INTEGRATION MIX

1. $\displaystyle\int 6(4x+3)^{\frac{1}{2}}\,dx = (4x+3)^{\frac{3}{2}} + C$

$\displaystyle\int 6(4x+3)^{\frac{1}{2}}\,dx =$ BY INSPECTION (LINEAR ADJUSTMENT) $= \frac{6}{6}(4x+3)^{\frac{3}{2}} + C$

$= \boxed{(4x+3)^{\frac{3}{2}} + C}$

2. $\displaystyle\int \frac{12}{\sqrt{3x+1}}\,dx = 8(3x+1)^{\frac{1}{2}} + C$

$\displaystyle\int \frac{12}{\sqrt{3x+1}}\,dx = \int 12(3x+1)^{-\frac{1}{2}}\,dx =$ BY INSPECTION (LINEAR ADJUSTMENT)

$= \frac{12}{\frac{3}{2}}(3x+1)^{\frac{1}{2}} + C = \boxed{8(3x+1)^{\frac{1}{2}} + C}$

3. $\displaystyle\int \frac{12x}{\sqrt{4x-1}}\,dx = \frac{1}{2}(4x-1)^{\frac{3}{2}} + \frac{3}{2}(4x-1)^{\frac{1}{2}} + C$

$\displaystyle\int \frac{12x}{\sqrt{4x-1}}\,dx = \dots$ BY SUBSTITUTION $\dots$

$u = 4x-1$
$\frac{du}{dx} = 4$
$dx = \frac{du}{4}$
$4x = u+1$

$\displaystyle = \int \frac{12x}{u^{\frac{1}{2}}}\left(\frac{du}{4}\right) = \int \frac{12x}{4u^{\frac{1}{2}}}\,du = \int \frac{3(4x)}{4u^{\frac{1}{2}}}\,du$

$\displaystyle = \int \frac{3(u+1)}{4u^{\frac{1}{2}}}\,du = \frac{3}{4}\int \frac{u+1}{u^{\frac{1}{2}}}\,du$

$\displaystyle = \frac{3}{4}\int \frac{u}{u^{\frac{1}{2}}} + \frac{1}{u^{\frac{1}{2}}}\,du = \frac{3}{4}\int u^{\frac{1}{2}} + u^{-\frac{1}{2}}\,du$

$\displaystyle = \frac{3}{4}\left[\frac{2}{3}u^{\frac{3}{2}} + 2u^{\frac{1}{2}}\right] + C = \frac{1}{2}u^{\frac{3}{2}} + \frac{3}{2}u^{\frac{1}{2}} + C$

$= \boxed{\frac{1}{2}(4x-1)^{\frac{3}{2}} + \frac{3}{2}(4x-1)^{\frac{1}{2}} + C}$

ALTERNATIVE SUBSTITUTION

$u = (4x-1)^{\frac{1}{2}}$
$u^2 = 4x-1$
$2u\frac{du}{dx} = 4$
$2u\,du = 4dx$
$dx = \frac{u\,du}{2}$
$4x = u^2+1$

$\displaystyle\int \frac{12x}{\sqrt{4x-1}}\,dx = \dots = \int \frac{12x}{u}\left(\frac{u\,du}{2}\right)$

$\displaystyle = \int \frac{12x\cancel{u}}{2\cancel{u}}\,du = \int \frac{3(4x)}{2}\,du$

$\displaystyle = \frac{3}{2}\int \underbrace{(u^2+1)}_{4x}\,du = \frac{3}{2}\left[\frac{1}{3}u^3 + u\right] + C$

$= \frac{1}{2}u^3 + \frac{3}{2}u + C$

$= \boxed{\frac{1}{2}(4x-1)^{\frac{3}{2}} + \frac{3}{2}(4x-1)^{\frac{1}{2}} + C}$

4. $\displaystyle\int \frac{12}{4x+1}\,dx = 3\ln|4x+1| + C$

$\displaystyle\int \frac{12}{4x+1}\,dx = \dots$ RECOGNIZING A STANDARD LOG DIFFERENTIATION (LINEAR ADJUSTMENT)

$= \frac{12}{4}\ln|4x+1| + C = \boxed{3\ln|4x+1| + C}$

5. $\displaystyle\int \frac{12x}{4x+1}\,dx = 3x - \frac{3}{4}\ln|4x+1| + C$

$$\int \frac{12x}{4x+1}\,dx = \dots \text{SUBSTITUTION} \dots = \int \frac{12x}{u}\left(\frac{du}{4}\right)$$

$u = 4x+1$

$\frac{du}{dx} = 4$

$dx = \frac{du}{4}$

$4x = u-1$

$$= \int \frac{12x}{4u}\,du = \int \frac{3(4x)}{4u}\,du$$

$$= \int \frac{3(u-1)}{4u}\,du = \frac{3}{4}\int \frac{u-1}{u}\,du$$

$$= \frac{3}{4}\int \frac{u}{u} - \frac{1}{u}\,du = \frac{3}{4}\int 1 - \frac{1}{u}\,du = \frac{3}{4}\Big[u - \ln|u|\Big] + C$$

$$= \frac{3}{4}u - \frac{3}{4}\ln|u| + C = \frac{3}{4}(4x+1) - \frac{3}{4}\ln|4x+1| + C$$

ALTERNATIVE BY MANIPULATION

$$\int \frac{12x}{4x+1}\,dx = \int \frac{3(4x+1)-3}{4x+1}\,dx$$

$$= \int \frac{3\cancel{(4x+1)}}{\cancel{4x+1}} - \frac{3}{4x+1}\,dx$$

$$= \int 3 - \frac{3}{4x+1}\,dx$$

$$= 3x - \frac{3}{4}\ln|4x+1| + C$$

6. $\displaystyle\int (\sin x + \cos x)^2\,dx = x - \frac{1}{2}\cos 2x + C$

$$\int (\sin x + \cos x)^2\,dx = \int \sin^2 x + 2\sin x\cos x + \cos^2 x\,dx$$

$$= \int 1 + \sin 2x\,dx$$

$$= x - \frac{1}{2}\cos 2x + C$$

7. $\displaystyle\int (2\sin x + \operatorname{cosec} x)^2\,dx = 6x - \sin 2x - \cot x + C$

$$\int (2\sin x + \operatorname{cosec} x)^2\,dx = \int 4\sin^2 x + 4\sin x\operatorname{cosec} x + \operatorname{cosec}^2 x\,dx$$

$$= \int 4\left(\tfrac{1}{2} - \tfrac{1}{2}\cos 2x\right) + 4\cancel{\sin x}\left(\frac{1}{\cancel{\sin x}}\right) + \operatorname{cosec}^2 x\,dx$$

$$= \int 2 - 2\cos 2x + 4 + \operatorname{cosec}^2 x$$

$$= \int 6 - 2\cos 2x + \operatorname{cosec}^2 x$$

$$= 6x - \sin 2x - \cot x + C$$

8. $\displaystyle\int \frac{5}{(3x-1)(2x+1)}\,dx = \ln\left|\frac{3x-1}{2x+1}\right| + C$

$$\int \frac{5}{(3x-1)(2x+1)}\,dx = \ldots \text{ BY PARTIAL FRACTIONS } \ldots$$

$$\boxed{\begin{aligned} \frac{5}{(3x-1)(2x+1)} &\equiv \frac{A}{3x-1} + \frac{B}{2x+1} \\ 5 &\equiv A(2x+1) + B(3x-1) \\ \bullet \text{ IF } x = \tfrac{1}{3} &\Rightarrow 5 = \tfrac{5}{3}A \\ &\Rightarrow \underline{A = 3} \\ \bullet \text{ IF } x = -\tfrac{1}{2} &\Rightarrow 5 = -\tfrac{5}{2}B \\ &\Rightarrow \underline{B = -2} \end{aligned}}$$

$$\ldots = \int \frac{3}{3x-1} - \frac{2}{2x+1}\,dx = \ln|3x-1| - \ln|2x+1| + C$$

$$= \underline{\ln\left|\frac{3x-1}{2x+1}\right| + C}$$

9. $\displaystyle\int 4x\sin 2x\,dx = -2x\cos 2x + \sin 2x + C$

$$\int 4x\sin 2x\,dx \ \ldots \text{ INTEGRATION BY PARTS } \ldots$$

$$\boxed{\begin{array}{c|c} 4x & 4 \\ \hline -\frac{1}{2}\cos 2x & \sin 2x \end{array}}$$

$$= -2x\cos 2x - \int -2\cos 2x\,dx$$

$$= -2x\cos 2x + \int 2\cos 2x\,dx$$

$$= \underline{-2x\cos 2x + \sin 2x + C}$$

10. $\displaystyle\int \frac{7}{4x}\,dx = \frac{7}{4}\ln|x| + C$

$$\int \frac{7}{4x}\,dx = \int \frac{7}{4} \times \frac{1}{x}\,dx = \ldots \text{ BY INSPECTION } \ldots = \underline{\frac{7}{4}\ln|x| + C}$$

11. $\displaystyle\int \left(x + \frac{2}{x}\right)^2 dx = \frac{1}{3}x^3 + 4x - \frac{4}{x} + C$

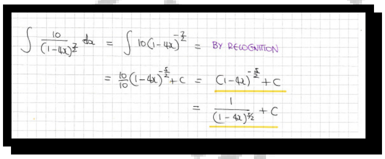
$$\int \left(x + \frac{2}{x}\right)^2 dx = \int x^2 + 2 \times \cancel{x} \times \frac{2}{\cancel{x}} + \frac{4}{x^2}\,dx$$

$$= \int x^2 + 4 + 4x^{-2}\,dx$$

$$= \frac{1}{3}x^3 + 4x - 4x^{-1} + C$$

$$= \underline{\frac{1}{3}x^3 + 4x - \frac{4}{x} + C}$$

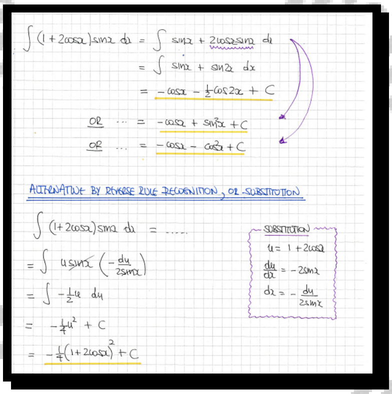
12. $\displaystyle\int \frac{10}{(1-4x)^{\frac{7}{2}}}\,dx = \frac{1}{(1-4x)^{\frac{5}{2}}} + C$


$\int \frac{10}{(1-4x)^{\frac{7}{2}}}\,dx = \int 10(1-4x)^{-\frac{7}{2}} =$ BY RECOGNITION

$= \frac{10}{10}(1-4x)^{-\frac{5}{2}} + C = (1-4x)^{-\frac{5}{2}} + C$

$= \frac{1}{(1-4x)^{5/2}} + C$

13. $\displaystyle\int (1+2\cos x)\sin x\,dx = \begin{bmatrix} -\cos x - \frac{1}{2}\cos 2x + C \\ -\cos x + \sin^2 x + C \\ -\cos x - \cos^2 x + C \\ -\frac{1}{4}(1+2\cos x)^2 + C \end{bmatrix}$


$\int (1+2\cos x)\sin x\,dx = \int \sin x + 2\cos x\sin x\,dx$

$= \int \sin x + \sin 2x\,dx$

$= -\cos x - \frac{1}{2}\cos 2x + C$

OR $\ldots = -\cos x + \sin^2 x + C$

OR $\ldots = -\cos x - \cos^2 x + C$

ALTERNATIVE BY REVERSE RULE RECOGNITION, OR SUBSTITUTION

$\int (1+2\cos x)\sin x\,dx = \ldots$

$= \int u\cancel{\sin x}\left(-\frac{du}{2\cancel{\sin x}}\right)$

$= \int -\frac{1}{2}u\,du$

$= -\frac{1}{4}u^2 + C$

$= -\frac{1}{4}(1+2\cos x)^2 + C$

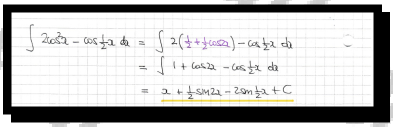
SUBSTITUTION

$u = 1 + 2\cos x$

$\frac{du}{dx} = -2\sin x$

$dx = -\frac{du}{2\sin x}$

14. $\displaystyle\int 2\cos^2 x - \cos\left(\tfrac{1}{2}x\right)dx = x + \tfrac{1}{2}\sin 2x - 2\sin\left(\tfrac{1}{2}x\right) + C$


$\int 2\cos^2 x - \cos\frac{1}{2}x\,dx = \int 2\left(\frac{1}{2}+\frac{1}{2}\cos 2x\right) - \cos\frac{1}{2}x\,dx$

$= \int 1 + \cos 2x - \cos\frac{1}{2}x\,dx$

$= x + \frac{1}{2}\sin 2x - 2\sin\frac{1}{2}x + C$

15. $\displaystyle\int \frac{14x}{x^2-4}\,dx = 7\ln\left|x^2-4\right|+C$

$$\int \frac{14x}{x^2-4}\,dx = 7\int \frac{2x}{x^2-4}\,dx = 7\ln|x^2-4|+C$$

$$\boxed{\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)|+C}$$

ALTERNATIVE BY SUBSTITUTION

$$\int \frac{14x}{x^2-4}\,dx = \dots \text{SUBSTITUTION} \dots = \int \frac{14x}{u}\left(\frac{du}{2x}\right)$$

$$= \int \frac{7}{u}\,du = 7\ln|u|+C$$

$$= 7\ln|x^2-4|+C$$

$$\boxed{u = x^2-4 \qquad \frac{du}{dx} = 2x \qquad dx = \frac{du}{2x}}$$

ALTERNATIVE BY PARTIAL FRACTIONS

FIRSTLY WE HAVE: $\dfrac{14x}{x^2-4} \equiv \dfrac{14x}{(x-2)(x+2)} \equiv \dfrac{A}{x-2}+\dfrac{B}{x+2}$

$$\boxed{14x \equiv A(x+2)+B(x-2)}$$

If $x=2$, $28 = 4A \Rightarrow A=7$

If $x=-2$, $-28 = -4B \Rightarrow B=7$

HENCE THE INTEGRAL BECOMES

$$\int \frac{14x}{x^2-4}\,dx = \int \frac{7}{x-2}+\frac{7}{x+2}\,dx = 7\ln|x-2|+7\ln|x+2|+C$$

$$= 7\left[\ln|x-2|+\ln|x+2|\right]+C$$

$$= 7\ln|x^2-4|+C$$

16. $\displaystyle\int \frac{9}{2x^2+x-1}\,dx = 3\ln\left|\frac{2x-1}{x+1}\right|+C$

$$\int \frac{9}{2x^2+x-1}\,dx = \int \frac{9}{(2x-1)(x+1)}\,dx \dots \text{BY PARTIAL FRACTIONS}$$

$$\frac{9}{(2x-1)(x+1)} \equiv \frac{A}{2x-1}+\frac{B}{x+1}$$

$$\boxed{9 \equiv A(x+1)+B(2x-1)}$$

• If $x=-1$: $9=-3B$, $B=-3$

• If $x=\frac{1}{2}$: $9=\frac{3}{2}A$, $A=6$

$$\dots = \int \frac{6}{2x-1}-\frac{3}{x+1}\,dx = 3\ln|2x-1|-3\ln|x+1|+C$$

$$= 3\ln\left|\frac{2x-1}{x+1}\right|+C$$

17. $\displaystyle\int (8x+1)\mathrm{e}^{-2x}\,dx = -\tfrac{1}{2}(8x+1)\mathrm{e}^{-2x}-2\mathrm{e}^{-2x}+C$

$$\int (8x+1)e^{-2x}\,dx = \dots \text{INTEGRATION BY PARTS}$$

$8x+1$	8
$-\frac{1}{2}e^{-2x}$	e^{-2x}

$$\dots = -\tfrac{1}{2}(8x+1)e^{-2x} - \int -4e^{-2x}\,dx$$

$$= -\tfrac{1}{2}(8x+1)e^{-2x} + \int 4e^{-2x}\,dx$$

$$= -\tfrac{1}{2}(8x+1)e^{-2x} - 2e^{-2x} + C$$

18. $\displaystyle\int \left(e^{2x}+e^{-x}\right)^2\,dx=\frac{1}{4}e^{4x}+2e^{x}-\frac{1}{2}e^{-2x}+C$

$$\int \left(e^{2x}+e^{-x}\right)^2 dx = \int \left(e^{2x}\right)^2+2\left(e^{2x}\right)\left(e^{-x}\right)+\left(e^{-x}\right)^2 dx$$
$$= \int e^{4x}+2e^{x}+e^{-2x}\,dx$$
$$= \frac{1}{4}e^{4x}+2e^{x}-\frac{1}{2}e^{-2x}+C$$

19. $\displaystyle\int \frac{6}{(3x+2)^3}\,dx=-\frac{1}{(3x+2)^2}+C$

$$\int \frac{6}{(3x+2)^3}\,dx = \int 6(3x+2)^{-3}\,dx = \text{BY INSPECTION}$$
$$= \frac{6}{-6}(3x+2)^{-2}+C = -\frac{1}{(3x+2)^2}+C$$

20. $\displaystyle\int \frac{6x}{(3x+2)^3}\,dx=\frac{2}{3}\left[\frac{1}{(3x+2)^2}-\frac{1}{3x+2}\right]+C$

$$\int \frac{6x}{(3x+2)^3}\,dx = \dots$$

BY SUBSTITUTION

$u=3x+2$

$\frac{du}{dx}=3$

$dx=\frac{du}{3}$

$3x=u-2$

$6x=2u-4$

$$= \int \frac{6x}{u^3}\left(\frac{du}{3}\right)$$
$$= \int \frac{2u-4}{3u^3}\,du$$
$$= \int \frac{2u}{3u^3}-\frac{4}{3u^3}\,du$$
$$= \int \frac{2}{3}u^{-2}-\frac{4}{3}u^{-3}\,du$$
$$= -\frac{2}{3}u^{-1}+\frac{2}{3}u^{-2}+C$$
$$= \frac{2}{3}\left[\frac{1}{u^2}-\frac{1}{u}\right]+C$$
$$= \frac{2}{3}\left[\frac{1}{(3x+2)^2}-\frac{1}{3x+2}\right]+C$$

ALTERNATIVE BY PARTIAL FRACTIONS

$$\frac{6x}{(3x+2)^3} \equiv \frac{A}{(3x+2)^3}+\frac{B}{(3x+2)^2}+\frac{C}{3x+2}$$

$$\boxed{6x \equiv A + B(3x+2)+C(3x+2)^2}$$

- If $x=-\frac{2}{3}$: $-4=A$
- If $x=0$: $0=A+2B+4C$, $0=-4+2B+4C$, $4=2B+4C$, $2=B+4C$
- If $x=-1$: $-6=A-B+C$, $-6=-4-B+C$, $-2=-B+C$

ADDING

$5C=0$

$C=0$

&

$B=2$

$$\int \frac{6x}{(3x+2)^3}\,dx = \int \frac{2}{(3x+2)^2}-\frac{4}{(3x+2)^3}\,dx$$
$$= \int 2(3x+2)^{-2}-4(3x+2)^{-3}$$
$$= \frac{2}{-3}(3x+2)^{-1}-\frac{4}{-6}(3x+2)^{-2}+C$$
$$= -\frac{2}{3}\left[\frac{1}{3x+2}-\frac{1}{(3x+2)^2}\right]+C$$
$$= \frac{2}{3}\left[\frac{1}{(3x+2)^2}-\frac{1}{3x+2}\right]+C$$

21. $\displaystyle\int (\sin x - 2\cos x)\sin x \, dx = \left[\begin{array}{c} \frac{1}{2}x - \frac{1}{4}\sin 2x + \frac{1}{2}\cos 2x + C \\ \frac{1}{2}x - \frac{1}{4}\sin 2x + \cos^2 x + C \\ \frac{1}{2}x - \frac{1}{4}\sin 2x - \sin^2 x + C \end{array}\right]$

$$\int (\sin x - 2\cos x)\sin x \, dx = \int \sin^2 x \underbrace{- 2\cos x \sin x}_{} \, dx$$

$$= \int \left(\tfrac{1}{2} - \tfrac{1}{2}\cos 2x\right) - \sin 2x \, dx$$

$$= \tfrac{1}{2}x - \tfrac{1}{4}\sin 2x + \tfrac{1}{2}\cos 2x + C$$

This can also be integrated directly to either

$+\cos^2 x$

OR

$-\sin^2 x$

22. $\displaystyle\int \frac{(2+\cos x)\sin 2x}{2\cos x} \, dx = \left[\begin{array}{c} -\frac{1}{2}(2+\cos x)^2 + C \\ -2\cos x + \frac{1}{2}\sin^2 x + C \\ -2\cos x - \frac{1}{2}\cos^2 x + C \\ -2\cos x - \frac{1}{4}\cos 2x + C \end{array}\right]$

$$\int \frac{(2+\cos x)\sin 2x}{2\cos x} \, dx = \int \frac{(2+\cos x)(\cancel{2}\sin x \cancel{\cos x})}{\cancel{2}\cancel{\cos x}} \, dx$$

$$= \int (2+\cos x)\sin x \, dx = -\tfrac{1}{2}(2+\cos x)^2 + C$$

(BY REVERSE CHAIN RULE)

OR CONTINUING WITH TRIGONOMETRIC IDENTITIES

$$= \int 2\sin x + \cos x \sin x \, dx = \begin{cases} -2\cos x + \frac{1}{2}\sin^2 x + C \\ -2\cos x - \frac{1}{2}\cos^2 x + C \end{cases}$$

(BOTH BY RECOGNITION)

OR ANOTHER VARIATION

$$= \int 2\sin x + \tfrac{1}{2}(2\sin x\cos x) \, dx$$

$$= \int 2\sin x + \tfrac{1}{2}\sin 2x \, dx$$

$$= -2\cos x - \tfrac{1}{4}\cos 2x + C$$

23. $\displaystyle\int \frac{8x-3}{4x} \, dx = 2x - \frac{3}{4}\ln|x| + C$

$$\int \frac{8x-3}{4x} \, dx = \int \frac{8x}{4x} - \frac{3}{4x} \, dx = \int 2 - \tfrac{3}{4}\left(\tfrac{1}{x}\right) \, dx$$

$$= 2x - \tfrac{3}{4}\ln|x| + C$$

24. $\displaystyle\int \frac{6x^2}{x^3+8}\,dx = 2\ln\left|x^3+8\right| + C$

$$\int \frac{6x^2}{x^3+8}\,dx = 2\int \frac{3x^2}{x^3+8}\,dx \leftarrow \boxed{\int \frac{f'(x)}{f(x)}\,dx = \ln|x|+C}$$

$$= 2\ln|x^3+8| + C$$

ALTERNATIVE BY SUBSTITUTION

$$\int \frac{6x^2}{x^3+8}\,dx = \dots \text{SUBSTITUTION} \dots = \int \frac{\cancel{6x^2}^{\,2}}{u}\left(\frac{du}{\cancel{3x^2}}\right)$$

$$= \int \frac{2}{u}\,du = 2\ln|u| + C$$

$$= 2\ln|x^3+8| + C$$

$u = x^3+8$

$\frac{du}{dx} = 3x^2$

$dx = \frac{du}{3x^2}$

25. $\displaystyle\int (1+2\cos x)^3 \sin x\,dx = -\tfrac{1}{8}(1+2\cos x)^4 + C$

$$\int (1+2\cos x)^3 \sin x\,dx = \frac{1}{-8}(1+2\cos x)^4 + C$$

(BY REVERSE CHAIN RULE)

ALTERNATIVE BY SUBSTITUTION

$$\int (1+2\cos x)^3 \sin x\,dx \text{ (BY SUBSTITUTION)} \dots$$

$$= \int u^3 \cancel{\sin x}\left(\frac{du}{-2\cancel{\sin x}}\right) = \int -\tfrac{1}{2}u^3\,du$$

$$= -\tfrac{1}{8}u^4 + C = -\tfrac{1}{8}(1+2\cos x)^4 + C$$

$u = 1+2\cos x$

$\frac{du}{dx} = -2\sin x$

$dx = \frac{du}{-2\sin x}$

[THE SUBSTITUTION $u = 2\cos x$ OR $u = \cos x$ ALSO "WORK"]

26. $\displaystyle\int \cos\sqrt{x}\,dx = 2\sqrt{x}\sin\sqrt{x} + 2\cos\sqrt{x} + C$

$$\int \cos\sqrt{x}\,dx \dots \text{BY SUBSTITUTION} \dots$$

$$= \int \cos u\,(2u\,du) = \int 2u\cos u\,du$$

$u = \sqrt{x}$

$u^2 = x$

$2u\frac{du}{dx} = 1$

$2u\,du = dx$

INTEGRATION BY PARTS NEXT

$$= 2u\sin u - \int 2\sin u\,du$$

$$= 2u\sin u + 2\cos u + C$$

$$= 2\sqrt{x}\sin\sqrt{x} + 2\cos\sqrt{x} + C$$

$2u$	2
$\sin u$	$\cos u$

27. $\displaystyle\int \frac{3}{2x-1}\,dx = \tfrac{3}{2}\ln|2x-1| + C$

$$\int \frac{3}{2x-1}\,dx = \dots \text{BY INSPECTION} \dots = \frac{3}{2}\ln|2x-1| + C$$

28. $\displaystyle\int \frac{10}{(2x+1)^6}\,dx = -\frac{1}{(2x+1)^5} + C$

$$\int \frac{10}{(2x+1)^6}\,dx = \int 10(2x+1)^{-6}\,dx \ldots \text{BY INSPECTION} \ldots = -\frac{10}{10}(2x+1)^{-5} + C$$
$$= -(2x+1)^{-5} + C = -\frac{1}{(2x+1)^5} + C$$

29. $\displaystyle\int \frac{3x-1}{(2x+1)(x-2)}\,dx = \frac{1}{2}\ln|2x+1| + \ln|x-2| + C$

$$\int \frac{3x-1}{(2x+1)(x-2)}\,dx = \ldots \text{BY PARTIAL FRACTIONS}$$
$$\frac{3x-1}{(2x+1)(x-2)} \equiv \frac{A}{2x+1} + \frac{B}{x-2}$$
$$\boxed{3x-1 \equiv A(x-2) + B(2x+1)}$$
• If $x=2$, $5 = 5B$, $B=1$ • If $x=-\frac{1}{2}$, $-\frac{5}{2} = -\frac{5}{2}A$, $A=1$
$$= \int \frac{1}{2x+1} + \frac{1}{x-2}\,dx = \frac{1}{2}\ln|2x+1| + \ln|x-2| + C$$

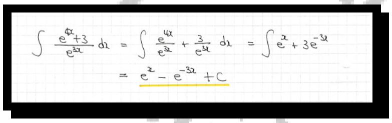
30. $\displaystyle\int (2+\sin x)^2\,dx = \frac{9}{2}x - 4\cos x - \frac{1}{4}\sin 2x + C$

$$\int (2+\sin x)^2\,dx = \int 4 + 4\sin x + \sin^2 x\,dx$$
$$= \int 4 + 4\sin x + \left(\frac{1}{2} - \frac{1}{2}\cos 2x\right)dx$$
$$= \int \frac{9}{2} + 4\sin x - \frac{1}{2}\cos 2x\,dx$$
$$= \frac{9}{2}x - 4\cos x - \frac{1}{4}\sin 2x + C$$

31. $\displaystyle\int 5(2x-3)^{\frac{1}{4}}\,dx = 2(2x-3)^{\frac{5}{4}} + C$

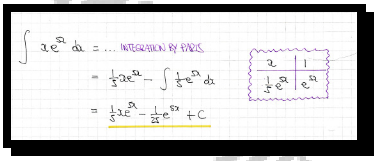
$$\int 5(2x-3)^{\frac{1}{4}}\,dx = \ldots \text{BY INSPECTION} \ldots = \frac{5}{\frac{5}{2}}(2x-3)^{\frac{5}{4}} + C$$
$$= 2(2x-3)^{\frac{5}{4}} + C$$

32. $\displaystyle\int \frac{e^{4x}+3}{e^{3x}}\,dx = e^{x} - e^{-3x} + C$



$$\int \frac{e^{4x}+3}{e^{3x}}\,dx = \int \frac{e^{4x}}{e^{3x}} + \frac{3}{e^{3x}}\,dx = \int e^{x} + 3e^{-3x}$$
$$= e^{x} - e^{-3x} + C$$

33. $\displaystyle\int x e^{5x}\,dx = \frac{1}{5}x e^{5x} - \frac{1}{25}e^{5x} + C$

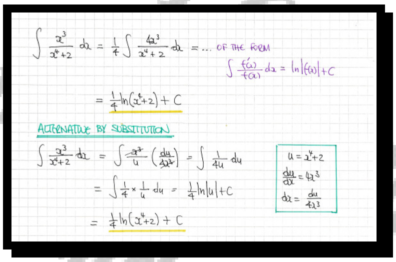


$\displaystyle\int x e^{5x}\,dx = \dots$ INTEGRATION BY PARTS

$$= \frac{1}{5}xe^{5x} - \int \frac{1}{5}e^{5x}\,dx$$
$$= \frac{1}{5}xe^{5x} - \frac{1}{25}e^{5x} + C$$

x	1
$\frac{1}{5}e^{5x}$	e^{5x}

34. $\displaystyle\int \frac{x^{3}}{x^{4}+2}\,dx = \frac{1}{4}\ln\left(x^{4}+2\right) + C$



$\displaystyle\int \frac{x^3}{x^4+2}\,dx = \frac{1}{4}\int \frac{4x^3}{x^4+2}\,dx = \dots$ OF THE FORM $\displaystyle\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + C$

$$= \frac{1}{4}\ln(x^4+2) + C$$

ALTERNATIVE BY SUBSTITUTION

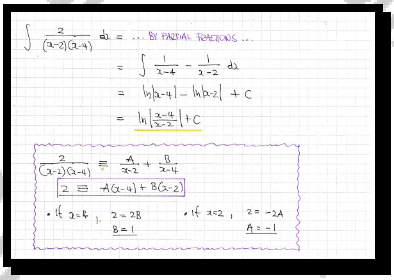
$$\int \frac{x^3}{x^4+2}\,dx = \int \frac{x^3}{u}\left(\frac{du}{4x^3}\right) = \int \frac{1}{4u}\,du$$
$$= \int \frac{1}{4}\times\frac{1}{u}\,du = \frac{1}{4}\ln|u| + C$$
$$= \frac{1}{4}\ln(x^4+2) + C$$

$u = x^4+2$

$\frac{du}{dx} = 4x^3$

$dx = \frac{du}{4x^3}$

35. $\displaystyle\int \frac{2}{(x-2)(x-4)}\,dx = \ln\left|\frac{x-4}{x-2}\right| + C$



$\displaystyle\int \frac{2}{(x-2)(x-4)}\,dx = \dots$ BY PARTIAL FRACTIONS ...

$$= \int \frac{1}{x-4} - \frac{1}{x-2}\,dx$$
$$= \ln|x-4| - \ln|x-2| + C$$
$$= \ln\left|\frac{x-4}{x-2}\right| + C$$

$$\frac{2}{(x-2)(x-4)} \equiv \frac{A}{x-2} + \frac{B}{x-4}$$
$$\boxed{2 \equiv A(x-4) + B(x-2)}$$

- If $x=4$, $2 = 2B$, $B = 1$
- If $x=2$, $2 = -2A$, $A = -1$

36. $\displaystyle\int \frac{3}{4x+1}\,dx = \frac{3}{4}\ln|4x+1| + C$

$\int \frac{3}{4x+1}\,dx = \ldots$ BY INSPECTION $\ldots = \frac{3}{4}\ln|4x+1| + C$

37. $\displaystyle\int \left(1+\frac{1}{x}\right)^2 dx = x + 2\ln|x| - \frac{1}{x} + C$

$$\int \left(1+\frac{1}{x}\right)^2 dx = \int 1^2 + 2\times1\times\frac{1}{x} + \left(\frac{1}{x}\right)^2 dx = \int 1 + \frac{2}{x} + \frac{1}{x^2}\,dx$$
$$= \int 1 + \frac{2}{x} + x^{-2}\,dx = x + 2\ln|x| - x^{-1} + C$$
$$= x + 2\ln|x| - \frac{1}{x} + C$$

38. $\displaystyle\int \frac{x}{\left(x^2-1\right)^3}\,dx = -\frac{1}{4\left(x^2-1\right)^2} + C$

$\int \frac{x}{(x^2-1)^3}\,dx = \ldots$ BY REVERSE CHAIN RULE $\ldots = \int x(x^2-1)^{-3}\,dx$

$\frac{d}{dx}(x^2-1) = 2x$

$$= -\frac{1}{4}(x^2-1)^{-2} + C \quad = -\frac{1}{4(x^2-1)^2} + C$$

ALTERNATIVE BY SUBSTITUTION

$$\int \frac{x}{(x^2-1)^3}\,dx = \int \frac{\cancel{x}}{u^3}\left(\frac{du}{2\cancel{x}}\right) = \int \frac{1}{2}u^{-3}\,du$$
$$= \frac{\frac{1}{2}}{-2}u^{-2} + C \quad = -\frac{1}{4}u^{-2}$$
$$= -\frac{1}{4(x^2-1)^2} + C$$

$u = x^2 - 1$

$\frac{du}{dx} = 2x$

$dx = \frac{du}{2x}$

39. $\displaystyle\int \cos x - \sin x\,dx = \sin x + \cos x + C$

$\int \cos x - \sin x\,dx = \sin x + \cos x + C$

40. $\int \sin x - \cos x \, dx = -\cos x - \sin x + C$

$$\int \sin x - \cos x \, dx = \underline{-\cos x - \sin x + C}$$

41. $\int \sin(4x+3) \, dx = -\frac{1}{4}\cos(4x+3) + C$

$$\int \sin(4x+3) \, dx = \ldots \text{BY INSPECTION} \ldots = \underline{-\tfrac{1}{4}\cos(4x+3) + C}$$

42. $\int \frac{x}{\sqrt{x+1}} \, dx = \frac{2}{3}(x+1)^{\frac{3}{2}} - 2(x+1)^{\frac{1}{2}} + C$

$$\int \frac{x}{\sqrt{x+1}} \, dx = \ldots \text{BY SUBSTITUTION} \ldots$$

$$= \int \frac{x}{u^{\frac{1}{2}}}(du) = \int \frac{u-1}{u^{\frac{1}{2}}} \, du = \int \frac{u}{u^{\frac{1}{2}}} - \frac{1}{u^{\frac{1}{2}}} \, du$$

$$= \int u^{\frac{1}{2}} - u^{-\frac{1}{2}} \, du = \frac{1}{\frac{3}{2}}u^{\frac{3}{2}} - \frac{1}{\frac{1}{2}}u^{\frac{1}{2}} + C$$

$$= \underline{\tfrac{2}{3}(x+1)^{\frac{3}{2}} - 2(x+1)^{\frac{1}{2}} + C}$$

$$\boxed{\begin{gathered} u = x+1 \\ \frac{du}{dx} = 1 \\ dx = du \\ x = u-1 \end{gathered}}$$

ALTERNATIVE SUBSTITUTION

$$\int \frac{x}{\sqrt{x+1}} \, dx = \int \frac{x}{\cancel{u}}(2\cancel{u}\,du) = \int 2x \, du$$

$$= \int 2(u^2-1) \, du = \int 2u^2 - 2 \, du$$

$$= \tfrac{2}{3}u^3 - 2u + C = \underline{\tfrac{2}{3}(x+1)^{\frac{3}{2}} - 2(x+1)^{\frac{1}{2}} + C}$$

$$\boxed{\begin{gathered} u = \sqrt{x+1} \\ u^2 = x+1 \\ 2u\frac{du}{dx} = 1 \\ dx = 2u \, du \\ x = u^2-1 \end{gathered}}$$

43. $\int \cos(5-2x) \, dx = -\frac{1}{2}\sin(5-2x) + C$

$$\int \cos(5-2x) \, dx = \underline{-\tfrac{1}{2}\sin(5-2x) + C}$$

44. $\displaystyle\int 3\sin 2x \, dx = -\frac{3}{2}\cos 2x + C$

$$\int 3\sin 2x \, dx = \underline{-\tfrac{3}{2}\cos 2x + C}$$

45. $\displaystyle\int \left(1+\sec^2 x\right)\sin x \, dx = -\cos x + \sec x + C$

$$\begin{aligned}\int \sin x\left(1+\sec^2 x\right) dx &= \int \sin x + \sin x\sec^2 x \, dx \\ &= \int \sin x + \frac{\sin x}{\cos x}\sec x \, dx \\ &= \int \sin x + \tan x\sec x \, dx \qquad \boxed{\tfrac{d}{dx}(\sec x) = \sec x\tan x} \\ &= \underline{-\cos x + \sec x + C}\end{aligned}$$

46. $\displaystyle\int \left(1-2\cos x\right)^2 \, dx = 3x - 4\sin x + \sin 2x + C$

$$\begin{aligned}\int (1-2\cos x)^2 dx &= \int 1 - 4\cos x + 4\cos^2 x \, dx \\ &= \int 1 - 4\cos x + 4\left(\tfrac{1}{2}+\tfrac{1}{2}\cos 2x\right) dx \\ &= \int 1 - 4\cos x + 2 + 2\cos 2x \, dx \\ &= \int 3 - 4\cos x + 2\cos 2x \, dx \\ &= \underline{3x - 4\sin x + \sin 2x + C}\end{aligned}$$

47. $\displaystyle\int \left(1+\cot^2 x\right)\sec^2 x \, dx = \tan x - \cot x + C$

$$\begin{aligned}\int \sec^2 x\left(1+\cot^2 x\right) dx &= \text{ALTHOUGH } 1+\cot^2 x \equiv \operatorname{cosec}^2 x\text{, MULTIPLY OUT} \\ &= \int \sec^2 x + \sec^2 x\cot^2 x \, dx \\ &= \int \sec^2 x + \frac{1}{\cancel{\cos^2 x}}\times\frac{\cancel{\cos^2 x}}{\sin^2 x} \, dx \\ &= \int \sec^2 x + \operatorname{cosec}^2 x \, dx \\ &= \underline{\tan x - \cot x + C}\end{aligned}$$

$$\boxed{\begin{aligned}&\tfrac{d}{dx}(\tan x) = \sec^2 x \\ &\tfrac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x\end{aligned}}$$

48. $\displaystyle\int 2x\cos 3x \, dx = \frac{2}{3}x\sin 3x + \frac{2}{9}\cos 3x + C$

$\displaystyle\int 2x\cos 3x \, dx = \ldots$ INTEGRATION BY PARTS

x	2
$\frac{1}{3}\sin 3x$	$\cos 3x$

$\displaystyle = (2x)\left(\tfrac{1}{3}\sin 3x\right) - \int \tfrac{2}{3}\sin 3x \, dx$

$\displaystyle = \tfrac{2}{3}x\sin 3x - \left[-\tfrac{2}{9}\cos 3x\right] + C$

$\displaystyle = \tfrac{2}{3}x\sin 3x + \tfrac{2}{9}\cos 3x + C$

49. $\displaystyle\int \frac{3}{(2+x)(1-x)} \, dx = \ln\left|\frac{x+2}{x-1}\right| + C$

$\displaystyle\int \frac{3}{(2+x)(1-x)} \, dx = \ldots$ BY PARTIAL FRACTIONS …

$\displaystyle \frac{3}{(2+x)(1-x)} \equiv \frac{A}{2+x} + \frac{B}{1-x}$

$3 \equiv A(1-x) + B(2+x)$

- If $x=1 \Rightarrow 3 = 3B \Rightarrow B = 1$
- If $x=-2$, $3 = 3A$, $A = 1$

$\displaystyle \ldots = \int \frac{1}{2+x} + \frac{1}{1-x} \, dx = \ln|2+x| - \ln|1-x| + C$

$\displaystyle = \ln\left|\frac{2+x}{1-x}\right| + C$

$\displaystyle = \ln\left|\frac{x+2}{x-1}\right| + C$

50. $\displaystyle\int 10(3x+1)^4 \, dx = \frac{2}{3}(3x+1)^5 + C$

$\displaystyle\int 10(3x+1)^4 \, dx = \ldots$ BY INSPECTION … $\displaystyle = \frac{10}{15}(3x+1)^5 + C$

$\displaystyle = \frac{2}{3}(3x+1)^5 + C$

51. $\displaystyle\int 6(2x+1)^{\frac{1}{2}} \, dx = 2(2x+1)^{\frac{3}{2}} + C$

$\displaystyle\int 6(2x+1)^{\frac{1}{2}} \, dx \ldots$ BY INSPECTION .. $\displaystyle = \frac{6}{3}(2x+1)^{\frac{3}{2}} + C$

$\displaystyle = 2(2x+1)^{\frac{3}{2}} + C$

52. $\displaystyle\int \frac{1}{\cos^2 x \tan^2 x}\, dx = -\cot x + C$

$$\int \frac{1}{\cos^2 x \tan^2 x}\, dx = \int \frac{1}{\cancel{\cos^2 x} \times \frac{\sin^2 x}{\cancel{\cos^2 x}}}\, dx = \int \operatorname{cosec}^2 x\, dx$$

$$= \underline{-\cot x + C}$$

ALTERNATIVE

$$\boxed{\int \sec^2 x (\tan x)^{-2}\, dx = -(\tan x)^{-1} + C}$$

$$\boxed{\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x}$$

53. $\displaystyle\int \cos x \sin x\, dx = \begin{bmatrix} -\frac{1}{4}\cos 2x + C \\ -\frac{1}{2}\cos^2 x + C \\ \frac{1}{2}\sin^2 x + C \end{bmatrix}$

$$\int \cos x \sin x\, dx = \int \tfrac{1}{2}(2\sin x \cos x)\, dx = \int \tfrac{1}{2}\sin 2x\, dx$$

$$= \underline{-\tfrac{1}{4}\cos 2x + C}$$

$$\int \cos x \sin x\, dx = \underline{\tfrac{1}{2}\sin^2 x + C} \qquad \text{SINCE } \frac{d}{dx}(\sin^2 x) = 2\sin x \cos x$$

$$\int \cos x \sin x\, dx = \underline{-\tfrac{1}{2}\cos^2 x + C} \qquad \text{SINCE } \frac{d}{dx}(\cos^2 x) = 2\cos x(-\sin x)$$

54. $\displaystyle\int \frac{2}{\cos^2 x}\, dx = 2\tan x + C$

$$\int \frac{2}{\cos^2 x}\, dx = \int 2\sec^2 x\, dx \ \ldots \text{ BY INSPECTION } \ldots = \underline{2\tan x + C}$$

55. $\displaystyle\int 2 + 2\tan^2 x\, dx = 2\tan x + C$

$$\int 2 + 2\tan^2 x\, dx = \int 2(1 + \tan^2 x)\, dx = \int 2\sec^2 x\, dx$$

$$= \ldots \text{ BY INSPECTION } \ldots = \underline{2\tan x + C}$$

56. $\displaystyle\int \frac{1+\cos x}{\sin^2 x}\,dx = -\cot x - \operatorname{cosec} x + C$

$$\int \frac{1+\cos x}{\sin^2 x}\,dx = \int \frac{1}{\sin^2 x} + \frac{\cos x}{\sin^2 x}\,dx = \int \operatorname{cosec}^2 x + \frac{\cos x}{\sin x}\times\frac{1}{\sin x}\,dx$$

$$= \int \operatorname{cosec}^2 x + \cot x \operatorname{cosec} x\,dx = -\cot x - \operatorname{cosec} x + C$$

$$\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x \qquad \frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$$

57. $\displaystyle\int \frac{(1+\cos x)^2}{\sin^2 x}\,dx = -x - 2\cot x - 2\operatorname{cosec} x + C$

$$\int \frac{(1+\cos x)^2}{\sin^2 x}\,dx = \int \frac{1+2\cos x+\cos^2 x}{\sin^2 x}\,dx$$

$$= \int \frac{1}{\sin^2 x} + \frac{2\cos x}{\sin x}\times\frac{1}{\sin x} + \frac{\cos^2 x}{\sin^2 x}\,dx = \int \operatorname{cosec}^2 x + 2\cot x \operatorname{cosec} x + \cot^2 x\,dx$$

$$= \int \operatorname{cosec}^2 x + 2\cot x \operatorname{cosec} x + (\operatorname{cosec}^2 x - 1)\,dx = \int 2\operatorname{cosec}^2 x + 2\cot x \operatorname{cosec} x - 1\,dx$$

$$\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x \ \ \& \ \ \frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$$

$$= -2\cot x - 2\operatorname{cosec} x - x + C$$

58. $\displaystyle\int x\sin 3x\,dx = -\frac{1}{3}x\cos 3x + \frac{1}{9}\sin 3x + C$

$\displaystyle\int x\sin 3x\,dx = \ldots$ INTEGRATION BY PARTS ...

x	1
$-\frac{1}{3}\cos 3x$	$\sin 3x$

$$= x\left(-\frac{1}{3}\cos 3x\right) - \int -\frac{1}{3}\cos 3x\,dx$$

$$= -\frac{1}{3}x\cos 3x + \int \frac{1}{3}\cos 3x\,dx$$

$$= -\frac{1}{3}x\cos 3x + \frac{1}{9}\sin 3x + C$$

59. $\displaystyle\int \frac{2x}{(2x+1)^3}\,dx = \frac{1}{4(2x+1)^2} - \frac{1}{2(2x+1)} + C$

$\displaystyle\int \frac{2x}{(2x+1)^3}\,dx = \ldots$ BY SUBSTITUTION ...

$u = 2x+1$
$\frac{du}{dx} = 2$
$dx = \frac{du}{2}$
$2x = u-1$

$$= \int \frac{2x}{u^3}\left(\frac{du}{2}\right) = \int \frac{2x}{2u^3}\,du$$

$$= \int \frac{u-1}{2u^3}\,du = \int \frac{u}{2u^3} - \frac{1}{2u^3}\,du$$

$$= \int \frac{1}{2}u^{-2} - \frac{1}{2}u^{-3}\,du = -\frac{1}{2}u^{-1} + \frac{1}{4}u^{-2} = \frac{1}{4}(2x+1)^{-2} - \frac{1}{2}(2x+1)^{-1} + C$$

$$= \frac{1}{4(2x+1)^2} - \frac{1}{2(2x+1)} + C$$

60. $\displaystyle\int (4-5x)^{-1}\,dx = -\frac{1}{5}\ln|4-5x| + C$

$$\int (4-5x)^{-1}\,dx = \int \frac{1}{4-5x}\,dx = -\frac{1}{5}\ln|4-5x| + C$$

61. $\displaystyle\int \frac{1}{4x}\,dx = \frac{1}{4}\ln|x| + C$

$$\int \frac{1}{4x}\,dx = \int \frac{1}{4}\times\frac{1}{x}\,dx = \frac{1}{4}\ln|x| + C$$

62. $\displaystyle\int \frac{1}{(x+1)(x+2)}\,dx = \ln\left|\frac{x+1}{x+2}\right| + C$

$$\int \frac{1}{(x+1)(x+2)}\,dx = \ldots \text{ BY PARTIAL FRACTIONS } \ldots$$

$$\frac{1}{(x+1)(x+2)} \equiv \frac{A}{x+1} + \frac{B}{x+2}$$

$$\boxed{1 \equiv A(x+2) + B(x+1)}$$

- if $x=-1 \Rightarrow 1 = A \Rightarrow A=1$
- if $x=-2 \Rightarrow 1 = -B \Rightarrow B=-1$

$$\ldots = \int \frac{1}{x+1} - \frac{1}{x+2}\,dx = \ln|x+1| - \ln|x+2| + C = \ln\left|\frac{x+1}{x+2}\right| + C$$

63. $\displaystyle\int \frac{x+1}{x}\,dx = x + \ln|x| + C$

$$\int \frac{x+1}{x}\,dx = \int \frac{x}{x} + \frac{1}{x}\,dx = \int 1 + \frac{1}{x}\,dx = x + \ln|x| + C$$

64. $\displaystyle\int \frac{x}{x+1}\,dx = x - \ln|x+1| + C$

$$\int \frac{x}{x+1}\,dx = \ldots \text{ BY MANIPULATION } \ldots = \int \frac{(x+1)-1}{x+1}\,dx$$

$$= \int \frac{x+1}{x+1} - \frac{1}{x+1}\,dx = \int 1 - \frac{1}{x+1}\,dx = \underline{x - \ln|x+1| + C}$$

ALTERNATIVE BY SUBSTITUTION

$$\int \frac{x}{x+1}\,dx = \int \frac{x}{u}(du) = \int \frac{u-1}{u}\,du$$

$$= \int \frac{u}{u} - \frac{1}{u}\,du = \int 1 - \frac{1}{u}\,du$$

$$= u - \ln|u| + C$$

$$= \underline{(x+1) - \ln|x+1| + C}$$

$u = x+1$

$\frac{du}{dx} = 1$

$dx = du$

$x = u-1$

65. $\displaystyle\int \frac{4x}{\sqrt{1-2x^2}}\,dx = -2\sqrt{1-2x^2} + C$

$$\int \frac{4x}{\sqrt{1-2x^2}}\,dx = \int 4x(1-2x^2)^{-\frac{1}{2}}\,dx \ldots \text{ BY REVERSE CHAIN RULE} \ldots$$

$$= \frac{4x}{-2x}(1-2x^2)^{\frac{1}{2}} + C = \underline{-(1-2x^2)^{\frac{1}{2}} + C}$$

ALTERNATIVE BY SUBSTITUTION

$$\int \frac{4x}{\sqrt{1-2x^2}}\,dx \ldots = \int \frac{\cancel{4x}}{u^{\frac{1}{2}}}\left(\frac{du}{\cancel{-4x}}\right)$$

$$= \int -u^{-\frac{1}{2}}\,du = -\frac{1}{\frac{1}{2}}u^{\frac{1}{2}} + C$$

$$= \underline{-2(1-2x^2)^{\frac{1}{2}} + C}$$

$u = 1-2x^2$

$\frac{du}{dx} = -4x$

$dx = \frac{du}{-4x}$

THE SUBSTITUTION $u = \sqrt{1-2x^2}$ ALSO WORKS

66. $\displaystyle\int \frac{x+1}{9x^2-1}\,dx = \tfrac{2}{9}\ln|3x-1| - \tfrac{1}{9}\ln|3x+1| + C$

$$\int \frac{x+1}{9x^2-1}\,dx = \int \frac{x+1}{(3x-1)(3x+1)}\,dx = \text{ BY PARTIAL FRACTIONS} \ldots$$

$$\frac{x+1}{(3x-1)(3x+1)} \equiv \frac{A}{3x-1} + \frac{B}{3x+1}$$

$$\boxed{x+1 \equiv A(3x+1) + B(3x-1)}$$

- IF $x = \frac{1}{3}$, $\frac{4}{3} = 2A$, $\underline{A = \frac{2}{3}}$
- IF $x = -\frac{1}{3}$, $\frac{2}{3} = -2B$, $\underline{B = -\frac{1}{3}}$

$$\ldots = \int \frac{\frac{2}{3}}{3x-1} - \frac{\frac{1}{3}}{3x+1}\,dx = \frac{\frac{2}{3}}{3}\ln|3x-1| - \frac{\frac{1}{3}}{3}\ln|3x+1| + C$$

$$= \underline{\frac{2}{9}\ln|3x-1| - \frac{1}{9}\ln|3x+1| + C}$$

67. $\displaystyle\int x\sin 4x \, dx = \frac{1}{16}\sin 4x - \frac{1}{4}x\cos 4x + C$

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$\displaystyle\int x\sin 4x \, dx = \dots$ INTEGRATION BY PARTS ...

x	1
$-\frac{1}{4}\cos 4x$	$\sin 4x$

$= x\left(-\frac{1}{4}\cos 4x\right) - \displaystyle\int -\frac{1}{4}\cos 4x \, dx$

$= -\frac{1}{4}x\cos 4x + \displaystyle\int \frac{1}{4}\cos 4x \, dx$

$= -\frac{1}{4}x\cos 4x + \frac{1}{16}\sin 4x + C$

68. $\displaystyle\int \ln x \, dx = x\ln x - x + C$

$\displaystyle\int \ln x \, dx = \int 1 \times \ln x \, dx = \dots$ INTEGRATION BY PARTS

$\ln x$	$\frac{1}{x}$
x	1

$= x\ln x - \displaystyle\int x\left(\frac{1}{x}\right) dx$

$= x\ln x - \displaystyle\int 1 \, dx$

$= x\ln x - x + C$

(THE SUBSTITUTION $u = \ln x \Rightarrow x = e^u$, FOLLOWED BY INTEGRATION BY PARTS ALSO WORKS HERE)

69. $\displaystyle\int \frac{4}{(2x-7)^2} \, dx = -\frac{2}{2x-7} + C$

$\displaystyle\int \frac{4}{(2x-7)^2} \, dx = \int 4(2x-7)^{-2} \, dx = \dots$ BY INSPECTION ...

$= \frac{4}{-2}(2x-7)^{-1} + C = -2(2x-7)^{-1} + C = -\frac{2}{2x-7} + C$

70. $\displaystyle\int 4\cos^2 x \, dx = 2x + \sin 2x + C$

$\displaystyle\int 4\cos^2 x \, dx = \int 4\left(\frac{1}{2} + \frac{1}{2}\cos 2x\right) dx = \int 2 + 2\cos 2x \, dx$

$= 2x + \sin 2x + C$

71. $\displaystyle\int (1+\tan^2 x)\sec^2 x \, dx = \tan x + \frac{1}{3}\tan^3 x + C$

$$\int \sec^2 x(1+\tan^2 x)\,dx = \int \sec^2 x + \sec^2 x \tan^2 x \, dx = \dots \text{ BY RECOGNITION...}$$

$$= \tan x + \frac{1}{3}\tan^3 x + C$$

$$\frac{d}{dx}(\tan^3 x) = 3 \times \tan^2 x \times \sec^2 x$$

72. $\displaystyle\int (1+\tan x)\sec^2 x \, dx = \begin{bmatrix} \frac{1}{2}(1+\tan x)^2 + C \\ \tan x + \frac{1}{2}\tan^2 x + C \\ \tan x + \frac{1}{2}\sec^2 x + C \end{bmatrix}$

$$\int \sec^2 x(1+\tan x)\,dx = \dots \text{ BY REVERSE CHAIN RULE/RECOGNITION}$$

$$= \frac{1}{2}(1+\tan x)^2 + C$$

ALTERNATIVE

$$\int \sec^2 x(1+\tan x)\,dx = \int \sec^2 x + \sec^2 x \tan x \, dx \dots \text{ BY RECOGNITION..}$$

$$= \tan x + \frac{1}{2}\tan^2 x + C$$

$$\frac{d}{dx}(\tan^2 x) = 2\tan x \times \sec^2 x$$

ANOTHER VARIANT

$$\int \sec^2 x(1+\tan x)\,dx = \int \sec^2 x + \sec^2 x \tan x \, dx$$

$$= \int \sec^2 x + \sec x(\sec x \tan x)\,dx$$

$$= \tan x + \frac{1}{2}\sec^2 x + C$$

$$\frac{d}{dx}(\sec^2 x) = 2 \times \sec x \times \sec x \tan x$$

73. $\displaystyle\int \operatorname{cosec}^2(3x+1)\,dx = -\frac{1}{3}\cot(3x+1) + C$

$$\int \operatorname{cosec}^2(3x+1)\,dx = -\frac{1}{3}\cot(3x+1) + C$$

STANDARD DIFFERENTIATION $\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$

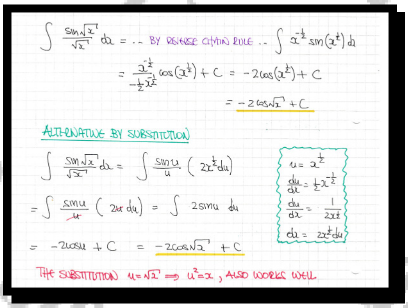
74. $\displaystyle\int 12\sec^2(2x+3)\,dx = 6\tan(2x+3) + C$

$$\int 12\sec^2(2x+3)\,dx = 6\tan(2x+3) + C$$

STANDARD DIFFERENTIATION RESULT

$$\frac{d}{dx}(\tan x) = \sec^2 x$$

75. $\displaystyle\int \frac{\sin\sqrt{x}}{\sqrt{x}}\,dx = -2\cos\sqrt{x} + C$


$\int \frac{\sin\sqrt{x}}{\sqrt{x}}\,dx = \dots$ BY REVERSE CHAIN RULE $\dots \int x^{-\frac{1}{2}}\sin(x^{\frac{1}{2}})\,dx$

$= \frac{x^{\frac{1}{2}}}{-\frac{1}{2}x^{\frac{1}{2}}}\cos(x^{\frac{1}{2}}) + C = -2\cos(x^{\frac{1}{2}}) + C$

$= -2\cos\sqrt{x} + C$

ALTERNATIVE BY SUBSTITUTION

$\int \frac{\sin\sqrt{x}}{\sqrt{x}}\,dx = \int \frac{\sin u}{u}\,(2x^{\frac{1}{2}}du)$

$= \int \frac{\sin u}{u}\,(2u\,du) = \int 2\sin u\,du$

$= -2\cos u + C = -2\cos\sqrt{x} + C$

$u = x^{\frac{1}{2}}$, $\frac{du}{dx} = \frac{1}{2}x^{-\frac{1}{2}}$, $\frac{du}{dx} = \frac{1}{2x^{\frac{1}{2}}}$, $dx = 2x^{\frac{1}{2}}du$

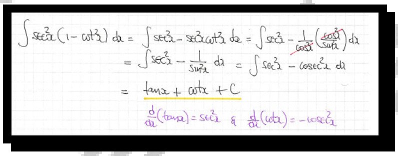
THE SUBSTITUTION $u = \sqrt{x} \Rightarrow u^2 = x$, ALSO WORKS WELL

76. $\displaystyle\int 6e^{2x+2}\,dx = 3e^{2x+2} + C$


$\int 6e^{2x+2}\,dx = \dots$ BY INSPECTION $\dots = \frac{6}{2}e^{2x+2} + C$

$= 3e^{2x+2} + C$

77. $\displaystyle\int \left(1-\cot^2 x\right)\sec^2 x\,dx = \tan x + \cot x + C$

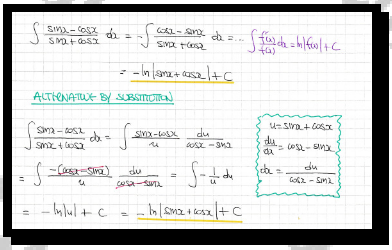

$\int \sec^2 x(1-\cot^2 x)\,dx = \int \sec^2 x - \sec^2 x\cot^2 x\,dx = \int \sec^2 x - \frac{1}{\cos^2 x}\left(\frac{\cos^2 x}{\sin^2 x}\right)dx$

$= \int \sec^2 x - \frac{1}{\sin^2 x}\,dx = \int \sec^2 x - \operatorname{cosec}^2 x\,dx$

$= \tan x + \cot x + C$

$\frac{d}{dx}(\tan x) = \sec^2 x$ & $\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$

78. $\displaystyle\int \frac{\sin x - \cos x}{\sin x + \cos x}\,dx = -\ln\left|\sin x + \cos x\right| + C$


$\int \frac{\sin x - \cos x}{\sin x + \cos x}\,dx = -\int \frac{\cos x - \sin x}{\sin x + \cos x}\,dx = \dots \int \frac{f'(x)}{f(x)}dx = \ln|f(x)| + C$

$= -\ln|\sin x + \cos x| + C$

ALTERNATIVE BY SUBSTITUTION

$\int \frac{\sin x - \cos x}{\sin x + \cos x}\,dx = \int \frac{\sin x - \cos x}{u}\,\frac{du}{\cos x - \sin x}$

$= \int \frac{-(\cos x - \sin x)}{u}\,\frac{du}{\cos x - \sin x} = \int -\frac{1}{u}\,du$

$= -\ln|u| + C = -\ln|\sin x + \cos x| + C$

$u = \sin x + \cos x$, $\frac{du}{dx} = \cos x - \sin x$, $dx = \frac{du}{\cos x - \sin x}$

79. $\displaystyle\int \sec x \tan x \sqrt{1+\sec x}\ dx = \frac{2}{3}(1+\sec x)^{\frac{3}{2}} + C$

$\displaystyle\int \sec x \tan x \sqrt{1+\sec x}\ dx = \int (\sec x+1)^{\frac{1}{2}}(\sec x \tan x)\ dx$

BY REVERSE CHAIN RULE AS $\frac{d}{dx}(\sec x) = \sec x \tan x$

$= \frac{1}{\frac{3}{2}}(\sec x+1)^{\frac{3}{2}} + C$

$= \frac{2}{3}(1+\sec x)^{\frac{3}{2}} + C$

ALTERNATIVE BY SUBSTITUTION

$\displaystyle\int \sec x \tan x \sqrt{1+\sec x}\ dx = \dots$

$= \displaystyle\int (\cancel{\sec x \tan x})\, u^{\frac{1}{2}} \frac{du}{\cancel{\sec x \tan x}} = \int u^{\frac{1}{2}}\ du$

$= \frac{2}{3}u^{\frac{3}{2}} + C = \frac{2}{3}(1+\sec x)^{\frac{3}{2}} + C$

$u = 1+\sec x$

$\frac{du}{dx} = \sec x \tan x$

$dx = \frac{du}{\sec x \tan x}$

(THE SUBSTITUTIONS $u = \sec x$ OR $u = (1+\sec x)^{\frac{1}{2}}$ ALSO WORK)

80. $\displaystyle\int \tan 2x \sec 2x\ dx = \frac{1}{2}\sec 2x + C$

$\displaystyle\int \tan 2x \sec 2x\ dx \dots$ STANDARD DIFFERENTIAL $= \frac{1}{2}\sec 2x + C$

$\frac{d}{dx}(\sec x) = \sec x \tan x$

81. $\displaystyle\int x^2 \ln x\ dx = \frac{1}{3}x^3 \ln|x| - \frac{1}{9}x^3 + C$

$\displaystyle\int x^2 \ln x\ dx = \dots$ INTEGRATION BY PARTS

$= \left(\frac{1}{3}x^3\right)(\ln|x|) - \displaystyle\int \left(\frac{1}{x}\right)\left(\frac{1}{3}x^3\right) dx$

$= \frac{1}{3}x^3 \ln|x| - \displaystyle\int \frac{1}{3}x^2\ dx$

$= \frac{1}{3}x^3 \ln|x| - \frac{1}{9}x^3 + C$

$\ln x$	$\frac{1}{x}$
$\frac{1}{3}x^3$	x^2

82. $\displaystyle\int \frac{6}{x^2-2x-8}\ dx = \ln\left|\frac{x-4}{x+2}\right| + C$

$\displaystyle\int \frac{6}{x^2-2x-8}\ dx = \int \frac{6}{(x+2)(x-4)}\ dx = \dots$ BY PARTIAL FRACTIONS

$\frac{6}{(x+2)(x-4)} \equiv \frac{A}{x+2} + \frac{B}{x-4}$

$\boxed{6 \equiv A(x-4) + B(x+2)}$

- IF $x=4 \Rightarrow 6 = 6B \Rightarrow B=1$
- IF $x=-2 \Rightarrow 6 = -6A \Rightarrow A=-1$

$= \displaystyle\int \frac{1}{x-4} - \frac{1}{x+2}\ dx = \ln|x-4| - \ln|x+2| + C = \ln\left|\frac{x-4}{x+2}\right| + C$

83. $\displaystyle\int 3\cot^2 x \ dx = -3x - 3\cot x + C$

$$\int 3\cot^2 x \, dx = \int 3(\operatorname{cosec}^2 x - 1)\, dx = \int 3\operatorname{cosec}^2 x - 3 \, dx$$

$\{1+\cot^2 x \equiv \operatorname{cosec}^2 x\}$

$$= -3\cot x - 3x + C$$

$\leftarrow$ SINCE $\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$

84. $\displaystyle\int \cos 2x \sin x \ dx = \begin{bmatrix} -\frac{2}{3}\cos^3 x + \cos x + C \\ -\frac{1}{6}\cos 3x + \frac{1}{2}\cos x + C \\ \frac{4}{3}\cos^3 x - \cos 2x \cos x + C \\ \frac{1}{3}\cos 2x \cos x + \frac{2}{3}\sin 2x \sin x + C \\ \frac{1}{3}\sin x \sin 2x + \frac{1}{3}\cos x + C \\ \frac{2}{3}\sin^2 x \cos x + \frac{1}{3}\cos x + C \\ \frac{1}{3}\left(1 + 2\sin^2 x\right)\cos x + C \\ -\frac{1}{4}\cos 3x + \frac{5}{4}\cos x + C \end{bmatrix}$

$$\int \cos 2x \sin x \, dx = \int (2\cos^2 x - 1)\sin x \, dx$$

$$= \int \underbrace{2\cos^2 x \sin x}_{} - \sin x \, dx = \ldots = -\frac{2}{3}\cos^3 x + \cos x + C$$

BY SUBSTITUTION $u = \cos x$

OR REVERSE CHAIN RULE SINCE $\frac{d}{dx}(\cos^3 x) = 3\cos^2 x(-\sin x)$

STANDARD ALTERNATIVE FOR THIS TYPE OF INTEGRAL

- $\sin(2x+x) = \sin 2x \cos x + \cos 2x \sin x$
- $\sin(2x-x) = \sin 2x \cos x - \cos 2x \sin x$

SUBTRACT

$$\sin(2x+x) - \sin(2x-x) = 2\cos 2x \sin x$$

$$\cos 2x \sin x = \frac{1}{2}\sin 3x - \frac{1}{2}\sin x$$

$$\int \cos 2x \sin x \, dx = \int \frac{1}{2}\sin 3x - \frac{1}{2}\sin x \, dx = -\frac{1}{6}\cos 3x + \frac{1}{2}\cos x + C$$

BY PARTS ONCE & RECOGNITION

$$\int \cos 2x \sin x \, dx = \ldots$$

$\cos 2x$	$-2\sin 2x$
$-\cos x$	$\sin x$

$$\ldots = -\cos 2x \cos x - \int (-2\sin 2x)(-\cos x)\, dx$$

$$= -\cos 2x \cos x - \int 2\sin 2x \cos x \, dx$$

$$= -\cos 2x \cos x - \int 2(2\sin x \cos x)\cos x \, dx$$

$$= -\cos 2x \cos x - \int 4\sin x \cos^2 x \, dx$$

$$= -\cos 2x \cos x + \frac{4}{3}\cos^3 x + C$$

SINCE $\frac{d}{dx}\left(\frac{4}{3}\cos^3 x\right) = \frac{4}{3} \times 3\cos^2 x \times (-\sin x)$

BY PARTS TWICE

$$\int \cos 2x \sin x \, dx = \ldots$$

$\cos 2x$	$-2\sin 2x$
$-\cos x$	$\sin x$

$$\ldots = -\cos x \cos 2x - \int (-2\sin 2x)(-\cos x)\, dx$$

$$= -\cos x \cos 2x - \int 2\sin 2x \cos x \, dx$$

$2\sin 2x$	$4\cos 2x$
$\sin x$	$\cos x$

$$= -\cos x \cos 2x - \left[2\sin 2x \sin x - \int 4\cos 2x \sin x \, dx\right]$$

$$= -\cos x \cos 2x - 2\sin 2x \sin x + 4\int \cos 2x \sin x \, dx$$

IF $\displaystyle\int \cos 2x \sin x \, dx = -\cos x \cos 2x - 2\sin 2x \sin x + 4\int \cos 2x \sin x \, dx$

$$\cos x \cos 2x + 2\sin 2x \sin x = 3\int \cos 2x \sin x \, dx$$

$$\therefore \int \cos 2x \sin x \, dx = \frac{1}{3}\cos x \cos 2x + \frac{2}{3}\sin 2x \sin x + C$$

$$= \frac{1}{3}\left[\cos 2x \cos x + \sin 2x \sin x\right] + \frac{1}{3}\sin 2x \sin x + C$$

$$= \frac{1}{3}\cos(2x - x) + \frac{1}{3}\sin 2x \sin x + C$$

$$= \frac{1}{3}\cos x + \frac{1}{3}\sin x \sin 2x + C$$

$$= \frac{1}{3}\cos x + \frac{2}{3}\sin^2 x \cos x + C$$

$$= \frac{1}{3}\cos x\left(1 + 2\sin^2 x\right) + C$$

USING THE IDENTITY $\sin 3x \equiv 3\sin x - 4\sin^3 x$

$$\int \cos 2x \sin x \, dx = \int (1 - 2\sin^2 x)\sin x \, dx = \int \sin x - 3\sin^3 x \, dx$$

$$= \int \sin x - 3\left(\frac{3}{4}\sin x - \frac{1}{4}\sin 3x\right) dx = \int \frac{3}{4}\sin 3x - \frac{5}{4}\sin x \, dx$$

$$= -\frac{1}{4}\cos 3x + \frac{5}{4}\cos x + C$$

85. $$\int \frac{\sqrt{x^2+4}}{x}\,dx = \sqrt{x^2+4} + \ln\left|\frac{\sqrt{x^2+4}-2}{\sqrt{x^2+4}+2}\right| + C$$

$\int \frac{\sqrt{x^2+4}}{x}\,dx = \dots$ SUBSTITUTION ...

$u = (x^2+4)^{\frac{1}{2}}$
$u^2 = x^2+4$
$2u\frac{du}{dx} = 2x$
$u\,du = x\,dx$
$dx = \frac{u}{x}\,du$
$x^2 = u^2-4$

$= \int \frac{u}{x}\left(\frac{u}{x}\,du\right) = \int \frac{u^2}{x^2}\,du$

$= \int \frac{u^2}{u^2-4}\,du = \int \frac{(u^2-4)+4}{u^2-4}\,du$

↑ IMPROPER FRACTION — LONG DIVIDE OR MANIPULATE

$= \int 1 + \frac{4}{u^2-4}\,du = \int 1 + \frac{4}{(u-2)(u+2)}\,du$

BY PARTIAL FRACTIONS

$\frac{4}{(u-2)(u+2)} \equiv \frac{A}{u-2} + \frac{B}{u+2}$

$\boxed{4 \equiv A(u+2) + B(u-2)}$

• IF $u=2$, $4 = 4A$, $A = 1$ • IF $u=-2$, $4 = -4B$, $B = -1$

$= \int 1 + \frac{1}{u-2} - \frac{1}{u+2}\,du = u + \ln|u-2| - \ln|u+2| + C$

$= u + \ln\left|\frac{u-2}{u+2}\right| + C = \sqrt{x^2+4} + \ln\left|\frac{\sqrt{x^2+4}-2}{\sqrt{x^2+4}+2}\right| + C$

86. $$\int 7(2x-3)^{\frac{5}{2}}\,dx = (2x-3)^{\frac{7}{2}} + C$$

$\int 7(2x-3)^{\frac{5}{2}}\,dx = \dots$ BY INSPECTION ... $= \frac{7}{7}(2x-3)^{\frac{7}{2}} + C$

$= (2x-3)^{\frac{7}{2}} + C$

87. $$\int \frac{x^2}{4-x^3}\,dx = -\tfrac{1}{3}\ln\left|4-x^3\right| + C$$

$\int \frac{x^2}{4-x^3}\,dx = -\frac{1}{3}\int \frac{-3x^2}{4-x^3}\,dx = \dots = -\frac{1}{3}\ln|4-x^3| + C$

↑ OF THE FORM $\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + C$

ALTERNATIVE BY SUBSTITUTION

$u = 4-x^3$
$\frac{du}{dx} = -3x^2$
$dx = \frac{du}{-3x^2}$

$\int \frac{x^2}{4-x^3}\,dx = \int \frac{\cancel{x^2}}{u}\left(-\frac{du}{3\cancel{x^2}}\right) = \int -\frac{1}{3u}\,du$

$= \int -\frac{1}{3} \times \frac{1}{u}\,du = -\frac{1}{3}\ln|u| + C$

$= -\frac{1}{3}\ln|4-x^3| + C$

88. $\displaystyle\int x\sin\left(\tfrac{1}{2}x\right)\,dx = -2x\cos\left(\tfrac{1}{2}x\right)+4\sin\left(\tfrac{1}{2}x\right)+C$

$\displaystyle\int x\sin\left(\tfrac{1}{2}x\right)dx = \ldots$ INTEGRATION BY PARTS

x	1
$-2\cos\left(\frac{1}{2}x\right)$	$\sin\left(\frac{1}{2}x\right)$

$= -2x\cos\left(\tfrac{1}{2}x\right) - \displaystyle\int -2\cos\left(\tfrac{1}{2}x\right)dx$

$= -2x\cos\left(\tfrac{1}{2}x\right) + \displaystyle\int 2\cos\left(\tfrac{1}{2}x\right)dx$

$= -2x\cos\left(\tfrac{1}{2}x\right) + 4\sin\left(\tfrac{1}{2}x\right) + C$

89. $\displaystyle\int \frac{4}{4x^2+4x+1}\,dx = -\frac{2}{2x+1}+C$

$\displaystyle\int \frac{4}{4x^2+4x+1}dx = \ldots$ THIS IS A PERFECT SQUARE $\ldots = \displaystyle\int \frac{4}{(2x+1)^2}dx$

$= \displaystyle\int 4(2x+1)^{-2}dx = -\tfrac{4}{2}(2x+1)^{-1} + C$

$= -\dfrac{2}{2x+1} + C$

90. $\displaystyle\int \frac{3}{\sqrt{4x+1}}\,dx = \tfrac{3}{2}\sqrt{4x+1}+C$

$\displaystyle\int \frac{3}{\sqrt{4x+1}}dx = \displaystyle\int 3(4x+1)^{-\frac{1}{2}} = \ldots$ BY INSPECTION $\ldots$

$= \tfrac{3}{2}(4x+1)^{\frac{1}{2}} + C = \tfrac{3}{2}\sqrt{4x+1} + C$

91. $\displaystyle\int \frac{x}{\sqrt{x-1}}\,dx = \tfrac{2}{3}(x-1)^{\frac{3}{2}}+2(x-1)^{\frac{1}{2}}+C$

$\displaystyle\int \frac{x}{\sqrt{x-1}}dx \ldots$ BY SUBSTITUTION $\ldots$

$u = x-1$, $\frac{du}{dx} = 1$, $dx = du$, $x = u+1$

$= \displaystyle\int \frac{u+1}{u^{\frac{1}{2}}}du = \displaystyle\int \frac{u}{u^{\frac{1}{2}}} + \frac{1}{u^{\frac{1}{2}}}du$

$= \displaystyle\int u^{\frac{1}{2}} + u^{-\frac{1}{2}}du = \tfrac{2}{3}u^{\frac{3}{2}} + 2u^{\frac{1}{2}} + C$

$= \tfrac{2}{3}(x-1)^{\frac{3}{2}} + 2(x-1)^{\frac{1}{2}} + C$

ALTERNATIVE SUBSTITUTION

$u = (x-1)^{\frac{1}{2}}$, $u^2 = x-1$, $2u\frac{du}{dx} = 1$, $dx = 2u\,du$, $x = u^2+1$

$\displaystyle\int \frac{x}{\sqrt{x-1}}dx = \displaystyle\int \frac{x}{\cancel{u}}(2\cancel{u}\,du)$

$= \displaystyle\int 2x\,du = \displaystyle\int 2(u^2+1)\,du$

$= \displaystyle\int 2u^2+2\,du = \tfrac{2}{3}u^3 + 2u + C$

$= \tfrac{2}{3}(x-1)^{\frac{3}{2}} + 2(x-1)^{\frac{1}{2}} + C$

92. $\displaystyle\int \frac{1}{3(x-2)^{\frac{1}{2}}}\,dx = \frac{2}{3}(x-2)^{\frac{1}{2}} + C$

$$\int \frac{1}{3(x-2)^{\frac{1}{2}}}\,dx = \dots \text{ BY INSPECTION } \dots = \int \frac{1}{3}(x-2)^{-\frac{1}{2}}\,dx$$
$$= \frac{\frac{1}{3}}{\frac{1}{2}}(x-2)^{\frac{1}{2}} + C = \frac{2}{3}(x-2)^{\frac{1}{2}} + C$$

93. $\displaystyle\int \frac{6x+3}{2x}\,dx = 3x + \frac{3}{2}\ln|x| + C$

$$\int \frac{6x+3}{2x}\,dx = \int \frac{6x}{2x} + \frac{3}{2x}\,dx = \int 3 + \frac{3}{2}\times\frac{1}{x}\,dx$$
$$= 3x + \frac{3}{2}\ln|x| + C$$

94. $\displaystyle\int \frac{4x+1}{2x-5}\,dx = 2x + \frac{11}{2}\ln|2x-5| + C$

$$\int \frac{4x+1}{2x-5}\,dx = \dots \text{ BY MANIPULATION TO SPLIT THE FRACTION}$$
$$= \int \frac{2(2x-5)+11}{2x-5}\,dx = \int \frac{2(2x-5)}{2x-5} + \frac{11}{2x-5}\,dx$$
$$= \int 2 + \frac{11}{2x-5}\,dx = 2x + \frac{11}{2}\ln|2x-5| + C$$

ALTERNATIVE BY SUBSTITUTION

$$\int \frac{4x+1}{2x-5}\,dx = \int \frac{4x+1}{u}\,\frac{du}{2}$$
$$= \int \frac{2(u+5)+1}{2u}\,du = \int \frac{2u+11}{2u}\,du$$
$$= \int \frac{2u}{2u} + \frac{11}{2u}\,du = \int 1 + \frac{11}{2}\times\frac{1}{u}\,du$$
$$= u + \frac{11}{2}\ln|u| + C = (2x-5) + \frac{11}{2}\ln|2x-5| + C$$
$$= 2x + \frac{11}{2}\ln|2x-5| + C$$

$u = 2x-5$
$\frac{du}{dx} = 2$
$dx = \frac{du}{2}$
$2x = u+5$

95. $\displaystyle\int \frac{4x}{x^2-1}\,dx = 2\ln\left|x^2-1\right| + C$

$$\int \frac{4x}{x^2-1}\,dx = 2\int \frac{2x}{x^2-1}\,dx = \dots = 2\ln|x^2-1| + C$$

OF THE FORM $\displaystyle\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + C$

ALTERNATIVE BY SUBSTITUTION

$$\int \frac{4x}{x^2-1}\,dx = \int \frac{\overset{2}{\cancel{4x}}}{u}\left(\frac{du}{\cancel{2x}}\right) = \int \frac{2}{u}\,du$$
$$= 2\ln|u| + C = 2\ln|x^2-1| + C$$

$u = x^2-1$
$\frac{du}{dx} = 2x$
$dx = \frac{du}{2x}$

PARTIAL FRACTION ALSO WORK HERE AFTER FACTORIZATION OF THE DENOMINATOR

96. $$\int \frac{x^2}{2x-1}\,dx = \left[\begin{array}{c} \frac{1}{16}(2x-1)^2+\frac{1}{4}(2x-1)+\frac{1}{8}\ln|2x-1|+C \\ \frac{1}{4}x^2+\frac{1}{4}x+\frac{1}{8}\ln|2x-1|+C \end{array}\right]$$

$\int \frac{x^2}{2x-1}\,dx = \ldots$ BY SUBSTITUTION ...

$u = 2x-1$

$\frac{du}{dx} = 2$

$dx = \frac{du}{2}$

$2x = u+1$

$4x^2 = u^2+2u+1$

$= \int \frac{x^2}{u}\left(\frac{du}{2}\right) = \int \frac{x^2}{2u}\,du = \int \frac{4x^2}{8u}\,du$

$= \int \frac{u^2+2u+1}{8u}\,du = \int \frac{u^2}{8u}+\frac{2u}{8u}+\frac{1}{8u}\,du$

$= \int \frac{1}{8}u+\frac{1}{4}+\frac{1}{8}\times\frac{1}{u}\,du = \frac{1}{16}u^2+\frac{1}{4}u+\frac{1}{8}\ln|u|+C$

$= \frac{1}{16}(2x-1)^2+\frac{1}{4}(2x-1)+\frac{1}{8}\ln|2x-1|+C$

ALTERNATIVE BY MANIPULATION

$\int \frac{x^2}{2x-1}\,dx = \frac{1}{4}\int \frac{4x^2}{2x-1}\,dx = \frac{1}{4}\int \frac{(4x^2-4x+1)+2(2x-1)+1}{2x-1}$

$= \frac{1}{4}\int \frac{(2x-1)^2+2(2x-1)+1}{2x-1}\,dx$

$= \frac{1}{4}\int (2x-1)+2+\frac{1}{2x-1}\,dx$

$= \frac{1}{4}\left[x^2+x+\frac{1}{2}\ln|2x-1|+C\right]$

$= \frac{1}{4}x^2+\frac{1}{4}x+\frac{1}{8}\ln|2x-1|+C$

97. $$\int \frac{1+\cos^4 x}{\cos^2 x}\,dx = \frac{1}{2}x+\frac{1}{4}\sin 2x+\tan x+C$$

$\int \frac{1+\cos^4 x}{\cos^2 x}\,dx = \int \frac{1}{\cos^2 x}+\frac{\cos^4 x}{\cos^2 x}\,dx = \int \sec^2 x+\cos^2 x\,dx$

$= \int \sec^2 x+\left(\frac{1}{2}+\frac{1}{2}\cos 2x\right)dx$

$= \int \frac{1}{2}+\frac{1}{2}\cos 2x+\sec^2 x\,dx$

$= \frac{1}{2}x+\frac{1}{4}\sin 2x+\tan x+C$

98. $$\int \frac{17-5x}{(2x+3)(2-x)^2}\,dx = \ln\left|\frac{2x+3}{2-x}\right|+\frac{1}{2-x}+C$$

$\int \frac{17-5x}{(2x+3)(2-x)^2}\,dx = \ldots$ BY PARTIAL FRACTIONS

$\frac{17-5x}{(2x+3)(2-x)^2} \equiv \frac{A}{2x+3}+\frac{B}{(2-x)^2}+\frac{C}{2-x}$

$\boxed{17-5x \equiv A(2-x)^2+B(2x+3)+C(2-x)(2x+3)}$

- IF $x=2$: $7=7B$, $B=1$
- IF $x=-\frac{3}{2}$: $\frac{49}{2}=\frac{49}{4}A$, $A=2$
- IF $x=0$: $17=4A+3B+6C$, $17=8+3+6C$, $6=6C$, $C=1$

$= \int \frac{2}{2x+3}+(2-x)^{-2}+\frac{1}{2-x}\,dx = \ln|2x+3|+(2-x)^{-1}-\ln|2-x|$

$= \ln\left|\frac{2x+3}{2-x}\right|+\frac{1}{2-x}+C$

99. $\displaystyle\int x\sin(2x-1)\ dx = -\frac{1}{2}x\cos(2x-1)+\frac{1}{4}\sin(2x-1)+C$

$\displaystyle\int x\sin(2x-1)\,dx = \ldots$ INTEGRATION BY PARTS …

x	1
$-\frac{1}{2}\cos(2x-1)$	$\sin(2x-1)$

$\displaystyle = -\frac{1}{2}x\cos(2x-1) - \int -\frac{1}{2}\cos(2x-1)\,dx$

$\displaystyle = -\frac{1}{2}x\cos(2x-1) + \int \frac{1}{2}\cos(2x-1)\,dx$

$\displaystyle = -\frac{1}{2}x\cos(2x-1) + \frac{1}{4}\sin(2x-1) + C$

100. $\displaystyle\int 4(3x-2)^3\ dx = \frac{1}{3}(3x-2)^4 + C$

$\displaystyle\int 4(3x-2)^3\,dx = \ldots$ BY INSPECTION … $\displaystyle = \frac{4}{12}(3x-2)^4 + C$

$\displaystyle = \frac{1}{3}(3x-2)^4 + C$

101. $\displaystyle\int \sqrt{x\sqrt{x}}\ dx = \frac{4}{7}x^{\frac{7}{4}} + C$

$\displaystyle\int \sqrt{x\sqrt{x}}\,dx = \int \left(x\cdot x^{\frac{1}{2}}\right)^{\frac{1}{2}}dx = \int \left(x^{\frac{3}{2}}\right)^{\frac{1}{2}}dx$

$\displaystyle = \int x^{\frac{3}{4}}\,dx = \frac{4}{7}x^{\frac{7}{4}} + C$

102. $\displaystyle\int \frac{1}{x^2\sqrt[3]{x^2}}\ dx = -\frac{3}{5}x^{-\frac{5}{3}} + C$

$\displaystyle\int \frac{1}{x^2\sqrt[3]{x^2}}\,dx = \int \frac{1}{x^2\cdot x^{\frac{2}{3}}}\,dx = \int \frac{1}{x^{\frac{8}{3}}}\,dx$

$\displaystyle = \int x^{-\frac{8}{3}}\,dx = -\frac{3}{5}x^{-\frac{5}{3}} + C$

103. $\displaystyle\int \frac{3}{\sqrt{2-4x}}\,dx = -\frac{3}{2}(2-4x)^{\frac{1}{2}} + C$

$$\int \frac{3}{\sqrt{2-4x}}\,dx = \int 3(2-4x)^{-\frac{1}{2}}\,dx = \dots \text{ BY INSPECTION}$$
$$= -\frac{3}{2}(2-4x)^{\frac{1}{2}} + C$$

104. $\displaystyle\int \frac{1}{1+\cos 2x}\,dx = \begin{bmatrix} \frac{1}{2}\tan x + C \\ \frac{1}{2}\operatorname{cosec} 2x - \frac{1}{2}\cot 2x + C \end{bmatrix}$

$$\int \frac{1}{1+\cos 2x}\,dx = \int \frac{1}{1+(2\cos^2 x - 1)}\,dx = \int \frac{1}{2\cos^2 x}\,dx$$
$$= \int \frac{1}{2}\sec^2 x\,dx = \frac{1}{2}\tan x + C$$

ALTERNATIVE

$$\int \frac{1}{1+\cos 2x}\,dx = \int \frac{1(1-\cos 2x)}{(1+\cos 2x)(1-\cos 2x)}\,dx = \int \frac{1-\cos 2x}{1-\cos^2 2x}\,dx$$
$$= \int \frac{1-\cos 2x}{\sin^2 2x}\,dx = \int \frac{1}{\sin^2 2x} - \frac{\cos 2x}{\sin^2 2x}\,dx = \int \operatorname{cosec}^2 2x - \frac{\cos 2x}{\sin 2x}\frac{1}{\sin 2x}\,dx$$
$$= \int \operatorname{cosec}^2 2x - \cot 2x \operatorname{cosec} 2x\,dx = -\frac{1}{2}\cot 2x + \frac{1}{2}\operatorname{cosec} 2x + C$$

$\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$ & $\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$

105. $\displaystyle\int \frac{2}{2x-x^2}\,dx = \ln\left|\frac{x}{2-x}\right| + C$

$$\int \frac{2}{2x-x^2}\,dx = \int \frac{2}{x(2-x)}\,dx = \dots \text{ BY PARTIAL FRACTIONS}$$

$$\frac{2}{x(2-x)} \equiv \frac{A}{x} + \frac{B}{2-x}$$
$$\boxed{2 \equiv A(2-x) + Bx}$$

- If $x=2$: $2 = 2B$, $B=1$
- If $x=0$: $2 = 2A$, $A=1$

$$= \int \frac{1}{x} + \frac{1}{2-x}\,dx = \ln|x| - \ln|2-x| + C = \ln\left|\frac{x}{2-x}\right| + C$$

106. $\displaystyle\int \frac{\ln x}{x^2}\,dx = -\frac{1}{x} - \frac{\ln|x|}{x} + C$

$\displaystyle\int \frac{\ln x}{x^2}\,dx = \int x^{-2}\ln x\,dx = \dots$ INTEGRATION BY PARTS

$\ln x$	$\frac{1}{x}$
$-x^{-1}$	x^{-2}

$= -x^{-1}\ln|x| - \int -x^{-1}\left(\frac{1}{x}\right)dx$

$= -\frac{1}{x}\ln|x| + \int x^{-2}\,dx$

$= -\frac{1}{x}\ln|x| - x^{-1} + C$

$= -\frac{1}{x}\ln|x| - \frac{1}{x} + C$

107. $\displaystyle\int \frac{3x^2}{x^3+1}\,dx = \ln\left|x^3+1\right| + C$

$\displaystyle\int \frac{3x^2}{x^3+1}\,dx = \dots$ OF THE FORM $\displaystyle\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + C$

$= \ln|x^3+1| + C$

THE SUBSTITUTION $u = x^3+1$ ALSO WORKS WELL

108. $\displaystyle\int \frac{2x+1}{2x-1}\,dx = x + \ln|2x-1| + C$

$\displaystyle\int \frac{2x+1}{2x-1}\,dx =$ BY MANIPULATION & SPLITTING

$= \displaystyle\int \frac{(2x-1)+2}{(2x-1)}\,dx = \int \frac{2x-1}{2x-1} + \frac{2}{2x-1}\,dx$

$= \displaystyle\int 1 + \frac{2}{2x-1}\,dx = x + \ln|2x-1| + C$

ALTERNATIVE BY SUBSTITUTION

$u = 2x-1$

$\frac{du}{dx} = 2$

$dx = \frac{du}{2}$

$2x = u+1$

$\displaystyle\int \frac{2x+1}{2x-1}\,dx = \int \frac{2x+1}{u}\,\frac{du}{2}$

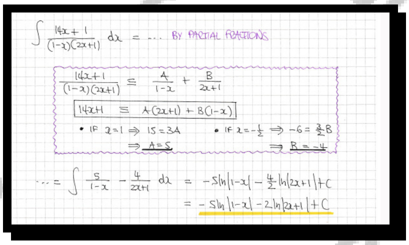
$= \displaystyle\int \frac{2x+1}{2u}\,du = \int \frac{u+2}{2u}\,du$

$= \displaystyle\int \frac{u}{2u} + \frac{2}{2u}\,du = \int \frac{1}{2} + \frac{1}{u}\,du$

$= \frac{1}{2}u + \ln|u| + C = \frac{1}{2}(2x-1) + \ln|2x-1| + C$

$= x + \ln|2x-1| + C$

109. $$\int \frac{14x+1}{(1-x)(2x+1)}\,dx = -5\ln|1-x| - 2\ln|2x+1| + C$$



$\int \frac{14x+1}{(1-x)(2x+1)}\,dx = \dots$ BY PARTIAL FRACTIONS

$\frac{14x+1}{(1-x)(2x+1)} \equiv \frac{A}{1-x} + \frac{B}{2x+1}$

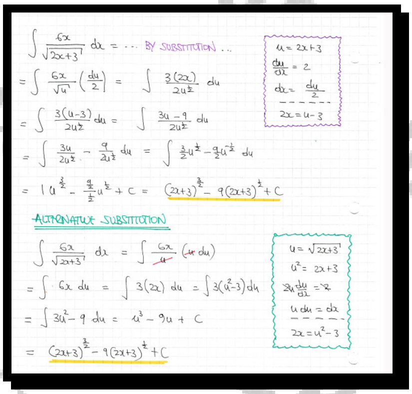
$\boxed{14x+1 \equiv A(2x+1) + B(1-x)}$

- IF $x=1 \Rightarrow 15 = 3A \Rightarrow A = 5$
- IF $x=-\frac{1}{2} \Rightarrow -6 = \frac{3}{2}B \Rightarrow B = -4$

$\dots = \int \frac{5}{1-x} - \frac{4}{2x+1}\,dx = -5\ln|1-x| - \frac{4}{2}\ln|2x+1| + C$

$= -5\ln|1-x| - 2\ln|2x+1| + C$

110. $$\int \frac{6x}{\sqrt{2x+3}}\,dx = (2x+3)^{\frac{3}{2}} - 9(2x+3)^{\frac{1}{2}} + C$$



$\int \frac{6x}{\sqrt{2x+3}}\,dx = \dots$ BY SUBSTITUTION ...

$u = 2x+3$, $\frac{du}{dx} = 2$, $dx = \frac{du}{2}$, $2x = u-3$

$= \int \frac{6x}{\sqrt{u}}\left(\frac{du}{2}\right) = \int \frac{3(2x)}{2u^{\frac{1}{2}}}\,du$

$= \int \frac{3(u-3)}{2u^{\frac{1}{2}}}\,du = \int \frac{3u-9}{2u^{\frac{1}{2}}}\,du$

$= \int \frac{3u}{2u^{\frac{1}{2}}} - \frac{9}{2u^{\frac{1}{2}}}\,du = \int \frac{3}{2}u^{\frac{1}{2}} - \frac{9}{2}u^{-\frac{1}{2}}\,du$

$= 1u^{\frac{3}{2}} - \frac{\frac{9}{2}}{\frac{1}{2}}u^{\frac{1}{2}} + C = (2x+3)^{\frac{3}{2}} - 9(2x+3)^{\frac{1}{2}} + C$

ALTERNATIVE SUBSTITUTION

$u = \sqrt{2x+3}$, $u^2 = 2x+3$, $2u\frac{du}{dx} = 2$, $u\,du = dx$, $2x = u^2-3$

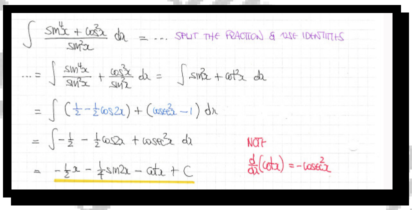
$\int \frac{6x}{\sqrt{2x+3}}\,dx = \int \frac{6x}{\cancel{u}}(\cancel{u}\,du)$

$= \int 6x\,du = \int 3(2x)\,du = \int 3(u^2-3)\,du$

$= \int 3u^2 - 9\,du = u^3 - 9u + C$

$= (2x+3)^{\frac{3}{2}} - 9(2x+3)^{\frac{1}{2}} + C$

111. $$\int \frac{\sin^4 x + \cos^2 x}{\sin^2 x}\,dx = -\frac{1}{2}x - \frac{1}{4}\sin 2x - \cot x + C$$



$\int \frac{\sin^4 x + \cos^2 x}{\sin^2 x}\,dx = \dots$ SPLIT THE FRACTION & USE IDENTITIES

$\dots = \int \frac{\sin^4 x}{\sin^2 x} + \frac{\cos^2 x}{\sin^2 x}\,dx = \int \sin^2 x + \cot^2 x\,dx$

$= \int \left(\frac{1}{2} - \frac{1}{2}\cos 2x\right) + \left(\operatorname{cosec}^2 x - 1\right)dx$

$= \int -\frac{1}{2} - \frac{1}{2}\cos 2x + \operatorname{cosec}^2 x\,dx$

$= -\frac{1}{2}x - \frac{1}{4}\sin 2x - \cot x + C$

NOTE $\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$

112. $\displaystyle\int \frac{2}{(x-4)\sqrt{x}}\,dx = \ln\left|\frac{\sqrt{x}-2}{\sqrt{x}+2}\right| + C$

$\displaystyle\int \frac{2}{(x-4)\sqrt{x}}\,dx = \dots$ BY SUBSTITUTION

$u=\sqrt{x}$
$u^2 = x$
$2u\frac{du}{dx}=1$
$2u\,du = dx$

$\displaystyle = \int \frac{2}{(u^2-4)\cancel{u}}(2\cancel{u}\,du) = \int \frac{4}{u^2-4}\,du$

$\displaystyle = \int \frac{4}{(u-2)(u+2)}\,du = \dots$ BY PARTIAL FRACTIONS

$\displaystyle \frac{4}{(u-2)(u+2)} \equiv \frac{A}{u-2} + \frac{B}{u+2}$

$4 \equiv A(u+2) + B(u-2)$

- IF $u=2 \Rightarrow 4 = 4A \Rightarrow \underline{A=1}$
- IF $u=-2 \Rightarrow 4 = -4B \Rightarrow \underline{B=-1}$

$\displaystyle \dots = \int \frac{1}{u-2} - \frac{1}{u+2}\,du = \ln|u-2| + \ln|u+2| + C$

$\displaystyle = \ln\left|\frac{u-2}{u+2}\right| + C = \ln\left|\frac{\sqrt{x}-2}{\sqrt{x}+2}\right| + C$

113. $\displaystyle\int \frac{1}{2+\sqrt{x-1}}\,dx = 2\sqrt{x-1} - 4\ln\left|2+\sqrt{x-1}\right| + C$

$\displaystyle\int \frac{1}{2+\sqrt{x-1}}\,dx = \dots$ BY SUBSTITUTION

$u = 2+\sqrt{x-1}$
$u-2 = \sqrt{x-1}$
$x-1 = (u-2)^2$
$x = 1+(u-2)^2$
$\frac{dx}{du} = 2(u-2)$
$dx = 2(u-2)\,du$

$\displaystyle = \int \frac{1}{u}\left(2(u-2)\,du\right) = \int \frac{2u-4}{u}\,du$

$\displaystyle = \int \frac{2u}{u} - \frac{4}{u}\,du = \int 2 - \frac{4}{u}\,du$

$= 2u - 4\ln|u| + C$

$= 2\left[2+\sqrt{x-1}\right] - 4\ln\left|2+\sqrt{x-1}\right| + C$

$= 2\sqrt{x-1} - 4\ln\left|2+\sqrt{x-1}\right| + C$

ALTERNATIVE SUBSTITUTION

$u = \sqrt{x-1}$
$u^2 = x-1$
$2u\frac{du}{dx} = 1$
$dx = 2u\,du$

$\displaystyle \int \frac{1}{2+\sqrt{x-1}}\,dx = \int \frac{1}{2+u}(2u\,du)$

$\displaystyle = \int \frac{2u}{u+2}\,du$ ← BY ANOTHER SUBSTITUTION $v=u+2$, IN ORDER TO SPLIT THE FRACTION, OR "MANIPULATION TO SPLIT"

$\displaystyle = \int \frac{2(u+2)-4}{u+2}\,du = \int \frac{2(u+2)}{u+2} - \frac{4}{u+2}\,du$

$\displaystyle = \int 2 - \frac{4}{u+2}\,du = 2u - 4\ln|u+2| + C$

$= 2\sqrt{x-1} - 4\ln\left|\sqrt{x-1}+2\right| + C$

114. $\displaystyle\int \frac{4}{\sqrt{6x-1}}\,dx = \frac{4}{3}\sqrt{6x-1} + C$

$\displaystyle\int \frac{4}{\sqrt{6x-1}}\,dx = \int 4(6x-1)^{-\frac{1}{2}}\,dx = \dots$ RECOGNITION

$= \frac{4}{3}(6x-1)^{\frac{1}{2}} + C = \frac{4}{3}\sqrt{6x-1} + C$

115. $\displaystyle\int \frac{3e^{2x}}{e^{2x}-1}\ dx = \frac{3}{2}\ln\left|e^{2x}-1\right| + C$

$$\int \frac{3e^{2x}}{e^{2x}-1}\,dx = 3\int \frac{e^{2x}}{e^{2x}-1}\,dx = \frac{3}{2}\int \frac{2e^{2x}}{e^{2x}-1}\,dx$$

CAN BE "MADE" IN THE FORM $\displaystyle\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)|+C$

$$= \frac{3}{2}\ln\left|e^{2x}-1\right|+C$$

ALTERNATIVE BY SUBSTITUTION

$$\int \frac{3e^{2x}}{e^{2x}-1}\,dx = \int \frac{3\cancel{e^{2x}}}{u}\left(\frac{du}{2\cancel{e^{2x}}}\right) = \int \frac{3}{2}\times\frac{1}{u}\,du$$

$$= \frac{3}{2}\ln|u|+C = \frac{3}{2}\ln\left|e^{2x}-1\right|+C$$

$u = e^{2x}-1$

$\frac{du}{dx} = 2e^{2x}$

$dx = \frac{du}{2e^{2x}}$

116. $\displaystyle\int x\sec^2 x\ \ dx = \begin{bmatrix} x\tan x - \ln|\sec x| + C \\ x\tan x + \ln|\cos x| + C \end{bmatrix}$

$$\int x\sec^2 x\,dx = \dots \quad \text{INTEGRATION BY PARTS}$$

x	1
$\tan x$	$\sec^2 x$

$$= x\tan x - \int \tan x\,dx$$

STANDARD RESULT

$$= x\tan x - \ln|\sec x|+C$$

$$= x\tan x - \int \frac{\sin x}{\cos x}\,dx$$

$$= x\tan x + \int \frac{-\sin x}{\cos x}\,dx$$

$$= x\tan x + \ln|\cos x|+C$$

OF THE FORM "BOTTOM" DIFFERENTIATE TO "TOP"

117. $\displaystyle\int \operatorname{cosec} 2x \cot 2x\ \ dx = -\frac{1}{2}\operatorname{cosec} 2x + C$

$$\int \operatorname{cosec}2x\cot 2x\,dx = \dots \quad \text{STANDARD DIFFERENTIAL} \quad \frac{d}{dx}(\operatorname{cosec}x) = -\operatorname{cosec}x\cot x$$

$$= -\frac{1}{2}\operatorname{cosec}2x + C$$

118. $\displaystyle\int \tan^2 x\sec^2 x\ \ dx = \frac{1}{3}\tan^3 x + C$

$$\int \tan^2 x\sec^2 x\,dx = \dots \quad \text{BY REVERSE CHAIN RULE SINCE } \frac{d}{dx}(\tan x) = \sec^2 x$$

$$= \frac{1}{3}\tan^3 x + C$$

THE SUBSTITUTION $u = \tan x$ ALSO WORKS WELL

119. $\displaystyle\int \sin 2x \operatorname{cosec} x \ dx = 2\sin x + C$

$\int \sin 2x \operatorname{cosec} x \, dx = \dots$ SIMPLIFY & INTEGRATE
$= \int (\cancel{2\sin x}\cos x)\left(\frac{1}{\cancel{\sin x}}\right) dx$
$= \int 2\cos x \, dx$
$= 2\sin x + C$

120. $\displaystyle\int (2\cos x - 3\sin x)^2 \ dx = \frac{13}{2}x - \frac{5}{4}\sin 2x + 3\cos 2x + C$

$\int (2\cos x - 3\sin x)^2 \, dx = \dots$ EXPAND & USE IDENTITIES ...
$= \int 4\cos^2 x - 12\cos x\sin x + 9\sin^2 x \, dx$
$= \int 4\left(\frac{1}{2} + \frac{1}{2}\cos 2x\right) - 6(2\sin x\cos x) + 9\left(\frac{1}{2} - \frac{1}{2}\cos 2x\right) dx$
$= \int 2 + 2\cos 2x - 6\sin 2x + \frac{9}{2} - \frac{9}{2}\cos 2x \, dx$
$= \int \frac{13}{2} - \frac{5}{2}\cos 2x - 6\sin 2x \, dx = \frac{13}{2}x - \frac{5}{4}\sin 2x + 3\cos 2x + C$

121. $\displaystyle\int \frac{\sin x + \tan x}{\cos x} \ dx = \begin{bmatrix} \sec x + \ln|\sec x| + C \\ \sec x - \ln|\cos x| + C \end{bmatrix}$

$\int \frac{\sin x + \tan x}{\cos x} \, dx = \dots$ SPLIT & USE TRIGONOMETRIC RESULTS
$= \int \frac{\sin x}{\cos x} + \frac{\tan x}{\cos x} \, dx = \int \tan x + \tan x \sec x \, dx$
$= \ln|\sec x| + \sec x + C$
STANDARD RESULT $\qquad \frac{d}{dx}(\sec x) = \sec x \tan x$
$\left[\text{OR} \quad -\ln|\cos x| + \sec x + C\right]$
$\int \tan x \, dx = \int \frac{\sin x}{\cos x} \, dx = -\int \frac{-\sin x}{\cos x} \, dx$
$= -\ln|\cos x| + C$
AS IT IS OF THE FORM "BOTTOM" DIFFERENTIATES TO "TOP"

122. $\displaystyle\int \frac{1 + \sin x}{\cos^2 x} \ dx = \tan x + \sec x + C$

$\int \frac{1 + \sin x}{\cos^2 x} \, dx = \dots$ SPLIT THE FRACTION & USE TRIG RESULTS
$= \int \frac{1}{\cos^2 x} + \frac{\sin x}{\cos^2 x} \, dx = \int \sec^2 x + \frac{\sin x}{\cos x} \times \frac{1}{\cos x} \, dx$
$= \int \sec^2 x + \tan x \sec x \, dx = \tan x + \sec x + C$
$\frac{d}{dx}(\tan x) = \sec^2 x$ & $\frac{d}{dx}(\sec x) = \sec x \tan x$

123. $$\int e^{\sin x} \cos x \; dx = e^{\sin x} + C$$

$\int e^{\sin x} \cos x \, dx = \dots$ BY REVERSE CHAIN RULE SINCE $\frac{d}{dx}\left(e^{\sin x}\right) = e^{\sin x}\cos x$

$= e^{\sin x} + C$

THE SUBSTITUTION $u = \sin x$ ALSO WORKS WELL

124. $$\int x^2 \sin x \; dx = -x^2 \cos x + 2x \sin x + 2\cos x + C$$

$\int x^2 \sin x \, dx = \dots$ INTEGRATION BY PARTS TWICE

x^2	$2x$
$-\cos x$	$\sin x$

$= -x^2\cos x - \int -2x\cos x \, dx$

$= -x^2\cos x + \int 2x\cos x \, dx$

REPEATING THE PROCESS

$2x$	2
$\sin x$	$\cos x$

$= -x^2\cos x + \left[2x\sin x - \int 2\sin x \, dx\right]$

$= -x^2\cos x + 2x\sin x - \int 2\sin x \, dx$

$= -x^2\cos x + 2x\sin x + 2\cos x + C$

125. $$\int (2x+1)^3 \; dx = \frac{1}{8}(2x+1)^4 + C$$

$\int (2x+1)^3 \, dx = \dots$ BY RECOGNITION $\dots = \frac{1}{8}(2x+1)^4 + C$

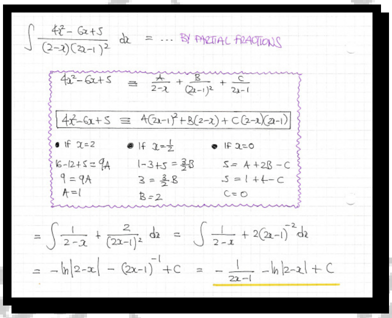
126. $$\int \frac{\tan^4 x}{\cos^2 x} \; dx = \frac{1}{5}\tan^5 x + C$$

$\int \frac{\tan^4 x}{\cos^2 x} \, dx = \int \tan^4 x \sec^2 x \, dx = \dots$ BY REVERSE CHAIN RULE AS $\frac{d}{dx}(\tan x) = \sec^2 x$

$= \frac{1}{5}\tan^5 x + C$

THE SUBSTITUTION $u = \tan x$ ALSO WORKS WELL

127. $$\int \frac{4x^2-6x+5}{(2-x)(2x-1)^2}\,dx = -\frac{1}{2x-1} - \ln|2-x| + C$$



$$\int \frac{4x^2-6x+5}{(2-x)(2x-1)^2}\,dx = \dots \text{ BY PARTIAL FRACTIONS}$$

$$\frac{4x^2-6x+5}{(2-x)(2x-1)^2} \equiv \frac{A}{2-x} + \frac{B}{(2x-1)^2} + \frac{C}{2x-1}$$

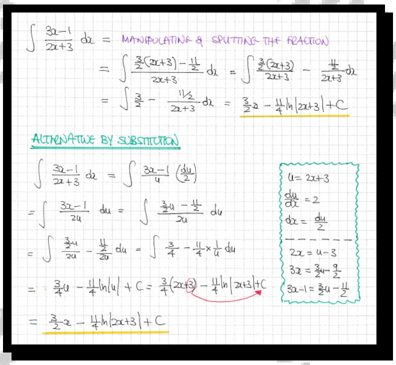
$$\boxed{4x^2-6x+5 \equiv A(2x-1)^2 + B(2-x) + C(2-x)(2x-1)}$$

- IF $x=2$: $16-12+5 = 9A$, $9 = 9A$, $A=1$
- IF $x=\frac{1}{2}$: $1-3+5 = \frac{3}{2}B$, $3 = \frac{3}{2}B$, $B=2$
- IF $x=0$: $5 = A+2B-C$, $5 = 1+4-C$, $C=0$

$$= \int \frac{1}{2-x} + \frac{2}{(2x-1)^2}\,dx = \int \frac{1}{2-x} + 2(2x-1)^{-2}\,dx$$

$$= -\ln|2-x| - (2x-1)^{-1} + C = -\frac{1}{2x-1} - \ln|2-x| + C$$

128. $$\int \frac{3x-1}{2x+3}\,dx = \frac{3}{2}x - \frac{11}{4}\ln|2x+3| + C$$



$$\int \frac{3x-1}{2x+3}\,dx = \text{ MANIPULATING \& SPLITTING THE FRACTION}$$

$$= \int \frac{\frac{3}{2}(2x+3) - \frac{11}{2}}{2x+3}\,dx = \int \frac{\frac{3}{2}(2x+3)}{2x+3} - \frac{\frac{11}{2}}{2x+3}\,dx$$

$$= \int \frac{3}{2} - \frac{11/2}{2x+3}\,dx = \frac{3}{2}x - \frac{11}{4}\ln|2x+3| + C$$

ALTERNATIVE BY SUBSTITUTION

$$\int \frac{3x-1}{2x+3}\,dx = \int \frac{3x-1}{u}\left(\frac{du}{2}\right)$$

$$= \int \frac{3x-1}{2u}\,du = \int \frac{\frac{3}{2}u - \frac{11}{2}}{2u}\,du$$

$$= \int \frac{\frac{3}{2}u}{2u} - \frac{\frac{11}{2}}{2u}\,du = \int \frac{3}{4} - \frac{11}{4}\times\frac{1}{u}\,du$$

$$= \frac{3}{4}u - \frac{11}{4}\ln|u| + C = \frac{3}{4}(2x+3) - \frac{11}{4}\ln|2x+3| + C$$

$$= \frac{3}{2}x - \frac{11}{4}\ln|2x+3| + C$$

$u = 2x+3$

$\frac{du}{dx} = 2$

$dx = \frac{du}{2}$

$2x = u-3$

$3x = \frac{3}{2}u - \frac{9}{2}$

$3x-1 = \frac{3}{2}u - \frac{11}{2}$

129. $$\int \frac{8x^2}{1-2x}\,dx = \begin{bmatrix} -2x^2-2x-\ln|1-2x|+C \\ 2(1-2x)-\frac{1}{2}(1-2x)^2-\ln|1-2x|+C \end{bmatrix}$$

$\int \frac{8x^2}{1-2x}\,dx = \dots$ BY MANIPULATION & SPLITTING THE FRACTION

$$= \int \frac{2(1-2x)^2-4(1-2x)+2}{1-2x}\,dx$$
$$= \int \frac{2(1-2x)^2}{1-2x} - \frac{4(1-2x)}{1-2x} + \frac{2}{1-2x}\,dx$$
$$= \int 2(1-2x) - 4 + \frac{2}{1-2x}\,dx$$
$$= \int -4x-2+\frac{2}{1-2x}\,dx$$
$$= -2x^2-2x-\ln|1-2x|+C$$

ALTERNATIVE MANIPULATION

$8x^2 = -4x(1-2x) - 2(1-2x) + 2$

ALTERNATIVE BY SUBSTITUTION

$$\int \frac{8x^2}{1-2x}\,dx = \int \frac{8x^2}{u}\left(-\frac{du}{2}\right)$$
$$= \int -\frac{4x^2}{u}\,du = \int -\frac{1-2u+u^2}{u}\,du$$
$$= \int -\frac{1}{u}+\frac{2u}{u}-\frac{u^2}{u}\,du$$
$$= \int -\frac{1}{u}+2-u\,du$$
$$= -\ln|u|+2u-\frac{1}{2}u^2+C$$
$$= -\ln|1-2x|+2(1-2x)-\frac{1}{2}(1-2x)^2+C$$
$$\dots = -\ln|1-2x|+2-4x-\frac{1}{2}+2x-2x^2+C$$
$$= -\ln|1-2x|-2x-2x^2+C \quad \text{(AS BEFORE)}$$

$u = 1-2x$

$\frac{du}{dx} = -2$

$dx = -\frac{du}{2}$

$2x = 1-u$

$4x^2 = 1-2u+u^2$

130. $$\int \frac{10}{(3x+1)^{\frac{3}{2}}}\,dx = -\frac{20}{3\sqrt{3x+1}}+C$$

$\int \frac{10}{(3x+1)^{\frac{3}{2}}}\,dx = \int 10(3x+1)^{-\frac{3}{2}}\,dx = \dots$ BY RECOGNITION

$$= \frac{10}{-\frac{3}{2}}(3x+1)^{-\frac{1}{2}}+C = -\frac{20}{3}(3x+1)^{-\frac{1}{2}}+C$$
$$= -\frac{20}{3\sqrt{3x+1}}+C$$

131. $$\int 5^x\,dx = \frac{5^x}{\ln 5}+C$$

$\int 5^x\,dx = \dots$ BY RECOGNITION SINCE $\frac{d}{dx}(5^x) = 5^x\ln 5$

$$= \frac{1}{\ln 5}5^x + C = \frac{5^x}{\ln 5}+C$$

132. $\displaystyle\int \sqrt{\sin x \cos^2 x}\ dx = \frac{2}{3}(\sin x)^{\frac{3}{2}} + C$

$$\int \sqrt{\sin x \cos^2 x}\ dx\ dx = \int (\sin x)^{\frac{1}{2}} \cos x\ dx$$
$$= \ldots \text{ BY REVERSE CHAIN RULE}$$
$$= \frac{2}{3}(\sin x)^{\frac{3}{2}} + C$$

THE SUBSTITUTION $u = \sin x$, OR $u = \sqrt{\sin x}$ ALSO WORKS

133. $\displaystyle\int (2x+1)\sin\left(x^2+x+1\right)\ dx = -\cos\left(x^2+x+1\right) + C$

$$\int (2x+1)\sin(x^2+x+1)\ dx = \ldots \text{ BY REVERSE CHAIN RULE SINCE } \frac{d}{dx}(x^2+x+1) = 2x+1$$
$$= -\cos(x^2+x+1) + C$$

(THE SUBSTITUTION $u = x^2+x+1$ ALSO WORKS)

134. $\displaystyle\int (2x+1)\left(x^2+x+1\right)\ dx = \begin{bmatrix} \frac{1}{2}\left(x^2+x+1\right)^2 + C \\ \frac{1}{2}x^4 + x^3 + \frac{3}{2}x^2 + x + C \end{bmatrix}$

$$\int (2x+1)(x^2+x+1)\ dx = \text{ BY REVERSE CHAIN RULE SINCE } \frac{d}{dx}(x^2+x+1) = 2x+1$$
$$= \frac{1}{2}(x^2+x+1)^2 + C$$

(THE SUBSTITUTION $u = x^2+x+1$, ALSO WORKS)

ALTERNATIVE BY EXPANSION $\ldots \frac{1}{2}x^4 + x^3 + \frac{3}{2}x^2 + x + C$

135. $\displaystyle\int (x+1)\cos\left(x^2+2x+1\right)\ dx = \frac{1}{2}\sin\left(x^2+2x+1\right) + C$

$$\int (x+1)\cos(x^2+2x+1) = \ldots \text{ BY REVERSE CHAIN RULE SINCE } \frac{d}{dx}(x^2+2x+1) = 2x+2 = 2(x+1)$$
$$= \frac{1}{2}\sin(x^2+2x+1) + C$$

(THE SUBSTITUTION $u = x^2+2x+1$, ALSO WORKS)

136. $$\int \frac{1}{2+\sqrt{x}}\,dx = 2\sqrt{x} - 4\ln\left(2+\sqrt{x}\right) + C$$

$\int \frac{1}{2+\sqrt{x}}\,dx = \dots$ BY SUBSTITUTION

$u = 2+\sqrt{x}$
$u-2 = \sqrt{x}$
$x = (u-2)^2$
$\frac{dx}{du} = 2(u-2)$
$dx = 2(u-2)\,du$

$= \int \frac{1}{u}\left(2(u-2)\,du\right) = \int \frac{2u-4}{u}\,du$

$= \int \frac{2u}{u} - \frac{4}{u}\,du = \int 2 - \frac{4}{u}\,du$

$= 2u - 4\ln|u| + C = 2\left[2+\sqrt{x}\right] - 4\ln(2+\sqrt{x}) + C$

$= 2\sqrt{x} - 4\ln(2+\sqrt{x}) + C$

(THE SUBSTITUTION $u=\sqrt{x}$ ALSO WORKS)

137. $$\int \frac{1}{e^{x}+e^{-x}+2}\,dx = -\frac{1}{e^{x}+1} + C$$

$\int \frac{1}{e^{x}+e^{-x}+2}\,dx = \dots$ BY SUBSTITUTION

$u = e^{x}$
$\frac{du}{dx} = e^{x}$
$\frac{du}{dx} = u$
$dx = \frac{du}{u}$

$= \int \frac{1}{u+\frac{1}{u}+2}\left(\frac{du}{u}\right) = \int \frac{1}{u^2+1+2u}\,du$

$= \int \frac{1}{u^2+2u+1}\,du = \int \frac{1}{(u+1)^2}\,du$

$= \int (u+1)^{-2}\,du = -(u+1)^{-1} + C = -\frac{1}{e^{x}+1} + C$

138. $$\int x^2\tan\left(x^3+1\right)\,dx = \tfrac{1}{3}\ln\left|\sec\left(x^3+1\right)\right| + C$$

$\int x^2\tan(x^3+1) = \dots$ BY REVERSE CHAIN RULE SINCE $\frac{d}{dx}(x^3+1) = x^2$

$= \frac{1}{3}\ln|\sec(x^3+1)| + C$

SINCE $\int \tan x\,dx = \ln|\sec x| + C$

(THE SUBSTITUTION $u = x^3+1$, ALSO WORKS)

139. $\displaystyle\int x^3 \ln\left(x^2+1\right)\, dx = \begin{bmatrix} \frac{1}{4}\left(x^2+1\right)\left(x^2-1\right)\ln\left(x^2+1\right)-\frac{1}{8}\left(x^2+1\right)\left(x^2-3\right)+C \\ \frac{1}{4}\left(x^4-1\right)\ln\left(x^2+1\right)-\frac{1}{8}x^2\left(x^2-2\right)+C \end{bmatrix}$

$\int x^3 \ln(x^2+1)\, dx = \dots$ BY SUBSTITUTION FIRST

$u = x^2+1$

$\frac{du}{dx} = 2x$

$dx = \frac{du}{2x}$

$x^2 = u-1$

$= \int x^3 \ln u \left(\frac{du}{2x}\right) = \int \frac{1}{2}x^2 \ln u \, du$

$= \int \frac{1}{2}(u-1)\ln u \, du$

PROCEED BY INTEGRATION BY PARTS

$\ln u$	$\frac{1}{u}$
$\frac{1}{4}(u-1)^2$	$\frac{1}{2}(u-1)$

$= \frac{1}{4}(u-1)^2\ln|u| - \int \frac{1}{4}\frac{(u-1)^2}{u}\, du$

$= \frac{1}{4}(u-1)^2\ln|u| - \int \frac{u^2-2u+1}{4u}\, du$

$= \frac{1}{4}(u-1)^2\ln|u| - \int \frac{1}{4}u - \frac{1}{2} + \frac{1}{4}\times\frac{1}{u}\, du$

$= \frac{1}{4}(u-1)^2\ln|u| - \frac{1}{8}u^2 + \frac{1}{2}u - \frac{1}{4}\ln|u| + C$

$= \frac{1}{4}\left[(u-1)^2-1\right]\ln|u| - \frac{1}{8}u^2 + \frac{1}{2}u + C$

$= \frac{1}{4}(u^2-2u)\ln|u| - \frac{u}{8}(u-4) + C$

$= \frac{1}{4}u(u-2)\ln|u| - \frac{1}{8}u(u-4) + C$

$= \frac{1}{4}(x^2+1)(x^2-1)\ln(x^2+1) - \frac{1}{8}(x^2+1)(x^2-3) + C$

$= \frac{1}{4}(x^4-1)\ln(x^2+1) - \frac{1}{8}(x^4-2x^2-3) + C$

$= \frac{1}{4}(x^4-1)\ln(x^2+1) - \frac{1}{8}x^2(x^2-2) + C$

140. $\displaystyle\int \sin^2 x \sec^2 x \, dx = -x + \tan x + C$

$\int \sin^2 x \sec^2 x\, dx = \int \sin^2 x \times \frac{1}{\cos^2 x}\, dx = \int \frac{\sin^2 x}{\cos^2 x}\, dx$

$= \int \tan^2 x\, dx = \int \sec^2 x - 1\, dx$

$= \tan x - x + C$

SINCE $\frac{d}{dx}(\tan x) = \sec^2 x$

141. $\displaystyle\int 3\sec^2 x \sin x \, dx = 3\sec x + C$

$\int 3\sec^2 x \sin x\, dx = \int 3\left(\frac{1}{\cos^2 x}\right)\sin x\, dx = \int \frac{3\sin x}{\cos x}\times\frac{1}{\cos x}\, dx$

$= \int 3\tan x \sec x\, dx = 3\sec x + C$

SINCE $\frac{d}{dx}(\sec x) = \sec x \tan x$

142. $\displaystyle\int \frac{1}{x(1+\ln x)^3}\ dx = -\frac{1}{2(1+\ln x)^2} + C$

$$\int \frac{1}{x(1+\ln x)^3}\,dx = \ldots \text{ BY REVERSE CHAIN RULE SINCE } \frac{d}{dx}(\ln x) = \frac{1}{x}$$

$$= \int \frac{1}{x}(1+\ln x)^{-3}\,dx$$

$$= \frac{1}{-2}(1+\ln x)^{-2} + C$$

$$= -\frac{1}{2(1+\ln x)^2} + C$$

ALTERNATIVE BY SUBSTITUTION

$$u = 1+\ln x \qquad \frac{du}{dx} = \frac{1}{x} \qquad dx = x\,du$$

$$\int \frac{1}{x(1+\ln x)^3}\,dx = \int \frac{1}{\cancel{x}u^3}(\cancel{x}\,du)$$

$$= \int \frac{1}{u^3}\,du = \int u^{-3}\,du = -\frac{1}{2}u^{-2} + C$$

$$= -\frac{1}{2u^2} + C = -\frac{1}{2(1+\ln x)^2} + C$$

143. $\displaystyle\int x^3 \ln x\ dx = \frac{1}{4}x^4 \ln|x| - \frac{1}{16}x^4 + C$

$$\int x^3 \ln x\,dx = \text{ INTEGRATION BY PARTS}$$

$\ln x$	$\frac{1}{x}$
$\frac{1}{4}x^4$	x^3

$$= \frac{1}{4}x^4\ln|x| - \int \frac{1}{4}x^3\,dx$$

$$= \frac{1}{4}x^4\ln|x| - \frac{1}{16}x^4 + C$$

144. $\displaystyle\int x\ln\left(2x^3\right)\ dx = \frac{1}{2}x^2 \ln\left|2x^3\right| - \frac{3}{4}x^2 + C$

$$\int x\ln(2x^3)\,dx = \ldots \text{ INTEGRATION BY PARTS}$$

$\ln(2x^3)$	$\frac{6x^2}{2x^3} = \frac{3}{x}$
$\frac{1}{2}x^2$	x

$$= \frac{1}{2}x^2\ln(2x^3) - \int \frac{1}{2}x^2\left(\frac{3}{x}\right)dx$$

$$= \frac{1}{2}x^2\ln(2x^3) - \int \frac{3}{2}x\,dx$$

$$= \frac{1}{2}x^2\ln(2x^3) - \frac{3}{4}x^2 + C$$

145. $\displaystyle\int 4 - \cos^4 x \sin x\ dx = 4x + \frac{1}{5}\cos^5 x + C$

$$\int 4 - \cos^4 x\sin x\,dx = \ldots \text{ BY REVERSE CHAIN RULE} \ldots$$

$$= 4x + \frac{1}{5}\cos^5 x + C$$

(THE SUBSTITUTION $u = \cos x$ ALSO WORKS WELL)

146. $\displaystyle\int \frac{\cos x}{\sin^3 x}\,dx = \begin{bmatrix} -\frac{1}{2}\cot^2 x + C \\ -\frac{1}{2}\operatorname{cosec}^2 x + C \end{bmatrix}$

$$\int \frac{\cos x}{\sin^3 x}\,dx = \int \cos x(\sin x)^{-3}\,dx = \dots \text{ BY REVERSE CHAIN RULE}$$

$$= -\frac{1}{2}(\sin x)^{-2} + C = \underline{-\frac{1}{2}\operatorname{cosec}^2 x + C}$$

VARIATIONS BY REVERSE CHAIN RULE

$$\int \frac{\cos x}{\sin^3 x}\,dx = \int \frac{\cos x}{\sin x} \times \frac{1}{\sin^2 x}\,dx = \int \cot x \operatorname{cosec}^2 x\,dx$$

$$= \begin{cases} \underline{-\frac{1}{2}\cot^2 x + C} & \leftarrow \frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x \\ \underline{-\frac{1}{2}\operatorname{cosec}^2 x + C} & \leftarrow \frac{d}{dx}(\operatorname{cosec}^2 x) = 2\operatorname{cosec} x(-\operatorname{cosec} x\cot x) \end{cases}$$

THE SUBSTITUTIONS, $u = \sin x$, $u = \cot x$, $u = \operatorname{cosec} x$ ALL WORK WELL

147. $\displaystyle\int \frac{4\sec^2 x}{\tan x}\,dx = \begin{bmatrix} 4\ln|\tan x| + C \\ 4\ln|\sec x| + 4\ln|\sin x| + C \end{bmatrix}$

$$\int \frac{4\sec^2 x}{\tan x}\,dx = \dots \text{ BY REVERSE CHAIN RULE}$$

$$= \underline{4\ln|\tan x| + C}$$

ALTERNATIVE BY SUBSTITUTION

$$\int \frac{4\sec^2 x}{\tan x}\,dx = \int \frac{4\cancel{\sec^2 x}}{u}\left(\frac{du}{\cancel{\sec^2 x}}\right)$$

$$= \int \frac{4}{u}\,du = 4\ln|u| + C$$

$$= \underline{4\ln|\tan x| + C}$$

$u = \tan x$

$\frac{du}{dx} = \sec^2 x$

$dx = \frac{du}{\sec^2 x}$

ALTERNATIVE BY TRIGONOMETRIC MANIPULATIONS

$$\int \frac{4\sec^2 x}{\tan x}\,dx = \int \frac{4(\tan^2 x + 1)}{\tan x}\,dx = \int \frac{4\tan^2 x}{\tan x} + \frac{4}{\tan x}\,dx$$

$$= \int 4\tan x + 4\cot x\,dx = \underline{4\ln|\sec x| + 4\ln|\sin x| + C}$$

$$\dots = \left[4\ln|\sec x \sin x| + C = 4\ln|\tan x| + C\right]$$

148. $\displaystyle\int \sec^2 x \tan x\sqrt{1+\tan x}\,dx = \frac{2}{5}(1+\tan x)^{\frac{5}{2}} - \frac{2}{3}(1+\tan x)^{\frac{3}{2}} + C$

$$\int \sec^2 x\tan x\sqrt{1+\tan x}\,dx = \dots \text{ BY SUBSTITUTION} \dots$$

$$= \int \cancel{\sec^2 x}\tan x\, u\,\frac{2u}{\cancel{\sec^2 x}}\,du = \int 2u^2\tan x\,du$$

$$= \int 2u^2(u^2-1)\,du = \int 2u^4 - 2u^2\,du$$

$$= \frac{2}{5}u^5 - \frac{2}{3}u^3 + C$$

$$= \underline{\frac{2}{5}(1+\tan x)^{\frac{5}{2}} - \frac{2}{3}(1+\tan x)^{\frac{3}{2}} + C}$$

$u = \sqrt{1+\tan x}$

$u^2 = 1 + \tan x$

$2u\frac{du}{dx} = \sec^2 x$

$dx = \frac{2u}{\sec^2 x}\,du$

$\tan x = u^2 - 1$

THE SUBSTITUTION $u = 1 + \tan x$ ALSO WORKS

149. $$\int \frac{\sqrt{1+2\tan x}}{\cos^2 x}\,dx = \frac{1}{3}\left(1+2\tan x\right)^{\frac{3}{2}} + C$$

$$\int \frac{\sqrt{1+2\tan x}}{\cos^2 x}\,dx = \dots$$ BY REVERSE CHAIN RULE

$$= \int (1+2\tan x)^{\frac{1}{2}}\sec^2 x\,dx = \frac{1}{3}(1+2\tan x)^{\frac{3}{2}} + C$$

ALTERNATIVE BY SUBSTITUTION

$$u = \sqrt{1+2\tan x}$$
$$u^2 = 1+2\tan x$$
$$2u\frac{du}{dx} = 2\sec^2 x$$
$$dx = \frac{u}{\sec^2 x}\,du$$

$$\int \frac{u}{\cos^2 x}\left(\frac{u}{\sec^2 x}\,du\right) = \int u^2\,du$$

$$= \frac{1}{3}u^3 + C = \frac{1}{3}(1+2\tan x)^{\frac{3}{2}} + C$$

THE SUBSTITUTION $u = 1+2\tan x$ ALSO WORKS

150. $$\int \tan^2 x\,dx = -x+\tan x + C$$

$$\int \tan^2 x\,dx = \int \sec^2 x - 1\,dx = \tan x - x + C$$

151. $$\int \frac{(1+\sin x)^2}{\cos^2 x}\,dx = -x+2\tan x+2\sec x + C$$

$$\int \frac{(1+\sin x)^2}{\cos^2 x}\,dx = \int \frac{1+2\sin x+\sin^2 x}{\cos^2 x}\,dx$$
$$= \int \frac{1}{\cos^2 x} + \frac{2\sin x}{\cos^2 x} + \frac{\sin^2 x}{\cos^2 x}\,dx$$
$$= \int \frac{1}{\cos^2 x} + \frac{2\sin x}{\cos x}\times\frac{1}{\cos x} + \frac{\sin^2 x}{\cos^2 x}\,dx$$
$$= \int \sec^2 x + 2\tan x\sec x + \tan^2 x\,dx$$
$$= \int \sec^2 x + 2\tan x\sec x + (\sec^2 x - 1)\,dx$$
$$= \int 2\sec^2 x + 2\tan x\sec x - 1\,dx$$
$$= 2\tan x + 2\sec x - x + C$$

152. $$\int \frac{\cos^2 x}{1+\sin x}\,dx = x+\cos x + C$$

$$\int \frac{\cos^2 x}{1+\sin x}\,dx = \int \frac{1-\sin^2 x}{1+\sin x}\,dx = \int \frac{(1-\sin x)\cancel{(1+\sin x)}}{\cancel{(1+\sin x)}}\,dx$$
$$= \int 1-\sin x\,dx = x+\cos x + C$$

VARIATION

$$\int \frac{\cos^2 x}{1+\sin x}\,dx = \int \frac{\cos^2 x(1-\sin x)}{(1+\sin x)(1-\sin x)}\,dx$$
$$= \int \frac{\cos^2 x(1-\sin x)}{1-\sin^2 x}\,dx = \int \frac{\cancel{\cos^2 x}(1-\sin x)}{\cancel{\cos^2 x}}\,dx$$
$$= \int 1-\sin x\,dx = x+\cos x + C$$

153. $$\int \frac{1}{1+\cos x}\,dx = \begin{bmatrix} \operatorname{cosec} x - \cot x + C \\ \tan\left(\tfrac{1}{2}x\right) + C \end{bmatrix}$$

$$\int \frac{1}{1+\cos x}\,dx = \int \frac{1-\cos x}{(1+\cos x)(1-\cos x)}\,dx = \int \frac{1-\cos x}{1-\cos^2 x}\,dx$$

$$= \int \frac{1-\cos x}{\sin^2 x}\,dx = \int \frac{1}{\sin^2 x} - \frac{\cos x}{\sin^2 x}\,dx = \int \operatorname{cosec}^2 x - \frac{\cos x}{\sin x}\frac{1}{\sin x}\,dx$$

$$= \int \operatorname{cosec}^2 x - \cot x \operatorname{cosec} x\,dx = \underline{-\cot x + \operatorname{cosec} x + C}$$

ALTERNATIVE USING THE DOUBLE ANGLE IDENTITIES

$$\int \frac{1}{1+\cos x}\,dx = \int \frac{1}{1+\left(2\cos^2\frac{x}{2}-1\right)}\,dx \qquad \begin{cases} \cos 2\theta \equiv 2\cos^2\theta - 1 \\ \cos\theta \equiv 2\cos^2\frac{\theta}{2} - 1 \end{cases}$$

$$= \int \frac{1}{2\cos^2\frac{x}{2}}\,dx = \int \tfrac{1}{2}\sec^2\frac{x}{2}\,dx$$

$$= \underline{\tan\frac{x}{2} + C}$$

NOTE: $\operatorname{cosec} x - \cot x = \frac{1}{\sin x} - \frac{\cos x}{\sin x} = \frac{1-\cos x}{\sin x} = \frac{1-\left(1-2\sin^2\frac{x}{2}\right)}{2\sin\frac{x}{2}\cos\frac{x}{2}}$

$$= \frac{2\sin^2\frac{x}{2}}{2\sin\frac{x}{2}\cos\frac{x}{2}} = \frac{\sin\frac{x}{2}}{\cos\frac{x}{2}} = \tan\frac{x}{2}$$

154. $$\int \frac{\cos x}{\sqrt{\sin x}}\,dx = 2\sqrt{\sin x} + C$$

$$\int \frac{\cos x}{\sqrt{\sin x}}\,dx = \int \cos x(\sin x)^{-\frac{1}{2}}\,dx = \dots$$ BY REVERSE CHAIN RULE...

$$= \underline{2(\sin x)^{\frac{1}{2}} + C}$$

THE SUBSTITUTION $u = \sin x$ OR $u = \sqrt{\sin x}$ BOTH WORK WELL

155. $$\int \frac{10x^4}{2x^{\frac{5}{2}}+1}\,dx = 2x^{\frac{5}{2}} - \ln\left(2x^{\frac{5}{2}}+1\right) + C$$

$$\int \frac{10x^4}{2x^{\frac{5}{2}}+1}\,dx = \dots$$ BY SUBSTITUTION

$$\begin{aligned} u &= 2x^{\frac{5}{2}}+1 \\ \frac{du}{dx} &= 5x^{\frac{3}{2}} \\ dx &= \frac{du}{5x^{\frac{3}{2}}} \\ 2x^{\frac{5}{2}} &= u-1 \end{aligned}$$

$$= \int \frac{10x^4}{u}\left(\frac{du}{5x^{\frac{3}{2}}}\right) = \int \frac{2x^{\frac{5}{2}}}{u}\,du$$

$$= \int \frac{u-1}{u}\,du = \int 1 - \frac{1}{u}\,du$$

$$= u - \ln|u| = \left(2x^{\frac{5}{2}}+1\right) - \ln\left(2x^{\frac{5}{2}}+1\right) + C$$

$$= \underline{2x^{\frac{5}{2}} - \ln\left(2x^{\frac{5}{2}}+1\right) + C}$$

156. $\displaystyle\int \sin\sqrt{x}\ dx = 2\sin\sqrt{x} - 2\sqrt{x}\cos\sqrt{x} + C$

$\displaystyle\int \sin\sqrt{x}\,dx = \ldots$ BY SUBSTITUTION ...

$u = \sqrt{x}$
$u^2 = x$
$2u\frac{du}{dx} = 1$
$2u\,du = dx$

$\displaystyle = \int \sin u\,(2u\,du) = \int 2u\sin u\,du$

INTEGRATION BY PARTS NEXT

$2u$	2
$-\cos u$	$\sin u$

$\displaystyle = -2u\cos u - \int -2\cos u\,du$

$\displaystyle = -2u\cos u + \int 2\cos u\,du$

$= -2u\cos u + 2\sin u + C$

$= 2\sin\sqrt{x} - 2\sqrt{x}\cos\sqrt{x} + C$

157. $$\int \frac{x^2}{1-2x}\ dx = \left[\begin{array}{c} -\frac{1}{16}(1-2x)^2 + \frac{1}{4}(1-2x) - \frac{1}{8}\ln|1-2x| + C \\ -\frac{1}{4}x^2 - \frac{1}{4}x - \frac{1}{8}\ln|1-2x| + C \end{array}\right.$$

$\displaystyle\int \frac{x^2}{1-2x}\,dx = \ldots$ BY SUBSTITUTION ...

$u = 1-2x$
$\frac{du}{dx} = -2$
$dx = -\frac{du}{2}$

$2x = 1-u$
$4x^2 = 1-2u+u^2$

$\displaystyle = \int \frac{x^2}{u}\left(\frac{du}{-2}\right) = \int \frac{x^2}{-2u}\,du$

$\displaystyle = \int -\frac{4x^2}{8u}\,du = -\frac{1}{8}\int \frac{4x^2}{u}\,du$

$\displaystyle = -\frac{1}{8}\int \frac{u^2-2u+1}{u}\,du = -\frac{1}{8}\int u - 2 + \frac{1}{u}\,du$

$\displaystyle = -\frac{1}{8}\left[\frac{1}{2}u^2 - 2u + \ln|u|\right] + C = -\frac{1}{16}(1-2x)^2 + \frac{1}{4}(1-2x) - \frac{1}{8}\ln|1-2x| + C$

ALTERNATIVE BY MANIPULATION & SPLIT

$\displaystyle\int \frac{x^2}{1-2x}\,dx = \int \frac{-\frac{1}{2}x(1-2x) - \frac{1}{4}(1-2x) + \frac{1}{4}}{1-2x}\,dx$

$\displaystyle = \int -\frac{1}{2}x - \frac{1}{4} + \frac{\frac{1}{4}}{1-2x}\,dx$

$= -\frac{1}{4}x^2 - \frac{1}{4}x - \frac{1}{8}\ln|1-2x| + C$

158. $\displaystyle\int \frac{12}{(1-2x)^5}\ dx = \frac{3}{2(1-2x)^4} + C$

$\displaystyle\int \frac{12}{(1-2x)^5}\,dx = \int 12(1-2x)^{-5} = \ldots$ BY RECOGNITION ...

$= \frac{12}{8}(1-2x)^{-4} + C = \frac{3}{2}(1-2x)^{-4} + C = \frac{3}{2(1-2x)^4} + C$

159. $$\int \frac{x^4+2x}{x^5+5x^2+8}\,dx = \frac{1}{5}\ln\left|x^5+5x^2+8\right| + C$$

$$\int \frac{x^4+2x}{x^5+5x^2+8}\,dx = \frac{1}{5}\int \frac{5x^4+10x}{x^5+5x^2+8}\,dx = \dots$$

...OF THE FORM $\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + C$ ...

$$= \frac{1}{5}\ln\left|x^5+5x^2+8\right| + C$$

ALTERNATIVE METHOD BY THE SUBSTITUTION, $u = x^5+5x^2+8$

160. $$\int \frac{x}{x-1}\,dx = x + \ln|x-1| + C$$

$\int \frac{x}{x-1}\,dx = \dots$ BY SUBSTITUTION ...

$u = x-1$
$\frac{du}{dx} = 1$
$du = dx$
$x = u+1$

$$= \int \frac{x}{u}\,(du) = \int \frac{u+1}{u}\,du = \int 1 + \frac{1}{u}\,du$$

$$= u + \ln|u| + C = (x-1) + \ln|x-1| + C$$

$$= x + \ln|x-1| + C$$

ALTERNATIVE BY MANIPULATIONS

$$\int \frac{x}{x-1}\,dx = \int \frac{(x-1)+1}{x-1}\,dx = \int \frac{x-1}{x-1} + \frac{1}{x-1}\,dx$$

$$= \int 1 + \frac{1}{x-1}\,dx = x + \ln|x-1| + C$$

161. $$\int \frac{1}{\left(1+\sqrt{x}\right)\sqrt{x}}\,dx = 2\ln\left(1+\sqrt{x}\right) + C$$

$\int \frac{1}{(1+\sqrt{x})\sqrt{x}}\,dx = \dots$ BY SUBSTITUTION ...

$u = \sqrt{x}$
$u^2 = x$
$2u\frac{du}{dx} = 1$
$2u\,du = dx$

$$= \int \frac{1}{(1+u)\cancel{u}}\,(2\cancel{u}\,du) = \int \frac{2}{1+u}\,du$$

$$= 2\ln|u+1| + C = 2\ln\left(\sqrt{x}+1\right) + C$$

CAN ALSO BY DONE BY "REVERSE CHAIN RULE" BY DIRECT RECOGNITION OF THE FUNCTION & ITS DERIVATIVE

162. $$\int \frac{2x^3+1}{x^4+2x}\,dx = \frac{1}{2}\ln\left|x^4+2x\right| + C$$

$$\int \frac{2x^3+1}{x^4+2x}\,dx = \frac{1}{2}\int \frac{4x^3+2}{x^4+2x}\,dx = \frac{1}{2}\ln\left|x^4+2x\right| + C$$

(THIS IS OF THE FORM $\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + C$)

THE SUBSTITUTION $u = x^4+2x$ ALSO WORKS WELL.

163. $$\int \frac{x+2}{x(x+1)}\,dx = \ln\left|\frac{x^2}{x+1}\right| + C$$

$$\int \frac{x+2}{x(x+1)}\,dx = \ldots \text{ BY PARTIAL FRACTIONS} \ldots$$

$$\frac{x+2}{x(x+1)} \equiv \frac{A}{x} + \frac{B}{x+1}$$

$$x+2 \equiv A(x+1) + Bx$$

- IF $x=0$: $2 = A$
- IF $x=-1$: $1=-B$, $B=-1$

$$= \int \frac{2}{x} - \frac{1}{x+1}\,dx$$
$$= 2\ln|x| - \ln|x+1| + C$$
$$= \ln x^2 - \ln|x+1| + C$$
$$= \ln\left|\frac{x^2}{x+1}\right| + C$$

164. $$\int \sin x \ln(\sec x)\,dx = \begin{bmatrix} -\left[1+\ln|\sec x|\right]\cos x \\ \left[-1+\ln|\cos x|\right]\cos x \end{bmatrix} + C$$

$$\int \sin x \ln(\sec x)\,dx = \ldots \text{ INTEGRATION BY PARTS}$$

$\ln(\sec x)$	$\tan x$
$-\cos x$	$\sin x$

$$= -\cos x \ln(\sec x) - \int -\cos x \tan x\,dx$$
$$= \cos x \ln(\cos x) + \int \sin x\,dx$$
$$= \cos x \ln(\cos x) - \cos x + C = -\cos x\left[\ln\sec x + 1\right] + C$$
$$= \cos x\left[\ln\cos x - 1\right] + C$$

165. $$\int \frac{(1+2\cos x)^2}{3\sin^2 x}\,dx = -\frac{5}{3}\cot x - \frac{4}{3}\operatorname{cosec} x - \frac{4}{3}x + C$$

$$\int \frac{(1+2\cos x)^2}{3\sin^2 x}\,dx = \int \frac{1+4\cos x+4\cos^2 x}{3\sin^2 x}\,dx$$
$$= \int \frac{1}{3}\operatorname{cosec}^2 x + \frac{4}{3}\frac{\cos x}{\sin x}\times\frac{1}{\sin x} + \frac{4\cos^2 x}{3\sin^2 x}\,dx$$
$$= \int \frac{1}{3}\operatorname{cosec}^2 x + \frac{4}{3}\cot x \operatorname{cosec} x + \frac{4}{3}\cot^2 x\,dx$$

$1+\cot^2\theta \equiv \operatorname{cosec}^2\theta$

$$= \int \frac{1}{3}\operatorname{cosec}^2 x + \frac{4}{3}\cot x \operatorname{cosec} x + \left(\frac{4}{3}\operatorname{cosec}^2 x - \frac{4}{3}\right)dx$$
$$= \int \frac{5}{3}\operatorname{cosec}^2 x + \frac{4}{3}\cot x \operatorname{cosec} x - \frac{4}{3}\,dx$$
$$= -\frac{5}{3}\cot x - \frac{4}{3}\operatorname{cosec} x - \frac{4}{3}x + C$$

$\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$ $\quad$ $\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$

166. $\displaystyle\int \frac{1}{x\ln x}\ dx = \ln\left|\ln|x|\right| + C$

$$\int \frac{1}{x\ln x}\,dx = \int \frac{1}{x}\times\frac{1}{\ln x}\,dx = \ldots \text{ BY REVERSE CHAIN RULE} \ldots$$

$$= \ln\left|\ln|x|\right| + C$$

ALTERNATIVE BY SUBSTITUTION

$$\int \frac{1}{x\ln x}\,dx = \int \frac{1}{xu}(x\,du) = \int \frac{1}{u}\,du$$

$$= \ln|u| + C$$

$$= \ln\left|\ln|x|\right| + C$$

$u = \ln x$

$\frac{du}{dx} = \frac{1}{x}$

$dx = x\,du$

167. $\displaystyle\int (2-3x)^{-2}\ dx = \frac{1}{3(2-3x)} + C$

$$\int (2-3x)^{-2}\,dx = \ldots \text{ BY RECOGNITION} \ldots$$

$$= \frac{1}{3}(2-3x)^{-1} + C = \frac{1}{3(2-3x)} + C$$

168. $\displaystyle\int 2\sec^2 x + \tfrac{1}{2}\sin 2x\ dx = 2\tan x - \tfrac{1}{4}\cos 2x + C$

$$\int 2\sec^2 x + \tfrac{1}{2}\sin 2x\,dx = \ldots \text{ BY RECOGNITION} \ldots$$

$$= 2\tan x - \tfrac{1}{4}\cos 2x + C$$

169. $\displaystyle\int \frac{\ln x}{\sqrt{x}}\ dx = 2\sqrt{x}\left[\ln|x| - 2\right] + C$

$$\int \frac{\ln x}{\sqrt{x}}\,dx = \int x^{-\frac{1}{2}}\ln x\,dx = \ldots \text{ INTEGRATION BY PARTS} \ldots$$

$$= 2x^{\frac{1}{2}}\ln|x| - \int 2x^{\frac{1}{2}}\left(\frac{1}{x}\right)dx$$

$$= 2x^{\frac{1}{2}}\ln|x| - \int 2x^{-\frac{1}{2}}\,dx$$

$$= 2x^{\frac{1}{2}}\ln|x| - 4x^{\frac{1}{2}} + C$$

$$= 2x^{\frac{1}{2}}\left[\ln|x| - 2\right] + C$$

$$= 2\sqrt{x}\left[\ln|x| - 2\right] + C$$

$\ln x$	$\frac{1}{x}$
$2x^{\frac{1}{2}}$	$x^{-\frac{1}{2}}$

170. $\displaystyle\int \frac{\cos^4 x}{\sin x}\ dx = \ln\left|\tan\left(\tfrac{1}{2}x\right)\right| + \cos x + \tfrac{1}{3}\cos^3 x + C$

$$\int \frac{\cos^4 x}{\sin x}\,dx = \int \frac{(1-\sin^2 x)^2}{\sin x}\,dx = \int \frac{1-2\sin^2 x+\sin^4 x}{\sin x}\,dx$$

$$= \int \operatorname{cosec} x - 2\sin x + \sin^3 x \,dx$$

MANIPULATE THIS FURTHER

$$= \int \operatorname{cosec} x - 2\sin x + \sin x(1-\cos^2 x)\,dx$$

$$= \int \operatorname{cosec} x - \sin x - \sin x\cos^2 x\,dx$$

STANDARD RESULTS

BY REVERSE CHAIN RULE

$$\frac{d}{dx}(\cos^3 x) = 3\cos^2 x(-\sin x)$$

$$= \ln\left|\tan\frac{x}{2}\right| + \cos x + \frac{1}{3}\cos^3 x + C$$

171. $\displaystyle\int 6\tan^2 x - \sec^2 x\ dx = 5\tan x - 6x + C$

$$\int 6\tan^2 x - \sec^2 x\,dx = \int 6(\sec^2 x - 1) - \sec^2 x\,dx$$

$$= \int 5\sec^2 x - 6\,dx$$

$$= 5\tan x - 6x + C$$

172. $\displaystyle\int 6\cos^4 x - 2\sin^2 x\ dx = \tfrac{1}{2}x + \tfrac{3}{2}\sin 2x + \tfrac{1}{8}\sin 4x + C$

$$\int 4\cos^4 x - 2\sin^2 x\,dx = \int 4(\cos^2 x)^2 - 2\left(\tfrac{1}{2}-\tfrac{1}{2}\cos 2x\right)dx$$

$$= \int 4\left(\tfrac{1}{2}+\tfrac{1}{2}\cos 2x\right)^2 - 1 + \cos 2x\,dx$$

$$= \int \cancel{1} + 2\cos 2x + \cos^2 2x \cancel{-1} + \cos 2x\,dx$$

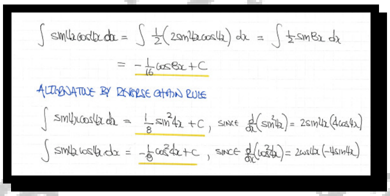
$$= \int 3\cos 2x + \cos^2 2x\,dx$$

$$= \int 3\cos 2x + \left(\tfrac{1}{2}+\tfrac{1}{2}\cos 4x\right)dx$$

$$= \int \tfrac{1}{2} + 3\cos 2x + \tfrac{1}{2}\cos 4x\,dx$$

$$= \tfrac{1}{2}x + \tfrac{3}{2}\sin 2x + \tfrac{1}{8}\sin 4x + C$$

173. $$\int \sin 4x \cos 4x \; dx = \begin{bmatrix} -\frac{1}{16}\cos 8x + C \\ \frac{1}{8}\sin^2 4x + C \\ -\frac{1}{8}\cos^2 4x + C \end{bmatrix}$$

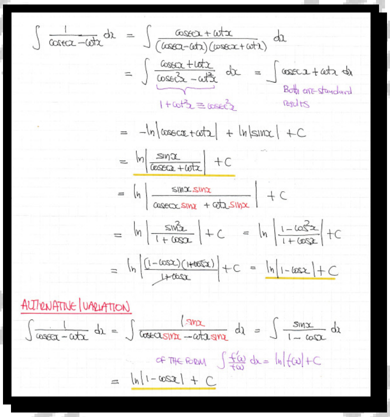


$$\int \sin 4x \cos 4x \, dx = \int \tfrac{1}{2}(2\sin 4x \cos 4x)\, dx = \int \tfrac{1}{2}\sin 8x \, dx$$
$$= \underline{-\tfrac{1}{16}\cos 8x + C}$$

ALTERNATIVE BY REVERSE CHAIN RULE

$$\int \sin 4x \cos 4x \, dx = \underline{\tfrac{1}{8}\sin^2 4x + C}, \text{ since } \tfrac{d}{dx}(\sin^2 4x) = 2\sin 4x(4\cos 4x)$$
$$\int \sin 4x \cos 4x \, dx = \underline{-\tfrac{1}{8}\cos^2 4x + C}, \text{ since } \tfrac{d}{dx}(\cos^2 4x) = 2\cos 4x(-4\sin 4x)$$

174. $$\int \frac{1}{\operatorname{cosec} x - \cot x} \; dx = \begin{bmatrix} \ln\left|\dfrac{\sin x}{\operatorname{cosec} x + \cot x}\right| + C \\ \ln|1-\cos x| + C \end{bmatrix}$$



$$\int \frac{1}{\operatorname{cosec} x - \cot x} dx = \int \frac{\operatorname{cosec} x + \cot x}{(\operatorname{cosec} x - \cot x)(\operatorname{cosec} x + \cot x)} dx$$
$$= \int \frac{\operatorname{cosec} x + \cot x}{\underbrace{\operatorname{cosec}^2 x - \cot^2 x}_{1+\cot^2 x \equiv \operatorname{cosec}^2 x}} dx = \int \operatorname{cosec} x + \cot x \, dx$$

Both are standard results

$$= -\ln|\operatorname{cosec} x + \cot x| + \ln|\sin x| + C$$
$$= \underline{\ln\left|\frac{\sin x}{\operatorname{cosec} x + \cot x}\right| + C}$$
$$= \ln\left|\frac{\sin x \sin x}{\operatorname{cosec} x \sin x + \cot x \sin x}\right| + C$$
$$= \ln\left|\frac{\sin^2 x}{1+\cos x}\right| + C = \ln\left|\frac{1-\cos^2 x}{1+\cos x}\right| + C$$
$$= \ln\left|\frac{(1-\cos x)\cancel{(1+\cos x)}}{\cancel{1+\cos x}}\right| + C = \underline{\ln|1-\cos x| + C}$$

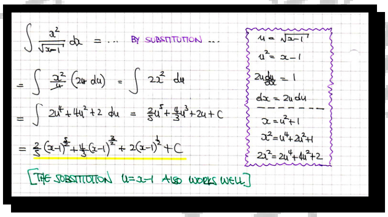
ALTERNATIVE | VARIATION

$$\int \frac{1}{\operatorname{cosec} x - \cot x} dx = \int \frac{1 \sin x}{\operatorname{cosec} x \sin x - \cot x \sin x} dx = \int \frac{\sin x}{1-\cos x} dx$$

OF THE FORM $\int \frac{f'(x)}{f(x)} dx = \ln|f(x)| + C$

$$= \underline{\ln|1-\cos x| + C}$$

175. $$\int \frac{x^2}{\sqrt{x-1}} \; dx = \tfrac{2}{5}(x-1)^{\frac{5}{2}} + \tfrac{4}{3}(x-1)^{\frac{3}{2}} + 2(x-1)^{\frac{1}{2}} + C$$



$$\int \frac{x^2}{\sqrt{x-1}} dx = \ldots \text{ BY SUBSTITUTION} \ldots$$
$$= \int \frac{x^2}{\cancel{u}}(2\cancel{u}\, du) = \int 2x^2 \, du$$
$$= \int 2u^4 + 4u^2 + 2 \, du = \tfrac{2}{5}u^5 + \tfrac{4}{3}u^3 + 2u + C$$
$$= \underline{\tfrac{2}{5}(x-1)^{\frac{5}{2}} + \tfrac{4}{3}(x-1)^{\frac{3}{2}} + 2(x-1)^{\frac{1}{2}} + C}$$

$u = \sqrt{x-1}$
$u^2 = x-1$
$2u\frac{du}{dx} = 1$
$dx = 2u\,du$
$x = u^2+1$
$x^2 = u^4+2u^2+1$
$2x^2 = 2u^4+4u^2+2$

[THE SUBSTITUTION $u=x-1$ ALSO WORKS WELL]

176. $$\int \frac{3e^{2x}}{\sqrt{e^x-1}}\,dx = 2\left(e^x-1\right)^{\frac{3}{2}} + 6\left(e^x-1\right)^{\frac{1}{2}} + C$$

$\int \frac{3e^{2x}}{\sqrt{e^x-1}}\,dx = \dots$ BY SUBSTITUTION ...

$= \int \frac{3e^{2x}}{\cancel{u}}\left(\frac{2\cancel{u}}{e^x}\,du\right) = \int \frac{6e^{2x}}{e^x}\,du$

$= \int 6e^x\,du = \int 6u^2+6\,du = 2u^3+6u+C$

$= 2(e^x-1)^{\frac{3}{2}} + 6(e^x-1)^{\frac{1}{2}} + C$

(THE SUBSTITUTION $u = e^x - 1$ ALSO WORKS WELL)

$u = \sqrt{e^x-1}$
$u^2 = e^x - 1$
$2u\frac{du}{dx} = e^x$
$dx = \frac{2u}{e^x}\,du$
$e^x = u^2+1$

177. $$\int \frac{1}{\left(2+\sqrt[3]{x}\right)\sqrt[3]{x^2}}\,dx = 3\ln\left|2+\sqrt[3]{x}\right| + C$$

$\int \frac{1}{\sqrt[3]{x^2}\,(2+\sqrt[3]{x})}\,dx = \dots$ BY REVERSE CHAIN RULE RECOGNITION OR SUBSTITUTION

$= \int \frac{1}{\cancel{u^2}(2+u)}(3\cancel{u^2}\,du) = \int \frac{3}{u+2}\,du$

$= 3\ln|u+2| + C = 3\ln\left|\sqrt[3]{x}+2\right| + C$

$u = \sqrt[3]{x}$
$u^3 = x$
$3u^2\frac{du}{dx} = 1$
$3u^2\,du = dx$
$u^2 = (\sqrt[3]{x})^2$
$u^2 = \sqrt[3]{x^2}$

178. $$\int \frac{4x^7}{x^4+1}\,dx = x^4 - \ln\left(x^4+1\right) + C$$

$\int \frac{4x^7}{x^4+1}\,dx = \dots$ BY SUBSTITUTION

$= \int \frac{4x^7}{u} \times \frac{du}{4x^3} = \int \frac{x^4}{u}\,du$

$= \int \frac{u-1}{u}\,du = \int 1 - \frac{1}{u}\,du$

$= u - \ln|u| + C = (x^4+1) - \ln(x^4+1) + C$

$= x^4 - \ln(x^4+1) + C$

$u = x^4+1$
$\frac{du}{dx} = 4x^3$
$dx = \frac{du}{4x^3}$
$x^4 = u - 1$

ALTERNATIVE BY MANIPULATION / LONG DIVISION

$\int \frac{4x^7}{x^4+1}\,dx = \int \frac{4x^3(x^4+1) - 4x^3}{x^4+1}\,dx = \int 4x^3 - \frac{4x^3}{x^4+1}\,dx$

$\int \frac{f'(x)}{f(x)}\,dx$

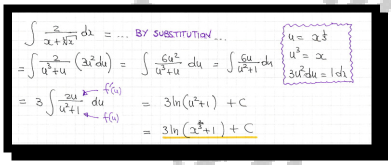
$= x^4 - \ln(x^4+1) + C$

179. $$\int \frac{x}{9x^2+1}\,dx = \frac{1}{18}\ln\left(9x^2+1\right) + C$$

$\int \frac{x}{9x^2+1}\,dx = \frac{1}{18}\int \frac{\overset{f'(x)}{18x}}{\underset{f(x)}{9x^2+1}}\,dx = \frac{1}{18}\ln(9x^2+1) + C$

[THE SUBSTITUTION $u = 9x^2+1$ ALSO WORKS WELL]

180. $$\int \frac{2}{x+\sqrt[3]{x}}\,dx = 3\ln\left|x+\sqrt[3]{x}\right| + C$$



$$\int \frac{2}{x+\sqrt[3]{x}}\,dx = \dots \text{ BY SUBSTITUTION} \dots$$

$$u = x^{\frac{1}{3}}, \quad u^3 = x, \quad 3u^2\,du = 1\,dx$$

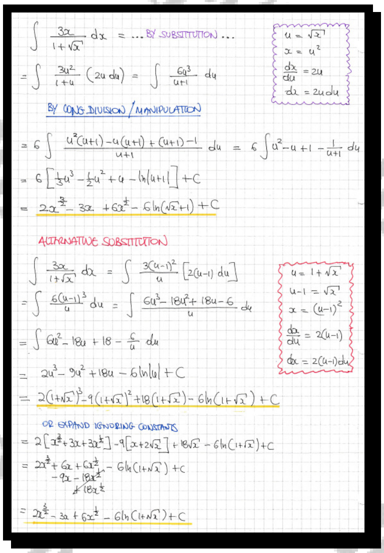
$$= \int \frac{2}{u^3+u}\left(3u^2\,du\right) = \int \frac{6u^2}{u^3+u}\,du = \int \frac{6u}{u^2+1}\,du$$

$$= 3\int \frac{2u}{u^2+1}\,du \qquad f'(u), \ f(u)$$

$$= 3\ln\left(u^2+1\right) + C$$

$$= 3\ln\left(x^{\frac{2}{3}}+1\right) + C$$

181. $$\int \frac{3x}{1+\sqrt{x}}\,dx = \begin{bmatrix} 2x^{\frac{3}{2}} - 3x + 6x^{\frac{1}{2}} - 6\ln\left|1+\sqrt{x}\right| + C \\ 2\left(1+\sqrt{x}\right)^3 - 9\left(1+\sqrt{x}\right)^2 + 18\left(1+\sqrt{x}\right) - 6\ln\left|1+\sqrt{x}\right| + C \end{bmatrix}$$



$$\int \frac{3x}{1+\sqrt{x}}\,dx = \dots \text{ BY SUBSTITUTION} \dots$$

$$u = \sqrt{x}, \quad x = u^2, \quad \frac{dx}{du} = 2u, \quad dx = 2u\,du$$

$$= \int \frac{3u^2}{1+u}\left(2u\,du\right) = \int \frac{6u^3}{u+1}\,du$$

BY LONG DIVISION / MANIPULATION

$$= 6\int \frac{u^2(u+1) - u(u+1) + (u+1) - 1}{u+1}\,du = 6\int u^2 - u + 1 - \frac{1}{u+1}\,du$$

$$= 6\left[\tfrac{1}{3}u^3 - \tfrac{1}{2}u^2 + u - \ln|u+1|\right] + C$$

$$= 2x^{\frac{3}{2}} - 3x + 6x^{\frac{1}{2}} - 6\ln\left(\sqrt{x}+1\right) + C$$

ALTERNATIVE SUBSTITUTION

$$u = 1+\sqrt{x}, \quad u-1 = \sqrt{x}, \quad x = (u-1)^2, \quad \frac{dx}{du} = 2(u-1), \quad dx = 2(u-1)\,du$$

$$\int \frac{3x}{1+\sqrt{x}}\,dx = \int \frac{3(u-1)^2}{u}\left[2(u-1)\,du\right]$$

$$= \int \frac{6(u-1)^3}{u}\,du = \int \frac{6u^3 - 18u^2 + 18u - 6}{u}\,du$$

$$= \int 6u^2 - 18u + 18 - \frac{6}{u}\,du$$

$$= 2u^3 - 9u^2 + 18u - 6\ln|u| + C$$

$$= 2\left(1+\sqrt{x}\right)^3 - 9\left(1+\sqrt{x}\right)^2 + 18\left(1+\sqrt{x}\right) - 6\ln\left(1+\sqrt{x}\right) + C$$

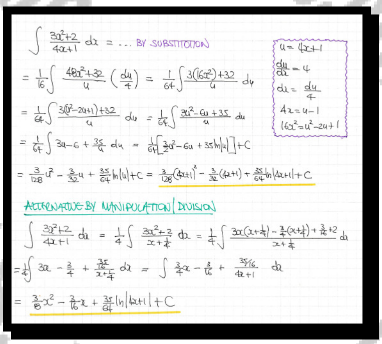
OR EXPAND IGNORING CONSTANTS

$$= 2\left[x^{\frac{3}{2}} + 3x + 3x^{\frac{1}{2}}\right] - 9\left[x + 2\sqrt{x}\right] + 18\sqrt{x} - 6\ln\left(1+\sqrt{x}\right) + C$$

$$= 2x^{\frac{3}{2}} + 6x + 6x^{\frac{1}{2}} - 9x - \cancel{18x^{\frac{1}{2}}} \cancel{+ 18x^{\frac{1}{2}}} - 6\ln\left(1+\sqrt{x}\right) + C$$

$$= 2x^{\frac{3}{2}} - 3x + 6x^{\frac{1}{2}} - 6\ln\left(1+\sqrt{x}\right) + C$$

182. $$\int \frac{3x^2+2}{4x+1}\,dx = \begin{bmatrix} \frac{3}{8}x^2 - \frac{3}{16}x + \frac{35}{64}\ln|4x+1| + C \\ \frac{3}{128}(4x+1)^2 - \frac{3}{32}(4x+1) + \frac{35}{64}\ln|4x+1| + C \end{bmatrix}$$



$$\int \frac{3x^2+2}{4x+1}\,dx = \dots \text{ BY SUBSTITUTION}$$

$$u = 4x+1, \quad \frac{du}{dx} = 4, \quad dx = \frac{du}{4}, \quad 4x = u-1, \quad 16x^2 = u^2-2u+1$$

$$= \frac{1}{16}\int \frac{48x^2+32}{u}\left(\frac{du}{4}\right) = \frac{1}{64}\int \frac{3(16x^2)+32}{u}\,du$$

$$= \frac{1}{64}\int \frac{3(u^2-2u+1)+32}{u}\,du = \frac{1}{64}\int \frac{3u^2-6u+35}{u}\,du$$

$$= \frac{1}{64}\int 3u - 6 + \frac{35}{u}\,du = \frac{1}{64}\left[\tfrac{3}{2}u^2 - 6u + 35\ln|u|\right] + C$$

$$= \frac{3}{128}u^2 - \frac{3}{32}u + \frac{35}{64}\ln|u| + C = \frac{3}{128}(4x+1)^2 - \frac{3}{32}(4x+1) + \frac{35}{64}\ln|4x+1| + C$$

ALTERNATIVE BY MANIPULATION / DIVISION

$$\int \frac{3x^2+2}{4x+1}\,dx = \frac{1}{4}\int \frac{3x^2+2}{x+\frac{1}{4}}\,dx = \frac{1}{4}\int \frac{3x\left(x+\frac{1}{4}\right) - \frac{3}{4}\left(x+\frac{1}{4}\right) + \frac{3}{16} + 2}{x+\frac{1}{4}}\,dx$$

$$= \frac{1}{4}\int 3x - \frac{3}{4} + \frac{\frac{35}{16}}{x+\frac{1}{4}}\,dx = \int \frac{3}{4}x - \frac{3}{16} + \frac{35/16}{4x+1}\,dx$$

$$= \frac{3}{8}x^2 - \frac{3}{16}x + \frac{35}{64}\ln|4x+1| + C$$

183. $\displaystyle\int \frac{3}{x}+\frac{4}{x^2}-\frac{2}{x^3}\ dx = \frac{1}{x^2}-\frac{4}{x}+3\ln|x| + C$

$\displaystyle\int \frac{3}{x}+\frac{4}{x^2}-\frac{2}{x^3}\ dx = \ldots$ BY INSPECTION $\ldots = \displaystyle\int \frac{3}{x}+4x^{-2}-2x^{-3}\ dx$

$= 3\ln|x| - 4x^{-1} + x^{-2} + C = \dfrac{1}{x^2}-\dfrac{4}{x}+3\ln|x|+C$

184. $\displaystyle\int 2\sin 2x\cos^2 x\ dx = -\cos^4 x + C$

$\displaystyle\int 2\sin 2x\cos^2 x\ dx = \int 2(2\sin x\cos x)\cos^2 x\ dx$

$= \displaystyle\int 4\sin x\cos^3 x\ dx$

$= \ldots$ BY REVERSE CHAIN RULE (RECOGNITION)

$= -\cos^4 x + C$

THE SUBSTITUTION $u = \cos x$ ALSO WORKS WELL

185. $\displaystyle\int \frac{10x^2-23x+11}{(2-3x)(2x-1)^2}\ dx = -\frac{1}{3}\ln|3x-2| - \frac{1}{2}\ln|2x-1| - \frac{2}{2x-1} + C$

$\displaystyle\int \frac{10x^2-23x+11}{(2-3x)(2x-1)^2}\ dx = \ldots$ BY PARTIAL FRACTIONS

$\dfrac{10x^2-23x+11}{(2-3x)(2x-1)^2} \equiv \dfrac{A}{2-3x}+\dfrac{B}{(2x-1)^2}+\dfrac{C}{2x-1}$

$\boxed{10x^2-23x+11 \equiv A(2x-1)^2 + B(2-3x) + C(2x-1)(2-3x)}$

- IF $x=\frac{1}{2}$: $\frac{5}{2}-\frac{23}{2}+11 = \frac{1}{2}B$; $5-23+22 = B$; $B=4$
- IF $x=\frac{2}{3}$: $\frac{40}{9}-\frac{46}{3}+11 = \frac{1}{9}A$; $40-138+99 = A$; $A=1$
- IF $x=0$: $11 = A+2B-2C$; $11 = 1+8-2C$; $2C=-2$; $C=-1$

$= \displaystyle\int \frac{1}{2-3x}-\frac{1}{2x-1}+4(2x-1)^{-2}\ dx = -\frac{1}{3}\ln|2-3x| - \frac{1}{2}\ln|2x-1| - 2(2x-1)^{-2}$

$= -\dfrac{1}{3}\ln|3x-2| - \dfrac{1}{2}\ln|2x-1| - \dfrac{2}{2x-1} + C$

186. $\displaystyle\int \frac{\sec x}{\sec x-\tan x}\ dx = \tan x + \sec x + C$

$\displaystyle\int \frac{\sec x}{\sec x-\tan x}\ dx = \int \frac{\sec x(\sec x+\tan x)}{(\sec x-\tan x)(\sec x+\tan x)}\ dx$

$= \displaystyle\int \frac{\sec^2 x+\sec x\tan x}{\sec^2 x-\tan^2 x}\ dx = \int \sec^2 x+\sec x\tan x\ dx$

$1+\tan^2 x \equiv \sec^2 x$

$= \tan x + \sec x + C$

187. $\displaystyle\int x\cos 6x \ dx = \frac{1}{6}x\sin 6x + \frac{1}{36}\cos 6x + C$

$\displaystyle\int x\cos 6x \, dx = \ldots$ INTEGRATION BY PARTS

x	1
$\frac{1}{6}\sin 6x$	$\cos 6x$

$= \frac{1}{6}x\sin 6x - \int \frac{1}{6}\sin 6x \, dx$

$= \frac{1}{6}x\sin 6x + \frac{1}{36}\cos 6x + C$

188. $\displaystyle\int \sin x\sin 3x \ dx = \begin{bmatrix} \frac{1}{4}\sin 2x - \frac{1}{8}\sin 4x + C \\ \frac{1}{8}\cos x\sin 3x - \frac{3}{8}\sin x\cos 3x + C \end{bmatrix}$

$\displaystyle\int \sin x\sin 3x \, dx = \ldots$ BY TRIG IDENTITIES

$= \int \frac{1}{2}\cos 2x - \frac{1}{2}\cos 4x \, dx$ (SEE BELOW)

$= \frac{1}{4}\sin 2x - \frac{1}{8}\sin 4x + C$

$\left.\begin{array}{l}\cos(3x+x) \equiv \cos 3x\cos x - \sin 3x\sin x \\ \cos(3x-x) \equiv \cos 3x\cos x + \sin 3x\sin x\end{array}\right\}$ SUBTRACT

$\cos 4x - \cos 2x \equiv -2\sin 3x\sin x$

$\sin 3x\sin x \equiv \frac{1}{2}\cos 2x - \frac{1}{2}\cos 4x$

ALTERNATIVE BY USING THE TRIPLE ANGLE FOR SINE

$\sin 3x = \sin(2x+x) = \sin 2x\cos x + \cos 2x\sin x$

$= (2\sin x\cos x)\cos x + (1-2\sin^2 x)\sin x$

$= 2\sin x\cos^2 x + \sin x - 2\sin^3 x$

$= 2\sin x(1-\sin^2 x) + \sin x - 2\sin^3 x$

$= 2\sin x - 2\sin^3 x + \sin x - 2\sin^3 x$

$= 3\sin x - 4\sin^3 x$

$\displaystyle\int \sin x\sin 3x \, dx = \int \sin x(3\sin x - 4\sin^3 x) \, dx$

$= \int 3\sin^2 x - 4\sin^4 x \, dx$

$= \int 3\left(\frac{1}{2} - \frac{1}{2}\cos 2x\right) - 4\left(\frac{1}{2} - \frac{1}{2}\cos 2x\right)^2 dx$

$= \int \frac{3}{2} - \frac{3}{2}\cos 2x - 4\left(\frac{1}{4} - \frac{1}{2}\cos 2x + \frac{1}{4}\cos^2 2x\right) dx$

$= \int \frac{3}{2} - \frac{3}{2}\cos 2x - 1 + 2\cos 2x - \cos^2 2x \, dx$

$= \int \cancel{\frac{1}{2}} + \frac{1}{2}\cos 2x - \left(\cancel{\frac{1}{2}} + \frac{1}{2}\cos 4x\right) dx$

$= \int \frac{1}{2}\cos 2x - \frac{1}{4}\cos 4x \, dx$

$= \frac{1}{4}\sin 2x - \frac{1}{8}\sin 4x + C$

ALTERNATIVE BY DOUBLE INTEGRATION BY PARTS

$\displaystyle\int \sin x\sin 3x \, dx = \ldots$ by parts

$\sin 3x$	$3\cos 3x$
$-\cos x$	$\sin x$

$= -\cos x\sin 3x - \int -\cos x\cos 3x \, dx$

$= -\cos x\sin 3x + \int 3\cos 3x\cos x \, dx$

By parts again

$3\cos 3x$	$-9\sin 3x$
$\sin x$	$\cos x$

$= -\cos x\sin 3x + \left[3\cos 3x\sin x - \int -9\sin 3x\sin x \, dx\right]$

$= -\cos x\sin 3x + 3\cos 3x\sin x + 9\int \sin 3x\sin x \, dx$

COLLECTING THE RESULTS

$\Rightarrow \int \sin x\sin 3x \, dx = -\cos x\sin 3x + 3\cos 3x\sin x + 9\int \sin 3x\sin x \, dx$

$\Rightarrow \cos x\sin 3x - 3\cos 3x\sin x = 8\int \sin 3x\sin x \, dx$

$\Rightarrow \int \sin x\sin 3x \, dx = \frac{1}{8}\cos x\sin 3x - \frac{3}{8}\cos 3x\sin x + C$

$= \frac{1}{8}(\cos x\sin 3x - \cos 3x\sin x) - \frac{1}{4}\cos 3x\sin x + C$

$= \frac{1}{8}\sin(3x-x) - \frac{1}{8}(2\cos 3x\sin x) + C$

$= \frac{1}{8}\sin 2x - \frac{1}{8}[\sin 4x - \sin 2x] + C$

$= \frac{1}{4}\sin 2x - \frac{1}{8}\sin 4x + C$ (AS BEFORE)

189. $\displaystyle\int 4\cos 3x + \frac{1}{2}\sin 3x \ dx = \frac{4}{3}\sin 3x - \frac{1}{6}\cos 3x + C$

$\displaystyle\int 4\cos 3x + \frac{1}{2}\sin 3x \, dx = \ldots$ BY INSPECTION $\ldots$

$= \frac{4}{3}\sin 3x - \frac{1}{6}\cos 3x + C$

190. $\displaystyle\int \sin^2 6x \ dx = \frac{1}{2}x - \frac{1}{24}\sin 12x + C$

$$\int \sin^2 6x \, dx = \int \frac{1}{2} - \frac{1}{2}\cos 12x \, dx = \frac{1}{2}x - \frac{1}{24}\sin 12x + C$$

$\cos 2\theta \equiv 1 - 2\sin^2\theta$
$\cos 12\theta \equiv 1 - 2\sin^2 6\theta$
$2\sin^2 6\theta \equiv 1 - \cos 12\theta$
$\sin^2 6\theta \equiv \frac{1}{2} - \frac{1}{2}\cos 12\theta$

191. $\displaystyle\int \frac{\cos 2x}{1-\cos^2 2x} \ dx = -\frac{1}{2}\operatorname{cosec} 2x + C$

$$\int \frac{\cos 2x}{1-\cos^2 2x}\,dx = \int \frac{\cos 2x}{\sin^2 2x}\,dx = \int \frac{\cos 2x}{\sin 2x} \times \frac{1}{\sin 2x}\,dx$$
$$= \int \cot 2x \operatorname{cosec} 2x \, dx = -\frac{1}{2}\operatorname{cosec} 2x + C$$

$\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$

ALTERNATIVE/VARIATION

$$\int \frac{\cos 2x}{1-\cos^2 2x}\,dx = \int \frac{\cos 2x}{\sin^2 2x}\,dx = \int (\sin 2x)^{-2}\cos 2x \, dx$$

... BY REVERSE CHAIN RULE OR THE SUBSTITUTION $u = \sin 2x$

$$= -\frac{1}{2}(\sin 2x)^{-1} + C = -\frac{1}{2}\operatorname{cosec} 2x + C$$

192. $\displaystyle\int \frac{1}{\cos^2 x \tan^2 x} \ dx = -\frac{1}{3}\cot^3 x + C$

$$\int \frac{1}{\cos^2 x \tan^4 x}\,dx = \int \frac{1}{\cos^2 x}(\tan x)^{-4}\,dx = \int (\tan x)^{-4}\sec^2 x \, dx$$

... BY REVERSE CHAIN RULE AS $\frac{d}{dx}(\tan x) = \sec^2 x$

$$= -\frac{1}{3}(\tan x)^{-3} + C = -\frac{1}{3}\cot^3 x + C$$

[THE SUBSTITUTION $u = \tan x$ WORK WELL ALSO]

193. $\displaystyle\int 3\cos^3 3x \ dx = \sin 3x - \frac{1}{3}\sin^3 3x + C$

$$\int 3\cos^3 3x \, dx = \int 3\cos 3x \cos^2 3x \, dx = \int 3\cos 3x (1 - \sin^2 3x)\,dx$$
$$= \int 3\cos 3x - \underbrace{3\cos 3x \sin^2 3x}_{\text{REVERSE CHAIN RULE (INSPECTION)}}\,dx$$
$$= \sin 3x - \frac{1}{3}\sin^3 3x + C$$

ALTERNATIVE BY SUBSTITUTION $u = \sin 3x$

$$\int 3\cos^3 3x \, dx = \int 3\cos^3 3x \left(\frac{du}{3\cos 3x}\right)$$
$$= \int \cos^2 3x \, du$$
$$= \int 1 - \sin^2 3x \, du$$
$$= \int 1 - u^2 \, du$$
$$= u - \frac{1}{3}u^3 + C = \sin 3x - \frac{1}{3}\sin^3 3x + C$$

$u = \sin 3x$
$\frac{du}{dx} = 3\cos 3x$
$dx = \frac{du}{3\cos 3x}$

194. $\displaystyle\int 3\sec^4 x \ dx = 3\tan x + \tan^3 x + C$

$$\int 3\sec^4 x \, dx = \int 3\sec^2 x \sec^2 x \, dx = \int 3\sec^2 x (1+\tan^2 x) \, dx$$

$$= \int 3\sec^2 x + \underbrace{3\sec^2 x \tan^2 x}_{\text{REVERSE CHAIN RULE OR } u=\tan x} \, dx$$

$$= 3\tan x + \tan^3 x + C$$

ALTERNATIVE BY THE SUBSTITUTION $u = \tan x$

$$\boxed{u = \tan x \quad \frac{du}{dx} = \sec^2 x \quad dx = \frac{du}{\sec^2 x}}$$

$$\int 3\sec^4 x \, dx = \int 3\sec^4 x \left(\frac{du}{\sec^2 x}\right) = \int 3\sec^2 x \, du$$

$$= \int 3(1+\tan^2 x) \, du = \int 3(1+u^2) \, du$$

$$= \int 3 + 3u^2 \, du = 3u + u^3 + C$$

$$= 3\tan x + \tan^3 x + C$$

195. $\displaystyle\int 2\sin x \cos 3x \ dx = \begin{bmatrix} \frac{1}{2}\cos 2x - \frac{1}{4}\cos 4x + C \\ -2\cos^4 x + 3\cos^2 x + C \\ -2\cos^4 x - 3\sin^2 x + C \\ -2\cos^4 x + \frac{3}{2}\cos 2x + C \\ \frac{1}{4}\cos x \cos 3x + \frac{3}{4}\sin x \sin 3x + C \end{bmatrix}$

$$\int 2\cos 3x \sin x \, dx = \dots \text{ BY TRIGONOMETRIC IDENTITIES (SEE BELOW)}$$

$$= \int \sin 4x - \sin 2x \, dx = -\frac{1}{4}\cos 4x + \frac{1}{2}\cos 2x + C$$

$$\sin 4x = \sin(3x+x) = \cancel{\sin 3x \cos x} + \cos 3x \sin x$$

$$\sin 2x = \sin(3x-x) = \cancel{\sin 3x \cos x} - \cos 3x \sin x \quad \Big) \text{ SUBTRACT}$$

$$\sin 4x - \sin 2x = 2\cos 3x \sin x$$

ALTERNATIVE BY USING THE TRIPLE ANGLE IDENTITY FOR COSINE

$$\cos 3x = \cos(2x+x) = \cos 2x \cos x - \sin 2x \sin x$$

$$= (2\cos^2 x - 1)\cos x - (2\sin x \cos x)\sin x$$

$$= 2\cos^3 x - \cos x - 2\sin^2 x \cos x$$

$$= 2\cos^3 x - \cos x - 2(1-\cos^2 x)\cos x$$

$$= 2\cos^3 x - \cos x - 2\cos x + 2\cos^3 x$$

$$= 4\cos^3 x - 3\cos x$$

$$\int 2\cos 3x \sin x \, dx = \int 2\sin x (4\cos^3 x - 3\cos x) \, dx$$

$$= \int 8\sin x \cos^3 x - 6\sin x \cos x \, dx$$

... BY REVERSE CHAIN RULE ...

$$= -2\cos^4 x + 3\cos^2 x + C$$

OR

$$-2\cos^4 x - 3\sin^2 x + C$$

OR

$$\dots = \int 8\cos^3 x \sin x - 3\sin 2x \, dx$$

$$= -2\cos^4 x + \frac{3}{2}\cos 2x + C$$

ALTERNATIVE BY DOUBLE INTEGRATION BY PARTS

$$\int 2\cos 3x \sin x \, dx = \dots \text{ integration by parts}$$

$2\cos 3x$	$-6\sin 3x$
$-\cos x$	$\sin x$

$$= -2\cos x \cos 3x - \int 6\sin 3x \cos x \, dx$$

.. by parts again for this integral

$6\sin 3x$	$18\cos 3x$
$\sin x$	$\cos x$

$$\Rightarrow \int 2\cos 3x \sin x \, dx = -2\cos x \cos 3x - \left[6\sin 3x \sin x - \int 18\cos 3x \sin x \, dx \right]$$

$$\Rightarrow \int 2\cos 3x \sin x \, dx = -2\cos x \cos 3x - 6\sin 3x \sin x + \int 18\cos 3x \sin x \, dx$$

$$\Rightarrow \int 2\cos 3x \sin x \, dx = -2\cos x \cos 3x - 6\sin 3x \sin x + 9\int 2\cos 3x \sin x \, dx$$

$$\Rightarrow 2\cos x \cos 3x + 6\sin 3x \sin x = 8\int 2\cos 3x \sin x \, dx$$

$$\Rightarrow \int 2\cos 3x \sin x = \frac{1}{4}\cos x \cos 3x + \frac{3}{4}\sin 3x \sin x + C$$

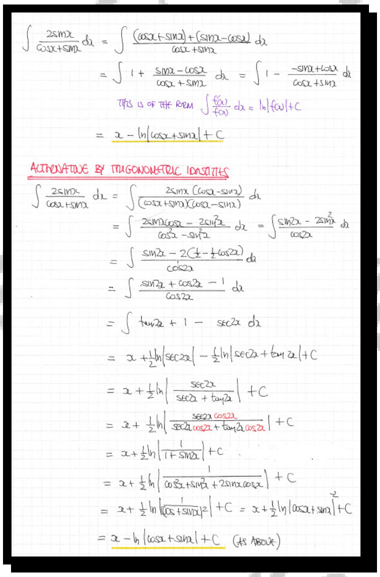
$$= \frac{1}{4}(\cos x \cos 3x + \sin 3x \sin x) + \frac{1}{2}\sin 3x \sin x + C$$

$$= \frac{1}{4}\cos(3x-x) + \frac{1}{4}[2\sin 3x \sin x] + C$$

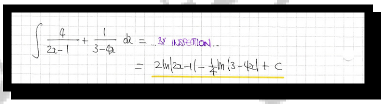
$$= \frac{1}{4}\cos 2x + \frac{1}{4}[\cos 2x - \cos 4x] + C$$

$$= \frac{1}{2}\cos 2x - \frac{1}{4}\cos 4x + C \quad \text{AS BEFORE}$$

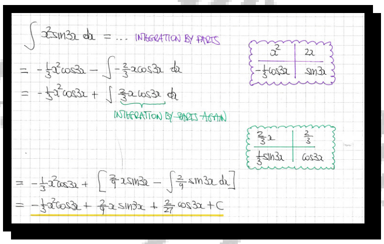
196. $$\int \frac{2\sin x}{\cos x+\sin x}\,dx = \begin{bmatrix} x-\ln\left|\cos x+\sin x\right| + C \\ x+\frac{1}{2}\ln\left|\sec 2x\right|-\frac{1}{2}\ln\left|\sec 2x+\tan 2x\right| + C \end{bmatrix}$$



197. $$\int \frac{4}{2x-1}+\frac{1}{3-4x}\,dx = 2\ln\left|2x-1\right|-\frac{1}{4}\ln\left|3-4x\right| + C$$



198. $$\int x^2\sin 3x\,dx = -\frac{1}{3}x^2\cos 3x+\frac{2}{9}x\sin 3x+\frac{2}{27}\cos 3x + C$$



199. $\displaystyle\int \operatorname{cosec}^2 2x \; dx = -\tfrac{1}{2}\cot 2x + C$

$\displaystyle\int \operatorname{cosec}^2 2x \, dx = \ldots$ BY INSPECTION AS $\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$

$= -\frac{1}{2}\cot 2x + C$

200. $\displaystyle\int \sin^3 2x \cos 2x \; dx = \tfrac{1}{8}\sin^4 2x + C$

$\displaystyle\int \sin^3 2x \cos 2x \, dx = \ldots$ BY REVERSE CHAIN RULE / INSPECTION

$= \frac{1}{8}\sin^4 2x + C$

[THE SUBSTITUTION $u = \sin 2x$ ALSO WORKS WELL]

201. $\displaystyle\int \cot^2 3x \; dx = -x - \tfrac{1}{3}\cot 3x + C$

$\displaystyle\int \cot^2 3x \, dx = \int (\operatorname{cosec}^2 3x - 1) \, dx = \ldots$

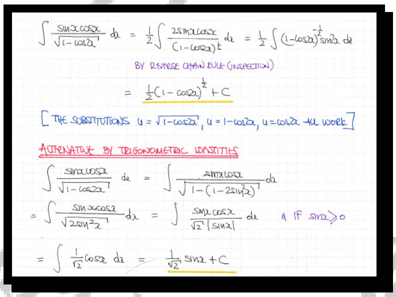
BY INSPECTION SINCE $\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$

$= -\frac{1}{3}\cot 3x - x + C$

202. $\displaystyle\int \frac{7}{3x} \; dx = \tfrac{7}{3}\ln|x| + C$

$\displaystyle\int \frac{7}{3x} \, dx = \int \frac{7}{3} \times \frac{1}{x} \, dx = \ldots$ BY INSPECTION $\ldots = \frac{7}{3}\ln|x| + C$

203. $$\int \frac{\sin x \cos x}{\sqrt{1-\cos 2x}}\, dx = \begin{bmatrix} \frac{1}{2}\sqrt{1-\cos 2x} + C \\ \frac{1}{\sqrt{2}}\sin x + C \end{bmatrix}$$



$$\int \frac{\sin x\cos x}{\sqrt{1-\cos 2x}}\,dx = \frac{1}{2}\int \frac{2\sin x\cos x}{(1-\cos 2x)^{\frac{1}{2}}}\,dx = \frac{1}{2}\int (1-\cos 2x)^{-\frac{1}{2}}\sin 2x\,dx$$

BY REVERSE CHAIN RULE (INSPECTION)

$$= \frac{1}{2}(1-\cos 2x)^{\frac{1}{2}} + C$$

[THE SUBSTITUTIONS $u=\sqrt{1-\cos 2x}$, $u = 1-\cos 2x$, $u=\cos 2x$ ALL WORK]

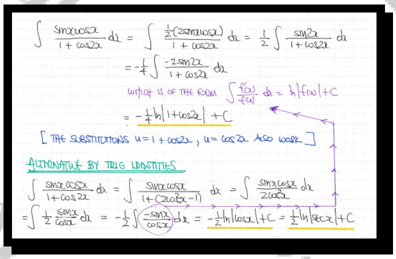
ALTERNATIVE BY TRIGONOMETRIC IDENTITIES

$$\int \frac{\sin x\cos x}{\sqrt{1-\cos 2x}}\,dx = \int \frac{\sin x\cos x}{\sqrt{1-(1-2\sin^2 x)}}\,dx$$

$$= \int \frac{\sin x\cos x}{\sqrt{2\sin^2 x}}\,dx = \int \frac{\sin x\cos x}{\sqrt{2}\,|\sin x|}\,dx \quad \text{\& IF } \sin x \geqslant 0$$

$$= \int \frac{1}{\sqrt{2}}\cos x\,dx = \frac{1}{\sqrt{2}}\sin x + C$$

204. $$\int \frac{\sin x \cos x}{1+\cos 2x}\, dx = \begin{bmatrix} -\frac{1}{4}\ln|1+\cos 2x| + C \\ -\frac{1}{2}\ln|\cos x| + C \\ \frac{1}{2}\ln|\sec x| + C \end{bmatrix}$$



$$\int \frac{\sin x\cos x}{1+\cos 2x}\,dx = \int \frac{\frac{1}{2}(2\sin x\cos x)}{1+\cos 2x}\,dx = \frac{1}{2}\int \frac{\sin 2x}{1+\cos 2x}\,dx$$

$$= -\frac{1}{4}\int \frac{-2\sin 2x}{1+\cos 2x}\,dx$$

WHICH IS OF THE FORM $\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + C$

$$= -\frac{1}{4}\ln|1+\cos 2x| + C$$

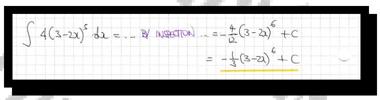
[THE SUBSTITUTIONS $u = 1+\cos 2x$, $u=\cos 2x$ ALSO WORK]

ALTERNATIVE BY TRIG IDENTITIES

$$\int \frac{\sin x\cos x}{1+\cos 2x}\,dx = \int \frac{\sin x\cos x}{1+(2\cos^2 x-1)}\,dx = \int \frac{\sin x\cos x}{2\cos^2 x}\,dx$$

$$= \int \frac{1}{2}\frac{\sin x}{\cos x}\,dx = -\frac{1}{2}\int \frac{-\sin x}{\cos x}\,dx = -\frac{1}{2}\ln|\cos x| + C = \frac{1}{2}\ln|\sec x| + C$$

205. $$\int 4(3-2x)^5\, dx = -\tfrac{1}{3}(3-2x)^6 + C$$



$$\int 4(3-2x)^5\,dx = \ldots \text{ BY INSPECTION } \ldots = -\frac{4}{12}(3-2x)^6 + C$$

$$= -\frac{1}{3}(3-2x)^6 + C$$

206. $\displaystyle\int \frac{1+\sin x}{\cos x}\, dx = \begin{bmatrix} \ln\left|\sec^2 x + \sec x \tan x\right| + C \\ -\ln\left|1-\sin x\right| + C \end{bmatrix}$

$$\int \frac{1+\sin x}{\cos x}\,dx = \int \frac{1}{\cos x} + \frac{\sin x}{\cos x}\,dx = \int \sec x + \tan x\,dx$$

..."STANDARD RESULTS"...

$$= \ln|\sec x + \tan x| + \ln|\sec x| + C$$

$$= \ln|\sec^2 x + \sec x \tan x| + C$$

OR TIDY FURTHER

$$= \ln\left|\frac{1}{\cos^2 x} + \frac{\sin x}{\cos^2 x}\right| + C = \ln\left|\frac{1+\sin x}{\cos^2 x}\right| + C$$

$$= \ln\left|\frac{1+\sin x}{1-\sin^2 x}\right| + C = \ln\left|\frac{\cancel{1+\sin x}}{(1-\sin x)\cancel{(1+\sin x)}}\right| + C$$

$$= \ln\left|\frac{1}{1-\sin x}\right| + C = -\ln|1-\sin x| + C$$

ALTERNATIVE BY TRIGONOMETRIC MANIPULATIONS

$$\int \frac{1+\sin x}{\cos x}\,dx = \int \frac{(1+\sin x)(1-\sin x)}{\cos x(1-\sin x)}\,dx = \int \frac{1-\sin^2 x}{\cos x(1-\sin x)}\,dx$$

$$= \int \frac{\cos^2 x}{\cos x(1-\sin x)}\,dx = \int \frac{\cos x}{1-\sin x}\,dx$$

$$= -\int \frac{-\cos x}{1-\sin x}\,dx = -\ln|1-\sin x| + C$$

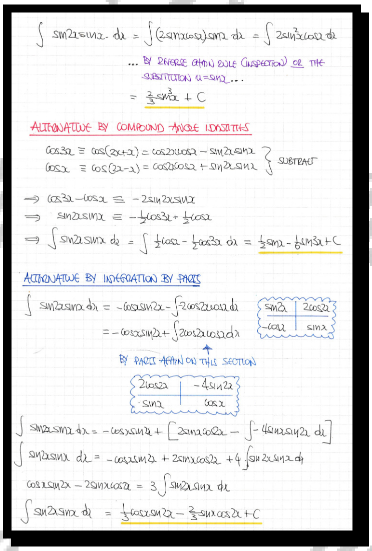
$$\left[\text{WHICH IS OF THE FORM } \int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + C\right]$$

207. $\displaystyle\int \sin 2x \sec x\;dx = -2\cos x + C$

$$\int \sin 2x \sec x\,dx = \int (2\sin x\cos x) \times \frac{1}{\cos x}\,dx = \int 2\sin x\,dx$$

$$= -2\cos x + C$$

208. $$\int \sin 2x \sin x \; dx = \begin{bmatrix} \frac{2}{3}\sin^3 x + C \\ \frac{1}{2}\sin x - \frac{1}{6}\sin 3x + C \\ \frac{1}{2}\cos x \sin 2x - \frac{2}{3}\sin x \cos 2x + C \end{bmatrix}$$



$\int \sin 2x \sin x \, dx = \int (2\sin x \cos x)\sin x \, dx = \int 2\sin^2 x \cos x \, dx$

... BY REVERSE CHAIN RULE (INSPECTION) OR THE SUBSTITUTION $u = \sin x$ ...

$= \frac{2}{3}\sin^3 x + C$

ALTERNATIVE BY COMPOUND ANGLE IDENTITIES

$\cos 3x \equiv \cos(2x+x) = \cos 2x \cos x - \sin 2x \sin x$
$\cos x \equiv \cos(2x-x) = \cos 2x \cos x + \sin 2x \sin x$ } SUBTRACT

$\Rightarrow \cos 3x - \cos x \equiv -2\sin 2x \sin x$
$\Rightarrow \sin 2x \sin x \equiv -\frac{1}{2}\cos 3x + \frac{1}{2}\cos x$
$\Rightarrow \int \sin 2x \sin x \, dx = \int \frac{1}{2}\cos x - \frac{1}{2}\cos 3x \, dx = \frac{1}{2}\sin x - \frac{1}{6}\sin 3x + C$

ALTERNATIVE BY INTEGRATION BY PARTS

$\int \sin 2x \sin x \, dx = -\cos x \sin 2x - \int -2\cos 2x \cos x \, dx$
$= -\cos x \sin 2x + \int 2\cos 2x \cos x \, dx$

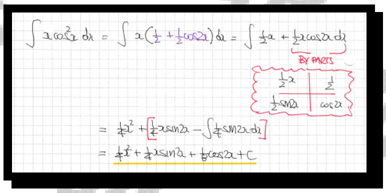
$\sin 2x$	$2\cos 2x$
$-\cos x$	$\sin x$

BY PARTS AGAIN ON THIS SECTION

$2\cos 2x$	$-4\sin 2x$
$\sin x$	$\cos x$

$\int \sin 2x \sin x \, dx = -\cos x \sin 2x + \left[2\sin x \cos 2x - \int -4\sin x \sin 2x \, dx\right]$
$\int \sin 2x \sin x \, dx = -\cos x \sin 2x + 2\sin x \cos 2x + 4\int \sin 2x \sin x \, dx$
$\cos x \sin 2x - 2\sin x \cos 2x = 3\int \sin 2x \sin x \, dx$
$\int \sin 2x \sin x \, dx = \frac{1}{3}\cos x \sin 2x - \frac{2}{3}\sin x \cos 2x + C$

209. $$\int x\cos^2 x \; dx = \frac{1}{4}x^2 + \frac{1}{4}x\sin 2x + \frac{1}{8}\cos 2x + C$$



$\int x\cos^2 x \, dx = \int x\left(\frac{1}{2} + \frac{1}{2}\cos 2x\right) dx = \int \frac{1}{2}x + \underbrace{\frac{1}{2}x\cos 2x}_{\text{BY PARTS}} \, dx$

$\frac{1}{2}x$	$\frac{1}{2}$
$\frac{1}{2}\sin 2x$	$\cos 2x$

$= \frac{1}{4}x^2 + \left[\frac{1}{4}x\sin 2x - \int \frac{1}{4}\sin 2x \, dx\right]$
$= \frac{1}{4}x^2 + \frac{1}{4}x\sin 2x + \frac{1}{8}\cos 2x + C$

210. $$\int \sin(x+1)^{\frac{1}{3}}\ dx = 3\left[2-(x+1)^{\frac{2}{3}}\right]\cos(x+1)^{\frac{1}{3}} + 6(x+1)^{\frac{1}{3}}\sin(x+1)^{\frac{1}{3}} + C$$

$\int \sin(x+1)^{\frac{1}{3}}\, dx = \ldots$ BY SUBSTITUTION FIRST …

$u=(x+1)^{\frac{1}{3}}$, $u^3 = x+1$, $x = u^3-1$, $\frac{dx}{du}=3u^2$, $dx = 3u^2\, du$

$= \int \sin u\,(3u^2\, du) = \int 3u^2 \sin u\, du$

BY PARTS NEXT

$$\begin{array}{c|c} 3u^2 & 6u \\ \hline -\cos u & \sin u \end{array}$$

$= -3u^2\cos u - \int -6u\cos u\, du$

$= -3u^2\cos u + \int 6u\cos u\, du$

BY PARTS AGAIN FOR THIS INTEGRAL

$$\begin{array}{c|c} 6u & 6 \\ \hline \sin u & \cos u \end{array}$$

$= -3u^2\cos u + \left[6u\sin u - \int 6\sin u\, du\right]$

$= -3u^2\cos u + 6u\sin u + 6\cos u + C$

$= (6-3u^2)\cos u + 6u\sin u + C$

$= \left[6-3(x+1)^{\frac{2}{3}}\right]\cos(x+1)^{\frac{1}{3}} + 6(x+1)^{\frac{1}{3}}\sin(x+1)^{\frac{1}{3}} + C$

$= 3\left[2-(x+1)^{\frac{2}{3}}\right]\cos(x+1)^{\frac{1}{3}} + 6(x+1)^{\frac{1}{3}}\sin(x+1)^{\frac{1}{3}} + C$

211. $$\int \frac{1-\ln x}{x^2}\ dx = \frac{\ln|x|}{x} + C$$

$\int \frac{1-\ln x}{x^2}\, dx = \int \frac{1}{x^2} - \frac{\ln x}{x^2}\, dx = \int \frac{1}{x^2} - \underbrace{x^{-2}\ln x}_{\text{BY PARTS}}\, dx$

$$\begin{array}{c|c} \ln x & \frac{1}{x} \\ \hline x^{-1} & -x^{-2} \end{array}$$

$= -\frac{1}{x} + \left[x^{-1}\ln|x| - \int \frac{1}{x}x^{-1}\, dx\right]$

$= -\frac{1}{x} + \frac{\ln|x|}{x} - \int \frac{1}{x^2}\, dx$

$= \cancel{-\frac{1}{x}} + \frac{\ln|x|}{x} + \cancel{\frac{1}{x}} + C$

$= \frac{\ln|x|}{x} + C$

ALTERNATIVE, STARTING WITH A SUBSTITUTION

$u = \ln x$, $\frac{du}{dx} = \frac{1}{x}$, $dx = x\, du$, $x = e^u$, $\frac{1}{x} = e^{-u}$

$\int \frac{1-\ln x}{x^2}\, dx = \int \frac{1-u}{x^2}\, x\, du$

$= \int \frac{1-u}{x}\, du$

$= \int (1-u)e^{-u}\, du$

BY PARTS

$$\begin{array}{c|c} 1-u & -1 \\ \hline -e^{-u} & e^{-u} \end{array}$$

$= -(1-u)e^{-u} - \int e^{-u}\, du$

$= (u-1)e^{-u} + e^{-u} + C$

$= ue^{-u} \cancel{-e^{-u}} \cancel{+e^{-u}} + C$

$= (\ln x) \times \frac{1}{x} + C$

$= \frac{\ln|x|}{x} + C$

212. $$\int 4xe^{-\frac{2}{3}x}\ dx = -3(2x+3)e^{-\frac{2}{3}x} + C$$

$\int 4xe^{-\frac{2}{3}x}\, dx = \ldots$ INTEGRATION BY PARTS …

$$\begin{array}{c|c} 4x & 4 \\ \hline -\frac{3}{2}e^{-\frac{2}{3}x} & e^{-\frac{2}{3}x} \end{array}$$

$= -6xe^{-\frac{2}{3}x} - \int -6e^{-\frac{2}{3}x}\, dx$

$= -6xe^{-\frac{2}{3}x} + \int 6e^{-\frac{2}{3}x}\, dx = -6xe^{-\frac{2}{3}x} - 9e^{-\frac{2}{3}x} + C$

$= -3(2x+3)e^{-\frac{2}{3}x} + C$

213. $\displaystyle\int \left(e^{x}+x\right)^{2}\,dx=\frac{1}{2}e^{2x}+2x\,e^{x}-2e^{x}+\frac{1}{3}x^{3}+C$

$$\int \left(e^{x}+x\right)^{2}dx=\int e^{2x}+\underbrace{2xe^{x}}_{\text{BY PARTS}}+x^{2}\,dx$$

$$\boxed{\begin{array}{c|c} 2x & 2 \\ \hline e^{x} & e^{x} \end{array}}$$

$$=\frac{1}{2}e^{2x}+\left[2xe^{x}-\int 2e^{x}\,dx\right]+\frac{1}{3}x^{3}$$

$$=\frac{1}{2}e^{2x}+2xe^{x}-2e^{x}+\frac{1}{3}x^{3}+C$$

214. $\displaystyle\int \frac{e^{4x}-e^{-x}}{e^{2x}}\,dx=\frac{1}{2}e^{2x}+\frac{1}{3}e^{-3x}+C$

$$\int \frac{e^{4x}-e^{-x}}{e^{2x}}\,dx=\int \frac{e^{4x}}{e^{2x}}-\frac{e^{-x}}{e^{2x}}\,dx=\int e^{2x}-e^{-3x}\,dx$$

$$=\frac{1}{2}e^{2x}+\frac{1}{3}e^{-3x}+C$$

215. $\displaystyle\int \frac{e^{\ln x}}{x}\,dx=x+C$

$$\int \frac{e^{\ln x}}{x}\,dx=\int \frac{x}{x}\,dx=\int 1\,dx=x+C$$

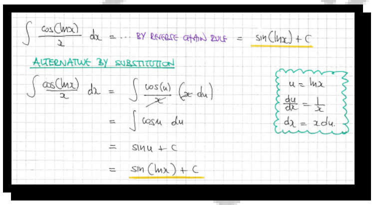
216. $\displaystyle\int \frac{1}{2(3x+1)^{4}}\,dx=-\frac{1}{18(3x+1)^{3}}+C$

$$\int \frac{1}{2(3x+1)^{4}}\,dx=\int \frac{1}{2}(3x+1)^{-4}\,dx=\dots \text{ BY INSPECTION}$$

$$=\frac{\frac{1}{2}}{-9}(3x+1)^{-3}+C=-\frac{1}{18}(3x+1)^{-3}+C$$

$$=-\frac{1}{18(3x+1)^{3}}+C$$

217. $\displaystyle\int \frac{\cos(\ln x)}{x}\,dx = \sin(\ln x) + C$


$\displaystyle\int \frac{\cos(\ln x)}{x}\,dx = \ldots$ BY REVERSE CHAIN RULE $= \sin(\ln x) + C$

ALTERNATIVE BY SUBSTITUTION

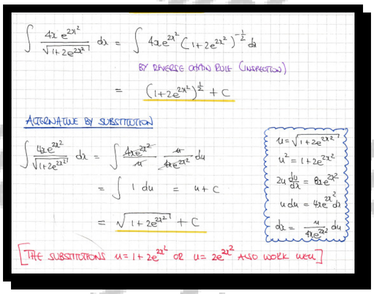
$\displaystyle\int \frac{\cos(\ln x)}{x}\,dx = \int \frac{\cos(u)}{\cancel{x}}(\cancel{x}\,du) = \int \cos u\,du = \sin u + C = \sin(\ln x) + C$

$u = \ln x$

$\frac{du}{dx} = \frac{1}{x}$

$dx = x\,du.$

218. $\displaystyle\int \frac{4x\mathrm{e}^{2x^2}}{\sqrt{1+2\mathrm{e}^{2x^2}}}\,dx = \sqrt{1+2\mathrm{e}^{2x^2}} + C$


$\displaystyle\int \frac{4x e^{2x^2}}{\sqrt{1+2e^{2x^2}}}\,dx = \int 4x e^{2x^2}\left(1+2e^{2x^2}\right)^{-\frac{1}{2}}dx$

BY REVERSE CHAIN RULE (INSPECTION)

$= \left(1+2e^{2x^2}\right)^{\frac{1}{2}} + C$

ALTERNATIVE BY SUBSTITUTION

$\displaystyle\int \frac{4x e^{2x^2}}{\sqrt{1+2e^{2x^2}}}\,dx = \int \frac{\cancel{4xe^{2x^2}}}{\cancel{u}}\,\frac{\cancel{u}}{\cancel{4xe^{2x^2}}}\,du = \int 1\,du = u + C = \sqrt{1+2e^{2x^2}} + C$

$u = \sqrt{1+2e^{2x^2}}$

$u^2 = 1+2e^{2x^2}$

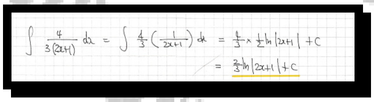
$2u\frac{du}{dx} = 8xe^{2x^2}$

$u\,du = 4xe^{2x^2}dx$

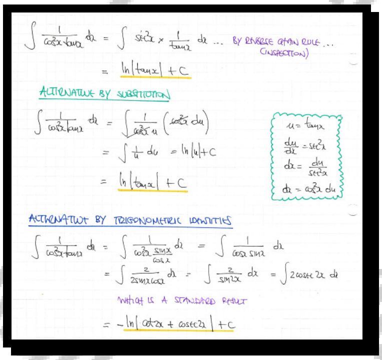
$dx = \frac{u}{4xe^{2x^2}}du$

[THE SUBSTITUTIONS $u = 1+2e^{2x^2}$ OR $u = 2e^{2x^2}$ ALSO WORK WELL]

219. $\displaystyle\int \frac{4}{3(2x+1)}\,dx = \frac{2}{3}\ln|2x+1| + C$


$\displaystyle\int \frac{4}{3(2x+1)}\,dx = \int \frac{4}{3}\left(\frac{1}{2x+1}\right)dx = \frac{4}{3}\times\frac{1}{2}\ln|2x+1| + C = \frac{2}{3}\ln|2x+1| + C$

220. $$\int \frac{1}{\cos^2 x \tan x}\, dx = \begin{bmatrix} \ln|\tan x| + C \\ -\ln|\cot 2x + \operatorname{cosec} 2x| + C \end{bmatrix}$$



$\int \frac{1}{\cos^2 x \tan x}\, dx = \int \sec^2 x \times \frac{1}{\tan x}\, dx \ldots$ BY REVERSE CHAIN RULE … (INSPECTION)

$= \ln|\tan x| + C$

ALTERNATIVE BY SUBSTITUTION

$\int \frac{1}{\cos^2 x \tan x}\, dx = \int \frac{1}{\cos^2 x\, u}\left(\cos^2 x\, du\right)$

$= \int \frac{1}{u}\, du = \ln|u| + C$

$= \ln|\tan x| + C$

$u = \tan x$

$\frac{du}{dx} = \sec^2 x$

$dx = \frac{du}{\sec^2 x}$

$dx = \cos^2 x\, du$

ALTERNATIVE BY TRIGONOMETRIC IDENTITIES

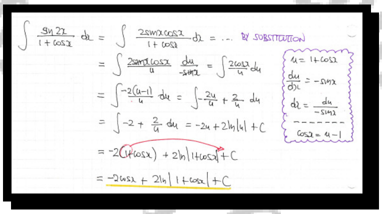
$\int \frac{1}{\cos^2 x \tan x}\, dx = \int \frac{1}{\cos^2 x \frac{\sin x}{\cos x}}\, dx = \int \frac{1}{\cos x \sin x}\, dx$

$= \int \frac{2}{2\sin x \cos x}\, dx = \int \frac{2}{\sin 2x}\, dx = \int 2\operatorname{cosec} 2x\, dx$

WHICH IS A STANDARD RESULT

$= -\ln|\cot 2x + \operatorname{cosec} 2x| + C$

221. $$\int \frac{\sin 2x}{1+\cos x}\, dx = -2\cos x + 2\ln|1+\cos x| + C$$



$\int \frac{\sin 2x}{1+\cos x}\, dx = \int \frac{2\sin x \cos x}{1+\cos x}\, dx = \ldots$ BY SUBSTITUTION

$= \int \frac{2\sin x \cos x}{u}\, \frac{du}{-\sin x} = \int \frac{2\cos x}{u}\, du$

$= \int \frac{-2(u-1)}{u}\, du = \int -\frac{2u}{u} + \frac{2}{u}\, du$

$= \int -2 + \frac{2}{u}\, du = -2u + 2\ln|u| + C$

$= -2(1+\cos x) + 2\ln|1+\cos x| + C$

$= -2\cos x + 2\ln|1+\cos x| + C$

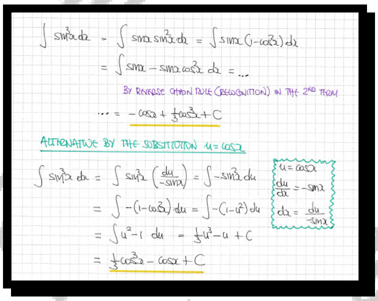
$u = 1+\cos x$

$\frac{du}{dx} = -\sin x$

$dx = \frac{du}{-\sin x}$

$\cos x = u-1$

222. $$\int \sin^3 x\, dx = \tfrac{1}{3}\cos^2 x - \cos x + C$$



$\int \sin^3 x\, dx = \int \sin x \sin^2 x\, dx = \int \sin x (1-\cos^2 x)\, dx$

$= \int \sin x - \sin x \cos^2 x\, dx = \ldots$

BY REVERSE CHAIN RULE (RECOGNITION) IN THE 2ND TERM

$\ldots = -\cos x + \frac{1}{3}\cos^3 x + C$

ALTERNATIVE BY THE SUBSTITUTION $u = \cos x$

$\int \sin^3 x\, dx = \int \sin^3 x \left(\frac{du}{-\sin x}\right) = \int -\sin^2 x\, du$

$= \int -(1-\cos^2 x)\, du = \int -(1-u^2)\, du$

$= \int u^2 - 1\, du = \frac{1}{3}u^3 - u + C$

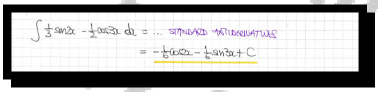
$= \frac{1}{3}\cos^3 x - \cos x + C$

$u = \cos x$

$\frac{du}{dx} = -\sin x$

$dx = \frac{du}{-\sin x}$

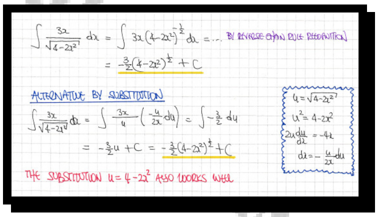
223. $$\int \tfrac{1}{3}\sin 2x - \tfrac{1}{2}\cos 3x \ dx = -\tfrac{1}{6}\cos 2x - \tfrac{1}{6}\sin 3x + C$$



$$\int \tfrac{1}{3}\sin 2x - \tfrac{1}{2}\cos 3x \, dx = \ldots$$ STANDARD ANTIDERIVATIVES

$$= -\tfrac{1}{6}\cos 2x - \tfrac{1}{6}\sin 3x + C$$

224. $$\int \frac{3x}{\sqrt{4-2x^2}} \ dx = -\frac{3}{2}\left(4-2x^2\right)^{\frac{1}{2}} + C$$



$$\int \frac{3x}{\sqrt{4-2x^2}}\,dx = \int 3x\left(4-2x^2\right)^{-\frac{1}{2}}dx = \ldots$$ BY REVERSE CHAIN RULE RECOGNITION

$$= -\tfrac{3}{2}\left(4-2x^2\right)^{\frac{1}{2}} + C$$

ALTERNATIVE BY SUBSTITUTION

$$\int \frac{3x}{\sqrt{4-2x^2}}\,dx = \int \frac{3x}{u}\left(-\frac{u}{2x}\,du\right) = \int -\tfrac{3}{2}\,du$$

$$= -\tfrac{3}{2}u + C = -\tfrac{3}{2}\left(4-2x^2\right)^{\frac{1}{2}} + C$$

$u = \sqrt{4-2x^2}$

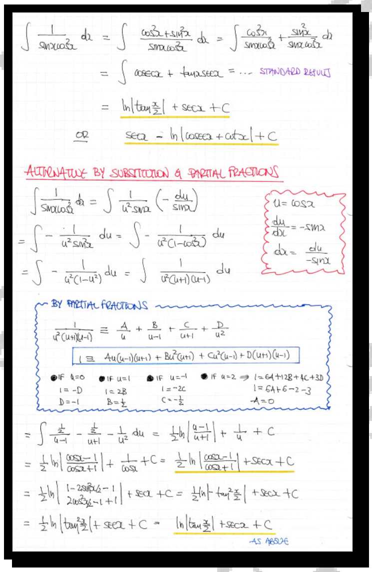
$u^2 = 4-2x^2$

$2u\frac{du}{dx} = -4x$

$dx = -\frac{u}{2x}\,du$

THE SUBSTITUTION $u = 4-2x^2$ ALSO WORKS WELL

225. $$\int \frac{1}{\sin x\cos^2 x} \ dx = \left[\begin{array}{c} \ln\left|\tan\left(\tfrac{1}{2}x\right)\right| + \sec x + C \\ \sec x - \ln\left|\operatorname{cosec} x + \cot x\right| + C \\ \tfrac{1}{2}\ln\left|\dfrac{\cos x - 1}{\cos x + 1}\right| + \sec x + C \end{array}\right]$$



$$\int \frac{1}{\sin x\cos^2 x}\,dx = \int \frac{\cos^2 x + \sin^2 x}{\sin x\cos^2 x}\,dx = \int \frac{\cos^2 x}{\sin x\cos^2 x} + \frac{\sin^2 x}{\sin x\cos^2 x}\,dx$$

$$= \int \operatorname{cosec} x + \tan x\sec x = \ldots$$ STANDARD RESULTS

$$= \ln\left|\tan\tfrac{x}{2}\right| + \sec x + C$$

OR $$\sec x - \ln\left|\operatorname{cosec} x + \cot x\right| + C$$

ALTERNATIVE BY SUBSTITUTION & PARTIAL FRACTIONS

$$\int \frac{1}{\sin x\cos^2 x}\,dx = \int \frac{1}{u^2\sin x}\left(-\frac{du}{\sin x}\right)$$

$$= \int -\frac{1}{u^2\sin^2 x}\,du = \int -\frac{1}{u^2(1-\cos^2 x)}\,du$$

$$= \int -\frac{1}{u^2(1-u^2)}\,du = \int \frac{1}{u^2(u+1)(u-1)}\,du$$

$u = \cos x$

$\frac{du}{dx} = -\sin x$

$dx = \frac{du}{-\sin x}$

BY PARTIAL FRACTIONS

$$\frac{1}{u^2(u+1)(u-1)} \equiv \frac{A}{u} + \frac{B}{u-1} + \frac{C}{u+1} + \frac{D}{u^2}$$

$$1 \equiv Au(u-1)(u+1) + Bu^2(u+1) + Cu^2(u-1) + D(u+1)(u-1)$$

- IF $u=0$: $1 = -D$, $D = -1$
- IF $u=1$: $1 = 2B$, $B = \frac{1}{2}$
- IF $u=-1$: $1 = -2C$, $C = -\frac{1}{2}$
- IF $u=2 \Rightarrow 1 = 6A + 12B + 4C + 3D$, $1 = 6A + 6 - 2 - 3$, $A = 0$

$$= \int \frac{\frac{1}{2}}{u-1} - \frac{\frac{1}{2}}{u+1} - \frac{1}{u^2}\,du = \tfrac{1}{2}\ln\left|\frac{u-1}{u+1}\right| + \frac{1}{u} + C$$

$$= \tfrac{1}{2}\ln\left|\frac{\cos x - 1}{\cos x + 1}\right| + \frac{1}{\cos x} + C = \tfrac{1}{2}\ln\left|\frac{\cos x - 1}{\cos x + 1}\right| + \sec x + C$$

$$= \tfrac{1}{2}\ln\left|\frac{1 - 2\sin^2\frac{x}{2} - 1}{2\cos^2\frac{x}{2} - 1 + 1}\right| + \sec x + C = \tfrac{1}{2}\ln\left|-\tan^2\tfrac{x}{2}\right| + \sec x + C$$

$$= \tfrac{1}{2}\ln\left|\tan^2\tfrac{x}{2}\right| + \sec x + C = \ln\left|\tan\tfrac{x}{2}\right| + \sec x + C$$

AS ABOVE

226. $\displaystyle\int \frac{3x}{4-2x^2}\,dx = -\frac{3}{4}\ln\left|4-2x^2\right| + C$

$$\int \frac{3x}{4-2x^2}\,dx = -\frac{3}{4}\int \frac{-4x}{4-2x^2}\,dx = \dots \quad \int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + C$$

$$= -\frac{3}{4}\ln|4-2x^2| + C$$

THE SUBSTITUTION $u = 4-2x^2$ ALSO WORKS WELL

227. $\displaystyle\int \frac{1-4x}{x(4x-\ln x)^{\frac{3}{2}}}\,dx = \frac{2}{\sqrt{4x-\ln x}} + C$

$$\int \frac{1-4x}{x(4x-\ln x)^{\frac{3}{2}}}\,dx = \int \left(\frac{1-4x}{x}\right)(4x-\ln x)^{-\frac{3}{2}}\,dx = \int -\left(4-\frac{1}{x}\right)(4x-\ln x)^{-\frac{3}{2}}\,dx$$

BY REVERSE CHAIN RULE / RECOGNITION

$$= \frac{-1}{-\frac{1}{2}}(4x-\ln x)^{-\frac{1}{2}} + C = \frac{2}{\sqrt{4x-\ln x}} + C$$

ALTERNATIVE BY SUBSTITUTION

$$\int \frac{1-4x}{x(4x-\ln x)^{\frac{3}{2}}}\,dx = \int \frac{1-4x}{x u^{3/2}}\left(\frac{x}{4x-1}\,du\right)$$

$$= \int -\frac{1}{u^{3/2}}\,du = \int -u^{-\frac{3}{2}}\,du = 2u^{-\frac{1}{2}} + C$$

$$= 2(4x-\ln x)^{-\frac{1}{2}} + C = \frac{2}{\sqrt{4x-\ln x}} + C$$

$u = 4x - \ln x$

$\frac{du}{dx} = 4 - \frac{1}{x}$

$\frac{du}{dx} = \frac{4x-1}{x}$

$dx = \frac{x}{4x-1}\,du$

THE SUBSTITUTION $u = \sqrt{4x-\ln x}$ ALSO WORKS WELL

228. $\displaystyle\int \frac{\sin x}{\cos^4 x}\,dx = \frac{1}{3}\sec^3 x + C$

$$\int \frac{\sin x}{\cos^4 x}\,dx = \int (\cos x)^{-4}\sin x\,dx = \dots \text{ BY REVERSE CHAIN RULE} \dots$$

$$= \frac{1}{3}(\cos x)^{-3} + C = \frac{1}{3\cos^3 x} + C = \frac{1}{3}\sec^3 x + C$$

ALTERNATIVE VARIATION OF REVERSE CHAIN RULE

$$\int \frac{\sin x}{\cos^4 x}\,dx = \int \frac{\sin x}{\cos x}\frac{1}{\cos^3 x}\,dx = \int \sec^3 x\tan x\,dx$$

$$= \int \sec^2 x(\sec x\tan x)\,dx = \frac{1}{3}\sec^3 x + C$$

$\frac{d}{dx}(\sec x) = \sec x\tan x$

THE SUBSTITUTIONS $u = \cos x$ OR $u = \sec x$ ALSO WORK WELL

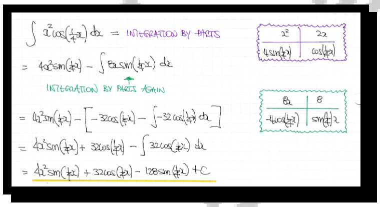
229. $\displaystyle\int \frac{\text{cosec}^2 x}{1+\cot x}\,dx = -\ln\left|1+\cot x\right| + C$

$$\int \frac{\text{cosec}^2 x}{1+\cot x}\,dx = -\int \frac{-\text{cosec}^2 x}{1+\cot x}\,dx = \dots \quad \int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + C$$

$$= -\ln|1+\cot x| + C$$

THE SUBSTITUTION $u = 1+\cot x$ ALSO WORKS WELL

230. $\displaystyle\int x^2\cos\left(\tfrac{1}{4}x\right)\,dx = 4x^2\sin\left(\tfrac{1}{4}x\right)+32\cos\left(\tfrac{1}{4}x\right)-128\sin\left(\tfrac{1}{4}x\right)+C$



$\int x^2\cos(\frac{1}{4}x)\,dx =$ INTEGRATION BY PARTS

x^2	$2x$
$4\sin(\frac{1}{4}x)$	$\cos(\frac{1}{4}x)$

$= 4x^2\sin(\frac{1}{4}x) - \int 8x\sin(\frac{1}{4}x)\,dx$

INTEGRATION BY PARTS AGAIN

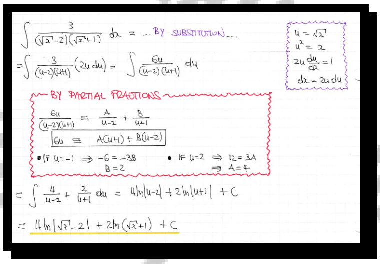
$8x$	8
$-4\cos(\frac{1}{4}x)$	$\sin(\frac{1}{4}x)$

$= 4x^2\sin(\frac{1}{4}x) - \left[-32\cos(\frac{1}{4}x) - \int -32\cos(\frac{1}{4}x)\,dx\right]$

$= 4x^2\sin(\frac{1}{4}x) + 32\cos(\frac{1}{4}x) - \int 32\cos(\frac{1}{4}x)\,dx$

$= 4x^2\sin(\frac{1}{4}x) + 32\cos(\frac{1}{4}x) - 128\sin(\frac{1}{4}x) + C$

231. $\displaystyle\int \frac{3}{\left(\sqrt{x}-2\right)\left(\sqrt{x}+1\right)}\,dx = 4\ln\left|\sqrt{x}-2\right|+2\ln\left(\sqrt{x}+1\right)+C$



$\int \frac{3}{(\sqrt{x}-2)(\sqrt{x}+1)}\,dx = \ldots$ BY SUBSTITUTION...

$u=\sqrt{x}$, $u^2=x$, $2u\frac{du}{dx}=1$, $dx=2u\,du$

$= \int \frac{3}{(u-2)(u+1)}(2u\,du) = \int \frac{6u}{(u-2)(u+1)}\,du$

BY PARTIAL FRACTIONS

$\frac{6u}{(u-2)(u+1)} \equiv \frac{A}{u-2}+\frac{B}{u+1}$

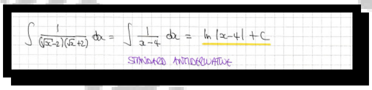
$\boxed{6u \equiv A(u+1)+B(u-2)}$

• IF $u=-1 \Rightarrow -6=-3B$, $B=2$ • IF $u=2 \Rightarrow 12=3A \Rightarrow A=4$

$= \int \frac{4}{u-2}+\frac{2}{u+1}\,du = 4\ln|u-2| + 2\ln|u+1| + C$

$= 4\ln|\sqrt{x}-2| + 2\ln(\sqrt{x}+1) + C$

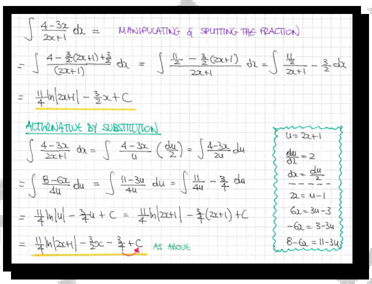
232. $\displaystyle\int \frac{1}{\left(\sqrt{x}-2\right)\left(\sqrt{x}+2\right)}\,dx = \ln|x-4|+C$



$\int \frac{1}{(\sqrt{x}-2)(\sqrt{x}+2)}\,dx = \int \frac{1}{x-4}\,dx = \ln|x-4|+C$

STANDARD ANTIDERIVATIVE

233. $\displaystyle\int \frac{4-3x}{2x+1}\,dx = \frac{11}{4}\ln|2x+1|-\frac{3}{2}x+C$



$\int \frac{4-3x}{2x+1}\,dx =$ MANIPULATING & SPLITTING THE FRACTION

$= \int \frac{4-\frac{3}{2}(2x+1)+\frac{3}{2}}{(2x+1)}\,dx = \int \frac{\frac{11}{2}-\frac{3}{2}(2x+1)}{2x+1}\,dx = \int \frac{\frac{11}{2}}{2x+1}-\frac{3}{2}\,dx$

$= \frac{11}{4}\ln|2x+1| - \frac{3}{2}x + C$

ALTERNATIVE BY SUBSTITUTION

$u=2x+1$, $\frac{du}{dx}=2$, $dx=\frac{du}{2}$; $2x=u-1$, $6x=3u-3$, $-6x=3-3u$, $8-6x=11-3u$

$\int \frac{4-3x}{2x+1}\,dx = \int \frac{4-3x}{u}\left(\frac{du}{2}\right) = \int \frac{4-3x}{2u}\,du$

$= \int \frac{8-6x}{4u}\,du = \int \frac{11-3u}{4u}\,du = \int \frac{11}{4u}-\frac{3}{4}\,du$

$= \frac{11}{4}\ln|u| - \frac{3}{4}u + C = \frac{11}{4}\ln|2x+1| - \frac{3}{4}(2x+1) + C$

$= \frac{11}{4}\ln|2x+1| - \frac{3}{2}x - \frac{3}{4} + C$ AS ABOVE

234. $\displaystyle\int \left(1+\frac{1}{x}\right)\sqrt{x}\ dx = \frac{2}{3}x^{\frac{3}{2}} + 2x^{\frac{1}{2}} + C$

$$\int \sqrt{x}\left(1+\tfrac{1}{x}\right)dx = \int x^{\frac{1}{2}}\left(1+x^{-1}\right)dx = \int x^{\frac{1}{2}} + x^{-\frac{1}{2}}\,dx$$
$$= \tfrac{2}{3}x^{\frac{3}{2}} + 2x^{\frac{1}{2}} + C$$

235. $\displaystyle\int \frac{\sec x}{\cos x - \sin x}\ dx = \ln\left|1-\tan x\right| + C$

$$\int \frac{\sec x}{\cos x - \sin x}\,dx = \int \frac{\sec x \sec x}{\cos x \sec x - \sin x \sec x}\,dx$$
$$= \int \frac{\sec^2 x}{1-\tan x}\,dx$$
$$\ldots \int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + C$$
$$= \ln|1-\tan x| + C$$

236. $\displaystyle\int \frac{x^2-2}{x^2-1}\ dx = x + \frac{1}{2}\ln\left|\frac{x+1}{x-1}\right| + C$

$$\int \frac{x^2-2}{x^2-1}\,dx = \int \frac{(x^2-1)-1}{x^2-1}\,dx = \int 1 - \frac{1}{x^2-1}\,dx$$

IMPROPER!

$$= \int 1 - \frac{1}{(x-1)(x+1)}\,dx = \ldots$$

PARTIAL FRACTIONS BY INSPECTION

$$= \int 1 - \left(\frac{\frac{1}{2}}{x-1} + \frac{-\frac{1}{2}}{x+1}\right)dx = \int 1 + \frac{\frac{1}{2}}{x+1} - \frac{\frac{1}{2}}{x-1}\,dx$$
$$= x + \tfrac{1}{2}\ln|x+1| - \tfrac{1}{2}\ln|x-1| + C = x + \tfrac{1}{2}\ln\left|\frac{x+1}{x-1}\right| + C$$

237. $\displaystyle\int \frac{1}{x^3-x^2}\ dx = \frac{1}{x} + \ln\left|\frac{x-1}{x}\right| + C$

$$\int \frac{1}{x^3-x^2}\,dx = \int \frac{1}{x^2(x-1)}\,dx = \ldots$$

BY PARTIAL FRACTIONS

$$\frac{1}{x^2(x-1)} \equiv \frac{A}{x^2} + \frac{B}{x-1} + \frac{C}{x}$$
$$\boxed{1 \equiv A(x-1) + Bx^2 + Cx(x-1)}$$

- IF $x=1$: $1 = B$
- IF $x=0$: $1 = -A$, $A = -1$
- IF $x=2$: $1 = A + 4B + 2C$, $1 = -1 + 4 + 2C$, $C = -1$

$$= \int \frac{1}{x-1} - \frac{1}{x} - \frac{1}{x^2}\,dx = \ln|x-1| - \ln|x| + \frac{1}{x} + C = \frac{1}{x} + \ln\left|\frac{x-1}{x}\right| + C$$

238. $\displaystyle\int \frac{6x^2}{2x^{\frac{3}{2}}-1}\,dx = 2x^{\frac{3}{2}} + \ln\left|2x^{\frac{3}{2}}-1\right| + C$

$$\int \frac{6x^2}{2x^{\frac{3}{2}}-1}\,dx = \int \frac{3x^{\frac{1}{2}}(2x^{\frac{3}{2}}-1)+3x^{\frac{1}{2}}}{(2x^{\frac{3}{2}}-1)}\,dx = \int 3x^{\frac{1}{2}} + \frac{3x^{\frac{1}{2}}}{2x^{\frac{3}{2}}-1}\,dx$$

$\int \frac{f'(x)}{f(x)}\,dx$

$$= \underline{2x^{\frac{3}{2}} + \ln|2x^{\frac{3}{2}}-1| + C}$$

ALTERNATIVE BY SUBSTITUTION

$$\int \frac{6x^2}{2x^{\frac{3}{2}}-1} = \int \frac{6x^2}{u}\frac{du}{3x^{\frac{1}{2}}} = \int \frac{2x^{\frac{3}{2}}}{u}\,du$$

$$= \int \frac{u+1}{u}\,du = \int 1+\frac{1}{u}\,du = u + \ln|u| + C$$

$$= (2x^{\frac{3}{2}}-1) + \ln|2x^{\frac{3}{2}}-1| + C = \underline{2x^{\frac{3}{2}} + \ln|2x^{\frac{3}{2}}-1| + C}$$

AS ABOVE

$u = 2x^{\frac{3}{2}}-1$

$\frac{du}{dx} = 3x^{\frac{1}{2}}$

$dx = \frac{du}{3x^{\frac{1}{2}}}$

$2x^{\frac{3}{2}} = u+1$

239. $\displaystyle\int \frac{1}{1+\sqrt{x-1}}\,dx = 2\sqrt{x-1} - 2\ln\left(1+\sqrt{x-1}\right) + C$

$$\int \frac{1}{1+\sqrt{x-1}}\,dx = \ldots \text{ BY SUBSTITUTION } \ldots$$

$$= \int \frac{1}{u}\left[2(u-1)\,du\right] = \int \frac{2u-2}{u}\,du$$

$$= \int 2 - \frac{2}{u}\,du = 2u - 2\ln|u| + C$$

$$= 2(1+\sqrt{x-1}) - 2\ln(1+\sqrt{x-1}) + C$$

$$= \underline{2\sqrt{x-1} - 2\ln(1+\sqrt{x-1}) + C}$$

$u = 1+\sqrt{x-1}$

$u-1 = \sqrt{x-1}$

$(u-1)^2 = x-1$

$x = 1+(u-1)^2$

$\frac{dx}{du} = 2(u-1)$

$dx = 2(u-1)\,du$

ALTERNATIVE SUBSTITUTION

$$\int \frac{1}{1+\sqrt{x-1}}\,dx = \int \frac{1}{1+u}(2u\,du) = \int \frac{2u}{u+1}\,du$$

MANIPULATE TO SPLIT TO AVOID A 2ND SUBSTITUTION

$$= \int \frac{2(u+1)-2}{u+1}\,du = \int 2 - \frac{2}{u+1}\,du$$

$$= 2u - 2\ln|u+1| + C = \underline{2\sqrt{x-1} - 2\ln(1+\sqrt{x-1}) + C}$$

$u = \sqrt{x-1}$

$u^2 = x-1$

$x = u^2+1$

$\frac{dx}{du} = 2u$

$dx = 2u\,du$

240. $\displaystyle\int \frac{x+1}{x-5}\,dx = x + 6\ln|x-5| + C$

$$\int \frac{x+1}{x-5}\,dx = \text{ BY MANIPULATION \& SPLIT}$$

$$= \int \frac{(x-5)+6}{x-5}\,dx = \int 1 + \frac{6}{x-5}\,dx$$

$$= \underline{x + 6\ln|x-5| + C}$$

ALTERNATIVE BY SUBSTITUTION $u = x-5$

$$\int \frac{x+1}{x-5}\,dx = \int \frac{x+1}{u}\,du = \int \frac{(u+5)+1}{u}$$

$$= \int \frac{u+6}{u}\,du = \int 1 + \frac{6}{u}\,du$$

$$= u + 6\ln|u| + C$$

$$= \underline{(x-5) + 6\ln|x-5| + C}$$

$u = x-5$

$\frac{du}{dx} = 1$

$dx = du$

$x = u+5$

241. $\displaystyle\int \frac{(x+2)^2}{3x}\,dx = \tfrac{1}{6}x^2 + \tfrac{4}{3}x + \tfrac{4}{3}\ln|x| + C$

$$\int \frac{(x+2)^2}{3x}\,dx = \int \frac{x^2+4x+4}{3x}\,dx = \int \tfrac{1}{3}x + \tfrac{4}{3} + \tfrac{4}{3x}\,dx$$

... STANDARD ANTIDERIVATIVES ...

$$= \tfrac{1}{6}x^2 + \tfrac{4}{3}x + \tfrac{4}{3}\ln|x| + C$$

242. $\displaystyle\int 4e^{-2x} - \tfrac{1}{3}\sin 3x\,dx = -2e^{-2x} + \tfrac{1}{9}\cos 3x + C$

$\displaystyle\int 4e^{-2x} - \tfrac{1}{3}\sin 3x\,dx =$ ... STANDARD ANTIDERIVATIVES ...

$$= -2e^{-2x} + \tfrac{1}{9}\cos 3x + C$$

243. $\displaystyle\int 2x\sec^2 2x\,dx = \begin{bmatrix} x\tan x + \tfrac{1}{2}\ln|\cos 2x| + C \\ x\tan x - \tfrac{1}{2}\ln|\sec 2x| + C \end{bmatrix}$

$\displaystyle\int 2x\sec^2 2x\,dx =$ ... INTEGRATION BY PARTS ...

$2x$	2
$\tfrac{1}{2}\tan 2x$	$\sec^2 2x$

$$= x\tan 2x - \int \tan 2x\,dx$$

STANDARD RESULT WHICH CAN EASILY BE PROVEN

$$\int \tan x\,dx = \ln|\sec x| + C = -\ln|\cos x| + C$$

$$= x\tan x + \tfrac{1}{2}\ln|\cos 2x| + C = x\tan x - \tfrac{1}{2}\ln|\sec 2x| + C$$

244. $\displaystyle\int \cos x\sin^8 x\,dx = \tfrac{1}{9}\sin^9 x + C$

$\displaystyle\int \cos x\sin^8 x\,dx =$ ... BY REVERSE CHAIN RULE (RECOGNITION)

$$= \tfrac{1}{9}\sin^9 x + C$$

[THE SUBSTITUTION $u = \sin x$ ALSO WORKS WELL]

245. $\displaystyle\int \frac{\sin^5 x}{\cos^7 x}\,dx = \frac{1}{6}\tan^6 x + C$

$$\int \frac{\sin^5 x}{\cos^7 x}\,dx = \int \frac{\sin^5 x}{\cos^5 x} \times \frac{1}{\cos^2 x}\,dx = \int \tan^5 x \sec^2 x\,dx$$

$= \dots$ BY REVERSE CHAIN RULE (RECOGNITION)

$= \underline{\frac{1}{6}\tan^6 x + C}$

[THE SUBSTITUTION $u = \tan x$, ALSO WORKS WELL]

246. $\displaystyle\int \tan 3x\,dx = \begin{bmatrix} \frac{1}{3}\ln|\sec x| + C \\ -\frac{1}{3}\ln|\cos x| + C \end{bmatrix}$

$$\int \tan 3x\,dx = \int \frac{\sin 3x}{\cos 3x}\,dx = -\frac{1}{3}\int \frac{-3\sin 3x}{\cos 3x}\,dx$$

$\displaystyle\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + C$

$= \underline{-\frac{1}{3}\ln|\cos 3x| + C} = \underline{\frac{1}{3}\ln|\sec 3x| + C}$

[THE SUBSTITUTION $u = \cos 3x$ OR $u = \sec 3x$ ALSO WORKS]

247. $\displaystyle\int \frac{1}{\sec x - 1}\,dx = -x - \cot x - \operatorname{cosec} x + C$

$$\int \frac{1}{\sec x - 1}\,dx = \int \frac{\sec x + 1}{(\sec x - 1)(\sec x + 1)}\,dx = \int \frac{\sec x + 1}{\sec^2 x - 1}\,dx$$

$= \displaystyle\int \frac{\sec x + 1}{\tan^2 x}$ $\quad [1 + \tan^2 x \equiv \sec^2 x]$

$= \displaystyle\int \frac{\sec x}{\tan^2 x} + \frac{1}{\tan^2 x}\,dx = \int \sec x \cot^2 x + \cot^2 x\,dx$

$= \displaystyle\int \frac{1}{\cos x}\frac{\cos^2 x}{\sin^2 x} + (\operatorname{cosec}^2 x - 1)\,dx$

$= \displaystyle\int \frac{\cos x}{\sin x}\frac{1}{\sin x} + \operatorname{cosec}^2 x - 1\,dx$

$= \displaystyle\int \cot x \operatorname{cosec} x + \operatorname{cosec}^2 x - 1\,dx$

$= \underline{-\operatorname{cosec} x - \cot x - x + C}$

$\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$ & $\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$

248. $\displaystyle\int \frac{\sin^2 x}{\cos^4 x}\,dx = \frac{1}{3}\tan^3 x + C$

$$\int \frac{\sin^2 x}{\cos^4 x}\,dx = \int \frac{\sin^2 x}{\cos^2 x}\frac{1}{\cos^2 x}\,dx = \int \tan^2 x \sec^2 x\,dx$$

BY REVERSE CHAIN RULE SINCE $\frac{d}{dx}(\tan x) = \sec^2 x$

$= \underline{\frac{1}{3}\tan^3 x + C}$

[THE SUBSTITUTION $u = \tan x$ ALSO WORKS WELL]

249. $\displaystyle\int \frac{\sin x \cos x}{1-\cos x}\, dx = \left[\begin{array}{c} \cos x + \ln|1-\cos x| + C \\ \cos x - \ln|\cot x + \operatorname{cosec} x| + C \end{array}\right]$

$$\int \frac{\sin x \cos x}{1-\cos x}\,dx = \dots \text{ BY SUBSTITUTION} \dots$$

$$\boxed{\begin{array}{l} u = 1-\cos x \\ \frac{du}{dx} = \sin x \\ dx = \frac{du}{\sin x} \\ \cos x = 1-u \end{array}}$$

$$= \int \frac{\cancel{\sin x}\cos x}{u}\,\frac{du}{\cancel{\sin x}} = \int \frac{1-u}{u}\,du$$

$$= \int \frac{1}{u} - 1\,du = \ln|u| - u + C$$

$$= \ln|1-\cos x| - (1-\cos x) + C$$

$$= \cos x + \ln|1-\cos x| + C$$

ALTERNATIVE BY TRIG MANIPULATIONS

$$\int \frac{\sin x\cos x}{1-\cos x}\,dx = \int \frac{\sin x\cos x(1+\cos x)}{(1-\cos x)(1+\cos x)}\,dx$$

$$= \int \frac{\sin x\cos x(1+\cos x)}{1-\cos^2 x}\,dx = \int \frac{\cancel{\sin x}\cos x(1+\cos x)}{\sin^2 x}\,dx$$

$$= \int \frac{\cos x + \cos^2 x}{\sin x}\,dx = \int \frac{\cos x + 1 - \sin^2 x}{\sin x}$$

$$= \int \cot x + \operatorname{cosec} x - \sin x\,dx = \dots \text{ STANDARD RESULTS}$$

$$= \underbrace{\ln|\sin x| - \ln|\operatorname{cosec} x + \cot x|}_{\ln|1-\cos x|} + \cos x + C$$

250. $\displaystyle\int \frac{(4x-1)^{-1}}{4}\, dx = -\frac{1}{16}\ln|4x-1| + C$

$$\int \frac{(4x-1)^{-1}}{4}\,dx = \int \frac{1}{4}\left(\frac{1}{4x-1}\right)dx \quad \text{STANDARD ANTIDERIVATIVE}$$

$$= \frac{1}{16}\ln|4x-1| + C$$

251. $\displaystyle\int \frac{e^{2x}-2e^{x}}{e^{x}+1}\, dx = e^{x} - 3\ln\left(e^{x}+1\right) + C$

$$\int \frac{e^{2x}-2e^x}{e^x+1}\,dx = \int \frac{e^x(e^x+1)-3e^x}{e^x+1}\,dx = \int e^x - \frac{3e^x}{e^x+1}\,dx$$

$$\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + C$$

$$= e^x - 3\ln(e^x+1) + C$$

ALTERNATIVE BY SUBSTITUTION

$$\boxed{\begin{array}{l} u = e^x+1 \\ \frac{du}{dx} = e^x \\ dx = \frac{du}{e^x} \\ e^x = u-1 \end{array}}$$

$$\int \frac{e^{2x}-2e^x}{e^x+1}\,dx = \int \frac{e^{2x}-2e^x}{u}\left(\frac{du}{e^x}\right)$$

$$= \int \frac{e^x-2}{u}\,du = \int \frac{(u-1)-2}{u}\,du$$

$$= \int \frac{u-3}{u}\,du = \int 1 - \frac{3}{u}\,du$$

$$= u - 3\ln|u| = e^x + 1 - 3\ln(e^x+1) + C$$

252. $\displaystyle\int \frac{2x^2-3x+2}{x-1}\,dx = x^2 - x + \ln|x-1| + C$

$\displaystyle\int \frac{2x^2-3x+2}{x-1}\,dx = \ldots$ BY LONG DIVISION OR MANIPULATION IN ORDER TO "SPLIT" THE FRACTION

$\displaystyle = \int \frac{2x(x-1)-(x-1)+1}{x-1}\,dx = \int 2x - 1 + \frac{1}{x-1}\,dx$

$= x^2 - x + \ln|x-1| + C$

[THE SUBSTITUTION $u = x-1$ ALSO WORKS WELL]

253. $\displaystyle\int 1-\cot^2 x\;\,dx = 2x + \cot x + C$

$\displaystyle\int 1-\cot^2 x\,dx = \int 1-(\operatorname{cosec}^2 x - 1)\,dx = \int 2 - \operatorname{cosec}^2 x\,dx$

$= 2x + \cot x + C$

$\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$

254. $\displaystyle\int \frac{x^2+1}{x^4-x^2}\,dx = \frac{1}{x} + \ln\left|\frac{x-1}{x+1}\right| + C$

$\displaystyle\int \frac{x^2+1}{x^4-x^2}\,dx = \int \frac{x^2+1}{x^2(x^2-1)}\,dx = \int \frac{x^2+1}{x^2(x+1)(x-1)}\,dx$

BY PARTIAL FRACTIONS

$\displaystyle\frac{x^2+1}{x^2(x+1)(x-1)} \equiv \frac{A}{x+1} + \frac{B}{x-1} + \frac{C}{x^2} + \frac{D}{x}$

$\boxed{x^2+1 \equiv Ax^2(x-1) + Bx^2(x+1) + C(x+1)(x-1) + Dx(x-1)(x+1)}$

- IF $x=0$: $1 = -C$, $C = -1$
- IF $x=1$: $2 = 2B$, $B = 1$
- IF $x=-1$: $2 = -2A$, $A = -1$
- IF $x=2$: $5 = 4A + 12B + 3C + 6D$, $5 = -4 + 12 - 3 + 6D$, $5 = 5 + 6D$, $D = 0$

$\displaystyle\ldots = \int \frac{1}{x-1} - \frac{1}{x+1} - \frac{1}{x^2}\,dx = \ln|x-1| - \ln|x+1| + \frac{1}{x} + C$

$\displaystyle = \frac{1}{x} + \ln\left|\frac{x-1}{x+1}\right| + C$

255. $\displaystyle\int \operatorname{cosec}^4 x \ dx = -\cot x - \tfrac{1}{3}\cot^3 x + C$

$$\int \operatorname{cosec}^4 x \, dx = \int \operatorname{cosec}^2 x \operatorname{cosec}^2 x \, dx = \int \operatorname{cosec}^2 x\left(1+\cot^2 x\right) dx$$
$$= \int \operatorname{cosec}^2 x + \operatorname{cosec}^2 x \cot^2 x \, dx$$

BY REVERSE CHAIN RULE (INSPECTION) AS $\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$

$$= -\cot x - \tfrac{1}{3}\cot^3 x + C$$

ALTERNATIVE BY SUBSTITUTION

$$\int \operatorname{cosec}^4 x \, dx = \int \operatorname{cosec}^4 x\left(\frac{du}{-\operatorname{cosec}^2 x}\right) = \int -\operatorname{cosec}^2 x \, du$$
$$= \int -\left(1+\cot^2 x\right) du = \int -1-u^2 \, du$$
$$= -u - \tfrac{1}{3}u^3 + C = -\cot x - \tfrac{1}{3}\cot^3 x + C$$

$u = \cot x$

$\frac{du}{dx} = -\operatorname{cosec}^2 x$

$dx = \frac{du}{-\operatorname{cosec}^2 x}$

256. $\displaystyle\int x^2 e^{\frac{1}{2}x} \ dx = 2x^2 e^{\frac{1}{2}x} - 8x e^{\frac{1}{2}x} + 16 e^{\frac{1}{2}x} + C$

$$\int x^2 e^{\frac{1}{2}x} dx = \dots \text{INTEGRATION BY PARTS}$$

x^2	$2x$
$2e^{\frac{1}{2}x}$	$e^{\frac{1}{2}x}$

$$= 2x^2 e^{\frac{1}{2}x} - \underbrace{\int 4x e^{\frac{1}{2}x} dx}_{\text{INTEGRATION BY PARTS AGAIN}}$$

$4x$	4
$2e^{\frac{1}{2}x}$	$e^{\frac{1}{2}x}$

$$= 2x^2 e^{\frac{1}{2}x} - \left[8x e^{\frac{1}{2}x} - \int 8 e^{\frac{1}{2}x} dx\right]$$
$$= 2x^2 e^{\frac{1}{2}x} - 8x e^{\frac{1}{2}x} + \int 8 e^{\frac{1}{2}x} dx$$
$$= 2x^2 e^{\frac{1}{2}x} - 8x e^{\frac{1}{2}x} + 16 e^{\frac{1}{2}x} + C = \left(2x^2 - 8x + 16\right) e^{\frac{1}{2}x} + C$$

257. $\displaystyle\int \left(e^x + 2e^{-x}\right)^2 \ dx = \tfrac{1}{2}e^{2x} + 4x - 2e^{-2x} + C$

$$\int \left(e^x + 2e^{-x}\right)^2 dx = \int \left(e^x\right)^2 + 2\left(e^x\right)\left(2e^{-x}\right) + \left(2e^{-x}\right)^2 dx$$
$$= \int e^{2x} + 4 + 4e^{-2x} dx$$
$$= \tfrac{1}{2}e^{2x} + 4x - 2e^{-2x} + C$$

258. $\displaystyle\int e^x \sin\left(e^x\right) \ dx = -\cos\left(e^x\right) + C$

$$\int e^x \sin\left(e^x\right) dx = \dots \text{BY REVERSE CHAIN RULE (INSPECTION)} \dots$$
$$= -\cos\left(e^x\right) + C$$

[THE SUBSTITUTION $u = e^x$ WORKS WELL ALSO]

259. $\displaystyle\int x\mathrm{e}^2 \, dx = \tfrac{1}{2}\mathrm{e}^2 x^2 + C$

$\int x e^2 \, dx = \frac{1}{2}x^2 e^2 + C = \frac{1}{2}e^2 x^2 + C$ [e^2 IS A CONSTANT]

260. $\displaystyle\int (2\cos x - 3)^2 \, dx = 11x + \sin 2x - 12\sin x + C$

$\int (2\cos x - 3)^2 \, dx = \int 4\cos^2 x - 12\cos x + 9 \, dx$

$= \int 4\left(\frac{1}{2} + \frac{1}{2}\cos 2x\right) - 12\cos x + 9 \, dx$

$= \int 11 + 2\cos 2x - 12\cos x \, dx$

$= 11x + \sin 2x - 12\sin x + C$

261. $\displaystyle\int \frac{x^2}{x-2} \, dx = \begin{bmatrix} \frac{1}{2}x^2 + 2x + 4\ln|x-2| + C \\ \frac{1}{2}(x-2)^2 + 4(x-2) + 4\ln|x-2| + C \end{bmatrix}$

$\int \frac{x^2}{x-2} \, dx = \ldots$ BY DIVISION/MANIPULATION IN ORDER TO SPLIT THE FRACTION

$= \int \frac{x(x-2) + 2(x-2) + 4}{x-2} \, dx = \int x + 2 + \frac{4}{x-2} \, dx$

$= \frac{1}{2}x^2 + 2x + 4\ln|x-2| + C$

ALTERNATIVE BY SUBSTITUTION

$u = x - 2$

$\frac{du}{dx} = 1$

$dx = du$

$x = u + 2$

$x^2 = (u+2)^2$

$x^2 = u^2 + 4u + 4$

$\int \frac{x^2}{x-2} \, dx = \int \frac{u^2 + 4u + 4}{u} \, du = \int u + 4 + \frac{4}{u} \, du$

$= \frac{1}{2}u^2 + 4u + 4\ln|u| + C$

$= \frac{1}{2}(x-2)^2 + 4(x-2) + 4\ln|x-2| + C$

262. $\displaystyle\int \frac{(x+1)\mathrm{e}^{\frac{1}{x}}}{x^3} \, dx = -\frac{1}{x}\mathrm{e}^{\frac{1}{x}} + C$

$\int \frac{(x+1)e^{\frac{1}{x}}}{x^3} \, dx = \ldots$ BY SUBSTITUTION ...

$u = \frac{1}{x}$

$\frac{du}{dx} = -\frac{1}{x^2}$

$dx = -x^2 \, du$

$= \int \frac{x+1}{x^3} e^u \left(-x^2 \, du\right) = \int -\left(\frac{x+1}{x}\right) e^u \, du$

$= \int -\left(1 + \frac{1}{x}\right) e^u \, du = \int -(1+u) e^u \, du$

INTEGRATION BY PARTS TO FOLLOW

$-(1+u)$	-1
e^u	e^u

$= -(1+u)e^u - \int -e^u \, du$

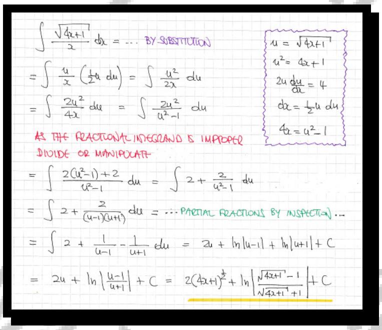
$= -(1+u)e^u + \int e^u \, du$

$= -(1+u)e^u + e^u + C$

$= e^u(-1 - u + 1) + C$

$= -ue^u + C = -\frac{1}{x}e^{\frac{1}{x}} + C$

263. $$\int \frac{\sqrt{4x+1}}{x}\,dx = 2(4x-1)^{\frac{1}{2}} + \ln\left|\frac{\sqrt{4x+1}-1}{\sqrt{4x+1}+1}\right| + C$$



$\int \frac{\sqrt{4x+1}}{x}\,dx = \dots$ BY SUBSTITUTION

$u = \sqrt{4x+1}$
$u^2 = 4x+1$
$2u\frac{du}{dx} = 4$
$dx = \frac{1}{2}u\,du$
$4x = u^2-1$

$= \int \frac{u}{x}\left(\frac{1}{2}u\,du\right) = \int \frac{u^2}{2x}\,du$

$= \int \frac{2u^2}{4x}\,du = \int \frac{2u^2}{u^2-1}\,du$

AS THE FRACTIONAL INTEGRAND IS IMPROPER DIVIDE OR MANIPULATE

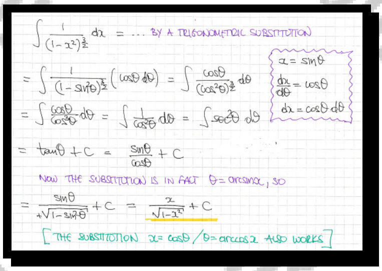
$= \int \frac{2(u^2-1)+2}{u^2-1}\,du = \int 2 + \frac{2}{u^2-1}\,du$

$= \int 2 + \frac{2}{(u-1)(u+1)}\,du = \dots$ PARTIAL FRACTIONS BY INSPECTION ...

$= \int 2 + \frac{1}{u-1} - \frac{1}{u+1}\,du = 2u + \ln|u-1| + \ln|u+1| + C$

$= 2u + \ln\left|\frac{u-1}{u+1}\right| + C = 2(4x+1)^{\frac{1}{2}} + \ln\left|\frac{\sqrt{4x+1}-1}{\sqrt{4x+1}+1}\right| + C$

264. $$\int \frac{1}{(1-x^2)^{\frac{3}{2}}}\,dx = \frac{x}{\sqrt{1-x^2}} + C$$



$\int \frac{1}{(1-x^2)^{\frac{3}{2}}}\,dx = \dots$ BY A TRIGONOMETRIC SUBSTITUTION

$x = \sin\theta$
$\frac{dx}{d\theta} = \cos\theta$
$dx = \cos\theta\,d\theta$

$= \int \frac{1}{(1-\sin^2\theta)^{\frac{3}{2}}}(\cos\theta\,d\theta) = \int \frac{\cos\theta}{(\cos^2\theta)^{\frac{3}{2}}}\,d\theta$

$= \int \frac{\cos\theta}{\cos^3\theta}\,d\theta = \int \frac{1}{\cos^2\theta}\,d\theta = \int \sec^2\theta\,d\theta$

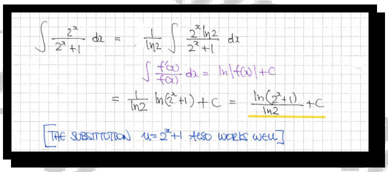
$= \tan\theta + C = \frac{\sin\theta}{\cos\theta} + C$

NOW THE SUBSTITUTION IS IN FACT $\theta = \arcsin x$, SO

$= \frac{\sin\theta}{+\sqrt{1-\sin^2\theta}} + C = \frac{x}{\sqrt{1-x^2}} + C$

[THE SUBSTITUTION $x = \cos\theta$ / $\theta = \arccos x$ ALSO WORKS]

265. $$\int \frac{2^x}{2^x+1}\,dx = \frac{\ln(2^x+1)}{\ln 2} + C$$



$\int \frac{2^x}{2^x+1}\,dx = \frac{1}{\ln 2}\int \frac{2^x\ln 2}{2^x+1}\,dx$

$\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + C$

$= \frac{1}{\ln 2}\ln(2^x+1) + C = \frac{\ln(2^x+1)}{\ln 2} + C$

[THE SUBSTITUTION $u = 2^x+1$ ALSO WORKS WELL]

266. $$\int (2x-1)\sqrt{2x-3}\ dx = \begin{bmatrix} \frac{1}{5}(2x-3)^{\frac{5}{2}} + \frac{2}{3}(2x-3)^{\frac{3}{2}} + C \\ \frac{1}{3}(2x-1)(2x-3)^{\frac{3}{2}} - \frac{2}{15}(2x-3)^{\frac{5}{2}} + C \end{bmatrix}$$

$\int (2x-1)\sqrt{2x-3}\,dx =$ ... BY SUBSTITUTION

$u = \sqrt{2x-3}$, $u^2 = 2x-3$, $2u\frac{du}{dx} = 2$, $dx = u\,du$, $2x = u^2+3$

$= \int (2x-1)u\,(u\,du) = \int u^2(u^2+3-1)\,du$

$= \int u^2(u^2+2)\,du = \int u^4 + 2u^2\,du$

$= \frac{1}{5}u^5 + \frac{2}{3}u^3 + C$

$= \frac{1}{5}(2x-3)^{\frac{5}{2}} + \frac{2}{3}(2x-3)^{\frac{3}{2}} + C$

[THE SUBSTITUTION $u = 2x-3$ ALSO WORKS]

ALTERNATIVE USING INTEGRATION BY PARTS

$2x-1$	2
$\frac{1}{3}(2x-3)^{\frac{3}{2}}$	$(2x-3)^{\frac{1}{2}}$

$\int (2x-1)\sqrt{2x-3}\,dx = \frac{1}{3}(2x-1)(2x-3)^{\frac{3}{2}} - \int \frac{2}{3}(2x-3)^{\frac{3}{2}}\,dx$

$= \frac{1}{3}(2x-1)(2x-3)^{\frac{3}{2}} - \frac{2}{15}(2x-3)^{\frac{5}{2}} + C$

ALTERNATIVE BY MANIPULATION

$\int (2x-1)\sqrt{2x-3}\,dx = \int [(2x-3)+2](2x-3)^{\frac{1}{2}}\,dx$

$= \int (2x-3)^{\frac{3}{2}} + 2(2x-3)^{\frac{1}{2}}\,dx = \frac{1}{5}(2x-3)^{\frac{5}{2}} + \frac{2}{3}(2x-3)^{\frac{3}{2}} + C$

AS ABOVE

267. $$\int \frac{9x^5}{\sqrt{x^3+1}}\ dx = \begin{bmatrix} 2\left(x^3+1\right)^{\frac{3}{2}} - 6\left(x^3+1\right)^{\frac{1}{2}} + C \\ 2x^3\left(x^3+1\right)^{\frac{1}{2}} - 4\left(x^3+1\right)^{\frac{3}{2}} + C \\ 2\left(x^3-2\right)\left(x^3+1\right)^{\frac{1}{2}} + C \end{bmatrix}$$

$\int \frac{9x^5}{\sqrt{x^3+1}}\,dx$ ... BY SUBSTITUTION

$u = \sqrt{x^3+1}$, $u^2 = x^3+1$, $2u\frac{du}{dx} = 3x^2$, $dx = \frac{2u}{3x^2}\,du$, $x^3 = u^2-1$, $6x^3 = 6u^2-6$

$= \int \frac{9x^5}{\cancel{u}}\left(\frac{2\cancel{u}}{3x^2}\,du\right) = \int 6x^3\,du$

$= \int 6u^2 - 6\,du = 2u^3 - 6u + C$

$= 2(x^3+1)^{\frac{3}{2}} - 6(x^3+1)^{\frac{1}{2}} + C$

$= 2(x^3+1)^{\frac{1}{2}}\left[(x^3+1) - 3\right] + C$

$= 2(x^3-2)(x^3+1)^{\frac{1}{2}} + C$

ALTERNATIVE BY INTEGRATION BY PARTS

$9x^3$	$27x^2$
$\frac{2}{3}(x^3+1)^{\frac{1}{2}}$	$x^2(x^3+1)^{-\frac{1}{2}}$

$\int \frac{9x^5}{\sqrt{x^3+1}}\,dx = \int (9x^3)\left(x^2(x^3+1)^{-\frac{1}{2}}\right)dx$

$= 6x^3(x^3+1)^{\frac{1}{2}} - \int 18x^2(x^3+1)^{\frac{1}{2}}\,dx$

$= 6x^3(x^3+1)^{\frac{1}{2}} - 4(x^3+1)^{\frac{3}{2}} + C$

$= 2(x^3+1)^{\frac{1}{2}}\left[3x^3 - 2(x^3+1)\right] + C$

$= 2(x^3+1)^{\frac{1}{2}}(x^3-2) + C$

ALTERNATIVE BY MANIPULATION & RECOGNITION

$\int \frac{9x^5}{\sqrt{x^3+1}}\,dx = \int \frac{9x^2(x^3+1) - 9x^2}{(x^3+1)^{\frac{1}{2}}}\,dx = \int 9x^2(x^3+1)^{\frac{1}{2}} - 9x^2(x^3+1)^{-\frac{1}{2}}\,dx$

$= 2(x^3+1)^{\frac{3}{2}} - 6(x^3+1)^{\frac{1}{2}} + C$ AS BY THE SUBSTITUTION METHOD

268. $$\int (3\sin x-\cos x)^2 \; dx = \begin{bmatrix} 5x-\sin 2x+\frac{3}{2}\cos 2x + C \\ 5x-\sin 2x-3\sin^2 x + C \\ 5x-\sin 2x+3\cos^2 x + C \end{bmatrix}$$

$\int (3\sin x-\cos x)^2 \, dx = \int 9\sin^2 x - 6\sin x\cos x + \cos^2 x \; dx$

$= \int 9\left(\frac{1}{2}-\frac{1}{2}\cos 2x\right) - 3\sin 2x + \left(\frac{1}{2}+\frac{1}{2}\cos 2x\right) dx$

$= \int 5 - 4\cos 2x - 3\sin 2x \; dx \; = \; 5x - 2\sin 2x + \frac{3}{2}\cos 2x + C$

VARIATION IF $\int -6\sin x\cos x \, dx = \begin{cases} -3\sin^2 x \\ 3\cos^2 x \end{cases}$

$\dots = \begin{cases} 5x - 2\sin 2x - 3\sin^2 x + C \\ 5x - 2\sin 2x + 3\cos^2 x + C \end{cases}$

269. $$\int \frac{4x}{x^2-10} \; dx = 2\ln\left|x^2-10\right| + C$$

$\int \frac{4x}{x^2-10} \, dx = 2\int \frac{2x}{x^2-10} \, dx = 2\ln|x^2-10| + C$

$\int \frac{f'(x)}{f(x)} \, dx = \ln|f(x)| + C$

[THE SUBSTITUTION $u=x^2-10$ (OR EVEN PARTIAL FRACTIONS!) WORK IN THIS QUESTION]

270. $$\int \frac{4\mathrm{e}^{3x}}{1-\mathrm{e}^{3x}} \; dx = -\frac{4}{3}\ln\left|1-\mathrm{e}^{3x}\right| + C$$

$\int \frac{4e^{3x}}{1-e^{3x}} \, dx = \frac{4}{-3}\int \frac{-3e^{3x}}{1-e^{3x}} \, dx = -\frac{4}{3}\ln|1-e^{3x}| + C$

$\int \frac{f'(x)}{f(x)} \, dx = \ln|f(x)| + C$

[THE SUBSTITUTION $u=1-e^{3x}$ ALSO WORKS WELL]

271. $$\int (1-\cos x)\sin x\cos x \; dx = \begin{bmatrix} \cos^3 x-\frac{1}{4}\cos 2x + C \\ \cos^3 x-\frac{1}{2}\cos^2 x + C \\ \cos^3 x+\frac{1}{2}\sin^2 x + C \end{bmatrix}$$

$\int \sin x\cos x(1-\cos x) \, dx = \int \sin x\cos x - \sin x\cos^2 x \; dx$

OR $\frac{1}{2}\sin 2x$

BY RECOGNITION (REVERSE CHAIN RULE)

$= \frac{1}{2}\sin^2 x + \cos^3 x + C = -\frac{1}{2}\cos^2 x + \cos^3 x + C = -\frac{1}{4}\cos 2x + \cos^3 x + C$

272. $\displaystyle\int (\tan x - 1)^2 \, dx = \begin{bmatrix} \tan x - 2\ln|\sec x| + C \\ \tan x + 2\ln|\cos x| + C \end{bmatrix}$

$$\int (\tan x - 1)^2 \, dx = \int \tan^2 x - 2\tan x + 1 \, dx$$

$$= \int \sec^2 x - 2\tan x \, dx = \text{... BOTH ARE STANDARD RESULTS}$$

$$= \tan x - 2\ln|\sec x| + C = \tan x + 2\ln|\cos x| + C$$

273. $\displaystyle\int \frac{1+e^x}{1-e^x} \, dx = x - 2\ln\left|1-e^x\right| + C$

$$\int \frac{1+e^x}{1-e^x} \, dx = \text{... BY SUBSTITUTION}$$

$u = 1 - e^x$

$\frac{du}{dx} = -e^x$

$dx = \frac{du}{-e^x}$

$e^x = 1 - u$

$1 + e^x = 2 - u$

$$\int \frac{2-u}{u} \, \frac{du}{-e^x} = \int \frac{2-u}{u} \, \frac{du}{u-1}$$

$$\int \frac{2-u}{u(u-1)} \, du \quad \text{... PARTIAL FRACTIONS BY INSPECTION (COVER UP)}$$

$$\int \frac{1}{u-1} - \frac{2}{u} \, du = \ln|u-1| - 2\ln|u| + C$$

$$= \ln|\cancel{1}-e^x\cancel{-1}| - 2\ln|1-e^x| + C = x - 2\ln|1-e^x| + C$$

ALTERNATIVE BY MANIPULATION

$$\int \frac{1+e^x}{1-e^x} \, dx = \int \frac{(1-e^x) + 2e^x}{1-e^x} \, dx = \int 1 + \frac{2e^x}{1-e^x} \, dx$$

$$= \int 1 - 2\left(\frac{-e^x}{1-e^x}\right) dx$$

$$\int \frac{f'(x)}{f(x)} \, dx = \ln|f(x)| + C$$

$$= x - 2\ln|1-e^x| + C$$

274. $\displaystyle\int \sin 2x \cos^4 2x \, dx = -\frac{1}{10}\cos^5 2x + C$

$$\int \sin 2x \cos^4 2x \, dx = \text{... BY REVERSE CHAIN RULE (INSPECTION) ...}$$

$$= -\frac{1}{10}\cos^5 2x + C$$

[THE SUBSTITUTION $u = \cos 2x$ ALSO WORKS WELL]

275. $$\int \frac{1}{\sin x \cos x}\, dx = \begin{bmatrix} \ln|\sin x| - \ln|\sec x| + C \\ \ln|\sin x| + \ln|\cos x| + C \\ \ln|\tan x| + C \\ -\ln|\cot 2x + \operatorname{cosec} 2x| + C \end{bmatrix}$$

$$\int \frac{1}{\sin x\cos x}\,dx = \int \frac{2}{2\sin x\cos x}\,dx = \int \frac{2}{\sin 2x}\,dx$$

$$= \int 2\operatorname{cosec}2x\,dx = -\ln|\cot 2x + \operatorname{cosec}2x| + C$$

or

$$= \ln|\tan x| + C$$

BOTH ARE "STANDARD" RESULTS

ALTERNATIVE MANIPULATION

$$\int \frac{1}{\sin x\cos x}\,dx = \int \frac{\cos^2 x + \sin^2 x}{\sin x\cos x}\,dx = \int \cot x + \tan x\,dx$$

$$= \ln|\sin x| + \ln|\sec x| + C = \ln|\sin x| - \ln|\cos x| + C$$

$$= \ln|\tan x| + C$$

276. $$\int \frac{1}{\sin^2 x\cos x}\, dx = -\cot x + \ln|\sec x + \tan x| + C$$

$$\int \frac{1}{\cos x\sin^2 x}\,dx = \int \frac{\cos^2 x + \sin^2 x}{\cos x\sin^2 x}\,dx = \int \frac{\cos x}{\sin^2 x} + \frac{1}{\cos x}\,dx$$

$$\int \cot x\operatorname{cosec}x + \sec x\,dx \ \ldots$$ STANDARD RESULTS ...

$$-\cot x + \ln|\sec x + \tan x| + C$$

277. $$\int \frac{1}{1-\sin x}\, dx = \tan x + \sec x + C$$

$$\int \frac{1}{1-\sin x}\,dx = \int \frac{1+\sin x}{(1-\sin x)(1+\sin x)}\,dx = \int \frac{1+\sin x}{1-\sin^2 x}\,dx$$

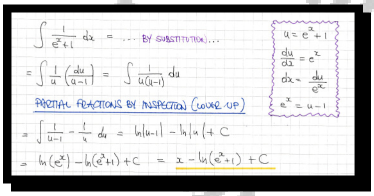
$$= \int \frac{1+\sin x}{\cos^2 x}\,dx = \int \frac{1}{\cos^2 x} + \frac{\sin x}{\cos^2 x}\,dx$$

$$= \int \sec^2 x + \frac{\sin x}{\cos x}\frac{1}{\cos x}\,dx = \int \sec^2 x + \sec x\tan x\,dx$$

STANDARD ANTIDERIVATIVES

$$= \tan x + \sec x + C$$

278. $\displaystyle\int \frac{1}{e^{x}+1}\,dx = x-\ln\left(e^{x}+1\right)+C$



$\int \frac{1}{e^x+1}\,dx = \dots$ BY SUBSTITUTION...

$u = e^x+1$

$\frac{du}{dx} = e^x$

$dx = \frac{du}{e^x}$

$e^x = u-1$

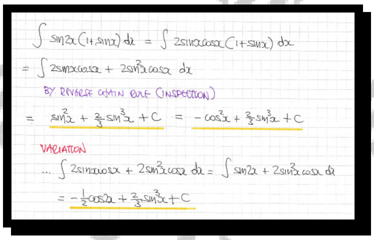
$= \int \frac{1}{u}\left(\frac{du}{u-1}\right) = \int \frac{1}{u(u-1)}\,du$

PARTIAL FRACTIONS BY INSPECTION (COVER UP)

$= \int \frac{1}{u-1} - \frac{1}{u}\,du = \ln|u-1| - \ln|u| + C$

$= \ln(e^x) - \ln(e^x+1) + C = x - \ln(e^x+1) + C$

279. $\displaystyle\int (1+\sin x)\sin 2x\;dx = \begin{bmatrix} \frac{2}{3}\sin^{3}x-\cos^{2}x+C \\ \frac{2}{3}\sin^{3}x+\sin^{2}x+C \\ \frac{2}{3}\sin^{3}x-\frac{1}{2}\cos 2x+C \end{bmatrix}$



$\int \sin 2x(1+\sin x)\,dx = \int 2\sin x\cos x(1+\sin x)\,dx$

$= \int 2\sin x\cos x + 2\sin^2 x\cos x\,dx$

BY REVERSE CHAIN RULE (INSPECTION)

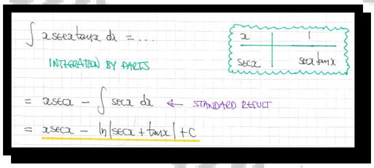
$= \sin^2 x + \frac{2}{3}\sin^3 x + C = -\cos^2 x + \frac{2}{3}\sin^3 x + C$

VARIATION

$\dots \int 2\sin x\cos x + 2\sin^2 x\cos x\,dx = \int \sin 2x + 2\sin^2 x\cos x\,dx$

$= -\frac{1}{2}\cos 2x + \frac{2}{3}\sin^3 x + C$

280. $\displaystyle\int x\sec x\tan x\;dx = x\sec x-\ln\left|\sec x+\tan x\right|+C$



$\int x\sec x\tan x\,dx = \dots$

INTEGRATION BY PARTS

x	1
$\sec x$	$\sec x\tan x$

$= x\sec x - \int \sec x\,dx \leftarrow$ STANDARD RESULT

$= x\sec x - \ln|\sec x + \tan x| + C$

281. $$\int \sin 3x \cos 2x \; dx = \begin{bmatrix} -\frac{1}{10}\cos 5x - \frac{1}{2}\cos x + C \\ \frac{2}{13}\sin 3x \sin 2x + \frac{3}{13}\cos 3x \cos 2x + C \end{bmatrix}$$

$\int \sin 3x \cos 2x \, dx = \ldots$ BY TRIG IDENTITIES

$\sin(3x+2x) = \sin 5x = \sin 3x \cos 2x + \cos 3x \sin 2x$
$\sin(3x-2x) = \sin x = \sin 3x \cos 2x - \cos 3x \sin 2x$
ADDING $\quad \sin 5x + \sin x = 2\sin 3x \cos 2x$

$= \int \frac{1}{2}\sin 5x + \frac{1}{2}\sin x \, dx = -\frac{1}{10}\cos 5x - \frac{1}{2}\cos x + C$

ALTERNATIVE BY DOUBLE INTEGRATION BY PARTS

$\int \sin 3x \cos 2x \, dx$

$\sin 3x$	$3\cos 3x$
$\frac{1}{2}\sin 2x$	$\cos 2x$

$= \frac{1}{2}\sin 3x \sin 2x - \underbrace{\int \frac{3}{2}\cos 3x \sin 2x \, dx}_{\text{BY PARTS AGAIN}}$

$\frac{3}{2}\cos 3x$	$-\frac{9}{2}\sin 3x$
$-\frac{1}{2}\cos 2x$	$\sin 2x$

$\Rightarrow \int \sin 3x \cos 2x \, dx = \frac{1}{2}\sin 3x \sin 2x - \left[-\frac{3}{4}\cos 3x \cos 2x + \frac{9}{4}\int \cos 2x \sin 3x \, dx\right]$

$\Rightarrow \int \sin 3x \cos 2x \, dx = \frac{1}{2}\sin 3x \sin 2x + \frac{3}{4}\cos 3x \cos 2x - \frac{9}{4}\int \sin 3x \cos 2x \, dx$

$\Rightarrow \frac{13}{4}\int \sin 3x \cos 2x \, dx = \frac{1}{2}\sin 3x \sin 2x + \frac{3}{4}\cos 3x \cos 2x$

$\Rightarrow \int \sin 3x \cos 2x \, dx = \frac{2}{13}\sin 3x \sin 2x + \frac{3}{13}\cos 3x \cos 2x + C$

282. $$\int \frac{1-2x}{x(2x-\ln x)^2} \; dx = \frac{1}{2x-\ln x} + C$$

$\int \frac{1-2x}{x(2x-\ln x)^2} \, dx = \ldots$ BY SUBSTITUTION ...

$u = 2x - \ln x$
$\frac{du}{dx} = 2 - \frac{1}{x}$
$\frac{du}{dx} = \frac{2x-1}{x}$
$\frac{du}{dx} = -\frac{1-2x}{x}$
$dx = -\frac{x}{1-2x}\,du$

$= \int \frac{\cancel{1-2x}}{\cancel{x}u^2}\left(-\frac{\cancel{x}}{\cancel{1-2x}}\,du\right) = \int -\frac{1}{u^2}\,du$

$= \frac{1}{u} + C = \frac{1}{2x-\ln x} + C$

283. $$\int x^2 e^{-\frac{1}{4}x} \; dx = -4e^{-\frac{1}{4}x}\left[x^2 + 8x + 32\right] + C$$

$\int x^2 e^{-\frac{1}{4}x} \, dx = \ldots$ INTEGRATION BY PARTS

x^2	$2x$
$-4e^{-\frac{1}{4}x}$	$e^{-\frac{1}{4}x}$

$= -4x^2 e^{-\frac{1}{4}x} - \int -8x e^{-\frac{1}{4}x} \, dx$

$= -4x^2 e^{-\frac{1}{4}x} + \int 8x e^{-\frac{1}{4}x} \, dx$ ... INTEGRATION BY PARTS AGAIN

$8x$	8
$-4e^{-\frac{1}{4}x}$	$e^{-\frac{1}{4}x}$

$= -4x^2 e^{-\frac{1}{4}x} + \left[-32x e^{-\frac{1}{4}x} - \int -32 e^{-\frac{1}{4}x} \, dx\right]$

$= -4x^2 e^{-\frac{1}{4}x} - 32x e^{-\frac{1}{4}x} + \int 32 e^{-\frac{1}{4}x} \, dx$

$= -4x^2 e^{-\frac{1}{4}x} - 32x e^{-\frac{1}{4}x} - 128 e^{-\frac{1}{4}x} + C$

$= -4e^{-\frac{1}{4}x}\left[x^2 + 8x + 32\right] + C$

284. $\displaystyle\int \frac{x+4}{x-4}\,dx = x + 8\ln|x-4| + C$

$$\int \frac{x+4}{x-4}\,dx = \int \frac{(x-4)+8}{(x-4)}\,dx = \int 1 + \frac{8}{x-4}\,dx$$
$$= x + 8\ln|x-4| + C$$

[THE SUBSTITUTION $u = x-4$ ALSO WORKS WELL]

285. $\displaystyle\int 3x^2\left(4-2x^3\right)^{\frac{3}{2}}\,dx = -\tfrac{1}{5}\left(4-2x^3\right)^{\frac{5}{2}} + C$

$$\int 3x^2(4-2x^3)^{\frac{3}{2}}\,dx = \text{ BY REVERSE CHAIN RULE (INSPECTION)} \ldots$$
$$= -\frac{1}{5}(4-2x^3)^{5/2} + C$$

[THE SUBSTITUTION $u = 4-2x^3$ OR $u = (4-2x^3)^{\frac{1}{2}}$ ALSO WORK WELL]

286. $\displaystyle\int x\sqrt{x+1}\,dx = \begin{bmatrix} \tfrac{2}{5}(x+1)^{\frac{5}{2}} - \tfrac{2}{3}(x+1)^{\frac{3}{2}} + C \\ \tfrac{2}{3}x(x+1)^{\frac{3}{2}} - \tfrac{4}{15}(x+1)^{\frac{5}{2}} + C \end{bmatrix}$

$$\int x\sqrt{x+1}\,dx = \ldots \text{ BY SUBSTITUTION } \ldots$$
$$= \int xu\,(2u\,du) = \int 2u^2x\,du$$
$$= \int 2u^2(u^2-1)\,du = \int 2u^4 - 2u^2\,du$$
$$= \frac{2}{5}u^5 - \frac{2}{3}u^3 + C = \frac{2}{5}(x+1)^{\frac{5}{2}} - \frac{2}{3}(x+1)^{\frac{3}{2}} + C$$

$u = \sqrt{x+1}$
$u^2 = x+1$
$x = u^2 - 1$
$\frac{dx}{du} = 2u$
$dx = 2u\,du$

ALTERNATIVE BY INTEGRATION BY PARTS

$$\int x\sqrt{x+1}\,dx = \frac{2}{3}x(x+1)^{\frac{3}{2}} - \int \frac{2}{3}(x+1)^{\frac{3}{2}}\,dx$$
$$= \frac{2}{3}x(x+1)^{\frac{3}{2}} - \frac{4}{15}(x+1)^{\frac{5}{2}} + C$$

x	1
$\frac{2}{3}(x+1)^{\frac{3}{2}}$	$(x+1)^{\frac{1}{2}}$

287. $\displaystyle\int \frac{x+1}{\sqrt[3]{x^2+2x+3}}\,dx = \tfrac{3}{4}\left(x^2+2x+3\right)^{\frac{2}{3}} + C$

$$\int \frac{x+1}{\sqrt[3]{x^2+2x+3}}\,dx = \int (x+1)(x^2+2x+3)^{-\frac{1}{3}}\,dx$$
$$= \frac{1}{2}\int (2x+2)(x^2+2x+3)^{-\frac{1}{3}}\,dx = \ldots \text{ BY REVERSE CHAIN RULE (INSPECTION)}$$
$$= \frac{1}{2}\times\frac{3}{2}(x^2+2x+3)^{\frac{2}{3}} + C = \frac{3}{4}(x^2+2x+3)^{\frac{2}{3}} + C$$

ALTERNATIVE BY SUBSTITUTION

$$\int \frac{x+1}{\sqrt[3]{x^2+2x+3}}\,dx = \int \frac{\cancel{x+1}}{u}\left(\frac{3u^2}{2\cancel{(x+1)}}\,du\right)$$
$$= \int \frac{3}{2}u\,du = \frac{3}{4}u^2 + C$$
$$= \frac{3}{4}(x^2+2x+3)^{\frac{2}{3}} + C$$

$u = (x^2+2x+3)^{\frac{1}{3}}$
$u^3 = x^2+2x+3$
$3u^2\frac{du}{dx} = 2x+2$
$dx = \frac{3u^2}{2x+2}\,du$
$dx = \frac{3u^2}{2(x+1)}\,du$

[THE SUBSTITUTION $u = x^2+2x+3$ ALSO WORKS WELL]

288. $$\int \sin 2x \cos x \, dx = \begin{bmatrix} -\frac{2}{3}\cos^3 x + C \\ -\frac{1}{6}\cos 3x - \frac{1}{2}\cos x + C \\ -\frac{1}{3}\left(\sin 2x \sin x + 2\cos 2x \cos x\right) + C \end{bmatrix}$$

$$\int \sin 2x \cos x \, dx = \int (2\sin x \cos x)\cos x \, dx = \int 2\sin x \cos^2 x \, dx$$

BY REVERSE CHAIN RULE (INSPECTION) $\ldots = -\frac{2}{3}\cos^3 x + C$

[THE SUBSTITUTION $u = \cos x$ ALSO WORKS WELL]

ALTERNATIVE BY TRIG IDENTITIES

$$\sin(2x + x) = \sin 3x = \sin 2x \cos x + \cos 2x \sin x$$
$$\sin(2x - x) = \sin x = \sin 2x \cos x - \cos 2x \sin x$$

ADDING $\sin 3x + \sin x = 2\sin 2x \cos x$

OR $\sin 2x \cos x = \frac{1}{2}\sin 3x + \frac{1}{2}\sin x$

$$\int \sin 2x \cos x \, dx = \int \frac{1}{2}\sin 3x + \frac{1}{2}\sin x \, dx$$
$$= -\frac{1}{6}\cos 3x - \frac{1}{2}\cos x + C$$

ALTERNATIVE USING INTEGRATION BY PARTS TWICE

$$\int \sin 2x \cos x \, dx = \sin 2x \sin x - \int 2\cos 2x \sin x \, dx$$

$\sin 2x$	$2\cos 2x$
$\sin x$	$\cos x$

$2\cos 2x$	$-4\sin 2x$
$-\cos x$	$\sin x$

$$\Rightarrow \int \sin 2x \cos x \, dx = \sin 2x \sin x - \left[-2\cos 2x \cos x - \int 4\sin 2x \cos x \, dx\right]$$
$$\Rightarrow \int \sin 2x \cos x \, dx = \sin 2x \sin x + 2\cos 2x \cos x + 4\int \sin 2x \cos x \, dx$$
$$\Rightarrow -3\int \sin 2x \cos x \, dx = \sin 2x \sin x + 2\cos 2x \cos x$$
$$\Rightarrow \int \sin 2x \cos x \, dx = -\frac{1}{3}\left[\sin 2x \sin x + 2\cos 2x \cos x\right]$$

289. $$\int \sin 2x \cos 2x \, dx = \begin{bmatrix} \frac{1}{4}\sin^2 2x + C \\ -\frac{1}{4}\cos^2 2x + C \\ -\frac{1}{8}\cos 4x + C \end{bmatrix}$$

$\int \sin 2x \cos 2x \, dx = \ldots$ BY REVERSE CHAIN RULE (INSPECTION)
OR THE SUBSTITUTION $u = \sin 2x$
OR THE SUBSTITUTION $u = \cos 2x$

$\ldots = \frac{1}{4}\sin^2 2x + C$ OR $-\frac{1}{4}\cos^2 2x + C$

ALTERNATIVE BY DOUBLE ANGLES

$$\int \sin 2x \cos 2x \, dx = \int \frac{1}{2}(2\sin 2x \cos 2x) \, dx = \int \frac{1}{2}\sin 4x \, dx$$
$$= -\frac{1}{8}\cos 4x + C$$

290. $$\int \frac{3^x}{3^x + 1} \, dx = \frac{\ln\left(3^x + 1\right)}{\ln 3} + C$$

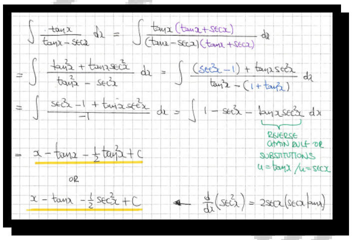
$$\int \frac{3^x}{3^x+1} \, dx = \frac{1}{\ln 3}\int \frac{3^x \ln 3}{3^x + 1} \, dx = \frac{1}{\ln 3}\ln(3^x + 1) + C$$

OF THE FORM $\int \frac{f'(x)}{f(x)} \, dx = \ln|f(x)| + C$

$$= \frac{\ln(3^x+1)}{\ln 3} + C$$

[THE SUBSTITUTION $u = 3^x + 1$ ALSO WORKS WELL]

291. $$\int \frac{\tan x}{\tan x - \sec x}\, dx = \begin{bmatrix} x - \tan x - \frac{1}{2}\tan^2 x + C \\ x - \tan x - \frac{1}{2}\sec^2 x + C \end{bmatrix}$$



$$\int \frac{\tan x}{\tan x - \sec x}\, dx = \int \frac{\tan x(\tan x + \sec x)}{(\tan x - \sec x)(\tan x + \sec x)}\, dx$$

$$= \int \frac{\tan^2 x + \tan x \sec x}{\tan^2 x - \sec^2 x}\, dx = \int \frac{(\sec^2 x - 1) + \tan x \sec x}{\tan^2 x - (1 + \tan^2 x)}\, dx$$

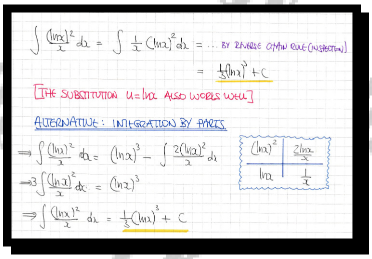
$$= \int \frac{\sec^2 x - 1 + \tan x \sec x}{-1}\, dx = \int 1 - \sec^2 x - \underbrace{\tan x \sec x}_{\text{REVERSE CHAIN RULE OR SUBSTITUTIONS } u = \tan x \,/\, u = \sec x}\, dx$$

$$= x - \tan x - \frac{1}{2}\tan^2 x + C$$

OR

$$x - \tan x - \frac{1}{2}\sec^2 x + C \quad \leftarrow \quad \frac{d}{dx}(\sec^2 x) = 2\sec x(\sec x \tan x)$$

292. $$\int \frac{(\ln x)^2}{x}\, dx = \frac{1}{3}(\ln x)^3 + C$$



$$\int \frac{(\ln x)^2}{x}\, dx = \int \frac{1}{x}(\ln x)^2\, dx = \ldots \text{ BY REVERSE CHAIN RULE (INSPECTION)}$$

$$= \frac{1}{3}(\ln x)^3 + C$$

[THE SUBSTITUTION $u = \ln x$ ALSO WORKS WELL]

ALTERNATIVE: INTEGRATION BY PARTS

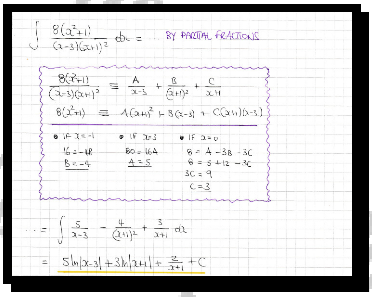
$(\ln x)^2$	$\frac{2\ln x}{x}$
$\ln x$	$\frac{1}{x}$

$$\Rightarrow \int \frac{(\ln x)^2}{x}\, dx = (\ln x)^3 - \int \frac{2(\ln x)^2}{x}\, dx$$

$$\Rightarrow 3\int \frac{(\ln x)^2}{x}\, dx = (\ln x)^3$$

$$\Rightarrow \int \frac{(\ln x)^2}{x}\, dx = \frac{1}{3}(\ln x)^3 + C$$

293. $$\int \frac{8(x^2+1)}{(x-3)(x+1)^2}\, dx = 5\ln|x-3| + 3\ln|x+1| + \frac{2}{x+1} + C$$



$$\int \frac{8(x^2+1)}{(x-3)(x+1)^2}\, dx = \ldots \text{ BY PARTIAL FRACTIONS}$$

$$\frac{8(x^2+1)}{(x-3)(x+1)^2} \equiv \frac{A}{x-3} + \frac{B}{(x+1)^2} + \frac{C}{x+1}$$

$$8(x^2+1) \equiv A(x+1)^2 + B(x-3) + C(x+1)(x-3)$$

- IF $x = -1$: $16 = -4B$, $B = -4$
- IF $x = 3$: $80 = 16A$, $A = 5$
- IF $x = 0$: $8 = A - 3B - 3C$, $8 = 5 + 12 - 3C$, $3C = 9$, $C = 3$

$$\ldots = \int \frac{5}{x-3} - \frac{4}{(x+1)^2} + \frac{3}{x+1}\, dx$$

$$= 5\ln|x-3| + 3\ln|x+1| + \frac{2}{x+1} + C$$

294. $\displaystyle\int \sqrt[3]{x\sqrt{\frac{1}{x}}}\ dx = \frac{6}{7}x^{\frac{7}{6}} + C$

$$\int \sqrt[3]{x\sqrt{\frac{1}{x}}}\ dx = \int \left[x\, x^{-\frac{1}{2}}\right]^{\frac{1}{3}} dx = \int \left(x^{\frac{1}{2}}\right)^{\frac{1}{3}} dx$$
$$= \int x^{\frac{1}{6}}\, dx = \frac{6}{7}x^{\frac{7}{6}} + C$$

295. $\displaystyle\int (x+1)\left(x^2+2x-1\right)^4\ dx = \frac{1}{10}\left(x^2+2x-1\right)^5 + C$

$$\int (x+1)(x^2+2x-1)^4\, dx = \frac{1}{2}\int (2x+2)(x^2+2x-1)^4\, dx$$

BY REVERSE CHAIN RULE (INSPECTION)

$$= \frac{1}{10}(x^2+2x-1)^5 + C$$

ALTERNATIVE BY SUBSTITUTION

$$\int (x+1)(x^2+2x-1)^4\, dx = \int \cancel{(x+1)}\,u^4\left(\frac{du}{2\cancel{(x+1)}}\right)$$
$$= \int \frac{1}{2}u^4\, du = \frac{1}{10}u^5 + C$$
$$= \frac{1}{10}(x^2+2x-1)^5 + C$$

$u = x^2+2x-1$

$\frac{du}{dx} = 2x+2$

$dx = \frac{du}{2(x+1)}$

296. $\displaystyle\int \sin x\cos^4 x\ dx = -\frac{1}{5}\cos^5 x + C$

$\int \sin x\cos^4 x\, dx = \dots$ BY REVERSE CHAIN RULE (INSPECTION)

$$= -\frac{1}{5}\cos^5 x + C$$

[THE SUBSTITUTION $u=\cos x$ ALSO WORKS WELL]

297. $\displaystyle\int \frac{2x+6}{x^2+6x+1}\ dx = \ln\left|x^2+6x+1\right| + C$

$\int \frac{2x+6}{x^2+6x+1}\, dx = \dots$ OF THE FORM $\int \frac{f'(x)}{f(x)}\, dx = \ln|f(x)| + C$

$$= \ln|x^2+6x+1| + C$$

298. $$\int \frac{1}{x(x-4)}\, dx = \frac{1}{4}\ln\left|\frac{x-4}{x}\right| + C$$

$\displaystyle\int \frac{1}{x(x-4)}\,dx = \ldots$ PARTIAL FRACTIONS BY INSPECTION (COVER UP)

$$= \int \frac{\frac{1}{4}}{x-4} - \frac{\frac{1}{4}}{x}\,dx = \frac{1}{4}\ln|x-4| - \frac{1}{4}\ln|x| + C$$

$$= \frac{1}{4}\ln\left|\frac{x-4}{x}\right| + C$$

299. $$\int \frac{\tan x}{\sqrt{1+\cos 2x}}\, dx = \pm\frac{1}{\sqrt{2}}\sec x + C$$

$$\int \frac{\tan x}{\sqrt{1+\cos 2x}}\,dx = \int \frac{\tan x}{\sqrt{1+2\cos^2 x-1}}\,dx = \int \frac{\tan x}{\sqrt{2\cos^2 x}}\,dx$$

NOW IF $\cos x > 0$

$$= \int \frac{\tan x}{\sqrt{2}\cos x}\,dx = \int \frac{1}{\sqrt{2}}\tan x\sec x\,dx = \frac{1}{\sqrt{2}}\sec x + C$$

AND IF $\cos x < 0$

$$= \int \frac{\tan x}{-\sqrt{2}\cos x}\,dx = \int -\frac{1}{\sqrt{2}}\tan x\sec x\,dx = -\frac{1}{\sqrt{2}}\sec x + C$$

COLLECTING RESULTS

$$\ldots = \pm\frac{1}{\sqrt{2}}\sec x + C$$

300. $$\int \cos^2 x\sin^2 x\, dx = \frac{1}{8}x - \frac{1}{32}\sin 4x + C$$

$$\int \cos^2 x\sin^2 x\,dx = \int\left(\tfrac{1}{2}+\tfrac{1}{2}\cos 2x\right)\left(\tfrac{1}{2}-\tfrac{1}{2}\cos 2x\right)dx$$

$$= \int \tfrac{1}{4} - \tfrac{1}{4}\cos^2 2x\,dx = \int \tfrac{1}{4} - \tfrac{1}{4}\left(\tfrac{1}{2}+\tfrac{1}{2}\cos 4x\right)dx$$

$$= \int \tfrac{1}{4} - \tfrac{1}{8} - \tfrac{1}{8}\cos 4x\,dx = \int \tfrac{1}{8} - \tfrac{1}{8}\cos 4x\,dx$$

$$= \tfrac{1}{8}x - \tfrac{1}{32}\sin 4x + C$$

ALTERNATIVE METHOD

$$\int \cos^2 x\sin^2 x\,dx = \int \tfrac{1}{4}(4\cos^2 x\sin^2 x)\,dx = \int \tfrac{1}{4}(2\cos x\sin x)^2\,dx$$

$$= \int \tfrac{1}{4}(\sin 2x)^2\,dx = \int \tfrac{1}{4}\sin^2 2x\,dx$$

$$= \int \tfrac{1}{4}\left(\tfrac{1}{2}-\tfrac{1}{2}\cos 4x\right)dx = \int \tfrac{1}{8} - \tfrac{1}{8}\cos 4x\,dx$$

$$= \tfrac{1}{8}x - \tfrac{1}{32}\sin 4x + C$$

301. $$\int \frac{3x^3+5x}{x^2+1}\,dx = \frac{3}{2}x^2 + \ln\left(x^2+1\right) + C$$

$\int \frac{3x^3+5x}{x^2+1}\,dx = \ldots$ BY ALGEBRAIC DIVISION OR MANIPULATION ...

$= \int \frac{3x(x^2+1)+2x}{x^2+1}\,dx = \int 3x + \frac{2x}{x^2+1}\,dx$

OF THE FORM $\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + C$

$= \frac{3}{2}x^2 + \ln(x^2+1) + C$

ALTERNATIVE BY DIRECT SUBSTITUTION

$\int \frac{3x^3+5x}{x^2+1}\,dx = \int \frac{3x^3+5x}{u}\left(\frac{du}{2x}\right)$

$= \int \frac{x(3x^2+5)}{2xu}\,du = \int \frac{3x^2+5}{2u}\,du$

$= \int \frac{3u+2}{2u}\,du = \int \frac{3}{2} + \frac{1}{u}\,du$

$= \frac{3}{2}u + \ln|u| + C = \frac{3}{2}(x^2+1) + \ln(x^2+1) + C$

$u = x^2+1$

$\frac{du}{dx} = 2x$

$dx = \frac{du}{2x}$

$x^2 = u-1$

$3x^2 = 3u-3$

$3x^2+5 = 3u+2$

302. $$\int \sin^2 2x\,dx = \frac{1}{2}x - \frac{1}{8}\sin 4x + C$$

$\int \sin^2 2x\,dx = \int \frac{1}{2} - \frac{1}{2}\cos 4x\,dx = \frac{1}{2}x - \frac{1}{8}\sin 4x + C$

$\left[\begin{array}{l}\cos 2\theta = 1 - 2\sin^2\theta \\ \cos 4\theta = 1 - 2\sin^2 2\theta\end{array}\right]$

303. $$\int \frac{\cos 2x}{\cos^2 x}\,dx = 2x - \tan x + C$$

$\int \frac{\cos 2x}{\cos^2 x}\,dx = \int \frac{2\cos^2 x - 1}{\cos^2 x}\,dx = \int 2 - \frac{1}{\cos^2 x}\,dx$

$= \int 2 - \sec^2 x\,dx = 2x - \tan x + C$

304. $$\int \sec^3 x \tan x\,dx = \frac{1}{3}\sec^3 x + C$$

$\int \sec^3 x \tan x\,dx = \ldots$ BY REVERSE CHAIN RULE (INSPECTION), SINCE

$\frac{d}{dx}(\sec^3 x) = 3\sec^2 x(\sec x \tan x) = 3\sec^3 x \tan x$

$= \frac{1}{3}\sec^3 x + C$

[THE SUBSTITUTION $u = \sec x$ ALSO WORKS WELL]

305. $\displaystyle\int x\left[(\ln x)^2 - 1\right]\, dx = \frac{1}{4}x^2\left[2\left(\ln|x|\right)^2 - 2\ln|x| + 1\right] + C$

$\int x\left[(\ln x)^2-1\right]dx = \ldots$ INTEGRATION BY PARTS ...

$(\ln x)^2-1$	$\frac{2}{x}\ln x$
$\frac{1}{2}x^2$	x

$= \frac{1}{2}x^2\left[(\ln|x|)^2-1\right] - \int x\ln x\, dx$ ← INTEGRATION BY PARTS AGAIN

$\ln x$	$\frac{1}{x}$
$\frac{1}{2}x^2$	x

$= \frac{1}{2}x^2\left[(\ln|x|)^2-1\right] - \left\{\frac{1}{2}x^2\ln|x| - \int \frac{1}{2}x\, dx\right\}$

$= \frac{1}{2}x^2\left[(\ln|x|)^2-1\right] - \frac{1}{2}x^2\ln|x| + \int \frac{1}{2}x\, dx$

$= \frac{1}{2}x^2\left[(\ln|x|)^2-1\right] - \frac{1}{2}x^2\ln|x| + \frac{1}{4}x^2 + C$

$= \frac{1}{4}x^2\left[2(\ln|x|)^2 - 2\ln|x| + 1\right] + C$

306. $\displaystyle\int \frac{x^2}{x^3+5}\, dx = \frac{1}{3}\ln\left|x^3+5\right| + C$

$\int \frac{x^2}{x^3+5}\,dx = \ldots$ OF THE FORM $\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + C$

$= \frac{1}{3}\int \frac{3x^2}{x^3+5}\,dx = \frac{1}{3}\ln|x^3+5| + C$

[THE SUBSTITUTION $u = x^3+5$ ALSO WORKS WELL]

307. $\displaystyle\int \frac{2x+1}{3x-1}\, dx = \frac{2}{3}x + \frac{5}{9}\ln|3x-1| + C$

$\int \frac{2x+1}{3x-1}\,dx =$ MANIPULATION & SPLIT

$= \int \frac{\frac{2}{3}(3x-1)+\frac{5}{3}}{(3x-1)}\,dx = \int \frac{2}{3} + \frac{\frac{5}{3}}{3x-1}\,dx$

$= \frac{2}{3}x + \frac{5}{9}\ln|3x-1| + C$

[THE SUBSTITUTION $u = 3x-1$ IS A VIABLE ALTERNATIVE]

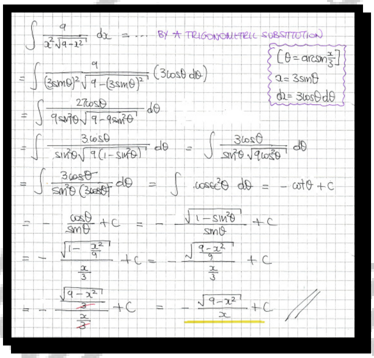
308. $\displaystyle\int x\left(3+x^2\right)^4\, dx = \frac{1}{10}\left(3+x^2\right)^5 + C$

$\int x(3+x^2)^4\,dx = \ldots$ BY REVERSE CHAIN RULE (INSPECTION)

$= \frac{1}{10}(3+x^2)^5 + C$

[THE SUBSTITUTION $u = 3+x^2$ ALSO WORKS WELL]

309. $$\int \frac{9}{x^2\sqrt{9-x^2}}\,dx = -\frac{\sqrt{9-x^2}}{x} + C$$

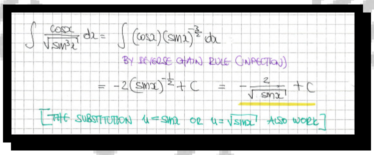


$$\int \frac{9}{x^2\sqrt{9-x^2}}\,dx = \dots$$ BY A TRIGONOMETRIC SUBSTITUTION

$\left[\theta = \arcsin\frac{x}{3}\right]$, $x = 3\sin\theta$, $dx = 3\cos\theta\,d\theta$

$$= \int \frac{9}{(3\sin\theta)^2\sqrt{9-(3\sin\theta)^2}}(3\cos\theta\,d\theta)$$
$$= \int \frac{27\cos\theta}{9\sin^2\theta\sqrt{9-9\sin^2\theta}}\,d\theta$$
$$= \int \frac{3\cos\theta}{\sin^2\theta\sqrt{9(1-\sin^2\theta)}}\,d\theta = \int \frac{3\cos\theta}{\sin^2\theta\sqrt{9\cos^2\theta}}\,d\theta$$
$$= \int \frac{\cancel{3\cos\theta}}{\sin^2\theta\,(\cancel{3\cos\theta})}\,d\theta = \int \csc^2\theta\,d\theta = -\cot\theta + C$$
$$= -\frac{\cos\theta}{\sin\theta} + C = -\frac{\sqrt{1-\sin^2\theta}}{\sin\theta} + C$$
$$= -\frac{\sqrt{1-\frac{x^2}{9}}}{\frac{x}{3}} + C = -\frac{\sqrt{\frac{9-x^2}{9}}}{\frac{x}{3}} + C$$
$$= -\frac{\frac{\sqrt{9-x^2}}{\cancel{3}}}{\frac{x}{\cancel{3}}} + C = -\frac{\sqrt{9-x^2}}{x} + C$$

310. $$\int \frac{\cos x}{\sqrt{\sin^3 x}}\,dx = -\frac{2}{\sqrt{\sin x}} + C$$



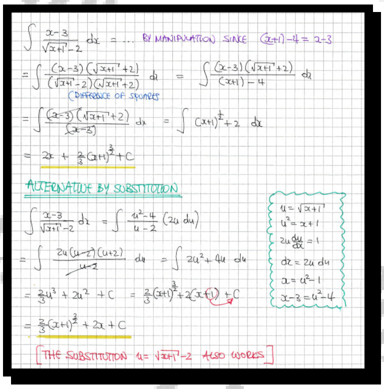
$$\int \frac{\cos x}{\sqrt{\sin^3 x}}\,dx = \int (\cos x)(\sin x)^{-\frac{3}{2}}\,dx$$

BY REVERSE CHAIN RULE (INSPECTION)

$$= -2(\sin x)^{-\frac{1}{2}} + C = -\frac{2}{\sqrt{\sin x}} + C$$

[THE SUBSTITUTION $u = \sin x$ OR $u = \sqrt{\sin x}$ ALSO WORK]

311. $$\int \frac{x-3}{\sqrt{x+1}-2}\,dx = 2x + \frac{2}{3}(x+1)^{\frac{3}{2}} + C$$



$$\int \frac{x-3}{\sqrt{x+1}-2}\,dx = \dots$$ BY MANIPULATION SINCE $(x+1)-4 = x-3$

$$= \int \frac{(x-3)(\sqrt{x+1}+2)}{(\sqrt{x+1}-2)(\sqrt{x+1}+2)}\,dx = \int \frac{(x-3)(\sqrt{x+1}+2)}{(x+1)-4}\,dx$$

(DIFFERENCE OF SQUARES)

$$= \int \frac{\cancel{(x-3)}(\sqrt{x+1}+2)}{\cancel{(x-3)}}\,dx = \int (x+1)^{\frac{1}{2}} + 2\,dx$$
$$= 2x + \frac{2}{3}(x+1)^{\frac{3}{2}} + C$$

ALTERNATIVE BY SUBSTITUTION

$u = \sqrt{x+1}$, $u^2 = x+1$, $2u\frac{du}{dx} = 1$, $dx = 2u\,du$, $x = u^2-1$, $x-3 = u^2-4$

$$\int \frac{x-3}{\sqrt{x+1}-2}\,dx = \int \frac{u^2-4}{u-2}(2u\,du)$$
$$= \int \frac{2u\cancel{(u-2)}(u+2)}{\cancel{u-2}}\,du = \int 2u^2 + 4u\,du$$
$$= \frac{2}{3}u^3 + 2u^2 + C = \frac{2}{3}(x+1)^{\frac{3}{2}} + 2(x+1) + C$$
$$= \frac{2}{3}(x+1)^{\frac{3}{2}} + 2x + C$$

[THE SUBSTITUTION $u = \sqrt{x+1}-2$ ALSO WORKS]

312. $$\int \frac{x-4}{x^2-4}\,dx = \begin{bmatrix} \frac{1}{2}\ln\left|x^2-4\right| + \ln\left|\frac{x+2}{x-2}\right| + C \\ \frac{3}{2}\ln\left|x+2\right| - \frac{1}{2}\ln\left|x-2\right| + C \end{bmatrix}$$

$$\int \frac{x-4}{x^2-4}\,dx = \int \frac{x}{x^2-4} - \frac{4}{x^2-4}\,dx$$ OR DIRECTLY INTO PARTIAL FRACTIONS

$$= \frac{1}{2}\int \frac{2x}{x^2-4}\,dx - \int \frac{4}{(x-2)(x+2)}\,dx$$ (PARTIAL FRACTIONS BY INSPECTION

$$= \frac{1}{2}\ln\left|x^2-4\right| - \int \frac{1}{x-2} - \frac{1}{x+2}\,dx = \frac{1}{2}\ln\left|x^2-4\right| - \ln\left|x-2\right| + \ln\left|x+2\right| + C$$

OR DIRECTLY INTO PARTIAL FRACTIONS BY INSPECTION

$$\int \frac{x-4}{x^2-4}\,dx = \int \frac{x-4}{(x-2)(x+2)}\,dx = \int \frac{-\frac{1}{2}}{x-2} + \frac{\frac{3}{2}}{x+2}\,dx$$

$$= \frac{3}{2}\ln\left|x+2\right| - \frac{1}{2}\ln\left|x-2\right| + C$$

313. $$\int \frac{2\sin x}{\cos x + \sin x}\,dx = \begin{bmatrix} x - \ln\left|\cos x + \sin x\right| + C \\ x + \frac{1}{2}\ln\left|\sec 2x\right| - \frac{1}{2}\ln\left|\sec 2x + \tan 2x\right| + C \end{bmatrix}$$

$$\int \frac{2\sin x}{\cos x + \sin x}\,dx = \dots$$ BY MANIPULATION

$$= \int \frac{\cos x + \sin x + \sin x - \cos x}{\cos x + \sin x}\,dx$$

$$= \int \frac{\cos x + \sin x}{\cos x + \sin x} - \frac{\sin x + \cos x}{\cos x + \sin x}\,dx \longrightarrow \int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + C$$

$$= x - \ln\left|\cos x + \sin x\right| + C$$

ALTERNATIVE BY TRIGONOMETRIC MANIPULATIONS

$$\int \frac{2\sin x}{\cos x + \sin x}\,dx = \int \frac{2\sin x(\cos x - \sin x)}{(\cos x + \sin x)(\cos x - \sin x)}\,dx$$

$$= \int \frac{2\sin x\cos x - 2\sin^2 x}{\cos^2 x - \sin^2 x}\,dx = \int \frac{\sin 2x + \overbrace{(1-2\sin^2 x)}^{\cos 2x} - 1}{\cos 2x}\,dx$$

$$= \int \frac{\sin 2x + \cos 2x - 1}{\cos 2x}\,dx = \int \tan 2x + 1 - \sec 2x\,dx$$

NOW THESE ARE STANDARD QUOTABLE INTEGRALS, WHICH OF COURSE CAN EASILY BE PROVEN (NOT HERE)

$$= \frac{1}{2}\ln\left|\sec 2x\right| + x - \frac{1}{2}\ln\left|\sec 2x + \tan 2x\right| + C$$

$$= x + \frac{1}{2}\ln\left|\frac{\sec 2x}{\sec 2x + \tan 2x}\right| + C = x - \frac{1}{2}\ln\left|\frac{\sec 2x + \tan 2x}{\sec 2x}\right| + C$$

$$= x - \frac{1}{2}\ln\left|1 + \tan 2x\cos 2x\right| + C$$

$$= x - \frac{1}{2}\ln\left|1 + \frac{\sin 2x}{\cancel{\cos 2x}}\cancel{\cos 2x}\right| + C = x - \frac{1}{2}\ln\left|1 + \sin 2x\right| + C$$

$$= x - \frac{1}{2}\ln\left|\cos^2 x + \sin^2 x + 2\sin x\cos x\right| + C$$

$$= x - \frac{1}{2}\ln\left|(\cos x + \sin x)^2\right| + C$$

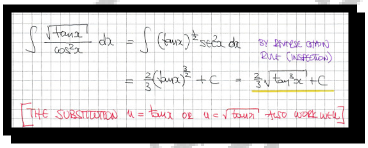
$$= x - \ln\left|\cos x + \sin x\right| + C$$ (AS ABOVE)

314. $$\int \left(1 - x^{-2}\right)^2\,dx = 2 + \frac{2}{x} - \frac{1}{3x^3} + C$$

$$\int (1-x^{-2})^2\,dx = \int 1 - 2x^{-2} + x^{-4}\,dx = x + 2x^{-1} - \frac{1}{3}x^{-3} + C$$

$$= x + \frac{2}{x} - \frac{1}{3x^3} + C$$

315. $\displaystyle\int \frac{\sqrt{\tan x}}{\cos^2 x}\,dx = \frac{2}{3}\sqrt{\tan^3 x} + C$

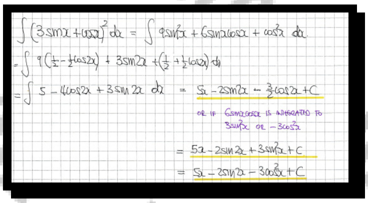


$\displaystyle\int \frac{\sqrt{\tan x}}{\cos^2 x}\,dx = \int (\tan x)^{\frac{1}{2}}\sec^2 x\,dx$ BY REVERSE CHAIN RULE (INSPECTION)

$= \frac{2}{3}(\tan x)^{\frac{3}{2}} + C = \frac{2}{3}\sqrt{\tan^3 x} + C$

[THE SUBSTITUTION $u = \tan x$ OR $u = \sqrt{\tan x}$ ALSO WORK WELL]

316. $\displaystyle\int (3\sin x + \cos x)^2\,dx = \begin{bmatrix} 5x - 2\sin 2x - \frac{3}{2}\cos 2x + C \\ 5x - 2\sin 2x + 3\sin^2 x + C \\ 5x - 2\sin 2x - 3\cos^2 x + C \end{bmatrix}$



$\displaystyle\int (3\sin x + \cos x)^2\,dx = \int 9\sin^2 x + 6\sin x\cos x + \cos^2 x\,dx$

$\displaystyle = \int 9\left(\tfrac{1}{2} - \tfrac{1}{2}\cos 2x\right) + 3\sin 2x + \left(\tfrac{1}{2} + \tfrac{1}{2}\cos 2x\right)dx$

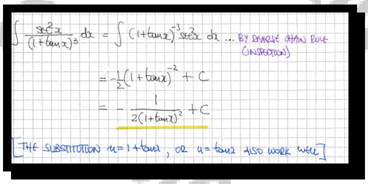
$\displaystyle = \int 5 - 4\cos 2x + 3\sin 2x\,dx = 5x - 2\sin 2x - \frac{3}{2}\cos 2x + C$

OR IF $6\sin x\cos x$ IS INTEGRATED TO $3\sin^2 x$ OR $-3\cos^2 x$

$= 5x - 2\sin 2x + 3\sin^2 x + C$

$= 5x - 2\sin 2x - 3\cos^2 x + C$

317. $\displaystyle\int \frac{\sec^2 x}{(1+\tan x)^3}\,dx = -\frac{1}{2(1+\tan x)^2} + C$



$\displaystyle\int \frac{\sec^2 x}{(1+\tan x)^3}\,dx = \int (1+\tan x)^{-3}\sec^2 x\,dx$ ... BY REVERSE CHAIN RULE (INSPECTION)

$= -\frac{1}{2}(1+\tan x)^{-2} + C$

$\displaystyle = -\frac{1}{2(1+\tan x)^2} + C$

[THE SUBSTITUTION $u = 1 + \tan x$, OR $u = \tan x$ ALSO WORK WELL]

318. $\displaystyle\int \frac{1}{\cos^2 x \sin^2 x}\, dx = \begin{bmatrix} -2\cot 2x + C \\ -\cot x + \tan x + C \end{bmatrix}$

$$\int \frac{1}{\cos^2 x \sin^2 x}\, dx = \int \frac{1}{\left(\frac{1}{2}+\frac{1}{2}\cos 2x\right)\left(\frac{1}{2}-\frac{1}{2}\cos 2x\right)}\, dx$$

$$= \int \frac{1}{\frac{1}{4}-\frac{1}{4}\cos^2 2x}\, dx = \int \frac{4}{1-\cos^2 2x}\, dx = \int \frac{4}{\sin^2 2x}\, dx$$

$$= \int 4\operatorname{cosec}^2 2x\, dx = \underline{-2\cot 2x + C}$$

ALTERNATIVE (VARIATION)

$$\int \frac{1}{\cos^2 x \sin^2 x}\, dx = \int \frac{4}{4\cos^2 x \sin^2 x}\, dx = \int \frac{4}{(2\cos x \sin x)^2}\, dx$$

$$= \int \frac{4}{(\sin 2x)^2}\, dx = \int 4\operatorname{cosec}^2 2x\, dx = \underline{-2\cot 2x + C}$$

ANOTHER ALTERNATIVE

$$\int \frac{1}{\cos^2 x \sin^2 x}\, dx = \int \frac{\cos^2 x + \sin^2 x}{\cos^2 x \sin^2 x}\, dx = \ldots$$ SPLITTING THE FRACTION ...

$$= \int \frac{1}{\sin^2 x} + \frac{1}{\cos^2 x}\, dx = \int \operatorname{cosec}^2 x + \sec^2 x\, dx$$

$$= \underline{-\cot x + \tan x + C}$$

319. $\displaystyle\int \cot 2x \; dx = \frac{1}{2}\ln|\sin 2x| + C$

$$\int \cot 2x\, dx = \int \frac{\cos 2x}{\sin 2x}\, dx = \frac{1}{2}\int \frac{2\cos 2x}{\sin 2x}\, dx$$

OF THE FORM $\int \frac{f'(x)}{f(x)}\, dx$

$$= \underline{\frac{1}{2}\ln|\sin 2x| + C}$$

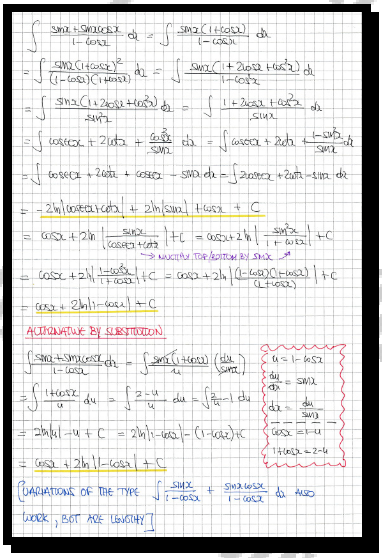
[THE SUBSTITUTION $u = \sin 2x$ ALSO WORKS WELL]

320. $\displaystyle\int 2^x \; dx = \frac{2^x}{\ln 2} + C$

$$\int 2^x\, dx = \frac{1}{\ln 2}\int 2^x \ln 2\, dx = \underline{\frac{2^x}{\ln 2} + C}$$

$$\frac{d}{dx}\left(2^x\right) = 2^x \ln 2$$

321. $$\int \frac{\sin x + \sin x \cos x}{1-\cos x}\, dx = \begin{bmatrix} \cos x + 2\ln\left|1-\cos x\right| + C \\ \cos x + 2\ln\left|\sin x\right| - 2\ln\left|\cot x + \operatorname{cosec} x\right| + C \end{bmatrix}$$



$$\int \frac{\sin x + \sin x\cos x}{1-\cos x}\,dx = \int \frac{\sin x(1+\cos x)}{1-\cos x}\,dx$$
$$= \int \frac{\sin x(1+\cos x)^2}{(1-\cos x)(1+\cos x)}\,dx = \int \frac{\sin x(1+2\cos x+\cos^2 x)}{1-\cos^2 x}\,dx$$
$$= \int \frac{\sin x(1+2\cos x+\cos^2 x)}{\sin^2 x}\,dx = \int \frac{1+2\cos x+\cos^2 x}{\sin x}\,dx$$
$$= \int \operatorname{cosec} x + 2\cot x + \frac{\cos^2 x}{\sin x}\,dx = \int \operatorname{cosec} x + 2\cot x + \frac{1-\sin^2 x}{\sin x}\,dx$$
$$= \int \operatorname{cosec} x + 2\cot x + \operatorname{cosec} x - \sin x\,dx = \int 2\operatorname{cosec} x + 2\cot x - \sin x\,dx$$
$$= -2\ln\left|\operatorname{cosec} x + \cot x\right| + 2\ln\left|\sin x\right| + \cos x + C$$
$$= \cos x + 2\ln\left|\frac{\sin x}{\operatorname{cosec} x + \cot x}\right| + C = \cos x + 2\ln\left|\frac{\sin^2 x}{1+\cos x}\right| + C$$

→ MULTIPLY TOP/BOTTOM BY $\sin x$

$$= \cos x + 2\ln\left|\frac{1-\cos^2 x}{1+\cos x}\right| + C = \cos x + 2\ln\left|\frac{(1-\cos x)(1+\cos x)}{(1+\cos x)}\right| + C$$
$$= \cos x + 2\ln\left|1-\cos x\right| + C$$

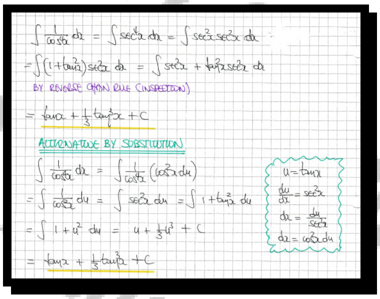
ALTERNATIVE BY SUBSTITUTION

$$\int \frac{\sin x + \sin x\cos x}{1-\cos x}\,dx = \int \frac{\sin x(1+\cos x)}{u}\left(\frac{du}{\sin x}\right)$$
$$= \int \frac{1+\cos x}{u}\,du = \int \frac{2-u}{u}\,du = \int \frac{2}{u} - 1\,du$$
$$= 2\ln|u| - u + C = 2\ln\left|1-\cos x\right| - (1-\cos x) + C$$
$$= \cos x + 2\ln\left|1-\cos x\right| + C$$

$u = 1-\cos x$
$\frac{du}{dx} = \sin x$
$dx = \frac{du}{\sin x}$
$\cos x = 1-u$
$1+\cos x = 2-u$

[VARIATIONS OF THE TYPE $\int \frac{\sin x}{1-\cos x} + \frac{\sin x\cos x}{1-\cos x}\,dx$ ALSO WORK, BUT ARE LENGTHY]

322. $$\int \frac{1}{\cos^4 x}\, dx = \tan x + \frac{1}{3}\tan^3 x + C$$



$$\int \frac{1}{\cos^4 x}\,dx = \int \sec^4 x\,dx = \int \sec^2 x \sec^2 x\,dx$$
$$= \int (1+\tan^2 x)\sec^2 x\,dx = \int \sec^2 x + \tan^2 x \sec^2 x\,dx$$

BY REVERSE CHAIN RULE (INSPECTION)

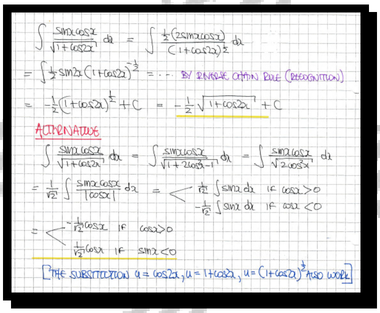
$$= \tan x + \frac{1}{3}\tan^3 x + C$$

ALTERNATIVE BY SUBSTITUTION

$$\int \frac{1}{\cos^4 x}\,dx = \int \frac{1}{\cos^4 x}(\cos^2 x\,du)$$
$$= \int \frac{1}{\cos^2 x}\,du = \int \sec^2 x\,du = \int 1+\tan^2 x\,du$$
$$= \int 1+u^2\,du = u + \frac{1}{3}u^3 + C$$
$$= \tan x + \frac{1}{3}\tan^3 x + C$$

$u = \tan x$
$\frac{du}{dx} = \sec^2 x$
$dx = \frac{du}{\sec^2 x}$
$dx = \cos^2 x\,du$

323. $$\int \frac{\sin x \cos x}{\sqrt{1+\cos 2x}}\, dx = \left[\begin{array}{c} -\frac{1}{2}\sqrt{1+\cos 2x}+C \\ \left[\begin{array}{ll} -\dfrac{\cos x}{\sqrt{2}}+C & \text{if} \quad \cos x>0 \\ \dfrac{\cos x}{\sqrt{2}}+C & \text{if} \quad \cos x<0 \end{array}\right] \end{array}\right]$$



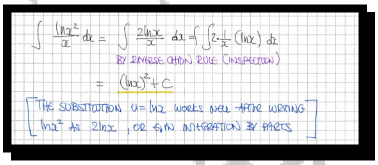
$$\int \frac{\sin x \cos x}{\sqrt{1+\cos 2x}}\, dx = \int \frac{\frac{1}{2}(2\sin x\cos x)}{(1+\cos 2x)^{\frac{1}{2}}}\, dx$$
$$= \int \tfrac{1}{2}\sin 2x\,(1+\cos 2x)^{-\frac{1}{2}} = \dots \quad \text{BY REVERSE CHAIN RULE (RECOGNITION)}$$
$$= -\tfrac{1}{2}(1+\cos 2x)^{\frac{1}{2}}+C = -\tfrac{1}{2}\sqrt{1+\cos 2x}+C$$

ALTERNATIVE

$$\int \frac{\sin x\cos x}{\sqrt{1+\cos 2x}}\, dx = \int \frac{\sin x\cos x}{\sqrt{1+2\cos^2 x-1}}\, dx = \int \frac{\sin x\cos x}{\sqrt{2\cos^2 x}}\, dx$$
$$= \frac{1}{\sqrt{2}}\int \frac{\sin x\cos x}{|\cos x|}\, dx = \begin{cases} \frac{1}{\sqrt{2}}\int \sin x\, dx & \text{IF } \cos x>0 \\ -\frac{1}{\sqrt{2}}\int \sin x\, dx & \text{IF } \cos x<0 \end{cases}$$
$$= \begin{cases} -\frac{1}{\sqrt{2}}\cos x & \text{IF } \cos x>0 \\ \frac{1}{\sqrt{2}}\cos x & \text{IF } \sin x<0 \end{cases}$$

[THE SUBSTITUTION $u=\cos 2x$, $u=1+\cos 2x$, $u=(1+\cos 2x)^{\frac{1}{2}}$ ALSO WORK]

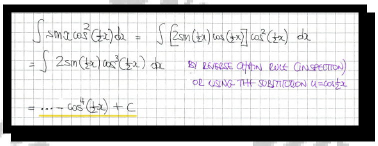
324. $$\int \frac{\ln x^2}{x}\, dx = \left(\ln|x|\right)^2 + C$$



$$\int \frac{\ln x^2}{x}\, dx = \int \frac{2\ln x}{x}\, dx = \int 2\cdot\frac{1}{x}(\ln x)\, dx$$
BY REVERSE CHAIN RULE (INSPECTION)
$$= (\ln x)^2 + C$$

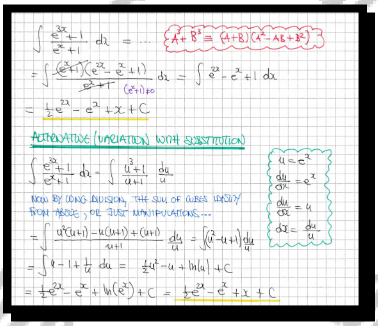
[THE SUBSTITUTION $u=\ln x$ WORKS WELL AFTER WRITING $\ln x^2$ AS $2\ln x$, OR EVEN INTEGRATION BY PARTS]

325. $$\int \sin x \cos^2\left(\tfrac{1}{2}x\right)\, dx = -\cos^4\left(\tfrac{1}{2}x\right) + C$$

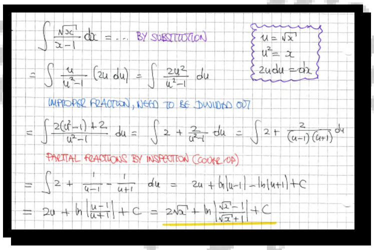


$$\int \sin x\cos^2\left(\tfrac{1}{2}x\right) dx = \int \left[2\sin\left(\tfrac{1}{2}x\right)\cos\left(\tfrac{1}{2}x\right)\right]\cos^2\left(\tfrac{1}{2}x\right)\, dx$$
$$= \int 2\sin\left(\tfrac{1}{2}x\right)\cos^3\left(\tfrac{1}{2}x\right)\, dx$$
BY REVERSE CHAIN RULE (INSPECTION) OR USING THE SUBSTITUTION $u=\cos\frac{1}{2}x$
$$= \dots -\cos^4\left(\tfrac{1}{2}x\right) + C$$

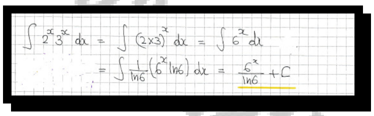
326. $$\int \frac{e^{3x}+1}{e^{x}+1}\,dx = \frac{1}{2}e^{2x} - e^{x} + x + C$$



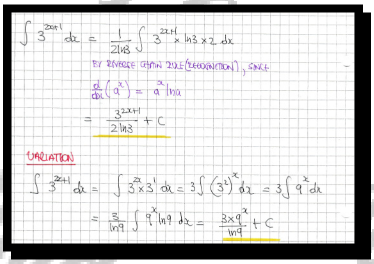
327. $$\int \frac{\sqrt{x}}{x-1}\,dx = 2\sqrt{x} + \ln\left|\frac{\sqrt{x}-1}{\sqrt{x}+1}\right| + C$$



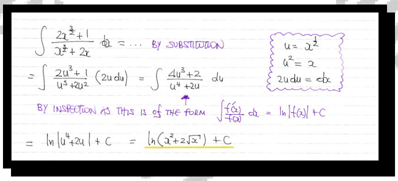
328. $$\int 2^{x}\,3^{x}\,dx = \frac{6^{x}}{\ln 6} + C$$



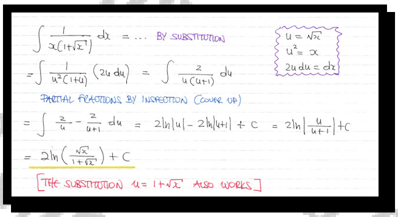
329. $$\int 3^{2x+1}\,dx = \begin{bmatrix} \dfrac{3^{2x+1}}{2\ln 3}+C \\ \dfrac{3\times 9^{x}}{\ln 9}+C \end{bmatrix}$$



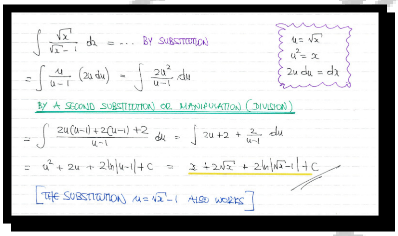
330. $$\int \frac{2x^{\frac{3}{2}}+1}{x^{\frac{5}{2}}+2x}\,dx = \ln\left(x^2+2\sqrt{x}\right)+C$$



331. $$\int \frac{1}{x\left(1+\sqrt{x}\right)}\,dx = 2\ln\left(\frac{\sqrt{x}}{1+\sqrt{x}}\right)+C$$



332. $$\int \frac{\sqrt{x}}{\sqrt{x}-1}\ dx = x + 2\sqrt{x} + 2\ln\left|\sqrt{x}-1\right| + C$$



$\int \frac{\sqrt{x}}{\sqrt{x}-1}\,dx = \ldots$ BY SUBSTITUTION

$u=\sqrt{x}$
$u^2 = x$
$2u\,du = dx$

$= \int \frac{u}{u-1}(2u\,du) = \int \frac{2u^2}{u-1}\,du$

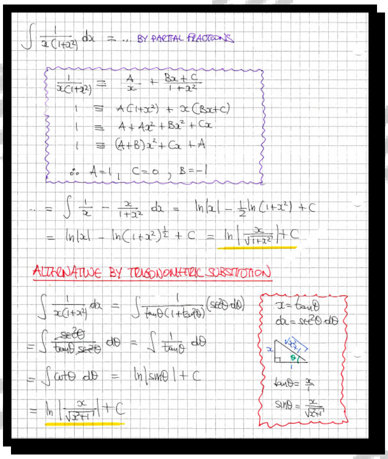
BY A SECOND SUBSTITUTION OR MANIPULATION (DIVISION)

$= \int \frac{2u(u-1)+2(u-1)+2}{u-1}\,du = \int 2u+2+\frac{2}{u-1}\,du$

$= u^2+2u+2\ln|u-1|+C = x+2\sqrt{x}+2\ln|\sqrt{x}-1|+C$

[THE SUBSTITUTION $u=\sqrt{x}-1$ ALSO WORKS]

333. $$\int \frac{1}{x\left(1+x^2\right)}\ dx = \ln\left|\frac{x}{\sqrt{x^2+1}}\right| + C$$



$\int \frac{1}{x(1+x^2)}\,dx = \ldots$ BY PARTIAL FRACTIONS

$\frac{1}{x(1+x^2)} \equiv \frac{A}{x} + \frac{Bx+C}{1+x^2}$

$1 \equiv A(1+x^2) + x(Bx+C)$

$1 \equiv A + Ax^2 + Bx^2 + Cx$

$1 \equiv (A+B)x^2 + Cx + A$

$\therefore A=1,\ C=0,\ B=-1$

$\ldots = \int \frac{1}{x} - \frac{x}{1+x^2}\,dx = \ln|x| - \frac{1}{2}\ln(1+x^2) + C$

$= \ln|x| - \ln(1+x^2)^{\frac{1}{2}} + C = \ln\left|\frac{x}{\sqrt{1+x^2}}\right| + C$

ALTERNATIVE BY TRIGONOMETRIC SUBSTITUTION

$\int \frac{1}{x(1+x^2)}\,dx = \int \frac{1}{\tan\theta(1+\tan^2\theta)}(\sec^2\theta\,d\theta)$

$= \int \frac{\sec^2\theta}{\tan\theta\sec^2\theta}\,d\theta = \int \frac{1}{\tan\theta}\,d\theta$

$= \int \cot\theta\,d\theta = \ln|\sin\theta| + C$

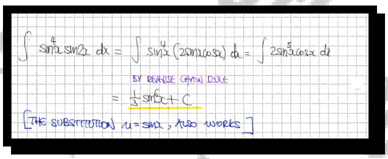
$= \ln\left|\frac{x}{\sqrt{x^2+1}}\right| + C$

$x=\tan\theta$
$dx = \sec^2\theta\,d\theta$

x, 1, $\sqrt{x^2+1}$, θ

$\tan\theta = \frac{x}{1}$
$\sin\theta = \frac{x}{\sqrt{x^2+1}}$

334. $$\int \sin^4 x \sin 2x\ \ dx = \tfrac{1}{3}\sin^6 x + C$$



$\int \sin^4 x \sin 2x\,dx = \int \sin^4 x(2\sin x\cos x)\,dx = \int 2\sin^5 x\cos x\,dx$

BY REVERSE CHAIN RULE

$= \frac{1}{3}\sin^6 x + C$

[THE SUBSTITUTION $u=\sin x$, ALSO WORKS]

335. $\displaystyle\int (\tan x + \cot x)^2 \ dx = \tan x - \cot x + C$

$\int (\tan x + \cot x)^2 \, dx = \int \tan^2 x + 2\tan x \cot x + \cot^2 x \ dx$

$= \int (\sec^2 x - 1) + 2\cancel{\tan x}\left(\frac{1}{\cancel{\tan x}}\right) + (\operatorname{cosec}^2 x - 1) \ dx$

$= \int \sec^2 x + \operatorname{cosec}^2 x \ dx = \underline{\tan x - \cot x + C}$

336. $\displaystyle\int \frac{x^2+2x-2}{x^2-2x+2} \ dx = x + \ln\left(x^2-2x+2\right) + C$

$\int \frac{x^2+2x-2}{x^2-2x+2} \, dx = \ldots$ BY DIVISION OR MANIPULATION

$= \int \frac{(x^2-2x+2)+4x-4}{x^2-2x+2} \, dx = \int 1 + \frac{4x-4}{x^2-2x+2} \, dx$

$= \int 1 + 2\left(\frac{2x-2}{x^2-2x+2}\right) dx = \underline{x + 2\ln|x^2-2x+2| + C}$

OF THE FORM $\int \frac{f'(x)}{f(x)} \, dx = \ln|f(x)| + C$

337. $\displaystyle\int x \sin x \cos x \ dx = -\tfrac{1}{4}x\cos 2x + \tfrac{1}{8}\sin 2x + C$

$\int x \sin x \cos x \, dx = \int x \times \frac{1}{2}(2\sin x \cos x) \, dx = \int \frac{1}{2}x \sin 2x \, dx$

INTEGRATION BY PARTS

$\frac{1}{2}x$	$\frac{1}{2}$
$-\frac{1}{2}\cos 2x$	$\sin 2x$

$= -\frac{1}{4}x\cos 2x - \int -\frac{1}{4}\cos 2x \, dx$

$= -\frac{1}{4}x\cos 2x + \int \frac{1}{4}\cos 2x \, dx$

$= \underline{-\frac{1}{4}x\cos 2x + \frac{1}{8}\sin 2x + C}$

338. $\displaystyle\int \sec x \tan^3 x \ dx = -\tfrac{1}{3}\sec^3 x - \sec x + C$

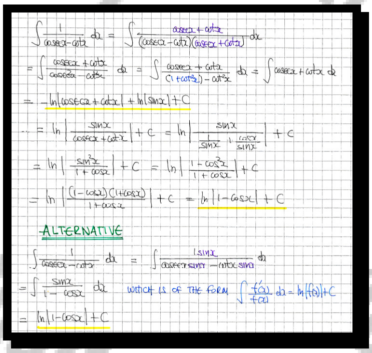
$\int \sec x \tan^3 x \, dx = \int \sec x \tan x (\tan^2 x) \, dx = \int \sec x \tan x (\sec^2 x - 1) \, dx$

$= \int \sec^3 x \tan x - \sec x \tan x \, dx =$ BY REVERSE CHAIN RULE (INSPECTION) AS $\frac{d}{dx}(\sec x) = \sec x \tan x$

$= \underline{\frac{1}{3}\sec^3 x - \sec x + C}$

[THE SUBSTITUTION $u = \sec x$ WORKS WELL TOO]

339. $$\int \frac{1}{\operatorname{cosec} x - \cot x}\, dx = \begin{bmatrix} \ln|\sin x| - \ln|\operatorname{cosec} x + \cot x| + C \\ \ln|1 - \cos x| + C \end{bmatrix}$$



$$\int \frac{1}{\operatorname{cosec}x - \cot x}\,dx = \int \frac{\operatorname{cosec}x + \cot x}{(\operatorname{cosec}x - \cot x)(\operatorname{cosec}x + \cot x)}\,dx$$

$$= \int \frac{\operatorname{cosec}x + \cot x}{\operatorname{cosec}^2 x - \cot^2 x}\,dx = \int \frac{\operatorname{cosec}x + \cot x}{(1+\cot^2 x) - \cot^2 x}\,dx = \int \operatorname{cosec}x + \cot x\,dx$$

$$= -\ln|\operatorname{cosec}x + \cot x| + \ln|\sin x| + C$$

$$\ldots = \ln\left|\frac{\sin x}{\operatorname{cosec}x + \cot x}\right| + C = \ln\left|\frac{\sin x}{\frac{1}{\sin x} + \frac{\cos x}{\sin x}}\right| + C$$

$$= \ln\left|\frac{\sin^2 x}{1+\cos x}\right| + C = \ln\left|\frac{1-\cos^2 x}{1+\cos x}\right| + C$$

$$= \ln\left|\frac{(1-\cos x)(1+\cos x)}{1+\cos x}\right| + C = \ln|1-\cos x| + C$$

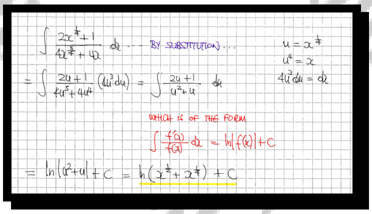
ALTERNATIVE

$$\int \frac{1}{\operatorname{cosec}x - \cot x}\,dx = \int \frac{1\sin x}{\operatorname{cosec}x\sin x - \cot x\sin x}\,dx$$

$$= \int \frac{\sin x}{1-\cos x}\,dx \quad \text{WHICH IS OF THE FORM } \int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + C$$

$$= \ln|1-\cos x| + C$$

340. $$\int \frac{2x^{\frac{1}{4}} + 1}{4x^{\frac{5}{4}} + 4x}\, dx = \ln\left(x^{\frac{1}{2}} + x^{\frac{1}{4}}\right) + C$$



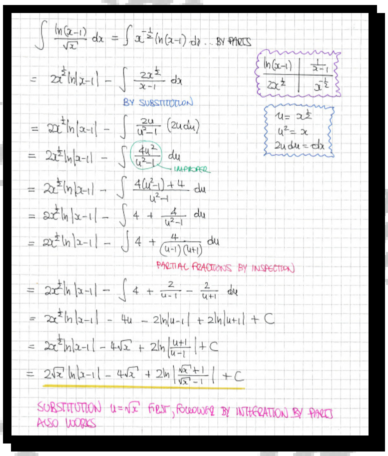
$$\int \frac{2x^{\frac{1}{4}}+1}{4x^{\frac{5}{4}}+4x}\,dx \ldots \text{BY SUBSTITUTION} \ldots \qquad u = x^{\frac{1}{4}},\quad u^4 = x,\quad 4u^3\,du = dx$$

$$= \int \frac{2u+1}{4u^5+4u^4}(4u^3\,du) = \int \frac{2u+1}{u^2+u}\,du$$

WHICH IS OF THE FORM $\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + C$

$$= \ln|u^2+u| + C = \ln\left(x^{\frac{1}{2}} + x^{\frac{1}{4}}\right) + C$$

341. $$\int \frac{\ln(x-1)}{\sqrt{x}}\, dx = 2\sqrt{x}\ln|x-1| - 4\sqrt{x} + 2\ln\left|\frac{\sqrt{x}+1}{\sqrt{x}-1}\right| + C$$



$$\int \frac{\ln(x-1)}{\sqrt{x}}\,dx = \int x^{-\frac{1}{2}}\ln(x-1)\,dx \ldots \text{BY PARTS}$$

$\ln(x-1)$	$\frac{1}{x-1}$
$2x^{\frac{1}{2}}$	$x^{-\frac{1}{2}}$

$$= 2x^{\frac{1}{2}}\ln|x-1| - \int \frac{2x^{\frac{1}{2}}}{x-1}\,dx$$

BY SUBSTITUTION $\qquad u = x^{\frac{1}{2}},\quad u^2 = x,\quad 2u\,du = dx$

$$= 2x^{\frac{1}{2}}\ln|x-1| - \int \frac{2u}{u^2-1}(2u\,du)$$

$$= 2x^{\frac{1}{2}}\ln|x-1| - \int \frac{4u^2}{u^2-1}\,du \quad \text{IMPROPER}$$

$$= 2x^{\frac{1}{2}}\ln|x-1| - \int \frac{4(u^2-1)+4}{u^2-1}\,du$$

$$= 2x^{\frac{1}{2}}\ln|x-1| - \int 4 + \frac{4}{u^2-1}\,du$$

$$= 2x^{\frac{1}{2}}\ln|x-1| - \int 4 + \frac{4}{(u-1)(u+1)}\,du$$

PARTIAL FRACTIONS BY INSPECTION

$$= 2x^{\frac{1}{2}}\ln|x-1| - \int 4 + \frac{2}{u-1} - \frac{2}{u+1}\,du$$

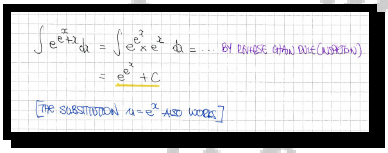
$$= 2x^{\frac{1}{2}}\ln|x-1| - 4u - 2\ln|u-1| + 2\ln|u+1| + C$$

$$= 2x^{\frac{1}{2}}\ln|x-1| - 4\sqrt{x} + 2\ln\left|\frac{u+1}{u-1}\right| + C$$

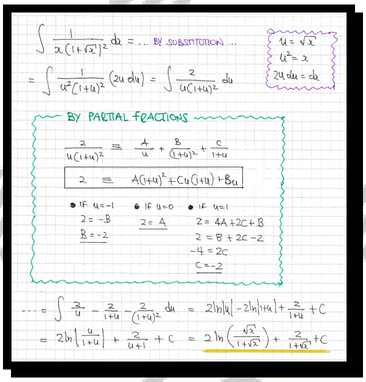
$$= 2\sqrt{x}\ln|x-1| - 4\sqrt{x} + 2\ln\left|\frac{\sqrt{x}+1}{\sqrt{x}-1}\right| + C$$

SUBSTITUTION $u=\sqrt{x}$ FIRST, FOLLOWED BY INTEGRATION BY PARTS ALSO WORKS

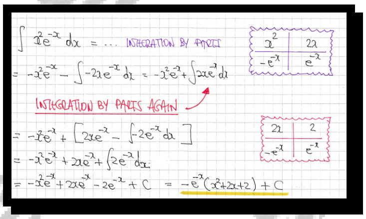
342. $\displaystyle\int e^{x+e^{x}} \, dx = e^{e^{x}} + C$



343. $\displaystyle\int \frac{1}{x\left(1+\sqrt{x}\right)^2} \, dx = 2\ln\left[\frac{\sqrt{x}}{1+\sqrt{x}}\right] + \frac{2}{1+\sqrt{x}} + C$



344. $\displaystyle\int x^2 e^{-x} \, dx = -e^{-x}\left(x^2+2x+2\right) + C$



345. $\displaystyle\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\ dx = 2e^{\sqrt{x}} + C$

$$\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx = \int x^{-\frac{1}{2}} e^{x^{\frac{1}{2}}}\,dx = \dots$$ BY REVERSE CHAIN RULE (INSPECTION)

$$= 2e^{x^{\frac{1}{2}}} + C = \underline{2e^{\sqrt{x}} + C}$$

[THE SUBSTITUTION $u = \sqrt{x}$ ALSO WORKS WELL]

346. $\displaystyle\int \sqrt{x}\,e^{\sqrt{x}}\ dx = 2e^{\sqrt{x}}\left(x + 2\sqrt{x} + 2\right) + C$

$$\int \sqrt{x}\,e^{\sqrt{x}}\,dx = \dots$$ BY SUBSTITUTION ...

$$\boxed{\begin{array}{l} u = \sqrt{x} \\ u^2 = x \\ 2u\,du = dx \end{array}}$$

$$= \int u e^{u}(2u\,du) = \underbrace{\int 2u^2 e^{u}\,du}_{\text{INTEGRATION BY PARTS}}$$

$$\boxed{\begin{array}{c|c} 2u^2 & 4u \\ \hline e^u & e^u \end{array}}$$

$$= 2u^2e^u - \underbrace{\int 4ue^u\,du}_{\text{INTEGRATION BY PARTS AGAIN}}$$

$$\boxed{\begin{array}{c|c} 4u & 4 \\ \hline e^u & e^u \end{array}}$$

$$= 2u^2e^u - \left[4ue^u - \int 4e^u\,du\right]$$

$$= 2u^2e^u - 4ue^u + \int 4e^u\,du$$

$$= 2u^2e^u - 4ue^u + 4e^u + C$$

$$= 2e^u(u^2 - 2u + 2) + C = \underline{2e^{\sqrt{x}}(x - 2\sqrt{x} + 2) + C}$$

347. $\displaystyle\int \frac{4x^2 - x + 1}{(x-1)(2x-1)}\ dx = 2x + 4\ln|x-1| - \frac{3}{2}\ln|2x-1| + C$

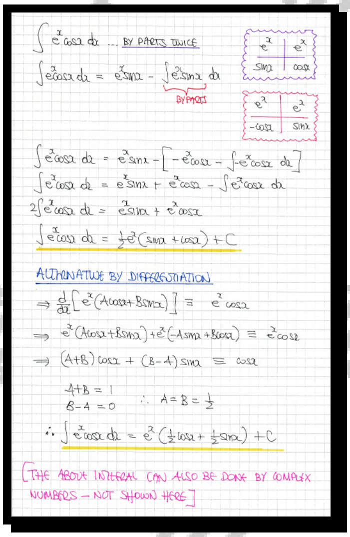
$$\int \frac{4x^2 - x + 1}{(x-1)(2x-1)}\,dx = \int \frac{4x^2 - x + 1}{2x^2 - 3x + 1}\,dx$$ ← IMPROPER

$$= \int \frac{2(2x^2 - 3x + 1) + 5x - 1}{2x^2 - 3x + 1}\,dx = \int 2 + \frac{5x - 1}{2x^2 - 3x + 1}\,dx$$

$$= \int 2 + \frac{5x - 1}{(x-1)(2x-1)}\,dx = \dots$$ PARTIAL FRACTIONS BY INSPECTION

$$= \int 2 + \frac{4}{x-1} - \frac{3}{2x-1}\,dx = \underline{2x + 4\ln|x-1| - \frac{3}{2}\ln|2x-1| + C}$$

348. $\displaystyle\int e^{x}\cos x \ dx = \frac{1}{2}e^{-x}\left(\sin x + \cos x\right) + C$



$\int e^{x}\cos x \, dx$ ... BY PARTS TWICE

$\int e^{x}\cos x \, dx = e^{x}\sin x - \underbrace{\int e^{x}\sin x \, dx}_{\text{BY PARTS}}$

e^{x}	e^{x}
$\sin x$	$\cos x$

e^{x}	e^{x}
$-\cos x$	$\sin x$

$\int e^{x}\cos x \, dx = e^{x}\sin x - \left[-e^{x}\cos x - \int -e^{x}\cos x \, dx\right]$

$\int e^{x}\cos x \, dx = e^{x}\sin x + e^{x}\cos x - \int e^{x}\cos x \, dx$

$2\int e^{x}\cos x \, dx = e^{x}\sin x + e^{x}\cos x$

$\int e^{x}\cos x \, dx = \frac{1}{2}e^{x}(\sin x + \cos x) + C$

ALTERNATIVE BY DIFFERENTIATION

$\Rightarrow \frac{d}{dx}\left[e^{x}(A\cos x + B\sin x)\right] \equiv e^{x}\cos x$

$\Rightarrow e^{x}(A\cos x + B\sin x) + e^{x}(-A\sin x + B\cos x) \equiv e^{x}\cos x$

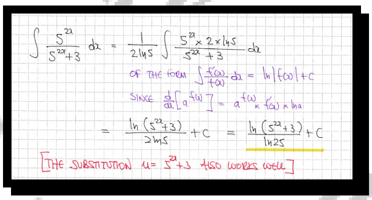
$\Rightarrow (A+B)\cos x + (B-A)\sin x \equiv \cos x$

$A+B = 1$
$B-A = 0$ $\quad \therefore A = B = \frac{1}{2}$

$\therefore \int e^{x}\cos x \, dx = e^{x}\left(\frac{1}{2}\cos x + \frac{1}{2}\sin x\right) + C$

[THE ABOVE INTEGRAL CAN ALSO BE DONE BY COMPLEX NUMBERS – NOT SHOWN HERE]

349. $\displaystyle\int \frac{5^{2x}}{5^{2x}+3} \ dx = \frac{\ln\left(5^{2x}+3\right)}{\ln 25} + C$



$\int \frac{5^{2x}}{5^{2x}+3} \, dx = \frac{1}{2\ln 5}\int \frac{5^{2x}\times 2\times \ln 5}{5^{2x}+3} \, dx$

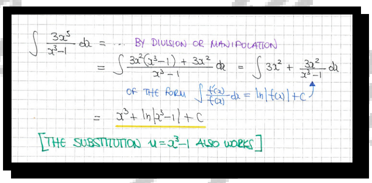
OF THE FORM $\int \frac{f'(x)}{f(x)} \, dx = \ln|f(x)| + C$

SINCE $\frac{d}{dx}\left[a^{f(x)}\right] = a^{f(x)} \times f'(x) \times \ln a$

$= \frac{\ln(5^{2x}+3)}{2\ln 5} + C = \frac{\ln(5^{2x}+3)}{\ln 25} + C$

[THE SUBSTITUTION $u = 5^{2x}+3$ ALSO WORKS WELL]

350. $\displaystyle\int \frac{3x^{5}}{x^{3}-1} \ dx = x^{3} + \ln\left|x^{3}-1\right| + C$



$\int \frac{3x^{5}}{x^{3}-1} \, dx = \ldots$ BY DIVISION OR MANIPULATION

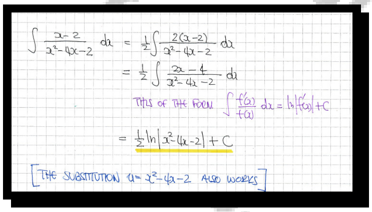
$= \int \frac{3x^{2}(x^{3}-1) + 3x^{2}}{x^{3}-1} \, dx = \int 3x^{2} + \frac{3x^{2}}{x^{3}-1} \, dx$

OF THE FORM $\int \frac{f'(x)}{f(x)} \, dx = \ln|f(x)| + C$

$= x^{3} + \ln|x^{3}-1| + C$

[THE SUBSTITUTION $u = x^{3}-1$ ALSO WORKS]

351. $\displaystyle\int \frac{x-2}{x^2-4x-2}\,dx = \frac{1}{2}\ln\left|x^2-4x-2\right| + C$



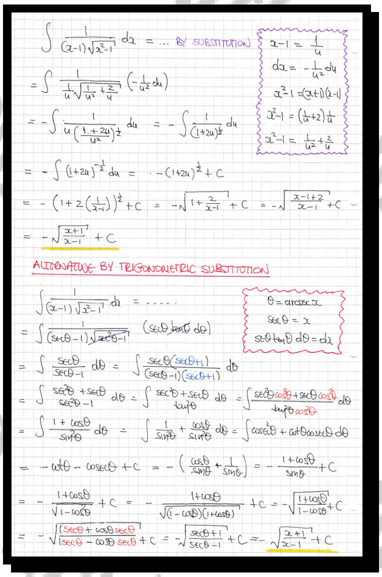
$$\int \frac{x-2}{x^2-4x-2}\,dx = \frac{1}{2}\int \frac{2(x-2)}{x^2-4x-2}\,dx$$
$$= \frac{1}{2}\int \frac{2x-4}{x^2-4x-2}\,dx$$

THIS OF THE FORM $\displaystyle\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)|+C$

$$= \frac{1}{2}\ln|x^2-4x-2| + C$$

[THE SUBSTITUTION $u = x^2-4x-2$ ALSO WORKS]

352. $\displaystyle\int \frac{1}{(x-1)\sqrt{x^2-1}}\,dx = -\sqrt{\frac{x+1}{x-1}} + C$



$\displaystyle\int \frac{1}{(x-1)\sqrt{x^2-1}}\,dx = \dots$ BY SUBSTITUTION

$x-1 = \frac{1}{u}$

$dx = -\frac{1}{u^2}\,du$

$x^2-1 = (x+1)(x-1)$

$x^2-1 = \left(\frac{1}{u}+2\right)\frac{1}{u}$

$x^2-1 = \frac{1}{u^2}+\frac{2}{u}$

$$= \int \frac{1}{\frac{1}{u}\sqrt{\frac{1}{u^2}+\frac{2}{u}}}\left(-\frac{1}{u^2}\,du\right)$$
$$= -\int \frac{1}{u\left(\frac{1+2u}{u^2}\right)^{\frac{1}{2}}}\,du = -\int \frac{1}{(1+2u)^{\frac{1}{2}}}\,du$$
$$= -\int (1+2u)^{-\frac{1}{2}}\,du = -(1+2u)^{\frac{1}{2}} + C$$
$$= -\left(1+2\left(\frac{1}{x-1}\right)\right)^{\frac{1}{2}} + C = -\sqrt{1+\frac{2}{x-1}} + C = -\sqrt{\frac{x-1+2}{x-1}} + C$$
$$= -\sqrt{\frac{x+1}{x-1}} + C$$

ALTERNATIVE BY TRIGONOMETRIC SUBSTITUTION

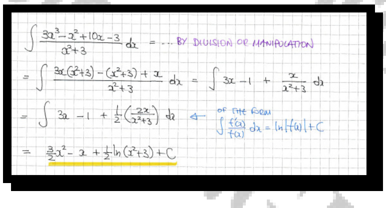
$\theta = \text{arcsec}\,x$

$\sec\theta = x$

$\sec\theta\tan\theta\,d\theta = dx$

$$\int \frac{1}{(x-1)\sqrt{x^2-1}}\,dx = \dots$$
$$= \int \frac{1}{(\sec\theta-1)\cancel{\sqrt{\sec^2\theta-1}}}\left(\sec\theta\cancel{\tan\theta}\,d\theta\right)$$
$$= \int \frac{\sec\theta}{\sec\theta-1}\,d\theta = \int \frac{\sec\theta(\sec\theta+1)}{(\sec\theta-1)(\sec\theta+1)}\,d\theta$$
$$= \int \frac{\sec^2\theta+\sec\theta}{\sec^2\theta-1}\,d\theta = \int \frac{\sec^2\theta+\sec\theta}{\tan^2\theta}\,d\theta = \int \frac{\sec^2\theta\cos^2\theta+\sec\theta\cos^2\theta}{\tan^2\theta\cos^2\theta}\,d\theta$$
$$= \int \frac{1+\cos\theta}{\sin^2\theta}\,d\theta = \int \frac{1}{\sin^2\theta}+\frac{\cos\theta}{\sin^2\theta}\,d\theta = \int \text{cosec}^2\theta + \cot\theta\,\text{cosec}\,\theta\,d\theta$$
$$= -\cot\theta - \text{cosec}\,\theta + C = -\left(\frac{\cos\theta}{\sin\theta}+\frac{1}{\sin\theta}\right) = -\frac{1+\cos\theta}{\sin\theta} + C$$
$$= -\frac{1+\cos\theta}{\sqrt{1-\cos^2\theta}} + C = -\frac{1+\cos\theta}{\sqrt{(1-\cos\theta)(1+\cos\theta)}} + C = -\sqrt{\frac{1+\cos\theta}{1-\cos\theta}} + C$$
$$= -\sqrt{\frac{\sec\theta+\cos\theta\sec\theta}{\sec\theta-\cos\theta\sec\theta}} + C = -\sqrt{\frac{\sec\theta+1}{\sec\theta-1}} + C = -\sqrt{\frac{x+1}{x-1}} + C$$

353. $\displaystyle\int \frac{3x^3-x^2+10x-3}{x^2+3}\,dx = \frac{3}{2}x^2 - x + \frac{1}{2}\ln\left(x^2+3\right) + C$



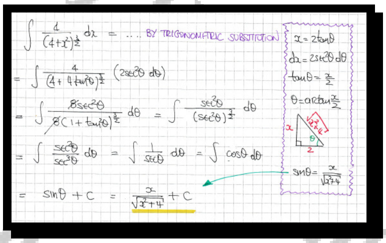
$\displaystyle\int \frac{3x^3-x^2+10x-3}{x^2+3}\,dx = \dots$ BY DIVISION OR MANIPULATION

$$= \int \frac{3x(x^2+3)-(x^2+3)+x}{x^2+3}\,dx = \int 3x-1+\frac{x}{x^2+3}\,dx$$
$$= \int 3x-1+\frac{1}{2}\left(\frac{2x}{x^2+3}\right)\,dx$$

OF THE FORM $\displaystyle\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)|+C$

$$= \frac{3}{2}x^2 - x + \frac{1}{2}\ln(x^2+3) + C$$

354. $$\int \frac{4}{\left(4+x^2\right)^{\frac{3}{2}}}\,dx = \frac{x}{\sqrt{4+x^2}} + C$$



$$\int \frac{4}{(4+x^2)^{\frac{3}{2}}}\,dx = \dots$$ BY TRIGONOMETRIC SUBSTITUTION

$$= \int \frac{4}{(4+4\tan^2\theta)^{\frac{3}{2}}}\left(2\sec^2\theta\,d\theta\right)$$

$$= \int \frac{8\sec^2\theta}{8(1+\tan^2\theta)^{\frac{3}{2}}}\,d\theta = \int \frac{\sec^2\theta}{(\sec^2\theta)^{\frac{3}{2}}}\,d\theta$$

$$= \int \frac{\sec^2\theta}{\sec^3\theta}\,d\theta = \int \frac{1}{\sec\theta}\,d\theta = \int \cos\theta\,d\theta$$

$$= \sin\theta + C = \frac{x}{\sqrt{x^2+4}} + C$$

$x = 2\tan\theta$

$dx = 2\sec^2\theta\,d\theta$

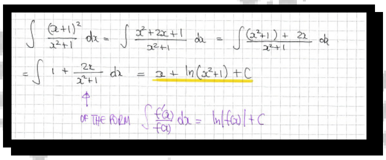
$\tan\theta = \frac{x}{2}$

$\theta = \arctan\frac{x}{2}$

x, 2, $\sqrt{x^2+4}$, θ

$\sin\theta = \frac{x}{\sqrt{x^2+4}}$

355. $$\int \frac{(x+1)^2}{x^2+1}\,dx = x + \ln\left(x^2+1\right) + C$$



$$\int \frac{(x+1)^2}{x^2+1}\,dx = \int \frac{x^2+2x+1}{x^2+1}\,dx = \int \frac{(x^2+1)+2x}{x^2+1}\,dx$$

$$= \int 1 + \frac{2x}{x^2+1}\,dx = x + \ln(x^2+1) + C$$

OF THE FORM $\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + C$

356. $$\int \frac{\cos x + \tan x}{1+\tan^2 x}\,dx = \sin x - \tfrac{1}{3}\sin^3 x + \tfrac{1}{2}\sin^2 x + C$$

$$\int \frac{\cos x + \tan x}{1+\tan^2 x}\,dx = \int \frac{\cos x + \tan x}{\sec^2 x}\,dx$$

$$= \int \frac{\cos x}{\sec^2 x} + \frac{\tan x}{\sec^2 x}\,dx = \int \cos^3 x + \tan x\cos^2 x\,dx$$

$$= \int \cos x\cos^2 x + \frac{\sin x}{\cos x}\cos^2 x\,dx = \int \cos x(1-\sin^2 x) + \sin x\cos x\,dx$$

$$= \int \cos x - \underbrace{\cos x\sin^2 x + \sin x\cos x}_{\text{BY REVERSE CHAIN RULE (RECOGNITION)}}\,dx$$

$$= \sin x - \tfrac{1}{3}\sin^3 x + \tfrac{1}{2}\sin^2 x + C$$

ALTERNATIVE BY SUBSTITUTION

$$\int \frac{\cos x + \tan x}{1+\tan^2 x}\,dx = \int \frac{\cos x + \tan x}{\sec^2 x}\,dx$$

$$= \int \frac{\cos x + \tan x}{\sec^2 x}\left(\frac{du}{\cos x}\right) = \int \frac{\cos x + \tan x}{\sec x}\,du$$

$$= \int \cos^2 x + \tan x\cos x\,du$$

$$= \int 1 - \sin^2 x + \frac{\sin x}{\cos x}\cos x\,du = \int 1 - \sin^2 x + \sin x\,du$$

$$= \int 1 - u^2 + u\,du = u - \tfrac{1}{3}u^3 + \tfrac{1}{2}u^2 + C$$

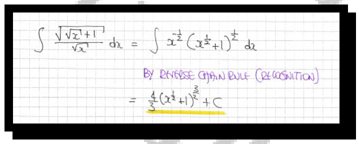
$$= \sin x - \tfrac{1}{3}\sin^3 x + \tfrac{1}{2}\sin^2 x + C$$

$u = \sin x$

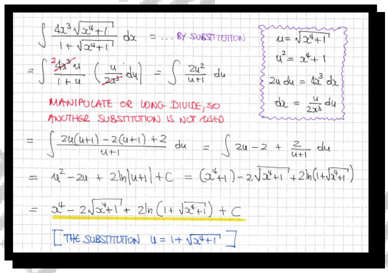
$\frac{du}{dx} = \cos x$

$dx = \frac{du}{\cos x}$

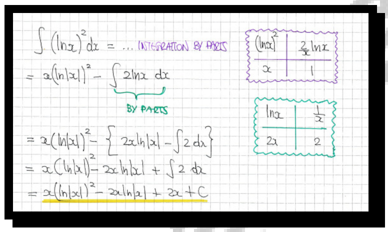
357. $$\int \frac{\sqrt{1+\sqrt{x}}}{\sqrt{x}}\, dx = \frac{4}{3}\left(1+\sqrt{x}\right)^{\frac{3}{2}} + C$$



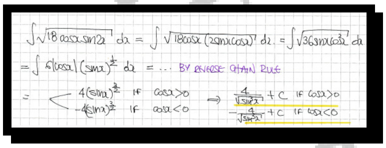
358. $$\int \frac{4x^3\sqrt{x^4+1}}{1+\sqrt{x^4+1}}\, dx = x^4 - 2\sqrt{x^4+1} + 2\ln\left(1+\sqrt{x^4+1}\right) + C$$



359. $$\int (\ln x)^2 \; dx = 2x + x\left(\ln|x|\right)^2 - 2x\ln|x| + C$$



360. $$\int \sqrt{18\cos x \sin 2x}\ dx = \begin{cases} \dfrac{4}{\sqrt{\sin^3 x}} + C & \text{if } \cos x > 0 \\ -\dfrac{4}{\sqrt{\sin^3 x}} + C & \text{if } \cos x > 0 \end{cases}$$

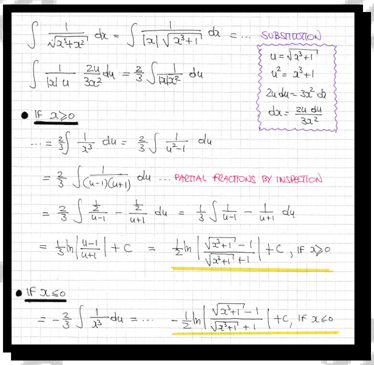


$$\int \sqrt{18\cos x \sin 2x}\ dx = \int \sqrt{18\cos x\,(2\sin x\cos x)}\ dx = \int \sqrt{36\sin x\cos^2 x}\ dx$$

$$= \int 6|\cos x|(\sin x)^{\frac{1}{2}}\ dx = \dots$$ BY REVERSE CHAIN RULE

$$= \begin{cases} 4(\sin x)^{\frac{3}{2}} & \text{IF } \cos x > 0 \\ -4(\sin x)^{\frac{3}{2}} & \text{IF } \cos x < 0 \end{cases} \Rightarrow \begin{cases} \dfrac{4}{\sqrt{\sin^3 x}} + C & \text{IF } \cos x > 0 \\ -\dfrac{4}{\sqrt{\sin^3 x}} + C & \text{IF } \cos x < 0 \end{cases}$$

361. $$\int \frac{1}{\sqrt{x^5 + x^2}}\ dx = \begin{cases} \dfrac{1}{2}\ln\left|\dfrac{\sqrt{x^3+1}-1}{\sqrt{x^3+1}+1}\right| + C & \text{if } x > 0 \\ -\dfrac{1}{2}\ln\left|\dfrac{\sqrt{x^3+1}-1}{\sqrt{x^3+1}+1}\right| + C & \text{if } x < 0 \end{cases}$$



$$\int \frac{1}{\sqrt{x^5+x^2}}\ dx = \int \frac{1}{|x|\sqrt{x^3+1}}\ dx = \dots$$ SUBSTITUTION

$u = \sqrt{x^3+1}$

$u^2 = x^3+1$

$2u\,du = 3x^2\,dx$

$dx = \frac{2u\,du}{3x^2}$

$$\int \frac{1}{|x|\,u}\,\frac{2u}{3x^2}\,du = \frac{2}{3}\int \frac{1}{|x|x^2}\ du$$

- IF $x \geqslant 0$

$$\dots = \frac{2}{3}\int \frac{1}{x^3}\ du = \frac{2}{3}\int \frac{1}{u^2-1}\ du$$

$$= \frac{2}{3}\int \frac{1}{(u-1)(u+1)}\ du$$ ... PARTIAL FRACTIONS BY INSPECTION

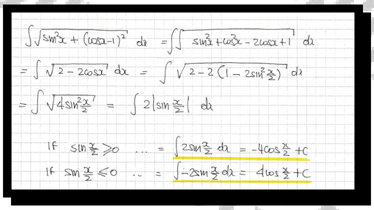
$$= \frac{2}{3}\int \frac{\frac{1}{2}}{u-1} - \frac{\frac{1}{2}}{u+1}\ du = \frac{1}{3}\int \frac{1}{u-1} - \frac{1}{u+1}\ du$$

$$= \frac{1}{3}\ln\left|\frac{u-1}{u+1}\right| + C = \frac{1}{2}\ln\left|\frac{\sqrt{x^3+1}-1}{\sqrt{x^3+1}+1}\right| + C, \text{ IF } x \geqslant 0$$

- IF $x \leqslant 0$

$$= -\frac{2}{3}\int \frac{1}{x^3}\,du = \dots -\frac{1}{2}\ln\left|\frac{\sqrt{x^3+1}-1}{\sqrt{x^3+1}+1}\right| + C, \text{ IF } x \leqslant 0$$

362. $$\int \sqrt{\sin^2 x + (\cos x - 1)^2}\ dx = \begin{cases} 4\cos\left(\frac{1}{2}x\right) + C & \text{if } \sin\left(\frac{1}{2}x\right) < 0 \\ -4\cos\left(\frac{1}{2}x\right) + C & \text{if } \sin\left(\frac{1}{2}x\right) > 0 \end{cases}$$



$$\int \sqrt{\sin^2 x + (\cos x - 1)^2}\ dx = \int \sqrt{\sin^2 x + \cos^2 x - 2\cos x + 1}\ dx$$

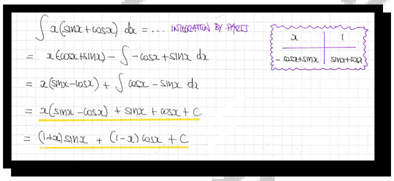
$$= \int \sqrt{2 - 2\cos x}\ dx = \int \sqrt{2 - 2\left(1 - 2\sin^2\frac{x}{2}\right)}\ dx$$

$$= \int \sqrt{4\sin^2\frac{x}{2}} = \int 2\left|\sin\frac{x}{2}\right|\ dx$$

IF $\sin\frac{x}{2} \geqslant 0$ ... $= \int 2\sin\frac{x}{2}\ dx = -4\cos\frac{x}{2} + C$

IF $\sin\frac{x}{2} \leqslant 0$ .. $= \int -2\sin\frac{x}{2}\ dx = 4\cos\frac{x}{2} + C$

363. $$\int x(\sin x + \cos x)\ dx = \begin{bmatrix} x(\sin x - \cos x) + \sin x + \cos x + C \\ (1+x)\sin x + (1-x)\cos x + C \end{bmatrix}$$

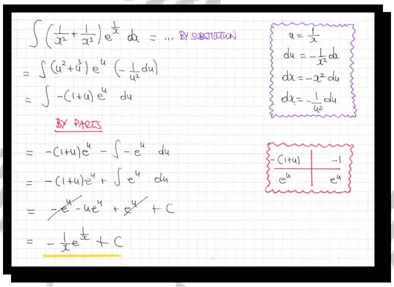


$$\int x(\sin x+\cos x)\,dx = \dots \quad \text{INTEGRATION BY PARTS}$$

x	1
$-\cos x+\sin x$	$\sin x+\cos x$

$$= x(-\cos x+\sin x) - \int -\cos x+\sin x\,dx$$
$$= x(\sin x-\cos x) + \int \cos x - \sin x\,dx$$
$$= x(\sin x-\cos x) + \sin x + \cos x + C$$
$$= (1+x)\sin x + (1-x)\cos x + C$$

364. $$\int \left(\frac{1}{x^2}+\frac{1}{x^3}\right)e^{\frac{1}{x}}\ dx = -\frac{1}{x}e^{\frac{1}{x}} + C$$



$$\int \left(\frac{1}{x^2}+\frac{1}{x^3}\right)e^{\frac{1}{x}}\,dx = \dots \quad \text{BY SUBSTITUTION}$$

$u=\frac{1}{x}$, $du = -\frac{1}{x^2}dx$, $dx = -x^2\,du$, $dx = -\frac{1}{u^2}du$

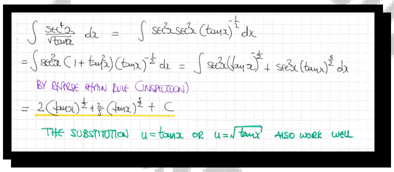
$$= \int (u^2+u^3)e^u\left(-\frac{1}{u^2}du\right)$$
$$= \int -(1+u)e^u\,du$$

BY PARTS

$-(1+u)$	-1
e^u	e^u

$$= -(1+u)e^u - \int -e^u\,du$$
$$= -(1+u)e^u + \int e^u\,du$$
$$= \cancel{-e^u} - ue^u + \cancel{e^u} + C$$
$$= -\frac{1}{x}e^{\frac{1}{x}} + C$$

365. $$\int \frac{\sec^4 x}{\sqrt{\tan x}}\ dx = 2(\tan x)^{\frac{1}{2}} + \frac{2}{5}(\tan x)^{\frac{5}{2}} + C$$



$$\int \frac{\sec^4 x}{\sqrt{\tan x}}\,dx = \int \sec^2 x\sec^2 x(\tan x)^{-\frac{1}{2}}\,dx$$
$$= \int \sec^2 x(1+\tan^2 x)(\tan x)^{-\frac{1}{2}}\,dx = \int \sec^2 x(\tan x)^{-\frac{1}{2}} + \sec^2 x(\tan x)^{\frac{3}{2}}\,dx$$

BY REVERSE CHAIN RULE (INSPECTION)

$$= 2(\tan x)^{\frac{1}{2}} + \frac{2}{5}(\tan x)^{\frac{5}{2}} + C$$

THE SUBSTITUTION $u=\tan x$ OR $u=\sqrt{\tan x}$ ALSO WORK WELL

366. $$\int \frac{\cos x}{(\cos x+\sin x)^3}\,dx = -\frac{1}{2(1+\tan x)^2}+C$$

$$\int \frac{\cos x}{(\cos x+\sin x)^3}\,dx = \int \frac{\cos x}{\cos^3 x\left(1+\frac{\sin x}{\cos x}\right)^3}\,dx$$

$$= \int \sec^2 x(1+\tan x)^{-3}\,dx = \dots$$ BY REVERSE CHAIN RULE (INSPECTION)

$$= -\frac{1}{2}(1+\tan x)^{-2}+C = -\frac{1}{2(1+\tan x)^2}+C$$

ALTERNATIVE BY SUBSTITUTION

$$\int \frac{\cos x}{(\cos x+\sin x)^3}\,dx = \dots$$

$u=\tan x$

$\frac{du}{dx}=\sec^2 x$

$dx=\frac{du}{\sec^2 x}$

$dx=\cos^2 x\,du$

$$= \int \frac{\cos x}{(\cos x+\sin x)^3}(\cos^2 x\,du)$$

$$= \int \frac{\cos^3 x}{(\cos x+\sin x)^3}\,du$$

$$= \int \left(\frac{\cos x}{\cos x+\sin x}\right)^3 du$$

$$= \int \left(\frac{\frac{\cos x}{\cos x}}{\frac{\cos x}{\cos x}+\frac{\sin x}{\cos x}}\right)^3 du = \int \left(\frac{1}{1+\tan x}\right)^3 du$$

$$= \int \frac{1}{(1+u)^3}\,du = \int (1+u)^{-3}\,du = -\frac{1}{2}(1+u)^{-2}+C$$

$$= -\frac{1}{(1+\tan x)^2}+C$$

367. $$\int \frac{20x}{4-x^2}\,dx = -10\ln\left|4-x^2\right|+C$$

$$\int \frac{20x}{4-x^2}\,dx = -10\int \frac{-2x}{4-x^2}\,dx = -10\ln|4-x^2|+C$$

OF THE FORM $\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)|+C$

THE SUBSTITUTION $u=4-x^2$ ALSO WORKS WELL, AS WELL AS PARTIAL FRACTIONS

368. $$\int \frac{\sqrt{x}}{1+\sqrt{x}}\,dx = x-2\sqrt{x}+2\ln\left(1+\sqrt{x}\right)+C$$

$$\int \frac{\sqrt{x}}{1+\sqrt{x}}\,dx = \dots$$ BY SUBSTITUTION

$u=1+\sqrt{x}$

$\sqrt{x}=u-1$

$x=(u-1)^2$

$dx=2(u-1)\,du$

$$= \int \frac{u-1}{u}(2(u-1)\,du) = \int \frac{2(u-1)^2}{u}\,du$$

$$= \int \frac{2u^2-4u+2}{u}\,du = \int 2u-4+\frac{2}{u}\,du$$

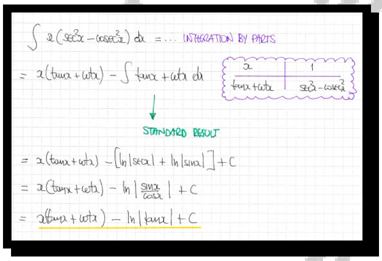
$$= u^2-4u+2\ln|u|+C$$

$$= (1+\sqrt{x})^2-4(1+\sqrt{x})+2\ln(1+\sqrt{x})+C$$

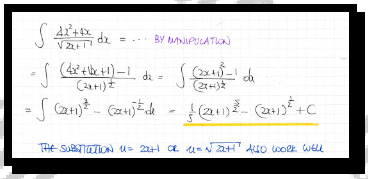
$$= 1+2\sqrt{x}+x-4-4\sqrt{x}+2\ln(1+\sqrt{x})+C$$

$$= x-2\sqrt{x}+2\ln(1+\sqrt{x})+C$$

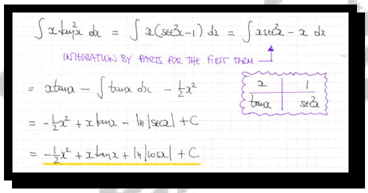
369. $\displaystyle\int x\left(\sec^2 x-\operatorname{cosec}^2 x\right)\,dx = x\left(\tan x+\cot x\right)-\ln\left|\tan x\right| + C$



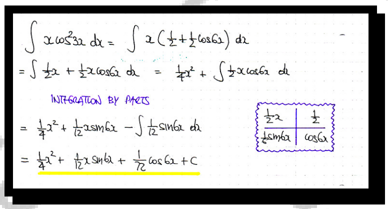
370. $\displaystyle\int \frac{4x^2+4x}{\sqrt{2x+1}}\,dx = \tfrac{1}{5}(2x+1)^{\frac{5}{2}}-(2x+1)^{\frac{1}{2}} + C$



371. $\displaystyle\int x\tan^2 x\,\,dx = -\frac{1}{2}x^2 + x\tan x + \ln\left|\cos x\right| + C$



372. $\displaystyle\int x\cos^2 3x\,\,dx = \frac{1}{4}x^2 + \frac{1}{12}x\sin 6x + \frac{1}{72}\cos 6x + C$



373. $\displaystyle\int \cos x\left(6\sin x - 2\sin 3x\right)^{\frac{2}{3}}\ dx = \frac{4}{3}\sin^3 x + C$

$\displaystyle\int \cos x\left(6\sin x - 2\sin 3x\right)^{\frac{2}{3}}\ dx$ ... BY TRIG IDENTITIES ...

$$\boxed{\begin{aligned}\sin 3x = \sin(2x+x) &= \sin 2x\cos x + \cos 2x\sin x\\ &= (2\sin x\cos x)\cos x + (1-2\sin^2 x)\sin x\\ &= 2\sin x\cos^2 x + \sin x - 2\sin^3 x\\ &= 2\sin x(1-\sin^2 x) + \sin x - 2\sin^3 x\\ &= 2\sin x - 2\sin^3 x + \sin x - 2\sin^3 x\\ &= 3\sin x - 4\sin^3 x\end{aligned}}$$

$$= \int \cos x\left[\cancel{6\sin x} - 2(\cancel{3\sin x} - 4\sin^3 x)\right]^{\frac{2}{3}} dx$$

$$= \int \cos x\left[8\sin^3 x\right]^{\frac{2}{3}} dx = \int \cos x\left[4\sin^2 x\right] dx$$

$$= \int 4\cos x\sin^2 x\ dx = \frac{4}{3}\sin^3 x + C$$

374. $\displaystyle\int \sqrt{x^2 - x^4}\ dx = \begin{cases} -\frac{1}{3}\left(1-x^2\right)^{\frac{3}{2}} + C & x > 0\\ \frac{1}{3}\left(1-x^2\right)^{\frac{3}{2}} + C & x < 0\end{cases}$

$$\int \sqrt{x^2 - x^4}\ dx = \int \sqrt{x^2(1-x^2)}\ dx = \int \sqrt{x^2}\,(1-x^2)^{\frac{1}{2}} dx$$

$$= \int |x|(1-x^2)^{\frac{1}{2}} = \begin{cases} -\frac{1}{3}(1-x^2)^{\frac{3}{2}} + C & \text{IF } x>0\\ \frac{1}{3}(1-x^2)^{\frac{3}{2}} + C & \text{IF } x<0\end{cases}$$

THE SUBSTITUTION $u = 1-x^2$ OR $u = \sqrt{1-x^2}$ MAY ALSO BE USED

375. $\displaystyle\int \frac{x+3}{\sqrt[3]{x^2+6x}}\ dx = \frac{3}{4}\left(x^2+6x\right)^{\frac{2}{3}} + C$

$$\int \frac{x+3}{\sqrt[3]{x^2+6x}}\ dx = \int (x+3)(x^2+6x)^{-\frac{1}{3}} dx = \frac{1}{2}\int (2x+6)(x^2+6x)^{-\frac{1}{3}} dx$$

= ... BY REVERSE CHAIN RULE (INSPECTION)

$$= \frac{1}{2}\times\frac{3}{2}(x^2+6x)^{\frac{2}{3}} + C = \frac{3}{4}(x^2+6x)^{\frac{2}{3}} + C$$

THE SUBSTITUTION $u = (x^2+6x)^{\frac{1}{3}}$ ALSO WORKS

376. $\displaystyle\int \frac{1-x}{1-\sqrt{x}}\,dx = \left[\begin{array}{c} -\frac{2}{3}\left(1-\sqrt{x}\right)^3+3\left(1-\sqrt{x}\right)^3-4\left(1-\sqrt{x}\right)+C \\ \frac{2}{3}x^{\frac{3}{2}}+x+C \end{array}\right]$

$\int \frac{1-x}{1-\sqrt{x}}\,dx = \ldots$ BY MANIPULATION (DIFFERENCE OF SQUARES)

$= \int \frac{\cancel{(1-\sqrt{x})}(1+\sqrt{x})}{\cancel{1-\sqrt{x}}}\,dx = \int 1+x^{\frac{1}{2}}\,dx = x+\frac{2}{3}x^{\frac{3}{2}}+C$

ALTERNATIVE BY SUBSTITUTION (LENGTHY)

$\int \frac{1-x}{1-\sqrt{x}}\,dx = \int \frac{1-(1-u)^2}{u}\left[-2(1-u)\,du\right]$

$u = 1-\sqrt{x}$
$\sqrt{x} = 1-u$
$x = (1-u)^2$
$dx = -2(1-u)\,du$

$= \int \frac{\cancel{1}-(\cancel{1}-2u+u^2)}{u}\left[2(u-1)\,du\right]$

$= \int \frac{2(u-1)(2u-u^2)}{u}\,du = \int 2(u-1)(2-u)\,du$

$= -\int 2(u-1)(u-2)\,du = \int -2u^2+6u-4\,du$

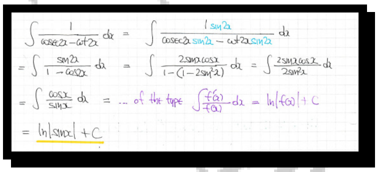
$= -\frac{2}{3}u^3+3u^2-4u+C = -\frac{2}{3}\left[1-\sqrt{x}\right]^3+3(1-\sqrt{x})^2-4(1-\sqrt{x})+C$

$= -\frac{2}{3}\left[\cancel{1}-3\sqrt{x}+3x-x^{\frac{3}{2}}\right]+3\left[\cancel{1}-2\sqrt{x}+x\right]\cancel{-4}+4\sqrt{x}+C$

$= \frac{2}{3}x^{\frac{3}{2}}-2x+\cancel{2\sqrt{x}}-\cancel{6\sqrt{x}}+3x+\cancel{4\sqrt{x}}+C$

$= \frac{2}{3}x^{\frac{3}{2}}+x+C$ (AS BEFORE)

377. $\displaystyle\int \frac{1}{\operatorname{cosec}2x-\cot 2x}\,dx = \ln\left|\sin x\right|+C$



$\int \frac{1}{\operatorname{cosec}2x-\cot 2x}\,dx = \int \frac{1\sin 2x}{\operatorname{cosec}2x\sin 2x-\cot 2x\sin 2x}\,dx$

$= \int \frac{\sin 2x}{1-\cos 2x}\,dx = \int \frac{2\sin x\cos x}{1-(1-2\sin^2 x)}\,dx = \int \frac{2\sin x\cos x}{2\sin^2 x}\,dx$

$= \int \frac{\cos x}{\sin x}\,dx = \ldots$ of the type $\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)|+C$

$= \ln|\sin x|+C$

378. $\displaystyle\int \sec^3 x\tan^5 x\,dx = \frac{1}{7}\sec^7 x-\frac{2}{5}\sec^5 x+\frac{1}{3}\sec^3 x+C$

$\int \sec^3 x\tan^5 x\,dx = \ldots$ BY MANIPULATION ...

$= \int \sec^3 x\tan^4 x\tan x\,dx = \int \sec^3 x(\sec^2 x-1)^2\tan x\,dx$

$= \int \sec^3 x(\sec^4 x-2\sec^2 x+1)\tan x\,dx = \int (\sec^7 x-2\sec^5 x+\sec^3 x)\tan x\,dx$

$= \int (\sec^6 x-2\sec^4 x+\sec^2 x)\sec x\tan x$

BY REVERSE CHAIN RULE AS $\frac{d}{dx}(\sec x) = \sec x\tan x$

$= \frac{1}{7}\sec^7 x-\frac{2}{5}\sec^5 x+\frac{1}{3}\sec^3 x+C$

ALTERNATIVE BY SUBSTITUTION

$\int \sec^3 x\tan^5 x\,dx = \int \sec^3 x\tan^5 x\,\frac{du}{\sec x\tan x}$

$u = \sec x$
$\frac{du}{dx} = \sec x\tan x$
$dx = \frac{du}{\sec x\tan x}$

$= \int \sec^2 x\tan^4 x\,du = \int \sec^2 x(\sec^2 x-1)^2\,du$

$= \int u^2(u^2-1)^2\,du = \int u^6-2u^4+u^2\,du$

$= \frac{1}{7}u^7-\frac{2}{5}u^5+\frac{1}{3}u^3+C = \frac{1}{7}\sec^7 x-\frac{2}{5}\sec^5 x+\frac{1}{3}\sec^3 x+C$

THE SUBSTITUTION $u = \cos x$ ALSO WORKS WELL