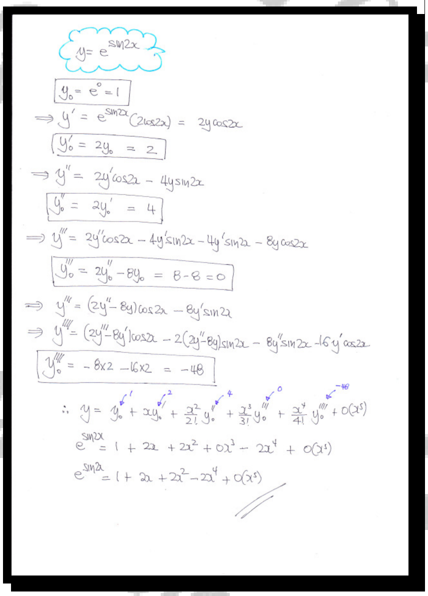
$\tan\left(x+\frac{\pi}{4}\right)=1+2x+\frac{x^2}{2}\times4+\frac{x^3}{6}\times16+\frac{x^4}{24}\times80+\frac{x^5}{120}\times512+O(x^6)$

$\tan\left(x+\frac{\pi}{4}\right)=1+2x+2x^2+\frac{8}{3}x^3+\frac{10}{3}x^4+\frac{64}{15}x^5+O(x^6)$

Question 3 (****+)

Find the Maclaurin expansion, up and including the term in x^4, for $y = \mathrm{e}^{\sin 2x}$.

$$\boxed{\mathrm{e}^{\sin 2x} = 1 + x + 2x^2 - 2x^4 + O\left(x^5\right)}$$



$y = e^{\sin 2x}$

$y_0 = e^0 = 1$

$\Rightarrow y' = e^{\sin 2x}(2\cos 2x) = 2y\cos 2x$

$y'_0 = 2y_0 = 2$

$\Rightarrow y'' = 2y'\cos 2x - 4y\sin 2x$

$y''_0 = 2y'_0 = 4$

$\Rightarrow y''' = 2y''\cos 2x - 4y'\sin 2x - 4y'\sin 2x - 8y\cos 2x$

$y'''_0 = 2y''_0 - 8y_0 = 8 - 8 = 0$

$\Rightarrow y''' = (2y'' - 8y)\cos 2x - 8y'\sin 2x$

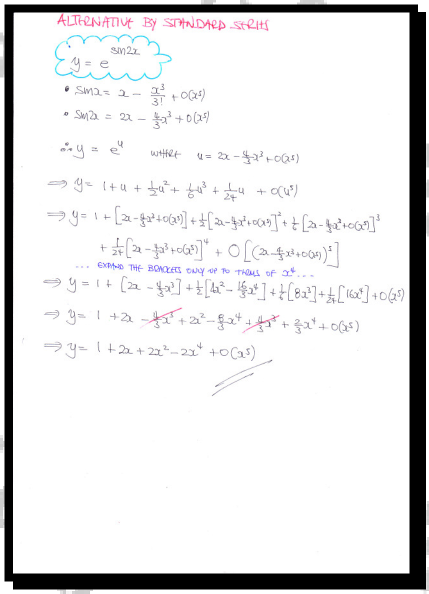
$\Rightarrow y'''' = (2y''' - 8y')\cos 2x - 2(2y'' - 8y)\sin 2x - 8y''\sin 2x - 16y'\cos 2x$

$y''''_0 = -8 \times 2 - 16 \times 2 = -48$

$\therefore y = y_0 + xy'_0 + \frac{x^2}{2!}y''_0 + \frac{x^3}{3!}y'''_0 + \frac{x^4}{4!}y''''_0 + O(x^5)$

$e^{\sin 2x} = 1 + 2x + 2x^2 + 0x^3 - 2x^4 + O(x^5)$

$e^{\sin 2x} = 1 + 2x + 2x^2 - 2x^4 + O(x^5)$



ALTERNATIVE BY STANDARD SERIES

$y = e^{\sin 2x}$

- $\sin x = x - \frac{x^3}{3!} + O(x^5)$
- $\sin 2x = 2x - \frac{4}{3}x^3 + O(x^5)$

$\therefore y = e^u$ where $u = 2x - \frac{4}{3}x^3 + O(x^5)$

$\Rightarrow y = 1 + u + \frac{1}{2}u^2 + \frac{1}{6}u^3 + \frac{1}{24}u + O(u^5)$

$\Rightarrow y = 1 + \left[2x - \frac{4}{3}x^3 + O(x^5)\right] + \frac{1}{2}\left[2x - \frac{4}{3}x^3 + O(x^5)\right]^2 + \frac{1}{6}\left[2x - \frac{4}{3}x^3 + O(x^5)\right]^3$
$+ \frac{1}{24}\left[2x - \frac{4}{3}x^3 + O(x^5)\right]^4 + O\left[\left(2x - \frac{4}{3}x^3 + O(x^5)\right)^5\right]$

... EXPAND THE BRACKETS ONLY UP TO TERMS OF x^4 ...

$\Rightarrow y = 1 + \left[2x - \frac{4}{3}x^3\right] + \frac{1}{2}\left[4x^2 - \frac{16}{3}x^4\right] + \frac{1}{6}\left[8x^3\right] + \frac{1}{24}\left[16x^4\right] + O(x^5)$

$\Rightarrow y = 1 + 2x - \frac{4}{3}x^3 + 2x^2 - \frac{8}{3}x^4 + \frac{4}{3}x^3 + \frac{2}{3}x^4 + O(x^5)$

$\Rightarrow y = 1 + 2x + 2x^2 - 2x^4 + O(x^5)$

Question 4 (****+)

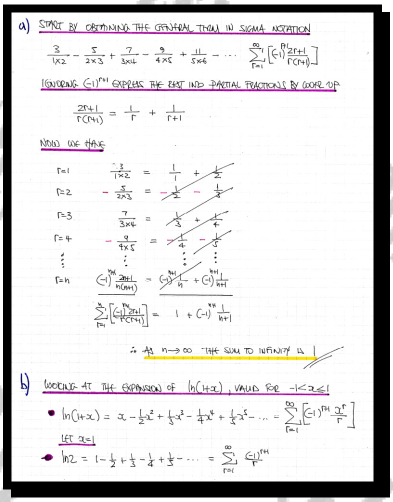
Consider the following infinite convergent series.

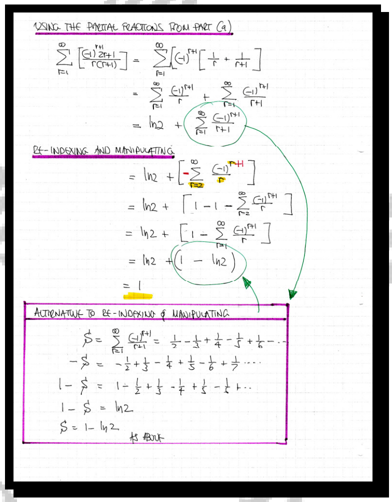
$$\frac{3}{1\times 2}-\frac{5}{2\times 3}+\frac{7}{3\times 4}-\frac{9}{4\times 5}+\frac{11}{5\times 6}-\ldots.$$

a) Use the method of differences, to find the sum of this series.

b) Verify the answer of part **(a)** by using a method based on the Maclaurin expansion of $\ln(1+x)$.

V, [ ], [1]





Question 5 (****+)

$$y = \ln\left(2 - e^{x}\right), \quad x < \ln 2 .$$

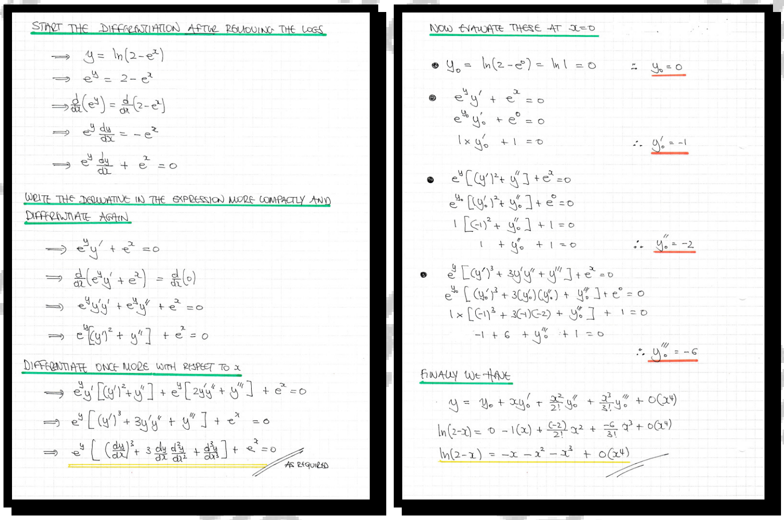
Show clearly that

$$e^{y}\left[\frac{d^{3}y}{dx^{3}} + 3\frac{dy}{dx}\frac{d^{2}y}{dx^{2}} + \left(\frac{dy}{dx}\right)^{3}\right] + e^{x} = 0 ,$$

and hence find the first 3 non zero terms in the Maclaurin expansion of

$$y = \ln\left(2 - e^{x}\right), \quad x < \ln 2 .$$

$\boxed{}$, $\boxed{y = \ln\left(2 - e^{x}\right) = -x - x^{2} - x^{3} + O\left(x^{4}\right)}$



START THE DIFFERENTIATION AFTER REMOVING THE LOGS

$\Rightarrow y = \ln(2-e^x)$
$\Rightarrow e^y = 2 - e^x$
$\Rightarrow \frac{d}{dx}(e^y) = \frac{d}{dx}(2-e^x)$
$\Rightarrow e^y \frac{dy}{dx} = -e^x$
$\Rightarrow e^y \frac{dy}{dx} + e^x = 0$

WRITE THE DERIVATIVE IN THE EXPRESSION MORE COMPACTLY AND DIFFERENTIATE AGAIN

$\Rightarrow e^y y' + e^x = 0$
$\Rightarrow \frac{d}{dx}(e^y y' + e^x) = \frac{d}{dx}(0)$
$\Rightarrow e^y y' y' + e^y y'' + e^x = 0$
$\Rightarrow e^y[(y')^2 + y''] + e^x = 0$

DIFFERENTIATE ONCE MORE WITH RESPECT TO x

$\Rightarrow e^y y'[(y')^2 + y''] + e^y[2y'y'' + y'''] + e^x = 0$
$\Rightarrow e^y[(y')^3 + 3y'y'' + y'''] + e^x = 0$
$\Rightarrow e^y\left[\left(\frac{dy}{dx}\right)^3 + 3\frac{dy}{dx}\frac{d^2y}{dx^2} + \frac{d^3y}{dx^3}\right] + e^x = 0$ AS REQUIRED

NOW EVALUATE THESE AT $x=0$

- $y_0 = \ln(2-e^0) = \ln 1 = 0 \quad \therefore y_0 = 0$
- $e^y y' + e^x = 0$, $e^{y_0} y_0' + e^0 = 0$, $1 \times y_0' + 1 = 0 \quad \therefore y_0' = -1$
- $e^y[(y')^2 + y''] + e^x = 0$, $e^{y_0}[(y_0')^2 + y_0''] + e^0 = 0$, $1[(-1)^2 + y_0''] + 1 = 0$, $1 + y_0'' + 1 = 0 \quad \therefore y_0'' = -2$
- $e^y[(y')^3 + 3y'y'' + y'''] + e^x = 0$, $e^{y_0}[(y_0')^3 + 3(y_0')(y_0'') + y_0'''] + e^0 = 0$, $1 \times [(-1)^3 + 3(-1)(-2) + y_0'''] + 1 = 0$, $-1 + 6 + y_0''' + 1 = 0 \quad \therefore y_0''' = -6$

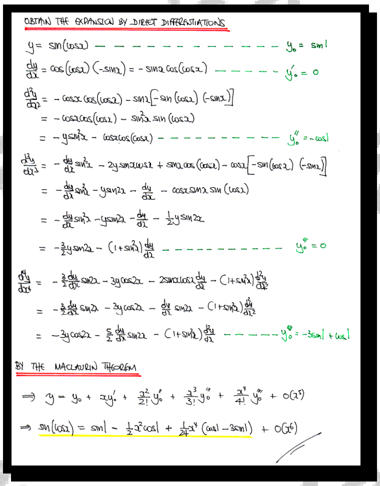
FINALLY WE HAVE

$y = y_0 + xy_0' + \frac{x^2}{2!}y_0'' + \frac{x^3}{3!}y_0''' + O(x^4)$
$\ln(2-x) = 0 - 1(x) + \frac{(-2)}{2!}x^2 + \frac{-6}{3!}x^3 + O(x^4)$
$\ln(2-x) = -x - x^2 - x^3 + O(x^4)$

Question 6 (****+)

Find the Maclaurin expansion, up and including the term in x^4, for $y = \sin(\cos x)$.

$$\sin(\cos x) = \sin 1 - \frac{1}{2}x^2 \cos 1 + x^4\left(\frac{1}{24}\cos 1 - \frac{1}{8}\sin 1\right) + O(x^6)$$

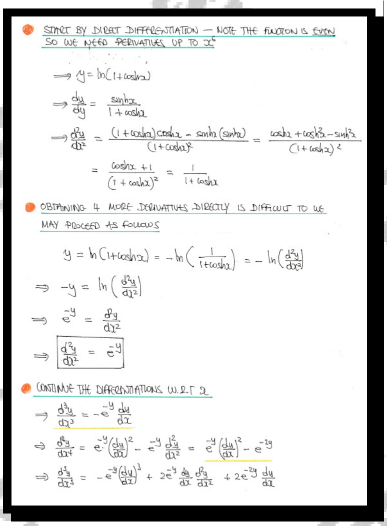


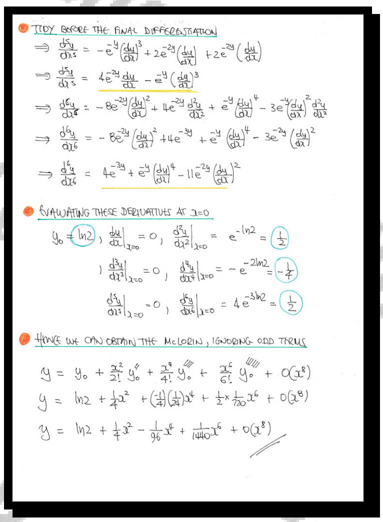
Question 7 (****+)

Find the first four non zero terms in the Maclaurin expansion of

$$y = \ln(1+\cosh x).$$

$$\ln(1+\cosh x) = \ln 2 + \frac{1}{4}x^2 - \frac{1}{96}x^4 + \frac{1}{1440}x^6 + O(x^8)$$





MACLAURIN EXPANSIONS 9 ENRICHMENT QUESTIONS

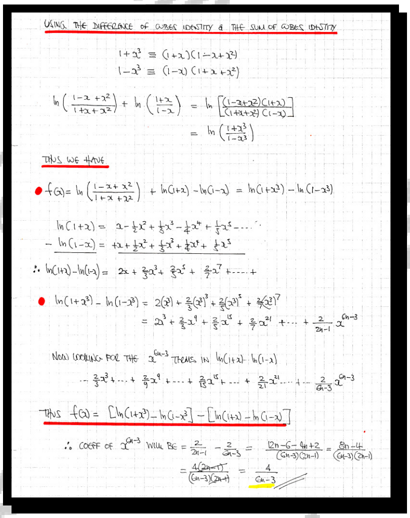
Question 1 (*****)

The curve with equation $y = f(x)$ is the solution of the differential equation

$$f(x) \equiv \ln\left(\frac{1-x+x^2}{1+x+x^2}\right).$$

Determine, in its simplest form, the coefficient of x^{6n-3}, $n \in \mathbb{N}$, in the Maclaurin series expansion of $f(x)$.

$\boxed{}$, $\boxed{\dfrac{4}{6n-3}}$



USING THE DIFFERENCE OF CUBES IDENTITY & THE SUM OF CUBES IDENTITY

$$1+x^3 \equiv (1+x)(1-x+x^2)$$
$$1-x^3 \equiv (1-x)(1+x+x^2)$$

$$\ln\left(\frac{1-x+x^2}{1+x+x^2}\right) + \ln\left(\frac{1+x}{1-x}\right) = \ln\left[\frac{(1-x+x^2)(1+x)}{(1+x+x^2)(1-x)}\right] = \ln\left(\frac{1+x^3}{1-x^3}\right)$$

THUS WE HAVE

- $f(x) = \ln\left(\frac{1-x+x^2}{1+x+x^2}\right) + \ln(1+x) - \ln(1-x) = \ln(1+x^3) - \ln(1-x^3)$

$$\ln(1+x) = x - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 - \tfrac{1}{4}x^4 + \tfrac{1}{5}x^5 - \dots$$
$$-\ln(1-x) = +x + \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 + \tfrac{1}{4}x^4 + \tfrac{1}{5}x^5$$

$$\therefore \ln(1+x) - \ln(1-x) = 2x + \tfrac{2}{3}x^3 + \tfrac{2}{5}x^5 + \tfrac{2}{7}x^7 + \dots +$$

- $\ln(1+x^3) - \ln(1-x^3) = 2(x^3) + \tfrac{2}{3}(x^3)^3 + \tfrac{2}{5}(x^3)^5 + \tfrac{2}{7}(x^3)^7$
$$= 2x^3 + \tfrac{2}{3}x^9 + \tfrac{2}{5}x^{15} + \tfrac{2}{7}x^{21} + \dots + \frac{2}{2n-1}x^{6n-3}$$

NOW LOOKING FOR THE x^{6n-3} TERMS IN $\ln(1+x) - \ln(1-x)$

$$\dots \tfrac{2}{3}x^3 + \dots + \tfrac{2}{9}x^9 + \dots + \tfrac{2}{15}x^{15} + \dots + \tfrac{2}{21}x^{21} \dots + \dots \frac{2}{6n-3}x^{6n-3}$$

THUS $f(x) = [\ln(1+x^3) - \ln(1-x^3)] - [\ln(1+x) - \ln(1-x)]$

$\therefore$ COEFF OF x^{6n-3} WILL BE $= \frac{2}{2n-1} - \frac{2}{6n-3} = \frac{12n-6-4n+2}{(6n-3)(2n-1)} = \frac{8n-4}{(6n-3)(2n-1)}$
$$= \frac{4(2n-1)}{(6n-3)(2n-1)} = \frac{4}{6n-3}$$

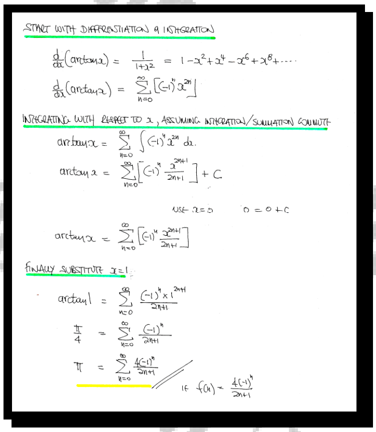
Question 2 (*****)

Find the Maclaurin expansion of $\arctan x$, and use it to show that

$$\pi = \sum_{n=0}^{\infty} f(n),$$

for some suitable function f.

$$\pi = \sum_{n=0}^{\infty} \frac{4(-1)n}{2n+1}$$



START WITH DIFFERENTIATION & INTEGRATION

$$\frac{d}{dx}(\arctan x) = \frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + x^8 + \dots$$

$$\frac{d}{dx}(\arctan x) = \sum_{n=0}^{\infty}\left[(-1)^n x^{2n}\right]$$

INTEGRATING WITH RESPECT TO x, ASSUMING INTEGRATION/SUMMATION COMMUTE

$$\arctan x = \sum_{n=0}^{\infty} \int (-1)^n x^{2n}\, dx$$

$$\arctan x = \sum_{n=0}^{\infty}\left[(-1)^n \frac{x^{2n+1}}{2n+1}\right] + C$$

USE $x=0$ $\quad 0 = 0 + C$

$$\arctan x = \sum_{n=0}^{\infty}\left[(-1)^n \frac{x^{2n+1}}{2n+1}\right]$$

FINALLY SUBSTITUTE $x=1$

$$\arctan 1 = \sum_{n=0}^{\infty} \frac{(-1)^n \times 1^{2n+1}}{2n+1}$$

$$\frac{\pi}{4} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$$

$$\pi = \sum_{n=0}^{\infty} \frac{4(-1)^n}{2n+1}$$

IF $f(n) = \frac{4(-1)^n}{2n+1}$

Question 3 (*****)

a) Use an appropriate integration method to evaluate the following integral.

$$\int_0^1 x^3 \arctan x \ dx .$$

b) Obtain an infinite series expansion for $\arctan x$ and use this series expansion to verify the answer obtained for the above integral in part **(a)**.

[*you may assume that integration and summation commute*]

$\boxed{}$, $\boxed{\frac{1}{6}}$

a) START BY INTEGRATION BY PARTS

$\arctan x$	$\frac{1}{1+x^2}$
$\frac{1}{4}x^4$	x^3

$$\int_0^1 x^3 \arctan x \, dx = \left[\tfrac{1}{4}x^4 \arctan x\right]_0^1 - \tfrac{1}{4}\int_0^1 \frac{x^4}{1+x^2}\,dx$$

$$= \tfrac{1}{4}\times\tfrac{\pi}{4} - 0 - \tfrac{1}{4}\int_0^1 \frac{x^4}{x^2+1}\,dx$$

$$= \frac{\pi}{16} - \frac{1}{4}\int_0^1 \frac{x^2(x^2+1)-(x^2+1)+1}{x^2+1}\,dx$$

$$= \frac{\pi}{16} - \frac{1}{4}\int_0^1 x^2 - 1 + \frac{1}{x^2+1}\,dx$$

$$= \frac{\pi}{16} - \frac{1}{4}\left[\tfrac{1}{3}x^3 - x + \arctan x\right]_0^1$$

$$= \frac{\pi}{16} - \frac{1}{4}\left[\left(\tfrac{1}{3} - 1 + \tfrac{\pi}{4}\right) - 0\right]$$

$$= \frac{\pi}{16} - \frac{1}{4}\left(-\tfrac{2}{3} + \tfrac{\pi}{4}\right)$$

$$= \frac{\pi}{16} + \frac{1}{6} - \frac{\pi}{16}$$

$$= \frac{1}{6}$$

b) NEED THE EXPANSION OF $\arctan x$

$$\frac{d}{dx}(\arctan x) = \frac{1}{1+x^2} = (1+x^2)^{-1} = 1 - x^2 + x^4 - x^6 + x^8 - \dots$$

INTEGRATE BOTH SIDES GIVES

$$\arctan x = x - \tfrac{1}{3}x^3 + \tfrac{1}{5}x^5 - \tfrac{1}{7}x^7 + \tfrac{1}{9}x^9 - \dots + C$$

$$\arctan 0 = 0 \implies C = 0$$

- $\arctan x = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \frac{1}{9}x^9 + \dots$
- $\arctan x = \sum_{n=0}^{\infty}\left[\frac{x^{2n+1}}{2n+1}(-1)^n\right]$

THUS WE NOW HAVE

$$\int_0^1 x^3 \arctan x \, dx = \int_0^1 x^3 \sum_{n=0}^{\infty}\left[\frac{(-1)^n x^{2n+1}}{2n+1}\right] dx$$

$$= \sum_{n=0}^{\infty}\left[\frac{(-1)^n}{2n+1}\int \frac{x^{2n+4}}{2n+1}\,dx\right]$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\left[\frac{x^{2n+5}}{2n+5}\right]_0^1$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)(2n+5)}\left[1^{2n+5} - 0^{2n+5}\right]$$

NEED TO SUM THIS SERIES BY PARTIAL FRACTIONS

$$\frac{1}{(2n+1)(2n+5)} \equiv \frac{\frac{1}{4}}{2n+1} - \frac{\frac{1}{4}}{2n+5} \quad \text{(BY INSPECTION)}$$

THUS WE NOW HAVE

$$\int_0^1 x^3 \arctan x \, dx = \frac{1}{4}\sum_{n=0}^{\infty}\left[\frac{(-1)^n}{2n+1} - \frac{(-1)^n}{2n+5}\right]$$

INVESTIGATE THE PATTERN

- $n=0$: $\frac{1}{1} - \frac{1}{5}$
- $n=1$: $\frac{-1}{3} + \frac{1}{7}$
- $n=2$: $\frac{1}{5} - \frac{1}{9}$
- $n=3$: $-\frac{1}{7} + \frac{1}{11}$
- $\vdots$
- $n=r-1$: $\frac{(-1)^{r-1}}{2r-1} - \frac{(-1)^{r-1}}{2r+3}$
- $n=r$: $\frac{(-1)^r}{2r+1} - \frac{(-1)^r}{2r+5}$

FINALLY WE HAVE THE RESULT

$$\int_0^1 x^3 \arctan x \, dx = \frac{1}{4}\lim_{r\to\infty}\sum_{n=0}^{r}\left[\frac{(-1)^n}{2n+1} - \frac{(-1)^n}{2n+5}\right]$$

$$= \frac{1}{4}\lim_{r\to\infty}\left[1 - \frac{1}{3} - \frac{(-1)^{r-1}}{2r+3} - \frac{(-1)^r}{2r+5}\right]$$

THESE TEND TO ZERO

$$= \tfrac{1}{4}\times\left(1 - \tfrac{1}{3}\right)$$

$$= \tfrac{1}{4}\times\tfrac{2}{3}$$

$$= \tfrac{1}{6} \quad \text{AS BEFORE}$$

Question 4 (*****)

It is given that

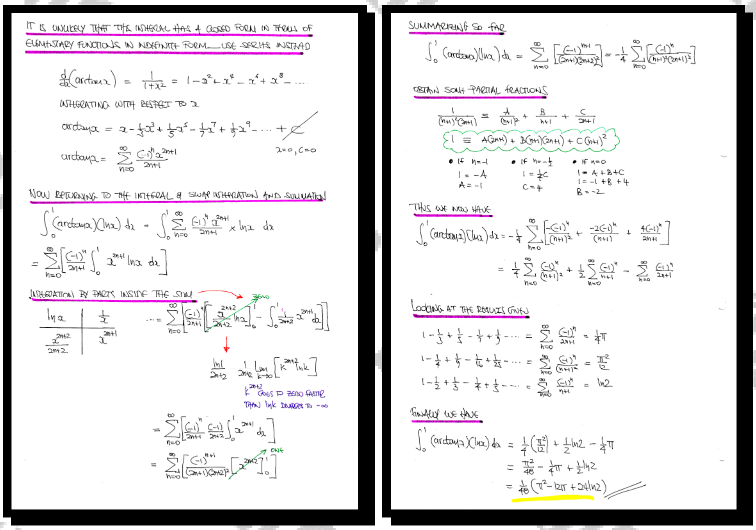
- $1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\frac{1}{11}+\ldots=\frac{1}{4}\pi$
- $1-\frac{1}{4}+\frac{1}{9}-\frac{1}{16}+\frac{1}{25}- \quad \ldots \ =\frac{1}{12}\pi^2$
- $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\ldots \ =\ln 2$

Assuming the following integral converges find its exact value.

$$\int_0^1 (\ln x)(\arctan x)\ dx .$$

[*you may assume that integration and summation commute*]

$\boxed{}$, $\boxed{\frac{1}{48}\left[\pi^2-12\pi+24\ln 2\right]}$

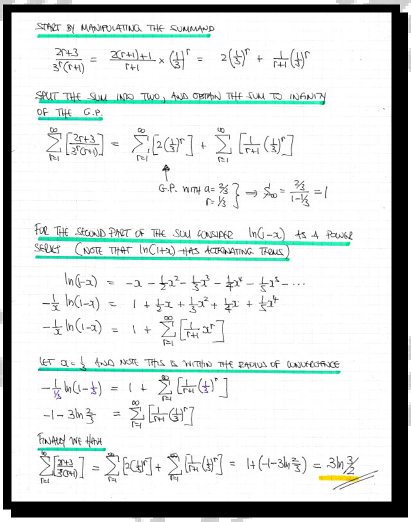


Question 5 **(*****)**

Show with detailed workings that

$$\sum_{r=1}^{\infty}\left[\frac{2r+3}{(r+1)3^r}\right] = 3\ln\left(\frac{3}{2}\right).$$

V , , proof


START BY MANIPULATING THE SUMMAND

$$\frac{2r+3}{3^r(r+1)} = \frac{2(r+1)+1}{r+1}\times\left(\frac{1}{3}\right)^r = 2\left(\frac{1}{3}\right)^r + \frac{1}{r+1}\left(\frac{1}{3}\right)^r$$

SPLIT THE SUM INTO TWO, AND OBTAIN THE SUM TO INFINITY OF THE G.P.

$$\sum_{r=1}^{\infty}\left[\frac{2r+3}{3^r(r+1)}\right] = \sum_{r=1}^{\infty}\left[2\left(\frac{1}{3}\right)^r\right] + \sum_{r=1}^{\infty}\left[\frac{1}{r+1}\left(\frac{1}{3}\right)^r\right]$$

G.P. WITH $a = \frac{2}{3}$, $r = \frac{1}{3}$ $\Rightarrow S_{\infty} = \frac{2/3}{1-1/3} = 1$

FOR THE SECOND PART OF THE SUM CONSIDER $\ln(1-x)$ AS A POWER SERIES (NOTE THAT $\ln(1+x)$ HAS ALTERNATING TERMS)

$$\ln(1-x) = -x - \frac{1}{2}x^2 - \frac{1}{3}x^3 - \frac{1}{4}x^4 - \frac{1}{5}x^5 - \dots$$

$$-\frac{1}{x}\ln(1-x) = 1 + \frac{1}{2}x + \frac{1}{3}x^2 + \frac{1}{4}x + \frac{1}{5}x^4$$

$$-\frac{1}{x}\ln(1-x) = 1 + \sum_{r=1}^{\infty}\left[\frac{1}{r+1}x^r\right]$$

LET $x = \frac{1}{3}$ AND NOTE THIS IS WITHIN THE RADIUS OF CONVERGENCE

$$-\frac{1}{1/3}\ln\left(1-\frac{1}{3}\right) = 1 + \sum_{r=1}^{\infty}\left[\frac{1}{r+1}\left(\frac{1}{3}\right)^r\right]$$

$$-1 - 3\ln\frac{2}{3} = \sum_{r=1}^{\infty}\left[\frac{1}{r+1}\left(\frac{1}{3}\right)^r\right]$$

FINALLY WE HAVE

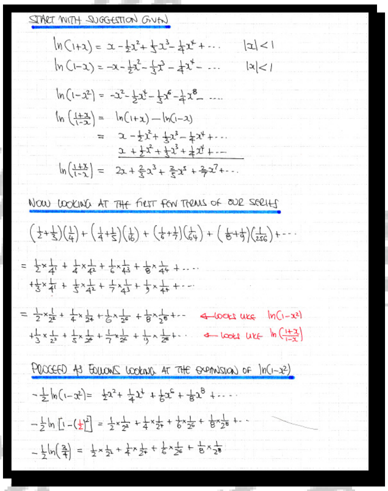
$$\sum_{r=1}^{\infty}\left[\frac{2r+3}{3^r(r+1)}\right] = \sum_{r=1}^{\infty}\left[2\left(\frac{1}{3}\right)^r\right] + \sum_{r=1}^{\infty}\left[\frac{1}{r+1}\left(\frac{1}{3}\right)^r\right] = 1 + \left(-1 - 3\ln\frac{2}{3}\right) = 3\ln\frac{3}{2}$$

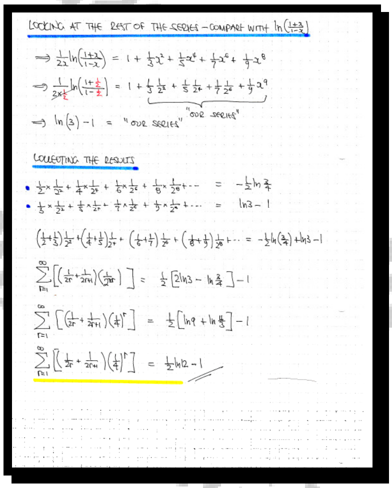
Question 6 (*****)

By considering the series expansions of $\ln\left(1-x^2\right)$ and $\ln\left(\frac{1+x}{1-x}\right)$, or otherwise, find the exact value of the following series.

$$\sum_{r=1}^{\infty}\left[\left(\frac{1}{2r}+\frac{1}{2r+1}\right)\left(\frac{1}{4}\right)^r\right].$$

$\boxed{}$, $\boxed{-1+\frac{1}{2}\ln 12}$





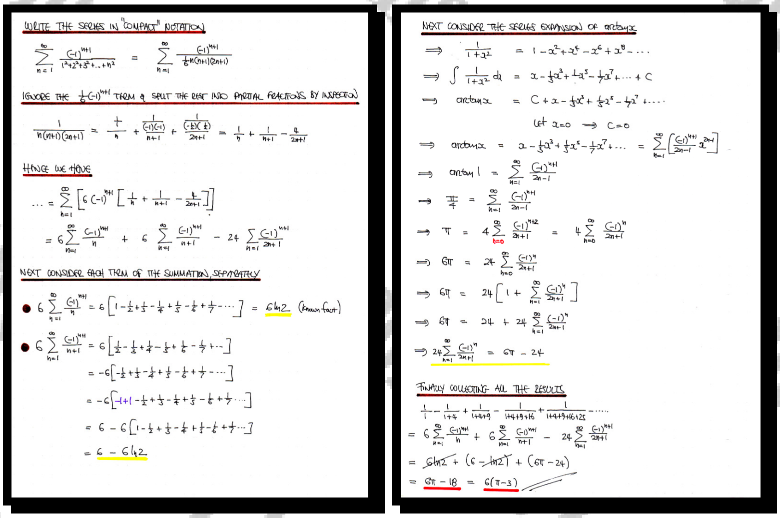
Question 7 (*****)

Find the sum to infinity of the following series.

$$\frac{1}{1}-\frac{1}{1+4}+\frac{1}{1+4+9}-\frac{1}{1+4+9+16}+\frac{1}{1+4+9+16+25}+\ldots$$

You may find the series expansion of $\arctan x$ useful in this question.

$\boxed{}$, $\boxed{6(\pi-3)}$



Question 8 (*****)

Find the sum to infinity of the following series.

$$1+\frac{1}{3\times 4}+\frac{1}{5\times 4^2}+\frac{1}{7\times 4^3}+\frac{1}{9\times 4^4}+\dots$$

$\boxed{\quad}$, $\boxed{\ln 3}$

METHOD A — USING SERIES EXPANSIONS

$$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + O(x^5)$$
$$\ln(1-x) = -x - \frac{1}{2}x^2 - \frac{1}{3}x^3 - \frac{1}{4}x^4 + O(x^5)$$

SUBTRACTING THE EXPANSIONS WE OBTAIN

$$\ln(1+x) - \ln(1-x) = 2x + \frac{2}{3}x^3 + \frac{2}{5}x^5 + O(x^7)$$
$$\ln\left(\frac{1+x}{1-x}\right) = 2\left[x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + O(x^9)\right]$$
$$\ln\left(\frac{1+x}{1-x}\right) = 2\sum_{k=0}^{\infty}\left(\frac{x^{2k+1}}{2k+1}\right)$$

NOW WITHIN THE RADIUS OF CONVERGENCE, LET $x = \frac{1}{2}$

$$\ln\left(\frac{1+\frac{1}{2}}{1-\frac{1}{2}}\right) = 2\sum_{k=0}^{\infty}\frac{\left(\frac{1}{2}\right)^{2k+1}}{2k+1}$$
$$\ln\left(\frac{3/2}{1/2}\right) = 2\sum_{k=0}^{\infty}\left[\frac{1}{(2k+1)\,2^{2k+1}}\right]$$
$$\ln 3 = \sum_{k=0}^{\infty}\frac{2}{(2k+1)2^{2k+1}}$$
$$\sum_{k=0}^{\infty}\frac{1}{(2k+1)2^{2k}} = \ln 3$$
$$\sum_{k=0}^{\infty}\frac{1}{(2k+1)4^k} = \ln 3$$
$$\therefore\ 1+\frac{1}{3\times 4}+\frac{1}{5\times 4^2}+\frac{1}{7\times 4^3}+\frac{1}{9\times 4^4}+\dots = \ln 3$$

METHOD B — ALTERNATIVE TECHNIQUES

CONSIDER

$$\int_0^{\frac{1}{2}} x^{2k}\,dx = \left[\frac{1}{2k+1}x^{2k+1}\right]_0^{\frac{1}{2}} = \frac{1}{2k+1}\left[\left(\frac{1}{2}\right)^{2k+1} - 0\right] = \frac{1}{(2k+1)2^{2k+1}}$$
$$= \frac{1}{(2k+1)2^{2k}\times 2} = \frac{1}{2}\,\frac{1}{(2k+1)\times 4^k}$$

NOW CONSIDER THE INFINITE SUM GIVEN

$$1+\frac{1}{3\times 4}+\frac{1}{5\times 4^2}+\frac{1}{7\times 4^3}+\dots = \sum_{k=0}^{\infty}\left[\frac{1}{(2k+1)\times 4^k}\right]$$
$$= 2\times\frac{1}{2}\sum_{k=0}^{\infty}\left[\frac{1}{(2k+1)4^k}\right] = 2\sum_{k=0}^{\infty}\left[\frac{1}{2}\,\frac{1}{(2k+1)\times 4^k}\right] = 2\sum_{k=0}^{\infty}\left[\int_0^{\frac{1}{2}} x^{2k}\,dx\right]$$

INTERCHANGE SUMMATION & INTEGRATION

$$\dots = 2\int_0^{\frac{1}{2}}\left[\sum_{k=0}^{\infty} x^{2k}\right]dx = 2\int_0^{\frac{1}{2}}\left[1+x^2+x^4+x^6+\dots\right]dx$$
$$= 2\int_0^{\frac{1}{2}}\frac{1}{1-x^2}\,dx = \int_0^{\frac{1}{2}}\frac{2}{(1-x)(1+x)}\,dx$$
$$= \int_0^{\frac{1}{2}}\frac{1}{1+x}+\frac{1}{1-x}\,dx = \Big[\ln|1+x| - \ln|1-x|\Big]_0^{\frac{1}{2}}$$
$$= \left(\ln\frac{3}{2} - \ln\frac{1}{2}\right) - (\ln 1 - \ln 1) = \ln\frac{3/2}{1/2} = \ln 3$$

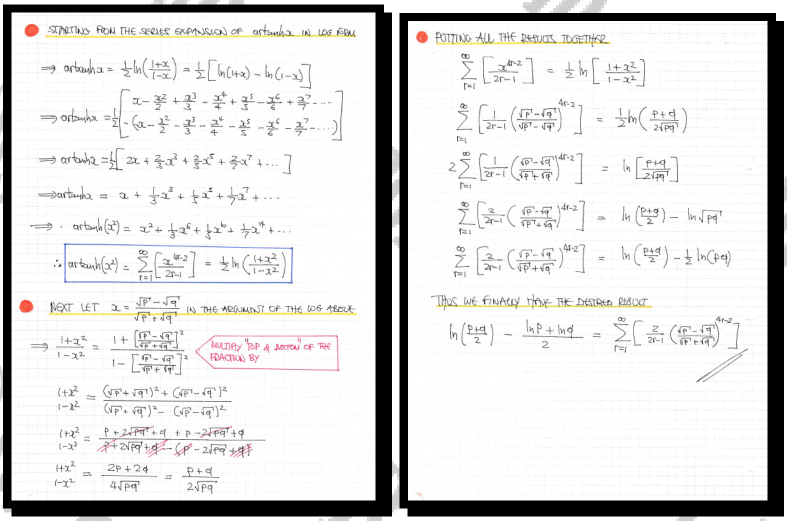
Question 9 (*****)

Given that p and q are positive, show that the natural logarithm of their arithmetic mean exceeds the arithmetic mean of their natural logarithms by

$$\sum_{r=1}^{\infty}\left[\frac{2}{2r-1}\left(\frac{\sqrt{p}-\sqrt{q}}{\sqrt{p}+\sqrt{q}}\right)^{4r-2}\right].$$

You may find the series expansion of $\operatorname{artanh}\left(x^2\right)$ useful in this question.

[ ], [proof]


STARTING FROM THE SERIES EXPANSION OF $\operatorname{artanh}x$ IN LOG FORM

$$\Rightarrow \operatorname{artanh}x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) = \frac{1}{2}\left[\ln(1+x) - \ln(1-x)\right]$$

$$\Rightarrow \operatorname{artanh}x = \frac{1}{2}\left[\begin{array}{l} x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} + \frac{x^7}{7} - \dots \\ -\left(x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \frac{x^5}{5} - \frac{x^6}{6} - \frac{x^7}{7} - \dots\right)\end{array}\right]$$

$$\Rightarrow \operatorname{artanh}x = \frac{1}{2}\left[2x + \frac{2}{3}x^3 + \frac{2}{5}x^5 + \frac{2}{7}x^7 + \dots\right]$$

$$\Rightarrow \operatorname{artanh}x = x + \frac{1}{3}x^3 + \frac{1}{5}x^5 + \frac{1}{7}x^7 + \dots$$

$$\Rightarrow \operatorname{artanh}(x^2) = x^2 + \frac{1}{3}x^6 + \frac{1}{5}x^{10} + \frac{1}{7}x^{14} + \dots$$

$$\therefore \operatorname{artanh}(x^2) = \sum_{r=1}^{\infty}\left[\frac{x^{4r-2}}{2r-1}\right] = \frac{1}{2}\ln\left(\frac{1+x^2}{1-x^2}\right)$$

NEXT LET $x = \frac{\sqrt{p}-\sqrt{q}}{\sqrt{p}+\sqrt{q}}$ IN THE ARGUMENT OF THE LOG ABOVE

$$\Rightarrow \frac{1+x^2}{1-x^2} = \frac{1+\left[\frac{\sqrt{p}-\sqrt{q}}{\sqrt{p}+\sqrt{q}}\right]^2}{1-\left[\frac{\sqrt{p}-\sqrt{q}}{\sqrt{p}+\sqrt{q}}\right]^2}$$

MULTIPLY "TOP & BOTTOM" OF THE FRACTION BY

$$\frac{1+x^2}{1-x^2} = \frac{(\sqrt{p}+\sqrt{q})^2 + (\sqrt{p}-\sqrt{q})^2}{(\sqrt{p}+\sqrt{q})^2 - (\sqrt{p}-\sqrt{q})^2}$$

$$\frac{1+x^2}{1-x^2} = \frac{p + 2\sqrt{pq} + q + p - 2\sqrt{pq} + q}{p + 2\sqrt{pq} + q - (p - 2\sqrt{pq} + q)}$$

$$\frac{1+x^2}{1-x^2} = \frac{2p+2q}{4\sqrt{pq}} = \frac{p+q}{2\sqrt{pq}}$$

PUTTING ALL THE RESULTS TOGETHER

$$\sum_{r=1}^{\infty}\left[\frac{x^{4r-2}}{2r-1}\right] = \frac{1}{2}\ln\left[\frac{1+x^2}{1-x^2}\right]$$

$$\sum_{r=1}^{\infty}\left[\frac{1}{2r-1}\left(\frac{\sqrt{p}-\sqrt{q}}{\sqrt{p}-\sqrt{q}}\right)^{4r-2}\right] = \frac{1}{2}\ln\left(\frac{p+q}{2\sqrt{pq}}\right)$$

$$2\sum_{r=1}^{\infty}\left[\frac{1}{2r-1}\left(\frac{\sqrt{p}-\sqrt{q}}{\sqrt{p}+\sqrt{q}}\right)^{4r-2}\right] = \ln\left[\frac{p+q}{2\sqrt{pq}}\right]$$

$$\sum_{r=1}^{\infty}\left[\frac{2}{2r-1}\left(\frac{\sqrt{p}-\sqrt{q}}{\sqrt{p}+\sqrt{q}}\right)^{4r-2}\right] = \ln\left(\frac{p+q}{2}\right) - \ln\sqrt{pq}$$

$$\sum_{r=1}^{\infty}\left[\frac{2}{2r-1}\left(\frac{\sqrt{p}-\sqrt{q}}{\sqrt{p}+\sqrt{q}}\right)^{4r-2}\right] = \ln\left(\frac{p+q}{2}\right) - \frac{1}{2}\ln(pq)$$

THUS WE FINALLY HAVE THE DESIRED RESULT

$$\ln\left(\frac{p+q}{2}\right) - \frac{\ln p + \ln q}{2} = \sum_{r=1}^{\infty}\left[\frac{2}{2r-1}\left(\frac{\sqrt{p}-\sqrt{q}}{\sqrt{p}+\sqrt{q}}\right)^{4r-2}\right]$$

TAYLOR SERIES EXPANSIONS 4 BASIC QUESTIONS

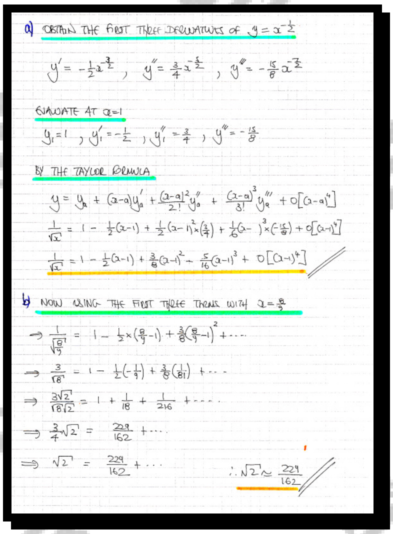
Question 1 (***)

$$y = \frac{1}{\sqrt{x}}, \quad x > 0$$

a) Find the first four terms in the Taylor expansion of y about $x = 1$.

b) Use the first **three** terms of the expansion found in part **(a)**, with $x = \frac{8}{9}$ to show that $\sqrt{2} \approx \frac{229}{162}$.

$\boxed{}$, $\boxed{y = 1 - \frac{1}{2}(x-1) + \frac{3}{8}(x-1)^2 - \frac{5}{16}(x-1)^3 + O\left((x-1)^4\right)}$



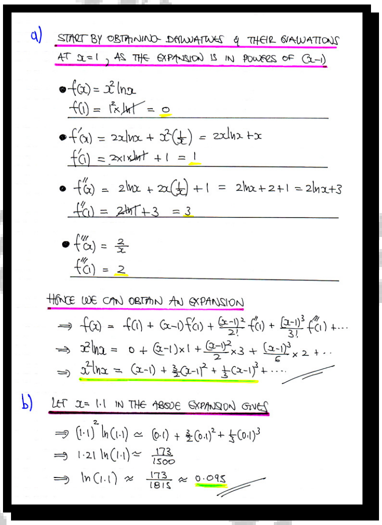
Question 2 (***)

$$f(x) = x^2 \ln x, \ x > 0$$

a) Find the first three non zero terms in the Taylor expansion of $f(x)$, in powers of $(x-1)$.

b) Use the first three terms of the expansion to show $\ln 1.1 \approx 0.095$.

$$f(x) = (x-1) + \tfrac{3}{2}(x-1)^2 + \tfrac{1}{3}(x-1)^3 + O\left((x-1)^4\right)$$



a) START BY OBTAINING DERIVATIVES & THEIR EVALUATIONS AT $x=1$, AS THE EXPANSION IS IN POWERS OF $(x-1)$

- $f(x) = x^2 \ln x$
  $f(1) = 1^2 \times \ln 1 = 0$
- $f'(x) = 2x\ln x + x^2\left(\frac{1}{x}\right) = 2x\ln x + x$
  $f'(1) = 2 \times 1 \times \ln 1 + 1 = 1$
- $f''(x) = 2\ln x + 2x\left(\frac{1}{x}\right) + 1 = 2\ln x + 2 + 1 = 2\ln x + 3$
  $f''(1) = 2\ln 1 + 3 = 3$
- $f'''(x) = \frac{2}{x}$
  $f'''(1) = 2$

HENCE WE CAN OBTAIN AN EXPANSION

$\Rightarrow f(x) = f(1) + (x-1)f'(1) + \frac{(x-1)^2}{2!}f''(1) + \frac{(x-1)^3}{3!}f'''(1) + \ldots$

$\Rightarrow x^2\ln x = 0 + (x-1)\times 1 + \frac{(x-1)^2}{2}\times 3 + \frac{(x-1)^3}{6}\times 2 + \ldots$

$\Rightarrow x^2\ln x = (x-1) + \frac{3}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 + \ldots$

b) LET $x = 1.1$ IN THE ABOVE EXPANSION GIVES

$\Rightarrow (1.1)^2\ln(1.1) \simeq (0.1) + \frac{3}{2}(0.1)^2 + \frac{1}{3}(0.1)^3$

$\Rightarrow 1.21\ln(1.1) \simeq \frac{173}{1500}$

$\Rightarrow \ln(1.1) \approx \frac{173}{1815} \approx 0.095$

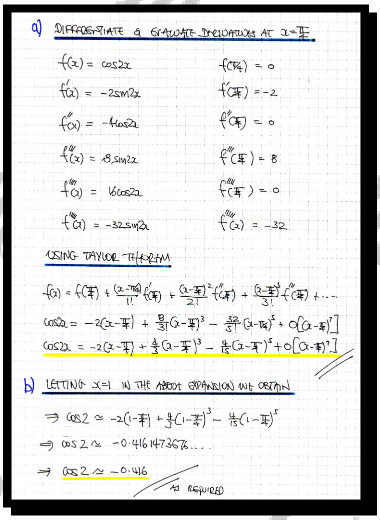
Question 3 (***)

$$f(x) = \cos 2x .$$

a) Find the first three non zero terms in the Taylor expansion of $f(x)$, in powers of $\left(x - \frac{\pi}{4}\right)$.

b) Use the first three terms of the expansion to show $\cos 2 \approx -0.416$.

$$f(x) = -2\left(x-\frac{\pi}{4}\right) + \frac{4}{3}\left(x-\frac{\pi}{4}\right)^3 - \frac{4}{15}\left(x-\frac{\pi}{4}\right)^5 + O\left(\left(x-\frac{\pi}{4}\right)^7\right)$$



a) DIFFERENTIATE & EVALUATE DERIVATIVES AT $x=\frac{\pi}{4}$

$f(x) = \cos 2x$ $\quad f(\frac{\pi}{4}) = 0$

$f'(x) = -2\sin 2x$ $\quad f'(\frac{\pi}{4}) = -2$

$f''(x) = -4\cos 2x$ $\quad f''(\frac{\pi}{4}) = 0$

$f'''(x) = 8\sin 2x$ $\quad f'''(\frac{\pi}{4}) = 8$

$f''''(x) = 16\cos 2x$ $\quad f''''(\frac{\pi}{4}) = 0$

$f'''''(x) = -32\sin 2x$ $\quad f'''''(x) = -32$

USING TAYLOR THEOREM

$f(x) = f(\frac{\pi}{4}) + \frac{(x-\frac{\pi}{4})}{1!}f'(\frac{\pi}{4}) + \frac{(x-\frac{\pi}{4})^2}{2!}f''(\frac{\pi}{4}) + \frac{(x-\frac{\pi}{4})^3}{3!}f'''(\frac{\pi}{4}) + \dots$

$\cos 2x = -2(x-\frac{\pi}{4}) + \frac{8}{3!}(x-\frac{\pi}{4})^3 - \frac{32}{5!}(x-\frac{\pi}{4})^5 + O[(x-\frac{\pi}{4})^7]$

$\cos 2x = -2(x-\frac{\pi}{4}) + \frac{4}{3}(x-\frac{\pi}{4})^3 - \frac{4}{15}(x-\frac{\pi}{4})^5 + O[(x-\frac{\pi}{4})^7]$

b) LETTING $x=1$ IN THE ABOVE EXPANSION WE OBTAIN

$\Rightarrow \cos 2 \simeq -2(1-\frac{\pi}{4}) + \frac{4}{3}(1-\frac{\pi}{4})^3 - \frac{4}{15}(1-\frac{\pi}{4})^5$

$\Rightarrow \cos 2 \simeq -0.416147367\dots$

$\Rightarrow \cos 2 \simeq -0.416$

AS REQUIRED

Question 4 (***)

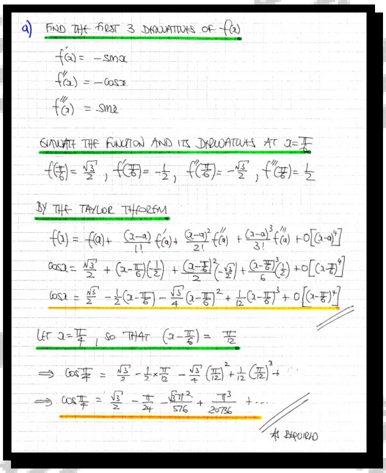
$$f(x) = \cos x\,.$$

a) Find the first four terms in the Taylor expansion of $f(x)$, in ascending powers of $\left(x - \frac{\pi}{6}\right)$.

b) Use the expansion of part **(a)** to show that

$$\cos\frac{\pi}{4} \approx \frac{\sqrt{3}}{2} - \frac{\pi}{24} - \frac{\sqrt{3}\pi^2}{576} - \frac{\pi^3}{20736}\,.$$

$$f(x) = \frac{\sqrt{3}}{2} - \frac{1}{2}\left(x-\frac{\pi}{6}\right) - \frac{\sqrt{3}}{4}\left(x-\frac{\pi}{6}\right)^2 + \frac{1}{12}\left(x-\frac{\pi}{6}\right)^3 + O\left(\left(x-\frac{\pi}{6}\right)^4\right)$$



a) FIND THE FIRST 3 DERIVATIVES OF $f(x)$

$f'(x) = -\sin x$

$f''(x) = -\cos x$

$f'''(x) = \sin x$

EVALUATE THE FUNCTION AND ITS DERIVATIVES AT $x = \frac{\pi}{6}$

$f\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$, $f'\left(\frac{\pi}{6}\right) = -\frac{1}{2}$, $f''\left(\frac{\pi}{6}\right) = -\frac{\sqrt{3}}{2}$, $f'''\left(\frac{\pi}{6}\right) = \frac{1}{2}$

BY THE TAYLOR THEOREM

$f(x) = f(a) + \frac{(x-a)}{1!}f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a) + O\left[(x-a)^4\right]$

$\cos x = \frac{\sqrt{3}}{2} + \left(x-\frac{\pi}{6}\right)\left(-\frac{1}{2}\right) + \frac{\left(x-\frac{\pi}{6}\right)^2}{2}\left(-\frac{\sqrt{3}}{2}\right) + \frac{\left(x-\frac{\pi}{6}\right)^3}{6}\left(\frac{1}{2}\right) + O\left[\left(x-\frac{\pi}{6}\right)^4\right]$

$\cos x = \frac{\sqrt{3}}{2} - \frac{1}{2}\left(x-\frac{\pi}{6}\right) - \frac{\sqrt{3}}{4}\left(x-\frac{\pi}{6}\right)^2 + \frac{1}{12}\left(x-\frac{\pi}{6}\right)^3 + O\left[\left(x-\frac{\pi}{6}\right)^4\right]$

LET $x = \frac{\pi}{4}$, SO THAT $\left(x - \frac{\pi}{6}\right) = \frac{\pi}{12}$

$\Rightarrow \cos\frac{\pi}{4} = \frac{\sqrt{3}}{2} - \frac{1}{2}\times\frac{\pi}{12} - \frac{\sqrt{3}}{4}\left(\frac{\pi}{12}\right)^2 + \frac{1}{12}\left(\frac{\pi}{12}\right)^3 + \dots$

$\Rightarrow \cos\frac{\pi}{4} = \frac{\sqrt{3}}{2} - \frac{\pi}{24} - \frac{\sqrt{3}\pi^2}{576} + \frac{\pi^3}{20736} + \dots$

AS REQUIRED

TAYLOR SERIES EXPANSIONS 3 STANDARD QUESTIONS

Question 1 (***+)

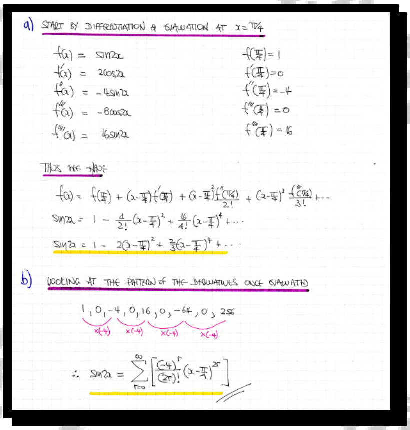
$$f(x) \equiv \sin 2x, \quad x \in \mathbb{R}.$$

a) Determine, in exact simplified form, the first 3 non zero terms, in the Taylor expansion of $f(x)$, centred at $x = \frac{1}{4}\pi$.

b) Write the **entire** expansion of $f(x)$, as a simplified expression in Σ notation.

$$f(x) = 1 - 2\left(x - \tfrac{1}{4}\pi\right)^2 + \tfrac{2}{3}\left(x - \tfrac{1}{4}\pi\right)^4 + \ldots,$$

$$f(x) = \sum_{r=0}^{\infty}\left[\frac{(-4)^r}{(2r)!}\left(x - \frac{1}{4}\pi\right)^{2r}\right]$$


a) START BY DIFFERENTIATION & EVALUATION AT $x = \pi/4$

$f(x) = \sin 2x$ $\quad f(\frac{\pi}{4}) = 1$

$f'(x) = 2\cos 2x$ $\quad f'(\frac{\pi}{4}) = 0$

$f''(x) = -4\sin 2x$ $\quad f''(\frac{\pi}{4}) = -4$

$f'''(x) = -8\cos 2x$ $\quad f'''(\frac{\pi}{4}) = 0$

$f''''(x) = 16\sin 2x$ $\quad f''''(\frac{\pi}{4}) = 16$

THUS WE HAVE

$f(x) = f(\frac{\pi}{4}) + (x-\frac{\pi}{4})f'(\frac{\pi}{4}) + (x-\frac{\pi}{4})^2\frac{f''(\pi/4)}{2!} + (x-\frac{\pi}{4})^3\frac{f'''(\pi/4)}{3!} + \ldots$

$\sin 2x = 1 - \frac{4}{2!}(x-\frac{\pi}{4})^2 + \frac{16}{4!}(x-\frac{\pi}{4})^4 + \ldots$

$\sin 2x = 1 - 2(x-\frac{\pi}{4})^2 + \frac{2}{3}(x-\frac{\pi}{4})^4 + \ldots$

b) LOOKING AT THE PATTERN OF THE DERIVATIVES ONCE EVALUATED

1, 0, −4, 0, 16, 0, −64, 0, 256

×(−4) ×(−4) ×(−4) ×(−4)

$\therefore \sin 2x = \sum_{r=0}^{\infty}\left[\frac{(-4)^r}{(2r)!}(x-\frac{\pi}{4})^{2r}\right]$

Question 2 (****)

$$y = \tan x\,.$$

a) Show that

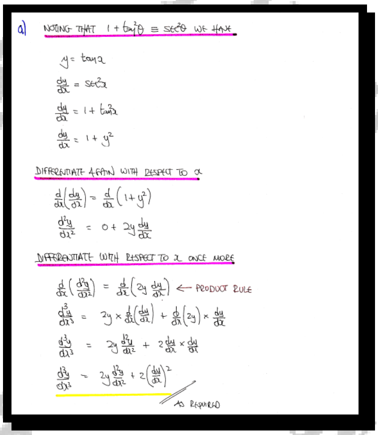
$$\frac{d^3y}{dx^3} = 2y\frac{d^2y}{dx^2} + 2\left(\frac{dy}{dx}\right)^2.$$

b) Determine the first four terms in the Taylor expansion of $\tan x$, in ascending powers of $\left(x - \frac{\pi}{4}\right)$.

c) Hence deduce that

$$\tan\frac{5\pi}{18} \approx 1 + \frac{\pi}{18} + \frac{\pi^2}{648} + \frac{\pi^3}{17496}.$$

$$\boxed{}, \quad \boxed{y = 1 + 2\left(x - \frac{\pi}{4}\right) + 2\left(x - \frac{\pi}{4}\right)^2 + \frac{8}{3}\left(x - \frac{\pi}{4}\right)^3 + O\left(\left(x - \frac{\pi}{4}\right)^4\right)}$$


a) NOTING THAT $1 + \tan^2\theta \equiv \sec^2\theta$ WE HAVE

$y = \tan x$

$\frac{dy}{dx} = \sec^2 x$

$\frac{dy}{dx} = 1 + \tan^2 x$

$\frac{dy}{dx} = 1 + y^2$

DIFFERENTIATE AGAIN WITH RESPECT TO x

$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(1 + y^2\right)$

$\frac{d^2y}{dx^2} = 0 + 2y\frac{dy}{dx}$

DIFFERENTIATE WITH RESPECT TO x ONCE MORE

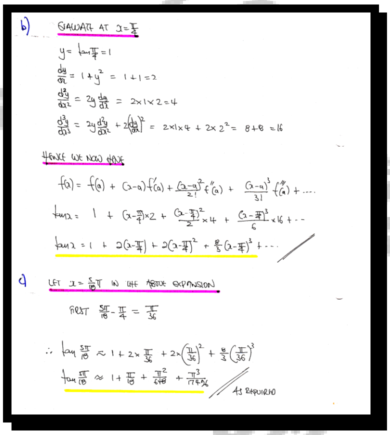
$\frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d}{dx}\left(2y\frac{dy}{dx}\right)$ ← PRODUCT RULE

$\frac{d^3y}{dx^3} = 2y \times \frac{d}{dx}\left(\frac{dy}{dx}\right) + \frac{d}{dx}(2y) \times \frac{dy}{dx}$

$\frac{d^3y}{dx^3} = 2y\frac{d^2y}{dx^2} + 2\frac{dy}{dx} \times \frac{dy}{dx}$

$\frac{d^3y}{dx^3} = 2y\frac{d^2y}{dx^2} + 2\left(\frac{dy}{dx}\right)^2$

AS REQUIRED


b) EVALUATE AT $x = \frac{\pi}{4}$

$y = \tan\frac{\pi}{4} = 1$

$\frac{dy}{dx} = 1 + y^2 = 1 + 1 = 2$

$\frac{d^2y}{dx^2} = 2y\frac{dy}{dx} = 2 \times 1 \times 2 = 4$

$\frac{d^3y}{dx^3} = 2y\frac{d^2y}{dx^2} + 2\left(\frac{dy}{dx}\right)^2 = 2 \times 1 \times 4 + 2 \times 2^2 = 8 + 8 = 16$

HENCE WE NOW HAVE

$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a) + \dots$

$\tan x = 1 + \left(x - \frac{\pi}{4}\right) \times 2 + \frac{\left(x - \frac{\pi}{4}\right)^2}{2} \times 4 + \frac{\left(x - \frac{\pi}{4}\right)^3}{6} \times 16 + \dots$

$\tan x = 1 + 2\left(x - \frac{\pi}{4}\right) + 2\left(x - \frac{\pi}{4}\right)^2 + \frac{8}{3}\left(x - \frac{\pi}{4}\right)^3 + \dots$

c) LET $x = \frac{5}{18}\pi$ IN THE ABOVE EXPANSION

FIRST $\frac{5\pi}{18} - \frac{\pi}{4} = \frac{\pi}{36}$

$\therefore \tan\frac{5\pi}{18} \approx 1 + 2 \times \frac{\pi}{36} + 2 \times \left(\frac{\pi}{36}\right)^2 + \frac{8}{3}\left(\frac{\pi}{36}\right)^3$

$\tan\frac{5\pi}{18} \approx 1 + \frac{\pi}{18} + \frac{\pi^2}{648} + \frac{\pi^3}{17496}$

AS REQUIRED

Question 3 (****)

$$y = \tan^2 x .$$

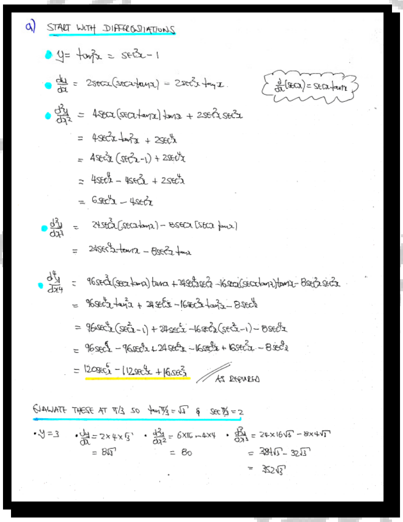
a) Show that

$$\frac{d^4 y}{dx^4} = 120 \sec^6 x - 120 \sec^4 x + 16 \sec^2 x .$$

b) Determine the first 5 terms in the Taylor expansion of $\tan^2 x$, in ascending powers of $\left(x - \frac{\pi}{3}\right)$.

V , ,

$$y = 3 + 8\sqrt{3}\left(x - \frac{\pi}{3}\right) + 40\left(x - \frac{\pi}{3}\right)^2 + \frac{176}{3}\left(x - \frac{\pi}{3}\right)^3 + \frac{728}{3}\left(x - \frac{\pi}{3}\right)^4 + O\left(\left(x - \frac{\pi}{3}\right)^5\right)$$

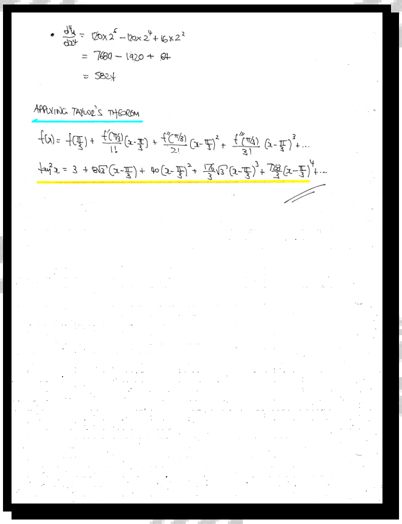


a) START WITH DIFFERENTIATIONS

- $y = \tan^2 x = \sec^2 x - 1$
- $\frac{dy}{dx} = 2\sec x(\sec x \tan x) = 2\sec^2 x \tan x$ [$\frac{d}{dx}(\sec x) = \sec x \tan x$]
- $\frac{d^2y}{dx^2} = 4\sec x(\sec x \tan x)\tan x + 2\sec^2 x \sec^2 x$
  $= 4\sec^2 x \tan^2 x + 2\sec^4 x$
  $= 4\sec^2 x(\sec^2 x - 1) + 2\sec^4 x$
  $= 4\sec^4 x - 4\sec^2 x + 2\sec^4 x$
  $= 6\sec^4 x - 4\sec^2 x$
- $\frac{d^3y}{dx^3} = 24\sec^3 x(\sec x \tan x) - 8\sec x(\sec x \tan x)$
  $= 24\sec^4 x \tan x - 8\sec^2 x \tan x$
- $\frac{d^4y}{dx^4} = 96\sec^3 x(\sec x \tan x)\tan x + 24\sec^4 x \sec^2 x - 16\sec x(\sec x \tan x)\tan x - 8\sec^2 x \sec^2 x$
  $= 96\sec^4 x \tan^2 x + 24\sec^6 x - 16\sec^2 x \tan^2 x - 8\sec^4 x$
  $= 96\sec^4 x(\sec^2 x - 1) + 24\sec^6 x - 16\sec^2 x(\sec^2 x - 1) - 8\sec^4 x$
  $= 96\sec^6 x - 96\sec^4 x + 24\sec^6 x - 16\sec^4 x + 16\sec^2 x - 8\sec^4 x$
  $= 120\sec^6 x - 112\sec^4 x + 16\sec^2 x$ // AS REQUIRED

EVALUATE THESE AT $\pi/3$ SO $\tan\frac{\pi}{3} = \sqrt{3}$ & $\sec\frac{\pi}{3} = 2$

- $y = 3$
- $\frac{dy}{dx} = 2 \times 4 \times \sqrt{3} = 8\sqrt{3}$
- $\frac{d^2y}{dx^2} = 6 \times 16 - 4 \times 4 = 80$
- $\frac{d^3y}{dx^3} = 24 \times 16\sqrt{3} - 8 \times 4\sqrt{3} = 384\sqrt{3} - 32\sqrt{3} = 352\sqrt{3}$



- $\frac{d^4y}{dx^4} = 120 \times 2^6 - 120 \times 2^4 + 16 \times 2^2$
  $= 7680 - 1920 + 64$
  $= 5824$

APPLYING TAYLOR'S THEOREM

$f(x) = f(\frac{\pi}{3}) + \frac{f'(\pi/3)}{1!}(x - \frac{\pi}{3}) + \frac{f''(\pi/3)}{2!}(x - \frac{\pi}{3})^2 + \frac{f'''(\pi/3)}{3!}(x - \frac{\pi}{3})^3 + \ldots$

$\tan^2 x = 3 + 8\sqrt{3}(x - \frac{\pi}{3}) + 40(x - \frac{\pi}{3})^2 + \frac{176}{3}\sqrt{3}(x - \frac{\pi}{3})^3 + \frac{728}{3}(x - \frac{\pi}{3})^4 + \ldots$ //

O.D.E.
TAYLOR SERIES
EXPANSIONS
3 BASIC
QUESTIONS

Question 1 (**+)

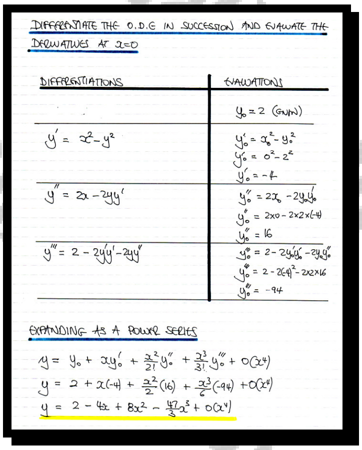
A curve has equation $y = f(x)$ which satisfies the differential equation

$$\frac{dy}{dx} = x^2 - y^2,$$

subject to the condition $x = 0$, $y = 2$.

Determine the first 4 terms in the infinite series expansion of $y = f(x)$ in ascending powers of x.

$\boxed{}$, $\boxed{y = 2 - 4x + 8x^2 - \frac{47}{3}x^3 + O\left(x^4\right)}$



DIFFERENTIATE THE O.D.E IN SUCCESSION AND EVALUATE THE DERIVATIVES AT $x=0$

DIFFERENTIATIONS	EVALUATIONS
	$y_0 = 2$ (GIVEN)
$y' = x^2 - y^2$	$y_0' = x_0^2 - y_0^2$ $y_0' = 0^2 - 2^2$ $y_0' = -4$
$y'' = 2x - 2yy'$	$y_0'' = 2x_0 - 2y_0y_0'$ $y_0'' = 2\times0 - 2\times2\times(-4)$ $y_0'' = 16$
$y''' = 2 - 2y'y' - 2yy''$	$y_0''' = 2 - 2y_0'y_0' - 2y_0y_0''$ $y_0''' = 2 - 2(-4)^2 - 2\times2\times16$ $y_0''' = -94$

EXPANDING AS A POWER SERIES

$y = y_0 + xy_0' + \frac{x^2}{2!}y_0'' + \frac{x^3}{3!}y_0''' + O(x^4)$

$y = 2 + x(-4) + \frac{x^2}{2}(16) + \frac{x^3}{6}(-94) + O(x^4)$

$y = 2 - 4x + 8x^2 - \frac{47}{3}x^3 + O(x^4)$

Question 2 (***)

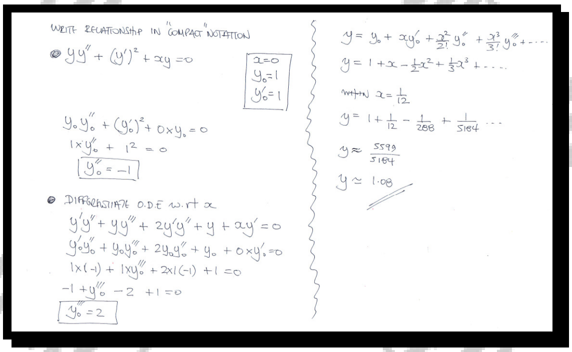
A curve has an equation $y = f(x)$ that satisfies the differential equation

$$y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 + xy = 0,$$

subject to the conditions $x = 0$, $y = 1$, $\frac{dy}{dx} = 1$.

By using the first four terms in the expansion of $y = f(x)$ in ascending powers of x, show that $y \approx 1.08$ at $x = \frac{1}{12}$.

proof



WRITE RELATIONSHIP IN "COMPACT" NOTATION

- $yy'' + (y')^2 + xy = 0$

$x = 0$, $y_0 = 1$, $y'_0 = 1$

$y_0 y''_0 + (y'_0)^2 + 0 \times y_0 = 0$

$1 \times y''_0 + 1^2 = 0$

$y''_0 = -1$

- DIFFERENTIATE O.D.E w.r.t x

$y'y'' + yy''' + 2y'y'' + y + xy' = 0$

$y'_0 y''_0 + y_0 y'''_0 + 2y'_0 y''_0 + y_0 + 0 \times y'_0 = 0$

$1 \times (-1) + 1 \times y'''_0 + 2 \times 1(-1) + 1 = 0$

$-1 + y'''_0 - 2 + 1 = 0$

$y'''_0 = 2$

$y = y_0 + xy'_0 + \frac{x^2}{2!}y''_0 + \frac{x^3}{3!}y'''_0 + \dots$

$y = 1 + x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + \dots$

WITH $x = \frac{1}{12}$

$y = 1 + \frac{1}{12} - \frac{1}{288} + \frac{1}{5184} \dots$

$y \approx \frac{5599}{5184}$

$y \simeq 1.08$

Question 3 (***)

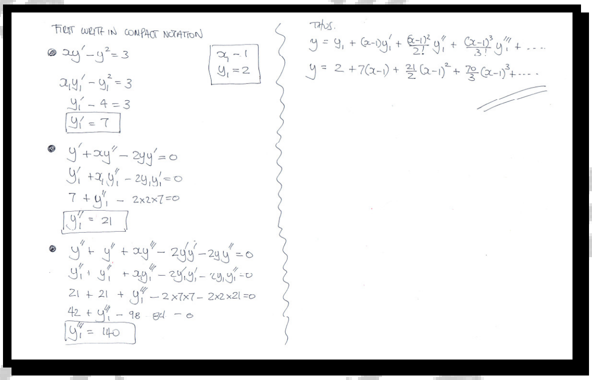
A curve has an equation $y = f(x)$ that satisfies the differential equation

$$x\frac{dy}{dx} - y^2 = 3, \; x \neq 0,$$

subject to the condition $y = 2$ at $x = 1$.

Find the first four terms in the expansion of $y = f(x)$ as powers of $(x-1)$.

$$\boxed{y = 2 - 7(x-1) + \tfrac{21}{2}(x-1)^2 + \tfrac{70}{3}(x-1)^3 \, O\left((x-1)^4\right)}$$


FIRST WRITE IN COMPACT NOTATION

- $xy' - y^2 = 3$ $\quad \boxed{x_1 = 1, \; y_1 = 2}$

$x_1 y_1' - y_1^2 = 3$

$y_1' - 4 = 3$

$\boxed{y_1' = 7}$

- $y' + xy'' - 2yy' = 0$

$y_1' + x_1 y_1'' - 2y_1 y_1' = 0$

$7 + y_1'' - 2 \times 2 \times 7 = 0$

$\boxed{y_1'' = 21}$

- $y'' + y'' + xy''' - 2y'y' - 2yy'' = 0$

$y_1'' + y_1'' + x_1 y_1''' - 2y_1' y_1' - 2y_1 y_1'' = 0$

$21 + 21 + y_1''' - 2 \times 7 \times 7 - 2 \times 2 \times 21 = 0$

$42 + y_1''' - 98 - 84 = 0$

$\boxed{y_1''' = 140}$

THUS

$y = y_1 + (x-1)y_1' + \frac{(x-1)^2}{2!}y_1'' + \frac{(x-1)^3}{3!}y_1''' + \dots$

$y = 2 + 7(x-1) + \frac{21}{2}(x-1)^2 + \frac{70}{3}(x-1)^3 + \dots$

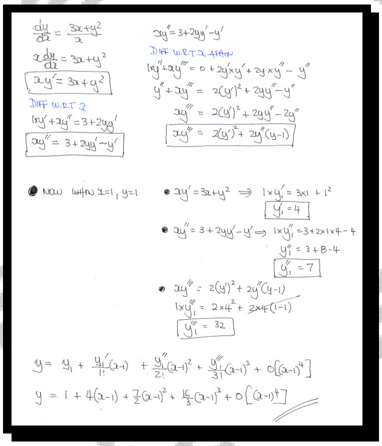
O.D.E.
TAYLOR SERIES
EXPANSIONS
3 STANDARD
QUESTIONS

Question 1 (***+)

$$\frac{dy}{dx}=\frac{3x+y^2}{x}, \quad x\neq 0.$$

Given that $y=1$ at $x=1$, find a series solution for the above differential equation in ascending powers of $(x-1)$, up and including the terms in $(x-1)^3$.

$$\boxed{y=1+4(x-1)+\tfrac{7}{2}(x-1)^2+\tfrac{16}{3}(x-1)^3+O\left[(x-1)^4\right]}$$



$\frac{dy}{dx}=\frac{3x+y^2}{x}$

$x\frac{dy}{dx}=3x+y^2$

$xy'=3x+y^2$

DIFF W.R.T x

$1\times y'+xy''=3+2yy'$

$xy''=3+2yy'-y'$

$xy''=3+2yy'-y'$

DIFF W.R.T x AGAIN

$1\times y''+xy'''=0+2y'\times y'+2y\times y''-y''$

$y''+xy'''=2(y')^2+2yy''-y''$

$xy'''=2(y')^2+2yy''-2y''$

$xy'''=2(y')^2+2y''(y-1)$

• NOW WITH $x=1$, $y=1$

• $xy'=3x+y^2 \Rightarrow 1\times y_1'=3\times1+1^2$

$y_1'=4$

• $xy''=3+2yy'-y' \Rightarrow 1\times y_1''=3+2\times1\times4-4$

$y_1''=3+8-4$

$y_1''=7$

• $xy'''=2(y')^2+2y''(y-1)$

$1\times y_1'''=2\times4^2+2\times4(1-1)$

$y_1'''=32$

$y=y_1+\frac{y_1'}{1!}(x-1)+\frac{y_1''}{2!}(x-1)^2+\frac{y_1'''}{3!}(x-1)^3+O[(x-1)^4]$

$y=1+4(x-1)+\frac{7}{2}(x-1)^2+\frac{16}{3}(x-1)^3+O[(x-1)^4]$

Question 2 (*+)**

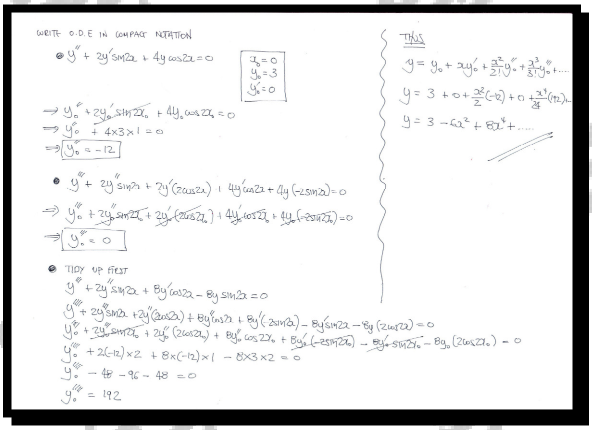
A curve has an equation $y = f(x)$ that satisfies the differential equation

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx}\sin 2x + 4y\cos 2x = 0,$$

subject to the conditions $y = 3$, $\frac{dy}{dx} = 0$ at $x = 0$.

Find a series solution for $f(x)$ up and including the term in x^4 .

$$\boxed{y = 3 - 6x^2 + 8x^4 + O\left(x^6\right)}$$



WRITE O.D.E IN COMPACT NOTATION

- $y'' + 2y'\sin 2x + 4y\cos 2x = 0$

$x_0 = 0$, $y_0 = 3$, $y'_0 = 0$

$\Rightarrow y''_0 + 2y'_0\sin 2x_0 + 4y_0\cos 2x_0 = 0$

$\Rightarrow y''_0 + 4\times 3\times 1 = 0$

$\Rightarrow \boxed{y''_0 = -12}$

- $y''' + 2y''\sin 2x + 2y'(2\cos 2x) + 4y'\cos 2x + 4y(-2\sin 2x) = 0$

$\Rightarrow y'''_0 + 2y''_0\sin 2x_0 + 2y'_0(2\cos 2x_0) + 4y'_0\cos 2x_0 + 4y_0(-2\sin 2x_0) = 0$

$\Rightarrow \boxed{y'''_0 = 0}$

- TIDY UP FIRST

$y''' + 2y''\sin 2x + 8y'\cos 2x - 8y\sin 2x = 0$

$y'''' + 2y'''\sin 2x + 2y''(2\cos 2x) + 8y''\cos 2x + 8y'(-2\sin 2x) - 8y'\sin 2x - 8y(2\cos 2x) = 0$

$y''''_0 + 2y'''_0\sin 2x_0 + 2y''_0(2\cos 2x_0) + 8y''_0\cos 2x_0 + 8y'_0(-2\sin 2x_0) - 8y'_0\sin 2x_0 - 8y_0(2\cos 2x_0) = 0$

$y''''_0 + 2(-12)\times 2 + 8\times(-12)\times 1 - 8\times 3\times 2 = 0$

$y''''_0 - 48 - 96 - 48 = 0$

$y''''_0 = 192$

THUS

$y = y_0 + xy'_0 + \frac{x^2}{2!}y''_0 + \frac{x^3}{3!}y'''_0 + \dots$

$y = 3 + 0 + \frac{x^2}{2}(-12) + 0 + \frac{x^4}{24}(192) + \dots$

$y = 3 - 6x^2 + 8x^4 + \dots$

Question 3 (***+)

A curve has an equation $y = f(x)$ that satisfies the differential equation

$$e^{-x}\frac{d^2y}{dx^2} = 2y\frac{dy}{dx} + y^2 + 1$$

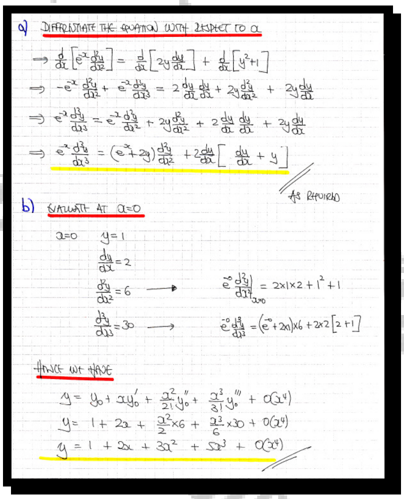
with $y = 1$, $\frac{dy}{dx} = 2$ at $x = 0$.

a) Show clearly that

$$e^{-x}\frac{d^3y}{dx^3} = \left(2y + e^{-x}\right)\frac{d^2y}{dx^2} + 2\frac{dy}{dx}\left(y + \frac{dy}{dx}\right).$$

b) Find a series solution for $f(x)$, up and including the term in x^3.

$\boxed{}$, $\boxed{y = 1 + 2x + 3x^2 + 5x^3 + O(x^4)}$


a) DIFFERENTIATE THE EQUATION WITH RESPECT TO x

$$\Rightarrow \frac{d}{dx}\left[e^{-x}\frac{d^2y}{dx^2}\right] = \frac{d}{dx}\left[2y\frac{dy}{dx}\right] + \frac{d}{dx}\left[y^2+1\right]$$

$$\Rightarrow -e^{-x}\frac{d^2y}{dx^2} + e^{-x}\frac{d^3y}{dx^3} = 2\frac{dy}{dx}\frac{dy}{dx} + 2y\frac{d^2y}{dx^2} + 2y\frac{dy}{dx}$$

$$\Rightarrow e^{-x}\frac{d^3y}{dx^3} = e^{-x}\frac{d^2y}{dx^2} + 2y\frac{d^2y}{dx^2} + 2\frac{dy}{dx}\frac{dy}{dx} + 2y\frac{dy}{dx}$$

$$\Rightarrow e^{-x}\frac{d^3y}{dx^3} = (e^{-x}+2y)\frac{d^2y}{dx^2} + 2\frac{dy}{dx}\left[\frac{dy}{dx} + y\right]$$

AS REQUIRED

b) EVALUATE AT x=0

$x=0$, $y=1$

$\frac{dy}{dx} = 2$

$\frac{d^2y}{dx^2} = 6 \longrightarrow e^{-0}\left.\frac{d^2y}{dx^2}\right|_{x=0} = 2\times1\times2 + 1^2 + 1$

$\frac{d^3y}{dx^3} = 30 \longrightarrow e^{-0}\frac{d^3y}{dx^3} = (e^{-0}+2\times1)\times6 + 2\times2[2+1]$

HENCE WE HAVE

$$y = y_0 + xy_0' + \frac{x^2}{2!}y_0'' + \frac{x^3}{3!}y_0''' + O(x^4)$$

$$y = 1 + 2x + \frac{x^2}{2}\times6 + \frac{x^3}{6}\times30 + O(x^4)$$

$$y = 1 + 2x + 3x^2 + 5x^3 + O(x^4)$$

MIXED SERIES EXPANSIONS 3 QUESTIONS

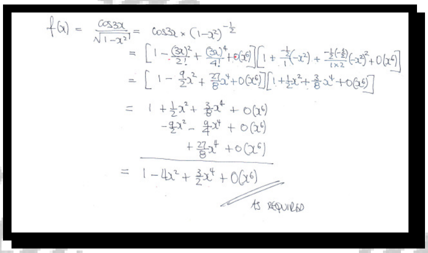
Question 1 (***+)

$$f(x) = \frac{\cos 3x}{\sqrt{1-x^2}}, \quad |x| < 1.$$

Show clearly that

$$f(x) \approx 1 - 4x^2 + \frac{3}{2}x^4.$$

proof



$$f(x) = \frac{\cos 3x}{\sqrt{1-x^2}} = \cos 3x \times (1-x^2)^{-\frac{1}{2}}$$
$$= \left[1 - \frac{(3x)^2}{2!} + \frac{(3x)^4}{4!} + O(x^6)\right]\left[1 + \frac{-\frac{1}{2}}{1}(-x^2) + \frac{-\frac{1}{2}(-\frac{3}{2})}{1\times 2}(-x^2)^2 + O(x^6)\right]$$
$$= \left[1 - \frac{9}{2}x^2 + \frac{27}{8}x^4 + O(x^6)\right]\left[1 + \frac{1}{2}x^2 + \frac{3}{8}x^4 + O(x^6)\right]$$
$$= 1 + \frac{1}{2}x^2 + \frac{3}{8}x^4 + O(x^6)$$
$$- \frac{9}{2}x^2 - \frac{9}{4}x^4 + O(x^6)$$
$$+ \frac{27}{8}x^4 + O(x^6)$$
$$= 1 - 4x^2 + \frac{3}{2}x^4 + O(x^6)$$

AS REQUIRED

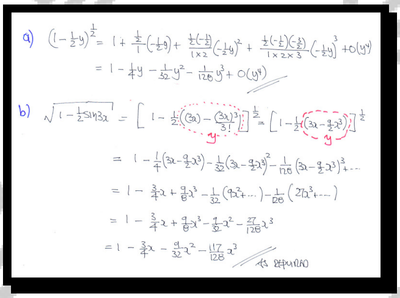
Question 2 (***+)

a) Find the first four terms in the series expansion of $\left(1 - \frac{1}{2}y\right)^{\frac{1}{2}}$.

b) By considering the first two non zero terms in the expansion of $\sin 3x$ and the answer from part (a), show that

$$\sqrt{1 - \frac{1}{2}\sin 3x} \approx 1 - \frac{3}{4}x - \frac{9}{32}x^2 + \frac{117}{128}x^3.$$

$$1 - \frac{1}{4}y - \frac{1}{32}y^2 - \frac{1}{128}y^3 + O(y^4)$$



a) $$\left(1-\frac{1}{2}y\right)^{\frac{1}{2}} = 1 + \frac{\frac{1}{2}}{1}\left(-\frac{1}{2}y\right) + \frac{\frac{1}{2}(-\frac{1}{2})}{1\times 2}\left(-\frac{1}{2}y\right)^2 + \frac{\frac{1}{2}(-\frac{1}{2})(-\frac{3}{2})}{1\times 2\times 3}\left(-\frac{1}{2}y\right)^3 + O(y^4)$$
$$= 1 - \frac{1}{4}y - \frac{1}{32}y^2 - \frac{1}{128}y^3 + O(y^4)$$

b) $$\sqrt{1-\frac{1}{2}\sin 3x} = \left[1 - \frac{1}{2}\left((3x) - \frac{(3x)^3}{3!}\right)\right]^{\frac{1}{2}} = \left[1 - \frac{1}{2}\left(3x - \frac{9}{2}x^3\right)\right]^{\frac{1}{2}}$$
y y
$$= 1 - \frac{1}{4}\left(3x - \frac{9}{2}x^3\right) - \frac{1}{32}\left(3x - \frac{9}{2}x^3\right)^2 - \frac{1}{128}\left(3x - \frac{9}{2}x^3\right)^3 + \dots$$
$$= 1 - \frac{3}{4}x + \frac{9}{8}x^3 - \frac{1}{32}(9x^2 + \dots) - \frac{1}{128}(27x^3 + \dots)$$
$$= 1 - \frac{3}{4}x + \frac{9}{8}x^3 - \frac{9}{32}x^2 - \frac{27}{128}x^3$$
$$= 1 - \frac{3}{4}x - \frac{9}{32}x^2 - \frac{117}{128}x^3$$

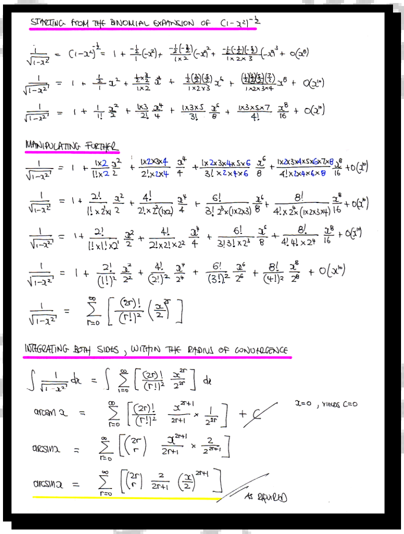
AS REQUIRED

Question 3 (*****)

By considering a suitable binomial expansion, show that

$$\arcsin x = \sum_{r=0}^{\infty}\left[\binom{2r}{r}\frac{2}{2r+1}\left(\frac{x}{2}\right)^{2r+1}\right].$$

proof



STARTING FROM THE BINOMIAL EXPANSION OF $(1-x^2)^{-\frac{1}{2}}$

$$\frac{1}{\sqrt{1-x^2}} = (1-x^2)^{-\frac{1}{2}} = 1 + \frac{-\frac{1}{2}}{1}(-x^2) + \frac{-\frac{1}{2}(-\frac{3}{2})}{1\times2}(-x^2)^2 + \frac{-\frac{1}{2}(-\frac{3}{2})(-\frac{5}{2})}{1\times2\times3}(-x^2)^3 + O(x^8)$$

$$\frac{1}{\sqrt{1-x^2}} = 1 + \frac{\frac{1}{2}}{1}x^2 + \frac{\frac{1}{2}\times\frac{3}{2}}{1\times2}x^4 + \frac{\frac{1}{2}(\frac{3}{2})(\frac{5}{2})}{1\times2\times3}x^6 + \frac{(\frac{1}{2})(\frac{3}{2})(\frac{5}{2})(\frac{7}{2})}{1\times2\times3\times4}x^8 + O(x^{10})$$

$$\frac{1}{\sqrt{1-x^2}} = 1 + \frac{1}{1!}\frac{x^2}{2} + \frac{1\times3}{2!}\frac{x^4}{4} + \frac{1\times3\times5}{3!}\frac{x^6}{8} + \frac{1\times3\times5\times7}{4!}\frac{x^8}{16} + O(x^{10})$$

MANIPULATING FURTHER

$$\frac{1}{\sqrt{1-x^2}} = 1 + \frac{1\times2}{1!\times2}\frac{x^2}{2} + \frac{1\times2\times3\times4}{2!\times2\times4}\frac{x^4}{4} + \frac{1\times2\times3\times4\times5\times6}{3!\times2\times4\times6}\frac{x^6}{8} + \frac{1\times2\times3\times4\times5\times6\times7\times8}{4!\times2\times4\times6\times8}\frac{x^8}{16} + O(x^{10})$$

$$\frac{1}{\sqrt{1-x^2}} = 1 + \frac{2!}{1!\times2^1\times1}\frac{x^2}{2} + \frac{4!}{2!\times2^2(1\times2)}\frac{x^4}{4} + \frac{6!}{3!\,2^3\times(1\times2\times3)}\frac{x^6}{8} + \frac{8!}{4!\times2^4\times(1\times2\times3\times4)}\frac{x^8}{16} + O(x^{10})$$

$$\frac{1}{\sqrt{1-x^2}} = 1 + \frac{2!}{1!\times1!\times2^1}\frac{x^2}{2} + \frac{4!}{2!\times2!\times2^2}\frac{x^4}{4} + \frac{6!}{3!\,3!\times2^3}\frac{x^6}{8} + \frac{8!}{4!\,4!\times2^4}\frac{x^8}{16} + O(x^{10})$$

$$\frac{1}{\sqrt{1-x^2}} = 1 + \frac{2!}{(1!)^2}\frac{x^2}{2^2} + \frac{4!}{(2!)^2}\frac{x^4}{2^4} + \frac{6!}{(3!)^2}\frac{x^6}{2^6} + \frac{8!}{(4!)^2}\frac{x^8}{2^8} + O(x^{10})$$

$$\frac{1}{\sqrt{1-x^2}} = \sum_{r=0}^{\infty}\left[\frac{(2r)!}{(r!)^2}\left(\frac{x}{2}\right)^{2r}\right]$$

INTEGRATING BOTH SIDES, WITHIN THE RADIUS OF CONVERGENCE

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \int \sum_{r=0}^{\infty}\left[\frac{(2r)!}{(r!)^2}\frac{x^{2r}}{2^{2r}}\right]dx$$

$$\arcsin x = \sum_{r=0}^{\infty}\left[\frac{(2r)!}{(r!)^2}\frac{x^{2r+1}}{2r+1}\times\frac{1}{2^{2r}}\right] + C$$

$x=0$, YIELDS $C=0$

$$\arcsin x = \sum_{r=0}^{\infty}\left[\binom{2r}{r}\frac{x^{2r+1}}{2r+1}\times\frac{2}{2^{2r+1}}\right]$$

$$\arcsin x = \sum_{r=0}^{\infty}\left[\binom{2r}{r}\frac{2}{2r+1}\left(\frac{x}{2}\right)^{2r+1}\right]$$

AS REQUIRED