INTEGRATION

BY PARTIAL FRACTIONS

Question 1

Carry out each of the following integrations.

1. $$\int \frac{17-4x}{(x-2)(x+1)}\,dx = 3\ln|x-2| - 7\ln|x+1| + C$$

2. $$\int \frac{2-x}{(x+1)(2x-1)}\,dx = \frac{1}{2}\ln|2x-1| - \ln|x+1| + C$$

3. $$\int \frac{4}{(x-2)(2-3x)}\,dx = \ln\left|\frac{3x-2}{x-2}\right| + C$$

4. $$\int \frac{5x-7}{(x-1)(5x-3)}\,dx = 2\ln|5x-3| - \ln|x-1| + C$$

5. $$\int \frac{18x-1}{(2x+1)(3x-1)}\,dx = 2\ln|2x+1| + \ln|3x-1| + C$$

6. $$\int \frac{3x-5}{x(1-x)}\,dx = 2\ln|x-1| - 5\ln|x| + C$$

7. $$\int \frac{7x-19}{x^2-2x-15}\,dx = 5\ln|x+3| + 2\ln|x-5| + C$$

8. $$\int \frac{x^2+14x+1}{(x+3)(x-5)(x+7)}\,dx = \ln\left|\frac{x^2-2x-15}{x+7}\right| + C$$

9. $$\int \frac{7x+4}{(x-2)(x+1)^2}\,dx = 2\ln\left|\frac{x-2}{x+1}\right| - \frac{1}{x+1} + C$$

10. $$\int \frac{2x^2+x+8}{(x-2)(x+1)^2}\,dx = 2\ln|x-2| + \frac{3}{x+1} + C$$

1. $\displaystyle\int \frac{17-4x}{(x-2)(x+1)}\,dx = \int \frac{3}{x-2} - \frac{7}{x+1}\,dx$

$= 3\ln|x-2| - 7\ln|x+1| + C$

BY PARTIAL FRACTIONS

$\frac{17-4x}{(x-2)(x+1)} \equiv \frac{A}{x-2} + \frac{B}{x+1}$

$17-4x \equiv A(x+1) + B(x-2)$

- IF $x=2$, $9=3A \Rightarrow A=3$
- IF $x=-1$, $21=-3B \Rightarrow B=-7$

2. $\displaystyle\int \frac{2-x}{(x+1)(2x-1)}\,dx = \int \frac{1}{2x-1} - \frac{1}{x+1}\,dx$

$= \frac{1}{2}\ln|2x-1| - \ln|x+1| + C$

BY PARTIAL FRACTIONS

$\frac{2-x}{(x+1)(2x-1)} \equiv \frac{A}{x+1} + \frac{B}{2x-1}$

$2-x \equiv A(2x-1) + B(x+1)$

- IF $x=-1 \Rightarrow 3=-3A \Rightarrow A=-1$
- IF $x=0 \Rightarrow 2=-A+B \Rightarrow B=1$

3. $\displaystyle\int \frac{4}{(x-2)(2-3x)}\,dx = \int \frac{-3}{2-3x} - \frac{1}{x-2}\,dx$

$= \ln|2-3x| - \ln|x-2| + C$

$= \ln|3x-2| - \ln|x-2| + C$

$= \ln\left|\frac{3x-2}{x-2}\right| + C$

BY PARTIAL FRACTIONS

$\frac{4}{(x-2)(2-3x)} \equiv \frac{A}{x-2} + \frac{B}{2-3x}$

$4 \equiv A(2-3x) + B(x-2)$

- IF $x=2$, $4=-4A \Rightarrow A=-1$
- IF $x=0$, $4=2A-2B$

  $4=-2-2B$

  $2B=-6$

  $B=-3$

4. $\displaystyle\int \frac{5x-7}{(x-1)(5x-3)}\,dx = \int \frac{10}{5x-3} - \frac{1}{x-1}\,dx$

$= 2\ln|5x-3| - \ln|x-1| + C$

BY PARTIAL FRACTION

$\frac{5x-7}{(x-1)(5x-3)} \equiv \frac{A}{x-1} + \frac{B}{5x-3}$

$5x-7 \equiv A(5x-3) + B(x-1)$

- IF $x=1$, $-2=2A \Rightarrow A=-1$
- IF $x=\frac{3}{5}$, $-4=-\frac{2}{5}B \Rightarrow B=10$

5. $\displaystyle\int \frac{18x-1}{(2x+1)(3x-1)}\,dx$

$= \displaystyle\int \frac{4}{2x+1} + \frac{3}{3x-1}\,dx$

$= 2\ln|2x+1| + \ln|3x+1| + C$

BY PARTIAL FRACTIONS

$\frac{18x-1}{(2x+1)(3x-1)} \equiv \frac{A}{2x+1} + \frac{B}{3x-1}$

$18x-1 \equiv A(3x-1) + B(2x+1)$

- IF $x=\frac{1}{3}$, $5=\frac{5}{3}B \Rightarrow B=3$
- IF $x=-\frac{1}{2}$, $-10=-\frac{5}{2}A \Rightarrow A=4$

6. $\displaystyle\int \frac{3x-5}{x(1-x)}\,dx = \int -\frac{2}{1-x} - \frac{5}{x}\,dx$

$= 2\ln|1-x| - 5\ln|x| + C$

$= 2\ln|x-1| - 5\ln|x| + C$

BY PARTIAL FRACTIONS

$\frac{3x-5}{x(1-x)} \equiv \frac{A}{x} + \frac{B}{1-x}$

$3x-5 \equiv A(1-x) + Bx$

- IF $x=1$, $-2=B \Rightarrow B=-2$
- IF $x=0$, $-5=A \Rightarrow A=-5$

7. $\displaystyle\int \frac{7x-19}{x^2-2x-15}\,dx = \int \frac{7x-19}{(x+3)(x-5)}\,dx$

$= \displaystyle\int \frac{5}{x+3} + \frac{2}{x}\,dx$

$= 5\ln|x+3| + 2\ln|x-5| + C$

BY PARTIAL FRACTIONS

$\frac{7x-19}{(x+3)(x-5)} \equiv \frac{A}{x+3} + \frac{B}{x-5}$

$7x-19 \equiv A(x-5) + B(x+3)$

- IF $x=5$, $16=8B \Rightarrow B=2$
- IF $x=-3$, $-40=-8A \Rightarrow A=5$

8. $\displaystyle\int \frac{x^2+14x+1}{(x+3)(x-5)(x+7)}\,dx$

$= \displaystyle\int \frac{1}{x+3} - \frac{1}{x+7} + \frac{1}{x-5}\,dx$

$= \ln|x+3| - \ln|x+7| + \ln|x-5| + C$

$= \ln\left|\frac{x^2-2x-15}{x+7}\right| + C$

BY PARTIAL FRACTIONS

$\frac{x^2+14x+1}{(x+3)(x-5)(x+7)} \equiv \frac{A}{x+3} + \frac{B}{x-5} + \frac{C}{x+7}$

$x^2+14x+1 \equiv A(x-5)(x+7) + B(x+3)(x+7) + C(x+3)(x-5)$

- IF $x=5$, $96=8\times12B \Rightarrow B=1$
- IF $x=-7$, $-48=(-4)(-12)C \Rightarrow C=-1$
- IF $x=-3$, $-32=-8\times4A \Rightarrow A=1$

9. $\displaystyle\int \frac{7x+4}{(x-2)(x+1)^2}\,dx = \int \frac{2}{x-2} + (x+1)^{-2} + \frac{2}{x+1}\,dx$

$= 2\ln|x-2| - (x+1)^{-1} + 2\ln|x+1| + C$

$= 2\ln\left|\frac{x-2}{x+1}\right| - \frac{1}{x+1} + C$

BY PARTIAL FRACTIONS

$\frac{7x+4}{(x-2)(x+1)^2} \equiv \frac{A}{x-2} + \frac{B}{(x+1)^2} + \frac{C}{x+1}$

$7x+4 \equiv A(x+1)^2 + B(x-2) + C(x+1)(x-2)$

- IF $x=2$, $18=9A \Rightarrow A=2$
- IF $x=-1$, $-3=-3B \Rightarrow B=1$
- IF $x=0$, $4=A-2B-2C$

  $4=2-2-2C$

  $2C=4$

  $C=2$

10. $\displaystyle\int \frac{2x^2+x+8}{(x-2)(x+1)^2}\,dx = \int \frac{2}{x-2} - 3(x+1)^{-2}\,dx$

$= 2\ln|x-2| + 3(x+1)^{-1} + C$

$= 2\ln|x-2| + \frac{3}{x+1} + C$

BY PARTIAL FRACTIONS

$\frac{2x^2+x+8}{(x-2)(x+1)^2} \equiv \frac{A}{x-2} + \frac{B}{(x+1)^2} + \frac{C}{x+1}$

$2x^2+x+8 \equiv A(x+1)^2 + B(x-2) + C(x+1)(x-2)$

- IF $x=2$, $18=9A \Rightarrow A=2$
- IF $x=-1$, $9=-3B \Rightarrow B=-3$
- IF $x=0$, $8=A-2B-2C$

  $8=2+6-2C$

  $2C=0$

  $C=0$

Question 2

Carry out each of the following integrations.

1. $\displaystyle\int \frac{3x-1}{(2x+1)(x-2)}\,dx = \frac{1}{2}\ln|2x+1| + \ln|x-2| + C$

2. $\displaystyle\int \frac{2}{(x-2)(x-4)}\,dx = \ln\left|\frac{x-4}{x-2}\right| + C$

3. $\displaystyle\int \frac{3}{(2+x)(1-x)}\,dx = \ln\left|\frac{2+x}{1-x}\right| + C$

4. $\displaystyle\int \frac{1}{(x+1)(x+2)}\,dx = \ln\left|\frac{x+1}{x+2}\right| + C$

5. $\displaystyle\int \frac{x+1}{9x^2-1}\,dx = \frac{2}{9}\ln|3x-1| - \frac{1}{9}\ln|3x+1| + C$

6. $\displaystyle\int \frac{6}{x^2-2x-8}\,dx = \ln\left|\frac{x-4}{x+2}\right| + C$

7. $\displaystyle\int \frac{17-5x}{(2x+3)(2-x)^2}\,dx = \ln\left|\frac{2x+3}{2-x}\right| + \frac{1}{2-x} + C$

8. $\displaystyle\int \frac{14x+1}{(1-x)(2x+1)}\,dx = -5\ln|1-x| - 2\ln|2x+1| + C$

9. $\displaystyle\int \frac{4x^2-6x+5}{(2-x)(2x-1)^2}\,dx = -\frac{1}{2x-1} - \ln|2-x| + C$

10. $\displaystyle\int \frac{x+2}{x(x-1)}\,dx = 3\ln|x-1| - 2\ln|x| + C$

1. $\displaystyle\int \frac{3x-1}{(2x+1)(x-2)}\,dx = \ldots$ PARTIAL FRACTIONS $= \ldots$

$\dfrac{3x-1}{(2x+1)(x-2)} \equiv \dfrac{A}{2x+1} + \dfrac{B}{x-2}$

$3x-1 \equiv A(x-2) + B(2x+1)$

If $x=2 \Rightarrow 5 = 5B \Rightarrow B=1$

If $x=-\frac{1}{2} \Rightarrow -\frac{5}{2} = -\frac{5}{2}A \Rightarrow A=1$

$\displaystyle = \int \frac{1}{2x+1} + \frac{1}{x-2}\,dx = \frac{1}{2}\ln|2x+1| + \ln|x-2| + C$

2. $\displaystyle\int \frac{2}{(x-2)(x-4)}\,dx = \ldots$ PARTIAL FRACTIONS

$\displaystyle = \int \frac{1}{x-4} - \frac{1}{x-2}\,dx$

$= \ln|x-4| - \ln|x-2| + C$

$= \ln\left|\dfrac{x-4}{x-2}\right| + C$

$\dfrac{2}{(x-2)(x-4)} \equiv \dfrac{A}{x-2} + \dfrac{B}{x-4}$

$2 \equiv A(x-4) + B(x-2)$

If $x=4 \Rightarrow 2 = 2B \Rightarrow B=1$

If $x=2 \Rightarrow 2 = -2A \Rightarrow A=-1$

3. $\displaystyle\int \frac{3}{(2+x)(1-x)}\,dx = \ldots$ PARTIAL FRACTIONS

$\displaystyle = \int \frac{1}{2+x} + \frac{1}{1-x}\,dx$

$= \ln|2+x| - \ln|1-x| + C$

$= \ln\left|\dfrac{x+2}{1-x}\right| + C$

$\dfrac{3}{(2+x)(1-x)} \equiv \dfrac{A}{2+x} + \dfrac{B}{1-x}$

$3 \equiv A(1-x) + B(2+x)$

If $x=1 \Rightarrow 3 = 3B \Rightarrow B=1$

If $x=-2 \Rightarrow 3 = 3A \Rightarrow A=1$

4. $\displaystyle\int \frac{1}{(x+1)(x+2)}\,dx = \ldots$ PARTIAL FRACTIONS

$\displaystyle = \int \frac{1}{x+1} - \frac{1}{x+2}\,dx$

$= \ln|x+1| - \ln|x+2| + C$

$= \ln\left|\dfrac{x+1}{x+2}\right| + C$

$\dfrac{1}{(x+1)(x+2)} \equiv \dfrac{A}{x+1} + \dfrac{B}{x+2}$

$1 \equiv A(x+2) + B(x+1)$

If $x=-2 \Rightarrow 1 = -B \Rightarrow B=-1$

If $x=-1 \Rightarrow 1 = A \Rightarrow A=1$

5. $\displaystyle\int \frac{x+1}{9x^2-1}\,dx = \int \frac{x+1}{(3x-1)(3x+1)}\,dx$

$\displaystyle = \int \frac{\frac{2}{3}}{3x-1} - \frac{\frac{1}{3}}{3x+1}\,dx$

$= \frac{2}{9}\ln|3x-1| - \frac{1}{9}\ln|3x+1| + C$

BY PARTIAL FRACTIONS

$\dfrac{x+1}{(3x-1)(3x+1)} \equiv \dfrac{A}{3x+1} + \dfrac{B}{3x-1}$

$x+1 \equiv A(3x-1) + B(3x+1)$

- If $x=\frac{1}{3} \Rightarrow \frac{4}{3} = 2B \Rightarrow B=\frac{2}{3}$
- If $x=-\frac{1}{3} \Rightarrow \frac{2}{3} = -2A \Rightarrow A=-\frac{1}{3}$

6. $\displaystyle\int \frac{6}{x^2-2x-8}\,dx = \int \frac{6}{(x-4)(x+2)}\,dx$

$\displaystyle = \int \frac{1}{x-4} - \frac{1}{x+2}\,dx$

$= \ln|x-4| - \ln|x+2| + C$

$= \ln\left|\dfrac{x-4}{x+2}\right| + C$

$\dfrac{6}{(x-4)(x+2)} \equiv \dfrac{A}{x-4} + \dfrac{B}{x+2}$

$6 \equiv A(x+2) + B(x-4)$

- If $x=4 \Rightarrow 6 = 6A \Rightarrow A=1$
- If $x=-2 \Rightarrow 6 = -6B \Rightarrow B=-1$

7. $\displaystyle\int \frac{17-5x}{(2x+3)(2-x)^2}\,dx =$

$\displaystyle = \int \frac{2}{2x+3} + (2-x)^{-2} + \frac{1}{2-x}\,dx$

$= \ln|2x+3| + (2-x)^{-1} - \ln|2-x| + C$

$= \ln\left|\dfrac{2x+3}{2-x}\right| + \dfrac{1}{2-x} + C$

$\dfrac{17-5x}{(2x+3)(2-x)^2} \equiv \dfrac{A}{2x+3} + \dfrac{B}{(2-x)^2} + \dfrac{C}{2-x}$

$17-5x \equiv A(2-x)^2 + B(2x+3) + C(2x+3)(2-x)$

- If $x=2 \Rightarrow 7 = 7B \Rightarrow B=1$
- If $x=-\frac{3}{2} \Rightarrow 17+\frac{15}{2} = \frac{49}{4}A$
  $\Rightarrow 68+30 = 49A$
  $\Rightarrow A=2$
- If $x=0 \Rightarrow 17 = 4A+3B+6C$
  $\Rightarrow 17 = 8+3+6C$
  $\Rightarrow C=1$

8. $\displaystyle\int \frac{14x+1}{(1-x)(2x+1)}\,dx =$

$\displaystyle \int \frac{5}{1-x} - \frac{4}{2x+1}\,dx =$

$-5\ln|1-x| - 2\ln|2x+1| + C$

$\dfrac{14x+1}{(1-x)(2x+1)} \equiv \dfrac{A}{1-x} + \dfrac{B}{2x+1}$

$14x+1 \equiv A(2x+1) + B(1-x)$

- If $x=1 \Rightarrow 15 = 3A \Rightarrow A=5$
- If $x=-\frac{1}{2} \Rightarrow -6 = \frac{3}{2}B \Rightarrow B=-4$

9. $\displaystyle\int \frac{4x^2-6x+5}{(2-x)(2x-1)^2}\,dx$

$\displaystyle = \int \frac{1}{2-x} + 2(2x-1)^{-2}\,dx$

$= -\ln|2-x| - (2x-1)^{-1} + C$

$= -\ln|2-x| - \dfrac{1}{2x-1} + C$

$\dfrac{4x^2-6x+5}{(2-x)(2x-1)^2} \equiv \dfrac{A}{2-x} + \dfrac{B}{(2x-1)^2} + \dfrac{C}{2x-1}$

$4x^2-6x+5 \equiv A(2x-1)^2 + B(2-x) + C(2x-1)(2-x)$

- If $x=2 \Rightarrow 16-12+5 = 9A \Rightarrow A=1$
- If $x=\frac{1}{2} \Rightarrow 1-3+5 = \frac{3}{2}B \Rightarrow B=2$
- If $x=0 \Rightarrow 5 = A+2B-2C$
  $5 = 1+4-2C$
  $C=0$

10. $\displaystyle\int \frac{x+2}{x(x-1)}\,dx$

$\displaystyle = \int \frac{3}{x-1} - \frac{2}{x}\,dx$

$= 3\ln|x-1| - 2\ln x + C$

$\dfrac{x+2}{x(x-1)} \equiv \dfrac{A}{x} + \dfrac{B}{x-1}$

$x+2 \equiv A(x-1) + Bx$

- If $x=0 \Rightarrow 2 = -A$
- If $x=1 \Rightarrow 3 = B$

Question 3

Carry out each of the following integrations.

1. $\displaystyle\int \frac{10x^2-23x+11}{(2-3x)(2x-1)^2}\,dx = -\frac{2}{2x-1}-\frac{1}{3}\ln|2-3x|-\frac{1}{2}\ln|2x-1|+C$

2. $\displaystyle\int \frac{1}{x^2(x-1)}\,dx = \frac{1}{x}+\ln\left|\frac{x-1}{x}\right|+C$

3. $\displaystyle\int \frac{8(x^2+1)}{(x-3)(x+1)^2}\,dx = 5\ln|x-3|+3\ln|x+1|+\frac{4}{x+1}+C$

4. $\displaystyle\int \frac{1}{x(x-2)}\,dx = \frac{1}{2}\ln\left|\frac{x-2}{x}\right|+C$

5. $\displaystyle\int \frac{1}{x^2-4}\,dx = \frac{1}{4}\ln\left|\frac{x-2}{x+2}\right|+C$

6. $\displaystyle\int \frac{4x^2-x+1}{(x-1)(2x-1)}\,dx = 2x+4\ln|x-1|-\frac{3}{2}\ln|2x-1|+C$

7. $\displaystyle\int \frac{2}{x(x^2-1)}\,dx = \ln\left|\frac{x^2-1}{x^2}\right|+C$

8. $\displaystyle\int \frac{2x^2+5x-1}{x^3+x^2-2x}\,dx = 2\ln|x-1|+\frac{1}{2}\ln\left|\frac{x}{x+2}\right|+C$

9. $\displaystyle\int \frac{1}{x^2-4}\,dx = \frac{1}{4}\ln\left|\frac{x-2}{x+2}\right|+C$

10. $\displaystyle\int \frac{2}{2x-x^2}\,dx = \ln\left|\frac{x}{2-x}\right|+C$

1. $\int \frac{10x^2-23x+11}{(2-3x)(2x-1)^2}\,dx$ = BY PARTIAL FRACTIONS

$\frac{10x^2-23x+11}{(2-3x)(2x-1)^2} \equiv \frac{A}{2-3x} + \frac{B}{(2x-1)^2} + \frac{C}{2x-1}$

$10x^2-23x+11 \equiv A(2x-1)^2 + B(2-3x) + C(2x-1)(2-3x)$

- If $x=\frac{1}{2} \Rightarrow \frac{5}{2}-\frac{23}{2}+11 = \frac{1}{2}B \Rightarrow \boxed{B=4}$
- If $x=\frac{2}{3} \Rightarrow \frac{40}{9}-\frac{46}{3}+11 = \frac{1}{9}A \Rightarrow \boxed{A=1}$
- If $x=0 \Rightarrow \left.\begin{array}{l} 11 = A+2B-2C \\ 11 = 1+8-2C \end{array}\right\} \Rightarrow \boxed{C=-1}$

$= \int \frac{1}{2-3x} + 4(2x-1)^{-2} - \frac{1}{2x-1}\,dx$

$= -\frac{1}{3}\ln|2-3x| - 2(2x-1)^{-1} - \frac{1}{2}\ln|2x-1| + C$

$= -\frac{1}{3}\ln|2-3x| - \frac{2}{2x-1} - \frac{1}{2}\ln|2x-1| + C$

2. $\int \frac{1}{x^2(x-1)}\,dx$ = BY PARTIAL FRACTIONS

$\frac{1}{x^2(x-1)} \equiv \frac{A}{x^2} + \frac{B}{x} + \frac{C}{x-1}$

$1 \equiv A(x-1) + Bx(x-1) + Cx^2$

- If $x=1 \Rightarrow \boxed{1=C}$
- If $x=0 \Rightarrow 1=-A \Rightarrow \boxed{A=-1}$
- If $x=2 \Rightarrow \left.\begin{array}{l} 1 = A+2B+4C \\ 1 = -1+2B+4 \end{array}\right\} \Rightarrow \boxed{B=-1}$

$= \int -x^{-2} - \frac{1}{x} + \frac{1}{x-1}\,dx$

$= x^{-1} - \ln|x| + \ln|x-1| + C$

$= \frac{1}{x} + \ln\left|\frac{x-1}{x}\right| + C$

3. $\int \frac{8(x^2+1)}{(x-3)(x+1)^2}\,dx$ = BY PARTIAL FRACTIONS

$\frac{8(x^2+1)}{(x-3)(x+1)^2} \equiv \frac{A}{x-3} + \frac{B}{(x+1)^2} + \frac{C}{x+1}$

$8(x^2+1) \equiv A(x+1)^2 + B(x-3) + C(x+1)(x-3)$

- If $x=-1 \Rightarrow 16=-4B \Rightarrow \boxed{B=-4}$
- If $x=3 \Rightarrow 80=16A \Rightarrow \boxed{A=5}$
- If $x=0 \Rightarrow \left.\begin{array}{l} 8 = A-3B-3C \\ 8 = 5+12-3C \end{array}\right\} \Rightarrow \boxed{C=3}$

$= \int \frac{5}{x-3} - 4(x+1)^{-2} + \frac{3}{x+1}\,dx$

$= 5\ln|x-3| + 4(x+1)^{-1} + 3\ln|x+1| + C$

$= 5\ln|x-3| + 3\ln|x+1| + \frac{4}{x+1} + C$

4. $\int \frac{1}{x(x-2)}\,dx$ = BY PARTIAL FRACTIONS

$\frac{1}{x(x-2)} \equiv \frac{A}{x} + \frac{B}{x-2}$

$1 \equiv A(x-2) + Bx$

If $x=0 \Rightarrow 1=-2A \Rightarrow A=-\frac{1}{2}$

If $x=2 \Rightarrow 1=2B \Rightarrow B=\frac{1}{2}$

$\int \frac{-\frac{1}{2}}{x} + \frac{\frac{1}{2}}{x-2}\,dx = -\frac{1}{2}\ln|x| + \frac{1}{2}\ln|x-2| + C$

$= \frac{1}{2}\left[\ln|x-2| - \ln|x|\right] + C = \frac{1}{2}\ln\left|\frac{x-2}{x}\right| + C$

5. $\int \frac{1}{x^2-4}\,dx = \int \frac{1}{(x+2)(x-2)}\,dx$ = BY PARTIAL FRACTIONS

$\frac{1}{(x+2)(x-2)} \equiv \frac{A}{x+2} + \frac{B}{x-2}$

$1 \equiv A(x-2) + B(x+2)$

- If $x=2$ $4B=1 \Rightarrow B=\frac{1}{4}$
- If $x=-2$ $-4A=1 \Rightarrow A=-\frac{1}{4}$

$= \int \frac{\frac{1}{4}}{x-2} - \frac{\frac{1}{4}}{x+2}\,dx = \frac{1}{4}\ln|x-2| - \frac{1}{4}\ln|x+2| + C$

$= \frac{1}{4}\left[\ln|x-2| - \ln|x+2|\right] + C = \frac{1}{4}\ln\left|\frac{x-2}{x+2}\right| + C$

6. $\int \frac{4x^2-x+1}{(x-1)(2x-1)}\,dx$ BY PARTIAL FRACTIONS

$\frac{4x^2-x+1}{(x-1)(2x-1)} \equiv A + \frac{B}{x-1} + \frac{C}{2x-1}$

$4x^2-x+1 \equiv A(x-1)(2x-1) + B(2x-1) + C(x-1)$

- If $x=1 \Rightarrow \boxed{4=B}$
- If $x=\frac{1}{2} \Rightarrow \frac{3}{2}=-\frac{1}{2}C \Rightarrow \boxed{C=-3}$
- If $x=0 \Rightarrow \left.\begin{array}{l} 1 = A-B-C \\ 1 = A-4+3 \end{array}\right\} \Rightarrow \boxed{A=2}$

$= \int 2 + \frac{4}{x-1} - \frac{3}{2x-1}\,dx$

$= 2x + 4\ln|x-1| - \frac{3}{2}\ln|2x-1| + C$

7. $\int \frac{2}{x(x^2-1)}\,dx = \int \frac{2}{x(x+1)(x-1)}\,dx$

$\frac{2}{x(x+1)(x-1)} \equiv \frac{A}{x} + \frac{B}{x+1} + \frac{C}{x-1}$

$2 \equiv A(x+1)(x-1) + Bx(x-1) + Cx(x+1)$

- If $x=1$ $2=2C \Rightarrow \boxed{C=1}$
- If $x=-1$ $2=2B \Rightarrow \boxed{B=1}$
- If $x=0$ $2=-A \Rightarrow \boxed{A=-2}$

$= \int -\frac{2}{x} + \frac{1}{x+1} + \frac{1}{x-1}\,dx$

$= -2\ln|x| + \ln|x+1| + \ln|x-1| + C$

$= \ln|x^2-1| - \ln|x^2| + C = \ln\left|\frac{x^2-1}{x^2}\right| + C$

8. $\int \frac{2x^2+5x-1}{x^3+x^2-2x}\,dx = \int \frac{2x^2+5x-1}{x(x^2+x-2)}\,dx$

$\frac{2x^2+5x-1}{x(x-1)(x+2)} \equiv \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+2}$

$2x^2+5x-1 \equiv A(x-1)(x+2) + Bx(x+2) + Cx(x-1)$

- If $x=1 \Rightarrow 6=3B \Rightarrow \boxed{B=2}$
- If $x=0 \Rightarrow -1=-2A \Rightarrow \boxed{A=\frac{1}{2}}$
- If $x=-2 \Rightarrow -3=6C \Rightarrow \boxed{C=-\frac{1}{2}}$

$= \int \frac{2x^2+5x-1}{x(x-1)(x+2)}\,dx = \int \frac{2}{x-1} + \frac{\frac{1}{2}}{x} - \frac{\frac{1}{2}}{x+2}\,dx$

$= 2\ln|x-1| + \frac{1}{2}\ln|x| - \frac{1}{2}\ln|x+2| + C$

$= 2\ln|x-1| + \frac{1}{2}\ln\left|\frac{x}{x+2}\right| + C$

9. $\int \frac{1}{x^2-4}\,dx = \int \frac{1}{(x-2)(x+2)}\,dx$

$\frac{1}{(x-2)(x+2)} \equiv \frac{A}{x-2} + \frac{B}{x+2}$

$1 \equiv A(x+2) + B(x-2)$

- If $x=2 \Rightarrow 1=4A \Rightarrow A=\frac{1}{4}$
- If $x=-2 \Rightarrow 1=-4B \Rightarrow B=-\frac{1}{4}$

$= \int \frac{\frac{1}{4}}{x-2} - \frac{\frac{1}{4}}{x+2} = \frac{1}{4}\ln|x-2| - \frac{1}{4}\ln|x+2| + C$

$= \frac{1}{4}\ln\left|\frac{x-2}{x+2}\right| + C$

10. $\int \frac{2}{2x-x^2}\,dx = \int \frac{2}{x(2-x)}\,dx$

$\frac{2}{x(2-x)} \equiv \frac{A}{x} + \frac{B}{2-x}$

$2 \equiv A(2-x) + Bx$

If $x=0 \Rightarrow 2=2A \Rightarrow \boxed{A=1}$

If $x=2 \Rightarrow 2=2B \Rightarrow \boxed{B=1}$

$= \int \frac{1}{x} + \frac{1}{2-x}\,dx$

$= \ln x - \ln|2-x| + C$

$= \ln\left|\frac{x}{2-x}\right| + C$

Question 4

Carry out each of the following integrations.

1. $\displaystyle\int_0^1 \frac{3x}{(x+1)(x-2)}\,dx = -\ln 2$

2. $\displaystyle\int_{\frac{1}{6}}^{\frac{1}{3}} \frac{14x+1}{(2x+1)(1-x)}\,dx = 3\ln\left(\frac{5}{4}\right)$

3. $\displaystyle\int_0^{\frac{1}{2}} \frac{1}{(1-x)(1+x)^2}\,dx = \frac{1}{6}+\frac{1}{4}\ln 3$

4. $\displaystyle\int_2^6 \frac{5x+3}{(2x-3)(x+2)}\,dx = \ln 54$

5. $\displaystyle\int_0^{\frac{1}{2}} \frac{3-5x}{(1-x)(2-3x)}\,dx = \frac{4}{3}\ln 2$

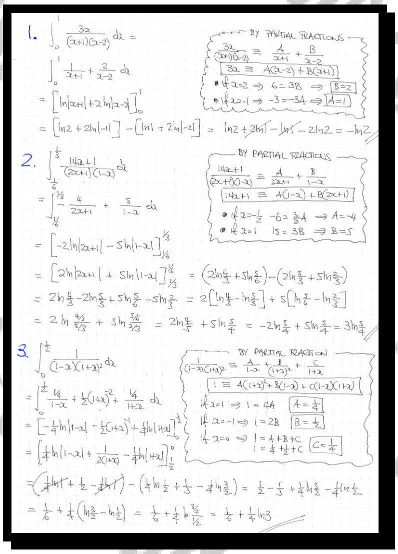
6. $\displaystyle\int_{-1}^1 \frac{9+4x^2}{9-4x^2}\,dx = -2+3\ln 5$

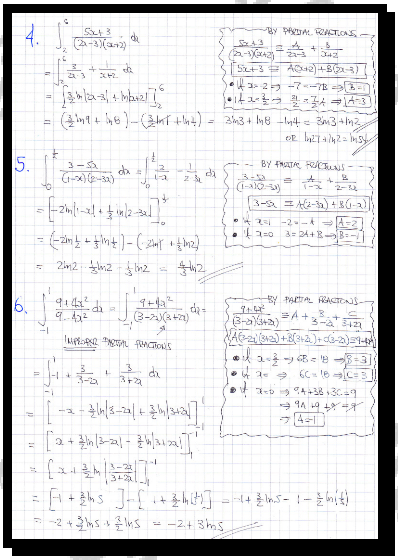
7. $\displaystyle\int_0^1 \frac{18-4x-x^2}{(4-3x)(1+x)^2}\,dx = \frac{7}{3}\ln 2+\frac{3}{2}$

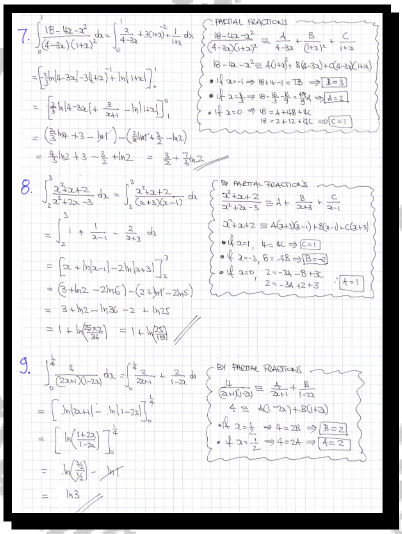
8. $\displaystyle\int_2^3 \frac{x^2+x+2}{x^2+2x-3}\,dx = 1+\ln\left(\frac{25}{18}\right)$

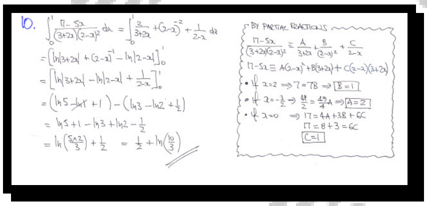
9. $\displaystyle\int_0^{\frac{1}{4}} \frac{4}{(2x+1)(1-2x)}\,dx = \ln 3$

10. $\displaystyle\int_0^1 \frac{17-5x}{(3+2x)(2-x)^2}\,dx = \frac{1}{2}+\ln\left(\frac{10}{3}\right)$









Question 5

Carry out each of the following integrations.

1. $\displaystyle\int_4^9 \frac{5x^2-8x+1}{2x(x-1)^2}\,dx = \ln\left(\frac{32}{3}\right)-\frac{5}{24}$

2. $\displaystyle\int_0^1 \frac{x^2}{x^2-4}\,dx = 1-\ln 3$

3. $\displaystyle\int_0^5 \frac{1}{(x+1)(x+2)(x+3)}\,dx = \ln\left(\frac{8}{7}\right)$

4. $\displaystyle\int_0^1 \frac{10}{(x+1)(x+3)(2x+1)}\,dx = 3\ln 3-3\ln 2$

5. $\displaystyle\int_0^4 \frac{13-2x}{(x+4)(2x+1)}\,dx = 4\ln 3-3\ln 2$

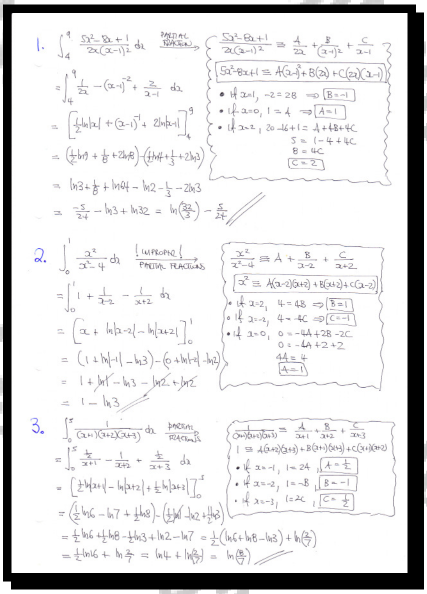
6. $\displaystyle\int_2^6 \frac{2x^2-x+11}{(x+2)(2x-3)}\,dx = 4+4\ln 3-3\ln 2$

7. $\displaystyle\int_0^2 \frac{25x+1}{(2x-1)(x+1)^2}\,dx = \frac{16}{3}$

8. $\displaystyle\int_5^8 \frac{2x^2}{x^2-16}\,dx = 6+4\ln 3$

9. $\displaystyle\int_2^3 \frac{x^2-3x+5}{(4-x)(1-x)^2}\,dx = \frac{1}{2}+\ln 2$

10. $\displaystyle\int_0^2 \frac{4x^3-12x^2-22x-3}{(4-x)(2x+1)}\,dx = \frac{1}{2}\ln\left(\frac{5}{64}\right)-6$



1. $\int_4^9 \frac{5x^2-8x+1}{2x(x-1)^2}\,dx$ $\xrightarrow{\text{PARTIAL FRACTION}}$ $\frac{5x^2-8x+1}{2x(x-1)^2} \equiv \frac{A}{2x} + \frac{B}{(x-1)^2} + \frac{C}{x-1}$

$5x^2-8x+1 \equiv A(x-1)^2 + B(2x) + C(2x)(x-1)$

- If $x=1$, $-2 = 2B \Rightarrow B=-1$
- If $x=0$, $1 = A \Rightarrow A=1$
- If $x=2$, $20-16+1 = A+4B+4C$; $5 = 1-4+4C$; $8 = 4C$; $C=2$

$= \int_4^9 \frac{1}{2x} - (x-1)^{-2} + \frac{2}{x-1}\,dx$

$= \left[\frac{1}{2}\ln|x| + (x-1)^{-1} + 2\ln|x-1|\right]_4^9$

$= \left(\frac{1}{2}\ln 9 + \frac{1}{8} + 2\ln 8\right) - \left(\frac{1}{2}\ln 4 + \frac{1}{3} + 2\ln 3\right)$

$= \ln 3 + \frac{1}{8} + \ln 64 - \ln 2 - \frac{1}{3} - 2\ln 3$

$= -\frac{5}{24} - \ln 3 + \ln 32 = \ln\left(\frac{32}{3}\right) - \frac{5}{24}$

2. $\int_0^1 \frac{x^2}{x^2-4}\,dx$ $\xrightarrow[\text{PARTIAL FRACTIONS}]{\text{! IMPROPER !}}$ $\frac{x^2}{x^2-4} \equiv A + \frac{B}{x-2} + \frac{C}{x+2}$

$x^2 \equiv A(x-2)(x+2) + B(x+2) + C(x-2)$

- If $x=2$, $4 = 4B \Rightarrow B=1$
- If $x=-2$, $4 = -4C \Rightarrow C=-1$
- If $x=0$, $0 = -4A+2B-2C$; $0 = -4A+2+2$; $4A = 4$; $A=1$

$= \int_0^1 1 + \frac{1}{x-2} - \frac{1}{x+2}\,dx$

$= \left[x + \ln|x-2| - \ln|x+2|\right]_0^1$

$= (1 + \ln|-1| - \ln 3) - (0 + \ln|-2| - \ln 2)$

$= 1 + \ln 1 - \ln 3 - \ln 2 + \ln 2$

$= 1 - \ln 3$

3. $\int_0^5 \frac{1}{(x+1)(x+2)(x+3)}\,dx$ $\xrightarrow{\text{PARTIAL FRACTIONS}}$ $\frac{1}{(x+1)(x+2)(x+3)} \equiv \frac{A}{x+1} + \frac{B}{x+2} + \frac{C}{x+3}$

$1 \equiv A(x+2)(x+3) + B(x+1)(x+3) + C(x+1)(x+2)$

- If $x=-1$, $1 = 2A$, $A = \frac{1}{2}$
- If $x=-2$, $1 = -B$, $B=-1$
- If $x=-3$, $1 = 2C$, $C = \frac{1}{2}$

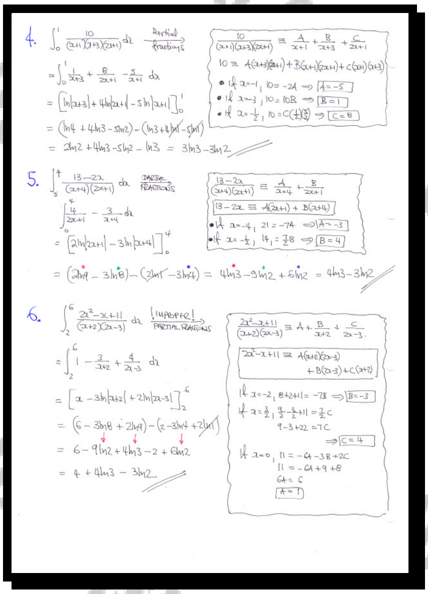
$= \int_0^5 \frac{\frac{1}{2}}{x+1} - \frac{1}{x+2} + \frac{\frac{1}{2}}{x+3}\,dx$

$= \left[\frac{1}{2}\ln|x+1| - \ln|x+2| + \frac{1}{2}\ln|x+3|\right]_0^5$

$= \left(\frac{1}{2}\ln 6 - \ln 7 + \frac{1}{2}\ln 8\right) - \left(\frac{1}{2}\ln 1 - \ln 2 + \frac{1}{2}\ln 3\right)$

$= \frac{1}{2}\ln 6 + \frac{1}{2}\ln 8 - \frac{1}{2}\ln 3 + \ln 2 - \ln 7 = \frac{1}{2}(\ln 6 + \ln 8 - \ln 3) + \ln\left(\frac{2}{7}\right)$

$= \frac{1}{2}\ln 16 + \ln\frac{2}{7} = \ln 4 + \ln\left(\frac{2}{7}\right) = \ln\left(\frac{8}{7}\right)$



4. $\int_0^1 \frac{10}{(x+1)(x+3)(2x+1)}\,dx$ $\xrightarrow{\text{Partial fractions}}$ $\frac{10}{(x+1)(x+3)(2x+1)} \equiv \frac{A}{x+1} + \frac{B}{x+3} + \frac{C}{2x+1}$

$10 \equiv A(x+3)(2x+1) + B(x+1)(2x+1) + C(x+1)(x+3)$

- If $x=-1$, $10 = -2A \Rightarrow A=-5$
- If $x=-3$, $10 = 10B \Rightarrow B=1$
- If $x=-\frac{1}{2}$, $10 = C\left(\frac{1}{2}\right)\left(\frac{5}{2}\right) \Rightarrow C=8$

$= \int_0^1 \frac{1}{x+3} + \frac{8}{2x+1} - \frac{5}{x+1}\,dx$

$= \left[\ln|x+3| + 4\ln|2x+1| - 5\ln|x+1|\right]_0^1$

$= (\ln 4 + 4\ln 3 - 5\ln 2) - (\ln 3 + 4\ln 1 - 5\ln 1)$

$= 2\ln 2 + 4\ln 3 - 5\ln 2 - \ln 3 = 3\ln 3 - 3\ln 2$

5. $\int_0^4 \frac{13-2x}{(x+4)(2x+1)}\,dx$ $\xrightarrow{\text{PARTIAL FRACTIONS}}$ $\frac{13-2x}{(x+4)(2x+1)} \equiv \frac{A}{x+4} + \frac{B}{2x+1}$

$13-2x \equiv A(2x+1) + B(x+4)$

- If $x=-4$, $21 = -7A \Rightarrow A=-3$
- If $x=-\frac{1}{2}$, $14 = \frac{7}{2}B \Rightarrow B=4$

$\int_0^4 \frac{4}{2x+1} - \frac{3}{x+4}\,dx$

$= \left[2\ln|2x+1| - 3\ln|x+4|\right]_0^4$

$= (2\ln 9 - 3\ln 8) - (2\ln 1 - 3\ln 4) = 4\ln 3 - 9\ln 2 + 6\ln 2 = 4\ln 3 - 3\ln 2$

6. $\int_2^6 \frac{2x^2-x+11}{(x+2)(2x-3)}\,dx$ $\xrightarrow[\text{PARTIAL FRACTIONS}]{\text{! IMPROPER !}}$ $\frac{2x^2-x+11}{(x+2)(2x-3)} \equiv A + \frac{B}{x+2} + \frac{C}{2x-3}$

$2x^2-x+11 \equiv A(x+2)(2x-3) + B(2x-3) + C(x+2)$

If $x=-2$, $8+2+11 = -7B \Rightarrow B=-3$

If $x=\frac{3}{2}$, $\frac{9}{2} - \frac{3}{2} + 11 = \frac{7}{2}C$; $9-3+22 = 7C$; $\Rightarrow C=4$

If $x=0$, $11 = -6A - 3B + 2C$; $11 = -6A + 9 + 8$; $6A = 6$; $A=1$

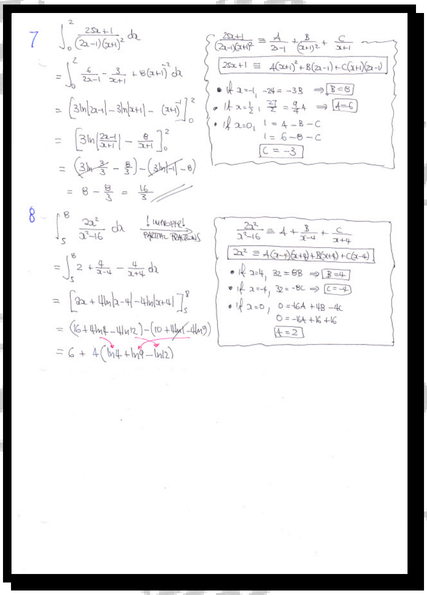
$= \int_2^6 1 - \frac{3}{x+2} + \frac{4}{2x-3}\,dx$

$= \left[x - 3\ln|x+2| + 2\ln|2x-3|\right]_2^6$

$= (6 - 3\ln 8 + 2\ln 9) - (2 - 3\ln 4 + 2\ln 1)$

$= 6 - 9\ln 2 + 4\ln 3 - 2 + 6\ln 2$

$= 4 + 4\ln 3 - 3\ln 2$



7. $\int_0^2 \frac{25x+1}{(2x-1)(x+1)^2}\,dx$ $\quad \frac{25x+1}{(2x-1)(x+1)^2} \equiv \frac{A}{2x-1} + \frac{B}{(x+1)^2} + \frac{C}{x+1}$

$25x+1 \equiv A(x+1)^2 + B(2x-1) + C(x+1)(2x-1)$

- If $x=-1$, $-24 = -3B \Rightarrow B=8$
- If $x=\frac{1}{2}$, $\frac{27}{2} = \frac{9}{4}A \Rightarrow A=6$
- If $x=0$, $1 = A-B-C$; $1 = 6-8-C$; $C=-3$

$= \int_0^2 \frac{6}{2x-1} - \frac{3}{x+1} + 8(x+1)^{-2}\,dx$

$= \left[3\ln|2x-1| - 3\ln|x+1| - 8(x+1)^{-1}\right]_0^2$

$= \left[3\ln\left|\frac{2x-1}{x+1}\right| - \frac{8}{x+1}\right]_0^2$

$= \left(3\ln\frac{3}{3} - \frac{8}{3}\right) - \left(3\ln|-1| - 8\right)$

$= 8 - \frac{8}{3} = \frac{16}{3}$

8. $\int_5^8 \frac{2x^2}{x^2-16}\,dx$ $\xrightarrow[\text{PARTIAL FRACTIONS}]{\text{! IMPROPER !}}$ $\frac{2x^2}{x^2-16} \equiv A + \frac{B}{x-4} + \frac{C}{x+4}$

$2x^2 \equiv A(x-4)(x+4) + B(x+4) + C(x-4)$

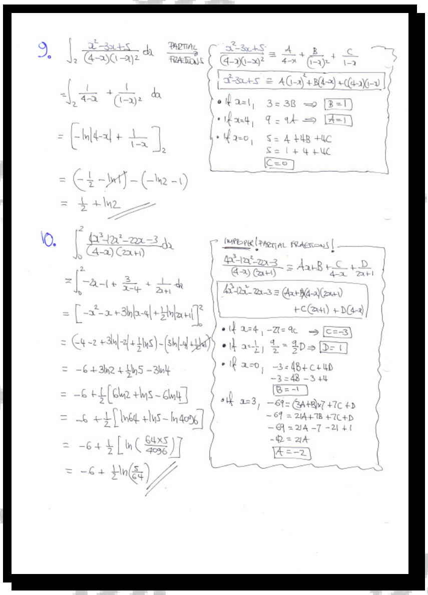
- If $x=4$, $32 = 8B \Rightarrow B=4$
- If $x=-4$, $32 = -8C \Rightarrow C=-4$
- If $x=0$, $0 = -16A + 4B - 4C$; $0 = -16A + 16 + 16$; $A=2$

$= \int_5^8 2 + \frac{4}{x-4} - \frac{4}{x+4}\,dx$

$= \left[2x + 4\ln|x-4| - 4\ln|x+4|\right]_5^8$

$= (16 + 4\ln 4 - 4\ln 12) - (10 + 4\ln 1 - 4\ln 9)$

$= 6 + 4(\ln 4 + \ln 9 - \ln 12)$



9. $\int_2 \frac{x^2-3x+5}{(4-x)(1-x)^2}\,dx$ $\xrightarrow{\text{PARTIAL FRACTIONS}}$ $\frac{x^2-3x+5}{(4-x)(1-x)^2} \equiv \frac{A}{4-x} + \frac{B}{(1-x)^2} + \frac{C}{1-x}$

$x^2-3x+5 \equiv A(1-x)^2 + B(4-x) + C(4-x)(1-x)$

- If $x=1$, $3 = 3B \Rightarrow B=1$
- If $x=4$, $9 = 9A \Rightarrow A=1$
- If $x=0$, $5 = A+4B+4C$; $5 = 1+4+4C$; $C=0$

$= \int_2 \frac{1}{4-x} + \frac{1}{(1-x)^2}\,dx$

$= \left[-\ln|4-x| + \frac{1}{1-x}\right]_2$

$= \left(-\frac{1}{2} - \ln 1\right) - (-\ln 2 - 1)$

$= \frac{1}{2} + \ln 2$

10. $\int_0^2 \frac{4x^3-12x^2-22x-3}{(4-x)(2x+1)}\,dx$ $\quad$ IMPROPER! PARTIAL FRACTIONS!

$\frac{4x^3-12x^2-22x-3}{(4-x)(2x+1)} \equiv Ax+B + \frac{C}{4-x} + \frac{D}{2x+1}$

$4x^3-12x^2-22x-3 \equiv (Ax+B)(4-x)(2x+1) + C(2x+1) + D(4-x)$

- If $x=4$, $-27 = 9C \Rightarrow C=-3$
- If $x=-\frac{1}{2}$, $\frac{9}{2} = \frac{9}{2}D \Rightarrow D=1$
- If $x=0$, $-3 = 4B + C + 4D$; $-3 = 4B - 3 + 4$; $B=-1$
- If $x=3$, $-69 = (3A+B)\times 7 + 7C + D$; $-69 = 21A + 7B + 7C + D$; $-69 = 21A - 7 - 21 + 1$; $-42 = 21A$; $A=-2$

$= \int_0^2 -2x - 1 + \frac{3}{x-4} + \frac{1}{2x+1}\,dx$

$= \left[-x^2 - x + 3\ln|x-4| + \frac{1}{2}\ln|2x+1|\right]_0^2$

$= \left(-4 - 2 + 3\ln|-2| + \frac{1}{2}\ln 5\right) - \left(3\ln|-4| + \frac{1}{2}\ln 1\right)$

$= -6 + 3\ln 2 + \frac{1}{2}\ln 5 - 3\ln 4$

$= -6 + \frac{1}{2}\left[6\ln 2 + \ln 5 - 6\ln 4\right]$

$= -6 + \frac{1}{2}\left[\ln 64 + \ln 5 - \ln 4096\right]$

$= -6 + \frac{1}{2}\left[\ln\left(\frac{64 \times 5}{4096}\right)\right]$

$= -6 + \frac{1}{2}\ln\left(\frac{5}{64}\right)$