93 EXAM QUESTIONS ON INVERSE TRIGONOMETRIC FUNCTIONS

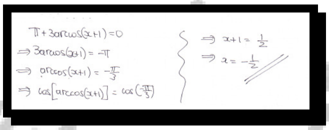
22 BASIC QUESTIONS

Question 1 **(**+)**

Solve the following trigonometric equation

$$\pi + 3\arccos(x+1) = 0 .$$

$$\boxed{x = -\frac{1}{2}}$$



$\pi + 3\arccos(x+1) = 0$

$\Rightarrow 3\arccos(x+1) = -\pi$

$\Rightarrow \arccos(x+1) = -\frac{\pi}{3}$

$\Rightarrow \cos[\arccos(x+1)] = \cos(\frac{-\pi}{3})$

$\Rightarrow x+1 = \frac{1}{2}$

$\Rightarrow x = -\frac{1}{2}$

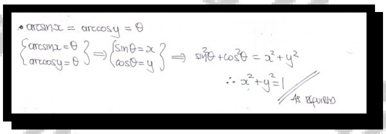
Question 2 **(***)**

It is given that $\arcsin x = \arccos y$.

Show, by a clear method, that

$$x^2 + y^2 = 1 .$$

$\boxed{}$, $\boxed{\text{proof}}$



- $\arcsin x = \arccos y = \theta$

$\begin{cases} \arcsin x = \theta \\ \arccos y = \theta \end{cases} \Rightarrow \begin{cases} \sin\theta = x \\ \cos\theta = y \end{cases} \Rightarrow \sin^2\theta + \cos^2\theta = x^2 + y^2$

$\therefore x^2 + y^2 = 1$

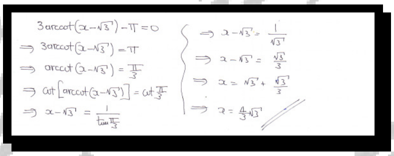
AS REQUIRED

Question 3 **(***)**

Solve the following trigonometric equation

$$3\operatorname{arccot}\left(x - \sqrt{3}\right) - \pi = 0 .$$

$$\boxed{x = \frac{4}{3}\sqrt{3}}$$



$3\operatorname{arccot}(x-\sqrt{3}) - \pi = 0$

$\Rightarrow 3\operatorname{arccot}(x-\sqrt{3}) = \pi$

$\Rightarrow \operatorname{arccot}(x-\sqrt{3}) = \frac{\pi}{3}$

$\Rightarrow \cot[\operatorname{arccot}(x-\sqrt{3})] = \cot\frac{\pi}{3}$

$\Rightarrow x-\sqrt{3} = \frac{1}{\tan\frac{\pi}{3}}$

$\Rightarrow x-\sqrt{3} = \frac{1}{\sqrt{3}}$

$\Rightarrow x-\sqrt{3} = \frac{\sqrt{3}}{3}$

$\Rightarrow x = \sqrt{3} + \frac{\sqrt{3}}{3}$

$\Rightarrow x = \frac{4}{3}\sqrt{3}$

Question 4 (***+)

A curve C is defined by the equation

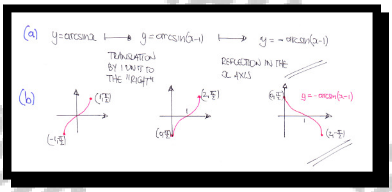
$$y = -\arcsin(x-1), \ 0 \le x \le 2 .$$

a) Describe the 2 geometric transformations that map the graph of $\arcsin x$ onto the graph of C.

b) Sketch the graph of C.

The sketch must include the coordinates of any points where the graph of C meets the coordinate axes and the coordinates of the endpoints of C.

C3, translation by 1 unit to the right, followed by reflection in the x axis



(a) $y=\arcsin x \longmapsto y=\arcsin(x-1) \longmapsto y=-\arcsin(x-1)$

TRANSLATION BY 1 UNIT TO THE "RIGHT"

REFLECTION IN THE x AXIS

(b)

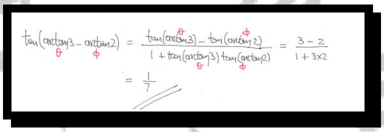
Question 5 (***+)

Simplify, showing all steps in the calculation, the following expression

$$\tan(\arctan 3 - \arctan 2),$$

giving the final answer as an exact fraction.

$$\boxed{\frac{1}{7}}$$



$$\tan(\underset{\theta}{\arctan 3} - \underset{\phi}{\arctan 2}) = \frac{\tan(\overset{\theta}{\arctan 3}) - \tan(\overset{\phi}{\arctan 2})}{1+\tan(\underset{\theta}{\arctan 3})\tan(\underset{\phi}{\arctan 2})} = \frac{3-2}{1+3\times 2}$$

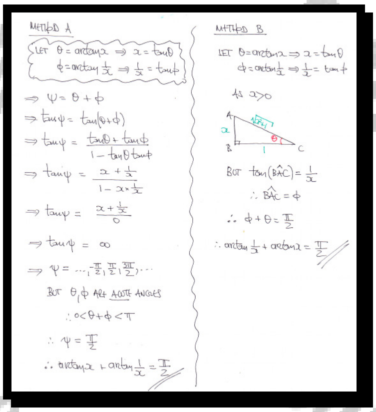
$$= \frac{1}{7}$$

Question 6 **(***+)**

Show clearly that if $x > 0$

$$\arctan x + \arctan\left(\frac{1}{x}\right) = \frac{\pi}{2}.$$

$\boxed{\text{proof}}$



METHOD A

LET $\theta = \arctan x \Rightarrow x = \tan\theta$
$\phi = \arctan\frac{1}{x} \Rightarrow \frac{1}{x} = \tan\phi$

$\Rightarrow \psi = \theta + \phi$

$\Rightarrow \tan\psi = \tan(\theta+\phi)$

$\Rightarrow \tan\psi = \frac{\tan\theta + \tan\phi}{1 - \tan\theta\tan\phi}$

$\Rightarrow \tan\psi = \frac{x + \frac{1}{x}}{1 - x \times \frac{1}{x}}$

$\Rightarrow \tan\psi = \frac{x + \frac{1}{x}}{0}$

$\Rightarrow \tan\psi = \infty$

$\Rightarrow \psi = \dots, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \dots$

BUT θ, ϕ ARE ACUTE ANGLES

$\therefore 0 < \theta + \phi < \pi$

$\therefore \psi = \frac{\pi}{2}$

$\therefore \arctan x + \arctan\frac{1}{x} = \frac{\pi}{2}$

METHOD B

LET $\theta = \arctan x \Rightarrow x = \tan\theta$
$\phi = \arctan\frac{1}{x} \Rightarrow \frac{1}{x} = \tan\phi$

AS $x > 0$

BUT $\tan(B\hat{A}C) = \frac{1}{x}$

$\therefore B\hat{A}C = \phi$

$\therefore \phi + \theta = \frac{\pi}{2}$

$\therefore \arctan\frac{1}{x} + \arctan x = \frac{\pi}{2}$

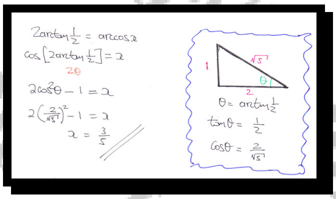
Question 7 **(***+)**

Solve the equation

$$2\arctan\left(\frac{1}{2}\right) = \arccos x,$$

showing clearly all the workings.

$\boxed{x = \frac{3}{5}}$



$2\arctan\frac{1}{2} = \arccos x$

$\cos\left[\underbrace{2\arctan\frac{1}{2}}_{2\theta}\right] = x$

$2\cos^2\theta - 1 = x$

$2\left(\frac{2}{\sqrt{5}}\right)^2 - 1 = x$

$x = \frac{3}{5}$

$\theta = \arctan\frac{1}{2}$

$\tan\theta = \frac{1}{2}$

$\cos\theta = \frac{2}{\sqrt{5}}$

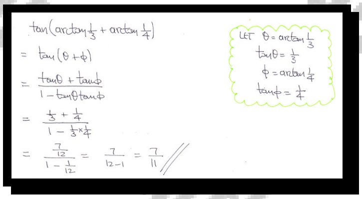
Question 8 **(***+)**

Simplify, showing all steps in the calculation, the expression

$$\tan\left[\arctan\frac{1}{3}+\arctan\frac{1}{4}\right],$$

giving the final answer as an exact fraction.

$$\boxed{\frac{7}{11}}$$



$\tan\left(\arctan\frac{1}{3}+\arctan\frac{1}{4}\right)$

$= \tan(\theta+\phi)$

$= \dfrac{\tan\theta+\tan\phi}{1-\tan\theta\tan\phi}$

$= \dfrac{\frac{1}{3}+\frac{1}{4}}{1-\frac{1}{3}\times\frac{1}{4}}$

$= \dfrac{\frac{7}{12}}{1-\frac{1}{12}} = \dfrac{7}{12-1} = \dfrac{7}{11}$ //

LET $\theta = \arctan\frac{1}{3}$

$\tan\theta = \frac{1}{3}$

$\phi = \arctan\frac{1}{4}$

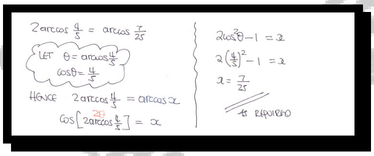
$\tan\phi = \frac{1}{4}$

Question 9 **(***+)**

Show clearly that

$$2\arccos\left(\frac{4}{5}\right)=\arccos\left(\frac{7}{25}\right).$$

$$\boxed{\text{proof}}$$



$2\arccos\frac{4}{5} = \arccos\frac{7}{25}$

LET $\theta = \arccos\frac{4}{5}$

$\cos\theta = \frac{4}{5}$

HENCE $2\arccos\frac{4}{5} = \arccos x$

$\cos\left[\overset{2\theta}{2\arccos\frac{4}{5}}\right] = x$

$2\cos^2\theta - 1 = x$

$2\left(\frac{4}{5}\right)^2 - 1 = x$

$x = \frac{7}{25}$

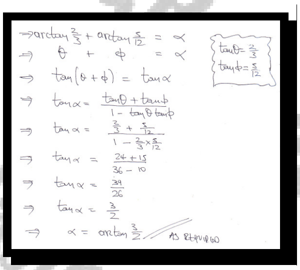
// AS REQUIRED

Question 10 (***+)

Show clearly that

$$\arctan\frac{2}{3}+\arctan\frac{5}{12}=\arctan\frac{3}{2}.$$

proof



$\Rightarrow \arctan\frac{2}{3}+\arctan\frac{5}{12} = \alpha$

$\Rightarrow \theta + \phi = \alpha$

$\Rightarrow \tan(\theta+\phi) = \tan\alpha$

$\Rightarrow \tan\alpha = \frac{\tan\theta+\tan\phi}{1-\tan\theta\tan\phi}$

$\Rightarrow \tan\alpha = \frac{\frac{2}{3}+\frac{5}{12}}{1-\frac{2}{3}\times\frac{5}{12}}$

$\Rightarrow \tan\alpha = \frac{24+15}{36-10}$

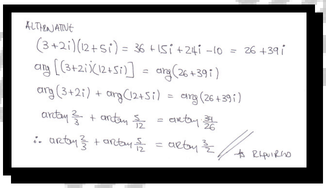
$\Rightarrow \tan\alpha = \frac{39}{26}$

$\Rightarrow \tan\alpha = \frac{3}{2}$

$\Rightarrow \alpha = \arctan\frac{3}{2}$ // AS REQUIRED

$\tan\theta=\frac{2}{3}$

$\tan\phi=\frac{5}{12}$



ALTERNATIVE

$(3+2i)(12+5i) = 36+15i+24i-10 = 26+39i$

$\arg[(3+2i)(12+5i)] = \arg(26+39i)$

$\arg(3+2i)+\arg(12+5i) = \arg(26+39i)$

$\arctan\frac{2}{3}+\arctan\frac{5}{12} = \arctan\frac{39}{26}$

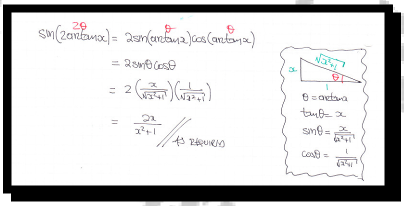
$\therefore \arctan\frac{2}{3}+\arctan\frac{5}{12} = \arctan\frac{3}{2}$ // AS REQUIRED

Question 11 (***+)

Show clearly that

$$\sin(2\arctan x)=\frac{2x}{x^2+1}.$$

proof



$\sin(2\overset{2\theta}{\arctan x}) = 2\sin(\overset{\theta}{\arctan x})\cos(\overset{\theta}{\arctan x})$

$= 2\sin\theta\cos\theta$

$= 2\left(\frac{x}{\sqrt{x^2+1}}\right)\left(\frac{1}{\sqrt{x^2+1}}\right)$

$= \frac{2x}{x^2+1}$ // AS REQUIRED

$\theta=\arctan x$

$\tan\theta = x$

$\sin\theta = \frac{x}{\sqrt{x^2+1}}$

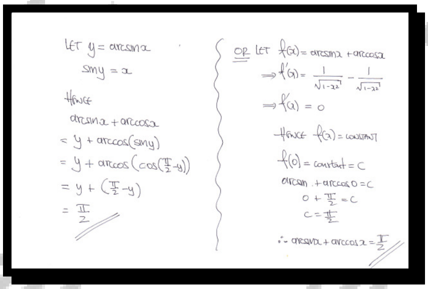
$\cos\theta = \frac{1}{\sqrt{x^2+1}}$

Question 12 (***+)

Prove the trigonometric identity

$$\arcsin x + \arccos x = \frac{\pi}{2}.$$

proof



LET $y = \arcsin x$

$\sin y = x$

HENCE

$\arcsin x + \arccos x$

$= y + \arccos(\sin y)$

$= y + \arccos\left(\cos\left(\frac{\pi}{2} - y\right)\right)$

$= y + \left(\frac{\pi}{2} - y\right)$

$= \frac{\pi}{2}$

OR LET $f(x) = \arcsin x + \arccos x$

$\Rightarrow f'(x) = \frac{1}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-x^2}}$

$\Rightarrow f'(x) = 0$

HENCE $f(x) = $ CONSTANT

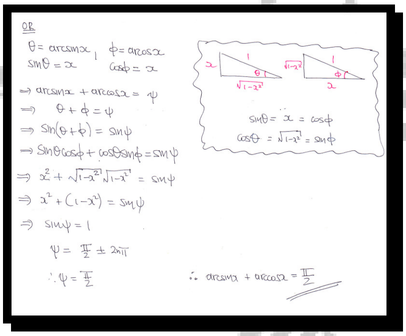
$f(0) = \text{constant} = C$

$\arcsin 0 + \arccos 0 = C$

$0 + \frac{\pi}{2} = C$

$C = \frac{\pi}{2}$

$\therefore \arcsin x + \arccos x = \frac{\pi}{2}$



OR

$\theta = \arcsin x$, $\phi = \arccos x$

$\sin\theta = x$ $\quad$ $\cos\phi = x$

$\Rightarrow \arcsin x + \arccos x = \psi$

$\Rightarrow \theta + \phi = \psi$

$\Rightarrow \sin(\theta + \phi) = \sin\psi$

$\Rightarrow \sin\theta\cos\phi + \cos\theta\sin\phi = \sin\psi$

$\Rightarrow x^2 + \sqrt{1-x^2}\sqrt{1-x^2} = \sin\psi$

$\Rightarrow x^2 + (1-x^2) = \sin\psi$

$\Rightarrow \sin\psi = 1$

$\psi = \frac{\pi}{2} \pm 2n\pi$

$\therefore \psi = \frac{\pi}{2}$

$\sin\theta = x = \cos\phi$

$\cos\theta = \sqrt{1-x^2} = \sin\phi$

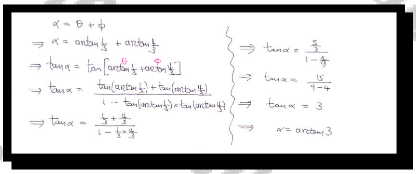
$\therefore \arcsin x + \arccos x = \frac{\pi}{2}$

Question 13 (***+)

Show clearly that

$$\arctan\frac{1}{3} + \arctan\frac{4}{3} = \arctan 3.$$

proof



$\alpha = \theta + \phi$

$\Rightarrow \alpha = \arctan\frac{1}{3} + \arctan\frac{4}{3}$

$\Rightarrow \tan\alpha = \tan\left[\overset{\theta}{\arctan\frac{1}{3}} + \overset{\phi}{\arctan\frac{4}{3}}\right]$

$\Rightarrow \tan\alpha = \dfrac{\tan(\arctan\frac{1}{3}) + \tan(\arctan\frac{4}{3})}{1 - \tan(\arctan\frac{1}{3}) \times \tan(\arctan\frac{4}{3})}$

$\Rightarrow \tan\alpha = \dfrac{\frac{1}{3} + \frac{4}{3}}{1 - \frac{1}{3} \times \frac{4}{3}}$

$\Rightarrow \tan\alpha = \dfrac{\frac{5}{3}}{1 - \frac{4}{9}}$

$\Rightarrow \tan\alpha = \dfrac{15}{9-4}$

$\Rightarrow \tan\alpha = 3$

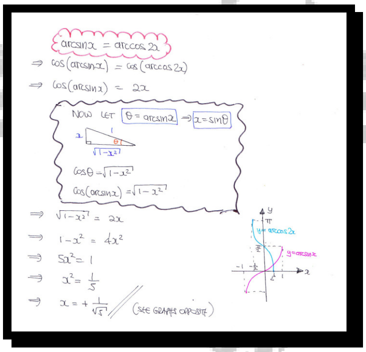
$\Rightarrow \alpha = \arctan 3$

Question 14 **(***+)**

Solve the trigonometric equation

$$\arcsin x = \arccos 2x\,.$$

$$\boxed{x = \frac{1}{\sqrt{5}}}$$



$\arcsin x = \arccos 2x$

$\Rightarrow \cos(\arcsin x) = \cos(\arccos 2x)$

$\Rightarrow \cos(\arcsin x) = 2x$

NOW LET $\boxed{\theta = \arcsin x} \Rightarrow \boxed{x = \sin\theta}$

$\cos\theta = \sqrt{1-x^2}$

$\cos(\arcsin x) = \sqrt{1-x^2}$

$\Rightarrow \sqrt{1-x^2} = 2x$

$\Rightarrow 1-x^2 = 4x^2$

$\Rightarrow 5x^2 = 1$

$\Rightarrow x^2 = \frac{1}{5}$

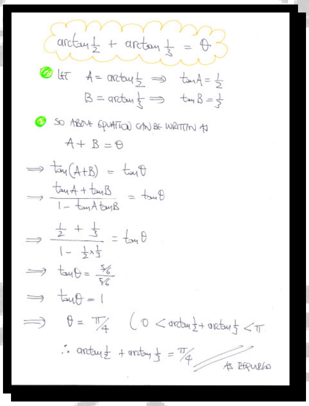
$\Rightarrow x = +\frac{1}{\sqrt{5}}$ // (SEE GRAPHS OPPOSITE)

Question 15 **(***+)**

Using a detailed method, show that

$$\arctan\frac{1}{2} + \arctan\frac{1}{3} = \frac{1}{4}\pi\,.$$

$$\boxed{\text{proof}}$$



$\arctan\frac{1}{2} + \arctan\frac{1}{3} = \theta$

- LET $A = \arctan\frac{1}{2} \Rightarrow \tan A = \frac{1}{2}$
  $B = \arctan\frac{1}{3} \Rightarrow \tan B = \frac{1}{3}$
- SO ABOVE EQUATION CAN BE WRITTEN AS
  $A + B = \theta$

$\Rightarrow \tan(A+B) = \tan\theta$

$\Rightarrow \frac{\tan A + \tan B}{1 - \tan A \tan B} = \tan\theta$

$\Rightarrow \frac{\frac{1}{2} + \frac{1}{3}}{1 - \frac{1}{2}\times\frac{1}{3}} = \tan\theta$

$\Rightarrow \tan\theta = \frac{5/6}{5/6}$

$\Rightarrow \tan\theta = 1$

$\Rightarrow \theta = \pi/4 \quad (0 < \arctan\frac{1}{2} + \arctan\frac{1}{3} < \pi$

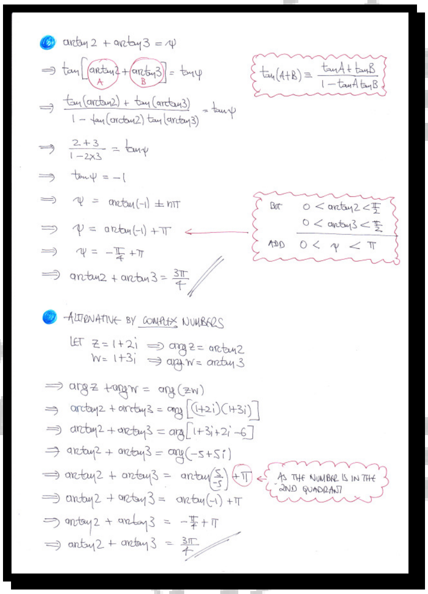
$\therefore \arctan\frac{1}{2} + \arctan\frac{1}{3} = \pi/4$ // AS REQUIRED

Question 16 (***+)

Show, by detailed workings, that

$$\arctan 2 + \arctan 3 = \frac{3\pi}{4}.$$

proof



$\arctan 2 + \arctan 3 = \psi$

$\Rightarrow \tan\left[\underbrace{\arctan 2}_{A} + \underbrace{\arctan 3}_{B}\right] = \tan\psi$

$\tan(A+B) \equiv \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$

$\Rightarrow \dfrac{\tan(\arctan 2) + \tan(\arctan 3)}{1 - \tan(\arctan 2)\tan(\arctan 3)} = \tan\psi$

$\Rightarrow \dfrac{2+3}{1-2\times 3} = \tan\psi$

$\Rightarrow \tan\psi = -1$

$\Rightarrow \psi = \arctan(-1) \pm n\pi$

$\Rightarrow \psi = \arctan(-1) + \pi \leftarrow$

BUT $0 < \arctan 2 < \frac{\pi}{2}$

$0 < \arctan 3 < \frac{\pi}{2}$

ADD $0 < \psi < \pi$

$\Rightarrow \psi = -\frac{\pi}{4} + \pi$

$\Rightarrow \arctan 2 + \arctan 3 = \frac{3\pi}{4}$ //

ALTERNATIVE BY COMPLEX NUMBERS

LET $z = 1+2i \Rightarrow \arg z = \arctan 2$

$w = 1+3i \Rightarrow \arg w = \arctan 3$

$\Rightarrow \arg z + \arg w = \arg(zw)$

$\Rightarrow \arctan 2 + \arctan 3 = \arg\left[(1+2i)(1+3i)\right]$

$\Rightarrow \arctan 2 + \arctan 3 = \arg\left[1+3i+2i-6\right]$

$\Rightarrow \arctan 2 + \arctan 3 = \arg(-5+5i)$

$\Rightarrow \arctan 2 + \arctan 3 = \arctan\left(\frac{5}{-5}\right) + \pi \leftarrow$ AS THE NUMBER IS IN THE 2ND QUADRANT

$\Rightarrow \arctan 2 + \arctan 3 = \arctan(-1) + \pi$

$\Rightarrow \arctan 2 + \arctan 3 = -\frac{\pi}{4} + \pi$

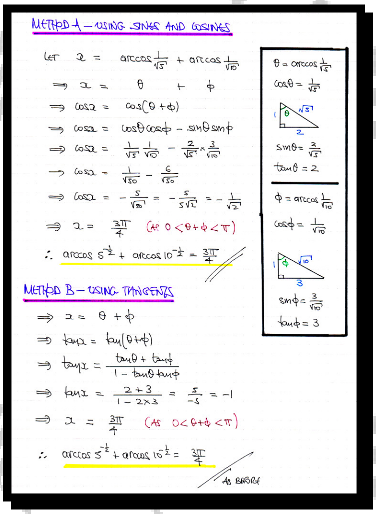
$\Rightarrow \arctan 2 + \arctan 3 = \frac{3\pi}{4}$ //

Question 17 (***+)

Use a detailed method to show that

$$\arccos\left(5^{-\frac{1}{2}}\right)+\arccos\left(10^{-\frac{1}{2}}\right)=\frac{3\pi}{4}.$$

$\boxed{}$, $\boxed{\text{proof}}$



METHOD A – USING SINES AND COSINES

LET $x = \arccos\frac{1}{\sqrt{5}} + \arccos\frac{1}{\sqrt{10}}$

$\Rightarrow x = \theta + \phi$

$\Rightarrow \cos x = \cos(\theta+\phi)$

$\Rightarrow \cos x = \cos\theta\cos\phi - \sin\theta\sin\phi$

$\Rightarrow \cos x = \frac{1}{\sqrt{5}}\frac{1}{\sqrt{10}} - \frac{2}{\sqrt{5}}\times\frac{3}{\sqrt{10}}$

$\Rightarrow \cos x = \frac{1}{\sqrt{50}} - \frac{6}{\sqrt{50}}$

$\Rightarrow \cos x = -\frac{5}{\sqrt{50}} = -\frac{5}{5\sqrt{2}} = -\frac{1}{\sqrt{2}}$

$\Rightarrow x = \frac{3\pi}{4}$ (AS $0<\theta+\phi<\pi$)

$\therefore \arccos 5^{-\frac{1}{2}} + \arccos 10^{-\frac{1}{2}} = \frac{3\pi}{4}$

$\theta = \arccos\frac{1}{\sqrt{5}}$

$\cos\theta = \frac{1}{\sqrt{5}}$

$\sin\theta = \frac{2}{\sqrt{5}}$

$\tan\theta = 2$

$\phi = \arccos\frac{1}{\sqrt{10}}$

$\cos\phi = \frac{1}{\sqrt{10}}$

$\sin\phi = \frac{3}{\sqrt{10}}$

$\tan\phi = 3$

METHOD B – USING TANGENTS

$\Rightarrow x = \theta + \phi$

$\Rightarrow \tan x = \tan(\theta+\phi)$

$\Rightarrow \tan x = \frac{\tan\theta + \tan\phi}{1-\tan\theta\tan\phi}$

$\Rightarrow \tan x = \frac{2+3}{1-2\times 3} = \frac{5}{-5} = -1$

$\Rightarrow x = \frac{3\pi}{4}$ (AS $0<\theta+\phi<\pi$)

$\therefore \arccos 5^{-\frac{1}{2}} + \arccos 10^{-\frac{1}{2}} = \frac{3\pi}{4}$

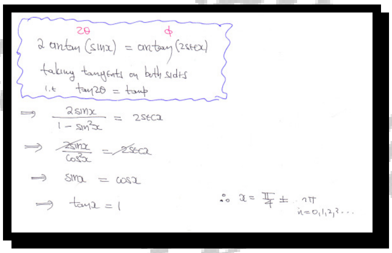
AS BEFORE

Question 18 (***+)

Find the general solution of the following trigonometric equation

$$2\arctan(\sin x)=\arctan(\sec x).$$

$\boxed{}$, $\boxed{x=\frac{\pi}{4}+k\pi,\ \ k\in\mathbb{Z}}$



$\overset{2\theta}{2\arctan(\sin x)} = \overset{\phi}{\arctan(2\sec x)}$

taking tangents on both sides

i.e $\tan 2\theta = \tan\phi$

$\Rightarrow \frac{2\sin x}{1-\sin^2 x} = 2\sec x$

$\Rightarrow \frac{\cancel{2}\sin x}{\cos^2 x} = \cancel{2}\sec x$

$\Rightarrow \sin x = \cos x$

$\Rightarrow \tan x = 1$

$\therefore x = \frac{\pi}{4} \pm n\pi$

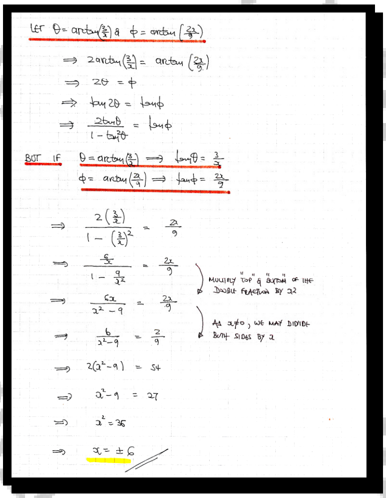
$n = 0,1,2,3\ldots$

Question 19 **(***+)**

Solve the following trigonometric equation

$$2\arctan\left(\frac{3}{x}\right) = \arctan\left(\frac{6x}{25}\right).$$

[illegible], $\boxed{x = \pm 6}$



LET $\theta = \arctan\left(\frac{3}{x}\right)$ & $\phi = \arctan\left(\frac{2x}{9}\right)$

$$\Rightarrow 2\arctan\left(\frac{3}{x}\right) = \arctan\left(\frac{2x}{9}\right)$$

$$\Rightarrow 2\theta = \phi$$

$$\Rightarrow \tan 2\theta = \tan\phi$$

$$\Rightarrow \frac{2\tan\theta}{1-\tan^2\theta} = \tan\phi$$

BUT IF $\theta = \arctan\left(\frac{3}{x}\right) \Longrightarrow \tan\theta = \frac{3}{x}$

$\phi = \arctan\left(\frac{2x}{9}\right) \Rightarrow \tan\phi = \frac{2x}{9}$

$$\Rightarrow \frac{2\left(\frac{3}{x}\right)}{1-\left(\frac{3}{x}\right)^2} = \frac{2x}{9}$$

$$\Rightarrow \frac{\frac{6}{x}}{1-\frac{9}{x^2}} = \frac{2x}{9}$$

MULTIPLY "TOP" & "BOTTOM" OF THE DOUBLE FRACTION BY x^2

$$\Rightarrow \frac{6x}{x^2-9} = \frac{2x}{9}$$

AS $x \neq 0$, WE MAY DIVIDE BOTH SIDES BY x

$$\Rightarrow \frac{6}{x^2-9} = \frac{2}{9}$$

$$\Rightarrow 2(x^2-9) = 54$$

$$\Rightarrow x^2-9 = 27$$

$$\Rightarrow x^2 = 36$$

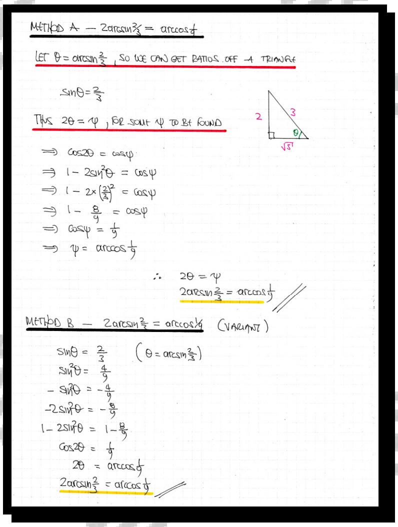
$$\Rightarrow x = \pm 6$$

Question 20 **(***+)**

Prove that

$$2\arcsin\left(\frac{2}{3}\right)=\arccos\left(\frac{1}{9}\right).$$

V, [illegible], proof



METHOD A — $2\arcsin\frac{2}{3} = \arccos\frac{1}{9}$

LET $\theta = \arcsin\frac{2}{3}$, SO WE CAN GET RATIOS OFF A TRIANGLE

$\sin\theta = \frac{2}{3}$

THUS $2\theta = \psi$, FOR SOME ψ TO BE FOUND

$\Rightarrow \cos2\theta = \cos\psi$

$\Rightarrow 1 - 2\sin^2\theta = \cos\psi$

$\Rightarrow 1 - 2\times\left(\frac{2}{3}\right)^2 = \cos\psi$

$\Rightarrow 1 - \frac{8}{9} = \cos\psi$

$\Rightarrow \cos\psi = \frac{1}{9}$

$\Rightarrow \psi = \arccos\frac{1}{9}$

$\therefore \quad 2\theta = \psi$

$2\arcsin\frac{2}{3} = \arccos\frac{1}{9}$ //

METHOD B — $2\arcsin\frac{2}{3} = \arccos\frac{1}{9}$ (VARIANT)

$\sin\theta = \frac{2}{3}$ $\left(\theta = \arcsin\frac{2}{3}\right)$

$\sin^2\theta = \frac{4}{9}$

$-\sin^2\theta = -\frac{4}{9}$

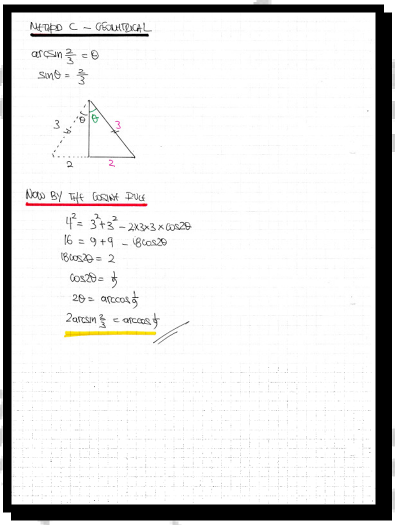
$-2\sin^2\theta = -\frac{8}{9}$

$1 - 2\sin^2\theta = 1 - \frac{8}{9}$

$\cos2\theta = \frac{1}{9}$

$2\theta = \arccos\frac{1}{9}$

$2\arcsin\frac{2}{3} = \arccos\frac{1}{9}$ //



METHOD C – GEOMETRICAL

$\arcsin\frac{2}{3} = \theta$

$\sin\theta = \frac{2}{3}$

NOW BY THE COSINE RULE

$4^2 = 3^2 + 3^2 - 2\times3\times3\times\cos2\theta$

$16 = 9 + 9 - 18\cos2\theta$

$18\cos2\theta = 2$

$\cos2\theta = \frac{1}{9}$

$2\theta = \arccos\frac{1}{9}$

$2\arcsin\frac{2}{3} = \arccos\frac{1}{9}$ //

Question 21 (***+)

Differentiate with respect to x

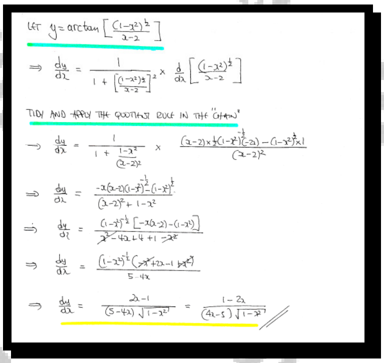
$$\arctan\left[\frac{\sqrt{1-x^2}}{x-2}\right].$$

Give a simplified answer in the form

$$\frac{A+Bx}{(Cx+D)\sqrt{1-x^2}},$$

where A, B, C and D are integers to be found.

$\boxed{\text{FP2-I}}$, $\boxed{A=1}$, $\boxed{B=-2}$, $\boxed{C=4}$, $\boxed{D=-5}$



LET $y = \arctan\left[\frac{(1-x^2)^{\frac{1}{2}}}{x-2}\right]$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{1+\left[\frac{(1-x^2)^{\frac{1}{2}}}{x-2}\right]^2} \times \frac{d}{dx}\left[\frac{(1-x^2)^{\frac{1}{2}}}{x-2}\right]$$

TIDY AND APPLY THE QUOTIENT RULE IN THE "CHAIN"

$$\rightarrow \frac{dy}{dx} = \frac{1}{1+\frac{1-x^2}{(x-2)^2}} \times \frac{(x-2)\times\frac{1}{2}(1-x^2)^{-\frac{1}{2}}(-2x)-(1-x^2)^{\frac{1}{2}}\times 1}{(x-2)^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-x(x-2)(1-x^2)^{-\frac{1}{2}}-(1-x^2)^{\frac{1}{2}}}{(x-2)^2+1-x^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{(1-x^2)^{-\frac{1}{2}}\left[-x(x-2)-(1-x^2)\right]}{\cancel{x^2}-4x+4+1\cancel{-x^2}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{(1-x^2)^{-\frac{1}{2}}(\cancel{-x^2}+2x-1\cancel{+x^2})}{5-4x}$$

$$\Rightarrow \frac{dy}{dx} = \frac{2x-1}{(5-4x)\sqrt{1-x^2}} = \frac{1-2x}{(4x-5)\sqrt{1-x^2}} \quad //$$

Question 22 (***+)

Differentiate with respect to x

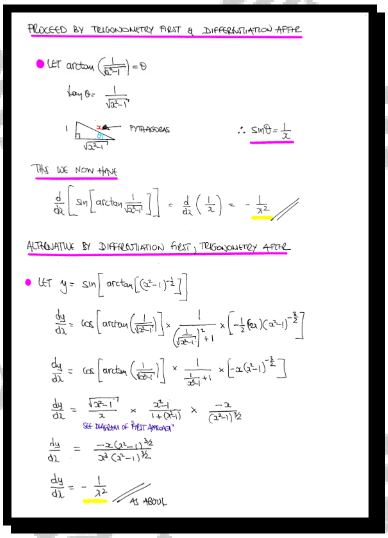
$$\sin\left[\arctan\left[\frac{1}{\sqrt{1-x^2}}\right]\right].$$

Give a simplified answer in the form

$$\frac{A}{x^n},$$

where A and n are integers to be found.

$\boxed{}$, $\boxed{A=-1}$, $\boxed{n=2}$



PROCEED BY TRIGONOMETRY FIRST & DIFFERENTIATION AFTER

- LET $\arctan\left(\frac{1}{\sqrt{x^2-1}}\right)=\theta$

$\tan\theta=\frac{1}{\sqrt{x^2-1}}$

1 x θ $\sqrt{x^2-1}$ ← PYTHAGORAS

$\therefore \sin\theta=\frac{1}{x}$

THUS WE NOW HAVE

$$\frac{d}{dx}\left[\sin\left[\arctan\frac{1}{\sqrt{x^2-1}}\right]\right]=\frac{d}{dx}\left(\frac{1}{x}\right)=-\frac{1}{x^2}$$

ALTERNATIVE BY DIFFERENTIATION FIRST, TRIGONOMETRY AFTER

- LET $y=\sin\left[\arctan\left[(x^2-1)^{-\frac{1}{2}}\right]\right]$

$$\frac{dy}{dx}=\cos\left[\arctan\left(\frac{1}{\sqrt{x^2-1}}\right)\right]\times\frac{1}{\left(\frac{1}{\sqrt{x^2-1}}\right)^2+1}\times\left[-\frac{1}{2}(2x)(x^2-1)^{-\frac{3}{2}}\right]$$

$$\frac{dy}{dx}=\cos\left[\arctan\left(\frac{1}{\sqrt{x^2-1}}\right)\right]\times\frac{1}{\frac{1}{x^2-1}+1}\times\left[-x(x^2-1)^{-\frac{3}{2}}\right]$$

$$\frac{dy}{dx}=\frac{\sqrt{x^2-1}}{x}\times\frac{x^2-1}{1+(x^2-1)}\times\frac{-x}{(x^2-1)^{3/2}}$$

SEE DIAGRAM OF "FIRST APPROACH"

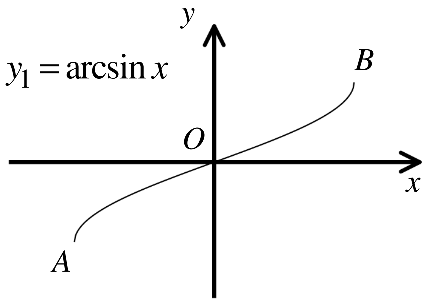
$$\frac{dy}{dx}=\frac{-x(x^2-1)^{3/2}}{x^3(x^2-1)^{3/2}}$$

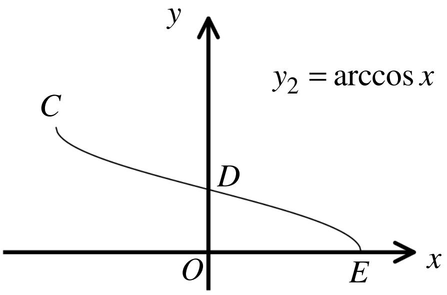
$$\frac{dy}{dx}=-\frac{1}{x^2}$$

AS ABOVE

19 STANDARD QUESTIONS

Question 1 (****)





The diagrams above shows the graphs of $y_1 = \arcsin x$ and $y_2 = \arccos x$.

The graph of y_1 has endpoints at A and B.

The graph of y_2 has endpoints at C and E, and D is the point where the graph of y_2 crosses the y axis .

a) State the coordinates of A, B, C, D and E.

The graph of y_2 can be obtained from the graph of y_1 by a series of two geometric transformations which can be carried out in a specific order.

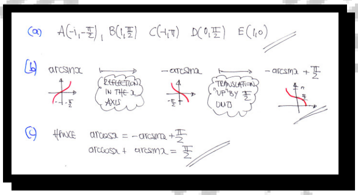
b) Describe the two geometric transformations.

c) Deduce using valid arguments that

$$\arcsin x + \arccos x = \text{constant},$$

stating the exact value of this constant.

$\boxed{A\left(-1,\frac{\pi}{2}\right)}$, $\boxed{B\left(1,\frac{\pi}{2}\right)}$, $\boxed{C(-1,\pi)}$, $\boxed{D\left(0,\frac{\pi}{2}\right)}$, $\boxed{E(1,0)}$, $\boxed{\text{constant } = \frac{\pi}{2}}$



(a) $A\left(-1,-\frac{\pi}{2}\right)$, $B\left(1,\frac{\pi}{2}\right)$ $C(-1,\pi)$ $D\left(0,\frac{\pi}{2}\right)$ $E(1,0)$

(b) $\arcsin x \xrightarrow{\text{REFLECTION IN THE } x \text{ AXIS}} -\arcsin x \xrightarrow{\text{TRANSLATION "UP" BY } \frac{\pi}{2} \text{ UNITS}} -\arcsin x + \frac{\pi}{2}$

(c) HENCE $\arccos x = -\arcsin x + \frac{\pi}{2}$

$\arccos x + \arcsin x = \frac{\pi}{2}$

Question 2 (****)

$$y = \arcsin x\,, \quad -1 \le x \le 1\,.$$

a) Show that

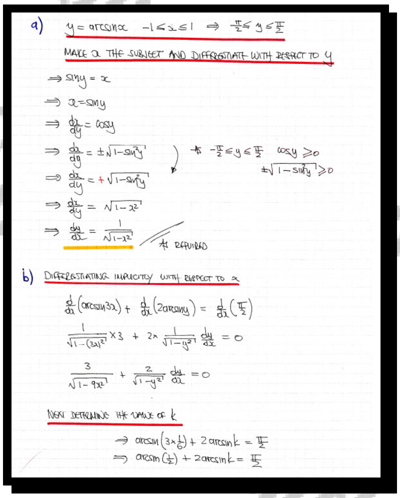
$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}\,.$$

The point $P\left(\frac{1}{6}, k\right)$, where k is a constant lies on the curve with equation

$$\arcsin 3x + 2\arcsin y = \frac{\pi}{2}\,, \ |x| \le \tfrac{1}{3}\,, \ |y| \le 1\,.$$

b) Find the value of the gradient at P.

$\boxed{}\,, \ \boxed{-\frac{3}{2}}$



a) $y = \arcsin x \quad -1 \le x \le 1 \Rightarrow -\frac{\pi}{2} \le y \le \frac{\pi}{2}$

MAKE x THE SUBJECT AND DIFFERENTIATE WITH RESPECT TO y

$\Rightarrow \sin y = x$

$\Rightarrow x = \sin y$

$\Rightarrow \frac{dx}{dy} = \cos y$

$\Rightarrow \frac{dx}{dy} = \pm\sqrt{1-\sin^2 y}$ AS $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$ $\cos y \geqslant 0$ $\pm\sqrt{1-\sin^2 y} \geqslant 0$

$\Rightarrow \frac{dx}{dy} = +\sqrt{1-\sin^2 y}$

$\Rightarrow \frac{dx}{dy} = \sqrt{1-x^2}$

$\Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$ // AS REQUIRED

b) DIFFERENTIATING IMPLICITLY WITH RESPECT TO x

$\frac{d}{dx}(\arcsin 3x) + \frac{d}{dx}(2\arcsin y) = \frac{d}{dx}\left(\frac{\pi}{2}\right)$

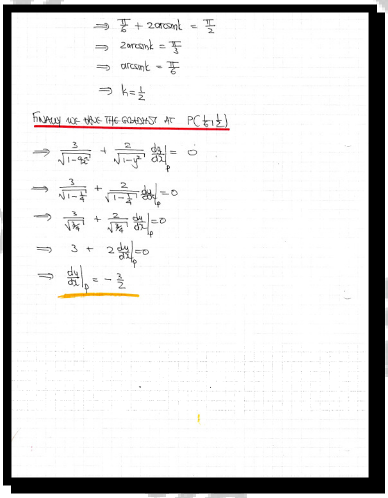
$\frac{1}{\sqrt{1-(3x)^2}} \times 3 + 2 \times \frac{1}{\sqrt{1-y^2}}\frac{dy}{dx} = 0$

$\frac{3}{\sqrt{1-9x^2}} + \frac{2}{\sqrt{1-y^2}}\frac{dy}{dx} = 0$

NEXT DETERMINE THE VALUE OF k

$\Rightarrow \arcsin\left(3 \times \frac{1}{6}\right) + 2\arcsin k = \frac{\pi}{2}$

$\Rightarrow \arcsin\left(\frac{1}{2}\right) + 2\arcsin k = \frac{\pi}{2}$



$\Rightarrow \frac{\pi}{6} + 2\arcsin k = \frac{\pi}{2}$

$\Rightarrow 2\arcsin k = \frac{\pi}{3}$

$\Rightarrow \arcsin k = \frac{\pi}{6}$

$\Rightarrow k = \frac{1}{2}$

FINALLY WE HAVE THE GRADIENT AT $P\left(\frac{1}{6}, \frac{1}{2}\right)$

$\Rightarrow \frac{3}{\sqrt{1-9x^2}} + \frac{2}{\sqrt{1-y^2}}\left.\frac{dy}{dx}\right|_P = 0$

$\Rightarrow \frac{3}{\sqrt{1-\frac{1}{4}}} + \frac{2}{\sqrt{1-\frac{1}{4}}}\left.\frac{dy}{dx}\right|_P = 0$

$\Rightarrow \frac{3}{\sqrt{3/4}} + \frac{2}{\sqrt{3/4}}\left.\frac{dy}{dx}\right|_P = 0$

$\Rightarrow 3 + 2\left.\frac{dy}{dx}\right|_P = 0$

$\Rightarrow \left.\frac{dy}{dx}\right|_P = -\frac{3}{2}$

Question 3 (****)

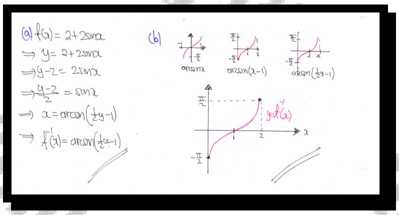
$$f(x) = 2 + 2\sin x,\ -\pi \le x \le \pi .$$

a) Find an expression for $f^{-1}(x)$.

b) Sketch the graph of $f^{-1}(x)$.

The sketch must include the coordinates of any points where the graph of $f^{-1}(x)$ meet the coordinate axes as well as the coordinates of its endpoints.

$$\boxed{f^{-1}(x) = \arcsin\left(\tfrac{1}{2}x - 1\right)}$$



(a) $f(x) = 2 + 2\sin x$

$\Rightarrow y = 2 + 2\sin x$

$\Rightarrow y - 2 = 2\sin x$

$\Rightarrow \frac{y-2}{2} = \sin x$

$\Rightarrow x = \arcsin\left(\frac{1}{2}y - 1\right)$

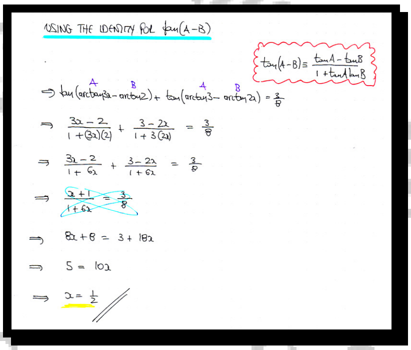
$\Rightarrow f^{-1}(x) = \arcsin\left(\frac{1}{2}x - 1\right)$

Question 4 (****)

Solve the following trigonometric equation

$$\tan(\arctan 3x - \arctan 2) + \tan(\arctan 3 - \arctan 2x) = \frac{3}{8} .$$

$$\boxed{x = \frac{1}{2}}$$



USING THE IDENTITY FOR tan(A−B)

$\tan(A-B) \equiv \frac{\tan A - \tan B}{1 + \tan A \tan B}$

$\Rightarrow \tan(\overset{A}{\arctan 3x} - \overset{B}{\arctan 2}) + \tan(\overset{A}{\arctan 3} - \overset{B}{\arctan 2x}) = \frac{3}{8}$

$\Rightarrow \frac{3x-2}{1+(3x)(2)} + \frac{3-2x}{1+3(2x)} = \frac{3}{8}$

$\Rightarrow \frac{3x-2}{1+6x} + \frac{3-2x}{1+6x} = \frac{3}{8}$

$\Rightarrow \frac{x+1}{1+6x} = \frac{3}{8}$

$\Rightarrow 8x + 8 = 3 + 18x$

$\Rightarrow 5 = 10x$

$\Rightarrow x = \frac{1}{2}$

Question 5 (****)

A curve has equation

$$y = \pi - \arccos(x+1),\ -2 \le x \le 0.$$

a) Describe geometrically the 3 transformations that map the graph of

$$y = \arccos x,\ -1 \le x \le 1,$$

onto the graph of

$$y = \pi - \arccos(x+1),\ -2 \le x \le 0.$$

b) Sketch the graph of

$$y = \pi - \arccos(x+1),\ -2 \le x \le 0.$$

The sketch must include the coordinates of any points where the graph meets the coordinate axes.

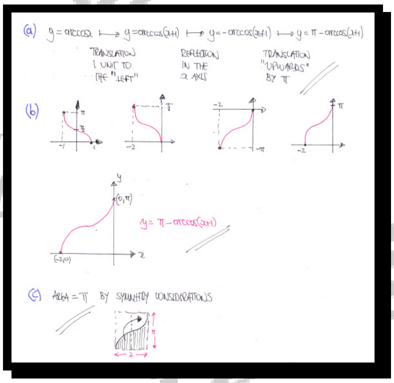
c) Use symmetry arguments to find the area of the finite region bounded by

$$y = \pi - \arccos(x+1),\ -2 \le x \le 0,$$

and the coordinate axes.

$\boxed{\quad}$, $\boxed{\text{translation by 1 unit to the right, followed by reflection in the } x \text{ axis}}$,

$\boxed{\text{area} = \pi}$



(a) $y = \arccos x \longmapsto y = \arccos(x+1) \longmapsto y = -\arccos(x+1) \longmapsto y = \pi - \arccos(x+1)$

TRANSLATION 1 UNIT TO THE "LEFT"

REFLECTION IN THE x AXIS

TRANSLATION "UPWARDS" BY π

(b)

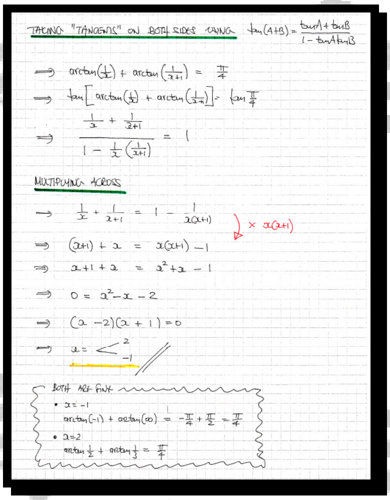
(c) AREA $= \pi$ BY SYMMETRY CONSIDERATIONS

Question 6 (****)

Solve the following trigonometric equation

$$\arctan\left(\frac{1}{x}\right)+\arctan\left(\frac{1}{x+1}\right)=\frac{\pi}{4}.$$

$\boxed{x=-1,\ 2}$



TAKING "TANGENTS" ON BOTH SIDES USING $\tan(A+B)=\frac{\tan A+\tan B}{1-\tan A\tan B}$

$\Longrightarrow \arctan\left(\frac{1}{x}\right)+\arctan\left(\frac{1}{x+1}\right)=\frac{\pi}{4}$

$\Longrightarrow \tan\left[\arctan\left(\frac{1}{x}\right)+\arctan\left(\frac{1}{x+1}\right)\right]=\tan\frac{\pi}{4}$

$\Longrightarrow \dfrac{\frac{1}{x}+\frac{1}{x+1}}{1-\frac{1}{x}\left(\frac{1}{x+1}\right)}=1$

MULTIPLYING ACROSS

$\longrightarrow \frac{1}{x}+\frac{1}{x+1}=1-\frac{1}{x(x+1)}$ $\quad\times\ x(x+1)$

$\Longrightarrow (x+1)+x=x(x+1)-1$

$\Longrightarrow x+1+x=x^2+x-1$

$\Longrightarrow 0=x^2-x-2$

$\Longrightarrow (x-2)(x+1)=0$

$\Longrightarrow x=\left\langle\begin{matrix}2\\-1\end{matrix}\right.$ //

BOTH ARE FINE

- $x=-1$

  $\arctan(-1)+\arctan(\infty)=-\frac{\pi}{4}+\frac{\pi}{2}=\frac{\pi}{4}$

- $x=2$

  $\arctan\frac{1}{2}+\arctan\frac{1}{3}=\frac{\pi}{4}$

Question 7 (****)

$$f(x) = -2 + 2\tan\left(\tfrac{1}{2}x\right), \quad -\pi \le x \le \pi .$$

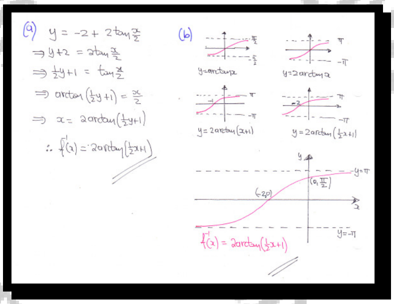
a) Find an expression for $f^{-1}(x)$.

b) Sketch the graph of $f^{-1}(x)$.

The sketch must include …

- …the equations of the asymptotes of $f^{-1}(x)$
- …the coordinates of any points where the graph of $f^{-1}(x)$ meets the coordinate axes.

$$\boxed{f^{-1}(x) = 2\arctan\left(\tfrac{1}{2}x + 1\right)}$$



(a) $y = -2 + 2\tan\frac{x}{2}$

$\Rightarrow y + 2 = 2\tan\frac{x}{2}$

$\Rightarrow \frac{1}{2}y + 1 = \tan\frac{x}{2}$

$\Rightarrow \arctan\left(\frac{1}{2}y + 1\right) = \frac{x}{2}$

$\Rightarrow x = 2\arctan\left(\frac{1}{2}y + 1\right)$

$\therefore f^{-1}(x) = 2\arctan\left(\frac{1}{2}x + 1\right)$

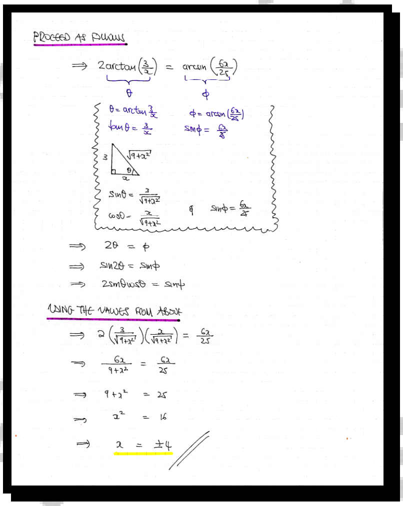
(b)

Question 8 (****)

Solve the following trigonometric equation

$$2\arctan\left(\frac{3}{x}\right) = \arcsin\left(\frac{6x}{25}\right).$$

$\boxed{\text{MP2}}$, $\boxed{x = \pm 4}$



PROCEED AS FOLLOWS

$$\Rightarrow \underbrace{2\arctan\left(\frac{3}{x}\right)}_{\theta} = \underbrace{\arcsin\left(\frac{6x}{25}\right)}_{\phi}$$

$\theta = \arctan\frac{3}{x}$ $\phi = \arcsin\left(\frac{6x}{25}\right)$

$\tan\theta = \frac{3}{x}$ $\sin\phi = \frac{6x}{25}$

$\sin\theta = \frac{3}{\sqrt{9+x^2}}$

$\cos\theta = \frac{x}{\sqrt{9+x^2}}$ & $\sin\phi = \frac{6x}{25}$

$\Rightarrow 2\theta = \phi$

$\Rightarrow \sin 2\theta = \sin\phi$

$\Rightarrow 2\sin\theta\cos\theta = \sin\phi$

USING THE VALUES FROM ABOVE

$\Rightarrow 2\left(\frac{3}{\sqrt{9+x^2}}\right)\left(\frac{x}{\sqrt{9+x^2}}\right) = \frac{6x}{25}$

$\Rightarrow \frac{6x}{9+x^2} = \frac{6x}{25}$

$\Rightarrow 9+x^2 = 25$

$\Rightarrow x^2 = 16$

$\Rightarrow x = \pm 4$

Question 9 **(****)**

The curves C_1 and C_2 have respective equations

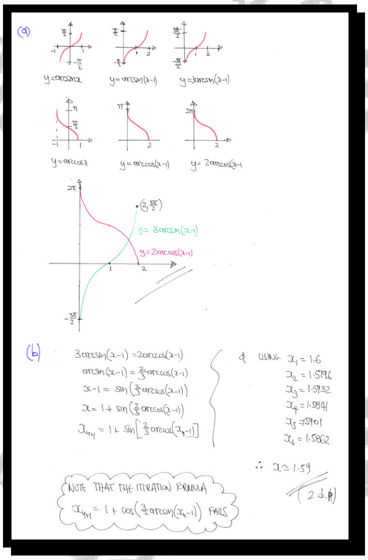
$$y_1 = 3\arcsin(x-1) \quad \text{and} \quad y_1 = 2\arccos(x-1).$$

a) Sketch in the same diagram the graphs of C_1 and C_2.

The sketch must include the coordinates of any points where the graphs of C_1 and C_2 meet the coordinate axes as well as the coordinates of the endpoints of the curves.

b) Use a suitable iteration formula of the form $x_{n+1} = f(x_n)$ with $x_1 = 1.6$ to find the x coordinate of the point of intersection between the graphs of C_1 and C_2.

$\boxed{x \approx 1.59}$



(b)

$3\arcsin(x-1) = 2\arccos(x-1)$

$\arcsin(x-1) = \frac{2}{3}\arccos(x-1)$

$x-1 = \sin\left(\frac{2}{3}\arccos(x-1)\right)$

$x = 1 + \sin\left(\frac{2}{3}\arccos(x-1)\right)$

$x_{n+1} = 1 + \sin\left[\frac{2}{3}\arccos(x_n-1)\right]$

USING $x_1 = 1.6$

$x_2 = 1.5796$

$x_3 = 1.5932$

$x_4 = 1.5841$

$x_5 = 1.5901$

$x_6 = 1.5862$

$\therefore x \simeq 1.59$ (2 d.p.)

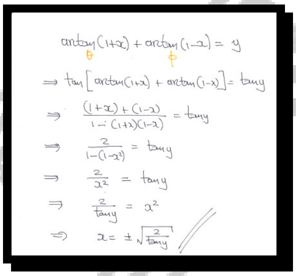
NOTE THAT THE ITERATION FORMULA $x_{n+1} = 1 + \cos\left(\frac{3}{2}\arcsin(x_n-1)\right)$ FAILS

Question 10 (****)

Make x the subject of the equation

$$\arctan(1+x)+\arctan(1-x)=y\,.$$

$$\boxed{x=\pm\sqrt{\frac{2}{\tan y}}}$$



$$\underset{\theta}{\arctan}(1+x)+\underset{\phi}{\arctan}(1-x)=y$$

$$\Rightarrow \tan\left[\arctan(1+x)+\arctan(1-x)\right]=\tan y$$

$$\Rightarrow \frac{(1+x)+(1-x)}{1-(1+x)(1-x)}=\tan y$$

$$\Rightarrow \frac{2}{1-(1-x^2)}=\tan y$$

$$\Rightarrow \frac{2}{x^2}=\tan y$$

$$\Rightarrow \frac{2}{\tan y}=x^2$$

$$\Rightarrow x=\pm\sqrt{\frac{2}{\tan y}}$$

Question 11 (****)

It is given that

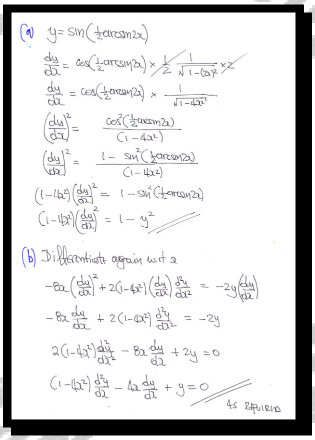
$$\frac{d}{du}(\arcsin u) = \frac{1}{\sqrt{1-u^2}}, \quad |u| \le 1.$$

Hence show that if $y = \sin\left(\frac{1}{2}\arcsin 2x\right)$, then …

a) … $\left(1-4x^2\right)\left(\frac{dy}{dx}\right)^2 = 1-y^2$.

b) … $\left(1-4x^2\right)\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + y = 0$.

$\boxed{}$, $\boxed{\text{proof}}$



(a) $y = \sin\left(\frac{1}{2}\arcsin 2x\right)$

$\frac{dy}{dx} = \cos\left(\frac{1}{2}\arcsin 2x\right) \times \cancel{\frac{1}{2}}\frac{1}{\sqrt{1-(2x)^2}} \times \cancel{2}$

$\frac{dy}{dx} = \cos\left(\frac{1}{2}\arcsin 2x\right) \times \frac{1}{\sqrt{1-4x^2}}$

$\left(\frac{dy}{dx}\right)^2 = \frac{\cos^2\left(\frac{1}{2}\arcsin 2x\right)}{(1-4x^2)}$

$\left(\frac{dy}{dx}\right)^2 = \frac{1-\sin^2\left(\frac{1}{2}\arcsin 2x\right)}{(1-4x^2)}$

$(1-4x^2)\left(\frac{dy}{dx}\right)^2 = 1-\sin^2\left(\frac{1}{2}\arcsin 2x\right)$

$(1-4x^2)\left(\frac{dy}{dx}\right)^2 = 1-y^2$

(b) Differentiate again w.r.t x

$-8x\left(\frac{dy}{dx}\right)^2 + 2(1-4x^2)\left(\frac{dy}{dx}\right)\frac{d^2y}{dx^2} = -2y\left(\frac{dy}{dx}\right)$

$-8x\frac{dy}{dx} + 2(1-4x^2)\frac{d^2y}{dx^2} = -2y$

$2(1-4x^2)\frac{d^2y}{dx^2} - 8x\frac{dy}{dx} + 2y = 0$

$(1-4x^2)\frac{d^2y}{dx} - 4x\frac{dy}{dx} + y = 0$

AS REQUIRED

Question 12 (****)

$$y = \arcsin x \,, \ -1 \le x \le 1 .$$

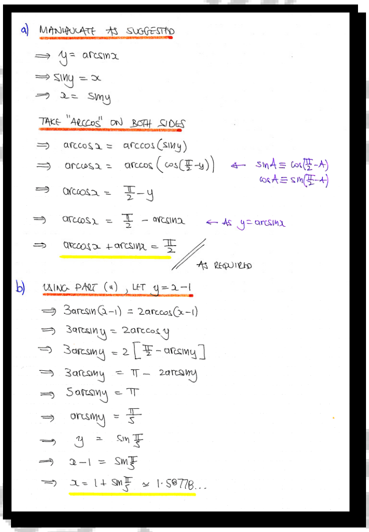
a) By expressing $\arccos x$ in terms of y, show that

$$\arcsin x + \arccos x = \frac{\pi}{2} .$$

b) Hence, or otherwise, solve the equation

$$3\arcsin(x-1) = 2\arccos(x-1) .$$

$$\boxed{x = 1 + \sin\left(\frac{\pi}{5}\right) \approx 1.5878}$$



a) MANIPULATE AS SUGGESTED

$\Rightarrow y = \arcsin x$

$\Rightarrow \sin y = x$

$\Rightarrow x = \sin y$

TAKE "ARCCOS" ON BOTH SIDES

$\Rightarrow \arccos x = \arccos(\sin y)$

$\Rightarrow \arccos x = \arccos\left(\cos\left(\frac{\pi}{2}-y\right)\right)$ ← $\sin A \equiv \cos\left(\frac{\pi}{2}-A\right)$, $\cos A \equiv \sin\left(\frac{\pi}{2}-A\right)$

$\Rightarrow \arccos x = \frac{\pi}{2} - y$

$\Rightarrow \arccos x = \frac{\pi}{2} - \arcsin x$ ← AS $y = \arcsin x$

$\Rightarrow \arccos x + \arcsin x = \frac{\pi}{2}$ // AS REQUIRED

b) USING PART (a), LET $y = x-1$

$\Rightarrow 3\arcsin(x-1) = 2\arccos(x-1)$

$\Rightarrow 3\arcsin y = 2\arccos y$

$\Rightarrow 3\arcsin y = 2\left[\frac{\pi}{2} - \arcsin y\right]$

$\Rightarrow 3\arcsin y = \pi - 2\arcsin y$

$\Rightarrow 5\arcsin y = \pi$

$\Rightarrow \arcsin y = \frac{\pi}{5}$

$\Rightarrow y = \sin\frac{\pi}{5}$

$\Rightarrow x-1 = \sin\frac{\pi}{5}$

$\Rightarrow x = 1 + \sin\frac{\pi}{5} \simeq 1.58778\ldots$

Question 13 (****)

A curve has equation

$$y = \arcsin 2x\,,\ -\frac{1}{2} \le x \le \frac{1}{2}\,,\ -\frac{\pi}{2} \le y \le \frac{\pi}{2}\,.$$

a) By finding $\frac{dx}{dy}$ and using an appropriate trigonometric identity show that

$$\frac{dy}{dx} = \frac{2}{\sqrt{1-4x^2}}\,.$$

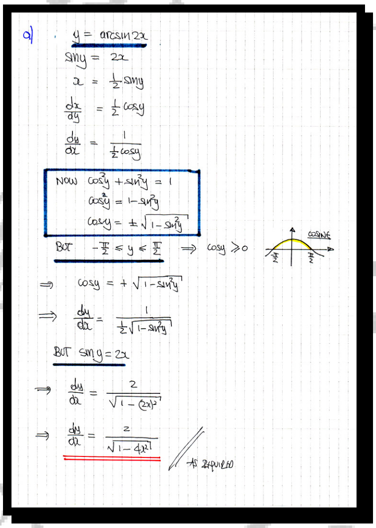
b) Show further that …

i. $\ldots \frac{d^2y}{dx^2} = \frac{Ax}{\left(1-4x^2\right)^{\frac{3}{2}}}$,

ii. $\ldots \frac{d^3y}{dx^3} = \frac{Bx^2+C}{\left(1-4x^2\right)^{\frac{5}{2}}}$,

where A, B and C are constants to be found.

$\boxed{}$, $\boxed{\text{proof}}$



a) $y = \arcsin 2x$

$\sin y = 2x$

$x = \frac{1}{2}\sin y$

$\frac{dx}{dy} = \frac{1}{2}\cos y$

$\frac{dy}{dx} = \frac{1}{\frac{1}{2}\cos y}$

NOW $\cos^2 y + \sin^2 y = 1$

$\cos^2 y = 1 - \sin^2 y$

$\cos y = \pm\sqrt{1-\sin^2 y}$

BUT $-\frac{\pi}{2} \leqslant y \leqslant \frac{\pi}{2} \Rightarrow \cos y \geqslant 0$

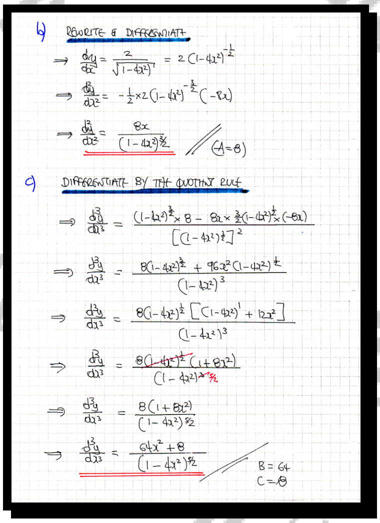
$\Rightarrow \cos y = +\sqrt{1-\sin^2 y}$

$\Rightarrow \frac{dy}{dx} = \frac{1}{\frac{1}{2}\sqrt{1-\sin^2 y}}$

BUT $\sin y = 2x$

$\Rightarrow \frac{dy}{dx} = \frac{2}{\sqrt{1-(2x)^2}}$

$\Rightarrow \frac{dy}{dx} = \frac{2}{\sqrt{1-4x^2}}$ // AS REQUIRED



b) REWRITE & DIFFERENTIATE

$\Rightarrow \frac{dy}{dx} = \frac{2}{\sqrt{1-4x^2}} = 2(1-4x^2)^{-\frac{1}{2}}$

$\Rightarrow \frac{d^2y}{dx^2} = -\frac{1}{2}\times 2(1-4x^2)^{-\frac{3}{2}}(-8x)$

$\Rightarrow \frac{d^2y}{dx^2} = \frac{8x}{(1-4x^2)^{3/2}}$ // $(A=8)$

c) DIFFERENTIATE BY THE QUOTIENT RULE

$\Rightarrow \frac{d^3y}{dx^3} = \frac{(1-4x^2)^{\frac{3}{2}}\times 8 - 8x\times\frac{3}{2}(1-4x^2)^{\frac{1}{2}}\times(-8x)}{\left[(1-4x^2)^{\frac{3}{2}}\right]^2}$

$\Rightarrow \frac{d^3y}{dx^3} = \frac{8(1-4x^2)^{\frac{3}{2}} + 96x^2(1-4x^2)^{\frac{1}{2}}}{(1-4x^2)^3}$

$\Rightarrow \frac{d^3y}{dx^3} = \frac{8(1-4x^2)^{\frac{1}{2}}\left[(1-4x^2)^1 + 12x^2\right]}{(1-4x^2)^3}$

$\Rightarrow \frac{d^3y}{dx^3} = \frac{8\cancel{(1-4x^2)^{\frac{1}{2}}}(1+8x^2)}{(1-4x^2)^{\cancel{3}\,5/2}}$

$\Rightarrow \frac{d^3y}{dx^3} = \frac{8(1+8x^2)}{(1-4x^2)^{5/2}}$

$\Rightarrow \frac{d^3y}{dx^3} = \frac{64x^2+8}{(1-4x^2)^{5/2}}$ //

$B = 64$
$C = 8$

Question 14 (****)

$$y = \arcsin x\,, \quad -\frac{1}{2} \le x \le \frac{1}{2}\,, \quad -\frac{\pi}{2} \le y \le \frac{\pi}{2}\,.$$

a) By finding $\frac{dx}{dy}$ and using an appropriate trigonometric identity show that

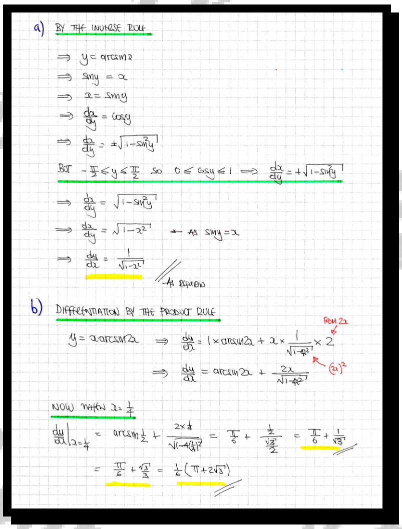
$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}\,.$$

A curve C has equation

$$y = x\arcsin 2x\,, \quad -\frac{1}{2} \le x \le \frac{1}{2}\,, \quad -\frac{\pi}{2} \le y \le \frac{\pi}{2}\,.$$

b) Find the exact value of $\frac{dy}{dx}$ at the point on C where $x = \frac{1}{4}$.

$$\boxed{}\,, \quad \boxed{\frac{1}{6}\left(\pi + 2\sqrt{3}\right)}$$



a) BY THE INVERSE RULE

$\Rightarrow y = \arcsin x$

$\Rightarrow \sin y = x$

$\Rightarrow x = \sin y$

$\Rightarrow \frac{dx}{dy} = \cos y$

$\Rightarrow \frac{dx}{dy} = \pm\sqrt{1-\sin^2 y}$

BUT $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$ SO $0 \le \cos y \le 1 \Rightarrow \frac{dx}{dy} = +\sqrt{1-\sin^2 y}$

$\Rightarrow \frac{dx}{dy} = \sqrt{1-\sin^2 y}$

$\Rightarrow \frac{dx}{dy} = \sqrt{1-x^2}$ ← AS $\sin y = x$

$\Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$ // AS REQUIRED

b) DIFFERENTIATION BY THE PRODUCT RULE

$y = x\arcsin 2x \Rightarrow \frac{dy}{dx} = 1 \times \arcsin 2x + x \times \frac{1}{\sqrt{1-4x^2}} \times 2$ ← FROM $2x$ ← $(2x)^2$

$\Rightarrow \frac{dy}{dx} = \arcsin 2x + \frac{2x}{\sqrt{1-4x^2}}$

NOW WHEN $x = \frac{1}{4}$

$\left.\frac{dy}{dx}\right|_{x=\frac{1}{4}} = \arcsin\frac{1}{2} + \frac{2\times\frac{1}{4}}{\sqrt{1-4\left(\frac{1}{4}\right)^2}} = \frac{\pi}{6} + \frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}} = \frac{\pi}{6} + \frac{1}{\sqrt{3}}$ //

$= \frac{\pi}{6} + \frac{\sqrt{3}}{3} = \frac{1}{6}\left(\pi + 2\sqrt{3}\right)$ //

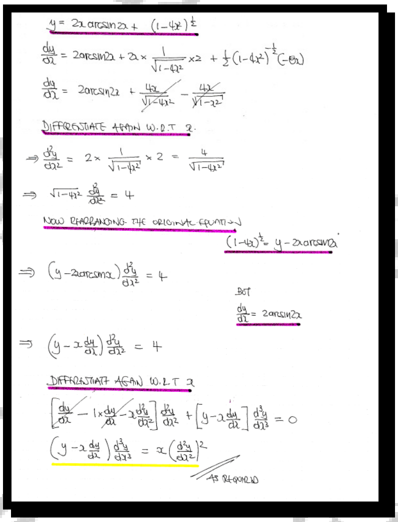
Question 15 (****)

$$y = 2x \arcsin 2x + \sqrt{1-4x^2}\ ,\ -\tfrac{1}{2} \le x \le \tfrac{1}{2}.$$

Show clearly that

$$\frac{d^3y}{dx^3}\left(y - x\frac{dy}{dx}\right) = x\left(\frac{d^2y}{dx^2}\right)^2 .$$

$\boxed{}$, $\boxed{\text{proof}}$



$$y = 2x\arcsin 2x + (1-4x)^{\frac{1}{2}}$$

$$\frac{dy}{dx} = 2\arcsin 2x + 2x \times \frac{1}{\sqrt{1-4x^2}} \times 2 + \frac{1}{2}(1-4x^2)^{-\frac{1}{2}}(-8x)$$

$$\frac{dy}{dx} = 2\arcsin 2x + \cancel{\frac{4x}{\sqrt{1-4x^2}}} - \cancel{\frac{4x}{\sqrt{1-x^2}}}$$

DIFFERENTIATE AGAIN W.R.T x.

$$\Rightarrow \frac{d^2y}{dx^2} = 2 \times \frac{1}{\sqrt{1-4x^2}} \times 2 = \frac{4}{\sqrt{1-4x^2}}$$

$$\Rightarrow \sqrt{1-4x^2}\,\frac{d^2y}{dx^2} = 4$$

NOW REARRANGING THE ORIGINAL EQUATION

$$(1-4x)^{\frac{1}{2}} = y - 2x\arcsin 2x$$

$$\Rightarrow (y - 2x\arcsin x)\frac{d^2y}{dx^2} = 4$$

BUT

$$\frac{dy}{dx} = 2\arcsin 2x$$

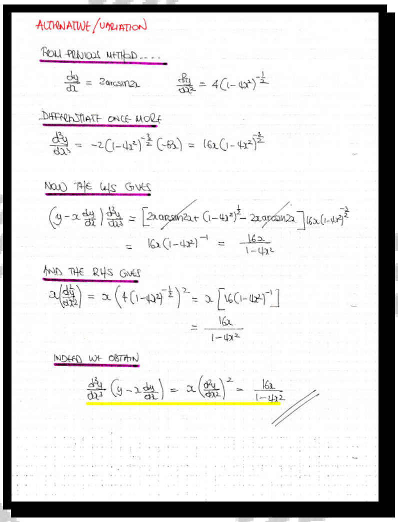
$$\Rightarrow \left(y - x\frac{dy}{dx}\right)\frac{d^2y}{dx^2} = 4$$

DIFFERENTIATE AGAIN W.R.T x

$$\left[\cancel{\frac{dy}{dx}} - 1\times\cancel{\frac{dy}{dx}} - x\frac{d^2y}{dx^2}\right]\frac{d^2y}{dx^2} + \left[y - x\frac{dy}{dx}\right]\frac{d^3y}{dx^3} = 0$$

$$\left(y - x\frac{dy}{dx}\right)\frac{d^3y}{dx^3} = x\left(\frac{d^2y}{dx^2}\right)^2$$

// AS REQUIRED



ALTERNATIVE/VARIATION

FROM PREVIOUS METHOD....

$$\frac{dy}{dx} = 2\arcsin 2x \qquad \frac{d^2y}{dx^2} = 4(1-4x^2)^{-\frac{1}{2}}$$

DIFFERENTIATE ONCE MORE

$$\frac{d^3y}{dx^3} = -2(1-4x^2)^{-\frac{3}{2}}(-8x) = 16x(1-4x^2)^{-\frac{3}{2}}$$

NOW THE LHS GIVES

$$\left(y - x\frac{dy}{dx}\right)\frac{d^3y}{dx^3} = \left[\cancel{2x\arcsin 2x} + (1-4x^2)^{\frac{1}{2}} - \cancel{2x\arcsin 2x}\right]16x(1-4x^2)^{-\frac{3}{2}}$$

$$= 16x(1-4x^2)^{-1} = \frac{16x}{1-4x^2}$$

AND THE RHS GIVES

$$x\left(\frac{d^2y}{dx^2}\right) = x\left(4(1-4x^2)^{-\frac{1}{2}}\right)^2 = x\left[16(1-4x^2)^{-1}\right]$$

$$= \frac{16x}{1-4x^2}$$

INDEED WE OBTAIN

$$\frac{d^3y}{dx^3}\left(y - x\frac{dy}{dx}\right) = x\left(\frac{d^2y}{dx^2}\right)^2 = \frac{16x}{1-4x^2}$$

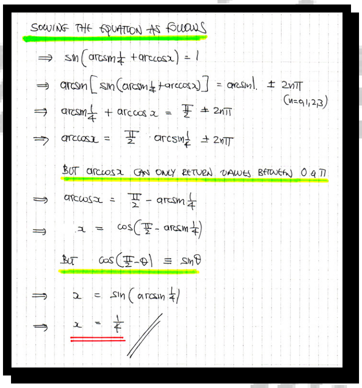
//

Question 16 **(****)**

Use trigonometric algebra to solve the equation

$$\sin\left[\arcsin\frac{1}{4}+\arccos x\right]=1.$$

$$\boxed{x=\frac{1}{4}}$$



SOLVING THE EQUATION AS FOLLOWS

$\Rightarrow \sin\left(\arcsin\frac{1}{4}+\arccos x\right)=1$

$\Rightarrow \arcsin\left[\sin\left(\arcsin\frac{1}{4}+\arccos x\right)\right]=\arcsin 1. \pm 2n\pi$ $(n=0,1,2,3)$

$\Rightarrow \arcsin\frac{1}{4}+\arccos x=\frac{\pi}{2}\pm 2n\pi$

$\Rightarrow \arccos x=\frac{\pi}{2}\cdot\arcsin\frac{1}{4}\pm 2n\pi$

BUT $\arccos x$ CAN ONLY RETURN VALUES BETWEEN 0 & π

$\Rightarrow \arccos x=\frac{\pi}{2}-\arcsin\frac{1}{4}$

$\Rightarrow x=\cos\left(\frac{\pi}{2}-\arcsin\frac{1}{4}\right)$

BUT $\cos\left(\frac{\pi}{2}-\theta\right)\equiv\sin\theta$

$\Rightarrow x=\sin\left(\arcsin\frac{1}{4}\right)$

$\Rightarrow x=\frac{1}{4}$ //

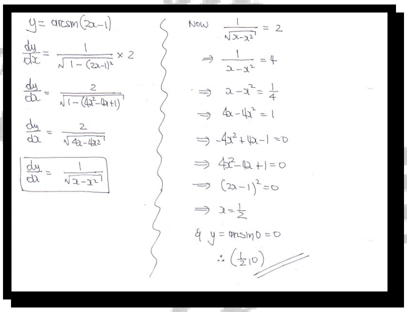
Question 17 **(****)**

The curve C has equation

$$y=\arcsin(2x-1),\ -0\le x\le 1.$$

Find the coordinates of the point on C, whose gradient is 2.

$$\boxed{\left(\frac{1}{2},0\right)}$$



$y=\arcsin(2x-1)$

$\frac{dy}{dx}=\frac{1}{\sqrt{1-(2x-1)^2}}\times 2$

$\frac{dy}{dx}=\frac{2}{\sqrt{1-(4x^2-4x+1)}}$

$\frac{dy}{dx}=\frac{2}{\sqrt{4x-4x^2}}$

$\boxed{\frac{dy}{dx}=\frac{1}{\sqrt{x-x^2}}}$

Now $\frac{1}{\sqrt{x-x^2}}=2$

$\Rightarrow \frac{1}{x-x^2}=4$

$\Rightarrow x-x^2=\frac{1}{4}$

$\Rightarrow 4x-4x^2=1$

$\Rightarrow -4x^2+4x-1=0$

$\Rightarrow 4x^2-4x+1=0$

$\Rightarrow (2x-1)^2=0$

$\Rightarrow x=\frac{1}{2}$

& $y=\arcsin 0=0$

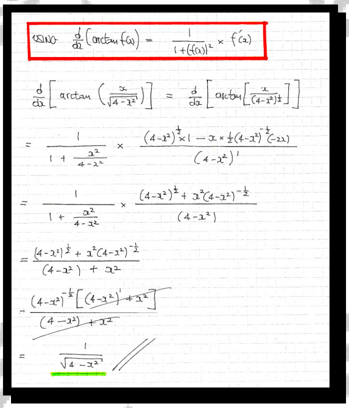
$\therefore \left(\frac{1}{2},0\right)$ //

Question 18 (****)

Find a simplified expression for

$$\frac{d}{dx}\left[\arctan\left(\frac{x}{\sqrt{4-x^2}}\right)\right]$$

$$\boxed{\frac{d}{dx}\left[\arctan\left(\frac{x}{\sqrt{4-x^2}}\right)\right]=\frac{1}{\sqrt{4-x^2}}}$$



USING $\frac{d}{dx}\left(\arctan f(x)\right) = \frac{1}{1+(f(x))^2} \times f'(x)$

$$\frac{d}{dx}\left[\arctan\left(\frac{x}{\sqrt{4-x^2}}\right)\right] = \frac{d}{dx}\left[\arctan\left[\frac{x}{(4-x^2)^{\frac{1}{2}}}\right]\right]$$

$$= \frac{1}{1+\frac{x^2}{4-x^2}} \times \frac{(4-x^2)^{\frac{1}{2}}\times 1 - x \times \frac{1}{2}(4-x^2)^{-\frac{1}{2}}(-2x)}{(4-x^2)^1}$$

$$= \frac{1}{1+\frac{x^2}{4-x^2}} \times \frac{(4-x^2)^{\frac{1}{2}} + x^2(4-x^2)^{-\frac{1}{2}}}{(4-x^2)}$$

$$= \frac{(4-x^2)^{\frac{1}{2}} + x^2(4-x^2)^{-\frac{1}{2}}}{(4-x^2)+x^2}$$

$$= \frac{(4-x^2)^{-\frac{1}{2}}\left[\cancel{(4-x^2)^1 + x^2}\right]}{\cancel{(4-x^2)+x^2}}$$

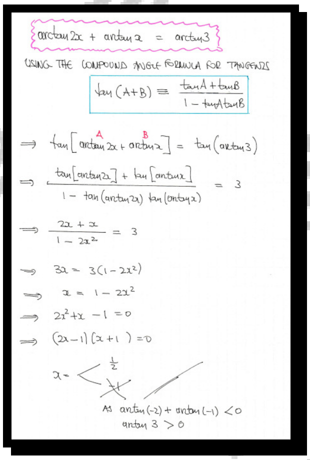
$$= \frac{1}{\sqrt{4-x^2}} \quad //$$

Question 19 (****)

Solve the following trigonometric equation.

$$\arctan 2x + \arctan x = \arctan 3, \qquad x \in \mathbb{R}.$$

$$\boxed{x=\frac{1}{2}}$$



$\arctan 2x + \arctan x = \arctan 3$

USING THE COMPOUND ANGLE FORMULA FOR TANGENTS

$$\tan(A+B) \equiv \frac{\tan A + \tan B}{1-\tan A \tan B}$$

$$\Rightarrow \tan\left[\overset{A}{\arctan 2x} + \overset{B}{\arctan x}\right] = \tan(\arctan 3)$$

$$\Rightarrow \frac{\tan[\arctan 2x] + \tan[\arctan x]}{1-\tan(\arctan 2x)\tan(\arctan x)} = 3$$

$$\Rightarrow \frac{2x+x}{1-2x^2} = 3$$

$$\Rightarrow 3x = 3(1-2x^2)$$

$$\Rightarrow x = 1-2x^2$$

$$\Rightarrow 2x^2+x-1=0$$

$$\Rightarrow (2x-1)(x+1)=0$$

$$x = \left\langle \begin{array}{l} \frac{1}{2} \\ \cancel{-1} \end{array} \right. \quad //$$

AS $\arctan(-2) + \arctan(-1) < 0$

$\arctan 3 > 0$

16

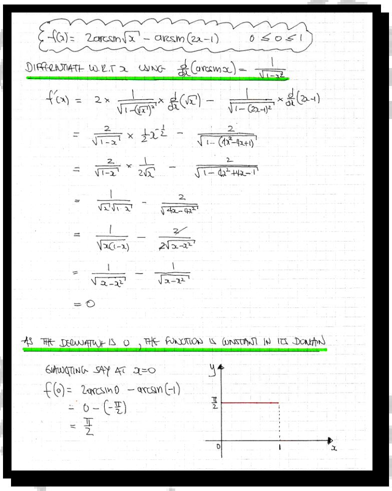
HARD QUESTIONS

Question 1 (****+)

$$f(x) = 2\arcsin\sqrt{x} - \arcsin(2x-1), \quad 0 \le x \le 1.$$

By considering $f'(x)$ sketch the graph of $f(x)$.

$\boxed{\text{[illegible]}}$, $\boxed{\text{graph}}$



$f(x) = 2\arcsin\sqrt{x} - \arcsin(2x-1) \qquad 0 \le 0 \le 1$

DIFFERENTIATE W.R.T x USING $\frac{d}{dx}(\arcsin x) = \frac{1}{\sqrt{1-x^2}}$

$$f'(x) = 2 \times \frac{1}{\sqrt{1-(\sqrt{x})^2}} \times \frac{d}{dx}(\sqrt{x}) - \frac{1}{\sqrt{1-(2x-1)^2}} \times \frac{d}{dx}(2x-1)$$

$$= \frac{2}{\sqrt{1-x}} \times \frac{1}{2}x^{-\frac{1}{2}} - \frac{2}{\sqrt{1-(4x^2-4x+1)}}$$

$$= \frac{2}{\sqrt{1-x}} \times \frac{1}{2\sqrt{x}} - \frac{2}{\sqrt{1-4x^2+4x-1}}$$

$$= \frac{1}{\sqrt{x}\sqrt{1-x}} - \frac{2}{\sqrt{4x-4x^2}}$$

$$= \frac{1}{\sqrt{x(1-x)}} - \frac{\cancel{2}}{\cancel{2}\sqrt{x-x^2}}$$

$$= \frac{1}{\sqrt{x-x^2}} - \frac{1}{\sqrt{x-x^2}}$$

$$= 0$$

AS THE DERIVATIVE IS 0, THE FUNCTION IS CONSTANT IN ITS DOMAIN

EVALUATING SAY AT $x=0$

$$f(0) = 2\arcsin 0 - \arcsin(-1)$$

$$= 0 - \left(-\frac{\pi}{2}\right)$$

$$= \frac{\pi}{2}$$

Question 2 (****+)

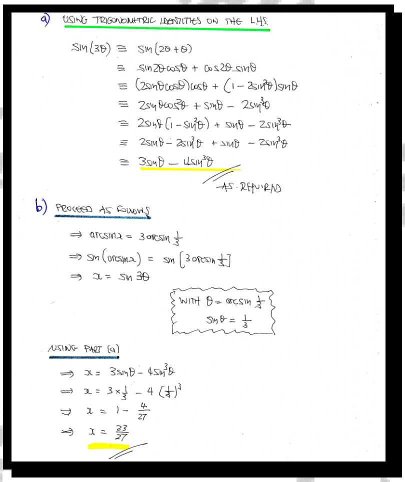
$$\sin 3\theta \equiv 3\sin\theta - 4\sin^3\theta .$$

a) Prove the validity of the above trigonometric identity by considering the expansion of $\sin(2\theta+\theta)$.

b) Hence or otherwise solve the equation

$$\arcsin x = 3\arcsin\left(\frac{1}{3}\right).$$

$$\boxed{x = \frac{23}{27}}$$



a) USING TRIGONOMETRIC IDENTITIES ON THE L.H.S.

$$\sin(3\theta) \equiv \sin(2\theta+\theta)$$
$$\equiv \sin 2\theta\cos\theta + \cos 2\theta\sin\theta$$
$$\equiv (2\sin\theta\cos\theta)\cos\theta + (1-2\sin^2\theta)\sin\theta$$
$$\equiv 2\sin\theta\cos^2\theta + \sin\theta - 2\sin^3\theta$$
$$\equiv 2\sin\theta(1-\sin^2\theta) + \sin\theta - 2\sin^3\theta$$
$$\equiv 2\sin\theta - 2\sin^3\theta + \sin\theta - 2\sin^3\theta$$
$$\equiv 3\sin\theta - 4\sin^3\theta$$

AS REQUIRED

b) PROCEED AS FOLLOWS

$$\Rightarrow \arcsin x = 3\arcsin\frac{1}{3}$$
$$\Rightarrow \sin(\arcsin x) = \sin\left[3\arcsin\frac{1}{3}\right]$$
$$\Rightarrow x = \sin 3\theta$$

WITH $\theta = \arcsin\frac{1}{3}$

$\sin\theta = \frac{1}{3}$

USING PART (a)

$$\Rightarrow x = 3\sin\theta - 4\sin^3\theta$$
$$\Rightarrow x = 3\times\frac{1}{3} - 4\left(\frac{1}{3}\right)^3$$
$$\Rightarrow x = 1 - \frac{4}{27}$$
$$\Rightarrow x = \frac{23}{27}$$

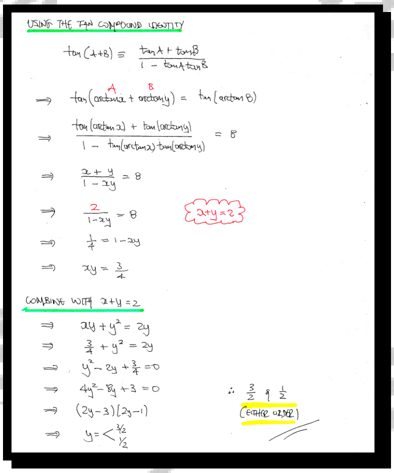
Question 3 (****+)

Solve the following simultaneous equations

$$\arctan x + \arctan y = \arctan 8$$

$$x + y = 2 .$$

$\boxed{x = \frac{1}{2}, y = \frac{3}{2}, \text{ in either order}}$



USING THE TAN COMPOUND IDENTITY

$$\tan(A+B) \equiv \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

$$\Rightarrow \tan(\overset{A}{\arctan x} + \overset{B}{\arctan y}) = \tan(\arctan 8)$$

$$\Rightarrow \frac{\tan(\arctan x) + \tan(\arctan y)}{1 - \tan(\arctan x)\tan(\arctan y)} = 8$$

$$\Rightarrow \frac{x+y}{1-xy} = 8$$

$$\Rightarrow \frac{2}{1-xy} = 8 \qquad x+y=2$$

$$\Rightarrow \frac{1}{4} = 1 - xy$$

$$\Rightarrow xy = \frac{3}{4}$$

COMBINE WITH $x+y=2$

$$\Rightarrow xy + y^2 = 2y$$

$$\Rightarrow \frac{3}{4} + y^2 = 2y$$

$$\Rightarrow y^2 - 2y + \frac{3}{4} = 0$$

$$\Rightarrow 4y^2 - 8y + 3 = 0$$

$$\Rightarrow (2y-3)(2y-1)$$

$$\Rightarrow y = <\begin{matrix} 3/2 \\ 1/2 \end{matrix}$$

$\therefore \frac{3}{2}$ & $\frac{1}{2}$

(EITHER ORDER)

Question 4 (****+)

$$f(x) = \sec x, \qquad 0 \le x < \frac{\pi}{2} \cup \frac{\pi}{2} < x \le \pi .$$

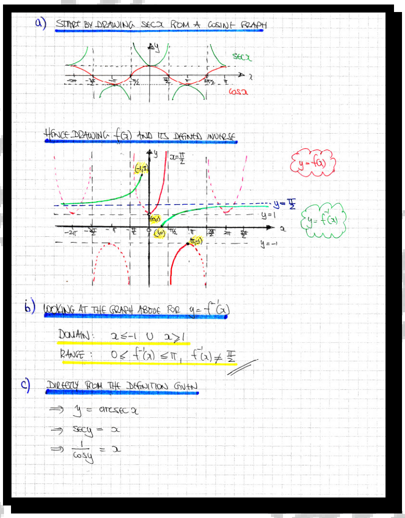
a) Sketch in the same diagram the graphs of $f(x)$ and $f^{-1}(x) = \operatorname{arcsec} x$.

b) State the domain and range of $f^{-1}(x) = \operatorname{arcsec} x$.

c) Show clearly that $\operatorname{arcsec} x = \arccos\left(\frac{1}{x}\right)$.

d) Show further that $\frac{d}{dx}(\operatorname{arcsec} x) = \frac{1}{\sqrt{x^4 - x^2}}$.

$\boxed{}$, $\boxed{\text{domain: } x \le -1 \cup x \ge 1}$, $\boxed{\text{range: } 0 \le f^{-1}(x) \le \pi,\ f^{-1}(x) \ne \frac{\pi}{2}}$



a) START BY DRAWING SECx FROM A COSINE GRAPH

HENCE DRAWING f(x) AND ITS DEFINED INVERSE

b) LOOKING AT THE GRAPH ABOVE FOR $y = f^{-1}(x)$

DOMAIN: $x \le -1 \cup x \ge 1$

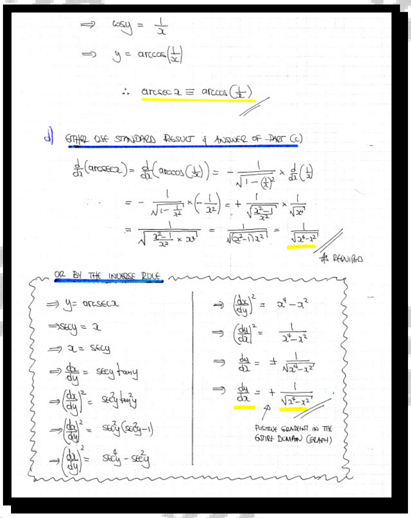
RANGE: $0 \le f^{-1}(x) \le \pi,\ f^{-1}(x) \ne \frac{\pi}{2}$

c) DIRECTLY FROM THE DEFINITION GIVEN

$\Rightarrow y = \operatorname{arcsec} x$

$\Rightarrow \sec y = x$

$\Rightarrow \frac{1}{\cos y} = x$



$\Rightarrow \cos y = \frac{1}{x}$

$\Rightarrow y = \arccos\left(\frac{1}{x}\right)$

$\therefore \operatorname{arcsec} x \equiv \arccos\left(\frac{1}{x}\right)$

d) EITHER USE STANDARD RESULT & ANSWER OF PART (c)

$$\frac{d}{dx}(\operatorname{arcsec} x) = \frac{d}{dx}\left(\arccos\left(\frac{1}{x}\right)\right) = -\frac{1}{\sqrt{1-\left(\frac{1}{x}\right)^2}} \times \frac{d}{dx}\left(\frac{1}{x}\right)$$

$$= -\frac{1}{\sqrt{1-\frac{1}{x^2}}} \times \left(-\frac{1}{x^2}\right) = +\frac{1}{\sqrt{\frac{x^2-1}{x^2}}} \times \frac{1}{\sqrt{x^4}}$$

$$= \frac{1}{\sqrt{\frac{x^2-1}{x^2} \times x^4}} = \frac{1}{\sqrt{(x^2-1)x^2}} = \frac{1}{\sqrt{x^4-x^2}}$$

AS REQUIRED

OR BY THE INVERSE RULE

$\Rightarrow y = \operatorname{arcsec} x$

$\Rightarrow \sec y = x$

$\Rightarrow x = \sec y$

$\Rightarrow \frac{dx}{dy} = \sec y \tan y$

$\Rightarrow \left(\frac{dx}{dy}\right)^2 = \sec^2 y \tan^2 y$

$\Rightarrow \left(\frac{dx}{dy}\right)^2 = \sec^2 y(\sec^2 y - 1)$

$\Rightarrow \left(\frac{dx}{dy}\right)^2 = \sec^4 y - \sec^2 y$

$\Rightarrow \left(\frac{dx}{dy}\right)^2 = x^4 - x^2$

$\Rightarrow \left(\frac{dy}{dx}\right)^2 = \frac{1}{x^4-x^2}$

$\Rightarrow \frac{dy}{dx} = \pm\frac{1}{\sqrt{x^4-x^2}}$

$\Rightarrow \frac{dy}{dx} = +\frac{1}{\sqrt{x^4-x^2}}$

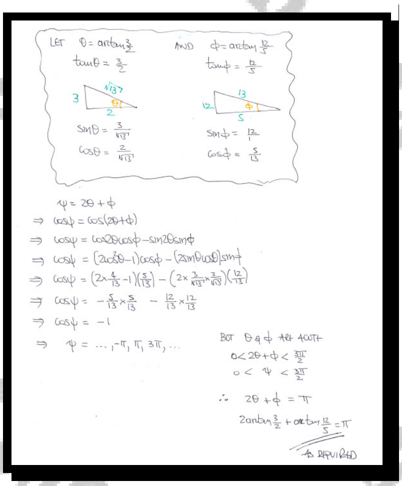
POSITIVE GRADIENT IN THE ENTIRE DOMAIN (GRAPH)

Question 5 **(****+)**

Show clearly that

$$2\arctan\left(\frac{3}{2}\right)+\arctan\left(\frac{12}{5}\right)=\pi\,.$$

V, proof



LET $\theta=\arctan\frac{3}{2}$ AND $\phi=\arctan\frac{12}{5}$

$\tan\theta=\frac{3}{2}$ $\tan\phi=\frac{12}{5}$

$\sin\theta=\frac{3}{\sqrt{13}}$ $\sin\phi=\frac{12}{1}$

$\cos\theta=\frac{2}{\sqrt{13}}$ $\cos\phi=\frac{5}{13}$

$\psi=2\theta+\phi$

$\Rightarrow \cos\psi=\cos(2\theta+\phi)$

$\Rightarrow \cos\psi=\cos2\theta\cos\phi-\sin2\theta\sin\phi$

$\Rightarrow \cos\psi=(2\cos^2\theta-1)\cos\phi-(2\sin\theta\cos\theta)\sin\phi$

$\Rightarrow \cos\psi=\left(2\times\frac{4}{13}-1\right)\left(\frac{5}{13}\right)-\left(2\times\frac{3}{\sqrt{13}}\times\frac{2}{\sqrt{13}}\right)\left(\frac{12}{13}\right)$

$\Rightarrow \cos\psi=-\frac{5}{13}\times\frac{5}{13}-\frac{12}{13}\times\frac{12}{13}$

$\Rightarrow \cos\psi=-1$

$\Rightarrow \psi=\dots,-\pi,\pi,3\pi,\dots$

BUT θ & ϕ ARE ACUTE

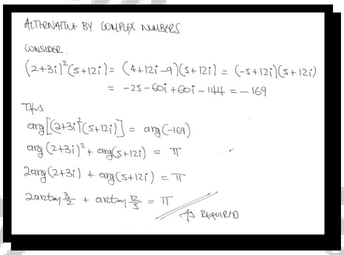
$0<2\theta+\phi<\frac{3\pi}{2}$

$0<\psi<\frac{3\pi}{2}$

$\therefore 2\theta+\phi=\pi$

$2\arctan\frac{3}{2}+\arctan\frac{12}{5}=\pi$

AS REQUIRED



ALTERNATIVE BY COMPLEX NUMBERS

CONSIDER

$(2+3i)^2(5+12i)=(4+12i-9)(5+12i)=(-5+12i)(5+12i)$

$=-25-60i+60i-144=-169$

THUS

$\arg\left[(2+3i)^2(5+12i)\right]=\arg(-169)$

$\arg(2+3i)^2+\arg(5+12i)=\pi$

$2\arg(2+3i)+\arg(5+12i)=\pi$

$2\arctan\frac{3}{2}+\arctan\frac{12}{5}=\pi$

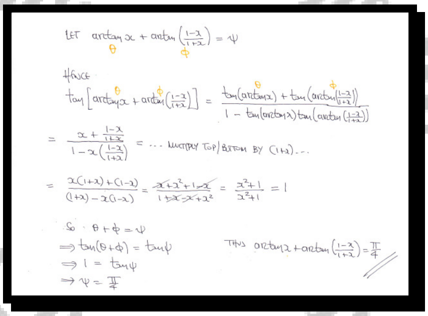
AS REQUIRED

Question 6 **(****+)**

Show clearly that

$$\arctan x+\arctan\left(\frac{1-x}{1+x}\right)=\frac{\pi}{4}\,.$$

proof



LET $\underset{\theta}{\arctan x}+\underset{\phi}{\arctan\left(\frac{1-x}{1+x}\right)}=\psi$

HENCE

$\tan\left[\overset{\theta}{\arctan x}+\overset{\phi}{\arctan\left(\frac{1-x}{1+x}\right)}\right]=\frac{\tan(\overset{\theta}{\arctan x})+\tan\left(\overset{\phi}{\arctan\left(\frac{1-x}{1+x}\right)}\right)}{1-\tan(\arctan x)\tan\left(\arctan\left(\frac{1-x}{1+x}\right)\right)}$

$=\frac{x+\frac{1-x}{1+x}}{1-x\left(\frac{1-x}{1+x}\right)}=\dots$ MULTIPLY TOP/BOTTOM BY $(1+x)\dots$

$=\frac{x(1+x)+(1-x)}{(1+x)-x(1-x)}=\frac{\cancel{x}+x^2+1\cancel{-x}}{1\cancel{+x}\cancel{-x}+x^2}=\frac{x^2+1}{x^2+1}=1$

SO $\theta+\phi=\psi$

$\Rightarrow \tan(\theta+\phi)=\tan\psi$

$\Rightarrow 1=\tan\psi$

$\Rightarrow \psi=\frac{\pi}{4}$

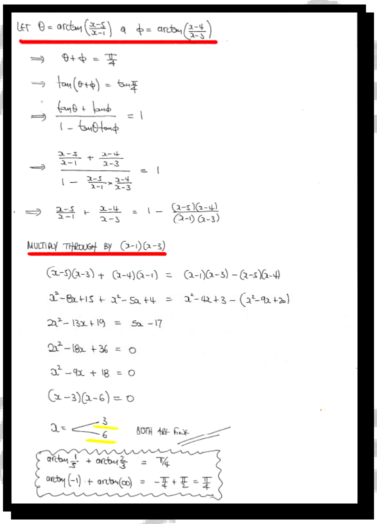
THUS $\arctan x+\arctan\left(\frac{1-x}{1+x}\right)=\frac{\pi}{4}$

Question 7 **(****+)**

Solve the following trigonometric equation.

$$\arctan\left(\frac{x-5}{x-1}\right)+\arctan\left(\frac{x-4}{x-3}\right)=\frac{\pi}{4},\ x\in\mathbb{R}.$$

$$\boxed{x=3 \ \cup \ x=6}$$



LET $\theta = \arctan\left(\frac{x-5}{x-1}\right)$ & $\phi = \arctan\left(\frac{x-4}{x-3}\right)$

$\Rightarrow \theta+\phi = \frac{\pi}{4}$

$\Rightarrow \tan(\theta+\phi) = \tan\frac{\pi}{4}$

$\Rightarrow \dfrac{\tan\theta+\tan\phi}{1-\tan\theta\tan\phi} = 1$

$\Rightarrow \dfrac{\frac{x-5}{x-1}+\frac{x-4}{x-3}}{1-\frac{x-5}{x-1}\times\frac{x-4}{x-3}} = 1$

$\Rightarrow \frac{x-5}{x-1}+\frac{x-4}{x-3} = 1-\frac{(x-5)(x-4)}{(x-1)(x-3)}$

MULTIPLY THROUGH BY $(x-1)(x-3)$

$(x-5)(x-3)+(x-4)(x-1) = (x-1)(x-3)-(x-5)(x-4)$

$x^2-8x+15+x^2-5x+4 = x^2-4x+3-(x^2-9x+20)$

$2x^2-13x+19 = 5x-17$

$2x^2-18x+36 = 0$

$x^2-9x+18 = 0$

$(x-3)(x-6) = 0$

$x = \begin{cases}3\\6\end{cases}$ BOTH ARE FINE

$\arctan\frac{1}{5}+\arctan\frac{2}{3} = \pi/4$

$\arctan(-1)+\arctan(\infty) = -\frac{\pi}{4}+\frac{\pi}{2} = \frac{\pi}{4}$

Question 8 (****+)

$$y = \arccos x\,,\ -1 \le x \le 1\,,\ 0 \le y \le \pi\,.$$

a) By writing $y = \arccos x$ as $x = \cos y$, show that

$$\frac{dy}{dx} = -\frac{1}{\sqrt{1-x^2}}\,.$$

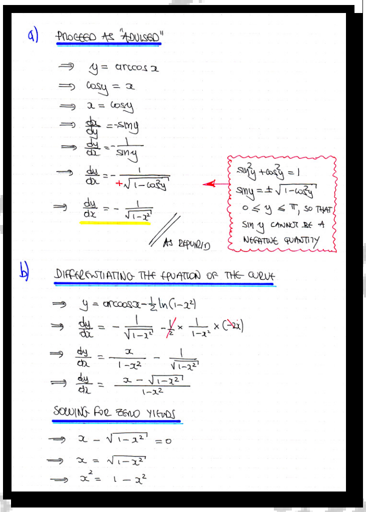
The curve C has equation

$$y = \arccos x - \tfrac{1}{2}\ln\left(1-x^2\right),\ x > 0\,.$$

b) Show that the y coordinate of the stationary point of C is

$$\frac{1}{4}(\pi + \ln 4)\,.$$

$\boxed{}$, $\boxed{\text{proof}}$



a) PROCEED AS "ADVISED"

$\Rightarrow\ y = \arccos x$

$\Rightarrow\ \cos y = x$

$\Rightarrow\ x = \cos y$

$\Rightarrow\ \frac{dx}{dy} = -\sin y$

$\Rightarrow\ \frac{dy}{dx} = -\frac{1}{\sin y}$

$\Rightarrow\ \frac{dy}{dx} = -\frac{1}{+\sqrt{1-\cos^2 y}}$

$\Rightarrow\ \frac{dy}{dx} = -\frac{1}{\sqrt{1-x^2}}$ // AS REQUIRED

$\sin^2 y + \cos^2 y = 1$

$\sin y = \pm\sqrt{1-\cos^2 y}$

$0 \le y \le \pi$, SO THAT $\sin y$ CANNOT BE A NEGATIVE QUANTITY

b) DIFFERENTIATING THE EQUATION OF THE CURVE

$\Rightarrow\ y = \arccos x - \frac{1}{2}\ln(1-x^2)$

$\Rightarrow\ \frac{dy}{dx} = -\frac{1}{\sqrt{1-x^2}} - \cancel{\frac{1}{2}} \times \frac{1}{1-x^2} \times (-\cancel{2}x)$

$\Rightarrow\ \frac{dy}{dx} = \frac{x}{1-x^2} - \frac{1}{\sqrt{1-x^2}}$

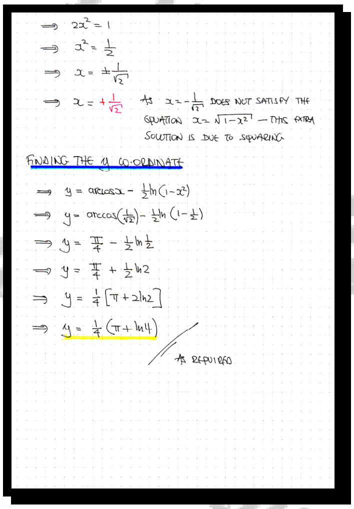
$\Rightarrow\ \frac{dy}{dx} = \frac{x - \sqrt{1-x^2}}{1-x^2}$

SOLVING FOR ZERO YIELDS

$\Rightarrow\ x - \sqrt{1-x^2} = 0$

$\Rightarrow\ x = \sqrt{1-x^2}$

$\Rightarrow\ x^2 = 1 - x^2$



$\Rightarrow\ 2x^2 = 1$

$\Rightarrow\ x^2 = \frac{1}{2}$

$\Rightarrow\ x = \pm\frac{1}{\sqrt{2}}$

$\Rightarrow\ x = +\frac{1}{\sqrt{2}}$ AS $x = -\frac{1}{\sqrt{2}}$ DOES NOT SATISFY THE EQUATION $x = \sqrt{1-x^2}$ — THIS EXTRA SOLUTION IS DUE TO SQUARING

FINDING THE y CO-ORDINATE

$\Rightarrow\ y = \arccos x - \frac{1}{2}\ln(1-x^2)$

$\Rightarrow\ y = \arccos\left(\frac{1}{\sqrt{2}}\right) - \frac{1}{2}\ln\left(1-\frac{1}{2}\right)$

$\Rightarrow\ y = \frac{\pi}{4} - \frac{1}{2}\ln\frac{1}{2}$

$\Rightarrow\ y = \frac{\pi}{4} + \frac{1}{2}\ln 2$

$\Rightarrow\ y = \frac{1}{4}\left[\pi + 2\ln 2\right]$

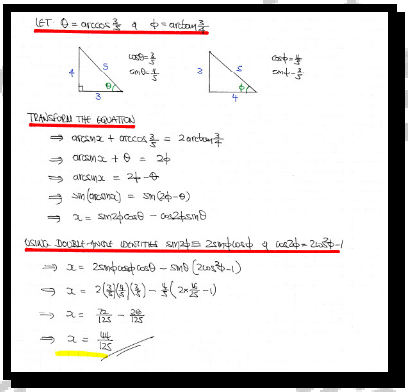
$\Rightarrow\ y = \frac{1}{4}(\pi + \ln 4)$ // AS REQUIRED

Question 9 (****+)

Solve the following trigonometric equation

$$\arcsin x + \arccos\frac{3}{5} = 2\arctan\frac{3}{4}.$$

$\boxed{}$, $\boxed{x = \frac{44}{125}}$



LET $\theta = \arccos\frac{3}{5}$ & $\phi = \arctan\frac{3}{4}$

$\cos\theta = \frac{3}{5}$

$\sin\theta = \frac{4}{5}$

$\cos\phi = \frac{4}{5}$

$\sin\phi = \frac{3}{5}$

TRANSFORM THE EQUATION

$\Rightarrow \arcsin x + \arccos\frac{3}{5} = 2\arctan\frac{3}{4}$

$\Rightarrow \arcsin x + \theta = 2\phi$

$\Rightarrow \arcsin x = 2\phi - \theta$

$\Rightarrow \sin(\arcsin x) = \sin(2\phi - \theta)$

$\Rightarrow x = \sin 2\phi\cos\theta - \cos 2\phi\sin\theta$

USING DOUBLE-ANGLE IDENTITIES $\sin 2\phi \equiv 2\sin\phi\cos\phi$ & $\cos 2\phi = 2\cos^2\phi - 1$

$\Rightarrow x = 2\sin\phi\cos\phi\cos\theta - \sin\theta\left(2\cos^2\phi - 1\right)$

$\Rightarrow x = 2\left(\frac{3}{5}\right)\left(\frac{4}{5}\right)\left(\frac{3}{5}\right) - \frac{4}{5}\left(2\times\frac{16}{25} - 1\right)$

$\Rightarrow x = \frac{72}{125} - \frac{28}{125}$

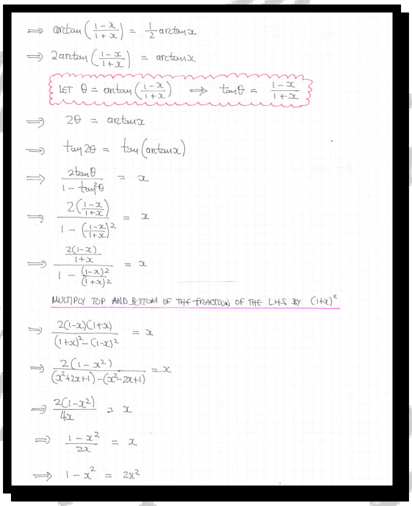
$\Rightarrow x = \frac{44}{125}$

Question 10 (****+)

Find the solution of the equation

$$\arctan\left(\frac{1-x}{1+x}\right) = \frac{1}{2}\arctan x.$$

$\boxed{}$, $\boxed{x = \frac{\sqrt{3}}{3}}$



$\Rightarrow \arctan\left(\frac{1-x}{1+x}\right) = \frac{1}{2}\arctan x$

$\Rightarrow 2\arctan\left(\frac{1-x}{1+x}\right) = \arctan x$

LET $\theta = \arctan\left(\frac{1-x}{1+x}\right) \Rightarrow \tan\theta = \frac{1-x}{1+x}$

$\Rightarrow 2\theta = \arctan x$

$\Rightarrow \tan 2\theta = \tan(\arctan x)$

$\Rightarrow \frac{2\tan\theta}{1-\tan^2\theta} = x$

$\Rightarrow \frac{2\left(\frac{1-x}{1+x}\right)}{1-\left(\frac{1-x}{1+x}\right)^2} = x$

$\Rightarrow \frac{\frac{2(1-x)}{1+x}}{1-\frac{(1-x)^2}{(1+x)^2}} = x$

MULTIPLY TOP AND BOTTOM OF THE FRACTION OF THE L.H.S BY $(1+x)^2$

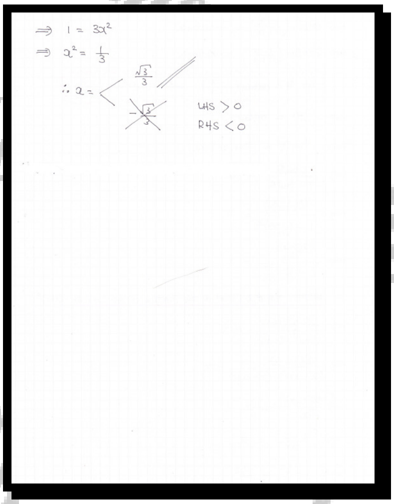
$\Rightarrow \frac{2(1-x)(1+x)}{(1+x)^2-(1-x)^2} = x$

$\Rightarrow \frac{2(1-x^2)}{(x^2+2x+1)-(x^2-2x+1)} = x$

$\Rightarrow \frac{2(1-x^2)}{4x} = x$

$\Rightarrow \frac{1-x^2}{2x} = x$

$\Rightarrow 1-x^2 = 2x^2$



$\Rightarrow 1 = 3x^2$

$\Rightarrow x^2 = \frac{1}{3}$

$\therefore x = \frac{\sqrt{3}}{3}$, $\cancel{-\frac{\sqrt{3}}{3}}$

LHS > 0

RHS < 0

Question 11 (****+)

The functions f and g are defined by

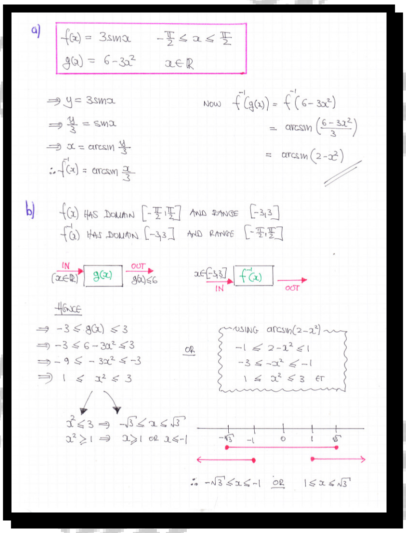
$$f(x) \equiv 3\sin x\,,\ x \in \mathbb{R}\,,\ -\frac{\pi}{2} \le x \le \frac{\pi}{2}$$

$$g(x) \equiv 6 - 3x^2\,,\ x \in \mathbb{R}\,.$$

a) Find an expression for $f^{-1}g(x)$.

b) Determine the domain of $f^{-1}g(x)$.

$$\boxed{}\,,\ \boxed{f^{-1}g(x) = \arcsin\left(2 - x^2\right)}\,,\ \boxed{-\sqrt{3} \le x \le -1 \ \text{ or } \ 1 \le x \le \sqrt{3}}$$



a) $\boxed{\begin{array}{ll} f(x) = 3\sin x & -\frac{\pi}{2} \le x \le \frac{\pi}{2} \\ g(x) = 6 - 3x^2 & x \in \mathbb{R} \end{array}}$

$\Rightarrow y = 3\sin x$

$\Rightarrow \frac{y}{3} = \sin x$

$\Rightarrow x = \arcsin \frac{y}{3}$

$\therefore f^{-1}(x) = \arcsin \frac{x}{3}$

NOW $f^{-1}(g(x)) = f^{-1}(6 - 3x^2)$

$= \arcsin\left(\frac{6 - 3x^2}{3}\right)$

$= \arcsin(2 - x^2)$ //

b) $f(x)$ HAS DOMAIN $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ AND RANGE $[-3, 3]$

$f^{-1}(x)$ HAS DOMAIN $[-3, 3]$ AND RANGE $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

HENCE

$\Rightarrow -3 \le g(x) \le 3$

$\Rightarrow -3 \le 6 - 3x^2 \le 3$

$\Rightarrow -9 \le -3x^2 \le -3$

$\Rightarrow 1 \le x^2 \le 3$

OR

USING $\arcsin(2 - x^2)$

$-1 \le 2 - x^2 \le 1$

$-3 \le -x^2 \le -1$

$1 \le x^2 \le 3$ ETC

$x^2 \le 3 \Rightarrow -\sqrt{3} \le x \le \sqrt{3}$

$x^2 \ge 1 \Rightarrow x \ge 1$ OR $x \le -1$

$\therefore -\sqrt{3} \le x \le -1$ OR $1 \le x \le \sqrt{3}$

Question 12 (****+)

$$y = \arctan x\,,\ \ x \in \mathbb{R}\,.$$

a) By writing the above equation in the form $x = g\left(y\right)$, show that

$$\frac{d}{dx}\left(\arctan x\right) = \frac{1}{1+x^2}\,.$$

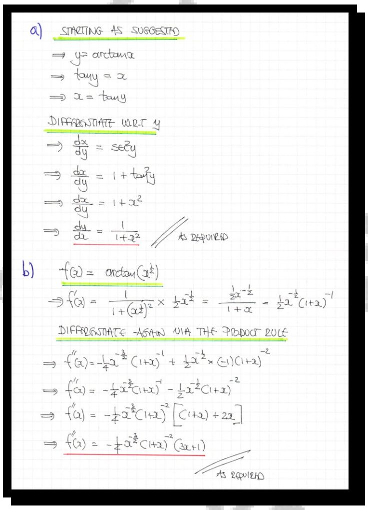
The function f is defined as

$$f\left(x\right) = \arctan\sqrt{x}\,,\ \ x \in \mathbb{R}\,,\ x \geq 0\,.$$

b) Show further that

$$f''\left(x\right) = -\frac{1}{4}x^{-\frac{3}{2}}\left(3x+1\right)\left(x+1\right)^{-2}\,.$$

$\boxed{}$, $\boxed{\text{proof}}$



a) STARTING AS SUGGESTED

$\Rightarrow y = \arctan x$

$\Rightarrow \tan y = x$

$\Rightarrow x = \tan y$

DIFFERENTIATE W.R.T y

$\Rightarrow \frac{dx}{dy} = \sec^2 y$

$\Rightarrow \frac{dx}{dy} = 1 + \tan^2 y$

$\Rightarrow \frac{dx}{dy} = 1 + x^2$

$\Rightarrow \frac{dy}{dx} = \frac{1}{1+x^2}$ // AS REQUIRED

b) $f(x) = \arctan(x^{\frac{1}{2}})$

$\Rightarrow f'(x) = \frac{1}{1+(x^{\frac{1}{2}})^2} \times \frac{1}{2}x^{-\frac{1}{2}} = \frac{\frac{1}{2}x^{-\frac{1}{2}}}{1+x} = \frac{1}{2}x^{-\frac{1}{2}}(1+x)^{-1}$

DIFFERENTIATE AGAIN VIA THE PRODUCT RULE

$\Rightarrow f''(x) = -\frac{1}{4}x^{-\frac{3}{2}}(1+x)^{-1} + \frac{1}{2}x^{-\frac{1}{2}} \times (-1)(1+x)^{-2}$

$\Rightarrow f''(x) = -\frac{1}{4}x^{-\frac{3}{2}}(1+x)^{-1} - \frac{1}{2}x^{-\frac{1}{2}}(1+x)^{-2}$

$\Rightarrow f''(x) = -\frac{1}{4}x^{-\frac{3}{2}}(1+x)^{-2}\left[(1+x) + 2x\right]$

$\Rightarrow f''(x) = -\frac{1}{4}x^{-\frac{3}{2}}(1+x)^{-2}(3x+1)$

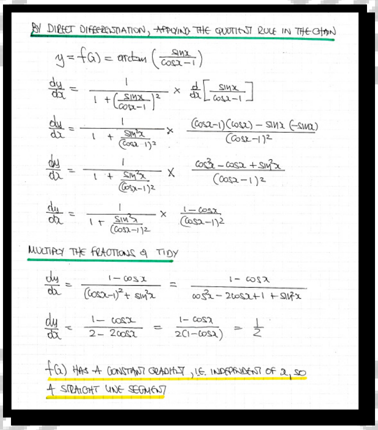
// AS REQUIRED

Question 13 **(****+)**

$$f(x) \equiv \arctan\left(\frac{\sin x}{\cos x - 1}\right),\ 0 < x < 2\pi .$$

Show that $f(x)$ represents a straight line segment.

V, , proof



BY DIRECT DIFFERENTIATION, APPLYING THE QUOTIENT RULE IN THE CHAIN

$$y = f(x) = \arctan\left(\frac{\sin x}{\cos x - 1}\right)$$

$$\frac{dy}{dx} = \frac{1}{1 + \left(\frac{\sin x}{\cos x - 1}\right)^2} \times \frac{d}{dx}\left[\frac{\sin x}{\cos x - 1}\right]$$

$$\frac{dy}{dx} = \frac{1}{1 + \frac{\sin^2 x}{(\cos x - 1)^2}} \times \frac{(\cos x - 1)(\cos x) - \sin x(-\sin x)}{(\cos x - 1)^2}$$

$$\frac{dy}{dx} = \frac{1}{1 + \frac{\sin^2 x}{(\cos x - 1)^2}} \times \frac{\cos^2 x - \cos x + \sin^2 x}{(\cos x - 1)^2}$$

$$\frac{dy}{dx} = \frac{1}{1 + \frac{\sin^2 x}{(\cos x - 1)^2}} \times \frac{1 - \cos x}{(\cos x - 1)^2}$$

MULTIPLY THE FRACTIONS & TIDY

$$\frac{dy}{dx} = \frac{1 - \cos x}{(\cos x - 1)^2 + \sin^2 x} = \frac{1 - \cos x}{\cos^2 x - 2\cos x + 1 + \sin^2 x}$$

$$\frac{dy}{dx} = \frac{1 - \cos x}{2 - 2\cos x} = \frac{1 - \cos x}{2(1 - \cos x)} = \frac{1}{2}$$

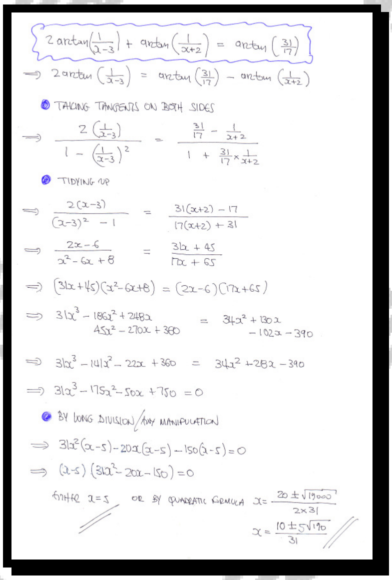
f(x) HAS A CONSTANT GRADIENT, I.E. INDEPENDENT OF x, SO A STRAIGHT LINE SEGMENT

Question 14 (****+)

$$2\arctan\left[\frac{1}{x-3}\right]+\arctan\left[\frac{1}{x+2}\right]=\arctan\left[\frac{31}{17}\right].$$

Show that $x=5$ is one of the solutions of the above trigonometric equation, and find in exact surd form the other two solutions.

$$\boxed{x=\frac{10\pm 5\sqrt{190}}{31}}$$



$$\boxed{2\arctan\left(\frac{1}{x-3}\right)+\arctan\left(\frac{1}{x+2}\right)=\arctan\left(\frac{31}{17}\right)}$$

$$\Longrightarrow 2\arctan\left(\frac{1}{x-3}\right)=\arctan\left(\frac{31}{17}\right)-\arctan\left(\frac{1}{x+2}\right)$$

- TAKING TANGENTS ON BOTH SIDES

$$\Longrightarrow \frac{2\left(\frac{1}{x-3}\right)}{1-\left(\frac{1}{x-3}\right)^2}=\frac{\frac{31}{17}-\frac{1}{x+2}}{1+\frac{31}{17}\times\frac{1}{x+2}}$$

- TIDYING UP

$$\Longrightarrow \frac{2(x-3)}{(x-3)^2-1}=\frac{31(x+2)-17}{17(x+2)+31}$$

$$\Longrightarrow \frac{2x-6}{x^2-6x+8}=\frac{31x+45}{17x+65}$$

$$\Longrightarrow (31x+45)(x^2-6x+8)=(2x-6)(17x+65)$$

$$\Longrightarrow \begin{matrix}31x^3-186x^2+248x\\ 45x^2-270x+360\end{matrix}=\begin{matrix}34x^2+130x\\ -102x-390\end{matrix}$$

$$\Longrightarrow 31x^3-141x^2-22x+360=34x^2+28x-390$$

$$\Longrightarrow 31x^3-175x^2-50x+750=0$$

- BY LONG DIVISION/ANY MANIPULATION

$$\Longrightarrow 31x^2(x-5)-20x(x-5)-150(x-5)=0$$

$$\Longrightarrow (x-5)(31x^2-20x-150)=0$$

EITHER $x=5$ // OR BY QUADRATIC FORMULA $x=\frac{20\pm\sqrt{19000}}{2\times31}$

$$x=\frac{10\pm5\sqrt{190}}{31}$$ //

Question 15 (****+)

$$y = \arccos x\,,\ x \in \mathbb{R}\,,\ -1 \le x \le 1\,.$$

a) By writing the above equation in the from $x = f\left(y\right)$, show that

$$\frac{d}{dx}\left(\arccos x\right) = -\frac{1}{\sqrt{1-x^2}}\,.$$

A curve has equation

$$y = \arccos\left(1-x^2\right),\quad x \in \mathbb{R}\,,\ 0 < x \le \sqrt{2}\,.$$

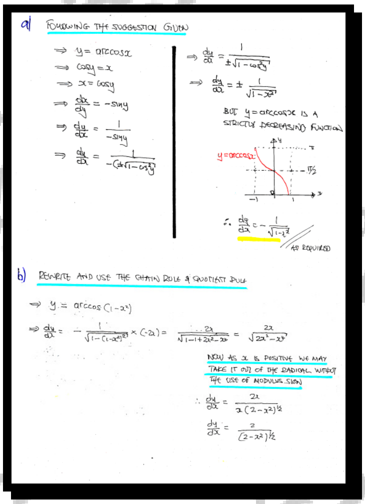
b) Show further that

$$\frac{d^2y}{dx^2} = \frac{2x}{\left(2-x^2\right)^{\frac{3}{2}}}\,.$$

c) Show clearly that

$$16\frac{d^3y}{dx^3} = 4x\frac{d^2y}{dx^2}\left(\frac{dy}{dx}\right)^2 + \left(2+x^2\right)\left(\frac{dy}{dx}\right)^5\,.$$

☐ , $\boxed{\text{proof}}$



a) FOLLOWING THE SUGGESTION GIVEN

$\Rightarrow y = \arccos x$

$\Rightarrow \cos y = x$

$\Rightarrow x = \cos y$

$\Rightarrow \frac{dx}{dy} = -\sin y$

$\Rightarrow \frac{dy}{dx} = \frac{1}{-\sin y}$

$\Rightarrow \frac{dy}{dx} = \frac{1}{-(\pm\sqrt{1-\cos^2 y})}$

$\Rightarrow \frac{dy}{dx} = \frac{1}{\pm\sqrt{1-\cos^2 y}}$

$\Rightarrow \frac{dy}{dx} = \pm\frac{1}{\sqrt{1-x^2}}$

BUT $y = \arccos x$ IS A STRICTLY DECREASING FUNCTION

$\therefore \frac{dy}{dx} = -\frac{1}{\sqrt{1-x^2}}$ // AS REQUIRED

b) REWRITE AND USE THE CHAIN RULE & QUOTIENT RULE

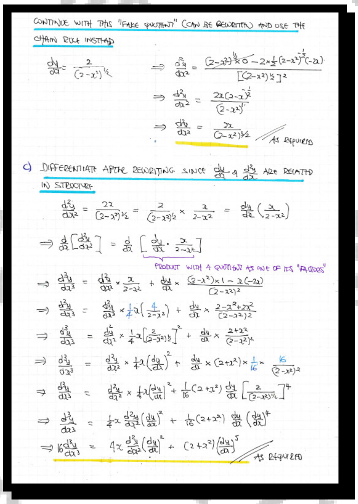
$\Rightarrow y = \arccos(1-x^2)$

$\Rightarrow \frac{dy}{dx} = -\frac{1}{\sqrt{1-(1-x^2)^2}} \times (-2x) = \frac{2x}{\sqrt{1-1+2x^2-x^4}} = \frac{2x}{\sqrt{2x^2-x^4}}$

NOW AS x IS POSITIVE WE MAY TAKE IT OUT OF THE RADICAL WITHOUT THE USE OF MODULUS SIGN

$\therefore \frac{dy}{dx} = \frac{2x}{x(2-x^2)^{\frac{1}{2}}}$

$\frac{dy}{dx} = \frac{2}{(2-x^2)^{\frac{1}{2}}}$



CONTINUE WITH THIS "FAKE QUOTIENT" (CAN BE REWRITTEN) AND USE THE CHAIN RULE INSTEAD

$\frac{dy}{dx} = \frac{2}{(2-x^2)^{\frac{1}{2}}}$

$\Rightarrow \frac{d^2y}{dx^2} = \frac{\cancel{(2-x^2)^{\frac{1}{2}} \times 0} - 2 \times \frac{1}{2}(2-x^2)^{-\frac{1}{2}}(-2x)}{[(2-x^2)^{\frac{1}{2}}]^2}$

$\Rightarrow \frac{d^2y}{dx^2} = \frac{2x(2-x^2)^{-\frac{1}{2}}}{(2-x^2)^1}$

$\Rightarrow \frac{d^2y}{dx^2} = \frac{2x}{(2-x^2)^{\frac{3}{2}}}$ // AS REQUIRED

c) DIFFERENTIATE AFTER REWRITING SINCE $\frac{dy}{dx}$ & $\frac{d^2y}{dx^2}$ ARE RELATED IN STRUCTURE

$\frac{d^2y}{dx^2} = \frac{2x}{(2-x^2)^{\frac{3}{2}}} = \frac{2}{(2-x^2)^{\frac{1}{2}}} \times \frac{x}{2-x^2} = \frac{dy}{dx}\left(\frac{x}{2-x^2}\right)$

$\Rightarrow \frac{d}{dx}\left[\frac{d^2y}{dx^2}\right] = \frac{d}{dx}\left[\underbrace{\frac{dy}{dx} \cdot \frac{x}{2-x^2}}_{\text{PRODUCT WITH A QUOTIENT AS ONE OF ITS "FACTORS"}}\right]$

$\Rightarrow \frac{d^3y}{dx^3} = \frac{d^2y}{dx^2} \times \frac{x}{2-x^2} + \frac{dy}{dx} \times \frac{(2-x^2)\times 1 - x(-2x)}{(2-x^2)^2}$

$\Rightarrow \frac{d^3y}{dx^3} = \frac{d^2y}{dx^2} \times \frac{1}{4}x\left(\frac{4}{2-x^2}\right) + \frac{dy}{dx} \times \frac{2-x^2+2x^2}{(2-x^2)^2}$

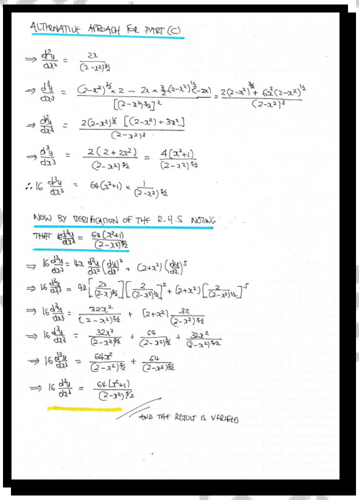
$\Rightarrow \frac{d^3y}{dx^3} = \frac{d^2y}{dx^2} \times \frac{1}{4}x\left[\frac{2}{(2-x^2)^{\frac{1}{2}}}\right]^2 + \frac{dy}{dx} \times \frac{2+x^2}{(2-x^2)^2}$

$\Rightarrow \frac{d^3y}{dx^3} = \frac{d^2y}{dx^2} \times \frac{1}{4}x\left(\frac{dy}{dx}\right)^2 + \frac{dy}{dx} \times (2+x^2) \times \frac{1}{16} \times \frac{16}{(2-x^2)^2}$

$\Rightarrow \frac{d^3y}{dx^3} = \frac{d^2y}{dx^2} \times \frac{1}{4}x\left(\frac{dy}{dx}\right)^2 + \frac{1}{16}(2+x^2)\frac{dy}{dx}\left[\frac{2}{(2-x^2)^{\frac{1}{2}}}\right]^4$

$\Rightarrow \frac{d^3y}{dx^3} = \frac{1}{4}x\frac{d^2y}{dx^2}\left(\frac{dy}{dx}\right)^2 + \frac{1}{16}(2+x^2)\frac{dy}{dx}\left(\frac{dy}{dx}\right)^4$

$\Rightarrow 16\frac{d^3y}{dx^3} = 4x\frac{d^2y}{dx^2}\left(\frac{dy}{dx}\right)^2 + (2+x^2)\left(\frac{dy}{dx}\right)^5$ // AS REQUIRED



ALTERNATIVE APPROACH FOR PART (C)

$\Rightarrow \frac{d^2y}{dx^2} = \frac{2x}{(2-x^2)^{\frac{3}{2}}}$

$\Rightarrow \frac{d^3y}{dx^3} = \frac{(2-x^2)^{\frac{3}{2}} \times 2 - 2x \times \frac{3}{2}(2-x^2)^{\frac{1}{2}}(-2x)}{[(2-x^2)^{\frac{3}{2}}]^2} = \frac{2(2-x^2)^{\frac{3}{2}} + 6x^2(2-x^2)^{\frac{1}{2}}}{(2-x^2)^3}$

$\Rightarrow \frac{d^3y}{dx^3} = \frac{2(2-x^2)^{\frac{1}{2}}[(2-x^2)+3x^2]}{(2-x^2)^3}$

$\Rightarrow \frac{d^3y}{dx^3} = \frac{2(2+2x^2)}{(2-x^2)^{\frac{5}{2}}} = \frac{4(x^2+1)}{(2-x^2)^{\frac{5}{2}}}$

$\therefore 16\frac{d^3y}{dx^3} = 64(x^2+1) \times \frac{1}{(2-x^2)^{\frac{5}{2}}}$

NOW BY VERIFICATION OF THE R.H.S NOTING THAT $16\frac{d^3y}{dx^3} = \frac{64(x^2+1)}{(2-x^2)^{\frac{5}{2}}}$

$\Rightarrow 16\frac{d^3y}{dx^3} = 4x\frac{d^2y}{dx^2}\left(\frac{dy}{dx}\right)^2 + (2+x^2)\left(\frac{dy}{dx}\right)^5$

$\Rightarrow 16\frac{d^3y}{dx^3} = 4x\left[\frac{2x}{(2-x^2)^{\frac{3}{2}}}\right]\left[\frac{2}{(2-x^2)^{\frac{1}{2}}}\right]^2 + (2+x^2)\left[\frac{2}{(2-x^2)^{\frac{1}{2}}}\right]^5$

$\Rightarrow 16\frac{d^3y}{dx^3} = \frac{32x^2}{(2-x^2)^{\frac{5}{2}}} + (2+x^2)\frac{32}{(2-x^2)^{\frac{5}{2}}}$

$\Rightarrow 16\frac{d^3y}{dx^3} = \frac{32x^2}{(2-x^2)^{\frac{5}{2}}} + \frac{64}{(2-x^2)^{\frac{5}{2}}} + \frac{32x^2}{(2-x^2)^{\frac{5}{2}}}$

$\Rightarrow 16\frac{d^3y}{dx^3} = \frac{64x^2}{(2-x^2)^{\frac{5}{2}}} + \frac{64}{(2-x^2)^{\frac{5}{2}}}$

$\Rightarrow 16\frac{d^3y}{dx^3} = \frac{64(x^2+1)}{(2-x^2)^{\frac{5}{2}}}$

// AND THE RESULT IS VERIFIED

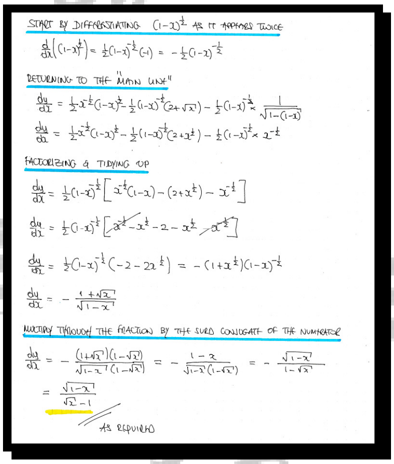
Question 16 (****+)

$$y = \left(2+\sqrt{x}\right)\sqrt{1-x} + \arcsin\sqrt{1-x}\ ,\ 0 \le x \le 1 .$$

Show with detailed workings that

$$\frac{dy}{dx} = \frac{\sqrt{1-x}}{\sqrt{x}-1} .$$

V, ☐, proof



START BY DIFFERENTIATING $(1-x)^{\frac{1}{2}}$ AS IT APPEARS TWICE

$$\frac{d}{dx}\left((1-x)^{\frac{1}{2}}\right) = \frac{1}{2}(1-x)^{-\frac{1}{2}}(-1) = -\frac{1}{2}(1-x)^{-\frac{1}{2}}$$

RETURNING TO THE "MAIN LINE"

$$\frac{dy}{dx} = \frac{1}{2}x^{-\frac{1}{2}}(1-x)^{\frac{1}{2}} - \frac{1}{2}(1-x)^{-\frac{1}{2}}(2+\sqrt{x}) - \frac{1}{2}(1-x)^{-\frac{1}{2}} \times \frac{1}{\sqrt{1-(1-x)}}$$

$$\frac{dy}{dx} = \frac{1}{2}x^{-\frac{1}{2}}(1-x)^{\frac{1}{2}} - \frac{1}{2}(1-x)^{-\frac{1}{2}}(2+x^{\frac{1}{2}}) - \frac{1}{2}(1-x)^{-\frac{1}{2}} \times x^{-\frac{1}{2}}$$

FACTORIZING & TIDYING UP

$$\frac{dy}{dx} = \frac{1}{2}(1-x)^{-\frac{1}{2}}\left[x^{-\frac{1}{2}}(1-x) - (2+x^{\frac{1}{2}}) - x^{-\frac{1}{2}}\right]$$

$$\frac{dy}{dx} = \frac{1}{2}(1-x)^{-\frac{1}{2}}\left[\cancel{x^{-\frac{1}{2}}} - x^{\frac{1}{2}} - 2 - x^{\frac{1}{2}} \cancel{- x^{-\frac{1}{2}}}\right]$$

$$\frac{dy}{dx} = \frac{1}{2}(1-x)^{-\frac{1}{2}}\left(-2-2x^{\frac{1}{2}}\right) = -(1+x^{\frac{1}{2}})(1-x)^{-\frac{1}{2}}$$

$$\frac{dy}{dx} = -\frac{1+\sqrt{x}}{\sqrt{1-x}}$$

MULTIPLY THROUGH THE FRACTION BY THE SURD CONJUGATE OF THE NUMERATOR

$$\frac{dy}{dx} = -\frac{(1+\sqrt{x})(1-\sqrt{x})}{\sqrt{1-x}\,(1-\sqrt{x})} = -\frac{1-x}{\sqrt{1-x}\,(1-\sqrt{x})} = -\frac{\sqrt{1-x}}{1-\sqrt{x}}$$

$$= \frac{\sqrt{1-x}}{\sqrt{x}-1}$$

// AS REQUIRED

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ENRICHMENT QUESTIONS

Question 1 (*****)

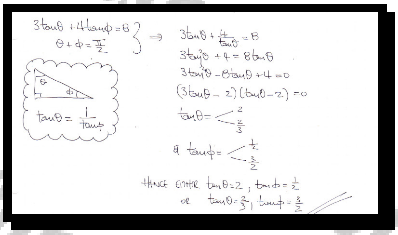
Given the simultaneous equations

$$3\tan\theta + 4\tan\varphi = 8$$

$$\theta + \varphi = \frac{\pi}{2},$$

find the possible value of $\tan\theta$ and the possible value of $\tan\varphi$.

$$\boxed{\left[\tan\theta, \tan\varphi\right] = \left[2, \frac{1}{2}\right] = \left[\frac{2}{3}, \frac{3}{2}\right]}$$



$$\left.\begin{array}{c} 3\tan\theta + 4\tan\phi = 8 \\ \theta + \phi = \frac{\pi}{2} \end{array}\right\} \Rightarrow 3\tan\theta + \frac{4}{\tan\theta} = 8$$

$$3\tan^2\theta + 4 = 8\tan\theta$$

$$3\tan^2\theta - 8\tan\theta + 4 = 0$$

$$(3\tan\theta - 2)(\tan\theta - 2) = 0$$

$$\tan\theta = \begin{cases} 2 \\ \frac{2}{3} \end{cases}$$

$$\& \ \tan\phi = \begin{cases} \frac{1}{2} \\ \frac{3}{2} \end{cases}$$

$$\tan\theta = \frac{1}{\tan\phi}$$

HENCE EITHER $\tan\theta = 2$, $\tan\phi = \frac{1}{2}$

or $\tan\theta = \frac{2}{3}$, $\tan\phi = \frac{3}{2}$

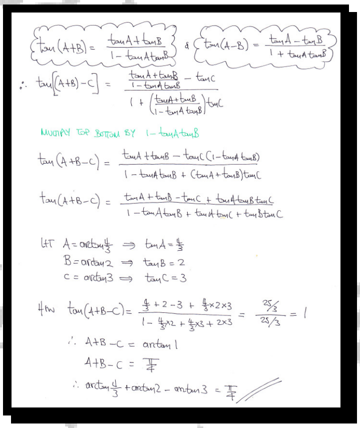
Question 2 (*****)

Simplify, showing all steps in the calculation, the expression

$$\arctan\frac{4}{3}+\arctan 2-\arctan 3,$$

giving the answer in terms of π.

$$\boxed{\frac{\pi}{4}}$$



$$\tan(A+B)=\frac{\tan A+\tan B}{1-\tan A\tan B} \quad \& \quad \tan(A-B)=\frac{\tan A-\tan B}{1+\tan A\tan B}$$

$$\therefore \tan\left[(A+B)-C\right]=\frac{\frac{\tan A+\tan B}{1-\tan A\tan B}-\tan C}{1+\left(\frac{\tan A+\tan B}{1-\tan A\tan B}\right)\tan C}$$

MULTIPLY TOP BOTTOM BY $1-\tan A\tan B$

$$\tan(A+B-C)=\frac{\tan A+\tan B-\tan C(1-\tan A\tan B)}{1-\tan A\tan B+(\tan A+\tan B)\tan C}$$

$$\tan(A+B-C)=\frac{\tan A+\tan B-\tan C+\tan A\tan B\tan C}{1-\tan A\tan B+\tan A\tan C+\tan B\tan C}$$

LET $A=\arctan\frac{4}{3} \Rightarrow \tan A=\frac{4}{3}$

$B=\arctan 2 \Rightarrow \tan B=2$

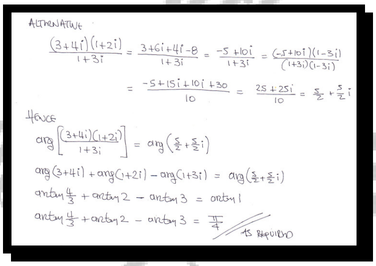
$C=\arctan 3 \Rightarrow \tan C=3$

THEN $\tan(A+B-C)=\frac{\frac{4}{3}+2-3+\frac{4}{3}\times2\times3}{1-\frac{4}{3}\times2+\frac{4}{3}\times3+2\times3}=\frac{25/3}{25/3}=1$

$\therefore A+B-C=\arctan 1$

$A+B-C=\frac{\pi}{4}$

$\therefore \arctan\frac{4}{3}+\arctan 2-\arctan 3=\frac{\pi}{4}$



ALTERNATIVE

$$\frac{(3+4i)(1+2i)}{1+3i}=\frac{3+6i+4i-8}{1+3i}=\frac{-5+10i}{1+3i}=\frac{(-5+10i)(1-3i)}{(1+3i)(1-3i)}$$

$$=\frac{-5+15i+10i+30}{10}=\frac{25+25i}{10}=\frac{5}{2}+\frac{5}{2}i$$

HENCE

$\arg\left[\frac{(3+4i)(1+2i)}{1+3i}\right]=\arg\left(\frac{5}{2}+\frac{5}{2}i\right)$

$\arg(3+4i)+\arg(1+2i)-\arg(1+3i)=\arg\left(\frac{5}{2}+\frac{5}{2}i\right)$

$\arctan\frac{4}{3}+\arctan 2-\arctan 3=\arctan 1$

$\arctan\frac{4}{3}+\arctan 2-\arctan 3=\frac{\pi}{4}$ AS REQUIRED

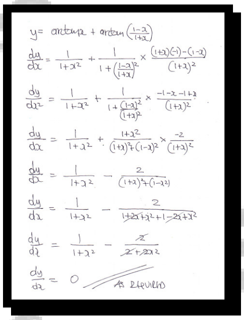
Question 3 (*****)

$$y = \arctan x + \arctan\left(\frac{1-x}{1+x}\right), \ x \in \mathbb{R}.$$

Without simplifying the above expression, use differentiation to show that for all values of x

$$\frac{dy}{dx} = 0.$$

$\boxed{\text{proof}}$



$$y = \arctan x + \arctan\left(\frac{1-x}{1+x}\right)$$

$$\frac{dy}{dx} = \frac{1}{1+x^2} + \frac{1}{1+\left(\frac{1-x}{1+x}\right)^2} \times \frac{(1+x)(-1)-(1-x)}{(1+x)^2}$$

$$\frac{dy}{dx^2} = \frac{1}{1+x^2} + \frac{1}{1+\frac{(1-x)^2}{(1+x)^2}} \times \frac{-1-x-1+x}{(1+x)^2}$$

$$\frac{dy}{dx} = \frac{1}{1+x^2} + \frac{1+x^2}{(1+x)^2+(1-x)^2} \times \frac{-2}{(1+x)^2}$$

$$\frac{dy}{dx} = \frac{1}{1+x^2} - \frac{2}{(1+x)^2+(1-x^2)}$$

$$\frac{dy}{dx} = \frac{1}{1+x^2} - \frac{2}{1+\cancel{2x}+x^2+1-\cancel{2x}+x^2}$$

$$\frac{dy}{dx} = \frac{1}{1+x^2} - \frac{\cancel{2}}{\cancel{2}+\cancel{2}x^2}$$

$$\frac{dy}{dx} = 0$$ As required

Question 4 (*****)

A curve C has equation

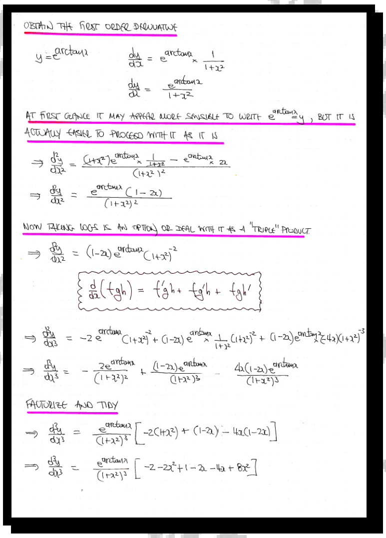
$$y = \mathrm{e}^{\arctan x}, \ x \in \mathbb{R}.$$

a) Show, with detailed workings, that

$$\frac{d^3y}{dx^3} = \frac{\left(6x^2 - 6x - 1\right)\mathrm{e}^{\arctan x}}{\left(1+x^2\right)^3}.$$

b) Deduce that C has a point of inflection, stating its coordinates.

$$\boxed{}, \ \boxed{\left(\frac{1}{2}, \mathrm{e}^{\arctan\frac{1}{2}}\right)}$$



OBTAIN THE FIRST ORDER DERIVATIVE

$y = e^{\arctan x}$ $\qquad \frac{dy}{dx} = e^{\arctan x} \times \frac{1}{1+x^2}$

$\frac{dy}{dx} = \frac{e^{\arctan x}}{1+x^2}$

AT FIRST GLANCE IT MAY APPEAR MORE SENSIBLE TO WRITE $e^{\arctan x} = y$, BUT IT IS ACTUALLY EASIER TO PROCEED WITH IT AS IT IS

$\Rightarrow \frac{d^2y}{dx^2} = \frac{\cancel{(1+x^2)} e^{\arctan x} \times \frac{1}{\cancel{1+x^2}} - e^{\arctan x} \times 2x}{(1+x^2)^2}$

$\Rightarrow \frac{d^2y}{dx^2} = \frac{e^{\arctan x}(1-2x)}{(1+x^2)^2}$

NOW TAKING LOGS IS AN OPTION, OR DEAL WITH IT AS A "TRIPLE" PRODUCT

$\Rightarrow \frac{d^2y}{dx^2} = (1-2x)e^{\arctan x}(1+x^2)^{-2}$

$$\boxed{\frac{d}{dx}(fgh) = f'gh + fg'h + fgh'}$$

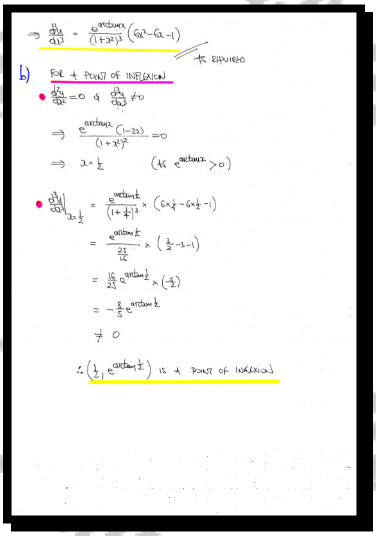
$\Rightarrow \frac{d^3y}{dx^3} = -2e^{\arctan x}(1+x^2)^{-2} + (1-2x)e^{\arctan x} \times \frac{1}{1+x^2}(1+x^2)^{-2} + (1-2x)e^{\arctan x} \times (-4x)(1+x^2)^{-3}$

$\Rightarrow \frac{d^3y}{dx^3} = -\frac{2e^{\arctan x}}{(1+x^2)^2} + \frac{(1-2x)e^{\arctan x}}{(1+x^2)^3} - \frac{4x(1-2x)e^{\arctan x}}{(1+x^2)^3}$

FACTORIZE AND TIDY

$\Rightarrow \frac{d^3y}{dx^3} = \frac{e^{\arctan x}}{(1+x^2)^3}\left[-2(1+x^2) + (1-2x) - 4x(1-2x)\right]$

$\Rightarrow \frac{d^3y}{dx^3} = \frac{e^{\arctan x}}{(1+x^2)^3}\left[-2 - 2x^2 + 1 - 2x - 4x + 8x^2\right]$



$\Rightarrow \frac{d^3y}{dx^3} = \frac{e^{\arctan x}}{(1+x^2)^3}(6x^2 - 6x - 1)$ // AS REQUIRED

b) FOR A POINT OF INFLEXION

- $\frac{d^2y}{dx^2} = 0$ & $\frac{d^3y}{dx^3} \neq 0$

$\Rightarrow \frac{e^{\arctan x}(1-2x)}{(1+x^2)^2} = 0$

$\Rightarrow x = \frac{1}{2}$ $\qquad$ (AS $e^{\arctan x} > 0$)

- $\left.\frac{d^3y}{dx^3}\right|_{x=\frac{1}{2}} = \frac{e^{\arctan\frac{1}{2}}}{(1+\frac{1}{4})^3} \times \left(6\times\frac{1}{4} - 6\times\frac{1}{2} - 1\right)$

$= \frac{e^{\arctan\frac{1}{2}}}{\frac{25}{16}} \times \left(\frac{3}{2} - 3 - 1\right)$

$= \frac{16}{25} e^{\arctan\frac{1}{2}} \times \left(-\frac{5}{2}\right)$

$= -\frac{8}{5} e^{\arctan\frac{1}{2}}$

$\neq 0$

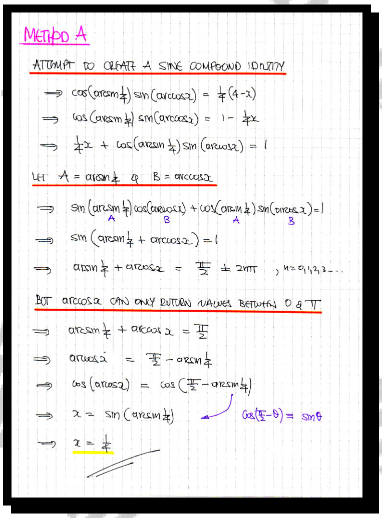
$\therefore \left(\frac{1}{2}, e^{\arctan\frac{1}{2}}\right)$ IS A POINT OF INFLEXION

Question 5 (*****)

Solve the following trigonometric equation

$$\cos\left(\arcsin\frac{1}{4}\right)\sin\left(\arccos x\right)=\frac{1}{4}(4-x)\ ,\quad x\in\mathbb{R}\,.$$

$\boxed{\quad}$, $\boxed{x=\frac{1}{4}}$



METHOD A

ATTEMPT TO CREATE A SINE COMPOUND IDENTITY

$\Rightarrow \cos\left(\arcsin\frac{1}{4}\right)\sin\left(\arccos x\right) = \frac{1}{4}(4-x)$

$\Rightarrow \cos\left(\arcsin\frac{1}{4}\right)\sin\left(\arccos x\right) = 1-\frac{1}{4}x$

$\Rightarrow \frac{1}{4}x + \cos\left(\arcsin\frac{1}{4}\right)\sin\left(\arccos x\right) = 1$

LET $A = \arcsin\frac{1}{4}$ & $B = \arccos x$

$\Rightarrow \sin\underbrace{\left(\arcsin\frac{1}{4}\right)}_{A}\cos\underbrace{\left(\arccos x\right)}_{B} + \cos\underbrace{\left(\arcsin\frac{1}{4}\right)}_{A}\sin\underbrace{\left(\arccos x\right)}_{B} = 1$

$\Rightarrow \sin\left(\arcsin\frac{1}{4} + \arccos x\right) = 1$

$\Rightarrow \arcsin\frac{1}{4} + \arccos x = \frac{\pi}{2} \pm 2n\pi \quad , n = 0,1,2,3\ldots$

BUT $\arccos x$ CAN ONLY RETURN VALUES BETWEEN 0 & π

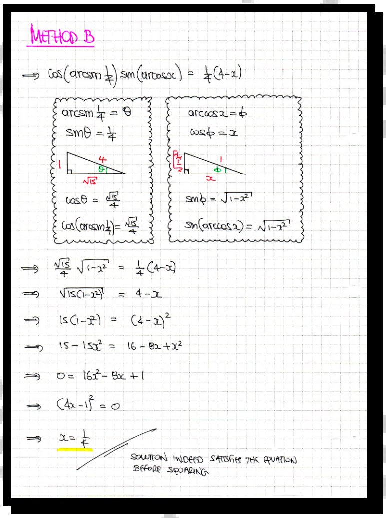
$\Rightarrow \arcsin\frac{1}{4} + \arccos x = \frac{\pi}{2}$

$\Rightarrow \arccos x = \frac{\pi}{2} - \arcsin\frac{1}{4}$

$\Rightarrow \cos\left(\arccos x\right) = \cos\left(\frac{\pi}{2} - \arcsin\frac{1}{4}\right)$

$\Rightarrow x = \sin\left(\arcsin\frac{1}{4}\right)$ $\quad \cos\left(\frac{\pi}{2}-\theta\right) \equiv \sin\theta$

$\Rightarrow x = \frac{1}{4}$



METHOD B

$\Rightarrow \cos\left(\arcsin\frac{1}{4}\right)\sin\left(\arccos x\right) = \frac{1}{4}(4-x)$

$\arcsin\frac{1}{4} = \theta$

$\sin\theta = \frac{1}{4}$

$\cos\theta = \frac{\sqrt{15}}{4}$

$\cos\left(\arcsin\frac{1}{4}\right) = \frac{\sqrt{15}}{4}$

$\arccos x = \phi$

$\cos\phi = x$

$\sin\phi = \sqrt{1-x^2}$

$\sin\left(\arccos x\right) = \sqrt{1-x^2}$

$\Rightarrow \frac{\sqrt{15}}{4}\sqrt{1-x^2} = \frac{1}{4}(4-x)$

$\Rightarrow \sqrt{15(1-x^2)} = 4-x$

$\Rightarrow 15(1-x^2) = (4-x)^2$

$\Rightarrow 15-15x^2 = 16-8x+x^2$

$\Rightarrow 0 = 16x^2-8x+1$

$\Rightarrow (4x-1)^2 = 0$

$\Rightarrow x = \frac{1}{4}$

SOLUTION INDEED SATISFIES THE EQUATION BEFORE SQUARING

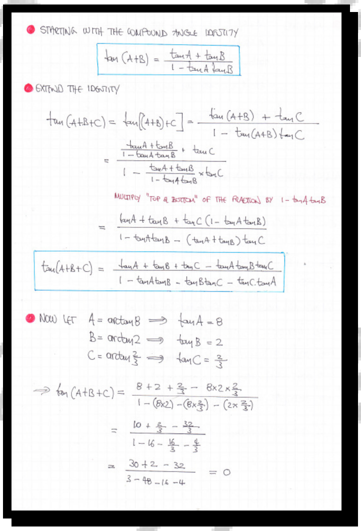
Question 6 (*****)

Simplify, showing all steps in the calculation, the expression

$$\arctan 8 + \arctan 2 + \arctan \frac{2}{3},$$

giving the answer in terms of π .

$\boxed{}$, $\boxed{\pi}$



- STARTING WITH THE COMPOUND ANGLE IDENTITY

$$\boxed{\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}}$$

- EXTEND THE IDENTITY

$$\tan(A+B+C) = \tan\left[(A+B)+C\right] = \frac{\tan(A+B) + \tan C}{1 - \tan(A+B)\tan C}$$

$$= \frac{\frac{\tan A + \tan B}{1 - \tan A \tan B} + \tan C}{1 - \frac{\tan A + \tan B}{1 - \tan A \tan B} \times \tan C}$$

MULTIPLY "TOP & BOTTOM" OF THE FRACTION BY $1 - \tan A \tan B$

$$= \frac{\tan A + \tan B + \tan C(1 - \tan A \tan B)}{1 - \tan A \tan B - (\tan A + \tan B)\tan C}$$

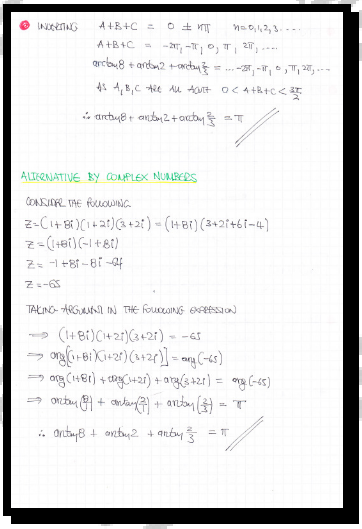
$$\boxed{\tan(A+B+C) = \frac{\tan A + \tan B + \tan C - \tan A \tan B \tan C}{1 - \tan A \tan B - \tan B \tan C - \tan C . \tan A}}$$

- NOW LET $A = \arctan 8 \Longrightarrow \tan A = 8$
  $B = \arctan 2 \Longrightarrow \tan B = 2$
  $C = \arctan \frac{2}{3} \Longrightarrow \tan C = \frac{2}{3}$

$$\Rightarrow \tan(A+B+C) = \frac{8 + 2 + \frac{2}{3} - 8 \times 2 \times \frac{2}{3}}{1 - (8 \times 2) - (8 \times \frac{2}{3}) - (2 \times \frac{2}{3})}$$

$$= \frac{10 + \frac{2}{3} - \frac{32}{3}}{1 - 16 - \frac{16}{3} - \frac{4}{3}}$$

$$= \frac{30 + 2 - 32}{3 - 48 - 16 - 4} = 0$$



- INVERTING $A + B + C = 0 \pm n\pi \quad n = 0, 1, 2, 3 \dots$

$$A + B + C = -2\pi, -\pi, 0, \pi, 2\pi, \dots$$

$$\arctan 8 + \arctan 2 + \arctan \tfrac{2}{3} = \dots -2\pi, -\pi, 0, \pi, 2\pi, \dots$$

AS A, B, C ARE ALL ACUTE $0 < A+B+C < \frac{3\pi}{2}$

$$\therefore \arctan 8 + \arctan 2 + \arctan \tfrac{2}{3} = \pi$$ //

ALTERNATIVE BY COMPLEX NUMBERS

CONSIDER THE FOLLOWING

$z = (1+8i)(1+2i)(3+2i) = (1+8i)(3+2i+6i-4)$

$z = (1+8i)(-1+8i)$

$z = -1 + 8i - 8i - 64$

$z = -65$

TAKING ARGUMENTS IN THE FOLLOWING EXPRESSION

$\Longrightarrow (1+8i)(1+2i)(3+2i) = -65$

$\Longrightarrow \arg\left[(1+8i)(1+2i)(3+2i)\right] = \arg(-65)$

$\Longrightarrow \arg(1+8i) + \arg(1+2i) + \arg(3+2i) = \arg(-65)$

$\Longrightarrow \arctan\left(\frac{8}{1}\right) + \arctan\left(\frac{2}{1}\right) + \arctan\left(\frac{2}{3}\right) = \pi$

$\therefore \arctan 8 + \arctan 2 + \arctan \frac{2}{3} = \pi$ //

Question 7 (*****)

It is given that

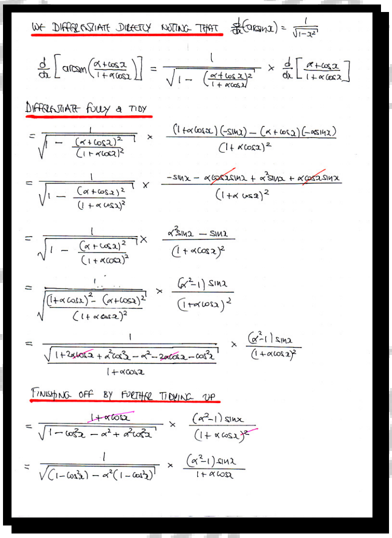
$$y = \arcsin\left[\frac{\alpha + \cos x}{1 + \alpha \cos x}\right],$$

where α is a constant.

Show that

$$\frac{dy}{dx} = -\frac{\sqrt{1-\alpha^2}}{1 + \alpha \cos x}.$$

$\boxed{\text{SP}}$, $\boxed{\text{proof}}$



WE DIFFERENTIATE DIRECTLY NOTING THAT $\frac{d}{dx}(\arcsin x) = \frac{1}{\sqrt{1-x^2}}$

$$\frac{d}{dx}\left[\arcsin\left(\frac{\alpha+\cos x}{1+\alpha\cos x}\right)\right] = \frac{1}{\sqrt{1-\left(\frac{\alpha+\cos x}{1+\alpha\cos x}\right)^2}} \times \frac{d}{dx}\left[\frac{\alpha+\cos x}{1+\alpha\cos x}\right]$$

DIFFERENTIATE FULLY & TIDY

$$= \frac{1}{\sqrt{1-\frac{(\alpha+\cos x)^2}{(1+\alpha\cos x)^2}}} \times \frac{(1+\alpha\cos x)(-\sin x)-(\alpha+\cos x)(-\alpha\sin x)}{(1+\alpha\cos x)^2}$$

$$= \frac{1}{\sqrt{1-\frac{(\alpha+\cos x)^2}{(1+\alpha\cos x)^2}}} \times \frac{-\sin x - \cancel{\alpha\cos x\sin x} + \alpha^2\sin x + \cancel{\alpha\cos x\sin x}}{(1+\alpha\cos x)^2}$$

$$= \frac{1}{\sqrt{1-\frac{(\alpha+\cos x)^2}{(1+\alpha\cos x)^2}}} \times \frac{\alpha^2\sin x - \sin x}{(1+\alpha\cos x)^2}$$

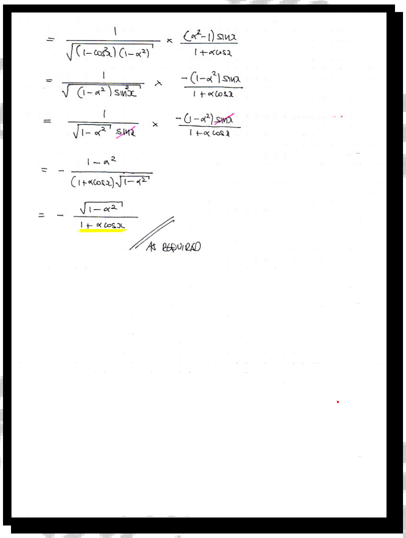
$$= \frac{1}{\sqrt{\frac{(1+\alpha\cos x)^2-(\alpha+\cos x)^2}{(1+\alpha\cos x)^2}}} \times \frac{(\alpha^2-1)\sin x}{(1+\alpha\cos x)^2}$$

$$= \frac{1}{\frac{\sqrt{1+\cancel{2\alpha\cos x}+\alpha^2\cos^2 x-\alpha^2-\cancel{2\alpha\cos x}-\cos^2 x}}{1+\alpha\cos x}} \times \frac{(\alpha^2-1)\sin x}{(1+\alpha\cos x)^2}$$

FINISHING OFF BY FURTHER TIDYING UP

$$= \frac{\cancel{1+\alpha\cos x}}{\sqrt{1-\cos^2 x-\alpha^2+\alpha^2\cos^2 x}} \times \frac{(\alpha^2-1)\sin x}{(1+\alpha\cos x)^{\cancel{2}}}$$

$$= \frac{1}{\sqrt{(1-\cos^2 x)-\alpha^2(1-\cos^2 x)}} \times \frac{(\alpha^2-1)\sin x}{1+\alpha\cos x}$$



$$= \frac{1}{\sqrt{(1-\cos^2 x)(1-\alpha^2)}} \times \frac{(\alpha^2-1)\sin x}{1+\alpha\cos x}$$

$$= \frac{1}{\sqrt{(1-\alpha^2)\sin^2 x}} \times \frac{-(1-\alpha^2)\sin x}{1+\alpha\cos x}$$

$$= \frac{1}{\sqrt{1-\alpha^2}\,\cancel{\sin x}} \times \frac{-(1-\alpha^2)\cancel{\sin x}}{1+\alpha\cos x}$$

$$= -\frac{1-\alpha^2}{(1+\alpha\cos x)\sqrt{1-\alpha^2}}$$

$$= -\frac{\sqrt{1-\alpha^2}}{1+\alpha\cos x}$$

// AS REQUIRED

Question 8 (*****)

The functions f and g are defined by

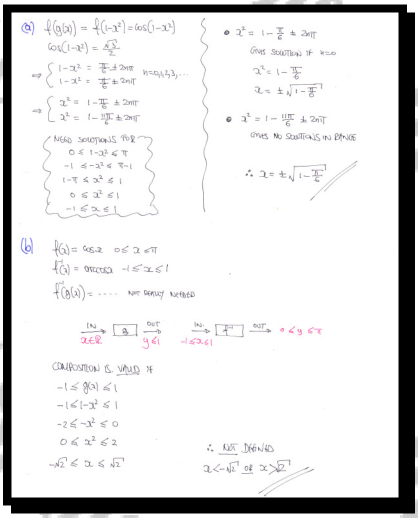
$$f(x) \equiv \cos x, \ x \in \mathbb{R}, \ 0 \le x \le \pi$$

$$g(x) \equiv 1 - x^2, \ x \in \mathbb{R}.$$

a) Solve the equation $fg(x) = \frac{1}{2}$.

b) Determine the values of x for which $f^{-1}g(x)$ is **not** defined.

$\boxed{x = \pm\sqrt{1-\frac{\pi}{6}}}$, $\boxed{x < -\sqrt{2} \ \text{ or } \ x > \sqrt{2}}$



(a) $f(g(x)) = f(1-x^2) = \cos(1-x^2)$

$\cos(1-x^2) = \frac{\sqrt{3}}{2}$

$\Rightarrow \begin{cases} 1-x^2 = \frac{\pi}{6} \pm 2n\pi \\ 1-x^2 = \frac{11\pi}{6} \pm 2n\pi \end{cases}$ $n = 0,1,2,3,\ldots$

$\Rightarrow \begin{cases} x^2 = 1-\frac{\pi}{6} \pm 2n\pi \\ x^2 = 1-\frac{11\pi}{6} \pm 2n\pi \end{cases}$

NEED SOLUTIONS FOR
$0 \le 1-x^2 \le \pi$
$-1 \le -x^2 \le \pi - 1$
$1-\pi \le x^2 \le 1$
$0 \le x^2 \le 1$
$-1 \le x \le 1$

- $x^2 = 1 - \frac{\pi}{6} \pm 2n\pi$

GIVES SOLUTION IF $n=0$

$x^2 = 1 - \frac{\pi}{6}$

$x = \pm\sqrt{1-\frac{\pi}{6}}$

- $x^2 = 1 - \frac{11\pi}{6} \pm 2n\pi$

GIVES NO SOLUTIONS IN RANGE

$\therefore x = \pm\sqrt{1-\frac{\pi}{6}}$

(b) $f(x) = \cos x \quad 0 \le x \le \pi$

$f^{-1}(x) = \arccos x \quad -1 \le x \le 1$

$f^{-1}(g(x)) = \ldots\ldots$ NOT REALLY NEEDED

$\xrightarrow[x \in \mathbb{R}]{\text{IN}} \boxed{g} \xrightarrow[y \le 1]{\text{OUT}} \quad \xrightarrow[-1 \le x \le 1]{\text{IN}} \boxed{f^{-1}} \xrightarrow{\text{OUT}} 0 \le y \le \pi$

COMPOSITION IS VALID IF

$-1 \le g(x) \le 1$

$-1 \le 1-x^2 \le 1$

$-2 \le -x^2 \le 0$

$0 \le x^2 \le 2$

$-\sqrt{2} \le x \le \sqrt{2}$

$\therefore$ NOT DEFINED

$x < -\sqrt{2}$ OR $x > \sqrt{2}$

Question 9 (*****)

The acute angles θ and φ satisfy the following equations

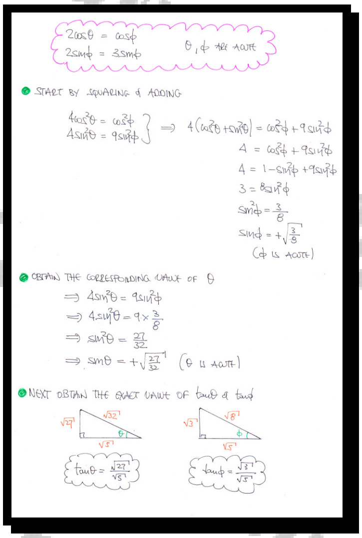
$$2\cos\theta = \cos\varphi$$

$$2\sin\theta = 3\sin\varphi .$$

Show clearly that

$$\theta + \varphi = \pi - \arctan\sqrt{15}$$

$\boxed{}$, $\boxed{\text{proof}}$



$2\cos\theta = \cos\phi$
$2\sin\phi = 3\sin\phi$ θ, ϕ ARE ACUTE

- START BY SQUARING & ADDING

$$\left.\begin{array}{l}4\cos^2\theta = \cos^2\phi\\ 4\sin^2\theta = 9\sin^2\phi\end{array}\right\} \Rightarrow 4(\cos^2\theta + \sin^2\theta) = \cos^2\phi + 9\sin^2\phi$$

$4 = \cos^2\phi + 9\sin^2\phi$

$4 = 1 - \sin^2\phi + 9\sin^2\phi$

$3 = 8\sin^2\phi$

$\sin^2\phi = \frac{3}{8}$

$\sin\phi = +\sqrt{\frac{3}{8}}$

(ϕ IS ACUTE)

- OBTAIN THE CORRESPONDING VALUE OF θ

$\Rightarrow 4\sin^2\theta = 9\sin^2\phi$

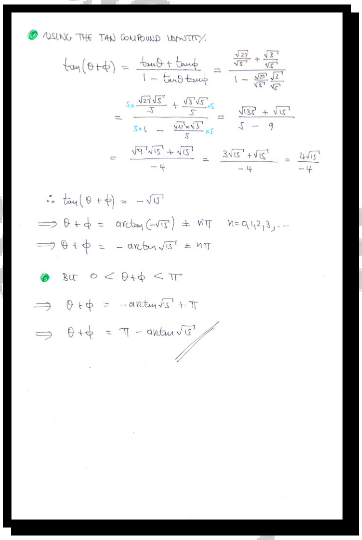
$\Rightarrow 4\sin^2\theta = 9 \times \frac{3}{8}$

$\Rightarrow \sin^2\theta = \frac{27}{32}$

$\Rightarrow \sin\theta = +\sqrt{\frac{27}{32}}$ (θ IS ACUTE)

- NEXT OBTAIN THE EXACT VALUE OF $\tan\theta$ & $\tan\phi$

$\tan\theta = \frac{\sqrt{27}}{\sqrt{5}}$ $\tan\phi = \frac{\sqrt{3}}{\sqrt{5}}$



- USING THE TAN COMPOUND IDENTITY.

$$\tan(\theta+\phi) = \frac{\tan\theta + \tan\phi}{1 - \tan\theta\tan\phi} = \frac{\frac{\sqrt{27}}{\sqrt{5}} + \frac{\sqrt{3}}{\sqrt{5}}}{1 - \frac{\sqrt{27}}{\sqrt{5}}\frac{\sqrt{3}}{\sqrt{5}}}$$

$$= \frac{5\times\frac{\sqrt{27}\sqrt{5}}{5} + \frac{\sqrt{3}\sqrt{5}}{5}\times5}{5\times1 - \frac{\sqrt{27}\times\sqrt{3}}{5}\times5} = \frac{\sqrt{135} + \sqrt{15}}{5 - 9}$$

$$= \frac{\sqrt{9}\sqrt{15} + \sqrt{15}}{-4} = \frac{3\sqrt{15} + \sqrt{15}}{-4} = \frac{4\sqrt{15}}{-4}$$

$\therefore \tan(\theta+\phi) = -\sqrt{15}$

$\Rightarrow \theta + \phi = \arctan(-\sqrt{15}) \pm n\pi \quad n = 0,1,2,3,\ldots$

$\Rightarrow \theta + \phi = -\arctan\sqrt{15} \pm n\pi$

- BUT $0 < \theta + \phi < \pi$

$\Rightarrow \theta + \phi = -\arctan\sqrt{15} + \pi$

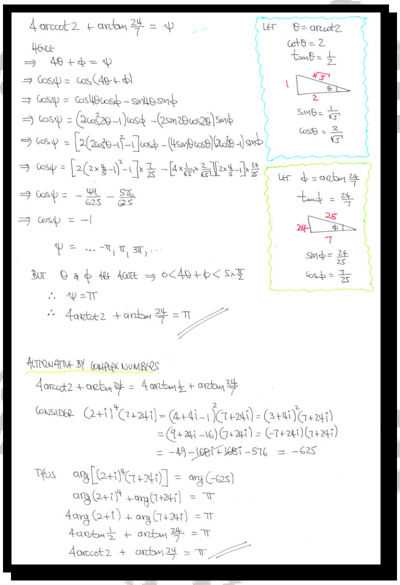
$\Rightarrow \theta + \phi = \pi - \arctan\sqrt{15}$ //

Question 10 (*****)

Show clearly that

$$4\operatorname{arccot} 2 + \arctan\left(\frac{24}{7}\right) = \pi \, .$$

, proof



$4\operatorname{arccot}2 + \arctan\frac{24}{7} = \psi$

HENCE

$\Rightarrow 4\theta + \phi = \psi$

$\Rightarrow \cos\psi = \cos(4\theta + \phi)$

$\Rightarrow \cos\psi = \cos4\theta\cos\phi - \sin4\theta\sin\phi$

$\Rightarrow \cos\psi = (2\cos^2 2\theta - 1)\cos\phi - (2\sin2\theta\cos2\theta)\sin\phi$

$\Rightarrow \cos\psi = \left[2(2\cos^2\theta-1)^2-1\right]\cos\phi - (4\sin\theta\cos\theta)(2\cos^2\theta-1)\sin\phi$

$\Rightarrow \cos\psi = \left[2\left(2\times\frac{4}{5}-1\right)^2-1\right]\times\frac{7}{25} - \left[4\times\frac{1}{\sqrt{5}}\times\frac{2}{\sqrt{5}}\right]\left[2\times\frac{4}{5}-1\right]\times\frac{24}{25}$

$\Rightarrow \cos\psi = -\frac{44}{625} - \frac{576}{625}$

$\Rightarrow \cos\psi = -1$

$\psi = \dots -\pi, \pi, 3\pi, \dots$

BUT θ & ϕ ARE ACUTE $\Rightarrow 0 < 4\theta + \phi < 5\times\frac{\pi}{2}$

$\therefore \psi = \pi$

$\therefore 4\operatorname{arccot}2 + \arctan\frac{24}{7} = \pi$

LET $\theta = \operatorname{arccot}2$

$\cot\theta = 2$

$\tan\theta = \frac{1}{2}$

$\sqrt{5}$

1

θ

2

$\sin\theta = \frac{1}{\sqrt{5}}$

$\cos\theta = \frac{2}{\sqrt{5}}$

LET $\phi = \arctan\frac{24}{7}$

$\tan\phi = \frac{24}{7}$

25

24

ϕ

7

$\sin\phi = \frac{24}{25}$

$\cos\phi = \frac{7}{25}$

ALTERNATIVE BY COMPLEX NUMBERS

$4\operatorname{arccot}2 + \arctan\frac{24}{7} = 4\arctan\frac{1}{2} + \arctan\frac{24}{7}$

CONSIDER $(2+i)^4(7+24i) = (4+4i-1)^2(7+24i) = (3+4i)^2(7+24i)$

$= (9+24i-16)(7+24i) = (-7+24i)(7+24i)$

$= -49 \cancel{-168i} + \cancel{168i} - 576 = -625$

THUS $\arg\left[(2+i)^4(7+24i)\right] = \arg(-625)$

$\arg(2+i)^4 + \arg(7+24i) = \pi$

$4\arg(2+i) + \arg(7+24i) = \pi$

$4\arctan\frac{1}{2} + \arctan\frac{24}{7} = \pi$

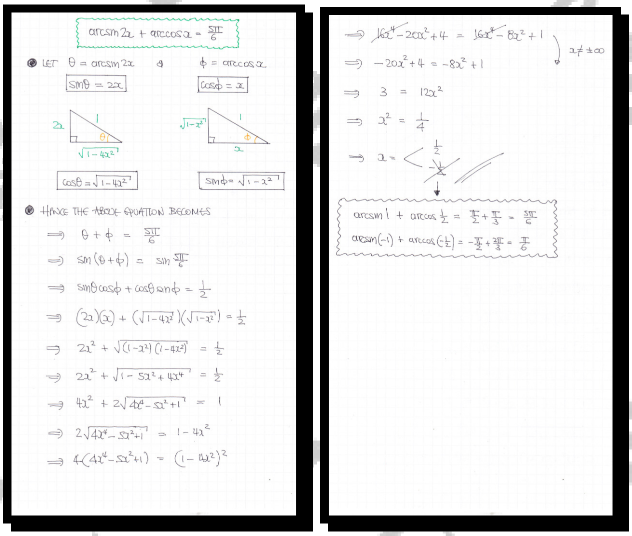
$4\operatorname{arccot}2 + \arctan\frac{24}{7} = \pi$

Question 11 (*****)

Solve the following trigonometric equation

$$\arcsin 2x + \arccos x = \frac{5\pi}{6}.$$

$\boxed{\text{S B}}$, $\boxed{x = \frac{1}{2}}$



$\arcsin 2x + \arccos x = \frac{5\pi}{6}$

- LET $\theta = \arcsin 2x$ & $\phi = \arccos x$

$\boxed{\sin\theta = 2x}$ $\boxed{\cos\phi = x}$

$\boxed{\cos\theta = \sqrt{1-4x^2}}$ $\boxed{\sin\phi = \sqrt{1-x^2}}$

- HENCE THE ABOVE EQUATION BECOMES

$\Rightarrow \theta + \phi = \frac{5\pi}{6}$

$\Rightarrow \sin(\theta+\phi) = \sin\frac{5\pi}{6}$

$\Rightarrow \sin\theta\cos\phi + \cos\theta\sin\phi = \frac{1}{2}$

$\Rightarrow (2x)(x) + (\sqrt{1-4x^2})(\sqrt{1-x^2}) = \frac{1}{2}$

$\Rightarrow 2x^2 + \sqrt{(1-x^2)(1-4x^2)} = \frac{1}{2}$

$\Rightarrow 2x^2 + \sqrt{1-5x^2+4x^4} = \frac{1}{2}$

$\Rightarrow 4x^2 + 2\sqrt{4x^4-5x^2+1} = 1$

$\Rightarrow 2\sqrt{4x^4-5x^2+1} = 1-4x^2$

$\Rightarrow 4(4x^4-5x^2+1) = (1-4x^2)^2$

$\Rightarrow \cancel{16x^4} - 20x^2 + 4 = \cancel{16x^4} - 8x^2 + 1$ $x \neq \pm\infty$

$\Rightarrow -20x^2 + 4 = -8x^2 + 1$

$\Rightarrow 3 = 12x^2$

$\Rightarrow x^2 = \frac{1}{4}$

$\Rightarrow x = \left\langle \begin{array}{l} \frac{1}{2} \\ \cancel{-\frac{1}{2}} \end{array}\right.$

$\arcsin 1 + \arccos\frac{1}{2} = \frac{\pi}{2} + \frac{\pi}{3} = \frac{5\pi}{6}$

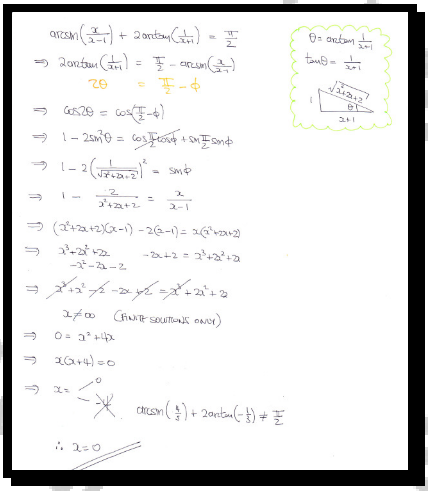
$\arcsin(-1) + \arccos(-\frac{1}{2}) = -\frac{\pi}{2} + \frac{2\pi}{3} = \frac{\pi}{6}$

Question 12 (*****)

Find the only finite solution of the equation

$$\arcsin\left(\frac{x}{x-1}\right)+2\arctan\left(\frac{1}{x+1}\right)=\frac{\pi}{2}.$$

$\boxed{x=0}$



$\arcsin\left(\frac{x}{x-1}\right)+2\arctan\left(\frac{1}{x+1}\right)=\frac{\pi}{2}$

$\Rightarrow 2\arctan\left(\frac{1}{x+1}\right)=\frac{\pi}{2}-\arcsin\left(\frac{x}{x-1}\right)$

$2\theta = \frac{\pi}{2}-\phi$

$\Rightarrow \cos 2\theta=\cos\left(\frac{\pi}{2}-\phi\right)$

$\Rightarrow 1-2\sin^2\theta=\cancel{\cos\frac{\pi}{2}\cos\phi}+\sin\frac{\pi}{2}\sin\phi$

$\Rightarrow 1-2\left(\frac{1}{\sqrt{x^2+2x+2}}\right)^2=\sin\phi$

$\Rightarrow 1-\frac{2}{x^2+2x+2}=\frac{x}{x-1}$

$\Rightarrow (x^2+2x+2)(x-1)-2(x-1)=x(x^2+2x+2)$

$\Rightarrow x^3+2x^2+2x-x^2-2x-2 \quad -2x+2=x^3+2x^2+2x$

$\Rightarrow \cancel{x^3}+x^2\cancel{-2}-2x\cancel{+2}=\cancel{x^3}+2x^2+2x$

$x \neq \infty$ (FINITE SOLUTIONS ONLY)

$\Rightarrow 0=x^2+4x$

$\Rightarrow x(x+4)=0$

$\Rightarrow x = 0,\ \cancel{-4}$. $\arcsin\left(\frac{4}{5}\right)+2\arctan\left(-\frac{1}{3}\right)\neq\frac{\pi}{2}$

$\therefore x=0$

$\theta=\arctan\frac{1}{x+1}$

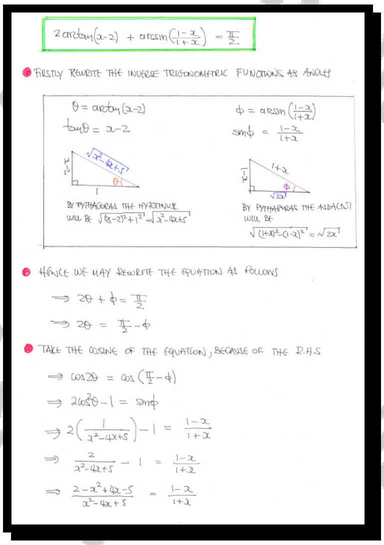
$\tan\theta=\frac{1}{x+1}$

Question 13 (*****)

Solve the trigonometric equation

$$2\arctan(x-2)+\arcsin\left(\frac{1-x}{1+x}\right)=\frac{\pi}{2},\ x\in\mathbb{R}.$$

$\boxed{\quad\quad}$, $\boxed{x=4}$



$$\boxed{2\arctan(x-2)+\arcsin\left(\frac{1-x}{1+x}\right)=\frac{\pi}{2}}$$

- FIRSTLY REWRITE THE INVERSE TRIGONOMETRIC FUNCTIONS AS ANGLES

$\theta=\arctan(x-2)$

$\tan\theta=x-2$

BY PYTHAGORAS THE HYPOTENUSE WILL BE $\sqrt{(x-2)^2+1^2}=\sqrt{x^2-4x+5}$

$\phi=\arcsin\left(\frac{1-x}{1+x}\right)$

$\sin\phi=\frac{1-x}{1+x}$

BY PYTHAGORAS THE ADJACENT WILL BE $\sqrt{(1+x)^2-(1-x)^2}=\sqrt{2x}$

- HENCE WE MAY REWRITE THE EQUATION AS FOLLOWS

$\Rightarrow 2\theta+\phi=\frac{\pi}{2}$

$\Rightarrow 2\theta=\frac{\pi}{2}-\phi$

- TAKE THE COSINE OF THE EQUATION, BECAUSE OF THE R.H.S

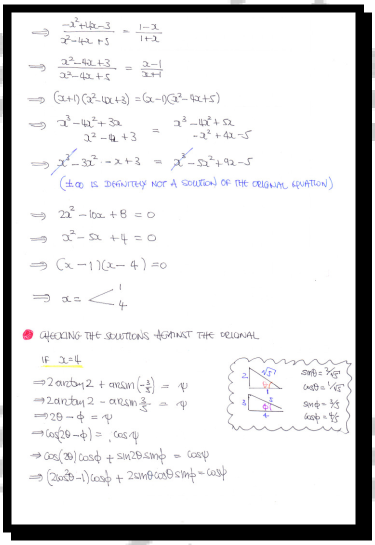
$\Rightarrow \cos 2\theta=\cos\left(\frac{\pi}{2}-\phi\right)$

$\Rightarrow 2\cos^2\theta-1=\sin\phi$

$\Rightarrow 2\left(\frac{1}{x^2-4x+5}\right)-1=\frac{1-x}{1+x}$

$\Rightarrow \frac{2}{x^2-4x+5}-1=\frac{1-x}{1+x}$

$\Rightarrow \frac{2-x^2+4x-5}{x^2-4x+5}=\frac{1-x}{1+x}$



$\Rightarrow \frac{-x^2+4x-3}{x^2-4x+5}=\frac{1-x}{1+x}$

$\Rightarrow \frac{x^2-4x+3}{x^2-4x+5}=\frac{x-1}{x+1}$

$\Rightarrow (x+1)(x^2-4x+3)=(x-1)(x^2-4x+5)$

$\Rightarrow \begin{matrix}x^3-4x^2+3x\\ x^2-4x+3\end{matrix}=\begin{matrix}x^3-4x^2+5x\\ -x^2+4x-5\end{matrix}$

$\Rightarrow \cancel{x^3}-3x^2-x+3=\cancel{x^3}-5x^2+9x-5$

($\pm\infty$ IS DEFINITELY NOT A SOLUTION OF THE ORIGINAL EQUATION)

$\Rightarrow 2x^2-10x+8=0$

$\Rightarrow x^2-5x+4=0$

$\Rightarrow (x-1)(x-4)=0$

$\Rightarrow x=\begin{cases}1\\4\end{cases}$

- CHECKING THE SOLUTIONS AGAINST THE ORIGINAL

IF $x=4$

$\Rightarrow 2\arctan 2+\arcsin\left(-\frac{3}{5}\right)=\psi$

$\Rightarrow 2\arctan 2-\arcsin\frac{3}{5}=\psi$

$\Rightarrow 2\theta-\phi=\psi$

$\Rightarrow \cos(2\theta-\phi)=\cos\psi$

$\Rightarrow \cos(2\theta)\cos\phi+\sin 2\theta\sin\phi=\cos\psi$

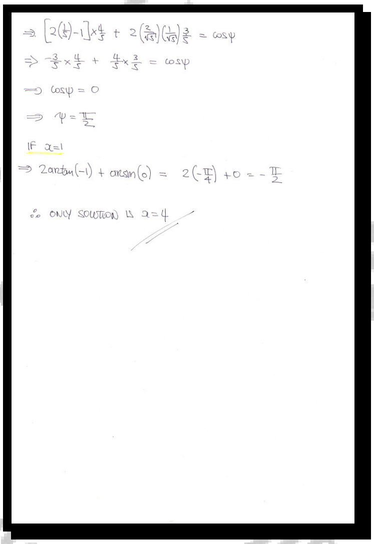
$\Rightarrow (2\cos^2\theta-1)\cos\phi+2\sin\theta\cos\theta\sin\phi=\cos\psi$

$\sin\theta=2/\sqrt{5}$

$\cos\theta=1/\sqrt{5}$

$\sin\phi=3/5$

$\cos\phi=4/5$



$\Rightarrow \left[2\left(\frac{1}{5}\right)-1\right]\times\frac{4}{5}+2\left(\frac{2}{\sqrt{5}}\right)\left(\frac{1}{\sqrt{5}}\right)\frac{3}{5}=\cos\psi$

$\Rightarrow -\frac{3}{5}\times\frac{4}{5}+\frac{4}{5}\times\frac{3}{5}=\cos\psi$

$\Rightarrow \cos\psi=0$

$\Rightarrow \psi=\frac{\pi}{2}$

IF $x=1$

$\Rightarrow 2\arctan(-1)+\arcsin(0)=2\left(-\frac{\pi}{4}\right)+0=-\frac{\pi}{2}$

$\therefore$ ONLY SOLUTION IS $x=4$

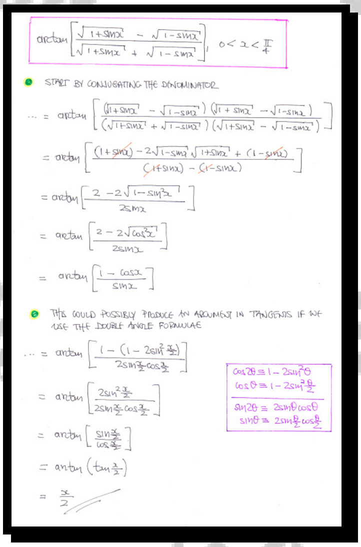
Question 14 (*****)

Use trigonometric algebra to fully simplify

$$\arctan\left[\frac{\sqrt{1+\sin x}-\sqrt{1-\sin x}}{\sqrt{1+\sin x}+\sqrt{1-\sin x}}\right],\ 0<x<\frac{\pi}{4},$$

giving the final answer in terms of x.

$\boxed{}$, $\boxed{\frac{1}{2}x}$



$$\boxed{\arctan\left[\frac{\sqrt{1+\sin x}-\sqrt{1-\sin x}}{\sqrt{1+\sin x}+\sqrt{1-\sin x}}\right],\ 0<x<\frac{\pi}{4}}$$

- START BY CONJUGATING THE DENOMINATOR

$$\dots=\arctan\left[\frac{(\sqrt{1+\sin x}-\sqrt{1-\sin x})(\sqrt{1+\sin x}-\sqrt{1-\sin x})}{(\sqrt{1+\sin x}+\sqrt{1-\sin x})(\sqrt{1+\sin x}-\sqrt{1-\sin x})}\right]$$

$$=\arctan\left[\frac{(1+\cancel{\sin x})-2\sqrt{1-\sin x}\sqrt{1+\sin x}+(1-\cancel{\sin x})}{(\cancel{1}+\sin x)-(\cancel{1}-\sin x)}\right]$$

$$=\arctan\left[\frac{2-2\sqrt{1-\sin^2 x}}{2\sin x}\right]$$

$$=\arctan\left[\frac{2-2\sqrt{\cos^2 x}}{2\sin x}\right]$$

$$=\arctan\left[\frac{1-\cos x}{\sin x}\right]$$

- THIS COULD POSSIBLY PRODUCE AN ARGUMENT IN TANGENTS IF WE USE THE DOUBLE ANGLE FORMULAE

$$\dots=\arctan\left[\frac{1-(1-2\sin^2\frac{x}{2})}{2\sin\frac{x}{2}\cos\frac{x}{2}}\right]$$

$$=\arctan\left[\frac{2\sin^2\frac{x}{2}}{2\sin\frac{x}{2}\cos\frac{x}{2}}\right]$$

$$=\arctan\left[\frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}\right]$$

$$=\arctan\left(\tan\frac{x}{2}\right)$$

$$=\frac{x}{2}$$

$\cos 2\theta\equiv 1-2\sin^2\theta$
$\cos\theta\equiv 1-2\sin^2\frac{\theta}{2}$
$\sin 2\theta\equiv 2\sin\theta\cos\theta$
$\sin\theta\equiv 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$

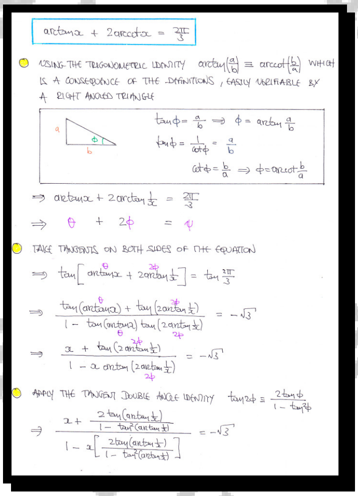
Question 15 (*****)

Use trigonometric algebra to solve the equation

$$\arctan x + 2\operatorname{arccot} x = \frac{2\pi}{3}.$$

You may assume that $\operatorname{arccot} x$ is the inverse function for the part of $\cot x$ for which $0 \le x \le \pi$.

$\boxed{}$, $\boxed{x = \sqrt{3}}$



$$\boxed{\arctan x + 2\operatorname{arccot} x = \frac{2\pi}{3}}$$

- USING THE TRIGONOMETRIC IDENTITY $\arctan\left(\frac{a}{b}\right) \equiv \operatorname{arccot}\left(\frac{b}{a}\right)$ WHICH IS A CONSEQUENCE OF THE DEFINITIONS, EASILY VERIFIABLE BY A RIGHT ANGLED TRIANGLE

$\tan\phi = \frac{a}{b} \Rightarrow \phi = \arctan\frac{a}{b}$

$\tan\phi = \frac{1}{\cot\phi} = \frac{a}{b}$

$\cot\phi = \frac{b}{a} \Rightarrow \phi = \operatorname{arccot}\frac{b}{a}$

$$\Rightarrow \arctan x + 2\arctan\frac{1}{x} = \frac{2\pi}{3}$$

$$\Rightarrow \theta + 2\phi = \psi$$

- TAKE TANGENTS ON BOTH SIDES OF THE EQUATION

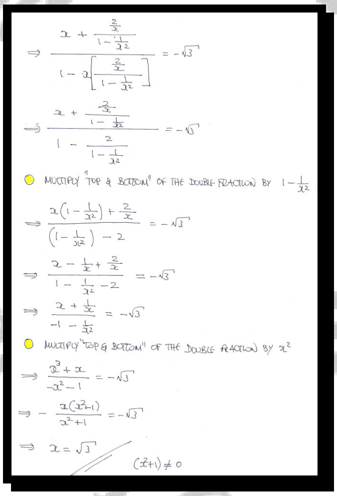
$$\Rightarrow \tan\left[\overset{\theta}{\arctan x} + \overset{2\phi}{2\arctan\frac{1}{x}}\right] = \tan\frac{2\pi}{3}$$

$$\Rightarrow \frac{\tan(\overset{\theta}{\arctan x}) + \tan(\overset{2\phi}{2\arctan\frac{1}{x}})}{1 - \tan(\underset{\theta}{\arctan x})\tan(\underset{2\phi}{2\arctan\frac{1}{x}})} = -\sqrt{3}$$

$$\Rightarrow \frac{x + \tan(\overset{2\phi}{2\arctan\frac{1}{x}})}{1 - x\arctan(\underset{2\phi}{2\arctan\frac{1}{x}})} = -\sqrt{3}$$

- APPLY THE TANGENT DOUBLE ANGLE IDENTITY $\tan 2\phi \equiv \frac{2\tan\phi}{1-\tan^2\phi}$

$$\Rightarrow \frac{x + \frac{2\tan(\arctan\frac{1}{x})}{1-\tan^2(\arctan\frac{1}{x})}}{1 - x\left[\frac{2\tan(\arctan\frac{1}{x})}{1-\tan^2(\arctan\frac{1}{x})}\right]} = -\sqrt{3}$$



$$\Rightarrow \frac{x + \frac{\frac{2}{x}}{1-\frac{1}{x^2}}}{1 - x\left[\frac{\frac{2}{x}}{1-\frac{1}{x^2}}\right]} = -\sqrt{3}$$

$$\Rightarrow \frac{x + \frac{\frac{2}{x}}{1-\frac{1}{x^2}}}{1 - \frac{2}{1-\frac{1}{x^2}}} = -\sqrt{3}$$

- MULTIPLY "TOP & BOTTOM" OF THE DOUBLE FRACTION BY $1 - \frac{1}{x^2}$

$$\Rightarrow \frac{x\left(1-\frac{1}{x^2}\right) + \frac{2}{x}}{\left(1-\frac{1}{x^2}\right) - 2} = -\sqrt{3}$$

$$\Rightarrow \frac{x - \frac{1}{x} + \frac{2}{x}}{1 - \frac{1}{x^2} - 2} = -\sqrt{3}$$

$$\Rightarrow \frac{x + \frac{1}{x}}{-1 - \frac{1}{x^2}} = -\sqrt{3}$$

- MULTIPLY "TOP & BOTTOM" OF THE DOUBLE FRACTION BY x^2

$$\Rightarrow \frac{x^3 + x}{-x^2 - 1} = -\sqrt{3}$$

$$\Rightarrow -\frac{x(x^2+1)}{x^2+1} = -\sqrt{3}$$

$$\Rightarrow x = \sqrt{3}$$

$(x^2+1) \neq 0$

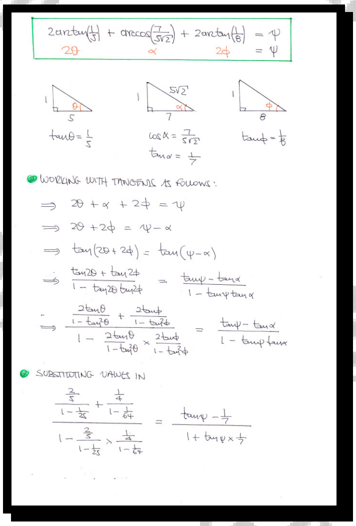
Question 16 (*****)

Use trigonometric algebra to fully simplify

$$2\arctan\left(\frac{1}{5}\right)+\arccos\left(\frac{7}{5\sqrt{2}}\right)+\arctan\left(\frac{1}{8}\right),$$

giving the final answer in terms of π.

$\boxed{}$, $\boxed{\frac{\pi}{4}}$



$$\boxed{\begin{array}{l} 2\arctan\left(\frac{1}{5}\right) + \arccos\left(\frac{7}{5\sqrt{2}}\right) + 2\arctan\left(\frac{1}{8}\right) = \psi \\ 2\theta \qquad\qquad \alpha \qquad\qquad 2\phi \qquad = \psi \end{array}}$$

$\tan\theta = \frac{1}{5}$ $\quad \cos\alpha = \frac{7}{5\sqrt{2}}$, $\tan\alpha = \frac{1}{7}$ $\quad \tan\phi = \frac{1}{8}$

- WORKING WITH TANGENTS AS FOLLOWS:

$\Rightarrow 2\theta + \alpha + 2\phi = \psi$

$\Rightarrow 2\theta + 2\phi = \psi - \alpha$

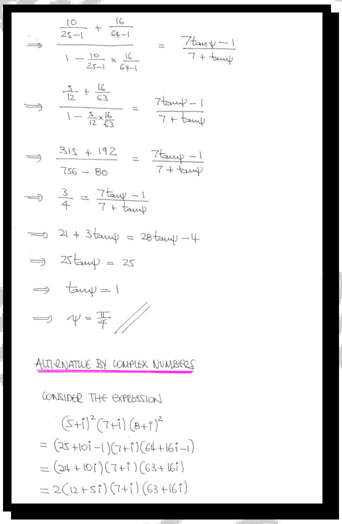
$\Rightarrow \tan(2\theta + 2\phi) = \tan(\psi - \alpha)$

$\Rightarrow \dfrac{\tan 2\theta + \tan 2\phi}{1 - \tan 2\theta \tan 2\phi} = \dfrac{\tan\psi - \tan\alpha}{1 - \tan\psi\tan\alpha}$

$\Rightarrow \dfrac{\frac{2\tan\theta}{1-\tan^2\theta} + \frac{2\tan\phi}{1-\tan^2\phi}}{1 - \frac{2\tan\theta}{1-\tan^2\theta} \times \frac{2\tan\phi}{1-\tan^2\phi}} = \dfrac{\tan\psi - \tan\alpha}{1 - \tan\psi\tan\alpha}$

- SUBSTITUTING VALUES IN

$\dfrac{\frac{\frac{2}{5}}{1-\frac{1}{25}} + \frac{\frac{1}{4}}{1-\frac{1}{64}}}{1 - \frac{\frac{2}{5}}{1-\frac{1}{25}} \times \frac{\frac{1}{4}}{1-\frac{1}{64}}} = \dfrac{\tan\psi - \frac{1}{7}}{1 + \tan\psi \times \frac{1}{7}}$



$\Rightarrow \dfrac{\frac{10}{25-1} + \frac{16}{64-1}}{1 - \frac{10}{25-1} \times \frac{16}{64-1}} = \dfrac{7\tan\psi - 1}{7 + \tan\psi}$

$\Rightarrow \dfrac{\frac{5}{12} + \frac{16}{63}}{1 - \frac{5}{12} \times \frac{16}{63}} = \dfrac{7\tan\psi - 1}{7 + \tan\psi}$

$\Rightarrow \dfrac{315 + 192}{756 - 80} = \dfrac{7\tan\psi - 1}{7 + \tan\psi}$

$\Rightarrow \dfrac{3}{4} = \dfrac{7\tan\psi - 1}{7 + \tan\psi}$

$\Rightarrow 21 + 3\tan\psi = 28\tan\psi - 4$

$\Rightarrow 25\tan\psi = 25$

$\Rightarrow \tan\psi = 1$

$\Rightarrow \psi = \frac{\pi}{4}$ //

ALTERNATIVE BY COMPLEX NUMBERS

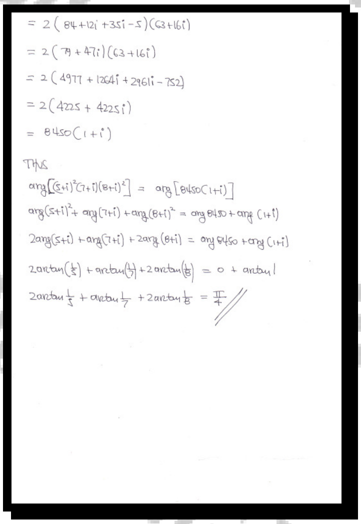
CONSIDER THE EXPRESSION

$(5+i)^2(7+i)(8+i)^2$

$= (25+10i-1)(7+i)(64+16i-1)$

$= (24+10i)(7+i)(63+16i)$

$= 2(12+5i)(7+i)(63+16i)$



$= 2(84+12i+35i-5)(63+16i)$

$= 2(79+47i)(63+16i)$

$= 2(4977+1264i+2961i-752)$

$= 2(4225+4225i)$

$= 8450(1+i)$

THUS

$\arg\left[(5+i)^2(7+i)(8+i)^2\right] = \arg\left[8450(1+i)\right]$

$\arg(5+i)^2 + \arg(7+i) + \arg(8+i)^2 = \arg 8450 + \arg(1+i)$

$2\arg(5+i) + \arg(7+i) + 2\arg(8+i) = \arg 8450 + \arg(1+i)$

$2\arctan\left(\frac{1}{5}\right) + \arctan\left(\frac{1}{7}\right) + 2\arctan\left(\frac{1}{8}\right) = 0 + \arctan 1$

$2\arctan\frac{1}{5} + \arctan\frac{1}{7} + 2\arctan\frac{1}{8} = \frac{\pi}{4}$ //

Question 17 (*****)

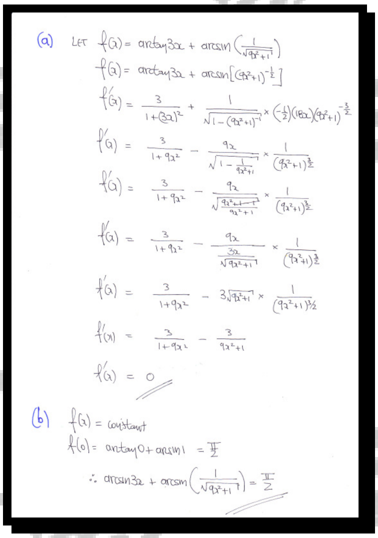
$$f(x) = \arctan(3x) + \arcsin\left(\frac{1}{\sqrt{9x^2+1}}\right), \quad x \in \mathbb{R}.$$

Show, by a detailed method, that …

a) … $f'(x) = 0$.

b) … $\arctan(3x) + \arcsin\left(\frac{1}{\sqrt{9x^2+1}}\right) \equiv k\pi$, stating the value of the constant k.

$\boxed{}$, $\boxed{k = \frac{1}{2}}$



(a) LET $f(x) = \arctan 3x + \arcsin\left(\frac{1}{\sqrt{9x^2+1}}\right)$

$f(x) = \arctan 3x + \arcsin\left[(9x^2+1)^{-\frac{1}{2}}\right]$

$f'(x) = \frac{3}{1+(3x)^2} + \frac{1}{\sqrt{1-(9x^2+1)^{-1}}} \times \left(-\frac{1}{2}\right)(18x)(9x^2+1)^{-\frac{3}{2}}$

$f'(x) = \frac{3}{1+9x^2} - \frac{9x}{\sqrt{1-\frac{1}{9x^2+1}}} \times \frac{1}{(9x^2+1)^{\frac{3}{2}}}$

$f'(x) = \frac{3}{1+9x^2} - \frac{9x}{\sqrt{\frac{9x^2\cancel{+1}\cancel{-1}}{9x^2+1}}} \times \frac{1}{(9x^2+1)^{\frac{3}{2}}}$

$f'(x) = \frac{3}{1+9x^2} - \frac{9x}{\frac{3x}{\sqrt{9x^2+1}}} \times \frac{1}{(9x^2+1)^{\frac{3}{2}}}$

$f'(x) = \frac{3}{1+9x^2} - 3\sqrt{9x^2+1} \times \frac{1}{(9x^2+1)^{3/2}}$

$f'(x) = \frac{3}{1+9x^2} - \frac{3}{9x^2+1}$

$f'(x) = 0$

(b) $f(x) = \text{constant}$

$f(0) = \arctan 0 + \arcsin 1 = \frac{\pi}{2}$

$\therefore \arcsin 3x + \arcsin\left(\frac{1}{\sqrt{9x^2+1}}\right) = \frac{\pi}{2}$

Question 18 (*****)

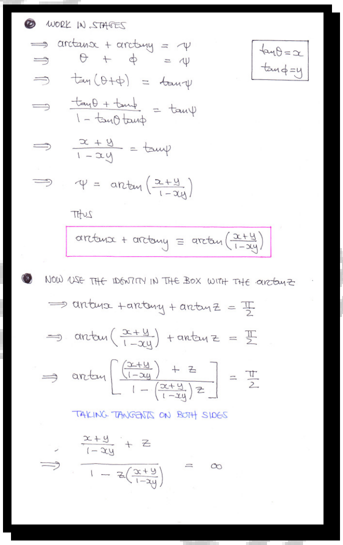
It given that

$$\arctan x + \arctan y + \arctan z = \frac{\pi}{2}.$$

Show that x, y and z satisfy the relationship

$$xy + yz + zx = 1.$$

$\boxed{}$, $\boxed{\text{proof}}$



● WORK IN STAGES

$$\Longrightarrow \arctan x + \arctan y = \psi$$
$$\Longrightarrow \theta + \phi = \psi$$
$$\Longrightarrow \tan(\theta+\phi) = \tan\psi$$
$$\Longrightarrow \frac{\tan\theta + \tan\phi}{1-\tan\theta\tan\phi} = \tan\psi$$
$$\Longrightarrow \frac{x+y}{1-xy} = \tan\psi$$
$$\Longrightarrow \psi = \arctan\left(\frac{x+y}{1-xy}\right)$$

$$\boxed{\begin{aligned}\tan\theta &= x\\ \tan\phi &= y\end{aligned}}$$

THUS

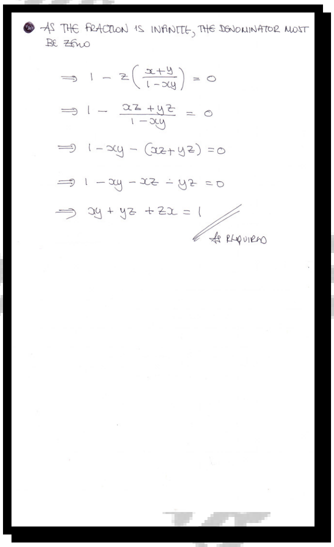
$$\boxed{\arctan x + \arctan y \equiv \arctan\left(\frac{x+y}{1-xy}\right)}$$

● NOW USE THE IDENTITY IN THE BOX WITH THE arctan z

$$\Longrightarrow \arctan x + \arctan y + \arctan z = \frac{\pi}{2}$$
$$\Longrightarrow \arctan\left(\frac{x+y}{1-xy}\right) + \arctan z = \frac{\pi}{2}$$
$$\Longrightarrow \arctan\left[\frac{\left(\frac{x+y}{1-xy}\right) + z}{1-\left(\frac{x+y}{1-xy}\right)z}\right] = \frac{\pi}{2}$$

TAKING TANGENTS ON BOTH SIDES

$$\Longrightarrow \frac{\frac{x+y}{1-xy} + z}{1 - z\left(\frac{x+y}{1-xy}\right)} = \infty$$



● AS THE FRACTION IS INFINITE, THE DENOMINATOR MUST BE ZERO

$$\Longrightarrow 1 - z\left(\frac{x+y}{1-xy}\right) = 0$$
$$\Longrightarrow 1 - \frac{xz+yz}{1-xy} = 0$$
$$\Longrightarrow 1 - xy - (xz+yz) = 0$$
$$\Longrightarrow 1 - xy - xz - yz = 0$$
$$\Longrightarrow xy + yz + zx = 1$$

AS REQUIRED

Question 19 (*****)

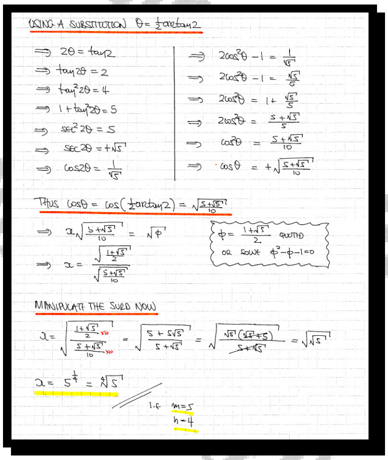
The positive solution of the quadratic equation $x^2 - x - 1 = 0$ is denoted by ϕ, and is commonly known as the golden section or golden number.

Solve the equation

$$x\cos\left(\tfrac{1}{2}\arctan 2\right) = \sqrt{\phi}\,, \quad x \in \mathbb{R}\ .$$

Give the answer in the form $\sqrt[n]{m}$, where m and n are positive integers.

V, ☐, $\boxed{x = \sqrt[4]{5}}$



USING A SUBSTITUTION $\theta = \frac{1}{2}\arctan 2$

$\Rightarrow 2\theta = \tan 2$

$\Rightarrow \tan 2\theta = 2$

$\Rightarrow \tan^2 2\theta = 4$

$\Rightarrow 1 + \tan^2 2\theta = 5$

$\Rightarrow \sec^2 2\theta = 5$

$\Rightarrow \sec 2\theta = +\sqrt{5}$

$\Rightarrow \cos 2\theta = \frac{1}{\sqrt{5}}$

$\Rightarrow 2\cos^2\theta - 1 = \frac{1}{\sqrt{5}}$

$\Rightarrow 2\cos^2\theta - 1 = \frac{\sqrt{5}}{5}$

$\Rightarrow 2\cos^2\theta = 1 + \frac{\sqrt{5}}{5}$

$\Rightarrow 2\cos^2\theta = \frac{5+\sqrt{5}}{5}$

$\Rightarrow \cos^2\theta = \frac{5+\sqrt{5}}{10}$

$\Rightarrow \cos\theta = +\sqrt{\frac{5+\sqrt{5}}{10}}$

THUS $\cos\theta = \cos\left(\frac{1}{2}\arctan 2\right) = \sqrt{\frac{5+\sqrt{5}}{10}}$

$\Rightarrow x\sqrt{\frac{5+\sqrt{5}}{10}} = \sqrt{\phi}$

$\Rightarrow x = \frac{\sqrt{\frac{1+\sqrt{5}}{2}}}{\sqrt{\frac{5+\sqrt{5}}{10}}}$

$\phi = \frac{1+\sqrt{5}}{2}$ QUOTED

OR SOLVE $\phi^2 - \phi - 1 = 0$

MANIPULATE THE SURD NOW

$$x = \sqrt{\frac{\frac{1+\sqrt{5}}{2}\times 10}{\frac{5+\sqrt{5}}{10}\times 10}} = \sqrt{\frac{5+5\sqrt{5}}{5+\sqrt{5}}} = \sqrt{\frac{\sqrt{5}(\cancel{\sqrt{5}+5})}{\cancel{5+\sqrt{5}}}} = \sqrt{\sqrt{5}}$$

$x = 5^{\frac{1}{4}} = \sqrt[4]{5}$

i.e. $m = 5$

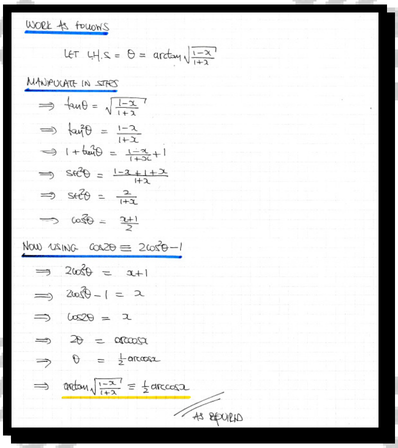
$n = 4$

Question 20 (*****)

Prove that if $|x| \le 1$

$$\arctan\left[\sqrt{\frac{1-x}{1+x}}\right] \equiv \frac{1}{2}\arccos x .$$

V, , proof



WORK AS FOLLOWS

LET L.H.S $= \theta = \arctan\sqrt{\frac{1-x}{1+x}}$

MANIPULATE IN STEPS

$\Rightarrow \tan\theta = \sqrt{\frac{1-x}{1+x}}$

$\Rightarrow \tan^2\theta = \frac{1-x}{1+x}$

$\Rightarrow 1+\tan^2\theta = \frac{1-x}{1+x}+1$

$\Rightarrow \sec^2\theta = \frac{1-x+1+x}{1+x}$

$\Rightarrow \sec^2\theta = \frac{2}{1+x}$

$\Rightarrow \cos^2\theta = \frac{x+1}{2}$

NOW USING $\cos 2\theta \equiv 2\cos^2\theta - 1$

$\Rightarrow 2\cos^2\theta = x+1$

$\Rightarrow 2\cos^2\theta - 1 = x$

$\Rightarrow \cos 2\theta = x$

$\Rightarrow 2\theta = \arccos x$

$\Rightarrow \theta = \frac{1}{2}\arccos x$

$\Rightarrow \arctan\sqrt{\frac{1-x}{1+x}} = \frac{1}{2}\arccos x$

AS REQUIRED

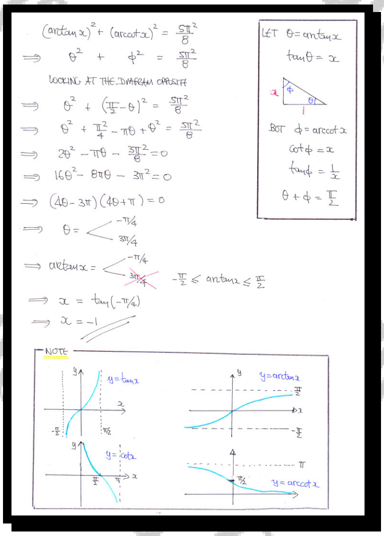
Question 21 (*****)

Use a trigonometric algebra to solve the following equation

$$(\arctan x)^2 + (\operatorname{arccot} x)^2 = \frac{5\pi^2}{8}.$$

You may assume that $y = \operatorname{arccot} x$ is the inverse function of $y = \cot x$, $0 \le x \le \pi$

$\boxed{}$, $\boxed{x = -1}$



$(\arctan x)^2 + (\operatorname{arccot} x)^2 = \frac{5\pi^2}{8}$

$\Rightarrow \quad \theta^2 + \phi^2 = \frac{5\pi^2}{8}$

LOOKING AT THE DIAGRAM OPPOSITE

$\Rightarrow \quad \theta^2 + \left(\frac{\pi}{2} - \theta\right)^2 = \frac{5\pi^2}{8}$

$\Rightarrow \quad \theta^2 + \frac{\pi^2}{4} - \pi\theta + \theta^2 = \frac{5\pi^2}{8}$

$\Rightarrow \quad 2\theta^2 - \pi\theta - \frac{3\pi^2}{8} = 0$

$\Rightarrow \quad 16\theta^2 - 8\pi\theta - 3\pi^2 = 0$

$\Rightarrow \quad (4\theta - 3\pi)(4\theta + \pi) = 0$

$\Rightarrow \quad \theta = \begin{cases} -\pi/4 \\ 3\pi/4 \end{cases}$

$\Rightarrow \quad \arctan x = \begin{cases} -\pi/4 \\ \cancel{3\pi/4} \end{cases} \quad -\frac{\pi}{2} \le \arctan x \le \frac{\pi}{2}$

$\Rightarrow \quad x = \tan(-\pi/4)$

$\Rightarrow \quad x = -1$

LET $\theta = \arctan x$

$\tan\theta = x$

BUT $\phi = \operatorname{arccot} x$

$\cot\phi = x$

$\tan\phi = \frac{1}{x}$

$\theta + \phi = \frac{\pi}{2}$

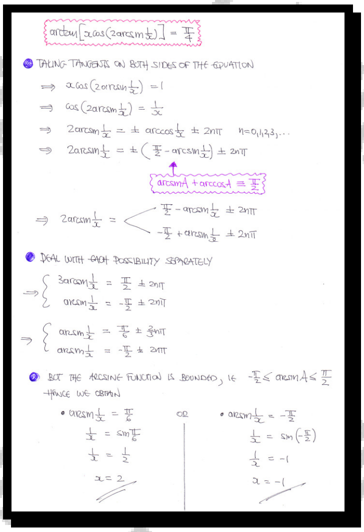
NOTE

Question 22 (*****)

Solve the following trigonometric equation

$$\arctan\left[x\cos\left(2\arcsin\tfrac{1}{x}\right)\right]=\tfrac{1}{4}\pi\ .$$

$\boxed{}$, $\boxed{x=-1,\quad x=2}$



$\boxed{\arctan\left[x\cos\left(2\arcsin\frac{1}{x}\right)\right]=\frac{\pi}{4}}$

- TAKING TANGENTS ON BOTH SIDES OF THE EQUATION

$$\Rightarrow x\cos\left(2\arcsin\frac{1}{x}\right)=1$$
$$\Rightarrow \cos\left(2\arcsin\frac{1}{x}\right)=\frac{1}{x}$$
$$\Rightarrow 2\arcsin\frac{1}{x}=\pm\arccos\frac{1}{x}\pm 2n\pi \qquad n=0,1,2,3,\dots$$
$$\Rightarrow 2\arcsin\frac{1}{x}=\pm\left(\frac{\pi}{2}-\arcsin\frac{1}{x}\right)\pm 2n\pi$$

$\boxed{\arcsin A+\arccos A\equiv\frac{\pi}{2}}$

$$\Rightarrow 2\arcsin\frac{1}{x}=\begin{cases}\frac{\pi}{2}-\arcsin\frac{1}{x}\pm 2n\pi\\ -\frac{\pi}{2}+\arcsin\frac{1}{x}\pm 2n\pi\end{cases}$$

- DEAL WITH EACH POSSIBILITY SEPARATELY

$$\Rightarrow\begin{cases}3\arcsin\frac{1}{x}=\frac{\pi}{2}\pm 2n\pi\\ \arcsin\frac{1}{x}=-\frac{\pi}{2}\pm 2n\pi\end{cases}$$

$$\Rightarrow\begin{cases}\arcsin\frac{1}{x}=\frac{\pi}{6}\pm\frac{2}{3}n\pi\\ \arcsin\frac{1}{x}=-\frac{\pi}{2}\pm 2n\pi\end{cases}$$

- BUT THE ARCSINE FUNCTION IS BOUNDED, IE $-\frac{\pi}{2}\leqslant\arcsin A\leqslant\frac{\pi}{2}$ HENCE WE OBTAIN

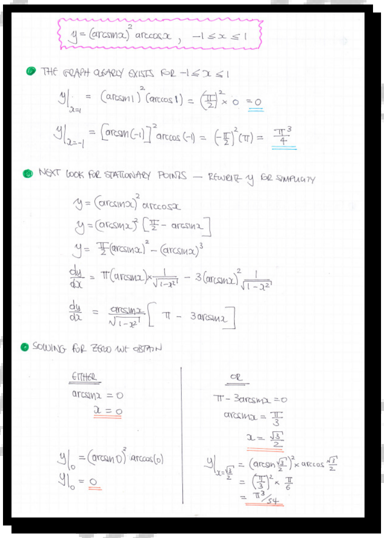
• $\arcsin\frac{1}{x}=\frac{\pi}{6}$	OR	• $\arcsin\frac{1}{x}=-\frac{\pi}{2}$
$\frac{1}{x}=\sin\frac{\pi}{6}$		$\frac{1}{x}=\sin\left(-\frac{\pi}{2}\right)$
$\frac{1}{x}=\frac{1}{2}$		$\frac{1}{x}=-1$
$x=2$		$x=-1$

Question 23 (*****)

On a clearly labelled set of axes, draw a detailed sketch of the graph of

$$y = (\arcsin x)^2 \arccos x,\ -1 \le x \le 1.$$

☐ , graph



$\boxed{y=(\arcsin x)^2 \arccos x\ ,\quad -1\leq x\leq 1}$

- THE GRAPH CLEARLY EXISTS FOR $-1\leq x\leq 1$

$y|_{x=1} = (\arcsin 1)^2(\arccos 1) = \left(\frac{\pi}{2}\right)^2 \times 0 = \underline{\underline{0}}$

$y|_{x=-1} = [\arcsin(-1)]^2 \arccos(-1) = \left(-\frac{\pi}{2}\right)^2(\pi) = \underline{\underline{\frac{\pi^3}{4}}}$

- NEXT LOOK FOR STATIONARY POINTS — REWRITE y FOR SIMPLICITY

$y=(\arcsin x)^2 \arccos x$

$y=(\arcsin x)^2\left[\frac{\pi}{2}-\arcsin x\right]$

$y=\frac{\pi}{2}(\arcsin x)^2-(\arcsin x)^3$

$\frac{dy}{dx}=\pi(\arcsin x)\times\frac{1}{\sqrt{1-x^2}}-3(\arcsin x)^2\frac{1}{\sqrt{1-x^2}}$

$\frac{dy}{dx}=\frac{\arcsin x}{\sqrt{1-x^2}}\left[\pi-3\arcsin x\right]$

- SOLVING FOR ZERO WE OBTAIN

EITHER

$\arcsin x = 0$

$\underline{\underline{x=0}}$

$y|_0 = (\arcsin 0)^2 \arccos(0)$

$y|_0 = \underline{\underline{0}}$

OR

$\pi - 3\arcsin x = 0$

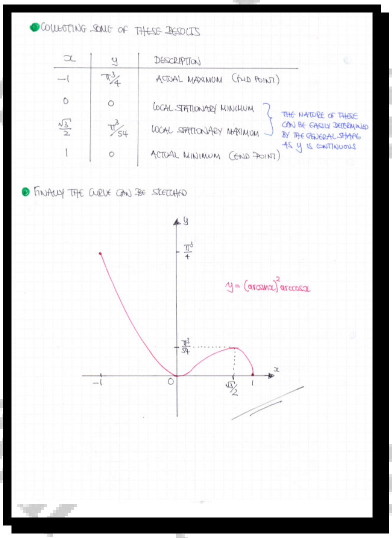
$\arcsin x = \frac{\pi}{3}$

$\underline{\underline{x=\frac{\sqrt{3}}{2}}}$

$y|_{x=\frac{\sqrt{3}}{2}} = \left(\arcsin\frac{\sqrt{3}}{2}\right)^2\times\arccos\frac{\sqrt{3}}{2}$

$= \left(\frac{\pi}{3}\right)^2\times\frac{\pi}{6}$

$= \underline{\underline{\pi^3/54}}$



- COLLECTING SOME OF THESE RESULTS

x	y	DESCRIPTION
-1	$\pi^3/4$	ACTUAL MAXIMUM (END POINT)
0	0	LOCAL STATIONARY MINIMUM
$\frac{\sqrt{3}}{2}$	$\pi^3/54$	LOCAL STATIONARY MAXIMUM
1	0	ACTUAL MINIMUM (END POINT)

THE NATURE OF THESE CAN BE EASILY DETERMINED BY THE GENERAL SHAPE AS y IS CONTINUOUS

- FINALLY THE CURVE CAN BE SKETCHED

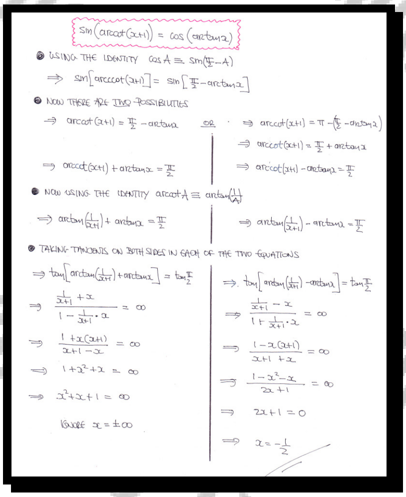
Question 24 (*****)

Solve the following trigonometric equation

$$\sin\left[\operatorname{arccot}(x+1)\right] = \cos(\arctan x).$$

You may assume that $y = \operatorname{arccot} x$ is the inverse function for $y = \cot x$, $0 \le x \le \pi$.

$\boxed{}$, $\boxed{x = -\frac{1}{2}}$



$\boxed{\sin(\operatorname{arccot}(x+1)) = \cos(\arctan x)}$

- USING THE IDENTITY $\cos A \equiv \sin(\frac{\pi}{2} - A)$

$\Rightarrow \sin[\operatorname{arcccot}(x+1)] = \sin[\frac{\pi}{2} - \arctan x]$

- NOW THERE ARE TWO POSSIBILITIES

$\Rightarrow \operatorname{arccot}(x+1) = \frac{\pi}{2} - \arctan x$

$\Rightarrow \operatorname{arccot}(x+1) + \arctan x = \frac{\pi}{2}$

OR

$\Rightarrow \operatorname{arccot}(x+1) = \pi - (\frac{\pi}{2} - \arctan x)$

$\Rightarrow \operatorname{arccot}(x+1) = \frac{\pi}{2} + \arctan x$

$\Rightarrow \operatorname{arccot}(x+1) - \arctan x = \frac{\pi}{2}$

- NOW USING THE IDENTITY $\operatorname{arccot} A \equiv \arctan(\frac{1}{A})$

$\Rightarrow \arctan(\frac{1}{x+1}) + \arctan x = \frac{\pi}{2}$

$\Rightarrow \arctan(\frac{1}{x+1}) - \arctan x = \frac{\pi}{2}$

- TAKING TANGENTS ON BOTH SIDES IN EACH OF THE TWO EQUATIONS

$\Rightarrow \tan[\arctan(\frac{1}{x+1}) + \arctan x] = \tan\frac{\pi}{2}$

$\Rightarrow \dfrac{\frac{1}{x+1} + x}{1 - \frac{1}{x+1} \cdot x} = \infty$

$\Rightarrow \dfrac{1 + x(x+1)}{x+1-x} = \infty$

$\Rightarrow 1 + x^2 + x = \infty$

$\Rightarrow x^2 + x + 1 = \infty$

IGNORE $x = \pm\infty$

$\Rightarrow \tan[\arctan(\frac{1}{x+1}) - \arctan x] = \tan\frac{\pi}{2}$

$\Rightarrow \dfrac{\frac{1}{x+1} - x}{1 + \frac{1}{x+1} \cdot x} = \infty$

$\Rightarrow \dfrac{1 - x(x+1)}{x+1+x} = \infty$

$\Rightarrow \dfrac{1 - x^2 - x}{2x+1} = \infty$

$\Rightarrow 2x + 1 = 0$

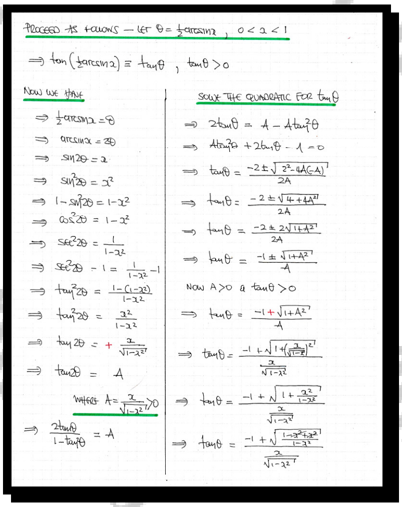
$\Rightarrow x = -\frac{1}{2}$

Question 25 (*****)

Prove that if $|x| \le 1$

$$\tan\left[\frac{1}{2}\arcsin x\right] \equiv \frac{1-\sqrt{1-x^2}}{x}.$$

V, ☐, proof



PROCEED AS FOLLOWS – LET $\theta = \frac{1}{2}\arcsin x$, $0 < x < 1$

$\Rightarrow \tan\left(\frac{1}{2}\arcsin x\right) \equiv \tan\theta$, $\tan\theta > 0$

NOW WE HAVE

$\Rightarrow \frac{1}{2}\arcsin x = \theta$

$\Rightarrow \arcsin x = 2\theta$

$\Rightarrow \sin 2\theta = x$

$\Rightarrow \sin^2 2\theta = x^2$

$\Rightarrow 1-\sin^2 2\theta = 1-x^2$

$\Rightarrow \cos^2 2\theta = 1-x^2$

$\Rightarrow \sec^2 2\theta = \frac{1}{1-x^2}$

$\Rightarrow \sec^2 2\theta - 1 = \frac{1}{1-x^2} - 1$

$\Rightarrow \tan^2 2\theta = \frac{1-(1-x^2)}{1-x^2}$

$\Rightarrow \tan^2 2\theta = \frac{x^2}{1-x^2}$

$\Rightarrow \tan 2\theta = + \frac{x}{\sqrt{1-x^2}}$

$\Rightarrow \tan 2\theta = A$

WHERE $A = \frac{x}{\sqrt{1-x^2}} > 0$

$\Rightarrow \frac{2\tan\theta}{1-\tan^2\theta} = A$

SOLVE THE QUADRATIC FOR $\tan\theta$

$\Rightarrow 2\tan\theta = A - A\tan^2\theta$

$\Rightarrow A\tan^2\theta + 2\tan\theta - 1 = 0$

$\Rightarrow \tan\theta = \frac{-2 \pm \sqrt{2^2-4A(-A)}}{2A}$

$\Rightarrow \tan\theta = \frac{-2 \pm \sqrt{4+4A^2}}{2A}$

$\Rightarrow \tan\theta = \frac{-2 \pm 2\sqrt{1+A^2}}{2A}$

$\Rightarrow \tan\theta = \frac{-1 \pm \sqrt{1+A^2}}{A}$

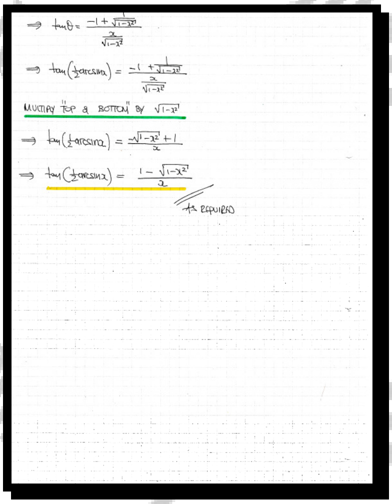
NOW $A>0$ & $\tan\theta > 0$

$\Rightarrow \tan\theta = \frac{-1 + \sqrt{1+A^2}}{A}$

$\Rightarrow \tan\theta = \frac{-1+\sqrt{1+\left(\frac{x}{\sqrt{1-x^2}}\right)^2}}{\frac{x}{\sqrt{1-x^2}}}$

$\Rightarrow \tan\theta = \frac{-1+\sqrt{1+\frac{x^2}{1-x^2}}}{\frac{x}{\sqrt{1-x^2}}}$

$\Rightarrow \tan\theta = \frac{-1+\sqrt{\frac{1-\cancel{x^2}+\cancel{x^2}}{1-x^2}}}{\frac{x}{\sqrt{1-x^2}}}$



$\Rightarrow \tan\theta = \frac{-1+\sqrt{1-x^2}}{\frac{x}{\sqrt{1-x^2}}}$

$\Rightarrow \tan\left(\frac{1}{2}\arcsin x\right) = \frac{-1+\frac{1}{\sqrt{1-x^2}}}{\frac{x}{\sqrt{1-x^2}}}$

MULTIPLY "TOP & BOTTOM" BY $\sqrt{1-x^2}$

$\Rightarrow \tan\left(\frac{1}{2}\arcsin x\right) = \frac{-\sqrt{1-x^2}+1}{x}$

$\Rightarrow \tan\left(\frac{1}{2}\arcsin x\right) = \frac{1-\sqrt{1-x^2}}{x}$

// AS REQUIRED

Question 26 (*****)

It is given that

$$(\arcsin x)^3 + (\arccos x)^3 = k\pi^3, \ |x| \le 1,$$

for some constant k.

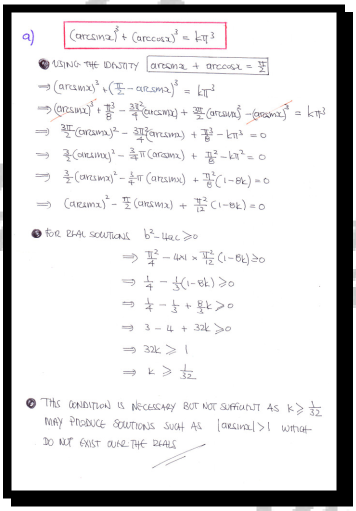
a) Show that a necessary but not sufficient condition for the above equation to have solutions is that

$$k \ge \frac{1}{32}.$$

b) Solve the equation given that it only has one solution.

c) Given instead that that $k = \frac{7}{96}$, find the two solutions of the equation, giving the answers in the form $x = \sin(a\pi)$, where $a \in \mathbb{Q}$.

$$\boxed{}, \ \boxed{x = \frac{\sqrt{2}}{2}}, \ \boxed{x = \sin\left(\frac{\pi}{12}\right), \quad x = \sin\left(\frac{5\pi}{12}\right)}$$



a) $\boxed{(\arcsin x)^3 + (\arccos x)^3 = k\pi^3}$

- USING THE IDENTITY $\boxed{\arcsin x + \arccos x = \frac{\pi}{2}}$

$$\Rightarrow (\arcsin x)^3 + \left(\frac{\pi}{2} - \arcsin x\right)^3 = k\pi^3$$

$$\Rightarrow \cancel{(\arcsin x)^3} + \frac{\pi^3}{8} - \frac{3\pi^2}{4}(\arcsin x) + \frac{3\pi}{2}(\arcsin x)^2 - \cancel{(\arcsin x)^3} = k\pi^3$$

$$\Rightarrow \frac{3\pi}{2}(\arcsin x)^2 - \frac{3\pi^2}{4}(\arcsin x) + \frac{\pi^3}{8} - k\pi^3 = 0$$

$$\Rightarrow \frac{3}{2}(\arcsin x)^2 - \frac{3}{4}\pi(\arcsin x) + \frac{\pi^2}{8} - k\pi^2 = 0$$

$$\Rightarrow \frac{3}{2}(\arcsin x)^2 - \frac{3}{4}\pi(\arcsin x) + \frac{\pi^2}{8}(1-8k) = 0$$

$$\Rightarrow (\arcsin x)^2 - \frac{\pi}{2}(\arcsin x) + \frac{\pi^2}{12}(1-8k) = 0$$

- FOR REAL SOLUTIONS $b^2 - 4ac \geqslant 0$

$$\Rightarrow \frac{\pi^2}{4} - 4\times1\times\frac{\pi^2}{12}(1-8k) \geqslant 0$$

$$\Rightarrow \frac{1}{4} - \frac{1}{3}(1-8k) \geqslant 0$$

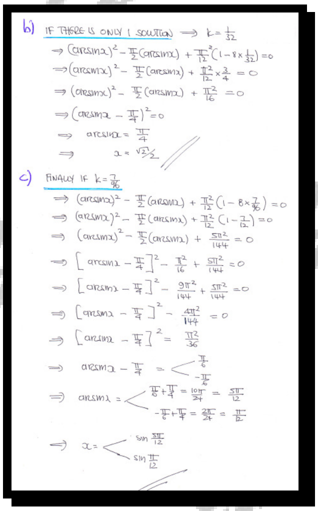
$$\Rightarrow \frac{1}{4} - \frac{1}{3} + \frac{8}{3}k \geqslant 0$$

$$\Rightarrow 3 - 4 + 32k \geqslant 0$$

$$\Rightarrow 32k \geqslant 1$$

$$\Rightarrow k \geqslant \frac{1}{32}$$

- THIS CONDITION IS NECESSARY BUT NOT SUFFICIENT AS $k \geqslant \frac{1}{32}$ MAY PRODUCE SOLUTIONS SUCH AS $|\arcsin x| > 1$ WHICH DO NOT EXIST OVER THE REALS



b) IF THERE IS ONLY 1 SOLUTION $\Rightarrow k = \frac{1}{32}$

$$\Rightarrow (\arcsin x)^2 - \frac{\pi}{2}(\arcsin x) + \frac{\pi^2}{12}\left(1 - 8\times\frac{1}{32}\right) = 0$$

$$\Rightarrow (\arcsin x)^2 - \frac{\pi}{2}(\arcsin x) + \frac{\pi^2}{12}\times\frac{3}{4} = 0$$

$$\Rightarrow (\arcsin x)^2 - \frac{\pi}{2}(\arcsin x) + \frac{\pi^2}{16} = 0$$

$$\Rightarrow \left(\arcsin x - \frac{\pi}{4}\right)^2 = 0$$

$$\Rightarrow \arcsin x = \frac{\pi}{4}$$

$$\Rightarrow x = \sqrt{2}/2$$

c) FINALLY IF $k = \frac{7}{96}$

$$\Rightarrow (\arcsin x)^2 - \frac{\pi}{2}(\arcsin x) + \frac{\pi^2}{12}\left(1 - 8\times\frac{7}{96}\right) = 0$$

$$\Rightarrow (\arcsin x)^2 - \frac{\pi}{2}(\arcsin x) + \frac{\pi^2}{12}\left(1 - \frac{7}{12}\right) = 0$$

$$\Rightarrow (\arcsin x)^2 - \frac{\pi}{2}(\arcsin x) + \frac{5\pi^2}{144} = 0$$

$$\Rightarrow \left[\arcsin x - \frac{\pi}{4}\right]^2 - \frac{\pi^2}{16} + \frac{5\pi^2}{144} = 0$$

$$\Rightarrow \left[\arcsin x - \frac{\pi}{4}\right]^2 - \frac{9\pi^2}{144} + \frac{5\pi^2}{144} = 0$$

$$\Rightarrow \left[\arcsin x - \frac{\pi}{4}\right]^2 - \frac{4\pi^2}{144} = 0$$

$$\Rightarrow \left[\arcsin x - \frac{\pi}{4}\right]^2 = \frac{\pi^2}{36}$$

$$\Rightarrow \arcsin x - \frac{\pi}{4} = \begin{cases} \frac{\pi}{6} \\ -\frac{\pi}{6} \end{cases}$$

$$\Rightarrow \arcsin x = \begin{cases} \frac{\pi}{6} + \frac{\pi}{4} = \frac{10\pi}{24} = \frac{5\pi}{12} \\ -\frac{\pi}{6} + \frac{\pi}{4} = \frac{2\pi}{24} = \frac{\pi}{12} \end{cases}$$

$$\Rightarrow x = \begin{cases} \sin\frac{5\pi}{12} \\ \sin\frac{\pi}{12} \end{cases}$$

Question 27 **(*****)**

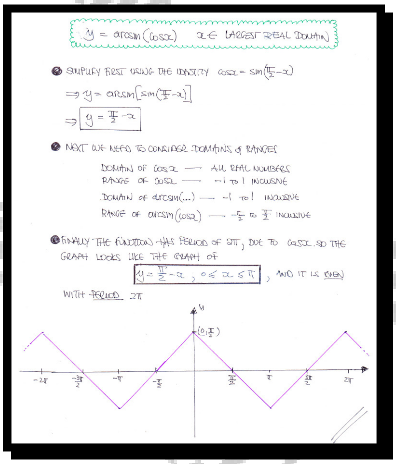
Sketch the graph of

$$f(x) = \arcsin(\cos x),$$

in the largest domain that the function is defined.

Indicate the coordinates of any intersections with the axes, and the coordinates of the cusps of the curve.

$\boxed{}$, $\boxed{\text{graph}}$



$y = \arcsin(\cos x)$ $x \in$ LARGEST REAL DOMAIN

- SIMPLIFY FIRST USING THE IDENTITY $\cos x = \sin\left(\frac{\pi}{2} - x\right)$

$\Rightarrow y = \arcsin\left[\sin\left(\frac{\pi}{2} - x\right)\right]$

$\Rightarrow \boxed{y = \frac{\pi}{2} - x}$

- NEXT WE NEED TO CONSIDER DOMAINS & RANGES

DOMAIN OF $\cos x$ — ALL REAL NUMBERS
RANGE OF $\cos x$ — -1 TO 1 INCLUSIVE
DOMAIN OF $\arcsin(\dots)$ — -1 TO 1 INCLUSIVE
RANGE OF $\arcsin(\cos x)$ — $-\frac{\pi}{2}$ TO $\frac{\pi}{2}$ INCLUSIVE

- FINALLY THE FUNCTION HAS PERIOD OF 2π, DUE TO $\cos x$. SO THE GRAPH LOOKS LIKE THE GRAPH OF

$\boxed{y = \frac{\pi}{2} - x,\ 0 \leqslant x \leqslant \pi}$, AND IT IS EVEN

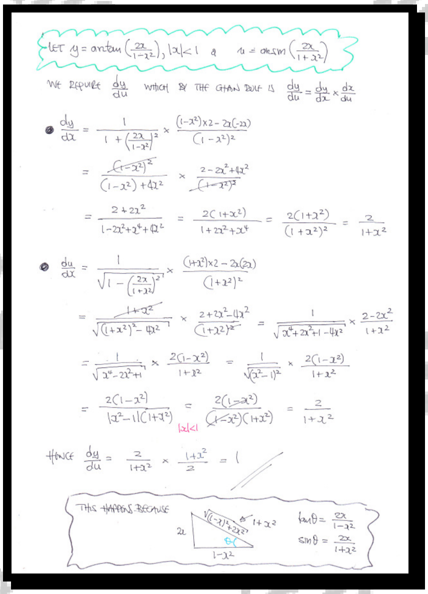
WITH PERIOD 2π

Question 28 (*****)

$$y = \arctan\left(\frac{2x}{1-x^2}\right), \ x \in \mathbb{R}.$$

Differentiate y with respect to $\arcsin\left(\frac{2x}{1+x^2}\right)$, fully simplifying the answer.

$\boxed{\quad\quad}$, $\boxed{1}$



LET $y = \arctan\left(\frac{2x}{1-x^2}\right), |x|<1$ & $u = \arcsin\left(\frac{2x}{1+x^2}\right)$

WE REQUIRE $\frac{dy}{du}$ WHICH BY THE CHAIN RULE IS $\frac{dy}{du} = \frac{dy}{dx} \times \frac{dx}{du}$

- $$\frac{dy}{dx} = \frac{1}{1+\left(\frac{2x}{1-x^2}\right)^2} \times \frac{(1-x^2)\times 2 - 2x(-2x)}{(1-x^2)^2}$$

$$= \frac{\cancel{(1-x^2)^2}}{(1-x^2)+4x^2} \times \frac{2-2x^2+4x^2}{\cancel{(1-x^2)^2}}$$

$$= \frac{2+2x^2}{1-2x^2+x^4+4x^2} = \frac{2(1+x^2)}{1+2x^2+x^4} = \frac{2(1+x^2)}{(1+x^2)^2} = \frac{2}{1+x^2}$$

- $$\frac{du}{dx} = \frac{1}{\sqrt{1-\left(\frac{2x}{1+x^2}\right)^2}} \times \frac{(1+x^2)\times 2 - 2x(2x)}{(1+x^2)^2}$$

$$= \frac{\cancel{1+x^2}}{\sqrt{(1+x^2)^2-4x^2}} \times \frac{2+2x^2-4x^2}{(1+x^2)^{\cancel{2}}} = \frac{1}{\sqrt{x^4+2x^2+1-4x^2}} \times \frac{2-2x^2}{1+x^2}$$

$$= \frac{1}{\sqrt{x^4-2x^2+1}} \times \frac{2(1-x^2)}{1+x^2} = \frac{1}{\sqrt{(x^2-1)^2}} \times \frac{2(1-x^2)}{1+x^2}$$

$$= \frac{2(1-x^2)}{|x^2-1|(1+x^2)} \underset{|x|<1}{=} \frac{2\cancel{(1-x^2)}}{\cancel{(1-x^2)}(1+x^2)} = \frac{2}{1+x^2}$$

HENCE $\frac{dy}{du} = \frac{2}{1+x^2} \times \frac{1+x^2}{2} = 1$ //

THIS HAPPENS BECAUSE

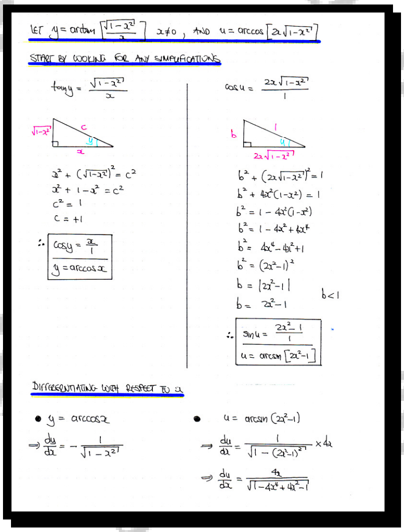
$\tan\theta = \frac{2x}{1-x^2}$

$\sin\theta = \frac{2x}{1+x^2}$

Question 29 (*****)

Differentiate $\arctan\left[\dfrac{\sqrt{1-x^2}}{x}\right]$ with respect to $\arccos\left[2x\sqrt{1-x^2}\right]$, and hence sketch the graph of the resulting gradient function.

$\boxed{}$, $\boxed{-\frac{1}{2}\operatorname{sign}(x)}$



LET $y = \arctan\left[\frac{\sqrt{1-x^2}}{x}\right]$ $x \neq 0$, AND $u = \arccos\left[2x\sqrt{1-x^2}\right]$

START BY LOOKING FOR ANY SIMPLIFICATIONS

$\tan y = \frac{\sqrt{1-x^2}}{x}$

$x^2 + (\sqrt{1-x^2})^2 = c^2$

$x^2 + 1 - x^2 = c^2$

$c^2 = 1$

$c = +1$

$\therefore \boxed{\begin{array}{c}\cos y = \frac{x}{1} \\ \hline y = \arccos x\end{array}}$

$\cos u = \frac{2x\sqrt{1-x^2}}{1}$

$b^2 + (2x\sqrt{1-x^2})^2 = 1$

$b^2 + 4x^2(1-x^2) = 1$

$b^2 = 1 - 4x^2(1-x^2)$

$b^2 = 1 - 4x^2 + 4x^4$

$b^2 = 4x^4 - 4x^2 + 1$

$b^2 = (2x^2-1)^2$

$b = |2x^2-1|$ $\quad b<1$

$b = 2x^2 - 1$

$\therefore \boxed{\begin{array}{c}\sin u = \frac{2x^2-1}{1} \\ \hline u = \arcsin\left[2x^2-1\right]\end{array}}$

DIFFERENTIATING WITH RESPECT TO x

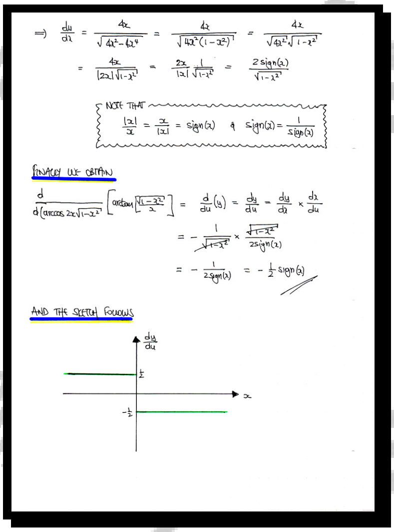
- $y = \arccos x$

$\Rightarrow \frac{dy}{dx} = -\frac{1}{\sqrt{1-x^2}}$

- $u = \arcsin(2x^2-1)$

$\Rightarrow \frac{du}{dx} = \frac{1}{\sqrt{1-(2x^2-1)^2}} \times 4x$

$\Rightarrow \frac{du}{dx} = \frac{4x}{\sqrt{1-4x^4+4x^2-1}}$



$\Rightarrow \frac{du}{dx} = \frac{4x}{\sqrt{4x^2-4x^4}} = \frac{4x}{\sqrt{4x^2(1-x^2)}} = \frac{4x}{\sqrt{4x^2}\sqrt{1-x^2}}$

$= \frac{4x}{|2x|\sqrt{1-x^2}} = \frac{2x}{|x|}\frac{1}{\sqrt{1-x^2}} = \frac{2\operatorname{sign}(x)}{\sqrt{1-x^2}}$

NOTE THAT

$\frac{|x|}{x} = \frac{x}{|x|} = \operatorname{sign}(x)$ & $\operatorname{sign}(x) = \frac{1}{\operatorname{sign}(x)}$

FINALLY WE OBTAIN

$\frac{d}{d(\arccos 2x\sqrt{1-x^2})}\left[\arctan\left[\frac{\sqrt{1-x^2}}{x}\right]\right] = \frac{d}{du}(y) = \frac{dy}{du} = \frac{dy}{dx} \times \frac{dx}{du}$

$= -\frac{1}{\cancel{\sqrt{1-x^2}}} \times \frac{\cancel{\sqrt{1-x^2}}}{2\operatorname{sign}(x)}$

$= -\frac{1}{2\operatorname{sign}(x)} = -\frac{1}{2}\operatorname{sign}(x)$

AND THE SKETCH FOLLOWS

Question 30 (*****)

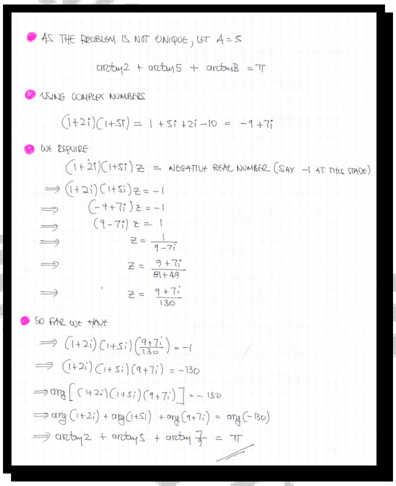
It is given that

$$\arctan 2 + \arctan A + \arctan B = \pi .$$

It is further given that A and B are distinct positive real numbers other than unity.

Determine a pair of possible values for A and B.

$\boxed{}$, $\boxed{5 \ \& \ \frac{7}{9}}$



- AS THE PROBLEM IS NOT UNIQUE, LET $A=5$

$$\arctan 2 + \arctan 5 + \arctan B = \pi$$

- USING COMPLEX NUMBERS

$$(1+2i)(1+5i) = 1+5i+2i-10 = -9+7i$$

- WE REQUIRE

$$(1+2i)(1+5i)z = \text{NEGATIVE REAL NUMBER (SAY } -1 \text{ AT THIS STAGE)}$$

$$\Rightarrow (1+2i)(1+5i)z = -1$$

$$\Rightarrow (-9+7i)z = -1$$

$$\Rightarrow (9-7i)z = 1$$

$$\Rightarrow z = \frac{1}{9-7i}$$

$$\Rightarrow z = \frac{9+7i}{81+49}$$

$$\Rightarrow z = \frac{9+7i}{130}$$

- SO FAR WE HAVE

$$\Rightarrow (1+2i)(1+5i)\left(\frac{9+7i}{130}\right) = -1$$

$$\Rightarrow (1+2i)(1+5i)(9+7i) = -130$$

$$\Rightarrow \arg\left[(1+2i)(1+5i)(9+7i)\right] = -130$$

$$\Rightarrow \arg(1+2i) + \arg(1+5i) + \arg(9+7i) = \arg(-130)$$

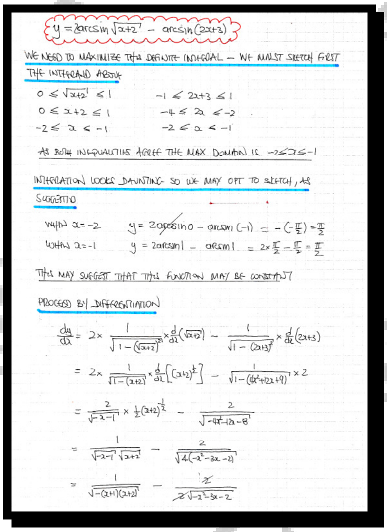
$$\Rightarrow \arctan 2 + \arctan 5 + \arctan \frac{7}{9} = \pi$$

Question 31 (*****)

By sketching the graph of the integrand, or otherwise, determine the maximum value of the following function

$$F(a,b) \equiv \int_a^b 2\arcsin\sqrt{x+2} - \arcsin(2x+3) \; dx .$$

$\boxed{\phantom{\frac{\pi}{2}}}$, $\boxed{\text{proof}}$



$$y = 2\arcsin\sqrt{x+2} - \arcsin(2x+3)$$

WE NEED TO MAXIMIZE THE DEFINITE INTEGRAL – WE MUST SKETCH FIRST THE INTEGRAND ABOVE

$0 \leqslant \sqrt{x+2} \leqslant 1$ $\qquad$ $-1 \leqslant 2x+3 \leqslant 1$

$0 \leqslant x+2 \leqslant 1$ $\qquad$ $-4 \leqslant 2x \leqslant -2$

$-2 \leqslant x \leqslant -1$ $\qquad$ $-2 \leqslant x \leqslant -1$

AS BOTH INEQUALITIES AGREE THE MAX DOMAIN IS $-2 \leqslant x \leqslant -1$

INTEGRATION LOOKS DAUNTING SO WE MAY OPT TO SKETCH, AS SUGGESTED

WHEN $x=-2$ $\qquad$ $y = 2\cancel{\arcsin 0} - \arcsin(-1) = -\left(-\frac{\pi}{2}\right) = \frac{\pi}{2}$

WHEN $x=-1$ $\qquad$ $y = 2\arcsin 1 - \arcsin 1 = 2\times\frac{\pi}{2} - \frac{\pi}{2} = \frac{\pi}{2}$

THIS MAY SUGGEST THAT THIS FUNCTION MAY BE CONSTANT

PROCEED BY DIFFERENTIATION

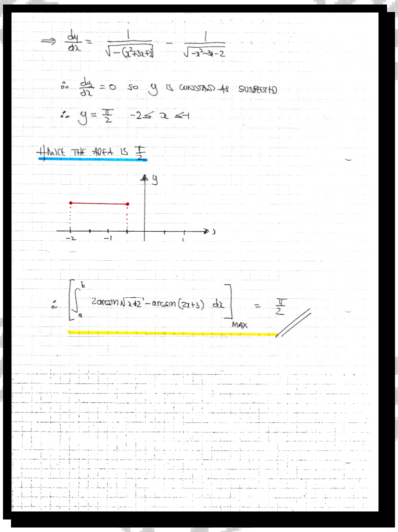
$$\frac{dy}{dx} = 2\times\frac{1}{\sqrt{1-\left(\sqrt{x+2}\right)^2}}\times\frac{d}{dx}\left(\sqrt{x+2}\right) - \frac{1}{\sqrt{1-(2x+3)^2}}\times\frac{d}{dx}(2x+3)$$

$$= 2\times\frac{1}{\sqrt{1-(x+2)}}\times\frac{d}{dx}\left[(x+2)^{\frac{1}{2}}\right] - \frac{1}{\sqrt{1-(4x^2+12x+9)}}\times 2$$

$$= \frac{2}{\sqrt{-x-1}}\times\frac{1}{2}(x+2)^{-\frac{1}{2}} - \frac{2}{\sqrt{-4x^2-12x-8}}$$

$$= \frac{1}{\sqrt{-x-1}\sqrt{x+2}} - \frac{2}{\sqrt{4(-x^2-3x-2)}}$$

$$= \frac{1}{\sqrt{-(x+1)(x+2)}} - \frac{\cancel{2}}{\cancel{2}\sqrt{-x^2-3x-2}}$$



$$\Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{-(x^2+3x+2)}} - \frac{1}{\sqrt{-x^2-3x-2}}$$

$\therefore \frac{dy}{dx} = 0$ SO y IS CONSTANT AS SUSPECTED

$\therefore y = \frac{\pi}{2} \quad -2 \leqslant x \leqslant -1$

HENCE THE AREA IS $\frac{\pi}{2}$

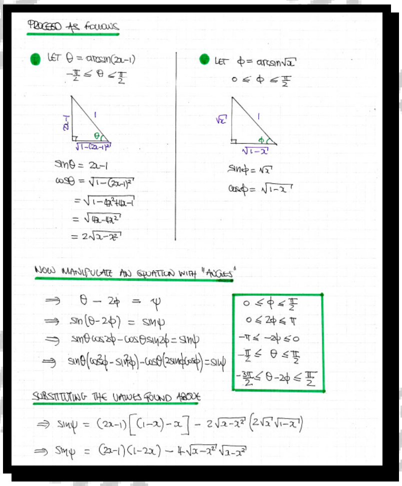
$$\therefore \left[\int_a^b 2\arcsin\sqrt{x+2} - \arcsin(2x+3) \; dx\right]_{\text{MAX}} = \frac{\pi}{2}$$

Question 32 (*****)

If $0 \le x \le 1$, simplify fully

$$\arcsin(2x-1) - 2\arcsin\sqrt{x}\ .$$

V, $\boxed{}$, $\boxed{-\frac{1}{2}\pi}$



PROCEED AS FOLLOWS

- LET $\theta = \arcsin(2x-1)$
  $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

$\sin\theta = 2x-1$
$\cos\theta = \sqrt{1-(2x-1)^2}$
$= \sqrt{1-4x^2+4x-1}$
$= \sqrt{4x-4x^2}$
$= 2\sqrt{x-x^2}$

- LET $\phi = \arcsin\sqrt{x}$
  $0 \le \phi \le \frac{\pi}{2}$

$\sin\phi = \sqrt{x}$
$\cos\phi = \sqrt{1-x}$

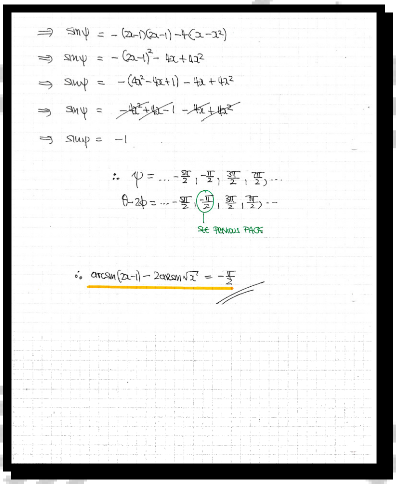
NOW MANIPULATE AN EQUATION WITH "ANGLES"

$\Rightarrow \theta - 2\phi = \psi$
$\Rightarrow \sin(\theta-2\phi) = \sin\psi$
$\Rightarrow \sin\theta\cos2\phi - \cos\theta\sin2\phi = \sin\psi$
$\Rightarrow \sin\theta(\cos^2\phi - \sin^2\phi) - \cos\theta(2\sin\phi\cos\phi) = \sin\psi$

$0 \le \phi \le \frac{\pi}{2}$
$0 \le 2\phi \le \pi$
$-\pi \le -2\phi \le 0$
$-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$
$-\frac{3\pi}{2} \le \theta - 2\phi \le \frac{\pi}{2}$

SUBSTITUTING THE VALUES FOUND ABOVE

$\Rightarrow \sin\psi = (2x-1)\left[(1-x)-x\right] - 2\sqrt{x-x^2}\left(2\sqrt{x}\sqrt{1-x}\right)$
$\Rightarrow \sin\psi = (2x-1)(1-2x) - 4\sqrt{x-x^2}\sqrt{x-x^2}$



$\Rightarrow \sin\psi = -(2x-1)(2x-1) - 4(x-x^2)$
$\Rightarrow \sin\psi = -(2x-1)^2 - 4x + 4x^2$
$\Rightarrow \sin\psi = -(4x^2-4x+1) - 4x + 4x^2$
$\Rightarrow \sin\psi = \cancel{-4x^2} + \cancel{4x} - 1 - \cancel{4x} + \cancel{4x^2}$
$\Rightarrow \sin\psi = -1$

$\therefore \psi = \ldots -\frac{5\pi}{2}, -\frac{\pi}{2}, \frac{3\pi}{2}, \frac{7\pi}{2}, \ldots$
$\theta - 2\phi = \ldots -\frac{5\pi}{2}, \boxed{-\frac{\pi}{2}}, \frac{3\pi}{2}, \frac{7\pi}{2}, \ldots$

SEE PREVIOUS PAGE

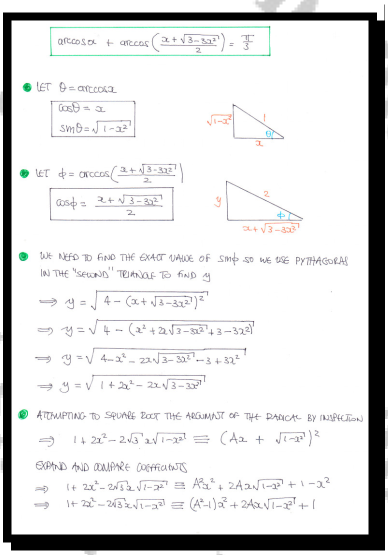
$\therefore \arcsin(2x-1) - 2\arcsin\sqrt{x} = -\frac{\pi}{2}$

Question 33 (*****)

Prove that for all x such that $-1 \le x \le 1$

$$\arccos x + \arccos\left[\frac{1}{2}\left(x+\sqrt{3-3x^2}\right)\right] = \frac{\pi}{3}.$$

$\boxed{}$, $\boxed{\text{proof}}$



$$\boxed{\arccos x + \arccos\left(\frac{x+\sqrt{3-3x^2}}{2}\right) = \frac{\pi}{3}}$$

- LET $\theta = \arccos x$

$$\boxed{\begin{aligned}\cos\theta &= x\\ \sin\theta &= \sqrt{1-x^2}\end{aligned}}$$

- LET $\phi = \arccos\left(\frac{x+\sqrt{3-3x^2}}{2}\right)$

$$\boxed{\cos\phi = \frac{x+\sqrt{3-3x^2}}{2}}$$

- WE NEED TO FIND THE EXACT VALUE OF $\sin\phi$ SO WE USE PYTHAGORAS IN THE "SECOND" TRIANGLE TO FIND y

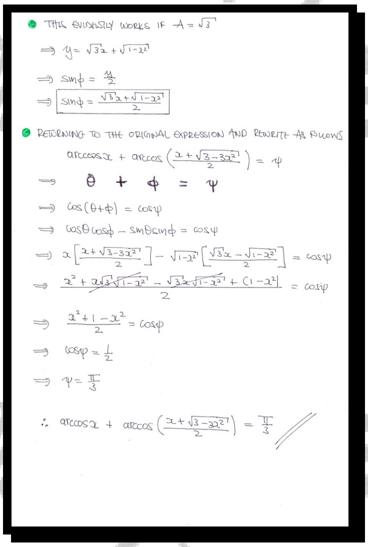
$$\Rightarrow y = \sqrt{4-\left(x+\sqrt{3-3x^2}\right)^2}$$
$$\Rightarrow y = \sqrt{4-\left(x^2+2x\sqrt{3-3x^2}+3-3x^2\right)}$$
$$\Rightarrow y = \sqrt{4-x^2-2x\sqrt{3-3x^2}-3+3x^2}$$
$$\Rightarrow y = \sqrt{1+2x^2-2x\sqrt{3-3x^2}}$$

- ATTEMPTING TO SQUARE ROOT THE ARGUMENT OF THE RADICAL BY INSPECTION

$$\Rightarrow 1+2x^2-2\sqrt{3}\,x\sqrt{1-x^2} \equiv \left(Ax+\sqrt{1-x^2}\right)^2$$

EXPAND AND COMPARE COEFFICIENTS

$$\Rightarrow 1+2x^2-2\sqrt{3}\,x\sqrt{1-x^2} \equiv A^2x^2+2Ax\sqrt{1-x^2}+1-x^2$$
$$\Rightarrow 1+2x^2-2\sqrt{3}\,x\sqrt{1-x^2} \equiv (A^2-1)x^2+2Ax\sqrt{1-x^2}+1$$



- THIS EVIDENTLY WORKS IF $A=\sqrt{3}$

$$\Rightarrow y = \sqrt{3}\,x+\sqrt{1-x^2}$$
$$\Rightarrow \sin\phi = \frac{y}{2}$$
$$\Rightarrow \boxed{\sin\phi = \frac{\sqrt{3}\,x+\sqrt{1-x^2}}{2}}$$

- RETURNING TO THE ORIGINAL EXPRESSION AND REWRITE AS FOLLOWS

$$\arccos x + \arccos\left(\frac{x+\sqrt{3-3x^2}}{2}\right) = \psi$$
$$\Rightarrow \theta + \phi = \psi$$
$$\Rightarrow \cos(\theta+\phi) = \cos\psi$$
$$\Rightarrow \cos\theta\cos\phi - \sin\theta\sin\phi = \cos\psi$$
$$\Rightarrow x\left[\frac{x+\sqrt{3-3x^2}}{2}\right] - \sqrt{1-x^2}\left[\frac{\sqrt{3}\,x-\sqrt{1-x^2}}{2}\right] = \cos\psi$$
$$\Rightarrow \frac{x^2+\cancel{x\sqrt{3}\sqrt{1-x^2}}-\cancel{\sqrt{3}\,x\sqrt{1-x^2}}+(1-x^2)}{2} = \cos\psi$$
$$\Rightarrow \frac{x^2+1-x^2}{2} = \cos\psi$$
$$\Rightarrow \cos\psi = \frac{1}{2}$$
$$\Rightarrow \psi = \frac{\pi}{3}$$

$$\therefore \arccos x + \arccos\left(\frac{x+\sqrt{3-3x^2}}{2}\right) = \frac{\pi}{3} \;//$$

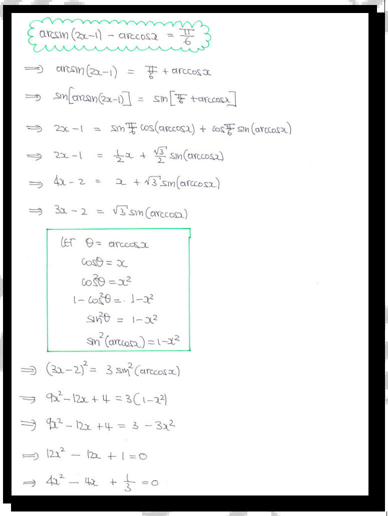
Question 34 (*****)

Find, in exact surd form, the only real solution of the following trigonometric equation

$$\arcsin(2x-1)-\arccos x=\frac{\pi}{6}.$$

The rejection of any additional solutions must be fully justified.

$\boxed{}\ ,\ \boxed{x=\frac{1}{2}-\frac{1}{6}\sqrt{6}}$



$\arcsin(2x-1)-\arccos x=\frac{\pi}{6}$

$\Rightarrow \arcsin(2x-1)=\frac{\pi}{6}+\arccos x$

$\Rightarrow \sin[\arcsin(2x-1)]=\sin[\frac{\pi}{6}+\arccos x]$

$\Rightarrow 2x-1=\sin\frac{\pi}{6}\cos(\arccos x)+\cos\frac{\pi}{6}\sin(\arccos x)$

$\Rightarrow 2x-1=\frac{1}{2}x+\frac{\sqrt{3}}{2}\sin(\arccos x)$

$\Rightarrow 4x-2=x+\sqrt{3}\sin(\arccos x)$

$\Rightarrow 3x-2=\sqrt{3}\sin(\arccos x)$

LET $\theta=\arccos x$
$\cos\theta=x$
$\cos^2\theta=x^2$
$1-\cos^2\theta=1-x^2$
$\sin^2\theta=1-x^2$
$\sin^2(\arccos x)=1-x^2$

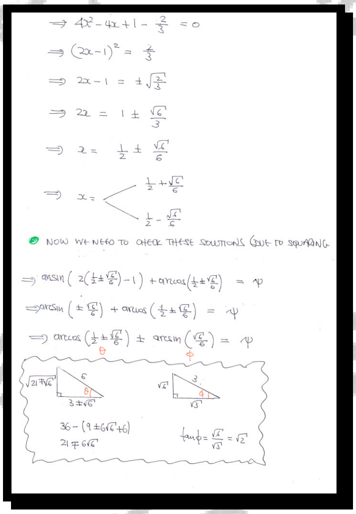
$\Rightarrow (3x-2)^2=3\sin^2(\arccos x)$

$\Rightarrow 9x^2-12x+4=3(1-x^2)$

$\Rightarrow 9x^2-12x+4=3-3x^2$

$\Rightarrow 12x^2-12x+1=0$

$\Rightarrow 4x^2-4x+\frac{1}{3}=0$



$\Rightarrow 4x^2-4x+1-\frac{2}{3}=0$

$\Rightarrow (2x-1)^2=\frac{2}{3}$

$\Rightarrow 2x-1=\pm\sqrt{\frac{2}{3}}$

$\Rightarrow 2x=1\pm\frac{\sqrt{6}}{3}$

$\Rightarrow x=\frac{1}{2}\pm\frac{\sqrt{6}}{6}$

$\Rightarrow x=\begin{cases}\frac{1}{2}+\frac{\sqrt{6}}{6}\\ \frac{1}{2}-\frac{\sqrt{6}}{6}\end{cases}$

- NOW WE NEED TO CHECK THESE SOLUTIONS (DUE TO SQUARING

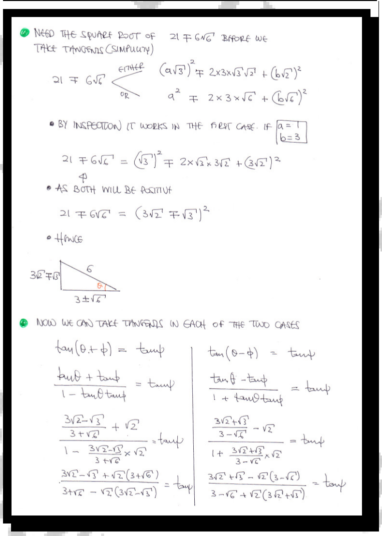
$\Rightarrow \arcsin\left(2\left(\frac{1}{2}\pm\frac{\sqrt{6}}{6}\right)-1\right)+\arccos\left(\frac{1}{2}\pm\frac{\sqrt{6}}{6}\right)=\psi$

$\Rightarrow \arcsin\left(\pm\frac{\sqrt{6}}{6}\right)+\arccos\left(\frac{1}{2}\pm\frac{\sqrt{6}}{6}\right)=\psi$

$\Rightarrow \underset{\theta}{\arccos\left(\frac{1}{2}\pm\frac{\sqrt{6}}{6}\right)}\pm\underset{\phi}{\arcsin\left(\frac{\sqrt{6}}{6}\right)}=\psi$

$36-(9\pm6\sqrt{6}+6)$
$21\mp6\sqrt{6}$

$\tan\phi=\frac{\sqrt{6}}{\sqrt{3}}=\sqrt{2}$



- NEED THE SQUARE ROOT OF $21\mp6\sqrt{6}$ BEFORE WE TAKE TANGENTS (SIMPLICITY)

$21\mp6\sqrt{6}$ EITHER $(a\sqrt{3})^2\mp2\times3\times\sqrt{3}\sqrt{2}+(b\sqrt{2})^2$ OR $a^2\mp2\times3\times\sqrt{6}+(b\sqrt{6})^2$

- BY INSPECTION IT WORKS IN THE FIRST CASE. IF $\boxed{\begin{matrix}a=1\\b=3\end{matrix}}$

$21\mp6\sqrt{6}=(\sqrt{3})^2\mp2\times\sqrt{3}\times3\sqrt{2}+(3\sqrt{2})^2$

- AS BOTH WILL BE POSITIVE

$21\mp6\sqrt{6}=(3\sqrt{2}\mp\sqrt{3})^2$

- HENCE

- NOW WE CAN TAKE TANGENTS IN EACH OF THE TWO CASES

$\tan(\theta+\phi)=\tan\psi$

$\frac{\tan\theta+\tan\phi}{1-\tan\theta\tan\phi}=\tan\psi$

$\frac{\frac{3\sqrt{2}-\sqrt{3}}{3+\sqrt{6}}+\sqrt{2}}{1-\frac{3\sqrt{2}-\sqrt{3}}{3+\sqrt{6}}\times\sqrt{2}}=\tan\psi$

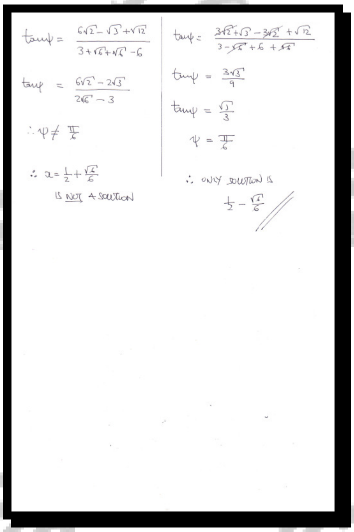
$\frac{3\sqrt{2}-\sqrt{3}+\sqrt{2}(3+\sqrt{6})}{3+\sqrt{6}-\sqrt{2}(3\sqrt{2}-\sqrt{3})}=\tan\psi$

$\tan(\theta-\phi)=\tan\psi$

$\frac{\tan\theta-\tan\phi}{1+\tan\theta\tan\phi}=\tan\psi$

$\frac{\frac{3\sqrt{2}+\sqrt{3}}{3-\sqrt{6}}-\sqrt{2}}{1+\frac{3\sqrt{2}+\sqrt{3}}{3-\sqrt{6}}\times\sqrt{2}}=\tan\psi$

$\frac{3\sqrt{2}+\sqrt{3}-\sqrt{2}(3-\sqrt{6})}{3-\sqrt{6}+\sqrt{2}(3\sqrt{2}+\sqrt{3})}=\tan\psi$



$\tan\psi=\frac{6\sqrt{2}-\sqrt{3}+\sqrt{12}}{3+\sqrt{6}+\sqrt{6}-6}$

$\tan\psi=\frac{6\sqrt{2}-2\sqrt{3}}{2\sqrt{6}-3}$

$\therefore\psi\neq\frac{\pi}{6}$

$\therefore x=\frac{1}{2}+\frac{\sqrt{6}}{6}$ IS NOT A SOLUTION

$\tan\psi=\frac{\cancel{3\sqrt{2}}+\sqrt{3}-\cancel{3\sqrt{2}}+\sqrt{12}}{3-\cancel{\sqrt{6}}+6+\cancel{\sqrt{6}}}$

$\tan\psi=\frac{3\sqrt{3}}{9}$

$\tan\psi=\frac{\sqrt{3}}{3}$

$\psi=\frac{\pi}{6}$

$\therefore$ ONLY SOLUTION IS

$\frac{1}{2}-\frac{\sqrt{6}}{6}$ //

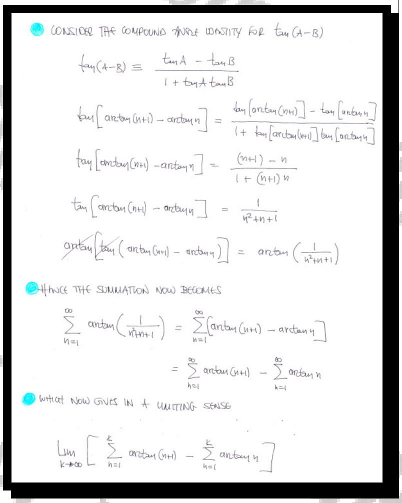
Question 35 (*****)

By considering the trigonometric identity for $\tan(A-B)$, with $A = \arctan(n+1)$ and $B = \arctan(n)$, sum the following series

$$\sum_{n=1}^{\infty} \arctan\left(\frac{1}{n^2+n+1}\right).$$

You may assume the series converges.

$\boxed{\quad}$, $\boxed{\dfrac{\pi}{4}}$



CONSIDER THE COMPOUND ANGLE IDENTITY FOR $\tan(A-B)$

$$\tan(A-B) \equiv \frac{\tan A - \tan B}{1+\tan A \tan B}$$

$$\tan\left[\arctan(n+1) - \arctan n\right] = \frac{\tan\left[\arctan(n+1)\right] - \tan\left[\arctan n\right]}{1+\tan\left[\arctan(n+1)\right]\tan\left[\arctan n\right]}$$

$$\tan\left[\arctan(n+1) - \arctan n\right] = \frac{(n+1)-n}{1+(n+1)n}$$

$$\tan\left[\arctan(n+1) - \arctan n\right] = \frac{1}{n^2+n+1}$$

$$\cancel{\arctan}\left[\cancel{\tan}\left(\arctan(n+1) - \arctan n\right)\right] = \arctan\left(\frac{1}{n^2+n+1}\right)$$

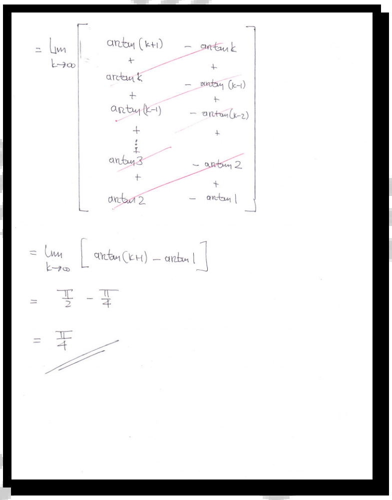
HENCE THE SUMMATION NOW BECOMES

$$\sum_{n=1}^{\infty} \arctan\left(\frac{1}{n^2+n+1}\right) = \sum_{n=1}^{\infty}\left[\arctan(n+1) - \arctan n\right]$$

$$= \sum_{n=1}^{\infty} \arctan(n+1) - \sum_{k=1}^{\infty} \arctan n$$

WHICH NOW GIVES IN A LIMITING SENSE

$$\lim_{k\to\infty}\left[\sum_{n=1}^{k} \arctan(n+1) - \sum_{n=1}^{k} \arctan n\right]$$



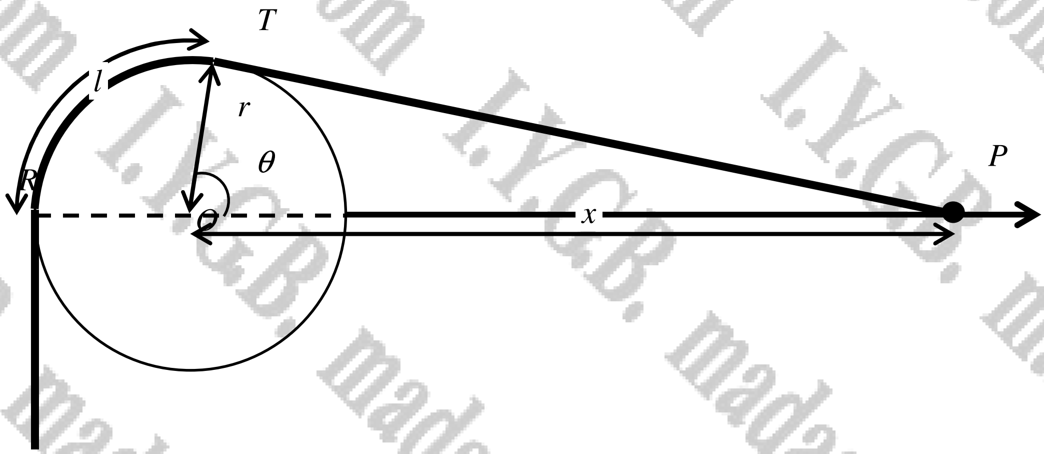
$$= \lim_{k\to\infty}\left[\begin{array}{cc} \arctan(k+1) & -\arctan k \\ + & + \\ \arctan k & -\arctan(k-1) \\ + & + \\ \arctan(k-1) & -\arctan(k-2) \\ + & + \\ \vdots & \\ \arctan 3 & -\arctan 2 \\ + & + \\ \arctan 2 & -\arctan 1 \end{array}\right]$$

$$= \lim_{k\to\infty}\left[\arctan(k+1) - \arctan 1\right]$$

$$= \frac{\pi}{2} - \frac{\pi}{4}$$

$$= \frac{\pi}{4}$$

Question 36 (*****)



A circular wheel of radius r and centre at the origin O of a positive x axis. A particle P is constrained to move on the positive x axis, so that the distance OP is x. The particle is connected to a taut cable which runs over the wheel and hangs vertically down on the other side of the wheel as shown in the figure above. The section of the cable RT, which is in contact with the wheel has length l. The section of the cable TP is a straight line.

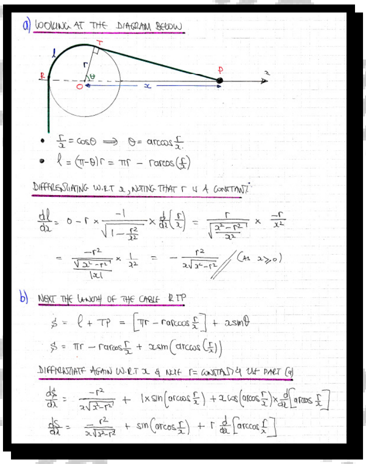
a) Given that the angle $TOP = \theta$ show that

$$\frac{dl}{dx} = -\frac{r^2}{x\sqrt{x^2 - r^2}}.$$

Let $s = l + |TP|$ and suppose that P is moving in the positive x direction with constant speed 2 units per unit time.

b) Find the rate at which s is increasing when P is at a distance of $2r$ from O.

$\boxed{\quad\quad}$, $\boxed{\sqrt{3}}$



a) LOOKING AT THE DIAGRAM BELOW

- $\frac{r}{x} = \cos\theta \implies \theta = \arccos\frac{r}{x}.$
- $l = (\pi - \theta)r = \pi r - r\arccos\left(\frac{r}{x}\right)$

DIFFERENTIATING W.R.T x, NOTING THAT r IS A CONSTANT.

$$\frac{dl}{dx} = 0 - r \times \frac{-1}{\sqrt{1 - \frac{r^2}{x^2}}} \times \frac{d}{dx}\left(\frac{r}{x}\right) = \frac{r}{\sqrt{\frac{x^2 - r^2}{x^2}}} \times \frac{-r}{x^2}$$

$$= \frac{-r^2}{\frac{\sqrt{x^2 - r^2}}{|x|}} \times \frac{1}{x^2} = -\frac{r^2}{x\sqrt{x^2 - r^2}} \quad // \text{(As } x \geqslant 0)$$

b) NEXT THE LENGTH OF THE CABLE RTP

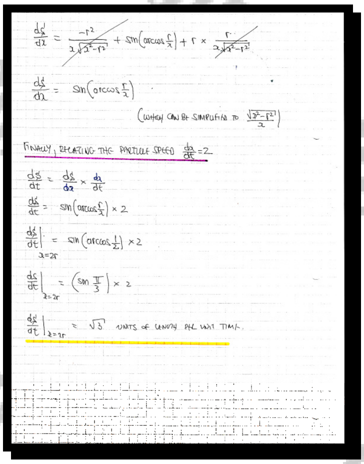
$$s = l + TP = \left[\pi r - r\arccos\frac{r}{x}\right] + x\sin\theta$$

$$s = \pi r - r\arccos\frac{r}{x} + x\sin\left(\arccos\left(\frac{r}{x}\right)\right)$$

DIFFERENTIATE AGAIN W.R.T x & NOTE r = CONSTANT & 1st PART (a)

$$\frac{ds}{dx} = \frac{-r^2}{x\sqrt{x^2 - r^2}} + 1 \times \sin\left(\arccos\frac{r}{x}\right) + x\cos\left(\arccos\frac{r}{x}\right) \times \frac{d}{dx}\left[\arccos\frac{r}{x}\right]$$

$$\frac{ds}{dx} = \frac{-r^2}{x\sqrt{x^2 - r^2}} + \sin\left(\arccos\frac{r}{x}\right) + r\frac{d}{dx}\left[\arccos\frac{r}{x}\right]$$



$$\frac{ds}{dx} = \cancel{\frac{-r^2}{x\sqrt{x^2 - r^2}}} + \sin\left(\arccos\frac{r}{x}\right) + r \times \cancel{\frac{r}{x\sqrt{x^2 - r^2}}}$$

$$\frac{ds}{dx} = \sin\left(\arccos\frac{r}{x}\right)$$

(WHICH CAN BE SIMPLIFIED TO $\frac{\sqrt{x^2 - r^2}}{x}$)

FINALLY, RELATING THE PARTICLE SPEED $\frac{dx}{dt} = 2$

$$\frac{ds}{dt} = \frac{ds}{dx} \times \frac{dx}{dt}$$

$$\frac{ds}{dt} = \sin\left(\arccos\frac{r}{x}\right) \times 2$$

$$\left.\frac{ds}{dt}\right|_{x=2r} = \sin\left(\arccos\frac{1}{2}\right) \times 2$$

$$\left.\frac{ds}{dt}\right|_{x=2r} = \left(\sin\frac{\pi}{3}\right) \times 2$$

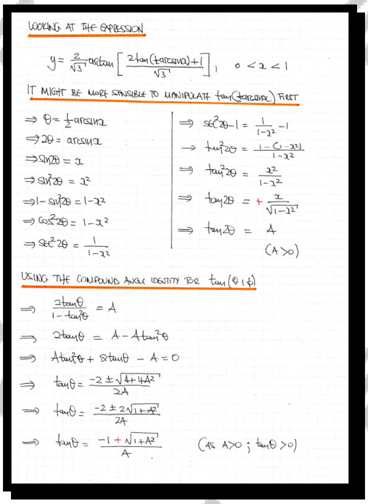
$$\left.\frac{ds}{dt}\right|_{x=2r} = \sqrt{3} \text{ UNITS OF LENGTH PER UNIT TIME.}$$

Question 37 (*****)

Prove that if $0<x<1$

$$\frac{d}{dx}\left[\frac{2}{\sqrt{3}}\arctan\left[\frac{1+2\tan\left(\frac{1}{2}\arcsin x\right)}{\sqrt{3}}\right]\right] \equiv \frac{1}{(2+x)\sqrt{1-x^2}}.$$

V , , proof



LOOKING AT THE EXPRESSION

$$y=\frac{2}{\sqrt{3}}\arctan\left[\frac{2\tan\left(\frac{1}{2}\arcsin x\right)+1}{\sqrt{3}}\right], \quad 0<x<1$$

IT MIGHT BE MORE SENSIBLE TO MANIPULATE $\tan\left(\frac{1}{2}\arcsin x\right)$ FIRST

$\Rightarrow \theta=\frac{1}{2}\arcsin x$

$\Rightarrow 2\theta=\arcsin x$

$\Rightarrow \sin 2\theta = x$

$\Rightarrow \sin^2 2\theta = x^2$

$\Rightarrow 1-\sin^2 2\theta = 1-x^2$

$\Rightarrow \cos^2 2\theta = 1-x^2$

$\Rightarrow \sec^2 2\theta = \frac{1}{1-x^2}$

$\Rightarrow \sec^2 2\theta - 1 = \frac{1}{1-x^2}-1$

$\Rightarrow \tan^2 2\theta = \frac{1-(1-x^2)}{1-x^2}$

$\Rightarrow \tan^2 2\theta = \frac{x^2}{1-x^2}$

$\Rightarrow \tan 2\theta = + \frac{x}{\sqrt{1-x^2}}$

$\Rightarrow \tan 2\theta = A$

$(A>0)$

USING THE COMPOUND ANGLE IDENTITY FOR $\tan(\theta+\phi)$

$\Rightarrow \frac{2\tan\theta}{1-\tan^2\theta} = A$

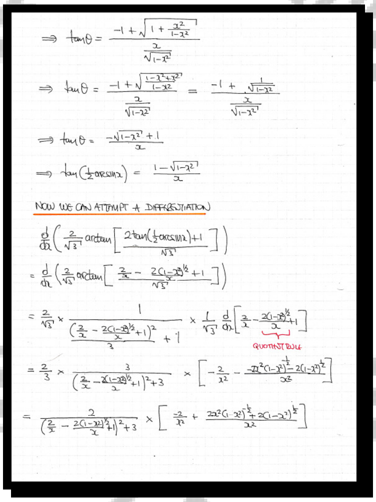
$\Rightarrow 2\tan\theta = A - A\tan^2\theta$

$\Rightarrow A\tan^2\theta + 2\tan\theta - A = 0$

$\Rightarrow \tan\theta = \frac{-2\pm\sqrt{4+4A^2}}{2A}$

$\Rightarrow \tan\theta = \frac{-2\pm 2\sqrt{1+A^2}}{2A}$

$\Rightarrow \tan\theta = \frac{-1+\sqrt{1+A^2}}{A}$ (AS $A>0$; $\tan\theta>0$)



$\Rightarrow \tan\theta = \frac{-1+\sqrt{1+\frac{x^2}{1-x^2}}}{\frac{x}{\sqrt{1-x^2}}}$

$\Rightarrow \tan\theta = \frac{-1+\sqrt{\frac{1-x^2+x^2}{1-x^2}}}{\frac{x}{\sqrt{1-x^2}}} = \frac{-1+\frac{1}{\sqrt{1-x^2}}}{\frac{x}{\sqrt{1-x^2}}}$

$\Rightarrow \tan\theta = \frac{-\sqrt{1-x^2}+1}{x}$

$\Rightarrow \tan\left(\frac{1}{2}\arcsin x\right) = \frac{1-\sqrt{1-x^2}}{x}$

NOW WE CAN ATTEMPT A DIFFERENTIATION

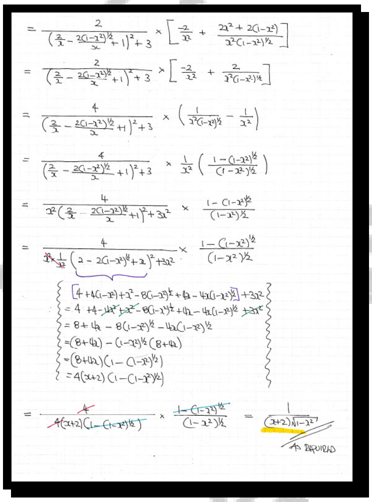
$\frac{d}{dx}\left(\frac{2}{\sqrt{3}}\arctan\left[\frac{2\tan\left(\frac{1}{2}\arcsin x\right)+1}{\sqrt{3}}\right]\right)$

$= \frac{d}{dx}\left(\frac{2}{\sqrt{3}}\arctan\left[\frac{\frac{2}{x}-\frac{2(1-x^2)^{\frac{1}{2}}}{x}+1}{\sqrt{3}}\right]\right)$

$= \frac{2}{\sqrt{3}} \times \frac{1}{\frac{\left(\frac{2}{x}-\frac{2(1-x^2)^{\frac{1}{2}}}{x}+1\right)^2}{3}+1} \times \frac{1}{\sqrt{3}}\frac{d}{dx}\left[\frac{2}{x}-\underbrace{\frac{2(1-x^2)^{\frac{1}{2}}}{x}}_{\text{QUOTIENT RULE}}+1\right]$

$= \frac{2}{3} \times \frac{3}{\left(\frac{2}{x}-\frac{2(1-x^2)^{\frac{1}{2}}}{x}+1\right)^2+3} \times \left[-\frac{2}{x^2}-\frac{-x^2(1-x^2)^{-\frac{1}{2}}-2(1-x^2)^{\frac{1}{2}}}{x^2}\right]$

$= \frac{2}{\left(\frac{2}{x}-\frac{2(1-x^2)^{\frac{1}{2}}}{x}+1\right)^2+3} \times \left[\frac{-2}{x^2}+\frac{2x^2(1-x^2)^{-\frac{1}{2}}+2(1-x^2)^{\frac{1}{2}}}{x^2}\right]$



$= \frac{2}{\left(\frac{2}{x}-\frac{2(1-x^2)^{\frac{1}{2}}}{x}+1\right)^2+3} \times \left[\frac{-2}{x^2}+\frac{2x^2+2(1-x^2)}{x^2(1-x^2)^{\frac{1}{2}}}\right]$

$= \frac{2}{\left(\frac{2}{x}-\frac{2(1-x^2)^{\frac{1}{2}}}{x}+1\right)^2+3} \times \left[\frac{-2}{x^2}+\frac{2}{x^2(1-x^2)^{\frac{1}{2}}}\right]$

$= \frac{4}{\left(\frac{2}{x}-\frac{2(1-x^2)^{\frac{1}{2}}}{x}+1\right)^2+3} \times \left(\frac{1}{x^2(1-x^2)^{\frac{1}{2}}}-\frac{1}{x^2}\right)$

$= \frac{4}{\left(\frac{2}{x}-\frac{2(1-x^2)^{\frac{1}{2}}}{x}+1\right)^2+3} \times \frac{1}{x^2}\left(\frac{1-(1-x^2)^{\frac{1}{2}}}{(1-x^2)^{\frac{1}{2}}}\right)$

$= \frac{4}{x^2\left(\frac{2}{x}-\frac{2(1-x^2)^{\frac{1}{2}}}{x}+1\right)^2+3x^2} \times \frac{1-(1-x^2)^{\frac{1}{2}}}{(1-x^2)^{\frac{1}{2}}}$

$= \frac{4}{\cancel{x^2}\times\frac{1}{\cancel{x^2}}\underbrace{\left(2-2(1-x^2)^{\frac{1}{2}}+x\right)^2+3x^2}} \times \frac{1-(1-x^2)^{\frac{1}{2}}}{(1-x^2)^{\frac{1}{2}}}$

$\left[4+4(1-x^2)+x^2-8(1-x^2)^{\frac{1}{2}}+4x-4x(1-x^2)^{\frac{1}{2}}\right]+3x^2$

$= 4+4-\cancel{4x^2}\cancel{+x^2}-8(1-x^2)^{\frac{1}{2}}+4x-4x(1-x^2)^{\frac{1}{2}}\cancel{+3x^2}$

$= 8+4x-8(1-x^2)^{\frac{1}{2}}-4x(1-x^2)^{\frac{1}{2}}$

$=(8+4x)-(1-x^2)^{\frac{1}{2}}(8+4x)$

$=(8+4x)\left(1-(1-x^2)^{\frac{1}{2}}\right)$

$=4(x+2)\left(1-(1-x^2)^{\frac{1}{2}}\right)$

$= \frac{\cancel{4}}{\cancel{4}(x+2)\cancel{\left(1-(1-x^2)^{\frac{1}{2}}\right)}} \times \frac{\cancel{1-(1-x^2)^{\frac{1}{2}}}}{(1-x^2)^{\frac{1}{2}}} = \frac{1}{(x+2)\sqrt{1-x^2}}$

AS REQUIRED