

Created by T. Madas

# 470

## TRIGONOMETRY

### EXAM QUESTIONS

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# 32

## BASIC QUESTIONS



**Question 1** (\*\*)

Given that  $\cos x = \sqrt{2} - 1$ , show clearly that

$$\cos 2x = 5 - 4\sqrt{2}.$$

proof

$$\begin{aligned}\cos 2x &= 2\cos^2 x - 1 \\ \cos 2x &= 2(\sqrt{2} - 1)^2 - 1 \\ \cos 2x &= 2(2 - 2\sqrt{2} + 1) - 1 \\ \cos 2x &= 4 - 4\sqrt{2} + 2 - 1 \\ \cos 2x &= 5 - 4\sqrt{2}\end{aligned}$$

**Question 2** (\*\*)

Show clearly that

$$\frac{\cos(x-y)}{\sin y \cos y} \equiv \frac{\cos x}{\sin y} + \frac{\sin x}{\cos y}.$$

proof

$$\begin{aligned}\text{LHS} &= \frac{\cos(x-y)}{\sin y \cos y} \\ &= \frac{\cos x \cos y + \sin x \sin y}{\sin y \cos y} \\ &= \frac{\cos x \cos y}{\sin y \cos y} + \frac{\sin x \sin y}{\sin y \cos y} \\ &= \frac{\cos x}{\sin y} + \frac{\sin x}{\cos y} \\ &= \text{RHS}\end{aligned}$$

**Question 3** (\*\*+)

Prove the validity of the trigonometric identity

$$\tan 2\theta \sec \theta \equiv 2 \sin \theta \sec 2\theta.$$

proof

$$\begin{aligned}\text{LHS} &= \tan 2\theta \sec \theta \\ &= \frac{\sin 2\theta}{\cos 2\theta} \times \frac{1}{\cos \theta} \\ &= \frac{\sin 2\theta}{\cos \theta \cos 2\theta} \\ &= \frac{2 \sin \theta \cos \theta}{\cos \theta \cos 2\theta} \times \sec 2\theta \\ &= 2 \sin \theta \sec 2\theta \\ &= \text{RHS}\end{aligned}$$

**Question 4** (\*\*+)

If  $\sin x = \frac{3}{5}$ , show clearly that

$$\sec 2x = \frac{25}{7}.$$

 , proof

$$\sec 2x = \frac{1}{\cos 2x} = \frac{1}{1-2\sin^2 x} = \frac{1}{1-2\left(\frac{3}{5}\right)^2} = \frac{1}{1-\frac{18}{25}} = \frac{25}{7}$$

**Question 5** (\*\*+)

Solve the following trigonometric equation

$$\cos x \cot x + \sin x + 2 \cot x = 0, \quad 0 \leq x < 360^\circ, \quad x \neq 0^\circ, 180^\circ.$$

$$x = 120^\circ, 240^\circ$$

$$\begin{aligned} \cos x \cot x + \sin x + 2 \cot x &= 0 \\ \Rightarrow \cos x \times \frac{\cos x}{\sin x} + \sin x + 2 \left( \frac{\cos x}{\sin x} \right) &= 0 \\ \Rightarrow \frac{\cos^2 x}{\sin x} + \sin x + \frac{2 \cos x}{\sin x} &= 0 \\ \Rightarrow \cos^2 x + \sin^2 x + 2 \cos x &= 0 \\ \Rightarrow 1 + 2 \cos x &= 0 \\ \Rightarrow 2 \cos x &= -1 \\ \Rightarrow \cos x &= -\frac{1}{2} \end{aligned} \quad \begin{aligned} \arccos\left(-\frac{1}{2}\right) &= 120^\circ \\ x &= 120^\circ + 360^\circ n \\ x &= 240^\circ + 360^\circ n \quad n=0,1,2,\dots \\ x_1 &= 120^\circ \\ x_2 &= 240^\circ \end{aligned}$$

## Question 6 (\*\*+)

Simplify fully the following trigonometric expression

$$\frac{\sqrt{2} \cos x^\circ - 2 \sin(45 - x)^\circ}{2 \sin(60 + x)^\circ - \sqrt{3} \cos x^\circ}$$

$$\sqrt{2}$$

$$\begin{aligned} \frac{\sqrt{2} \cos x - 2 \sin(45 - x)}{2 \sin(60 + x) - \sqrt{3} \cos x} &= \frac{\sqrt{2} \cos x - 2 [\sin 45 \cos x - \cos 45 \sin x]}{2 [\sin 60 \cos x + \cos 60 \sin x] - \sqrt{3} \cos x} \\ &= \frac{\sqrt{2} \cos x - 2 \left[ \frac{\sqrt{2}}{2} \cos x - \frac{\sqrt{2}}{2} \sin x \right]}{2 \left[ \frac{\sqrt{3}}{2} \cos x + \frac{1}{2} \sin x \right] - \sqrt{3} \cos x} = \frac{\sqrt{2} \cos x - \sqrt{2} \cos x + \sqrt{2} \sin x}{\sqrt{3} \cos x + \sin x - \sqrt{3} \cos x} \\ &= \frac{\sqrt{2} \sin x}{\sin x} = \sqrt{2} \end{aligned}$$

## Question 7 (\*\*+)

Solve the following trigonometric equation

$$\pi - 3 \arccos(x+1) = 0.$$

$$\boxed{\phantom{0}}, \quad x = -\frac{1}{2}$$

$$\begin{aligned} \pi - 3 \arccos(x+1) &= 0 \\ 3 \arccos(x+1) &= \pi \\ \arccos(x+1) &= \frac{\pi}{3} \\ \cos[\arccos(x+1)] &= \cos\left(\frac{\pi}{3}\right) \\ x+1 &= \frac{1}{2} \\ x &= -\frac{1}{2} \end{aligned}$$

**Question 8 (\*\*+)**

The angle  $x$  is acute so that  $\tan x = \frac{1}{2}$ .

- a) Find the exact value of  $\operatorname{cosec} x$ .

It is further given that  $\tan(x+y) = 2$ , where  $y$  is another angle.

- b) Determine the value of  $\tan y$ .

$$\boxed{\frac{1}{\sqrt{5}}}, \operatorname{cosec} x = \sqrt{5}, \tan y = \frac{3}{4}$$

**Question 9 (\*\*\*)**

$$\operatorname{cosec} \theta + 8 \cos \theta = 0, 0^\circ \leq \theta < 360^\circ.$$

Find the solutions of the above trigonometric equation, giving the answers in degrees correct to one decimal place.

$$\boxed{\theta = 97.2^\circ, 172.8^\circ, 277.2^\circ, 352.8^\circ}$$

## Question 10 (\*\*\*)

$$5 \sin 3x \cos x + 5 \cos 3x \sin x = 4, \quad 0 \leq x < \pi.$$

Use a compound angle trigonometric identity to find the solutions of the above trigonometric equation, giving the answers in radians correct to two decimal places.

$$\boxed{\phantom{0.23}}, \quad \boxed{x = 0.23^\circ, 0.55^\circ, 1.80^\circ, 2.12^\circ}$$

$5 \sin 3x \cos x + 5 \cos 3x \sin x = 4$   
 $\Rightarrow \sin(3x+x) = \frac{4}{5}$   
 $\Rightarrow \sin(4x) = \frac{4}{5}$   
 $\Rightarrow \sin \alpha = \frac{4}{5}$   
 $\alpha = 0.927 \pm 2\pi n$   
 $4x = 0.927 \pm 2\pi n$   
 $x = 0.232 \pm \frac{\pi n}{2}$   
 $x = 0.23^\circ, 0.55^\circ, 1.80^\circ, 2.12^\circ$

## Question 11 (\*\*\*)

Prove the validity of the trigonometric identity

$$\frac{1 + \tan^2 x}{1 - \tan^2 x} \equiv \sec 2x.$$

proof

$LHS = \frac{1 + \tan^2 x}{1 - \tan^2 x} = \frac{1 + \frac{\sin^2 x}{\cos^2 x}}{1 - \frac{\sin^2 x}{\cos^2 x}} = \frac{\frac{\cos^2 x + \sin^2 x}{\cos^2 x}}{\frac{\cos^2 x - \sin^2 x}{\cos^2 x}} = \frac{1}{\cos 2x} = RHS$

## Question 12 (\*\*\*)

Solve the following trigonometric equation

$$3 \operatorname{arccot}(x - \sqrt{3}) - \pi = 0.$$

$$x = \frac{4}{3}\sqrt{3}$$

$$\begin{aligned} 3 \operatorname{arccot}(x - \sqrt{3}) - \pi &= 0 \\ \Rightarrow 3 \operatorname{arccot}(x - \sqrt{3}) &= \pi \\ \Rightarrow \operatorname{arccot}(x - \sqrt{3}) &= \frac{\pi}{3} \\ \Rightarrow \cot[\operatorname{arccot}(x - \sqrt{3})] &= \cot \frac{\pi}{3} \\ \Rightarrow x - \sqrt{3} &= \frac{1}{\tan \frac{\pi}{3}} \\ &\Rightarrow x - \sqrt{3} = \frac{1}{\sqrt{3}} \\ &\Rightarrow x - \sqrt{3} = \frac{\sqrt{3}}{3} \\ &\Rightarrow x = \sqrt{3} + \frac{\sqrt{3}}{3} \\ &\Rightarrow x = \frac{3\sqrt{3}}{3} + \frac{\sqrt{3}}{3} \\ &\Rightarrow x = \frac{4\sqrt{3}}{3} \end{aligned}$$

## Question 13 (\*\*\*)

$$f(x) \equiv \sqrt{3} \sin x + \cos x, \quad 0 \leq x < 2\pi.$$

a) Express  $f(x)$  in the form  $R \cos(x - \alpha)$ ,  $R > 0$ ,  $0 < \alpha < \frac{\pi}{2}$ .

b) State the maximum value of  $f(x)$  and find the value of  $x$  for which this maximum value occurs.

c) Solve the equation

$$f(x) = \sqrt{3}.$$

$$\boxed{\phantom{000}}, \quad f(x) \equiv 2 \cos\left(x - \frac{\pi}{3}\right), \quad \boxed{f(x)_{\max} = 2}, \quad \boxed{x = \frac{\pi}{3}}, \quad \boxed{x = \frac{\pi}{6}, \frac{\pi}{2}}$$

$$\begin{aligned} \text{a) } f(x) &= \sqrt{3} \sin x + \cos x \equiv R \cos(x - \alpha) \\ &\equiv R \cos x \cos \alpha + R \sin x \sin \alpha \\ &\equiv (R \cos \alpha) \cos x + (R \sin \alpha) \sin x \\ \begin{cases} R \cos \alpha = 1 \\ R \sin \alpha = \sqrt{3} \end{cases} &\Rightarrow R = \sqrt{1^2 + (\sqrt{3})^2} = \sqrt{1 + 3} = 2 \\ &\Rightarrow \tan \alpha = \frac{\sqrt{3}}{1} \Rightarrow \alpha = \frac{\pi}{3} \\ \therefore f(x) &= 2 \cos\left(x - \frac{\pi}{3}\right) \\ \text{b) } f(x)_{\max} &= 2 \quad \text{IT OCCURS WHEN } \cos\left(x - \frac{\pi}{3}\right) = 1 \\ &\quad x - \frac{\pi}{3} = 0 \quad \Rightarrow x = \frac{\pi}{3} \quad (0 \leq x < 2\pi) \\ \text{c) } f(x) &= \sqrt{3} \\ 2 \cos\left(x - \frac{\pi}{3}\right) &= \sqrt{3} \\ \cos\left(x - \frac{\pi}{3}\right) &= \frac{\sqrt{3}}{2} \\ \operatorname{arccos}\left(\frac{\sqrt{3}}{2}\right) &= \frac{\pi}{6} \\ x - \frac{\pi}{3} &= \frac{\pi}{6} \pm 2n\pi \quad n = 0, 1, 2, \dots \\ x &= \frac{\pi}{3} + \frac{\pi}{6} \pm 2n\pi \\ x &= \frac{2\pi}{6} \pm 2n\pi \\ x &= \frac{\pi}{3} \pm 2n\pi \\ \therefore x &= \frac{\pi}{3}, \frac{7\pi}{3} \end{aligned}$$

## Question 14 (\*\*\*)

It is given that

$$\frac{1 + \cot^2 x}{\cot x \operatorname{cosec} x} \equiv \sec x.$$

- a) Prove the validity of the above trigonometric identity.
- b) Hence solve the equation

$$\frac{4(1 + \cot^2 x)}{\cot x \operatorname{cosec} x} = \tan^2 x + 5, \quad 0 \leq x < 2\pi,$$

giving the answers in terms of  $\pi$ .

$$\boxed{\phantom{000}}, \quad x = \frac{\pi}{3}, \frac{5\pi}{3}$$

(a)  $LHS = \frac{1 + \cot^2 x}{\cot x \operatorname{cosec} x} = \frac{\sec^2 x}{\cot x \operatorname{cosec} x} = \frac{\sec^2 x}{\frac{\cos x}{\sin x} \cdot \frac{1}{\sin x}} = \frac{\sec^2 x}{\frac{\cos x}{\sin^2 x}} = \sec^2 x \cdot \frac{\sin^2 x}{\cos x} = \frac{\sec^2 x \sin^2 x}{\cos x} = \frac{1}{\cos x} = \sec x$

(b)  $\frac{4(1 + \cot^2 x)}{\cot x \operatorname{cosec} x} = \tan^2 x + 5$   
 $\Rightarrow 4 \sec^2 x = \tan^2 x + 5$   
 $\Rightarrow 4 \sec^2 x - \tan^2 x = 5$   
 $\Rightarrow 4(1 + \tan^2 x) - \tan^2 x = 5$   
 $\Rightarrow 4 + 4\tan^2 x - \tan^2 x = 5$   
 $\Rightarrow 3\tan^2 x = 1$   
 $\Rightarrow \tan^2 x = \frac{1}{3}$   
 $\Rightarrow \tan x = \pm \frac{1}{\sqrt{3}}$   
 $x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$

## Question 15 (\*\*\*)

Two vectors  $\mathbf{a}$  and  $\mathbf{b}$  are given below.

$$\mathbf{a} = (\sin \theta)\mathbf{i} + (2 \cos 2\theta)\mathbf{j} + (\sin \theta)\mathbf{k} \quad \text{and} \quad \mathbf{a} = 3\mathbf{i} - \mathbf{j} - \mathbf{k}.$$

Find the values of  $\theta$ ,  $0 \leq \theta < 2\pi$ , for which  $\mathbf{a}$  is perpendicular to  $\mathbf{b}$ .

$$\boxed{\phantom{000000}}, \quad \theta = \frac{1}{6}\pi, \frac{5}{6}\pi, \frac{3}{2}\pi$$

• START BY FORMING A DOT PRODUCT  
 $\mathbf{a} = (\sin \theta, 2 \cos 2\theta, \sin \theta)$   
 $\mathbf{b} = (3, -1, -1)$   
 $\Rightarrow \mathbf{a} \cdot \mathbf{b} = 0$   
 $\Rightarrow (\sin \theta, 2 \cos 2\theta, \sin \theta) \cdot (3, -1, -1) = 0$   
 $\Rightarrow 3 \sin \theta - 2 \cos 2\theta - \sin \theta = 0$   
 $\Rightarrow 2 \sin \theta - 2 \cos 2\theta = 0$   
 $\Rightarrow \sin \theta - \cos 2\theta = 0$

• BY TRIGONOMETRIC IDENTITIES  
 $\Rightarrow \sin \theta - (1 - 2 \sin^2 \theta) = 0$   
 $\Rightarrow \sin \theta - 1 + 2 \sin^2 \theta = 0$   
 $\Rightarrow 2 \sin^2 \theta + \sin \theta - 1 = 0$   
 $\Rightarrow (2 \sin \theta - 1)(\sin \theta + 1) = 0$   
 $\Rightarrow \sin \theta = \frac{1}{2}$   
 $\Rightarrow \theta = \begin{cases} \frac{\pi}{6} + 2n\pi \\ \frac{5\pi}{6} + 2n\pi \\ \frac{7\pi}{6} + 2n\pi \\ \frac{11\pi}{6} + 2n\pi \end{cases} \quad n = 0, 1, 2, 3, \dots$

$\therefore \theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$



## Question 16 (\*\*\*)

It is given that

$$\frac{1 - \cos 2\theta}{\sin 2\theta} \equiv \tan \theta, \quad \theta \neq 90k^\circ, k \in \mathbb{Z}.$$

a) Prove the validity of the above trigonometric identity.

b) Hence show that

$$\tan 15^\circ = 2 - \sqrt{3}.$$

□, □ proof

(a) LHS =  $\frac{1 - \cos 2\theta}{\sin 2\theta} = \frac{1 - (1 - 2\sin^2 \theta)}{2\sin \theta \cos \theta} = \frac{2\sin^2 \theta}{2\sin \theta \cos \theta} = \frac{\sin \theta}{\cos \theta} = \tan \theta = RHS$

(b) Let  $\theta = 15^\circ$   
 $\tan \theta \equiv \frac{1 - \cos 2\theta}{\sin 2\theta}$   
 $\tan 15^\circ = \frac{1 - \cos 30^\circ}{\sin 30^\circ} = \frac{1 - \frac{\sqrt{3}}{2}}{\frac{1}{2}} = \frac{2 - \sqrt{3}}{1} = 2 - \sqrt{3}$

## Question 17 (\*\*\*)

Prove the validity of the trigonometric identity

$$\frac{1}{\sec x - \tan x} - \frac{1}{\sec x + \tan x} \equiv 2 \tan x.$$

□ proof

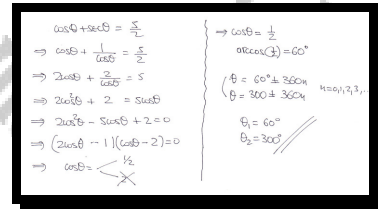
LHS =  $\frac{1}{\sec x - \tan x} - \frac{1}{\sec x + \tan x}$   
 $= \frac{(\sec x + \tan x) - (\sec x - \tan x)}{(\sec x - \tan x)(\sec x + \tan x)}$   
 $= \frac{\sec x + \tan x - \sec x + \tan x}{\sec^2 x - \tan^2 x}$   
 $= \frac{2\tan x}{(1 + \tan^2 x) - \tan^2 x}$   
 $= \frac{2\tan x}{1}$   
 $= 2\tan x = RHS$

## Question 18 (\*\*\*)

Solve the trigonometric equation

$$\cos \theta + \sec \theta = \frac{5}{2}, \quad 0^\circ \leq \theta < 360^\circ.$$

$$\theta = 60^\circ, 300^\circ$$



Handwritten solution for Question 18:

$$\begin{aligned} \cos \theta + \sec \theta &= \frac{5}{2} \\ \Rightarrow \cos \theta + \frac{1}{\cos \theta} &= \frac{5}{2} \\ \Rightarrow 2\cos \theta + \frac{2}{\cos \theta} &= 5 \\ \Rightarrow 2\cos^2 \theta + 2 &= 5\cos \theta \\ \Rightarrow 2\cos^2 \theta - 5\cos \theta + 2 &= 0 \\ \Rightarrow (2\cos \theta - 1)(\cos \theta - 2) &= 0 \\ \Rightarrow \cos \theta &= \frac{1}{2} \quad \text{or } \cos \theta = 2 \end{aligned}$$

Since  $\cos \theta = 2$  is not possible, we solve  $\cos \theta = \frac{1}{2}$ :

$$\Rightarrow \arccos\left(\frac{1}{2}\right) = 60^\circ$$

General solutions:

$$\begin{aligned} \theta &= 60^\circ \pm 360^\circ n \\ \theta &= 300^\circ \pm 360^\circ n \end{aligned}$$

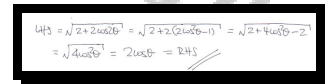
For  $n=0$ ,  $\theta = 60^\circ$  or  $300^\circ$ .

## Question 19 (\*\*\*)

Prove the validity of the trigonometric identity

$$\sqrt{2 + 2\cos 2\theta} \equiv 2\cos \theta.$$

proof



Handwritten proof for Question 19:

$$\begin{aligned} \text{LHS} &= \sqrt{2 + 2\cos 2\theta} = \sqrt{2 + 2(2\cos^2 \theta - 1)} = \sqrt{2 + 4\cos^2 \theta - 2} \\ &= \sqrt{4\cos^2 \theta} = 2\cos \theta = \text{RHS} \end{aligned}$$

## Question 20 (\*\*\*)

$$y \equiv 2\sqrt{2} \cos x + 2\sqrt{2} \sin x, \quad x \in \mathbb{R}.$$

a) Express  $y$  in the form  $R \sin(x + \alpha)$ ,  $R > 0$ ,  $0 < \alpha < \frac{\pi}{2}$ .

b) Solve the equation

$$y = 2 \quad \text{for} \quad 0 < x < 2\pi.$$

c) Write down the maximum value of  $y$ .

d) Find the smallest positive value of  $x$  for which this maximum value occurs.

$$\boxed{\frac{\pi}{4}}, \quad y \equiv 4 \sin\left(x + \frac{\pi}{4}\right), \quad x = \frac{7\pi}{12}, \frac{23\pi}{12}, \quad y_{\max} = 4, \quad x = \frac{\pi}{4}$$

(a)  $2\sqrt{2}\cos x + 2\sqrt{2}\sin x \equiv R\sin(x+\alpha)$   
 $\equiv R\sin x \cos \alpha + R\cos x \sin \alpha$   
 $\equiv (R\cos \alpha)\sin x + (R\sin \alpha)\cos x$   
 $R\cos \alpha = 2\sqrt{2}$   
 $R\sin \alpha = 2\sqrt{2}$   
 $R = \sqrt{(2\sqrt{2})^2 + (2\sqrt{2})^2} = 4$   
 $\tan \alpha = 1 \Rightarrow \alpha = \frac{\pi}{4}$   
 $\therefore 2\sqrt{2}\cos x + 2\sqrt{2}\sin x = 4\sin(x + \frac{\pi}{4})$

(b)  $4\sin(x + \frac{\pi}{4}) = 2$   
 $\Rightarrow \sin(x + \frac{\pi}{4}) = \frac{1}{2}$   
 $\sin^{-1}(\frac{1}{2}) = \frac{\pi}{6}$   
 $x + \frac{\pi}{4} = \frac{\pi}{6} + 2k\pi$   
 $x = -\frac{\pi}{12} + 2k\pi$   
 $x = \frac{23\pi}{12}$

(c) MAXIMUM VALUE IS 4  
 THIS MAX OCCURS WHEN  
 $\sin(x + \frac{\pi}{4}) = 1$   
 $x + \frac{\pi}{4} = \frac{\pi}{2}$   
 $x = \frac{\pi}{4}$

(d)  $x = \frac{7\pi}{12}$

**Question 21 (\*\*\*)**

Solve the following trigonometric equation

$$4 \sin x \cos x = 1, \quad 0 \leq x < \pi,$$

giving the answers in terms of  $\pi$ .

$$\boxed{\phantom{000}}, \quad x = \frac{\pi}{12}, \frac{5\pi}{12}$$

Handwritten solution for Question 21:

$$4 \sin x \cos x = 1$$

$$\Rightarrow 2 \sin(2x) = 1$$

$$\Rightarrow \sin(2x) = \frac{1}{2}$$

$$\Rightarrow 2x = \frac{\pi}{6} \text{ or } \frac{5\pi}{6}$$

$$\Rightarrow x = \frac{\pi}{12} \text{ or } \frac{5\pi}{12}$$

**Question 22 (\*\*\*)**

Solve the following trigonometric equation

$$\frac{\operatorname{cosec}^2 \theta \tan^2 \theta}{\cos \theta} + 8 = 0, \quad 0^\circ \leq \theta < 360^\circ.$$

$$\boxed{\theta = 120^\circ, 240^\circ}$$

Handwritten solution for Question 22:

$$\frac{\operatorname{cosec}^2 \theta \tan^2 \theta}{\cos \theta} + 8 = 0$$

$$\Rightarrow \frac{1}{\sin^2 \theta} \times \frac{\sin^2 \theta}{\cos^2 \theta} + 8 \cos \theta = 0$$

$$\Rightarrow \frac{1}{\cos^2 \theta} + 8 \cos \theta = 0$$

$$\Rightarrow 1 + 8 \cos^3 \theta = 0$$

$$\Rightarrow 8 \cos^3 \theta = -1$$

$$\Rightarrow \cos^3 \theta = -\frac{1}{8}$$

$$\Rightarrow \cos \theta = -\frac{1}{2}$$

$$\cos \theta = -\frac{1}{2}$$

$$\operatorname{cosec}(-\frac{1}{2}) = 120^\circ$$

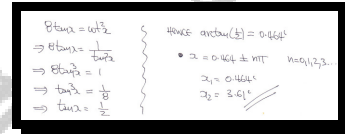
$$\theta = 120^\circ \text{ or } 240^\circ$$

## Question 23 (\*\*\*)

Solve the following trigonometric equation

$$8 \tan x = \cot^2 x, \quad 0 \leq x < 2\pi.$$

$$x = 0.464^\circ, 3.61^\circ$$



Handwritten solution for Question 23:

$$8 \tan x = \cot^2 x$$

$$\Rightarrow 8 \tan x = \frac{1}{\tan^2 x}$$

$$\Rightarrow 8 \tan^3 x = 1$$

$$\Rightarrow \tan^3 x = \frac{1}{8}$$

$$\Rightarrow \tan x = \frac{1}{2}$$

$$\Rightarrow x = \arctan\left(\frac{1}{2}\right) = 0.464^\circ$$

$$\bullet x = 0.464^\circ \pm \pi$$

$$x_1 = 0.464^\circ$$

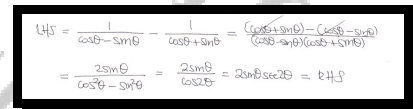
$$x_2 = 3.61^\circ$$

## Question 24 (\*\*\*)

Prove the validity of the following trigonometric identity

$$\frac{1}{\cos \theta - \sin \theta} - \frac{1}{\cos \theta + \sin \theta} \equiv 2 \sin \theta \sec 2\theta.$$

proof



Handwritten proof for Question 24:

$$LHS = \frac{1}{\cos \theta - \sin \theta} - \frac{1}{\cos \theta + \sin \theta} = \frac{(\cos \theta + \sin \theta) - (\cos \theta - \sin \theta)}{(\cos \theta - \sin \theta)(\cos \theta + \sin \theta)}$$

$$= \frac{2 \sin \theta}{\cos^2 \theta - \sin^2 \theta} = \frac{2 \sin \theta}{\cos 2\theta} = 2 \sin \theta \sec 2\theta = RHS$$

## Question 25 (\*\*\*)

It is given that

$$\frac{1 + \tan^2 x}{1 - \tan^2 x} \equiv \sec 2x.$$

- a) Prove the validity of the above trigonometric identity.
- b) Hence solve the equation

$$\frac{1 + \tan^2 x}{1 - \tan^2 x} + 2 = 0, \quad 0 \leq x < 2\pi,$$

giving the answers in terms of  $\pi$ .

$$x = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$$

(a) LHS =  $\frac{1 + \frac{\sin^2 x}{\cos^2 x}}{1 - \frac{\sin^2 x}{\cos^2 x}} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x - \sin^2 x} = \frac{1}{\cos 2x} = \sec 2x = \text{RHS}$

(b)  $\frac{1 + \tan^2 x}{1 - \tan^2 x} + 2 = 0$   
 $\Rightarrow \sec 2x + 2 = 0$   
 $\Rightarrow \sec 2x = -2$   
 $\Rightarrow \cos 2x = -\frac{1}{2}$   
 $\Rightarrow \cos(2x) = -\frac{1}{2}$   
 $2x = \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{8\pi}{3}, \frac{10\pi}{3}$   
 $x = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$

## Question 26 (\*\*\*)

$$f(x) \equiv \sin x + \sqrt{3} \cos x, \quad 0 \leq x < 2\pi.$$

a) Express  $f(x)$  in the form  $R \cos(x - \alpha)$ ,  $R > 0$ ,  $0 < \alpha < \frac{\pi}{2}$ .

b) State the minimum and maximum value of ...

i. ...  $f(x)$ .

ii. ...  $[f(x)]^2$ .

iii. ...  $\frac{1}{5 + f(x)}$ .

$$\boxed{\phantom{0}}, \quad \boxed{f(x) \equiv 2 \cos\left(x - \frac{\pi}{6}\right)}, \quad \boxed{[-2, 2]}, \quad \boxed{[0, 4]}, \quad \boxed{\left[\frac{1}{7}, \frac{1}{3}\right]}$$

(a)  $f(x) = \sqrt{3} \cos x + \sin x \equiv R \cos(x - \alpha)$   
 $\equiv R \cos x \cos \alpha + R \sin x \sin \alpha$   
 $\equiv (R \cos \alpha) \cos x + (R \sin \alpha) \sin x$   
 $R \cos \alpha = \sqrt{3}$   
 $R \sin \alpha = 1$   
 $R = \sqrt{(\sqrt{3})^2 + 1^2} = \sqrt{3+1} = \sqrt{4} = 2$   
 $\tan \alpha = \frac{1}{\sqrt{3}} \Rightarrow \alpha = \frac{\pi}{6}$   
 $\therefore f(x) = 2 \cos\left(x - \frac{\pi}{6}\right)$

(b)

	Min	Max	
$f(x)$	-2	2	$\leftarrow 2 \cos\left(x - \frac{\pi}{6}\right)$
$[f(x)]^2$	0	4	$\leftarrow 4 \cos^2\left(x - \frac{\pi}{6}\right)$
$\frac{1}{5 + f(x)}$	$\frac{1}{3}$	$\frac{1}{7}$	$\leftarrow \frac{1}{5 + 2 \cos\left(x - \frac{\pi}{6}\right)}$

## Question 27 (\*\*\*)

$$\cos^2 x + \sin^2 x \equiv 1.$$

a) Starting with the above identity prove that

$$1 + \tan^2 x \equiv \sec^2 x.$$

b) Hence, or otherwise, solve the following trigonometric equation

$$2 \tan^2 x + \sec^2 x = 5 \sec x, \quad 0 \leq x < 360^\circ.$$

$$\boxed{60^\circ}, \quad \boxed{x = 60^\circ, 300^\circ}$$

(a)  $\cos^2 \theta + \sin^2 \theta = 1$   
 $\frac{\cos^2 \theta}{\cos^2 \theta} + \frac{\sin^2 \theta}{\cos^2 \theta} = \frac{1}{\cos^2 \theta}$   
 $1 + \tan^2 \theta = \sec^2 \theta$

(b)  $2 \tan^2 x + \sec^2 x = 5 \sec x$   
 $\Rightarrow 2(\sec^2 x - 1) + \sec^2 x = 5 \sec x$   
 $\Rightarrow 2\sec^2 x - 2 + \sec^2 x = 5 \sec x$   
 $\Rightarrow 3\sec^2 x - 5 \sec x - 2 = 0$   
 $\Rightarrow (3\sec x + 1)(\sec x - 2) = 0$   
 $\Rightarrow \sec x = 2 \text{ or } \sec x = -\frac{1}{3}$   
 $\Rightarrow \cos x = \frac{1}{2} \text{ or } \cos x = -3$   
 $\Rightarrow \cos x = \frac{1}{2}$   
 $\Rightarrow x = 60^\circ \text{ or } 300^\circ$

## Question 28 (\*\*\*)

Prove the validity of the following trigonometric identity

$$\frac{1 + \cot^2 \theta}{2 \cot \theta} \equiv \operatorname{cosec} 2\theta.$$

$$\boxed{\quad}, \quad \boxed{\text{proof}}$$

LHS =  $\frac{1 + \cot^2 \theta}{2 \cot \theta} = \frac{\csc^2 \theta}{2 \cot \theta} = \frac{1/\sin^2 \theta}{\cos \theta/\sin \theta} = \frac{1}{\sin \theta \cos \theta} = \frac{1}{\frac{1}{2} \sin 2\theta} = \frac{2}{\sin 2\theta} = 2 \operatorname{cosec} 2\theta$

RHS =  $\operatorname{cosec} 2\theta$



## Question 29 (\*\*\*)

Prove the validity of the following trigonometric identity

$$\cot x - \tan x \equiv 2 \cot 2x.$$

□, proof

Handwritten proof for Question 29:

$$\begin{aligned} \text{LHS} &= \cot x - \tan x = \frac{\cos x}{\sin x} - \frac{\sin x}{\cos x} = \frac{\cos^2 x - \sin^2 x}{\sin x \cos x} = \frac{\cos 2x}{\frac{1}{2} \sin 2x} \\ &= \frac{2 \cos 2x}{\sin 2x} = 2 \cot 2x = \text{RHS} \end{aligned}$$

or

$$\begin{aligned} \text{RHS} &= 2 \cot 2x = \frac{2}{\tan 2x} = \frac{2}{\frac{\sin 2x}{\cos 2x}} = \frac{2 \cos 2x}{\sin 2x} = \frac{1 - \tan^2 x}{2 \tan x} \\ &= \frac{1}{\tan x} - \frac{\tan x}{2} = \cot x - \tan x = \text{LHS} \end{aligned}$$

## Question 30 (\*\*\*)

It is given that  $\arcsin x = \arccos y$ .

Show, by a clear method, that

$$x^2 + y^2 = 1.$$

□, proof

Handwritten proof for Question 30:

•  $\arcsin x = \arccos y = \theta$

$\begin{cases} \arcsin x = \theta \\ \arccos y = \theta \end{cases} \Rightarrow \begin{cases} \sin \theta = x \\ \cos \theta = y \end{cases} \Rightarrow \sin^2 \theta + \cos^2 \theta = x^2 + y^2$

$\therefore x^2 + y^2 = 1$   $\checkmark$  Q.E.D.

## Question 31 (\*\*\*)

$$6\sec^2 2x + 5 \tan 2x = 12, \quad 0 \leq \theta < \pi.$$

Find the solutions of the above trigonometric equation, giving the answers in radians correct to two decimal places.

$$\boxed{\phantom{000}}, \quad x = 0.29^\circ, 1.08^\circ, 1.86^\circ, 2.65^\circ$$

Handwritten solution for Question 31:

$$6\sec^2 2x + 5 \tan 2x = 12$$

$$\Rightarrow 6(1 + \tan^2 2x) + 5 \tan 2x = 12$$

$$\Rightarrow 6 \tan^2 2x + 5 \tan 2x - 6 = 0$$

By the quadratic formula or factorization

$$\Rightarrow \tan 2x = \frac{-5 \pm \sqrt{5^2 - 4(6)(-6)}}{2(6)}$$

$$\Rightarrow \tan 2x = \frac{-5}{12} \text{ or } \frac{7}{2}$$

$$\begin{cases} \arctan(-\frac{5}{12}) = -0.983^\circ \\ \arctan(\frac{7}{2}) = 0.983^\circ \end{cases}$$

or

$$2x = -0.983 + \pi n$$

$$2x = 0.983 + \pi n$$

$$x = -0.291 + \frac{\pi n}{2}$$

$$x = 0.291 + \frac{\pi n}{2}$$

$\therefore x_1 = 1.08^\circ$   
 $x_2 = 2.65^\circ$   
 $x_3 = 0.29^\circ$   
 $x_4 = 1.86^\circ$

## Question 32 (\*\*\*)

Solve the trigonometric equation

$$4 - 4 \cos 2\theta = \operatorname{cosec} \theta, \quad 0 \leq \theta < 2\pi,$$

giving the answers in terms of  $\pi$ .

$$\boxed{\phantom{000}}, \quad \theta = \frac{\pi}{6}, \frac{5\pi}{6}$$

Handwritten solution for Question 32:

$$4 - 4 \cos 2\theta = \operatorname{cosec} \theta$$

$$\Rightarrow 4 - 4(1 - 2\sin^2 \theta) = \frac{1}{\sin \theta}$$

$$\Rightarrow 4 - 4 + 8\sin^2 \theta = \frac{1}{\sin \theta}$$

$$\Rightarrow 8\sin^3 \theta = 1$$

$$\Rightarrow \sin^3 \theta = \frac{1}{8}$$

$$\Rightarrow \sin \theta = \frac{1}{2}$$

or

$$\operatorname{cosec}(\frac{\pi}{6}) = \frac{1}{\sin(\frac{\pi}{6})}$$

$$(\theta = \frac{\pi}{6} + 2\pi n)$$

$$(\theta = \frac{5\pi}{6} + 2\pi n)$$

$\therefore \theta = \frac{\pi}{6}$   
 $\theta = \frac{5\pi}{6}$

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# 192

## STANDARD QUESTIONS

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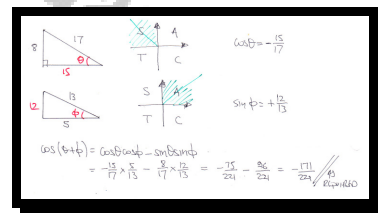
Question 1 (\*\*\*)

$$\sin \theta = \frac{8}{17} \quad \text{and} \quad \cos \varphi = \frac{5}{13}$$

If  $\theta$  is obtuse and  $\varphi$  is acute, show that

$$\cos(\theta + \varphi) = -\frac{171}{221}$$

, proof



Question 2 (\*\*\*)

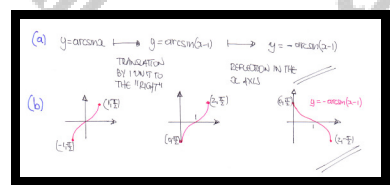
A curve  $C$  is defined by the equation

$$y = -\arcsin(x-1), \quad 0 \leq x \leq 2.$$

- Describe the 2 geometric transformations that map the graph of  $\arcsin x$  onto the graph of  $C$ .
- Sketch the graph of  $C$ .

The sketch must include the coordinates of any points where the graph of  $C$  meets the coordinate axes and the coordinates of the endpoints of  $C$ .

, translation by 1 unit to the right, followed by reflection in the  $x$  axis



## Question 3 (\*\*\*)

It is given that

$$\frac{\sec \theta}{\sec \theta - \cos \theta} \equiv \operatorname{cosec}^2 \theta, \quad \theta \neq \frac{n\pi}{2}, \quad n \in \mathbb{Z}.$$

a) Prove the validity of the above trigonometric identity.

b) Hence solve the equation

$$\frac{\sec \theta}{\sec \theta - \cos \theta} = 4(\operatorname{cosec} \theta - 1), \quad 0 \leq \theta < 2\pi,$$

giving the answers in terms of  $\pi$ .

$$\theta = \frac{\pi}{6}, \frac{5\pi}{6}$$

(a) LHS =  $\frac{\sec \theta}{\sec \theta - \cos \theta}$   
 $= \frac{\frac{1}{\cos \theta}}{\frac{1}{\cos \theta} - \cos \theta}$   
 $= \frac{1}{1 - \cos^2 \theta}$   
 $= \frac{1}{\sin^2 \theta}$   
 $= \operatorname{cosec}^2 \theta$   
 $= \text{RHS}$

(b)  $\frac{\sec \theta}{\sec \theta - \cos \theta} = 4(\operatorname{cosec} \theta - 1)$   
 $\Rightarrow \frac{1}{1 - \cos^2 \theta} = 4(\frac{1}{\sin \theta} - 1)$   
 $\Rightarrow \frac{1}{1 - \cos^2 \theta} = \frac{4 - 4\cos \theta}{\sin \theta}$   
 $\Rightarrow \frac{1}{1 - \cos^2 \theta} = \frac{4(1 - \cos \theta)}{\sin \theta}$   
 $\Rightarrow \frac{1}{1 + \cos \theta} = \frac{4}{\sin \theta}$   
 $\Rightarrow \sin \theta = 4(1 + \cos \theta)$   
 $\Rightarrow \sin^2 \theta = 16(1 + \cos \theta)^2$   
 $\Rightarrow 1 - \cos^2 \theta = 16(1 + \cos \theta)^2$   
 $\Rightarrow 1 - \cos^2 \theta = 16(1 + 2\cos \theta + \cos^2 \theta)$   
 $\Rightarrow 1 - \cos^2 \theta = 16 + 32\cos \theta + 16\cos^2 \theta$   
 $\Rightarrow 17\cos^2 \theta + 32\cos \theta + 15 = 0$   
 $\Rightarrow (17\cos \theta + 15)(\cos \theta + 1) = 0$   
 $\Rightarrow \cos \theta = -\frac{15}{17} \text{ or } \cos \theta = -1$   
 $\Rightarrow \theta = \arccos(-\frac{15}{17}) \text{ or } \theta = \pi$   
 $\Rightarrow \theta = 2.75 \text{ or } \theta = \pi$   
 $\Rightarrow \theta = 2.75 \text{ or } \theta = \pi$

**Question 4** (\*\*\*)

Solve each of the following trigonometric equations.

i.  $2\sec\theta - 1 = 2\sec\theta\sin^2\theta$ ,  $0^\circ \leq \theta < 180^\circ$ ,  $\theta \neq 90^\circ$

ii.  $4\cot^2x - 9\operatorname{cosec}x + 6 = 0$ ,  $0^\circ \leq x < 360^\circ$ ,  $x \neq 0^\circ, 180^\circ$

,  $\theta = 60^\circ$  ,  $x = 30^\circ, 150^\circ$

**Question 5** (\*\*\*)

Prove the validity of each of the following trigonometric identities.

a)  $\frac{1 - \tan^2\theta}{1 + \tan^2\theta} \equiv \cos 2\theta$ .

b)  $\frac{\sin\theta}{1 + \cos\theta} + \frac{1 + \cos\theta}{\sin\theta} \equiv 2\operatorname{cosec}\theta$ .

proof

## Question 6 (\*\*\*)

It is given that

$$\frac{1 + \cos 2\theta}{\sin 2\theta} \equiv \cot \theta, \quad \theta \neq \frac{k\pi}{2}, \quad k \in \mathbb{Z}.$$

a) Prove the validity of the above trigonometric identity.

b) Hence solve the equation

$$\operatorname{cosec} 4x + \cot 4x = 1, \quad 0 \leq x < 2\pi,$$

giving the answers in terms of  $\pi$ .

$$\boxed{\phantom{000}}, \quad x = \frac{\pi}{8}, \frac{5\pi}{8}, \frac{9\pi}{8}, \frac{13\pi}{8}$$

(a)  $\text{LHS} = \frac{1 + \cos 2\theta}{\sin 2\theta} = \frac{1 + (2\cos^2 \theta - 1)}{\sin 2\theta} = \frac{2\cos^2 \theta}{2\sin \theta \cos \theta} = \frac{\cos \theta}{\sin \theta} = \cot \theta = \text{RHS}$   
 (b)  $\operatorname{cosec} 4x + \cot 4x = 1$   
 $\Rightarrow \frac{1}{\sin 4x} + \frac{\cos 4x}{\sin 4x} = 1$   
 $\Rightarrow \frac{1 + \cos 4x}{\sin 4x} = 1$   
 $\Rightarrow 1 + \cos 4x = \sin 4x$   
 from part (a)  $\theta \mapsto 2x$   
 $\Rightarrow 2\cos^2 2x = 1$   
 $\Rightarrow \cos 2x = \pm \frac{1}{\sqrt{2}}$   
 $2x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$   
 $x = \frac{\pi}{8}, \frac{5\pi}{8}, \frac{9\pi}{8}, \frac{13\pi}{8}$

## Question 7 (\*\*\*)

$$y \equiv \sqrt{2} \cos \theta - \sqrt{6} \sin \theta, \quad 0 < \theta < 360^\circ.$$

- a) Express  $y$  in the form  $R \cos(\theta + \alpha)$ ,  $R > 0$ ,  $0 < \alpha < 90^\circ$ .
- b) Solve the equation  $y = 2$ .
- c) Write down the minimum value of ...

i. ...  $y^2$ .

ii. ...  $\frac{1}{y^2}$ .

$$\boxed{R = \sqrt{8}}, \quad \boxed{y \equiv \sqrt{8} \cos(\theta + 60^\circ)}, \quad \boxed{\theta = 255^\circ, 345^\circ}, \quad \boxed{\min = 0}, \quad \boxed{\min = \frac{1}{8}}$$

(a)  $\sqrt{2} \cos \theta - \sqrt{6} \sin \theta \equiv R \cos(\theta + \alpha)$   
 $\equiv R \cos \theta \cos \alpha - R \sin \theta \sin \alpha$   
 $\equiv (R \cos \alpha) \cos \theta - (R \sin \alpha) \sin \theta$   
 $R \cos \alpha = \sqrt{2} \Rightarrow R = \sqrt{4 + 2} = \sqrt{6}$   
 $R \sin \alpha = \sqrt{6} \Rightarrow \tan \alpha = \frac{\sqrt{6}}{\sqrt{6}} = 1 \Rightarrow \alpha = 60^\circ$   
 $\therefore y = \sqrt{8} \cos(\theta + 60^\circ)$

(b)  $\sqrt{2} \cos \theta - \sqrt{6} \sin \theta = 2$   
 $\Rightarrow \sqrt{8} \cos(\theta + 60^\circ) = 2$   
 $\Rightarrow \cos(\theta + 60^\circ) = \frac{\sqrt{2}}{2}$   
 $\Rightarrow \theta + 60^\circ = 45^\circ$   
 $\Rightarrow \theta = -15^\circ \pm 360^\circ$   
 $\Rightarrow \theta = 255^\circ, 345^\circ$

(c)  $y^2 = [\sqrt{8} \cos(\theta + 60^\circ)]^2 = 8 \cos^2(\theta + 60^\circ)$   
 $\therefore y^2_{\min} = 0$

(d) THE MINIMUM VALUE OF  $\frac{1}{y^2}$  OCCURS WHEN  $y^2$  IS MAXIMUM i.e. 8  
 $\therefore \left(\frac{1}{y^2}\right)_{\min} = \frac{1}{8}$



## Question 8 (\*\*\*)

$$\sin A = \frac{12}{13} \quad \text{and} \quad \cos B = \frac{4}{5}$$

If  $A$  is obtuse and  $B$  is acute, show clearly that

$$\sin(A+B) = \frac{33}{65}$$

☐ , ☐ proof

## Question 9 (\*\*\*)

By considering the compound angle identity for  $\tan(A+B)$ , with suitable values for  $A$  and  $B$ , show that

$$\cot 75^\circ = 2 - \sqrt{3}$$

☐ , ☐ proof

**Question 10** (\*\*\*)

It is given that

$$\tan \theta + \cot \theta \equiv 2 \operatorname{cosec} 2\theta, \quad \theta \neq \frac{k\pi}{2}, k \in \mathbb{Z}.$$

- a) Prove the validity of the above trigonometric identity.
- b) Hence find, in terms of  $\pi$ , the solutions of the equation

$$\tan \theta + \cot \theta = 4, \quad 0 \leq \theta < 2\pi,$$

giving the answers in terms of  $\pi$ .

$$\theta = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{13\pi}{12}, \frac{17\pi}{12}$$

**Question 11** (\*\*\*)

Solve the following trigonometric equation

$$\sin 2\theta = \tan \theta, \quad 0 \leq \theta \leq 180^\circ.$$

$$\theta = 0^\circ, 45^\circ, 135^\circ, 180^\circ$$

**Question 12** (\*\*\*)

It is given that

$$\cos(x+30^\circ) + \cos(x-30^\circ) \equiv \sqrt{3} \cos x.$$

- a) Prove the validity of the above trigonometric identity.
- b) Hence show that

$$\cos 75^\circ + \cos 15^\circ = \frac{1}{2}\sqrt{6}.$$

proof

(a) LHS =  $\cos(x+30^\circ) + \cos(x-30^\circ)$   
 $= \cos x \cos 30^\circ - \sin x \sin 30^\circ + \cos x \cos 30^\circ + \sin x \sin 30^\circ$   
 $= 2\cos x \cos 30^\circ = 2\cos x \times \frac{\sqrt{3}}{2} = \sqrt{3} \cos x = RHS$

(b) Let  $x=45^\circ$  in (a)  $\cos(45+30) + \cos(45-30) = \sqrt{3} \cos 45$   
 $\cos(75^\circ) + \cos(15^\circ) = \sqrt{3} \times \frac{\sqrt{2}}{2}$   
 $\cos 75^\circ + \cos 15^\circ = \frac{1}{2}\sqrt{6}$   $\checkmark$

**Question 13** (\*\*\*)

Prove the validity of each of the following trigonometric identities.

a)  $\left(\frac{1+\sin \theta}{\cos \theta}\right)^2 + \left(\frac{1-\sin \theta}{\cos \theta}\right)^2 \equiv 4 \tan^2 \theta + 2.$

b)  $\sin^2\left(\theta + \frac{\pi}{4}\right) + \sin^2\left(\theta - \frac{\pi}{4}\right) \equiv 1.$

proof

(a) LHS =  $\left(\frac{1+\sin \theta}{\cos \theta}\right)^2 + \left(\frac{1-\sin \theta}{\cos \theta}\right)^2 = \frac{1+2\sin \theta + \sin^2 \theta}{\cos^2 \theta} + \frac{1-2\sin \theta + \sin^2 \theta}{\cos^2 \theta}$   
 $= \frac{2+2\sin^2 \theta}{\cos^2 \theta} = \frac{2}{\cos^2 \theta} + \frac{2\sin^2 \theta}{\cos^2 \theta} = 2\sec^2 \theta + 2\tan^2 \theta$   
 $= 2(\tan^2 \theta + 1) + 2\tan^2 \theta = 4\tan^2 \theta + 2 = RHS$

(b) LHS =  $\sin^2\left(\theta + \frac{\pi}{4}\right) + \sin^2\left(\theta - \frac{\pi}{4}\right) = \left[\sin\left(\theta + \frac{\pi}{4}\right)\right]^2 + \left[\sin\left(\theta - \frac{\pi}{4}\right)\right]^2$   
 $= \left(\sin \theta \cos \frac{\pi}{4} + \cos \theta \sin \frac{\pi}{4}\right)^2 + \left(\sin \theta \cos \frac{\pi}{4} - \cos \theta \sin \frac{\pi}{4}\right)^2$   
 $= \left(\frac{\sqrt{2}}{2}(\sin \theta + \cos \theta)\right)^2 + \left(\frac{\sqrt{2}}{2}(\sin \theta - \cos \theta)\right)^2$   
 $= \frac{1}{2}(\sin^2 \theta + \cos^2 \theta + 2\sin \theta \cos \theta) + \frac{1}{2}(\sin^2 \theta + \cos^2 \theta - 2\sin \theta \cos \theta)$   
 $= \frac{1}{2}[2\sin^2 \theta + 2\cos^2 \theta] = \frac{1}{2}[2(\sin^2 \theta + \cos^2 \theta)] = \frac{1}{2}[2] = 1$   $\checkmark$

ALTERNATIVE: USE IDENTITY  $\sin A = \cos\left(\frac{\pi}{2} - A\right)$

## Question 14 (\*\*\*)

It is given that

$$\frac{1}{\operatorname{cosec} \theta - 1} + \frac{1}{\operatorname{cosec} \theta + 1} \equiv 2 \tan \theta \sec \theta.$$

a) Prove the validity of the above trigonometric identity.

b) Hence solve the equation

$$\frac{1}{\operatorname{cosec} \theta - 1} + \frac{1}{\operatorname{cosec} \theta + 1} = \frac{1}{2} \cot \theta \sec \theta, \quad 0 \leq \theta < 360^\circ.$$

$$\theta = 26.6^\circ, 153.4^\circ, 206.6^\circ, 333.4^\circ$$

(a) LHS =  $\frac{1}{\operatorname{cosec} \theta - 1} + \frac{1}{\operatorname{cosec} \theta + 1} = \frac{(\operatorname{cosec} \theta + 1) + (\operatorname{cosec} \theta - 1)}{(\operatorname{cosec} \theta - 1)(\operatorname{cosec} \theta + 1)}$   
 $= \frac{\operatorname{cosec} \theta + 1 + \operatorname{cosec} \theta - 1}{\operatorname{cosec}^2 \theta - 1} = \frac{2 \operatorname{cosec} \theta}{(\operatorname{cosec}^2 \theta - 1)}$   
 $= \frac{2 \operatorname{cosec} \theta}{\frac{\sin^2 \theta}{\cos^2 \theta} - 1} = \frac{2 \operatorname{cosec} \theta}{\frac{\sin^2 \theta - \cos^2 \theta}{\cos^2 \theta}} = \frac{2 \operatorname{cosec} \theta \cos^2 \theta}{\sin^2 \theta - \cos^2 \theta}$   
 $= \frac{2 \sin \theta \cos^2 \theta}{\sin^2 \theta - \cos^2 \theta} = \frac{2 \sin \theta \cos^2 \theta}{-\cos 2\theta} = \frac{2 \sin \theta \cos^2 \theta}{-\cos 2\theta}$   
 $= 2 \tan \theta \sec \theta = \text{RHS}$

(b)  $\frac{1}{\operatorname{cosec} \theta - 1} + \frac{1}{\operatorname{cosec} \theta + 1} = \frac{1}{2} \cot \theta \sec \theta$   
 $\Rightarrow 2 \tan \theta \sec \theta = \frac{1}{2} \cot \theta \sec \theta$   
 $\Rightarrow 4 \tan \theta \sec \theta = \cot \theta \sec \theta$   
 $\Rightarrow 4 \tan \theta \sec \theta - \cot \theta \sec \theta = 0$   
 $\Rightarrow \sec \theta (4 \tan \theta - \cot \theta) = 0$   
 $\Rightarrow \frac{1}{\cos \theta} (4 \tan \theta - \cot \theta) = 0$   
 $\Rightarrow 4 \tan \theta - \cot \theta = 0$   
 $\Rightarrow 4 \tan \theta = \cot \theta$   
 $\Rightarrow 4 \tan^2 \theta = 1$   
 $\Rightarrow \tan^2 \theta = \frac{1}{4}$   
 $\Rightarrow \tan \theta = \pm \frac{1}{2}$

•  $\tan^{-1}(\frac{1}{2}) = 26.6^\circ$   
 $\theta = 26.6^\circ, 386.4^\circ$   
 $\theta = 26.6^\circ, 386.4^\circ$

•  $\tan^{-1}(-\frac{1}{2}) = -26.6^\circ$   
 $\theta = -26.6^\circ, 333.4^\circ$   
 $\theta = 333.4^\circ, 66.6^\circ$

∴  $\theta = 26.6^\circ, 153.4^\circ, 206.6^\circ, 333.4^\circ$

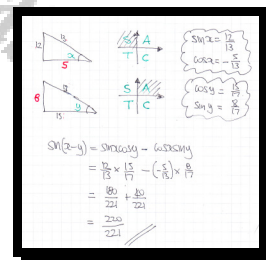
## Question 15 (\*\*\*)

$$\sin x = \frac{12}{13} \quad \text{and} \quad \cos y = \frac{15}{17}$$

If  $x$  is obtuse and  $y$  is acute, show clearly that

$$\sin(x - y) = \frac{220}{221}$$

 , proof



## Question 16 (\*\*\*)

Solve the following trigonometric equation

$$\frac{2 + \cos 2x}{3 + \sin^2 2x} = \frac{2}{5}, \quad \text{for } 0^\circ \leq x < 360^\circ,$$

$$x = 60^\circ, 120^\circ, 240^\circ, 300^\circ$$

## Question 17 (\*\*\*)

It is given that

$$\frac{2 \tan x}{1 + \tan^2 x} \equiv \sin 2x.$$

a) Prove the validity of the above trigonometric identity.

b) Use part (a) to show that

$$\tan 15^\circ = 2 - \sqrt{3}.$$

 ,  proof

(a)  $\frac{2 \tan x}{1 + \tan^2 x} = \frac{2 \sin x}{1 + \tan^2 x} = \frac{2 \sin x}{\frac{1 + \sin^2 x}{\cos^2 x}} = \frac{2 \sin x \cos^2 x}{1 + \sin^2 x} = \frac{2 \sin x \cos^2 x}{2 \cos^2 x} = \sin 2x$   
 (b)  $\tan 30^\circ = \frac{1}{3}$  in  $\frac{2 \tan 15^\circ}{1 + \tan^2 15^\circ} = \tan 30^\circ$   
 $\Rightarrow \frac{2 \tan 15^\circ}{1 + \tan^2 15^\circ} = \frac{1}{3}$   
 $\Rightarrow \frac{2t}{1+t^2} = \frac{1}{3} \quad (t = \tan 15^\circ)$   
 $\Rightarrow 4t = 1 + t^2$   
 $\Rightarrow 0 = t^2 - 4t + 1$   
 $\Rightarrow (t-2)^2 - 4 + 1 = 0$   
 $\Rightarrow (t-2)^2 = 3$   
 $\Rightarrow t-2 = \pm \sqrt{3}$   
 $\Rightarrow t = 2 \pm \sqrt{3}$   
 $\therefore \tan 15^\circ = 2 - \sqrt{3}$  (since  $\tan 15^\circ < 1$ )

## Question 18 (\*\*\*)

Solve the trigonometric equation

$$\frac{\sin x}{\cos x} + \frac{\cos x}{\sin x} = 4, \quad 0 \leq x < 360^\circ.$$

$$x = 15^\circ, 75^\circ, 195^\circ, 255^\circ$$

$\frac{\sin x}{\cos x} + \frac{\cos x}{\sin x} = 4$   
 $\Rightarrow \frac{\sin^2 x + \cos^2 x}{\sin x \cos x} = 4$   
 $\Rightarrow \frac{1}{\sin x \cos x} = 4$   
 $\Rightarrow \frac{2}{2 \sin x \cos x} = 4$   
 $\Rightarrow \frac{2}{\sin 2x} = 4$   
 $\Rightarrow \frac{1}{\sin 2x} = 2$   
 $\Rightarrow \sin 2x = \frac{1}{2}$   
 $\Rightarrow 2x = 30^\circ + 360^\circ n$   
 $\Rightarrow x = 15^\circ + 180^\circ n$   
 $\Rightarrow x = 15^\circ, 195^\circ$   
 $\Rightarrow 2x = 150^\circ + 360^\circ n$   
 $\Rightarrow x = 75^\circ + 180^\circ n$   
 $\Rightarrow x = 75^\circ, 255^\circ$

**Question 19** (\*\*\*)

Solve the following trigonometric equation

$$\tan(\theta + 45^\circ) = 1 - 2 \tan \theta, \quad 0 \leq \theta < 360^\circ.$$

$$x = 0, 180^\circ, 63.4^\circ, 243.4^\circ$$

Handwritten solution for Question 19:

$$\begin{aligned} \tan(\theta + 45) &= 1 - 2 \tan \theta \\ \frac{\tan \theta + \tan 45}{1 - \tan \theta \tan 45} &= 1 - 2 \tan \theta \\ \frac{\tan \theta + 1}{1 - \tan \theta} &= 1 - 2 \tan \theta \\ \frac{1 + T}{1 - T} &= 1 - 2T \quad (T = \tan \theta) \\ (1 + T) &= (1 - T)(1 - 2T) \\ 1 + T &= 1 - T - 2T + 2T^2 \\ 0 &= 2T^2 - 3T \\ 0 &= 3T(2T - 1) \\ T &= 0 \quad \text{or} \quad T = \frac{1}{2} \\ \theta &= 0^\circ \quad \text{or} \quad \theta = 63.4^\circ \\ \theta &= 180^\circ \quad \text{or} \quad \theta = 243.4^\circ \end{aligned}$$

**Question 20** (\*\*\*)

It is given that

$$\tan x \sec x + \operatorname{cosec} x \equiv \operatorname{cosec} x \sec^2 x, \quad x \neq \frac{n\pi}{2}, \quad n \in \mathbb{Z}.$$

a) Prove the validity of the above trigonometric identity.

b) Hence show that the equation

$$\tan x \sec x + \operatorname{cosec} x = \frac{1}{2} \sec^2 x$$

has no real solutions.

proof

Handwritten solution for Question 20:

(a) LHS =  $\tan x \sec x + \operatorname{cosec} x = \frac{\sin x}{\cos x} \cdot \frac{1}{\cos x} + \frac{1}{\sin x} = \frac{\sin^2 x}{\cos^2 x} + \frac{1}{\sin x}$   
 $= \frac{\sin^2 x + \cos^2 x}{\cos^2 x \sin x} = \frac{1}{\cos^2 x \sin x} = \frac{1}{\cos^2 x} \cdot \frac{1}{\sin x} = \sec^2 x \operatorname{cosec} x$   
 = RHS

(b)  $\tan x \sec x + \operatorname{cosec} x = \frac{1}{2} \sec^2 x$   
 $\Rightarrow \sec^2 x \operatorname{cosec} x = \frac{1}{2} \sec^2 x$   
 $\Rightarrow 2 \operatorname{cosec} x \sec^2 x = \sec^2 x$   
 $\Rightarrow 2 \operatorname{cosec} x \sec^2 x - \sec^2 x = 0$   
 $\Rightarrow \sec^2 x (2 \operatorname{cosec} x - 1) = 0$   
 $\Rightarrow \sec^2 x = 0 \quad \text{or} \quad \operatorname{cosec} x = \frac{1}{2}$   
 $\Rightarrow \frac{1}{\cos^2 x} = 0 \quad \text{or} \quad \frac{1}{\sin x} = \frac{1}{2}$   
 $\Rightarrow \cos^2 x = \infty \quad \text{or} \quad \sin x = 2$   
 $\therefore$  No real solutions

**Question 21** (\*\*\*)

Solve each of the following trigonometric equations.

i.  $\sin \varphi + \frac{1}{4} \sec \varphi = 0, \quad 0 \leq \varphi < \pi.$

ii.  $\cos 2y - 7 \cos y + 4 = 0, \quad 0 \leq y < 360^\circ.$

$$\varphi = \frac{7\pi}{12}, \frac{11\pi}{12}, \quad y = 60^\circ, 300^\circ$$

**Question 22** (\*\*\*)The acute angles  $\alpha$  and  $\beta$  satisfy the relationships

$$7 \cot^2 \alpha + 6 \cot \alpha = 1 \quad \text{and} \quad 6 \tan \beta = 8 + \sec^2 \beta.$$

a) Determine the value of  $\tan \alpha$  and the value of  $\tan \beta$ .

b) Show clearly that

$$\tan(\alpha + \beta) = -\frac{1}{2}.$$

$$\tan \alpha = \frac{1}{3}, \quad \tan \alpha = 7, \quad \tan \beta = 3$$



**Question 23** (\*\*\*)

Simplify, showing all steps in the calculation, the following expression

$$\tan(\arctan 3 - \arctan 2),$$

giving the final answer as an exact fraction.

$$\frac{1}{7}$$

$$\tan(\arctan 3 - \arctan 2) = \frac{\tan(\arctan 3) - \tan(\arctan 2)}{1 + \tan(\arctan 3)\tan(\arctan 2)} = \frac{3 - 2}{1 + 3 \times 2} = \frac{1}{7}$$

**Question 24** (\*\*\*)

Show clearly that if  $x > 0$

$$\arctan x + \arctan\left(\frac{1}{x}\right) = \frac{\pi}{2}.$$

proof

**Method A**

Let  $\theta = \arctan x \Rightarrow x = \tan \theta$   
 $\phi = \arctan \frac{1}{x} \Rightarrow \frac{1}{x} = \tan \phi$


$\Rightarrow \psi = \theta + \phi$   
 $\Rightarrow \tan \psi = \tan(\theta + \phi)$   
 $\Rightarrow \tan \psi = \frac{\tan \theta + \tan \phi}{1 - \tan \theta \tan \phi}$   
 $\Rightarrow \tan \psi = \frac{x + \frac{1}{x}}{1 - x \cdot \frac{1}{x}}$   
 $\Rightarrow \tan \psi = \frac{x + \frac{1}{x}}{0}$   
 $\Rightarrow \tan \psi = \infty$   
 $\Rightarrow \psi = \dots, \frac{\pi}{2}, \frac{3\pi}{2}, \dots$

But  $\theta, \phi$  are **acute angles**  
 $\therefore 0 < \theta + \phi < \pi$   
 $\therefore \psi = \frac{\pi}{2}$   
 $\therefore \arctan x + \arctan \frac{1}{x} = \frac{\pi}{2}$

**Method B**

Let  $\theta = \arctan x \Rightarrow x = \tan \theta$   
 $\phi = \arctan \frac{1}{x} \Rightarrow \frac{1}{x} = \tan \phi$

As  $x > 0$



But  $\tan(\widehat{BAC}) = \frac{1}{x}$   
 $\therefore \widehat{BAC} = \phi$   
 $\therefore \phi + \theta = \frac{\pi}{2}$   
 $\therefore \arctan \frac{1}{x} + \arctan x = \frac{\pi}{2}$

## Question 25 (\*\*\*)

Prove the validity of each of the following trigonometric identities.

a)  $\frac{\sec^2 \theta}{1 - \tan^2 \theta} \equiv \sec 2\theta.$

b)  $\frac{\cos 4\theta + \cos 2\theta}{\sin 4\theta - \sin 2\theta} \equiv \cot \theta.$

proof

Handwritten proof for Question 25a and 25b:

a)  $\text{LHS} = \frac{\sec^2 \theta}{1 - \tan^2 \theta} = \frac{\frac{1}{\cos^2 \theta}}{1 - \frac{\sin^2 \theta}{\cos^2 \theta}} = \frac{1}{\cos^2 \theta - \sin^2 \theta} = \frac{1}{\cos 2\theta} = \sec 2\theta = \text{RHS}$

b)  $\frac{\cos 4\theta + \cos 2\theta}{\sin 4\theta - \sin 2\theta} \equiv \cot \theta$

Using sum-to-product formulas:

$\cos A + \cos B = 2 \cos \frac{A+B}{2} \cos \frac{A-B}{2}$   
 $\sin A - \sin B = 2 \cos \frac{A+B}{2} \sin \frac{A-B}{2}$

$\text{LHS} = \frac{\cos 4\theta + \cos 2\theta}{\sin 4\theta - \sin 2\theta} = \frac{2 \cos \frac{4\theta+2\theta}{2} \cos \frac{4\theta-2\theta}{2}}{2 \cos \frac{4\theta+2\theta}{2} \sin \frac{4\theta-2\theta}{2}} = \frac{\cos 3\theta \cos \theta}{\cos 3\theta \sin \theta} = \cot \theta = \text{RHS}$

## Question 26 (\*\*\*)

It is given that

$$\frac{\sec^2 \theta}{1 - \tan^2 \theta} \equiv \sec 2\theta.$$

- a) Prove the validity of the above trigonometric identity.
- b) Hence, or otherwise, solve the equation

$$\frac{\sec^2 2x}{1 - \tan^2 2x} - 2 = 0, \quad 0 \leq x < \frac{\pi}{2},$$

giving the answers in terms of  $\pi$ .

$$x = \frac{\pi}{12}, \frac{5\pi}{12}$$

(a) LHS =  $\frac{\sec^2 \theta}{1 - \tan^2 \theta}$   
 $= \frac{1}{\cos^2 \theta} \cdot \frac{1}{1 - \frac{\sin^2 \theta}{\cos^2 \theta}}$   
 multiply top and bottom by  $\cos^2 \theta$   
 $= \frac{1}{\cos^2 \theta - \sin^2 \theta}$   
 $= \frac{1}{\cos 2\theta}$   
 $= \sec 2\theta$   
 $= \text{RHS}$

(b)  $\frac{\sec^2 2x}{1 - \tan^2 2x} - 2 = 0$   
 $\Rightarrow \frac{\sec^2 2x}{1 - \tan^2 2x} = 2$   
 $\Rightarrow \sec 2x = 2$   
 $\Rightarrow \cos 2x = \frac{1}{2}$   
 $\Rightarrow \cos(2x) = \frac{1}{2}$   
 $\Rightarrow 2x = \frac{\pi}{3} + 2k\pi$  or  $2x = \frac{5\pi}{3} + 2k\pi$   
 $\Rightarrow x = \frac{\pi}{6} + k\pi$  or  $x = \frac{5\pi}{6} + k\pi$   
 $\therefore x = \frac{\pi}{6}, \frac{5\pi}{6}$

**Question 27** (\*\*\*)

It is given that  $\theta$  is a reflex angle such that

$$\cos \theta = \frac{2}{3}.$$

Find the exact value of  $\sin 2\theta$ .

$$\sin 2\theta = -\frac{4\sqrt{5}}{9}$$

**Question 28** (\*\*\*)

It is given that

$$\frac{\operatorname{cosec} \theta}{\operatorname{cosec} \theta - \sin \theta} \equiv \sec^2 \theta, \quad \theta \neq \frac{n\pi}{2}, \quad n \in \mathbb{Z}.$$

- Prove the validity of the above trigonometric identity.
- Hence solve the equation

$$\frac{\operatorname{cosec} \theta}{\operatorname{cosec} \theta - \sin \theta} + 4(\sec \theta + 1) = 0, \quad 0 \leq \theta < 2\pi,$$

giving the answers in terms of  $\pi$ .

$$\theta = \frac{2\pi}{3}, \frac{4\pi}{3}$$

**Question 29** (\*\*\*)

Solve the following trigonometric equation

$$\sin 2\theta = \cot \theta, \quad 0 \leq \theta \leq 180^\circ.$$

$$\boxed{\phantom{000}}, \quad \theta = 45^\circ, 90^\circ, 135^\circ$$

Handwritten solution for Question 29:

$$\begin{aligned} \sin 2\theta &= \cot \theta \\ \Rightarrow 2 \sin \theta \cos \theta &= \frac{\cos \theta}{\sin \theta} \\ \Rightarrow 2 \sin \theta \cos \theta &= \cos \theta \\ \Rightarrow 2 \sin \theta \cos \theta - \cos \theta &= 0 \\ \Rightarrow \cos \theta (2 \sin \theta - 1) &= 0 \\ \text{either } \cos \theta &= 0 \\ \text{or } 2 \sin \theta - 1 &= 0 \\ \text{or } 2 \sin \theta &= 1 \\ \text{or } \sin \theta &= \frac{1}{2} \end{aligned}$$

For  $\cos \theta = 0$ :  
 $\theta = 90^\circ \pm 360^\circ n$   
 $\theta = 90^\circ$  (in  $0 \leq \theta \leq 180^\circ$ )

For  $\sin \theta = \frac{1}{2}$ :  
 $\theta = 30^\circ \pm 360^\circ n$   
 $\theta = 150^\circ \pm 360^\circ n$   
 $\theta = 30^\circ, 150^\circ$  (in  $0 \leq \theta \leq 180^\circ$ )

Final solutions:  $\theta = 30^\circ, 90^\circ, 150^\circ$

**Question 30** (\*\*\*)

It is given that

$$(\operatorname{cosec} \theta - \sin \theta) \sec^2 \theta \equiv \operatorname{cosec} \theta, \quad \theta \neq \frac{n\pi}{2}, n \in \mathbb{Z}.$$

a) Prove the validity of the above trigonometric identity.

b) Hence solve the equation

$$(\operatorname{cosec} \theta - \sin \theta) \sec^2 \theta = \sqrt{2}, \quad 0 \leq \theta < 2\pi,$$

giving the answers in terms of  $\pi$ .

$$\theta = \frac{\pi}{4}, \frac{3\pi}{4}$$

Handwritten solution for Question 30:

a) LHS =  $(\operatorname{cosec} \theta - \sin \theta) \sec^2 \theta = \left( \frac{1}{\sin \theta} - \sin \theta \right) \sec^2 \theta = \left( \frac{1 - \sin^2 \theta}{\sin \theta} \right) \sec^2 \theta$   
 $= \frac{\cos^2 \theta}{\sin \theta} \times \sec^2 \theta = \frac{\cos^2 \theta}{\sin \theta} \times \frac{1}{\cos^2 \theta} = \frac{1}{\sin \theta} = \operatorname{cosec} \theta = \text{RHS}$

b)  $(\operatorname{cosec} \theta - \sin \theta) \sec^2 \theta = \sqrt{2}$   
 $\Rightarrow \operatorname{cosec} \theta = \sqrt{2}$   
 $\Rightarrow \sin \theta = \frac{1}{\sqrt{2}}$   
 $\Rightarrow \operatorname{arcsin} \left( \frac{1}{\sqrt{2}} \right) = \frac{\pi}{4}$

Solutions:  $\theta = \frac{\pi}{4}, \frac{3\pi}{4}$

**Question 31** (\*\*\*)

The constants  $a$  and  $b$  are such so that

$$\tan a = \frac{1}{3} \quad \text{and} \quad \tan b = \frac{1}{7}.$$

Determine the exact value of  $\cot(a-b)$ , showing all the steps in the workings.

$$\cot(a-b) = \frac{11}{2}$$

$$\begin{aligned} \cot(a-b) &= \frac{1}{\tan(a-b)} = \frac{1}{\frac{\tan a - \tan b}{1 + \tan a \tan b}} = \frac{1 + \tan a \tan b}{\tan a - \tan b} \\ &= \frac{1 + \frac{1}{3} \times \frac{1}{7}}{\frac{1}{3} - \frac{1}{7}} = \frac{\frac{22}{21}}{\frac{2}{21}} = \frac{11}{2} \end{aligned}$$

**Question 32** (\*\*\*)

$$f(y) = 6 + 3\cos y + 4\sin y, \quad 0 < y < 2\pi.$$

- a) Express  $3\cos y + 4\sin y$  in the form  $a\cos(y-b)$ ,  $a > 0$ ,  $0 < b < \frac{\pi}{2}$ .

It is further given that for  $0 < y < 2\pi$

$$A \leq 2f(y) \leq B.$$

- b) Determine the value of each of the constants  $A$  and  $B$ .

$$\boxed{5}, \quad \boxed{3\cos y + 4\sin y \equiv 5\cos(y - 0.927^\circ)}, \quad \boxed{A=2}, \quad \boxed{B=22}$$

$$\begin{aligned} \text{(a)} \quad 3\cos y + 4\sin y &\equiv a\cos(y-b) \\ &\equiv a\cos y \cos b + a\sin y \sin b \\ &\equiv (a\cos b)\cos y + (a\sin b)\sin y \\ a\cos b &= 3 \\ a\sin b &= 4 \\ \therefore a &= \sqrt{3^2 + 4^2} = 5 \\ \tan b &= \frac{4}{3} \Rightarrow b = 0.927^\circ \\ \therefore 3\cos y + 4\sin y &\equiv 5\cos(y - 0.927^\circ) \end{aligned}$$

$$\begin{aligned} \text{(b)} \quad &\begin{cases} -5 \leq 3\cos y + 4\sin y \leq 5 \\ \quad \quad \quad (5\cos(y-0.927^\circ)) \end{cases} & \begin{cases} -2 \leq 12 + 6\cos(2y) + 8\sin(2y) \leq 22 \\ \quad \quad \quad (24\cos(\alpha)) \end{cases} \\ &\begin{cases} 1 \leq 6 + 3\cos y + 4\sin y \leq 11 \\ \quad \quad \quad (10\cos(\alpha)) \end{cases} & \begin{cases} \therefore A=2 \\ \quad \quad \quad B=22 \end{cases} \\ &\begin{cases} -2 \leq 12 + 6\cos y + 8\sin y \leq 22 \\ \quad \quad \quad (24\cos(\alpha)) \end{cases} \end{aligned}$$

## Question 33 (\*\*\*)

Use the compound angle identity for  $\tan(A+B)$ , with suitable values for  $A$  and  $B$ , to show that

$$\frac{1 + \tan 15^\circ}{1 - \tan 15^\circ} = \tan 60^\circ.$$

proof

Handwritten proof for Question 33:

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

Let  $A = 45^\circ$   $B = 15^\circ$

$$\tan(45+15) = \frac{\tan 45 + \tan 15}{1 - \tan 45 \tan 15}$$

$$\tan 60 = \frac{1 + \tan 15}{1 - \tan 15} \quad (\tan 45 = 1)$$

$$\tan 60 = \frac{1 + \tan 15}{1 - \tan 15}$$

## Question 34 (\*\*\*)

$$f(x) = \operatorname{cosec} x - \sin x, \quad 0 < x < 180^\circ.$$

Show that  $f(x) \geq 0$  for the entire domain of the function.

proof

Handwritten proof for Question 34:

$$f(x) = \operatorname{cosec} x - \sin x$$

$$f(x) = \frac{1}{\sin x} - \sin x$$

$$f(x) = \frac{1 - \sin^2 x}{\sin x}$$

$$f(x) = \frac{\cos^2 x}{\sin x}$$

Now  $y = \sin x$

$\sin x > 0$  if  $0 < x < 180^\circ$

$\cos^2 x > 0$

$\therefore f(x) > 0$

## Question 35 (\*\*\*)

Given that  $\cos x^\circ = \sin(x - 45)^\circ$ , show that

$$\tan x^\circ = 1 + \sqrt{2}.$$

 , proof

$$\begin{aligned} \Rightarrow \cos x &= \sin(x - 45) \\ \Rightarrow \cos x &= \sin x \cos 45 - \cos x \sin 45 \\ \Rightarrow \cos x &= \frac{\sqrt{2}}{2} \sin x - \frac{\sqrt{2}}{2} \cos x \\ \Rightarrow 2\cos x &= \sqrt{2} \sin x - \sqrt{2} \cos x \\ \Rightarrow \frac{2\cos x}{\sqrt{2}} &= \frac{\sqrt{2} \sin x}{\sqrt{2}} - \frac{\sqrt{2} \cos x}{\sqrt{2}} \\ \Rightarrow 2 &= \sqrt{2} \tan x - \sqrt{2} \\ \Rightarrow 2 + \sqrt{2} &= \sqrt{2} \tan x \\ \Rightarrow \tan x &= \frac{2 + \sqrt{2}}{\sqrt{2}} \\ \Rightarrow \tan x &= \sqrt{2} + 1 \end{aligned}$$

## Question 36 (\*\*\*)

$$f(x) \equiv \sin x + 2 \cos x.$$

a) Express  $f(x)$  in the form  $R \cos(x - \alpha)$ , where  $R > 0$ ,  $0 < \alpha < \frac{\pi}{2}$ .

b) Hence solve the equation

$$1 + 2 \cot x = \operatorname{cosec} x, \quad 0 < x < 2\pi.$$

$$f(x) = \sqrt{5} \cos(x - 0.464^\circ), \quad x = 1.57^\circ \cup x = 5.64^\circ$$

$$\begin{aligned} \text{a) } \sin x + 2 \cos x &\equiv R \cos(x - \alpha) \\ &\equiv R \cos x \cos \alpha + R \sin x \sin \alpha \\ &\equiv (R \cos \alpha) \cos x + (R \sin \alpha) \sin x \\ \therefore R \cos \alpha &= 2 \quad \text{SINE AND ADD} \quad R = \sqrt{2^2 + 1^2} \Rightarrow R = \sqrt{5} \\ R \sin \alpha &= 1 \quad \text{DIVIDE EQUATIONS} \quad \tan \alpha = \frac{1}{2} \Rightarrow \alpha = 0.464^\circ \\ \therefore f(x) &= \sqrt{5} \cos(x - 0.464^\circ) \\ \text{b) } 1 + 2 \cot x &= \operatorname{cosec} x \\ \Rightarrow 1 + \frac{2 \cos x}{\sin x} &= \frac{1}{\sin x} \\ \Rightarrow \sin x + 2 \cos x &= 1 \\ \Rightarrow \sqrt{5} \cos(x - 0.464^\circ) &= 1 \\ \Rightarrow \cos(x - 0.464^\circ) &= \frac{1}{\sqrt{5}} \\ \cos(x - 0.464^\circ) &= 0.4472 \\ x - 0.464^\circ &= 1.107 \pm 2\pi \\ x - 0.464^\circ &= 5.176 \pm 2\pi \\ x &= 1.57 \pm 2\pi \\ x &= 5.64 \pm 2\pi \\ \therefore x &= 1.57^\circ \\ &\text{or } 5.64^\circ \end{aligned}$$



## Question 37 (\*\*\*)

$$\frac{\sin 2x}{1 - \cos 2x} = \tan x, \quad 0 < x < 2\pi.$$

Find the solutions of the above trigonometric equation, giving the answers in radians in terms of  $\pi$ .

$$x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$$

Handwritten solution for Question 37:

$$\frac{\sin 2x}{1 - \cos 2x} = \tan x \Rightarrow \frac{2 \sin x \cos x}{1 - (1 - 2 \sin^2 x)} = \tan x$$

$$\Rightarrow \frac{2 \sin x \cos x}{2 \sin^2 x} = \tan x \Rightarrow \frac{\cos x}{\sin x} = \tan x$$

$$\Rightarrow \cot x = \tan x \Rightarrow \frac{1}{\tan x} = \tan x \Rightarrow \tan^2 x = 1$$

$$\Rightarrow \tan x = \pm 1 \Rightarrow x = \frac{\pi}{4} \pm n\pi, \quad n = 0, 1, 2, \dots$$

$$\therefore x_1 = \frac{\pi}{4}, x_2 = \frac{3\pi}{4}, x_3 = \frac{5\pi}{4}, x_4 = \frac{7\pi}{4}$$

## Question 38 (\*\*\*)

Given that

$$\tan\left(x + \frac{\pi}{4}\right) = 4 + \tan x,$$

find as an exact surd the exact value of  $\tan x$ .

$$\tan x = -2 \pm \sqrt{7}$$

Handwritten solution for Question 38:

$$\tan\left(x + \frac{\pi}{4}\right) = 4 + \tan x \Rightarrow \frac{\tan x + \tan \frac{\pi}{4}}{1 - \tan x \tan \frac{\pi}{4}} = 4 + \tan x$$

$$\Rightarrow \frac{\tan x + 1}{1 - \tan x} = 4 + \tan x \Rightarrow \frac{T+1}{1-T} = 4+T \quad (T = \tan x)$$

$$\Rightarrow T+1 = (4+T)(1-T) \Rightarrow T+1 = 4-4T+T-T^2 \Rightarrow T^2-4T+3=0$$

$$\Rightarrow (T+2)^2 = 7 \Rightarrow T+2 = \pm\sqrt{7} \Rightarrow T = -2 \pm \sqrt{7}$$

$$\therefore \tan x = -2 \pm \sqrt{7}$$

## Question 39 (\*\*\*)

$$\frac{1 + \tan^2 \theta}{1 - \tan^2 \theta} = 2, \quad 0 \leq \theta < 360.$$

Find the solutions of the above trigonometric equation, giving the answers in degrees.

$$\theta = 30^\circ, 150^\circ, 210^\circ, 330^\circ$$

Handwritten solution for Question 39:

$$\frac{1 + \tan^2 \theta}{1 - \tan^2 \theta} = 2 \Rightarrow \frac{1 + \frac{\sin^2 \theta}{\cos^2 \theta}}{1 - \frac{\sin^2 \theta}{\cos^2 \theta}} = 2$$

Multiply top & bottom of the fraction by  $\cos^2 \theta$

$$\Rightarrow \frac{\cos^2 \theta + \sin^2 \theta}{\cos^2 \theta - \sin^2 \theta} = 2 \Rightarrow \frac{1}{\cos 2\theta} = 2$$

$$\Rightarrow \cos 2\theta = \frac{1}{2}$$

$$\arccos\left(\frac{1}{2}\right) = 60^\circ$$

$$\begin{aligned} 2\theta &= 60^\circ \pm 360^\circ \quad \text{or} \quad 300^\circ \pm 360^\circ \\ 2\theta &= 300^\circ \pm 360^\circ \end{aligned}$$

$$\begin{aligned} \theta &= 30^\circ \pm 180^\circ \\ \theta &= 150^\circ \pm 180^\circ \end{aligned}$$

$$\begin{aligned} \theta &= 30^\circ \\ \theta &= 210^\circ \\ \theta &= 150^\circ \\ \theta &= 330^\circ \end{aligned}$$

## Question 40 (\*\*\*)

$$3 \sec 2\psi - 2 \cot 2\psi = 0, \quad 0 \leq \psi < 360.$$

Find the solutions of the above trigonometric equation, giving the answers in degrees.

$$\psi = 15^\circ, 75^\circ, 195^\circ, 255^\circ$$

Handwritten solution for Question 40:

$$3 \sec 2\psi - 2 \cot 2\psi = 0 \Rightarrow 3 \sec 2\psi = 2 \cot 2\psi$$

$$\Rightarrow \frac{3}{\cos 2\psi} = \frac{2 \cos 2\psi}{\sin 2\psi}$$

$$\Rightarrow 3 \sin 2\psi = 2 \cos^2 2\psi$$

$$\Rightarrow 3 \sin 2\psi = 2(1 - \sin^2 2\psi)$$

$$\Rightarrow 3 \sin 2\psi = 2 - 2 \sin^2 2\psi$$

$$\Rightarrow 2 \sin^2 2\psi + 3 \sin 2\psi - 2 = 0$$

$$\Rightarrow (2 \sin 2\psi - 1)(\sin 2\psi + 2) = 0$$

$$\sin 2\psi = \frac{1}{2}$$

$$\arcsin\left(\frac{1}{2}\right) = 30^\circ$$

$$\begin{aligned} 2\psi &= 30^\circ \pm 360^\circ \\ 2\psi &= 150^\circ \pm 360^\circ \end{aligned}$$

$$\begin{aligned} \psi &= 15^\circ \pm 180^\circ \\ \psi &= 75^\circ \pm 180^\circ \end{aligned}$$

$$\begin{aligned} \psi &= 15^\circ, 75^\circ, 195^\circ, 255^\circ \end{aligned}$$

## Question 41 (\*\*\*)

$$y = e^{-x} \sin(\sqrt{3}x), \quad x \in \mathbb{R}.$$

Find the exact values of the constants  $R$  and  $\alpha$  so that

$$\frac{dy}{dx} = R e^{-x} \cos(\sqrt{3}x + \alpha),$$

where  $R > 0$  and  $0 < \alpha < \frac{\pi}{2}$ .

$$\boxed{R = 2}, \quad \boxed{\alpha = \frac{\pi}{6}}$$

Handwritten solution for Question 41:

$$\begin{aligned}
 y &= e^{-x} \sin(\sqrt{3}x) \\
 \frac{dy}{dx} &= -e^{-x} \sin(\sqrt{3}x) + e^{-x} \sqrt{3} \cos(\sqrt{3}x) \\
 &= -e^{-x} \sin(\sqrt{3}x) + \sqrt{3} e^{-x} \cos(\sqrt{3}x) \\
 &= e^{-x} [-\sin(\sqrt{3}x) + \sqrt{3} \cos(\sqrt{3}x)] \\
 &= e^{-x} [\cos(\sqrt{3}x + \alpha)] \\
 &\equiv R \cos(\sqrt{3}x + \alpha) \\
 &\equiv R \cos(\sqrt{3}x) \cos \alpha - R \sin(\sqrt{3}x) \sin \alpha \\
 &\equiv R \cos \alpha \cos(\sqrt{3}x) - R \sin \alpha \sin(\sqrt{3}x)
 \end{aligned}$$

$\begin{aligned}
 R \cos \alpha &= \sqrt{3} \\
 R \sin \alpha &= 1
 \end{aligned} \Rightarrow R = \sqrt{(\sqrt{3})^2 + 1} = 2$

$\tan \alpha = \frac{1}{\sqrt{3}} \Rightarrow \alpha = \frac{\pi}{6}$


$\therefore y = e^{-x} \cos(\sqrt{3}x + \frac{\pi}{6})$   
 $y = 2 e^{-x} \cos[\sqrt{3}x + \frac{\pi}{6}]$

$R = 2$   
 $\alpha = \frac{\pi}{6}$

Solve the following trigonometric equation

$$\boxed{x = \frac{3}{5}}$$

$\cos \frac{1}{2} = \cos 2$   
 $\cos \left[ 2 \arctan \frac{1}{2} \right] = 2$   
 $26$   
 $2 \cos \theta - 1 = 2$   
 $2 \left( \frac{2}{\sqrt{5}} \right) - 1 = 2$   
 $2 = \frac{5}{2}$



$\theta = \arctan \frac{1}{2}$   
 $\tan \theta = \frac{1}{2}$   
 $\cos \theta = \frac{2}{\sqrt{5}}$

## Question 43 (\*\*\*)

The functions  $f$  and  $g$  are defined as

$$f(x) = 2\cos x + \sin x, \quad x \in \mathbb{R}.$$

$$g(x) = \frac{5}{x^2 + 5}, \quad x \in \mathbb{R}.$$

- a) Express  $f(x)$  in the form  $R\sin(x + \alpha)$ , where  $R > 0$  and  $0 < \alpha < \frac{\pi}{2}$ .
- b) Determine the range of  $gf(x)$ , showing clearly all the relevant workings.

$$\boxed{\phantom{000}}, \quad \boxed{f(x) = \sqrt{5} \sin(x + 1.107^c)}, \quad \boxed{\frac{1}{2} \leq gf(x) \leq 1}$$

$f(x) = 2\cos(x) + \sin(x) = R\sin(x + \alpha)$   
 $= R\sin(x)\cos(\alpha) + R\cos(x)\sin(\alpha)$   
 $= R\cos(\alpha)\sin(x) + R\sin(\alpha)\cos(x)$   
 Compare this with  $R\cos(\alpha) = 2$   
 $R\sin(\alpha) = 1$   
 $R^2 = 2^2 + 1^2 = 5$   
 $R = \sqrt{5}$   
 $\tan(\alpha) = \frac{1}{2}$   
 $\alpha = 1.107^c$   
 $\therefore f(x) = \sqrt{5}\sin(x + 1.107^c)$   
 (b)  $g(f(x)) = \frac{5}{\sin^2(x + 1.107^c) + 5}$   
 Now  $0 \leq \sin^2(x + 1.107^c) \leq 1$   
 $5 \leq \sin^2(x + 1.107^c) + 5 \leq 10$   
 $\frac{1}{10} \leq \frac{1}{\sin^2(x + 1.107^c) + 5} \leq \frac{1}{5}$   
 $\frac{1}{2} \leq \frac{5}{\sin^2(x + 1.107^c) + 5} \leq 1$   
 $\therefore \frac{1}{2} \leq g(f(x)) \leq 1$

**Question 44** (\*\*\*)

Simplify, showing clearly all the workings, the expression

$$\tan \left[ \arctan \frac{1}{3} + \arctan \frac{1}{4} \right],$$

giving the final answer as an exact fraction.

$$\boxed{\frac{7}{11}}$$

Handwritten solution for Question 44:

$$\begin{aligned} & \tan \left( \arctan \frac{1}{3} + \arctan \frac{1}{4} \right) \\ &= \tan \left( \theta + \phi \right) \\ &= \frac{\tan \theta + \tan \phi}{1 - \tan \theta \tan \phi} \\ &= \frac{\frac{1}{3} + \frac{1}{4}}{1 - \frac{1}{3} \times \frac{1}{4}} \\ &= \frac{\frac{7}{12}}{1 - \frac{1}{12}} = \frac{7}{12-1} = \frac{7}{11} \end{aligned}$$

Let  $\theta = \arctan \frac{1}{3}$   
 $\tan \theta = \frac{1}{3}$   
 $\phi = \arctan \frac{1}{4}$   
 $\tan \phi = \frac{1}{4}$

**Question 45** (\*\*\*)

Prove the validity of the following trigonometric identity

$$\frac{\sin 2\varphi}{\sin \varphi} - \frac{\cos 2\varphi}{\cos \varphi} \equiv \sec \varphi.$$

□, proof

Handwritten proof for Question 45:

$$\begin{aligned} \text{LHS} &= \frac{\sin 2\varphi}{\sin \varphi} - \frac{\cos 2\varphi}{\cos \varphi} \\ &= \frac{2\sin \varphi \cos \varphi}{\sin \varphi} - \frac{\cos^2 \varphi - 1}{\cos \varphi} \\ &= 2\cos \varphi - \frac{\cos^2 \varphi - 1}{\cos \varphi} \\ &= 2\cos \varphi - \frac{\cos^2 \varphi}{\cos \varphi} + \frac{1}{\cos \varphi} \\ &= \sec \varphi \\ &= \text{RHS} \end{aligned}$$

$$\begin{aligned} \text{LHS} &= \frac{\sin 2\varphi}{\sin \varphi} - \frac{\cos 2\varphi}{\cos \varphi} \\ &= \frac{\sin(2\varphi) + \cos(2\varphi)}{\sin \varphi \cos \varphi} \\ &= \frac{\sin(2\varphi) + \cos(2\varphi)}{\sin \varphi \cos \varphi} \\ &= \sec \varphi \\ &= \text{RHS} \end{aligned}$$

## Question 46 (\*\*\*)

It is given that

$$u = \sin \theta + \cos \theta, \quad v = \sin \theta - \cos \theta.$$

Use trigonometric identities to show that

$$u^2 + v^2 = 2.$$

proof

$u = \sin \theta + \cos \theta$   
 $v = \sin \theta - \cos \theta$

Method 1

$$\begin{aligned} u+v &= 2\sin \theta \\ u-v &= 2\cos \theta \end{aligned} \Rightarrow$$

$$\begin{aligned} (u+v)^2 &= 4\sin^2 \theta \\ (u-v)^2 &= 4\cos^2 \theta \end{aligned} \Rightarrow \text{ADD}$$

$$\begin{aligned} (u+v)^2 + (u-v)^2 &= 4\sin^2 \theta + 4\cos^2 \theta \\ u^2 + 2uv + v^2 + u^2 - 2uv + v^2 &= 4(\sin^2 \theta + \cos^2 \theta) \\ 2u^2 + 2v^2 &= 4 \\ u^2 + v^2 &= 2 \end{aligned}$$

OR - SQUARE & ADD FROM THE BEGINNING

$$\begin{aligned} u^2 &= (\sin \theta + \cos \theta)^2 = \sin^2 \theta + 2\sin \theta \cos \theta + \cos^2 \theta \\ v^2 &= (\sin \theta - \cos \theta)^2 = \sin^2 \theta - 2\sin \theta \cos \theta + \cos^2 \theta \\ \hline u^2 + v^2 &= 2\sin^2 \theta + 2\cos^2 \theta \\ u^2 + v^2 &= 2(\sin^2 \theta + \cos^2 \theta) \\ u^2 + v^2 &= 2 \end{aligned}$$

**Question 47** (\*\*\*)

Solve the trigonometric equation

$$(\operatorname{cosec} x - \sin x) \sec^2 x = 2, \quad 0 \leq x < \pi, \quad x \neq \frac{\pi}{2},$$

giving the answers in terms of  $\pi$ .

$$x = \frac{\pi}{6}, \frac{5\pi}{6}$$

$$\begin{aligned} (\operatorname{cosec} x - \sin x) \sec^2 x &= 2 \\ \Rightarrow \left( \frac{1}{\sin x} - \sin x \right) \sec^2 x &= 2 \\ \Rightarrow \frac{1 - \sin^2 x}{\sin x} \times \sec^2 x &= 2 \\ \Rightarrow \frac{\cos^2 x}{\sin x} \times \sec^2 x &= 2 \\ \Rightarrow \frac{\cos^2 x}{\sin x} \times \frac{1}{\sin^2 x} &= 2 \\ \Rightarrow \frac{1}{\sin x} &= 2 \\ \Rightarrow \sin x &= \frac{1}{2} \end{aligned}$$

$$\begin{aligned} \operatorname{cosec} \left( \frac{\pi}{6} \right) &= 2 \\ x &= \frac{\pi}{6} \text{ or } 2\pi \\ x &= \frac{5\pi}{6} \text{ or } 2\pi \end{aligned}$$

$$\therefore x_1 = \frac{\pi}{6}$$

$$x_2 = \frac{5\pi}{6}$$

**Question 48** (\*\*\*)

The angle  $\theta$  is such so that

$$\cot \theta = \frac{1}{3}.$$

Show clearly that

$$\cos \theta = \pm \frac{\sqrt{10}}{10}.$$

proof

$$\begin{aligned} \cot \theta &= \frac{1}{3} \\ \Rightarrow \tan \theta &= 3 \\ \Rightarrow 1 + \tan^2 \theta &= 10 \\ \Rightarrow \sec^2 \theta &= 10 \\ \Rightarrow \sec \theta &= \pm \sqrt{10} \\ \Rightarrow \cos \theta &= \pm \frac{1}{\sqrt{10}} \end{aligned}$$



**Question 49** (\*\*\*)

Use a detailed method to show that

$$\arccos\left(5^{-\frac{1}{2}}\right) + \arccos\left(10^{-\frac{1}{2}}\right) = \frac{3\pi}{4}.$$

 , proof

METHOD A - USING SINES AND COSINES

Let  $\alpha = \arccos \frac{1}{\sqrt{5}} + \arccos \frac{1}{\sqrt{10}}$   
 $\Rightarrow \alpha = \theta + \phi$   
 $\Rightarrow \cos \alpha = \cos(\theta + \phi)$   
 $\Rightarrow \cos \alpha = \cos \theta \cos \phi - \sin \theta \sin \phi$   
 $\Rightarrow \cos \alpha = \frac{1}{\sqrt{5}} \cdot \frac{1}{\sqrt{10}} - \frac{2}{\sqrt{5}} \cdot \frac{3}{\sqrt{10}}$   
 $\Rightarrow \cos \alpha = \frac{1}{\sqrt{50}} - \frac{6}{\sqrt{50}} = -\frac{5}{\sqrt{50}} = -\frac{1}{\sqrt{2}}$   
 $\Rightarrow \alpha = \frac{3\pi}{4}$  (At  $0 < \theta + \phi < \pi$ )  
 $\therefore \arccos \frac{1}{\sqrt{5}} + \arccos \frac{1}{\sqrt{10}} = \frac{3\pi}{4}$

METHOD B - USING TANGENTS

$\Rightarrow \alpha = \theta + \phi$   
 $\Rightarrow \tan \alpha = \tan(\theta + \phi)$   
 $\Rightarrow \tan \alpha = \frac{\tan \theta + \tan \phi}{1 - \tan \theta \tan \phi}$   
 $\Rightarrow \tan \alpha = \frac{\frac{2}{3} + \frac{3}{4}}{1 - \frac{2}{3} \times \frac{3}{4}} = \frac{\frac{17}{12}}{\frac{1}{4}} = -17$   
 $\Rightarrow \alpha = \frac{3\pi}{4}$  (At  $0 < \theta + \phi < \pi$ )  
 $\therefore \arccos \frac{1}{\sqrt{5}} + \arccos \frac{1}{\sqrt{10}} = \frac{3\pi}{4}$

Diagram 1:  $\theta = \arccos \frac{1}{\sqrt{5}}$   
 $\cos \theta = \frac{1}{\sqrt{5}}$   
 $\sin \theta = \frac{2}{\sqrt{5}}$   
 $\tan \theta = 2$   
 $\phi = \arccos \frac{1}{\sqrt{10}}$   
 $\cos \phi = \frac{1}{\sqrt{10}}$   
 $\sin \phi = \frac{3}{\sqrt{10}}$   
 $\tan \phi = 3$

Diagram 2:  $\tan \theta = 2$   
 $\tan \phi = 3$

**Question 50** (\*\*\*)

A relationship is defined as

$$x = \sin \theta \cos \theta, \quad 0 \leq \theta < 2\pi$$

$$y = 4 \cos^2 \theta, \quad 0 \leq \theta < 2\pi.$$

Use trigonometric identities to show that

$$16x^2 = y(4 - y).$$

proof

$\begin{cases} x = \sin \theta \cos \theta \\ y = 4 \cos^2 \theta \end{cases}$

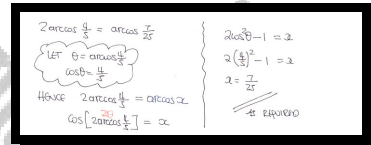
$\Rightarrow x^2 = \sin^2 \theta \cos^2 \theta$   
 $\Rightarrow x^2 = (1 - \cos^2 \theta) \cos^2 \theta$   
 $\Rightarrow 16x^2 = 4(1 - \cos^2 \theta) \cos^2 \theta$   
 $\Rightarrow 16x^2 = 4 \cos^2 \theta (4 - 4 \cos^2 \theta)$   
 $\Rightarrow 16x^2 = y(4 - y)$

**Question 51** (\*\*\*)

Show clearly that

$$2 \arccos\left(\frac{4}{5}\right) = \arccos\left(\frac{7}{25}\right).$$

proof



**Question 52** (\*\*\*)

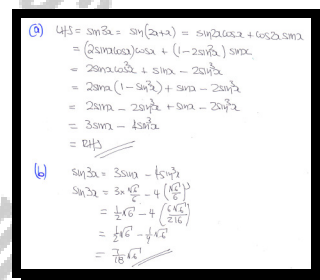
It is given that

$$\sin 3x \equiv 3 \sin x - 4 \sin^3 x.$$

- a) Prove the validity of the above trigonometric identity, by writing  $\sin 3x$  as  $\sin(2x + x)$ .

- b) Given that  $\sin x = \frac{\sqrt{6}}{6}$ , find the exact value of  $\sin 3x$ .

$$\sin 3x = \frac{7\sqrt{6}}{18}$$



**Question 53** (\*\*\*)

It is given that

$$4\operatorname{cosec}^2 2\theta - \sec^2 \theta \equiv \operatorname{cosec}^2 \theta, \quad \theta \neq \frac{n\pi}{2}, n \in \mathbb{Z}.$$

a) Prove the validity of the above trigonometric identity.

b) Hence show that if

$$4(\operatorname{cosec}^2 2\theta - 2) = \sec^2 \theta - 2\operatorname{cosec} \theta,$$

then either  $\sin \theta = \frac{1}{2}$  or  $\sin \theta = -\frac{1}{4}$ .

□, proof

Q) LHS =  $4\operatorname{cosec}^2 2\theta - \sec^2 \theta = \frac{4}{\sin^2 2\theta} - \frac{1}{\cos^2 \theta} = \frac{4}{(2\sin\theta\cos\theta)^2} - \frac{1}{\cos^2 \theta} = \frac{1}{\sin^2 \theta \cos^2 \theta} - \frac{1}{\cos^2 \theta}$

$= \frac{1 - \sin^2 \theta}{\sin^2 \theta \cos^2 \theta} = \frac{\cos^2 \theta}{\sin^2 \theta \cos^2 \theta} = \frac{1}{\sin^2 \theta} = \operatorname{cosec}^2 \theta = \text{RHS}$

(b)  $\Rightarrow 4(\operatorname{cosec}^2 2\theta - 2) = \sec^2 \theta - 2\operatorname{cosec} \theta$

$\Rightarrow 4\operatorname{cosec}^2 2\theta - 8 = \sec^2 \theta - 2\operatorname{cosec} \theta$

$\Rightarrow (4\operatorname{cosec}^2 2\theta - \sec^2 \theta) + 2\operatorname{cosec} \theta - 8$

$\Rightarrow \operatorname{cosec} \theta + 2\cos \theta - 8$

$\Rightarrow (\operatorname{cosec} \theta - 2)(\cos \theta + 4) = 0$

$\therefore \operatorname{cosec} \theta = 2 \quad \text{or} \quad \cos \theta = -4$

$\sin \theta = \frac{1}{2} \quad \text{or} \quad \sin \theta = -\frac{1}{4}$

**Question 54** (\*\*\*)

Show clearly that

$$\arctan \frac{2}{3} + \arctan \frac{5}{12} = \arctan \frac{3}{2}.$$

proof

$\Rightarrow \arctan \frac{2}{3} + \arctan \frac{5}{12} = \alpha$

$\Rightarrow \theta + \phi = \alpha$

$\Rightarrow \tan(\theta + \phi) = \tan \alpha$

$\Rightarrow \tan \alpha = \frac{\tan \theta + \tan \phi}{1 - \tan \theta \tan \phi}$

$\Rightarrow \tan \alpha = \frac{\frac{2}{3} + \frac{5}{12}}{1 - \frac{2}{3} \cdot \frac{5}{12}}$

$\Rightarrow \tan \alpha = \frac{26}{36 - 10}$

$\Rightarrow \tan \alpha = \frac{26}{26}$

$\Rightarrow \alpha = \arctan \frac{26}{26} = \arctan 1 = \frac{\pi}{4}$

ALTERNATIVE

$(3+2i)(12+5i) = 36 + 15i + 24i - 10 = 26 + 39i$

$\arg[(3+2i)(12+5i)] = \arg(26+39i)$

$\arg(3+2i) + \arg(12+5i) = \arg(26+39i)$

$\arctan \frac{2}{3} + \arctan \frac{5}{12} = \arctan \frac{39}{26}$

$\therefore \arctan \frac{2}{3} + \arctan \frac{5}{12} = \arctan \frac{3}{2}$

**Question 55** (\*\*\*)

Show clearly that

$$\sin(2 \arctan x) = \frac{2x}{x^2 + 1}.$$

**proof**

$$\begin{aligned} \sin(\alpha + \beta) &= \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta) \\ &= \sin(\alpha)\cos(\beta) \\ &= 2 \left( \frac{\alpha}{\sqrt{1+\alpha^2}} \right) \left( \frac{\beta}{\sqrt{1+\beta^2}} \right) \\ &= \frac{2\alpha\beta}{1+\alpha^2+\beta^2} \end{aligned}$$

**Question 56** (\*\*\*)

Prove the validity of the following trigonometric identity

$$\arcsin x + \arccos x = \frac{\pi}{2}.$$

**proof**

$$\begin{aligned} \text{let } y &= \arcsin x \\ \sin y &= x \\ \text{Hence } \arcsin x + \arccos x & \\ &= y + \arccos(\sin y) \\ &= y + \arccos(\cos(\frac{\pi}{2} - y)) \\ &= y + (\frac{\pi}{2} - y) \\ &= \frac{\pi}{2} \end{aligned}$$

Q8.

$\theta = \arcsin \alpha$      $\phi = \arcsin \alpha$   
 $\sin \theta = \alpha$      $\cos \phi = \alpha$

$\Rightarrow \arcsin \alpha + \arcsin \alpha = \psi$   
 $\Rightarrow \theta + \phi = \psi$   
 $\Rightarrow \sin(\theta + \phi) = \sin \psi$   
 $\Rightarrow \sin \theta \cos \phi + \cos \theta \sin \phi = \sin \psi$   
 $\Rightarrow \alpha + \sqrt{1-\alpha^2} \cdot \sqrt{1-\alpha^2} = \sin \psi$   
 $\Rightarrow \alpha^2 + (1-\alpha^2) = \sin^2 \psi$   
 $\Rightarrow \sin \psi = 1$   
 $\psi = \frac{\pi}{2} = 2\pi t$   
 $\therefore \psi = \frac{\pi}{2}$

$\therefore \arcsin \alpha + \arcsin \alpha = \frac{\pi}{2}$

Question 57 (\*\*\*)

Show clearly that

$$\arctan \frac{1}{3} + \arctan \frac{4}{3} = \arctan 3.$$

proof

$$\begin{aligned} \alpha &= \theta + \phi \\ \Rightarrow \alpha &= \arctan \frac{1}{3} + \arctan \frac{4}{3} \\ \Rightarrow \tan \alpha &= \tan \left[ \arctan \frac{1}{3} + \arctan \frac{4}{3} \right] \\ \Rightarrow \tan \alpha &= \frac{\tan(\arctan \frac{1}{3}) + \tan(\arctan \frac{4}{3})}{1 - \tan(\arctan \frac{1}{3}) \tan(\arctan \frac{4}{3})} \\ \Rightarrow \tan \alpha &= \frac{\frac{1}{3} + \frac{4}{3}}{1 - \frac{1}{3} \times \frac{4}{3}} \end{aligned} \quad \begin{aligned} \Rightarrow \tan \alpha &= \frac{\frac{5}{3}}{1 - \frac{4}{9}} \\ \Rightarrow \tan \alpha &= \frac{5}{1 - \frac{4}{9}} \\ \Rightarrow \tan \alpha &= 3 \\ \Rightarrow \alpha &= \arctan 3 \end{aligned}$$

Question 58 (\*\*\*)

Solve the trigonometric equation

$$\arcsin x = \arccos 2x.$$

$$x = \frac{1}{\sqrt{5}}$$

$$\begin{aligned} \arcsin x &= \arccos 2x \\ \Rightarrow \cos(\arcsin x) &= \cos(\arccos 2x) \\ \Rightarrow \cos(\arcsin x) &= 2x \\ \text{Sketch: let } \theta &= \arcsin x \Rightarrow x = \sin \theta \\ \cos \theta &= \sqrt{1-x^2} \\ \cos(\arcsin x) &= \sqrt{1-x^2} \\ \Rightarrow \sqrt{1-x^2} &= 2x \\ \Rightarrow 1-x^2 &= 4x^2 \\ \Rightarrow 5x^2 &= 1 \\ \Rightarrow x^2 &= \frac{1}{5} \\ \Rightarrow x &= \pm \frac{1}{\sqrt{5}} \quad (\text{see graph opposite}) \end{aligned}$$

**Question 59** (\*\*\*)

Using a detailed method, show that

$$\arctan \frac{1}{2} + \arctan \frac{1}{3} = \frac{1}{4} \pi .$$

proof

$\arctan \frac{1}{2} + \arctan \frac{1}{3} = \theta$   
 let  $A = \arctan \frac{1}{2} \Rightarrow \tan A = \frac{1}{2}$   
 $B = \arctan \frac{1}{3} \Rightarrow \tan B = \frac{1}{3}$   
 so  $A+B$  (acute) and  $A+B = \theta$   
 $A+B = \theta$   
 $\Rightarrow \tan(A+B) = \tan \theta$   
 $\Rightarrow \frac{\tan A + \tan B}{1 - \tan A \tan B} = \tan \theta$   
 $\Rightarrow \frac{\frac{1}{2} + \frac{1}{3}}{1 - \frac{1}{2} \cdot \frac{1}{3}} = \tan \theta$   
 $\Rightarrow \tan \theta = \frac{\frac{5}{6}}{\frac{5}{6}} = 1$   
 $\Rightarrow \theta = \frac{\pi}{4} \quad (0 < \arctan \frac{1}{2} + \arctan \frac{1}{3} < \pi)$   
 $\therefore \arctan \frac{1}{2} + \arctan \frac{1}{3} = \frac{\pi}{4}$  *As required*

**Question 60** (\*\*\*)Given that  $x$  is measured in radians, use small angle approximations to simplify the following expression.

$$\frac{\cos 7x - 1}{x \sin x}$$

$$\boxed{\phantom{000}}, \quad \boxed{-\frac{49}{2}}$$

USING THE APPROXIMATIONS FOR SMALL  $\theta$  IN RADIAN  
 $\bullet \sin \theta \approx \theta$   
 $\bullet \cos \theta \approx 1 - \frac{1}{2} \theta^2$   
 $\Rightarrow \frac{\cos 7x - 1}{x \sin x} \approx \frac{(1 - \frac{1}{2}(7x)^2) - 1}{x(7x)}$   
 $\approx \frac{-\frac{49}{2}x^2}{7x^2}$   
 $\approx -\frac{49}{2}$

## Question 61 (\*\*\*)

Given that  $x$  is measured in radians, use small angle approximations to simplify the following expression.

$$\frac{\cos^2(3x) - 1}{2x \sin\left(\frac{3}{4}x\right)}$$

 ,  -6

Using the approximations for small  $\theta$ , in radians

- $\sin \theta \approx \theta$
- $\cos \theta \approx 1 - \frac{1}{2}\theta^2$

$$\frac{\cos^2(3x) - 1}{2x \sin\left(\frac{3}{4}x\right)} \approx \frac{\left(1 - \frac{1}{2}(3x)^2\right)^2 - 1}{2x \left(\frac{3}{4}x\right)}$$

$$\approx \frac{\left(1 - \frac{9}{2}x^2\right)^2 - 1}{\frac{3}{2}x^2}$$

$$\approx \frac{\left(1 - 9x^2 + \frac{81}{4}x^4\right) - 1}{\frac{3}{2}x^2}$$

$$\approx \frac{-9x^2 + \frac{81}{4}x^4}{\frac{3}{2}x^2}$$

$$\approx -6$$

ALTERNATIVE

$$\frac{\cos^2(3x) - 1}{2x \sin\left(\frac{3}{4}x\right)} = \frac{1 - \cos^2(3x)}{2x \sin\left(\frac{3}{4}x\right)} = \frac{\sin^2(3x)}{2x \sin\left(\frac{3}{4}x\right)}$$

$$\approx \frac{(3x)^2}{2x \left(\frac{3}{4}x\right)}$$

$$\approx \frac{9x^2}{\frac{3}{2}x^2}$$

$$\approx -6$$

## Question 62 (\*\*\*)

Prove that

$$2\arcsin\left(\frac{2}{3}\right) = \arccos\left(\frac{1}{9}\right).$$


V, , proof

METHOD A - ZARCON'S = arccos f

LET  $\theta = \arcsin \frac{2}{3}$ , SO WE CAN GET RATIOS OFF A TRIANGLE

$\sin \theta = \frac{2}{3}$

THS  $2\theta = \varphi$ , WE JOIN UP TO GET FOUND



$\Rightarrow \cos 2\theta = \cos \varphi$   
 $\Rightarrow 1 - 2\sin^2 \theta = \cos \varphi$   
 $\Rightarrow 1 - 2 \times \left(\frac{2}{3}\right)^2 = \cos \varphi$   
 $\Rightarrow 1 - \frac{8}{9} = \cos \varphi$   
 $\Rightarrow \cos \varphi = \frac{1}{9}$   
 $\Rightarrow \varphi = \arccos \frac{1}{9}$


$\therefore 2\theta = \varphi$   
 $2\arcsin \frac{2}{3} = \arccos \frac{1}{9}$

METHOD B - ZARCON'S = arccos f (VARIANT)

$\sin \theta = \frac{2}{3}$  ( $\theta = \arcsin \frac{2}{3}$ )  
 $\sin 2\theta = \frac{4}{9}$   
 $-\sin \theta = -\frac{2}{3}$   
 $-2\sin \theta = -\frac{4}{3}$   
 $1 - 2\sin \theta = 1 - \frac{4}{3}$   
 $\cos 2\theta = \frac{1}{3}$   
 $2\theta = \arccos \frac{1}{3}$   
 $2\arcsin \frac{2}{3} = \arccos \frac{1}{3}$

METHOD C - CIRCUMFERENCE

$\arcsin \frac{2}{3} = \theta$   
 $\sin \theta = \frac{2}{3}$



HOW BY THE COSINE RULE

$1^2 = 3^2 + 3^2 - 2 \times 3 \times 3 \times \cos 2\theta$   
 $16 = 9 + 9 - 18 \cos 2\theta$   
 $18 \cos 2\theta = 2$   
 $\cos 2\theta = \frac{1}{9}$   
 $2\theta = \arccos \frac{1}{9}$   
 $2\arcsin \frac{2}{3} = \arccos \frac{1}{9}$



## Question 63 (\*\*\*)

Solve the following trigonometric equation

$$2 \arctan\left(\frac{3}{x}\right) = \arctan\left(\frac{6x}{25}\right).$$

$$\boxed{x = \pm 6}$$

Let  $\theta = \arctan\left(\frac{3}{x}\right)$  &  $\phi = \arctan\left(\frac{6x}{25}\right)$   
 $\Rightarrow 2\arctan\left(\frac{3}{x}\right) = \arctan\left(\frac{6x}{25}\right)$   
 $\Rightarrow 2\theta = \phi$   
 $\Rightarrow \tan 2\theta = \tan \phi$   
 $\Rightarrow \frac{2 \tan \theta}{1 - \tan^2 \theta} = \tan \phi$   
 But if  $\theta = \arctan\left(\frac{3}{x}\right) \Rightarrow \tan \theta = \frac{3}{x}$   
 $\phi = \arctan\left(\frac{6x}{25}\right) \Rightarrow \tan \phi = \frac{6x}{25}$   
 $\Rightarrow \frac{2\left(\frac{3}{x}\right)}{1 - \left(\frac{3}{x}\right)^2} = \frac{6x}{25}$   
 $\Rightarrow \frac{\frac{6}{x}}{1 - \frac{9}{x^2}} = \frac{6x}{25}$  (multiply top & bottom of the double fraction by  $x^2$ )  
 $\Rightarrow \frac{6x}{x^2 - 9} = \frac{6x}{25}$  (As x is 0, we have to be careful about signs by 2)  
 $\Rightarrow \frac{6x}{x^2 - 9} = \frac{6x}{25}$   
 $\Rightarrow 2(x^2 - 9) = 56$   
 $\Rightarrow x^2 - 9 = 27$   
 $\Rightarrow x^2 = 36$   
 $\Rightarrow x = \pm 6$

## Question 64 (\*\*\*)

Show, by detailed workings, that

$$\arctan 2 + \arctan 3 = \frac{3\pi}{4}.$$

proof

$\bullet \arctan 2 + \arctan 3 = \varphi$   
 $\Rightarrow \tan(\arctan 2 + \arctan 3) = \tan \varphi$   $\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$   
 $\Rightarrow \frac{\tan(\arctan 2) + \tan(\arctan 3)}{1 - \tan(\arctan 2) \tan(\arctan 3)} = \tan \varphi$   
 $\Rightarrow \frac{2+3}{1-2 \times 3} = \tan \varphi$   
 $\Rightarrow \tan \varphi = -1$   
 $\Rightarrow \varphi = \arctan(-1) \pm n\pi$   
 $\Rightarrow \varphi = \arctan(-1) + \pi$   $\text{or } \begin{matrix} 0 < \arctan 2 < \frac{\pi}{2} \\ 0 < \arctan 3 < \frac{\pi}{2} \\ \text{AND } 0 < \varphi < \pi \end{matrix}$   
 $\Rightarrow \varphi = -\frac{\pi}{4} + \pi$   
 $\Rightarrow \arctan 2 + \arctan 3 = \frac{3\pi}{4}$

$\bullet \text{ALTERNATIVE BY COMPLEX NUMBERS}$   
 Let  $z = 1+2i \Rightarrow \arg z = \arctan 2$   
 $w = 1+3i \Rightarrow \arg w = \arctan 3$   
 $\Rightarrow \arg z + \arg w = \arg(zw)$   
 $\Rightarrow \arctan 2 + \arctan 3 = \arg[(1+2i)(1+3i)]$   
 $\Rightarrow \arctan 2 + \arctan 3 = \arg[1+3i+2i-6]$   
 $\Rightarrow \arctan 2 + \arctan 3 = \arg(-5+5i)$   
 $\Rightarrow \arctan 2 + \arctan 3 = \arctan\left(\frac{5}{-5}\right) + \pi$   $\text{As the number is in the 2nd quadrant}$   
 $\Rightarrow \arctan 2 + \arctan 3 = \arctan(-1) + \pi$   
 $\Rightarrow \arctan 2 + \arctan 3 = -\frac{\pi}{4} + \pi$   
 $\Rightarrow \arctan 2 + \arctan 3 = \frac{3\pi}{4}$

## Question 65 (\*\*\*)

Find the general solution of the following trigonometric equation

$$2\arctan(\sin x) = \arctan(\sec x).$$

$$x = \frac{\pi}{4} + k\pi, \quad k \in \mathbb{Z}$$

$2\arctan(\sin x) = \arctan(\sec x)$   
 Taking tangents on both sides  
 $\text{i.e. } \tan 2\theta = \tan \phi$   
 $\Rightarrow \frac{2 \sin x}{1 - \sin^2 x} = \frac{1}{\cos x}$   
 $\Rightarrow \frac{2 \sin x}{\cos x} = \frac{1}{\cos x}$   
 $\Rightarrow \sin x = \frac{1}{2}$   
 $\Rightarrow \tan x = 1$   $\therefore x = \frac{\pi}{4} + n\pi$   
 $n = 0, 1, 2, \dots$

## Question 66 (\*\*\*\*)

$$f(x) = (x^2 + 1)(x - 1), \quad x \in \mathbb{R}.$$

a) Simplify  $f(x)$ .

b) Prove the validity of the trigonometric identity

$$\frac{\operatorname{cosec} \theta - \sin \theta}{\sec \theta - \cos \theta} \equiv \cot^3 \theta.$$

c) Hence, or otherwise, solve the equation

$$\frac{\operatorname{cosec} \theta - \sin \theta}{\sec \theta - \cos \theta} + \cot \theta = \operatorname{cosec}^2 \theta, \quad 0 \leq x < 2\pi,$$

giving the answers in terms of  $\pi$ .

$$x^3 - x^2 + x - 1, \quad x = \frac{\pi}{4}, \frac{5\pi}{4}$$

(a)  $(x^2+1)(x-1) = x^3 - x^2 + x - 1$   
 (b) 
$$\frac{\operatorname{cosec} \theta - \sin \theta}{\sec \theta - \cos \theta} = \frac{\frac{1}{\sin \theta} - \sin \theta}{\frac{1}{\cos \theta} - \cos \theta} = \frac{\frac{1 - \sin^2 \theta}{\sin \theta}}{\frac{1 - \cos^2 \theta}{\cos \theta}} = \frac{\cos^2 \theta}{\sin^2 \theta} = \cot^2 \theta$$
  
 (c) 
$$\frac{\operatorname{cosec} \theta - \sin \theta}{\sec \theta - \cos \theta} + \cot \theta = \operatorname{cosec}^2 \theta$$
  

$$\cot^2 \theta + \cot \theta = \operatorname{cosec}^2 \theta$$
  

$$\cot^2 \theta + \cot \theta = 1 + \cot^2 \theta$$
  

$$\cot^2 \theta - \cot^2 \theta + \cot \theta - 1 = 0$$
  

$$(\cot \theta + 1)(\cot \theta - 1) = 0$$
  

$$\cot \theta = 1 \quad \text{or} \quad \cot \theta = -1$$
  

$$\cot \theta = 1 \Rightarrow \theta = \frac{\pi}{4}, \frac{5\pi}{4}$$
  

$$\cot \theta = -1 \Rightarrow \theta = \frac{3\pi}{4}, \frac{7\pi}{4}$$
  

$$\theta = \frac{\pi}{4}, \frac{5\pi}{4}, \frac{3\pi}{4}, \frac{7\pi}{4}$$

**Question 67** (\*\*\*\*)

Solve the following trigonometric equation

$$2 \cot \theta - 3 \operatorname{cosec} \theta = 2 \sec \theta \operatorname{cosec} \theta, \quad 0 < \theta < 2\pi, \quad \theta \neq \frac{k\pi}{2}, k \in \mathbb{Z},$$

giving the answers in terms of  $\pi$ .

$$\theta = \frac{2\pi}{3}, \frac{4\pi}{3}$$

**Question 68** (\*\*\*\*)

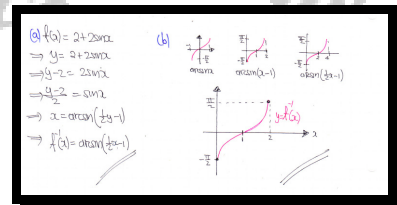
$$f(x) = 2 + 2 \sin x, \quad -\pi \leq x \leq \pi.$$

a) Find an expression for  $f^{-1}(x)$ .

b) Sketch the graph of  $f^{-1}(x)$ .

The sketch must include the coordinates of any points where the graph of  $f^{-1}(x)$  meet the coordinate axes as well as the coordinates of its endpoints.

$$f^{-1}(x) = \arcsin\left(\frac{1}{2}x - 1\right)$$



**Question 69** (\*\*\*\*)

Prove the validity of each of the following trigonometric identities.

a)  $2 \cot 2\theta + \tan \theta \equiv \cot \theta$ .

b)  $\frac{\sin 3x}{\sin x} - \frac{\cos 3x}{\cos x} \equiv 2$ .

proof

$$\begin{aligned} \text{LHS} &= 2 \cot 2\theta + \tan \theta \\ &= \frac{2}{\tan 2\theta} + \tan \theta \\ &= \frac{2}{\frac{\sin 2\theta}{\cos 2\theta}} + \frac{\sin \theta}{\cos \theta} \\ &= \frac{2 \cos 2\theta}{\sin 2\theta} + \frac{\sin \theta}{\cos \theta} \\ &= \frac{2(1 - \tan^2 \theta)}{2 \tan \theta} + \frac{\sin \theta}{\cos \theta} \\ &= \frac{1 - \tan^2 \theta}{\tan \theta} + \frac{\sin \theta}{\cos \theta} \\ &= \frac{1}{\tan \theta} - \frac{\tan^3 \theta}{\tan \theta} + \frac{\sin \theta}{\cos \theta} \\ &= \cot \theta - \tan^2 \theta + \frac{\sin \theta}{\cos \theta} \\ &= \cot \theta \end{aligned}$$

$$\begin{aligned} \text{RHS} &= \frac{\sin 3\theta}{\sin \theta} - \frac{\cos 3\theta}{\cos \theta} \\ &= \frac{\sin(2\theta + \theta)}{\sin \theta} - \frac{\cos(2\theta + \theta)}{\cos \theta} \\ &= \frac{\sin 2\theta \cos \theta + \cos 2\theta \sin \theta}{\sin \theta} - \frac{\cos 2\theta \cos \theta - \sin 2\theta \sin \theta}{\cos \theta} \\ &= \frac{\sin 2\theta \cos \theta}{\sin \theta} + \frac{\cos 2\theta \sin \theta}{\sin \theta} - \frac{\cos 2\theta \cos \theta}{\cos \theta} + \frac{\sin 2\theta \sin \theta}{\cos \theta} \\ &= \sin 2\theta + \cos 2\theta - \cos 2\theta + \sin 2\theta \\ &= 2 \sin 2\theta \\ &= 2 \end{aligned}$$

**Question 70** (\*\*\*\*)

Show that the following trigonometric equation

$$\tan 2\theta - 3 \cot \theta = 0, \quad 0 < \theta < 2\pi,$$

has six solutions in the interval  $0 < \theta < 2\pi$ , giving the answers in terms of  $\pi$ .

$$\theta = \frac{1}{3}\pi, \frac{1}{2}\pi, \frac{2}{3}\pi, \frac{4}{3}\pi, \frac{3}{2}\pi, \frac{5}{3}\pi$$

$$\tan 2\theta - 3 \cot \theta = 0$$

$$\frac{2 \tan \theta}{1 - \tan^2 \theta} - \frac{3}{\tan \theta} = 0$$

$$\frac{2 \tan^2 \theta - 3(1 - \tan^2 \theta)}{\tan \theta (1 - \tan^2 \theta)} = 0$$

$$2 \tan^2 \theta - 3 + 3 \tan^2 \theta = 0$$

$$5 \tan^2 \theta - 3 = 0$$

$$\tan^2 \theta = \frac{3}{5}$$

$$\tan \theta = \pm \sqrt{\frac{3}{5}}$$

$$\theta = \frac{1}{3}\pi, \frac{1}{2}\pi, \frac{2}{3}\pi, \frac{4}{3}\pi, \frac{3}{2}\pi, \frac{5}{3}\pi$$

## Question 71 (\*\*\*)

$$f(\theta^\circ) \equiv (\sqrt{3}+1)\cos 2\theta^\circ + (\sqrt{3}-1)\sin 2\theta^\circ.$$

a) Express  $f(\theta)$  in the form  $R\sin(2\theta + \alpha)$ ,  $R > 0$ ,  $0 \leq \theta^\circ < 90$ .

b) Solve the equation

$$f(\theta^\circ) = 2, \quad 0 \leq \theta^\circ < 360.$$

$$\boxed{\phantom{000}}, \quad \boxed{f(\theta^\circ) \equiv \sqrt{8} \sin(2\theta + 75)^\circ}, \quad \boxed{\theta = 30^\circ, 165^\circ, 210^\circ, 345^\circ}$$

(a)  $(\sqrt{3}+1)\cos 2\theta + (\sqrt{3}-1)\sin 2\theta \equiv R\sin(2\theta + \alpha)$   
 $\equiv R\cos\alpha\cos 2\theta + R\sin\alpha\sin 2\theta$   
 $\equiv (R\cos\alpha)\cos 2\theta + (R\sin\alpha)\sin 2\theta$   
 $\therefore \begin{cases} R\cos\alpha = \sqrt{3}+1 \\ R\sin\alpha = \sqrt{3}-1 \end{cases} \Rightarrow R^2 = (\sqrt{3}+1)^2 + (\sqrt{3}-1)^2 = 3 - 2\sqrt{3} + 1 + 3 + 2\sqrt{3} - 1 = 8$   
 $R = \sqrt{8}$   
 $\Rightarrow \tan\alpha = \frac{\sqrt{3}-1}{\sqrt{3}+1} \Rightarrow \alpha = 75^\circ$   
 $\therefore f(\theta) = \sqrt{8} \sin(2\theta + 75)$

(b)  $f(\theta) = 2$   
 $\sqrt{8} \sin(2\theta + 75) = 2$   
 $\sin(2\theta + 75) = \frac{2}{\sqrt{8}} = \frac{1}{\sqrt{2}}$   
 $\arcsin\left(\frac{1}{\sqrt{2}}\right) = 45^\circ$   
 $2\theta + 75 = 45 \pm 360n$   
 $2\theta + 75 = 135 \pm 360n$   
 $2\theta = -30 \pm 360n$   
 $\theta = -15 \pm 180n$   
 $\theta = 30 \pm 180n$   
 $\theta_1 = 15^\circ$   
 $\theta_2 = 345^\circ$   
 $\theta_3 = 30^\circ$   
 $\theta_4 = 210^\circ$

## Question 72 (\*\*\*\*)

It is given that

$$\frac{\sec x}{1 + \sec x} - \frac{\sec x}{1 - \sec x} \equiv 2 \operatorname{cosec}^2 x.$$

a) Prove the validity of the above trigonometric identity.

b) Hence, or otherwise, solve the equation

$$\frac{\sec x}{1 + \sec x} - \frac{\sec x}{1 - \sec x} = 16 \sin x, \quad 0 \leq x < 2\pi,$$

giving the answers in terms of  $\pi$ .

$$x = \frac{\pi}{6}, \frac{5\pi}{6}$$

(a) LHS =  $\frac{\sec x}{1 + \sec x} - \frac{\sec x}{1 - \sec x} = \frac{\sec x(1 - \sec x) - \sec x(1 + \sec x)}{(1 + \sec x)(1 - \sec x)}$   
 $= \frac{\sec x - \sec^2 x - \sec x - \sec^2 x}{1 - \sec^2 x} = \frac{-2\sec^2 x}{1 - \sec^2 x} = \frac{-2\sec^2 x}{1 - \frac{1}{\cos^2 x}}$   
 $= \frac{-2\sec^2 x}{\frac{\cos^2 x - 1}{\cos^2 x}} = \frac{-2\sec^2 x \cdot \cos^2 x}{\cos^2 x - 1} = \frac{-2}{\cos^2 x - 1} = \frac{2}{1 - \cos^2 x} = \frac{2}{\sin^2 x} = 2 \operatorname{cosec}^2 x = \text{RHS}$

(b)  $\frac{\sec x}{1 + \sec x} - \frac{\sec x}{1 - \sec x} = 16 \sin x \Rightarrow \sin x = \frac{1}{2}$   
 $\Rightarrow 2 \operatorname{cosec}^2 x = 16 \sin x$   
 $\Rightarrow 2 \sec^2 x = 16 \sin x$   
 $\Rightarrow \frac{1}{\sin^2 x} = 8 \sin x$   
 $\Rightarrow 8 \sin^3 x = 1$   
 $\Rightarrow \sin^3 x = \frac{1}{8}$   
 $\Rightarrow \sin x = \frac{1}{2}$   
 $\Rightarrow x = \frac{\pi}{6} \text{ or } \frac{5\pi}{6}$   
 $\Rightarrow x = \frac{\pi}{6} \text{ or } \frac{5\pi}{6}$

**Question 73** (\*\*\*\*)

Solve the trigonometric equation

$$\operatorname{cosec} \theta - \sin \theta + 2 \cos^2 \theta \cot \theta = 0, \quad 0 < \theta < 2\pi, \quad \theta \neq \pi,$$

giving the answers in terms of  $\pi$ .

$$\theta = \frac{\pi}{2}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{3\pi}{2}$$

**Question 74** (\*\*\*\*)

$$\sin \theta = \frac{5}{13} \quad \text{and} \quad \sin \varphi = -\frac{7}{25}.$$

If  $\theta$  is obtuse and  $\varphi$  is such so that  $180^\circ < \varphi < 270^\circ$ , show that

$$\sin(\theta + \varphi) = -\frac{36}{325}.$$

proof



**Question 75** (\*\*\*\*)

It is given that

$$\sin 2\theta \equiv \frac{2 \tan \theta}{1 + \tan^2 \theta}.$$

- a) Prove the validity of the above trigonometric identity.
- b) By using  $\theta = 15^\circ$  in the above identity, show that

$$\tan 15^\circ = 2 - \sqrt{3}.$$

proof

**Question 76** (\*\*\*\*)

Prove the validity of each of the following trigonometric identities.

a)  $\frac{2 \sin x \cos x - \cos x}{1 - \sin x + \sin^2 x - \cos^2 x} \equiv \cot x.$

b)  $\frac{\sec 2x - 1}{\sec 2x + 1} \equiv \tan^2 x.$

proof

**Question 77** (\*\*\*\*)

It is given that

$$\cos 3x \equiv 4\cos^3 x - 3\cos x.$$

- a) Prove the validity of the above trigonometric identity by writing  $\cos 3x$  as  $\cos(2x+x)$ .
- b) Hence, or otherwise, solve the trigonometric equation

$$8\cos^3 x - 6\cos x + 1 = 0, \quad 0 \leq x < 2\pi,$$

giving the answers in terms of  $\pi$ .

$$x = \frac{2\pi}{9}, \frac{4\pi}{9}, \frac{8\pi}{9}, \frac{10\pi}{9}, \frac{14\pi}{9}, \frac{16\pi}{9}$$

**Question 78** (\*\*\*\*)

It is given that

$$\cos \theta - \theta \sin \theta = 0.9994.$$

Given that the above equation has a solution that is numerically small, show by using a quadratic approximation that  $\theta = \pm 0.02^\circ$ .

proof

## Question 79 (\*\*\*\*)

$$\sin(A+B) \equiv \sin A \cos B + \cos A \sin B.$$

- a) Use the above trigonometric identity with suitable values for  $A$  and  $B$ , to show

$$\sin 75^\circ = \frac{\sqrt{6} + \sqrt{2}}{4}.$$

- b) Hence by using the trigonometric expansion of  $\cos(75^\circ + \alpha)$  with a suitable value for  $\alpha$ , show that

$$\cos 165^\circ = -\sin 75^\circ.$$

proof

$$\begin{aligned} \text{(a)} \quad \sin 75 &= \sin(45+30) = \sin 45 \cos 30 + \cos 45 \sin 30 \\ &= \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2} \cdot \frac{1}{2} = \frac{\sqrt{6}}{4} + \frac{\sqrt{2}}{4} = \frac{\sqrt{6} + \sqrt{2}}{4} \\ \text{(b)} \quad \text{Use } \alpha = 90 \\ \cos 165 &= \cos(90+75) = \cos 90 \cos 75 - \sin 90 \sin 75 \\ &= 0 \times \cos 75 - 1 \times \frac{\sqrt{6} + \sqrt{2}}{4} \\ &= -\frac{\sqrt{6} + \sqrt{2}}{4} = -\sin 75 \end{aligned}$$

## Question 80 (\*\*\*\*)

Use small angle approximations to show that if  $x$  is measured in radians then

$$\frac{1 + \cos x}{1 + \sin\left(\frac{1}{2}x\right)} \approx A + Bx,$$

where  $A$  and  $B$  are constants to be found.

,  $A = 2$ ,  $B = -1$

USING THE STANDARD APPROXIMATIONS FOR  $\sin x$  &  $\cos x$   

$$\frac{1 + \cos x}{1 + \sin\left(\frac{1}{2}x\right)} \approx \frac{1 + \left(1 - \frac{x^2}{2}\right)}{1 + \frac{x}{2}} = \frac{2 - \frac{x^2}{2}}{2 + x}$$

$$\approx \frac{2 - \frac{x^2}{2}}{2 + x} = \frac{(2 - \frac{x^2}{2})(2 - x)}{(2 + x)(2 - x)}$$

$$\approx \frac{2 - x}{2 - x} = 2 - x$$
 AS REQUESTED  $\therefore A = 2, B = -1$

## Question 81 (\*\*\*\*)

Solve each of the following trigonometric equations.

i.  $6 \tan x = \frac{2 - 3 \sec^2 x}{\tan x - 1}, \quad 0 \leq x < 2\pi, \quad x \neq \frac{\pi}{4}, \frac{5\pi}{4}.$

ii.  $\cos(3\theta - 60^\circ) = \cos(3\theta + 30^\circ), \quad 0 \leq \theta \leq 180^\circ.$

$\boxed{\phantom{0000}}, \quad \boxed{x \approx 0.322^c, \quad x \approx 3.46^c}, \quad \boxed{\theta = 5^\circ, 65^\circ, 125^\circ}$

**Method 1**

$$6 \tan x = \frac{2 - 3 \sec^2 x}{\tan x - 1} \quad 0 \leq x < 2\pi$$

$$\Rightarrow 6 \tan x (\tan x - 1) = 2 - 3 \sec^2 x$$

$$\Rightarrow 6 \tan^2 x - 6 \tan x = 2 - 3 \sec^2 x$$

$$\Rightarrow 6 \tan^2 x - 6 \tan x = 2 - 3(1 + \tan^2 x)$$

$$\Rightarrow 6 \tan^2 x - 6 \tan x = -1 - 3 \tan^2 x$$

$$\Rightarrow 9 \tan^2 x - 6 \tan x + 1 = 0$$

$$\Rightarrow (3 \tan x - 1)^2 = 0$$

$$\Rightarrow \tan x = \frac{1}{3}$$

$$x = \arctan\left(\frac{1}{3}\right) \pm n\pi \quad n=0,1,2,3,\dots$$

$$x = 0.32175 \dots \pm n\pi$$

$$x = \begin{cases} 0.321^c \\ 3.463^c \end{cases}$$

**Method 2**

$$\Rightarrow (\sqrt{3}+1) \tan 3\theta = \sqrt{3}-1$$

$$\Rightarrow \tan 3\theta = \frac{\sqrt{3}-1}{\sqrt{3}+1}$$

$$\Rightarrow 3\theta = \arctan\left(\frac{\sqrt{3}-1}{\sqrt{3}+1}\right) \pm 180n \quad n=0,1,2,3,\dots$$

$$\Rightarrow 3\theta = 15^\circ \pm 180n$$

$$\Rightarrow \theta = 5^\circ \pm 60n$$

$$\Rightarrow \theta = \begin{cases} 5^\circ \\ 65^\circ \\ 125^\circ \end{cases}$$

**Alternative Method**

$$\Rightarrow \cos(3\theta - 60^\circ) = \cos(3\theta + 30^\circ)$$

$$\Rightarrow 3\theta - 60^\circ = (3\theta + 30^\circ) \pm 360n$$

$$\Rightarrow 3\theta - 60^\circ = 360^\circ - (3\theta + 30^\circ) \pm 360n \quad n=0,1,2,3,\dots$$

$$\Rightarrow 6\theta = 390^\circ \pm 360n$$

$$\Rightarrow \theta = 65^\circ \pm 60n$$

$$\Rightarrow \theta = \begin{cases} 65^\circ \\ 5^\circ \\ 125^\circ \end{cases}$$

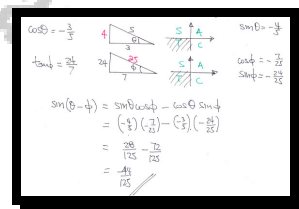
## Question 82 (\*\*\*\*)

$$\cos \theta = -\frac{3}{5} \quad \text{and} \quad \tan \varphi = \frac{24}{7}.$$

If  $\theta$  is reflex, and  $\varphi$  is also reflex, show that

$$\sin(\theta - \varphi) = -\frac{44}{125}.$$

proof

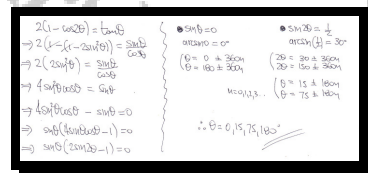


## Question 83 (\*\*\*\*)

Solve the trigonometric equation

$$2(1 - \cos 2\theta) = \tan \theta, \quad 0 \leq \theta \leq 180^\circ.$$

$$\theta = 0^\circ, 15^\circ, 75^\circ, 180^\circ$$



**Question 84** (\*\*\*\*)

A curve has equation

$$y = \pi - \arccos(x+1), \quad -2 \leq x \leq 0.$$

- a) Describe geometrically the 3 transformations that map the graph of

$$y = \arccos x, \quad -1 \leq x \leq 1,$$

onto the graph of

$$y = \pi - \arccos(x+1), \quad -2 \leq x \leq 0.$$

- b) Sketch the graph of

$$y = \pi - \arccos(x+1), \quad -2 \leq x \leq 0.$$

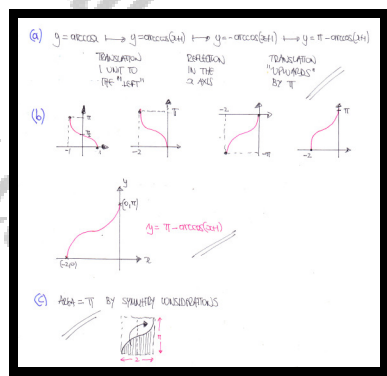
The sketch must include the coordinates of any points where the graph meets the coordinate axes.

- c) Use symmetry arguments to find the area of the finite region bounded by

$$y = \pi - \arccos(x+1), \quad -2 \leq x \leq 0,$$

and the coordinate axes.

, translation by 1 unit to the right, followed by reflection in the  $x$  axis,   
  area =  $\pi$



## Question 85 (\*\*\*\*)

It is given that

$$(\cos x + \sec x)^2 \equiv \cos^2 x + \tan^2 x + 3.$$

- a) Prove the validity of the above trigonometric identity.
- b) Hence solve the trigonometric equation

$$\cos^2 x + \tan^2 x = \frac{13}{4}, \quad 0 \leq x < 2\pi,$$

giving the answers in terms of  $\pi$ .

$$\boxed{\phantom{000}}, \quad x = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$$

1) SIMPLIFY FROM THE LEFT-HAND SIDE

$$\begin{aligned} \text{LHS} &= (\cos x + \sec x)^2 \\ &= \cos^2 x + 2\cos x \sec x + \sec^2 x \\ &= \cos^2 x + 2\cos x \left(\frac{1}{\cos x}\right) + \left(\frac{1}{\cos^2 x}\right) \\ &= \cos^2 x + 2 + \frac{1}{\cos^2 x} \\ &= \cos^2 x + \tan^2 x + 3 \\ &= \text{R.H.S.} \end{aligned}$$

2) CHECK THE ABOVE RESULT

$$\begin{aligned} \Rightarrow \cos^2 x + \tan^2 x &= \frac{13}{4} \\ \Rightarrow \cos^2 x + \frac{\sin^2 x}{\cos^2 x} + 3 &= \frac{13}{4} + 3 \\ \Rightarrow (\cos^2 x + \sec^2 x) &= \frac{25}{4} \\ \Rightarrow \cos^2 x + \sec^2 x &= \pm \frac{5}{2} \\ \Rightarrow \cos^2 x + \frac{1}{\cos^2 x} &= \pm \frac{5}{2} \\ \Rightarrow \cos^2 x + 1 &= \pm \frac{5}{2} \cos^2 x \\ \Rightarrow 2\cos^2 x + 1 &= \pm 5\cos^2 x \\ \Rightarrow 2\cos^2 x \pm 5\cos^2 x + 1 &= 0 \end{aligned}$$

3) FACTORISE THE QUADRATIC

$$\begin{aligned} \Rightarrow (2\cos^2 x + 1)(\cos^2 x + 2) &= 0 \\ \text{or} \\ (2\cos^2 x - 1)(\cos^2 x - 2) &= 0 \\ \Rightarrow \cos^2 x &= \frac{1}{2} \quad \text{or} \quad \cos^2 x = 2 \end{aligned}$$

4) SOLVE THE QUADRATIC

$$\begin{aligned} \cos^2 x &= \frac{1}{2} \quad \text{or} \quad \cos^2 x = 2 \\ \cos x &= \pm \frac{1}{\sqrt{2}} \quad \text{or} \quad \cos x = \pm \sqrt{2} \end{aligned}$$

5) FIND THE SOLUTIONS

$$\begin{aligned} \cos x &= \pm \frac{1}{\sqrt{2}} \quad \text{or} \quad \cos x = \pm \sqrt{2} \\ x &= \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4} \end{aligned}$$

## Question 86 (\*\*\*\*)

$$\cos(A+B) \equiv \cos A \cos B - \sin A \sin B$$

$$\cos(A-B) \equiv \cos A \cos B + \sin A \sin B$$

a) By using the above identities show that

$$\cos(A+B) + \cos(A-B) \equiv 2 \cos A \cos B.$$

b) Hence show that

$$\cos P + \cos Q \equiv 2 \cos\left(\frac{P+Q}{2}\right) \cos\left(\frac{P-Q}{2}\right).$$

c) Deduce that

$$\frac{\cos 4x + \cos 2x}{2 \cos 3x} \equiv \cos x.$$

proof

(a)  $\cos(A+B) = \cos A \cos B - \sin A \sin B$   
 $\cos(A-B) = \cos A \cos B + \sin A \sin B$   
 Add:  $\cos(A+B) + \cos(A-B) = 2 \cos A \cos B$   
 (b) Let  $P = A+B$ ,  $Q = A-B$   
 $A = \frac{P+Q}{2}$ ,  $B = \frac{P-Q}{2}$   
 Sub into (a)  $\Rightarrow \cos P + \cos Q = 2 \cos\left(\frac{P+Q}{2}\right) \cos\left(\frac{P-Q}{2}\right)$   
 (c) LHS =  $\frac{\cos 4x + \cos 2x}{2 \cos 3x} = \frac{2 \cos\left(\frac{4x+2x}{2}\right) \cos\left(\frac{4x-2x}{2}\right)}{2 \cos 3x}$   
 $= \frac{2 \cos 3x \cos x}{2 \cos 3x} = \cos x = \text{RHS}$



**Question 87** (\*\*\*\*)

Solve each of the following trigonometric equations.

i.  $\sec \theta = \frac{1 - \tan^2 \theta}{4 \sec \theta - 9}, \quad 0 \leq \theta < 2\pi.$

ii.  $2\cos\left(x+\frac{\pi}{2}\right)+\sin\left(x+\frac{\pi}{3}\right)=0, \quad 0 \leq x < 2\pi.$

$$\theta = \frac{\pi}{3}, \frac{5\pi}{3}, \quad x = \frac{\pi}{6}, \frac{7\pi}{6}$$

(7)  $\sec \theta = \frac{1 - \tan^2 \theta}{2 \cos \theta} = 1$   
 $\Rightarrow 4 \sec \theta - 1 \sec \theta = 1 - \tan^2 \theta$   
 $\Rightarrow 4 \sec \theta - \sec \theta = 1 - (\sec^2 \theta - 1)$   
 $\Rightarrow 3 \sec \theta - \sec \theta = 2 - 2 \sec \theta$   
 $\Rightarrow 5 \sec \theta - 4 \sec \theta = 2 - 2 \cdot 0$   
 $\Rightarrow (5 \sec \theta - 1) \sec \theta - 2 = 0$   
 $\Rightarrow \sec \theta = \frac{2}{5}$   
 $\Rightarrow \cos \theta = \frac{5}{2}$   
 $\therefore \omega_{\text{max}}(\xi) = \frac{\pi}{5}$

(8)  $\frac{\theta}{\pi} = \frac{\pi}{2} \pm n\pi$   
 $\frac{\theta}{\pi} = \frac{\pi}{2} \pm 2n\pi$        $n=0,1,2,\dots$   
 $\therefore \frac{\theta_1}{\pi} = \frac{\pi}{2}$   
 $\frac{\theta_2}{\pi} = \frac{5}{2}$

**Question 88** (\*\*\*\*)

Prove the validity of each of the following trigonometric identities.

a)  $\frac{\cos 2x - \cos x + 1}{\sin 2x - \sin x} \equiv \cot x.$

**b)**  $\tan\left(\theta + \frac{\pi}{4}\right) + \tan\left(\theta - \frac{\pi}{4}\right) \equiv 2 \tan 2\theta.$

proof

$$\begin{aligned} \text{(a)} \quad \frac{1}{\sin A} &= \frac{\cos A + i \sin A}{\sin A - i \sin A} = \frac{2 \cos A \cdot \cancel{1 - \cos A}}{2 \sin A (\cos A - \sin A)} = \frac{2 \cos A \cdot \cancel{(\cos A - i \sin A)}}{\sin A (2 \cos A - i)} \\ &= \frac{\cos A (2 \cos A - i)}{\sin A (2 \cos A - i)} = \cot A = \text{P.H.S.} \end{aligned}$$

$$\begin{aligned} \text{(b)} \quad \frac{1}{\sin \theta} &= \tan(\theta + \frac{\pi}{4}) + \tan(\theta - \frac{\pi}{4}) = \frac{\tan \theta + \tan \frac{\pi}{4}}{1 - \tan \theta \tan \frac{\pi}{4}} + \frac{\tan \theta - \tan \frac{\pi}{4}}{1 - \tan \theta \tan \frac{\pi}{4}} \\ &= \frac{\tan \theta + 1}{1 - \tan \theta} + \frac{\tan \theta - 1}{1 - \tan \theta} = \frac{(\tan \theta + 1)^2 + (\tan \theta - 1)(1 - \tan \theta)}{(1 - \tan \theta)(1 - \tan \theta)} \\ &= \frac{\cancel{\tan^2 \theta} + 2 \tan \theta + \cancel{1} + \cancel{\tan \theta} - \cancel{1} - \cancel{\tan \theta} + \cancel{1} + \tan^2 \theta}{1 - \tan^2 \theta} = \frac{4 \tan \theta}{1 - \tan^2 \theta} \\ &= 2 \left[ \frac{2 \tan \theta}{1 - \tan^2 \theta} \right] = 2 \tan 2\theta = \text{R.H.S.} \end{aligned}$$

**Question 89** (\*\*\*\*)

It is given that

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}.$$

- a) Use the above trigonometric identity to express  $\tan 2\theta$  in terms of  $\tan \theta$ .
- b) Hence determine the exact value of  $\tan 22.5^\circ$ , showing clearly all the relevant workings.

$$\tan 2\theta = \frac{2 \tan \theta}{1 - \tan^2 \theta}, \quad \tan 22.5^\circ = -1 + \sqrt{2}$$

(a)  $\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$   
 Let  $A=B=\theta$   
 $\Rightarrow \tan(\theta+\theta) = \frac{\tan \theta + \tan \theta}{1 - \tan \theta \tan \theta}$   
 $\Rightarrow \tan 2\theta = \frac{2 \tan \theta}{1 - \tan^2 \theta}$

(b) Let  $\theta = 22.5^\circ$   
 $\Rightarrow \tan 45^\circ = \frac{2 \tan 22.5^\circ}{1 - \tan^2 22.5^\circ}$   
 $\Rightarrow 1 = \frac{2T}{1-T^2}$  where  $T = \tan 22.5^\circ$   
 $\Rightarrow 1 - T^2 = 2T$   
 $\Rightarrow 0 = T^2 + 2T - 1$   
 $\Rightarrow 0 = (T+1)^2 - 2$   
 $\Rightarrow 2 = (T+1)^2$   
 $\Rightarrow \pm \sqrt{2} = T+1$   
 $\Rightarrow T = -1 \pm \sqrt{2}$   
 $\Rightarrow \tan 22.5^\circ = -1 + \sqrt{2}$  (since  $\tan 22.5^\circ > 0$ )

**Question 90** (\*\*\*\*)

Prove the validity of the trigonometric identity

$$\frac{1 - \cos 2x + \sin 2x}{1 + \cos 2x + \sin 2x} \equiv \tan x.$$

proof

$$\begin{aligned} \text{LHS} &= \frac{1 - \cos 2x + \sin 2x}{1 + \cos 2x + \sin 2x} = \frac{1 - (1 - 2\sin^2 x) + \sin 2x}{1 + (2\cos^2 x - 1) + \sin 2x} = \frac{2\sin^2 x + \sin 2x}{2\cos^2 x + \sin 2x} \\ &= \frac{2\sin^2 x + 2\sin x \cos x}{2\cos^2 x + 2\sin x \cos x} = \frac{2\sin x (\sin x + \cos x)}{2\cos x (\cos x + \sin x)} = \frac{\sin x}{\cos x} = \tan x = \text{RHS} \end{aligned}$$

## Question 91 (\*\*\*\*)

In this question it is given that the exact value of  $\tan 20^\circ = t$ .

a) Express  $\tan 25^\circ$  in terms of  $t$ .

b) By using the result of part (a) show that

$$\tan 25^\circ \tan 65^\circ = 1.$$

c) Show further that if

$$2 \cos(\theta^\circ + 20^\circ) = 5 \sin(\theta^\circ - 20^\circ),$$

then

$$\tan \theta = \frac{2+5t}{5+2t}.$$

$$\boxed{\phantom{000}}, \quad \tan 25^\circ = \frac{1-t}{1+t}$$

$\tan 25 = \tan(45-20) = \frac{\tan 45 - \tan 20}{1 + \tan 45 \tan 20} = \frac{1-t}{1+t}$   
 $\tan 65 = \tan(45+20) = \frac{\tan 45 + \tan 20}{1 - \tan 45 \tan 20} = \frac{1+t}{1-t}$   
 $\therefore \tan 25 \tan 65 = \frac{1-t}{1+t} \times \frac{1+t}{1-t} = 1$   
 (a)  $2 \cos(\theta+20) = 5 \sin(\theta-20)$   
 $\Rightarrow 2 \cos \theta \cos 20 - 2 \sin \theta \sin 20 = 5 \sin \theta \cos 20 - 5 \cos \theta \sin 20$   
 $\Rightarrow 2 \cos \theta \cos 20 - 2 \sin \theta \sin 20 = 5 \sin \theta \cos 20 - 5 \cos \theta \sin 20$   
 $\Rightarrow 2 - 2 \tan \theta \tan 20 = 5 \tan \theta - 5$   
 $\Rightarrow 2 + 5t = 5 \tan \theta + 2t \tan \theta$   
 $\Rightarrow (2+5t) = (5+2t) \tan \theta$   
 $\Rightarrow \tan \theta = \frac{2+5t}{5+2t}$

## Question 92 (\*\*\*)

$$f(x) = -2 + 2 \tan\left(\frac{1}{2}x\right), \quad -\pi \leq x \leq \pi.$$

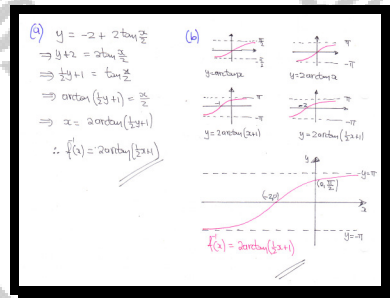
a) Find an expression for  $f^{-1}(x)$ .

b) Sketch the graph of  $f^{-1}(x)$ .

The sketch must include ...

- ...the equations of the asymptotes of  $f^{-1}(x)$
- ...the coordinates of any points where the graph of  $f^{-1}(x)$  meets the coordinate axes.

$$f^{-1}(x) = 2 \arctan\left(\frac{1}{2}x + 1\right)$$



## Question 93 (\*\*\*\*)

The function  $f$  is defined as

$$f(x) = \frac{1}{1 + \tan x}, \quad 0 \leq x < \frac{\pi}{2}.$$

- Use differentiation to show that  $f$  is a one to one function.
- Find a simplified expression for the inverse of  $f$ .
- Determine the range of  $f$ .

$$\boxed{\text{graph}}, \quad f^{-1}(x) = \arctan\left(\frac{1-x}{x}\right), \quad \boxed{0 < f(x) \leq 1}$$

(a)  $f(x) = \frac{1}{1+\tan x} = (1+\tan x)^{-1}$   
 $f'(x) = -(1+\tan x)^{-2} \times \sec^2 x$   
 $f'(x) = -\frac{\sec^2 x}{(1+\tan x)^2}$   
 Since  $f'(x) < 0$  for the entire domain, the function is decreasing, so the function is one-to-one.

(b)  $y = \frac{1}{1+\tan x}$   
 $1+\tan x = \frac{1}{y}$   
 $\tan x = \frac{1}{y} - 1$   
 $\tan x = \frac{1-y}{y}$   
 $x = \arctan\left(\frac{1-y}{y}\right)$   
 $\therefore f^{-1}(y) = \arctan\left(\frac{1-y}{y}\right)$

(c) The domain is  $0 \leq x < \frac{\pi}{2}$   
 $\tan x > 0$   
 $1+\tan x > 1$   
 $0 < \frac{1}{1+\tan x} \leq 1$   
 $\therefore$  Range  $0 < f(x) \leq 1$

## Question 94 (\*\*\*\*)

$$\frac{\cos\left(\frac{1}{2}x\right)}{1+\sin x} = 0.925.$$

Given that the above equation has a solution that is numerically small, find this solution by using a quadratic approximation.

*No credit will be given for solving a trigonometric equation.*

$$\boxed{\phantom{00000}}, \quad \boxed{x \approx 0.08}$$

TRIG. APPROXIMATIONS FOR SMALL ANGLES

$$\sin \theta \approx \theta$$

$$\cos \theta \approx 1 - \frac{\theta^2}{2}$$

$$\cos\left(\frac{\theta}{2}\right) \approx 1 - \frac{\left(\frac{\theta}{2}\right)^2}{2}$$

$$\approx 1 - \frac{\theta^2}{8}$$

THUS WE HAVE

$$\Rightarrow \frac{\cos \frac{x}{2}}{1 + \sin x} = 0.925$$

$$\Rightarrow \frac{1 - \frac{x^2}{8}}{1 + x} = 0.925$$

$$\Rightarrow \frac{8 - x^2}{8 + 8x} = 0.925$$

$$\Rightarrow 7.4 + 7.4x = 8 - x^2$$

$$\Rightarrow x^2 + 7.4x - 0.6 = 0$$

QUADRATIC FORMULA OR COMPLETING THE SQUARE

$$\Rightarrow (x + 3.7)^2 - 3.7^2 - 0.6 = 0$$

$$\Rightarrow (x + 3.7)^2 = 14.29$$

$$\Rightarrow x + 3.7 = \begin{matrix} 0.082 \dots \\ -3.782 \dots \end{matrix}$$

$$\Rightarrow x = \begin{matrix} 0.082 \dots \\ -3.782 \dots \end{matrix}$$

$\therefore x \approx 0.08$

## Question 95 (\*\*\*\*)

It is given that

$$\sin 3x = 3\sin x - 4\sin^3 x.$$

- a) Prove the validity of the above trigonometric identity, by writing  $\sin 3x$  as  $\sin(2x+x)$ .
- b) Hence, or otherwise, find the exact value of

$$\int_0^{\frac{\pi}{2}} \sin^3 x \, dx.$$

2
3

a)  $\sin 3x = \sin(2x+x) = \sin 2x \cos x + \cos 2x \sin x$   
 $= (2\sin x \cos x) \cos x + (1-2\sin^2 x) \sin x$   
 $= 2\sin x \cos^2 x + \sin x - 2\sin^3 x$   
 $= 2\sin x(1-\sin^2 x) + \sin x - 2\sin^3 x$   
 $= 2\sin x - 2\sin^3 x + \sin x - 2\sin^3 x$   
 $= 3\sin x - 4\sin^3 x$   
 $= \sin 3x$

b)  $\int_0^{\frac{\pi}{2}} \sin^3 x \, dx = \dots$   
 $\sin^3 x = \sin x - \sin x \cos^2 x$   
 $\int \sin^3 x \, dx = \int \sin x \, dx - \int \sin x \cos^2 x \, dx$   
 $= -\cos x - \left( -\frac{1}{3} \cos^3 x \right) + C$   
 $= -\cos x + \frac{1}{3} \cos^3 x + C$   
 $\int_0^{\frac{\pi}{2}} \sin^3 x \, dx = \left[ -\cos x + \frac{1}{3} \cos^3 x \right]_0^{\frac{\pi}{2}}$   
 $= \left( -\cos \frac{\pi}{2} + \frac{1}{3} \cos^3 \frac{\pi}{2} \right) - \left( -\cos 0 + \frac{1}{3} \cos^3 0 \right)$   
 $= \left( 0 + \frac{1}{3} \cdot 0 \right) - \left( -1 + \frac{1}{3} \cdot 1 \right)$   
 $= 1 - \frac{1}{3} = \frac{2}{3}$

## Question 96 (\*\*\*\*)

Solve the trigonometric equation

$$7\sin^2 x + \sin x \cos x = 6, \quad 0^\circ \leq x \leq 360^\circ,$$

giving the answers to the nearest degree.

	$x \approx 63^\circ, 108^\circ, 243^\circ, 288^\circ$
--	---

$7\sin^2 x + \sin x \cos x = 6$   
 $\frac{7\sin^2 x + \sin x \cos x}{\cos^2 x} = \frac{6}{\cos^2 x}$   
 $7\tan^2 x + \tan x = 6\sec^2 x$   
 $7\tan^2 x + \tan x = 6(1+\tan^2 x)$   
 $7\tan^2 x + \tan x = 6 + 6\tan^2 x$   
 $\tan^2 x + \tan x - 6 = 0$   
 $(\tan x - 2)(\tan x + 3) = 0$

$\tan x = -3$   
 $\arctan(-3) = -71.57^\circ$   
 $\alpha = 63.4^\circ \pm 180^\circ$   
 $\alpha = 71.6^\circ \pm 180^\circ$   
 $\therefore \alpha = 63^\circ, 243^\circ, 108^\circ, 288^\circ$

Alternative  
 $7\sin^2 x + \sin x \cos x = 6$   
 $14\sin^2 x + 2\sin x \cos x = 12$   
 $14\left(\frac{1}{2} - \frac{1}{2}\cos 2x\right) + \sin 2x = 12$   
 $7 - 7\cos 2x + \sin 2x = 12$   
 $\sin 2x - 7\cos 2x = 5$

Let  $\sin 2x = 5$   
 $\text{At } \sin(2x) = 5$   
 $\text{AND } \cos(2x) = 1$   
 $\text{APPROXIMATE}$

## Question 97 (\*\*\*\*)

$$f(x) = \sqrt{3} \cos x - \sin x, \quad x \in \mathbb{R}.$$

- a) Express  $f(x)$  in the form  $R \cos(x + \alpha)$ ,  $R > 0$ ,  $0 < \alpha < \frac{\pi}{2}$ .
- b) State the maximum value of  $f(x)$  and find the smallest positive value of  $x$  for which this maximum occurs.

The depth of the water,  $D$  metres, in a harbour is modelled by the equation

$$D = 13 + \sqrt{3} \cos\left(\frac{\pi t}{6}\right) - \sin\left(\frac{\pi t}{6}\right), \quad 0 \leq t < 24$$

where  $t$  is the time in hours measured since midnight.

- c) State the maximum depth of the water in the harbour and a time when this maximum depth occurs.
- d) Find the times when the depth of the water in the harbour is 12 metres.

$$\boxed{\phantom{000}}, \quad \boxed{\sqrt{3} \cos x - \sin x \equiv 2 \cos\left(x + \frac{\pi}{6}\right)}, \quad \boxed{\max = 2}, \quad \boxed{x = \frac{11\pi}{6}}, \quad \boxed{D_{\max} = 15},$$

$$\boxed{11:00/23:00}, \quad \boxed{03:00/07:00/15:00/19:00}$$

$f(x) = \sqrt{3} \cos x - \sin x \equiv R \cos(x + \alpha)$   
 $\equiv R \cos x \cos \alpha - R \sin x \sin \alpha$   
 $\equiv (R \cos \alpha) \cos x - (R \sin \alpha) \sin x$   
 $\therefore \begin{cases} R \cos \alpha = \sqrt{3} \\ R \sin \alpha = 1 \end{cases} \Rightarrow R = \sqrt{(\sqrt{3})^2 + 1^2} = 2$   
 $\tan \alpha = \frac{1}{\sqrt{3}} \Rightarrow \alpha = \frac{\pi}{6}$   
 $\therefore f(x) = 2 \cos\left(x + \frac{\pi}{6}\right)$   
**(b)**  $f(x) = 2$  if  $\cos\left(x + \frac{\pi}{6}\right) = 1$   
 $x + \frac{\pi}{6} = 0 \Rightarrow x = -\frac{\pi}{6}$   
 $x = \frac{11\pi}{6}$   
**(c)**  $D = 13 + \sqrt{3} \cos\left(\frac{\pi t}{6}\right) - \sin\left(\frac{\pi t}{6}\right)$   
 $D = 13 + 2 \cos\left(\frac{\pi t}{6} + \frac{\pi}{6}\right)$   
 $\therefore D_{\max} = 13 + 2 = 15$   
 $\cos\left(\frac{\pi t}{6} + \frac{\pi}{6}\right) = 1$   
 $\frac{\pi t}{6} + \frac{\pi}{6} = 0 \pm 2\pi n \Rightarrow \frac{\pi t}{6} = -\frac{\pi}{6} \pm 2\pi n$   
 $\frac{\pi t}{6} = -\frac{\pi}{6} \pm 2\pi n \Rightarrow t = -1 \pm 12n$   
 $t = 11$  or  $23$   
**(d)**  $D = 12 \Rightarrow 13 + 2 \cos\left(\frac{\pi t}{6} + \frac{\pi}{6}\right) = 12$   
 $\cos\left(\frac{\pi t}{6} + \frac{\pi}{6}\right) = -\frac{1}{2}$   
 $\frac{\pi t}{6} + \frac{\pi}{6} = \frac{2\pi}{3} \pm 2\pi n$   
 $\frac{\pi t}{6} = \frac{\pi}{2} \pm 2\pi n \Rightarrow t = 3 \pm 12n$   
 $t = 3$  or  $15$   
 $t = 7$  or  $19$   
 Times: 03:00, 07:00, 15:00, 19:00



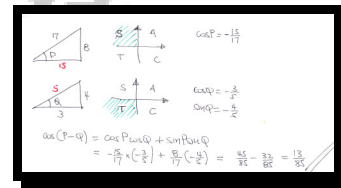
**Question 98** (\*\*\*\*)

$$\sin P = \frac{8}{17} \quad \text{and} \quad \tan Q = \frac{4}{3}.$$

If  $P$  is obtuse and  $Q$  is reflex, show that

$$\cos(P - Q) = \frac{13}{85}.$$

proof



**Question 99** (\*\*\*\*)

Prove the validity of each of the following trigonometric identities.

**a)**  $2 - 2 \tan x - \frac{2 \tan x}{\tan 2x} \equiv (1 - \tan x)^2.$

**b)**  $\frac{1 - \cos x}{1 + \cos x} \equiv \cot^2\left(\frac{x}{2}\right).$

proof

(a) 
$$\begin{aligned} \text{LHS} &= 2 - 2\sin\alpha - \frac{2\sin\alpha}{\tan\alpha} = 2 - 2\sin\alpha - \frac{2\sin\alpha}{1 - \tan^2\alpha} \\ &= 2 - 2\sin\alpha - \frac{2\sin\alpha}{1 - \tan^2\alpha} = 2 - 2\sin\alpha - (1 - \tan^2\alpha) \\ &= 1 - 2\sin\alpha + \tan^2\alpha = (1 - \sin\alpha)^2 = \text{RHS} \end{aligned}$$

(b) 
$$\begin{aligned} \text{LHS} &= \frac{1 + \cos\alpha}{1 - \cos\alpha} \cdot \frac{1 + 2\cos^2\alpha - 1}{1 - 2\sin^2\alpha} = \frac{2\cos^2\alpha}{2\cos^2\alpha} = \cot^2\frac{\alpha}{2} = \text{RHS} \end{aligned}$$

$$\begin{aligned} \cos 2\alpha &\equiv \cos^2\alpha - \sin^2\alpha \\ \cos 4\alpha &\equiv \cos^2 2\alpha - \sin^2 2\alpha \\ \cos 2\alpha &\equiv 1 - 2\sin^2\alpha \\ \cos 4\alpha &\equiv 1 - 2\sin^2 2\alpha \end{aligned}$$

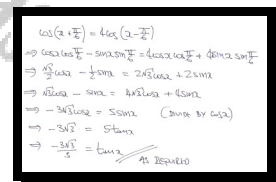
## Question 100 (\*\*\*\*)

$$\cos\left(x + \frac{\pi}{6}\right) = 4\cos\left(x - \frac{\pi}{6}\right).$$

Show by using an appropriate compound angle identity that

$$\tan x = -\frac{3}{5}\sqrt{3}.$$

proof



Handwritten proof for Question 100:

$$\begin{aligned}\cos\left(x + \frac{\pi}{6}\right) &= 4\cos\left(x - \frac{\pi}{6}\right) \\ \Rightarrow \cos x \cos \frac{\pi}{6} - \sin x \sin \frac{\pi}{6} &= 4\cos x \cos \frac{\pi}{6} + 4\sin x \sin \frac{\pi}{6} \\ \Rightarrow \frac{\sqrt{3}}{2}\cos x - \frac{1}{2}\sin x &= 2\sqrt{3}\cos x + 2\sin x \\ \Rightarrow \sqrt{3}\cos x - \sin x &= 4\sqrt{3}\cos x + 2\sin x \\ \Rightarrow -3\sqrt{3}\cos x &= 3\sin x \quad (\text{Divide by } \cos x) \\ \Rightarrow -3\sqrt{3} &= 3\tan x \\ \Rightarrow -\sqrt{3} &= \tan x\end{aligned}$$

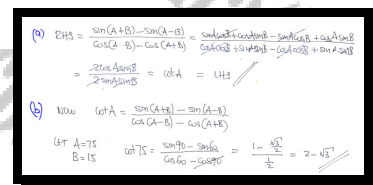
## Question 101 (\*\*\*\*)

Show clearly that ...

a) ...  $\cot A \equiv \frac{\sin(A+B) - \sin(A-B)}{\cos(A-B) - \cos(A+B)}.$

b) ...  $\cot 75^\circ = 2 - \sqrt{3},$   
(use part (a) with suitable values of  $A$  and  $B$ )

proof



Handwritten proof for Question 101:

(a)  $\cot A = \frac{\sin(A+B) - \sin(A-B)}{\cos(A-B) - \cos(A+B)}$

$$\begin{aligned}&= \frac{\sin A \cos B + \cos A \sin B - (\sin A \cos B - \cos A \sin B)}{\cos A \cos B + \sin A \sin B - (\cos A \cos B - \sin A \sin B)} \\&= \frac{2\cos A \sin B}{2\sin A \sin B} = \cot A\end{aligned}$$

(b) Now  $\cot A = \frac{\sin(A+B) - \sin(A-B)}{\cos(A-B) - \cos(A+B)}$

Let  $A = 75^\circ$   
 $B = 15^\circ$

$$\cot 75^\circ = \frac{\sin 90^\circ - \sin 60^\circ}{\cos 60^\circ - \cos 90^\circ} = \frac{1 - \frac{\sqrt{3}}{2}}{\frac{1}{2} - 0} = 2 - \sqrt{3}$$

**Question 102** (\*\*\*\*)

The curves  $C_1$  and  $C_2$  have respective equations

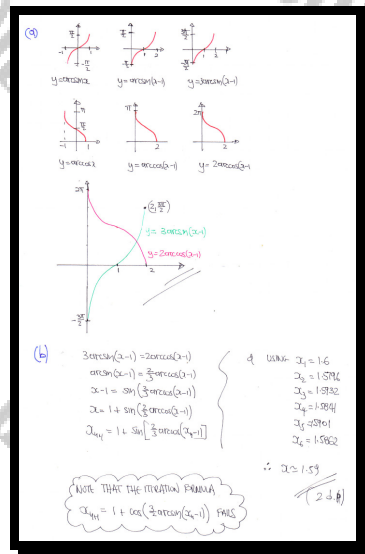
$$y_1 = 3\arcsin(x-1) \text{ and } y_1 = 2\arccos(x-1).$$

- a) Sketch in the same diagram the graph of  $C_1$  and the graph of  $C_2$ .

The sketch must include the coordinates of any points where the graphs of  $C_1$  and  $C_2$  meet the coordinate axes as well as the coordinates of the endpoints of the curves.

- b) Use a suitable iteration formula of the form  $x_{n+1} = f(x_n)$  with  $x_1 = 1.6$  to find the  $x$  coordinate of the point of intersection between the graph of  $C_1$  and the graph of  $C_2$ .

$$x \approx 1.59$$



**Question 103** (\*\*\*\*)

It is given that

$$\sin P + \sin Q \equiv 2 \sin\left(\frac{P+Q}{2}\right) \cos\left(\frac{P-Q}{2}\right).$$

- a) Prove the validity of the above trigonometric identity by using the compound angle identities for  $\sin(A+B)$  and  $\sin(A-B)$ .
- b) Hence, or otherwise, solve the trigonometric equation

$$\sin 7x + \sin x = 0, \quad 0 \leq x < \pi,$$

giving the answers in terms of  $\pi$ .

$$\boxed{\phantom{000000}}, \quad x = 0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}, \frac{3\pi}{4}, \frac{5\pi}{6}$$

**a) STARTING FROM THE COMPOUND ANGLE IDENTITIES**

$$\begin{aligned} \sin(A+B) &= \sin A \cos B + \cos A \sin B \\ \sin(A-B) &= \sin A \cos B - \cos A \sin B \end{aligned} \quad \text{ADDING}$$

$$\Rightarrow \sin(A+B) + \sin(A-B) = 2 \sin A \cos B$$

NOW LET IN THE L.H.S OF THE ABOVE EXPRESSION

$$\begin{aligned} A+B &= P & \text{OR} & & P &= A+B \\ A-B &= Q & & & Q &= A-B \end{aligned}$$

ADDING THE ABOVE

$$\Rightarrow 2A = P+Q \quad \Rightarrow A = \frac{P+Q}{2}$$

SUBTRACTING THE ABOVE

$$2B = P-Q \quad \Rightarrow B = \frac{P-Q}{2}$$

HENCE WE OBTAIN

$$\begin{aligned} \sin(A+B) + \sin(A-B) &= 2 \sin A \cos B \\ \sin P + \sin Q &= 2 \sin\left(\frac{P+Q}{2}\right) \cos\left(\frac{P-Q}{2}\right) \quad \text{Hence proved} \end{aligned}$$

**b) USING FACT (a) WITH  $P=7x$  &  $Q=x$**

$$\begin{aligned} \Rightarrow \sin 7x + \sin x &= 0 \\ \Rightarrow 2 \sin\left(\frac{7x+x}{2}\right) \cos\left(\frac{7x-x}{2}\right) &= 0 \\ \Rightarrow 2 \sin(4x) \cos(3x) &= 0 \end{aligned}$$

EITHER  $\sin 4x = 0$  OR  $\cos 3x = 0$

**ALTERNATIVE FOR PART (b) WITHOUT USING FACT (a)**

$$\begin{aligned} \Rightarrow \sin 7x + \sin x &= 0 \\ \Rightarrow \sin 7x &= -\sin x \\ \Rightarrow \sin 7x &= \sin(-x) \end{aligned}$$

$$\begin{aligned} \Rightarrow 7x &= -x + 2n\pi & \text{OR} & & 7x &= \pi - (-x) + 2n\pi \\ \Rightarrow 8x &= -x + 2n\pi & & & 8x &= \pi + x + 2n\pi \\ \Rightarrow 9x &= 2n\pi & & & 7x &= \pi + 2n\pi \end{aligned}$$

$$\Rightarrow x = \frac{2n\pi}{9} \quad \text{OR} \quad x = \frac{\pi + 2n\pi}{7}$$

WHICH GIVES THE SAME SOLUTIONS AS BEFORE

**Question 104** (\*\*\*\*)

Solve each of the following trigonometric equations.

i.  $\frac{5 + \tan^2 y}{\sec y} = 9 - \sec y, \quad 0 \leq y < 2\pi, \quad y \neq \frac{\pi}{2}, \frac{3\pi}{2}.$

ii.  $\sqrt{2} \cos\left(\theta + \frac{\pi}{4}\right) = \sin\left(\theta + \frac{\pi}{6}\right), \quad 0 \leq \theta < 2\pi.$

$$y = 1.32^\circ, 4.97^\circ, \quad \theta = \frac{\pi}{12}, \frac{13\pi}{12}$$

**Question 105** (\*\*\*\*)

Prove the validity of each of the following trigonometric identities.

a)  $(\cos x + \sin x)(\operatorname{cosec} x - \sec x) \equiv 2 \cot 2x.$

b)  $\frac{\cos x}{1 - \sin x} + \frac{1 - \sin x}{\cos x} \equiv \sec x.$

proof

It is given that

$$2\cos\left(x+\frac{\pi}{6}\right)=\sec\left(x+\frac{\pi}{2}\right), \quad 0\leq x\leq\pi,$$

giving the answers in terms of  $\pi$ .

$$\boxed{\phantom{000}}, \quad x = \frac{\pi}{2}, \frac{5\pi}{6}$$

(a)  $\cos(A+B) + \cos(A-B) = \cos(A+B) + \cos(A+B) + \sin(A+B) + \sin(A+B)$   
 $= 2\cos(A+B)$   
 $\therefore \cos(A+B) + \cos(A-B) \equiv 2\cos(A+B)$

(b)  $2\cos\left(2x + \frac{\pi}{4}\right) = \sec\left(2x + \frac{\pi}{4}\right)$   
 $\Rightarrow 2\cos\left(2x + \frac{\pi}{4}\right) = \frac{1}{\cos\left(2x + \frac{\pi}{4}\right)}$   
 $\Rightarrow 2\cos\left(2x + \frac{\pi}{4}\right) \cos\left(2x + \frac{\pi}{4}\right) = 1$   
 $2\cos^2\left(2x + \frac{\pi}{4}\right) = \cos\left(2x + \frac{\pi}{4}\right) \cos\left(2x + \frac{\pi}{4}\right)$   
 $\Rightarrow \cos\left(2x + \frac{\pi}{4}\right) = \cos\left(2x + \frac{\pi}{4}\right) \Rightarrow 1$   
 $\Rightarrow \cos\left(2x + \frac{\pi}{4}\right) = \frac{1}{2}$

General solution for  $\cos(x) = \frac{1}{2}$  is:  
 $x = 2n\pi \pm \frac{\pi}{3}$   
 $x = 2n\pi \pm \frac{2\pi}{3}$   
 $x = 2n\pi \pm \frac{\pi}{3}$   
 $x = 2n\pi \pm \frac{2\pi}{3}$

## Question 107 (\*\*\*\*)

It is given that

$$\cos 2x + \tan x \sin 2x \equiv 1, \quad x \neq 90^\circ(n+1), \quad n \in \mathbb{Z}.$$

- a) Prove the validity of the above trigonometric identity.
- b) Use the above result to solve the trigonometric equation

$$\tan x \sin 2x + 13 \cos x = 8, \quad 0 \leq x < 360^\circ.$$

$$x = 60^\circ, 300^\circ$$

(a) LHS =  $\cos 2x + \tan x \sin 2x$   
 $= \cos 2x + \frac{\sin x}{\cos x} \cdot 2 \sin x \cos x$   
 $= \cos 2x + 2 \sin^2 x$   
 $= \cos 2x + 2(1 - \cos^2 x)$   
 $= \cos 2x + 2 - 2 \cos^2 x$   
 $= 2 - 2 \cos^2 x + 2 \cos^2 x$   
 $= 2$   
 RHS = 2  
 LHS = RHS  
 Identity is proved.

(b) FROM PART (a)  
 $\tan x \sin 2x = 1 - \cos 2x$   
 Hence  
 $\Rightarrow \tan x \sin 2x + 13 \cos x = 8$   
 $\Rightarrow 1 - \cos 2x + 13 \cos x = 8$   
 $\Rightarrow 1 - (2 \cos^2 x - 1) + 13 \cos x = 8$   
 $\Rightarrow 2 - 2 \cos^2 x + 13 \cos x = 8$   
 $\Rightarrow 0 = 2 \cos^2 x - 13 \cos x + 6$   
 $\Rightarrow 0 = (2 \cos x - 1)(\cos x - 6)$   
 $\therefore \cos x = \frac{1}{2}$   
 $\cos x = \frac{1}{2} \Rightarrow x = 60^\circ$   
 $x = 60^\circ \pm 360^\circ$   
 $x = 300^\circ \pm 360^\circ$   
 $x_1 = 60^\circ$   
 $x_2 = 300^\circ$

## Question 108 (\*\*\*\*)

$$3\cos^2 x - \cos x = 1.99375$$

It is given that the above trigonometric equation has a solution that is numerically small.

Use small angle approximations to find this solution.

*No credit will be given for standard solution methods.*

$$\boxed{\phantom{00000}}, \quad \boxed{x \approx \pm 0.05}$$

**METHOD A**

Using the double angle formula:  $\cos(2x) = 2\cos^2 x - 1$

$$\Rightarrow 3\left(\frac{1 + \cos(2x)}{2}\right) - \cos x = 1.99375$$

$$\Rightarrow \frac{3}{2} + \frac{3}{2}\cos(2x) - \cos x = 1.99375$$

$$\Rightarrow 3 + 3\cos(2x) - 2\cos x = 1.99375 \times 2$$

Using a quadratic approximation for  $\cos x$  and  $\cos(2x)$

$$\cos x \approx 1 - \frac{x^2}{2}$$

$$\cos(2x) \approx 1 - \frac{(2x)^2}{2} \approx 1 - 2x^2$$

Substitute into equation:

$$\Rightarrow 3 + 3(1 - 2x^2) - 2\left(1 - \frac{x^2}{2}\right) = 1.99375 \times 2$$

$$\Rightarrow 3 + 3 - 6x^2 - 2 + x^2 = 1.99375 \times 2$$

$$\Rightarrow 4 - 5x^2 = 3.9875$$

$$\Rightarrow 5x^2 = 0.0125$$

$$\Rightarrow x^2 = 0.0025$$

$$\Rightarrow x = \pm 0.05$$

Small angle approx OK as  $0.05 \text{ rad} \approx 2.86^\circ$

**METHOD B**

$$3\cos^2 x - \cos x = 1.99375$$

$$3\cos^2 x - \cos x - 1.99375 = 0$$

By the quadratic formula:

$$\cos x = \frac{1 \pm \sqrt{1 - 4(3)(-1.99375)}}{6}$$

$$\cos x = \frac{1 \pm \sqrt{24.9375}}{6}$$

Using a quadratic approximation for  $\cos x$ :

$$1 - \frac{x^2}{2} = \frac{1 \pm \sqrt{24.9375}}{6}$$

$$-\frac{x^2}{2} = -1 + \frac{1 \pm \sqrt{24.9375}}{6}$$

$$x^2 = 2\left(1 - \frac{1 \pm \sqrt{24.9375}}{6}\right)$$

$$x^2 = \begin{cases} 0.00250187761 \dots \\ 3.3003456 \dots \end{cases}$$

$$x = \begin{cases} \pm 0.050 \dots \\ \pm 1.82 \dots \end{cases}$$

[We have to be careful!]



**Question 109** (\*\*\*\*)

It is given that

$$\frac{2 \cot \theta}{1 + \cot^2 \theta} \equiv \sin 2\theta.$$

- a) Prove the validity of the above trigonometric identity.
- b) Use the above result to solve the trigonometric equation

$$4 \cot^2 \theta + 1 = 2 \sin 2\theta (1 + \cot^2 \theta), \quad 0 \leq \theta < 360^\circ.$$

$$\boxed{\phantom{000}}, \quad \boxed{x \approx 63.4^\circ, 243.4^\circ}$$

(a) LHS =  $\frac{2 \cot \theta}{1 + \cot^2 \theta} = \frac{2 \cot \theta}{\sec^2 \theta} = 2 \cot \theta \sin^2 \theta = 2 \left( \frac{\cos \theta}{\sin \theta} \right) \sin^2 \theta$   
 $= 2 \cos \theta \sin \theta = \sin 2\theta$

(b)  $4 \cot^2 \theta + 1 = 2 \sin 2\theta (1 + \cot^2 \theta)$   
 $\Rightarrow 4 \cot^2 \theta + 1 = 2 \sin 2\theta (1 + \cot^2 \theta)$   
 $\Rightarrow 4 \cot^2 \theta - 4 \cot^2 \theta + 1 = 0$   
 $\Rightarrow (2 \cot \theta - 1)^2 = 0$   
 $\Rightarrow 2 \cot \theta = 1$   
 $\Rightarrow \cot \theta = \frac{1}{2}$   
 $\Rightarrow \tan \theta = 2$   
 $\Rightarrow \theta = 63.4^\circ$   
 $\therefore \theta_1 = 63.4^\circ$   
 $\theta_2 = 243.4^\circ$

**Question 110** (\*\*\*\*)If  $\sin(\theta + \alpha) = 2 \sin \theta$ , show clearly that

$$\tan \theta = \frac{\sin \alpha}{2 - \cos \alpha}.$$

proof

$\sin(\theta + \alpha) = 2 \sin \theta$   
 $\sin \theta \cos \alpha + \cos \theta \sin \alpha = 2 \sin \theta$   
 $\cos \theta \sin \alpha = 2 \sin \theta - \sin \theta \cos \alpha$   
 $\cos \theta \sin \alpha = \sin \theta (2 - \cos \alpha)$   
 $\frac{\cos \theta \sin \alpha}{\cos \theta} = \frac{\sin \theta (2 - \cos \alpha)}{\cos \theta}$   
 $\sin \alpha = \tan \theta (2 - \cos \alpha)$

$\therefore \tan \theta = \frac{\sin \alpha}{2 - \cos \alpha}$

## Question 111 (\*\*\*\*)

Given that the exact value of  $\tan 20^\circ = t$ , show that

$$\tan 10^\circ = \frac{-1 + \sqrt{t^2 + 1}}{t}$$

□, proof

PROCEED AS FOLLOWS

$$\tan 20^\circ = \tan(2 \times 10^\circ) = \frac{2 \tan 10^\circ}{1 - \tan^2 10^\circ}$$

$$t = \frac{2a}{1 - a^2} \quad \text{where } a = \tan 10^\circ$$

REARRANGING

$$\begin{aligned} \Rightarrow t(1 - a^2) &= 2a \\ \Rightarrow t - ta^2 &= 2a \\ \Rightarrow 0 &= ta^2 + 2a - t \\ \Rightarrow a^2 + \frac{2}{t}a - 1 &= 0 \end{aligned}$$

BY THE QUADRATIC FORMULA OR BY COMPLETING THE SQUARE

$$\begin{aligned} \Rightarrow \left(a + \frac{1}{t}\right)^2 - \left(\frac{1}{t}\right)^2 - 1 &= 0 \\ \Rightarrow \left(a + \frac{1}{t}\right)^2 &= 1 + \frac{1}{t^2} \\ \Rightarrow \left(a + \frac{1}{t}\right)^2 &= \frac{t^2 + 1}{t^2} \\ \Rightarrow a + \frac{1}{t} &= \pm \sqrt{\frac{t^2 + 1}{t^2}} \\ \Rightarrow a &= -\frac{1}{t} \pm \frac{\sqrt{t^2 + 1}}{t} \\ \Rightarrow a &= \frac{-1 \pm \sqrt{t^2 + 1}}{t} \end{aligned}$$

Now  $-1 - \sqrt{t^2 + 1} < 0$   
And  $a = \tan 10^\circ > 0$   
 $\therefore a = \frac{-1 + \sqrt{t^2 + 1}}{t}$   
 $\therefore \tan 10^\circ = \frac{-1 + \sqrt{t^2 + 1}}{t}$   
A.S. Required

## Question 112 (\*\*\*\*)

Use trigonometric algebra to solve the equation

$$\sin\left[\arcsin \frac{1}{4} + \arccos x\right] = 1.$$

$$x = \frac{1}{4}$$

SOLVING THE EQUATION AS FOLLOWS

$$\begin{aligned} \Rightarrow \sin\left(\arcsin \frac{1}{4} + \arccos x\right) &= 1 \\ \Rightarrow \arcsin\left[\sin\left(\arcsin \frac{1}{4} + \arccos x\right)\right] &= \arcsin 1 \pm 2n\pi \quad (n \in \mathbb{Z}) \\ \Rightarrow \arcsin \frac{1}{4} + \arccos x &= \frac{\pi}{2} \pm 2n\pi \\ \Rightarrow \arccos x &= \frac{\pi}{2} - \arcsin \frac{1}{4} \pm 2n\pi \end{aligned}$$

BUT  $\arccos x$  CAN ONLY RETURN VALUES BETWEEN 0 AND  $\pi$

$$\begin{aligned} \Rightarrow \arccos x &= \frac{\pi}{2} - \arcsin \frac{1}{4} \\ \Rightarrow x &= \cos\left(\frac{\pi}{2} - \arcsin \frac{1}{4}\right) \\ \text{BUT } \cos\left(\frac{\pi}{2} - \theta\right) &= \sin \theta \\ \Rightarrow x &= \sin\left(\arcsin \frac{1}{4}\right) \\ \Rightarrow x &= \frac{1}{4} \end{aligned}$$

## Question 113 (\*\*\*\*)

It is given that

$$\sin P + \sin Q \equiv 2 \sin\left(\frac{P+Q}{2}\right) \cos\left(\frac{P-Q}{2}\right).$$

- a) Prove the validity of the above trigonometric identity, by using the compound angle identities for  $\sin(A+B)$  and  $\sin(A-B)$ .
- b) Hence, or otherwise, solve the trigonometric equation

$$\sin 4\theta + \sin 2\theta = \cos \theta, \quad 0 \leq \theta < \pi,$$

giving the answers in terms of  $\pi$ .

$$\theta = \frac{\pi}{18}, \frac{5\pi}{18}, \frac{\pi}{2}, \frac{13\pi}{18}, \frac{17\pi}{18}$$

(a)  $\sin(A+B) = \sin A \cos B + \cos A \sin B$   
 $\sin(A-B) = \sin A \cos B - \cos A \sin B$   
 $\sin(A+B) + \sin(A-B) = 2 \sin A \cos B$  (\*)  
 Let  $\begin{cases} A+B=P \\ A-B=Q \end{cases}$  Add  $2A = P+Q \Rightarrow A = \frac{P+Q}{2}$   
 Subtract  $2B = P-Q \Rightarrow B = \frac{P-Q}{2}$   
 Hence (\*) results  $\sin P + \sin Q = 2 \sin\left(\frac{P+Q}{2}\right) \cos\left(\frac{P-Q}{2}\right)$  ✓

(b)  $\sin 4\theta + \sin 2\theta = \cos \theta$   
 $\Rightarrow 2 \sin\left(\frac{4\theta+2\theta}{2}\right) \cos\left(\frac{4\theta-2\theta}{2}\right) = \cos \theta$   
 $\Rightarrow 2 \sin 3\theta \cos \theta = \cos \theta$   
 $\Rightarrow 2 \sin 3\theta \cos \theta - \cos \theta = 0$   
 $\Rightarrow \cos \theta [2 \sin 3\theta - 1] = 0$   
 $\cos \theta = 0 \quad \text{or} \quad \sin 3\theta = \frac{1}{2}$   
 $\cos \theta = 0 \Rightarrow \theta = \frac{\pi}{2}, \frac{3\pi}{2}$   
 $\sin 3\theta = \frac{1}{2} \Rightarrow 3\theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$   
 $\theta = \frac{\pi}{18}, \frac{5\pi}{18}, \frac{7\pi}{18}, \frac{11\pi}{18}$

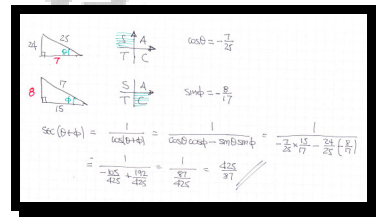
## Question 114 (\*\*\*\*)

$$\sin \theta = \frac{24}{25} \quad \text{and} \quad \cos \varphi = \frac{15}{17}.$$

If  $\theta$  is obtuse and  $\varphi$  is reflex, show clearly that

$$\sec(\theta + \varphi) = \frac{425}{87}.$$

proof



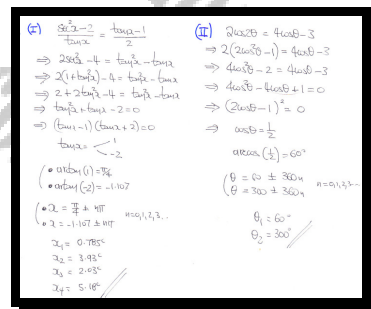
## Question 115 (\*\*\*\*)

Solve each of the following trigonometric equations.

i.  $\frac{\sec^2 x - 2}{\tan x} = \frac{\tan x - 1}{2}, \quad 0 \leq x < 2\pi, \quad x \neq \frac{\pi}{2}, \frac{3\pi}{2}.$

ii.  $2 \cos 2\theta = 4 \cos \theta - 3, \quad 0 \leq \theta < 360^\circ.$

,  $x = 0.785^\circ, 2.03^\circ, 3.93^\circ, 5.18^\circ, \theta = 60^\circ, 300^\circ$

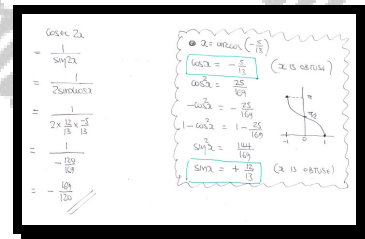


Question 116 (\*\*\*\*)

$$x = \arccos\left(-\frac{5}{13}\right).$$

Determine the exact value of  $\operatorname{cosec} 2x$ .

$$\boxed{-\frac{169}{120}}$$



Question 117 (\*\*\*\*)

$$f(x) = \sec x, \quad x \in \mathbb{R}, \quad 0 \leq x \leq 4\pi.$$

- a) Sketch the graph of  $f(x)$ , showing clearly the coordinates of any stationary points and equations of asymptotes.

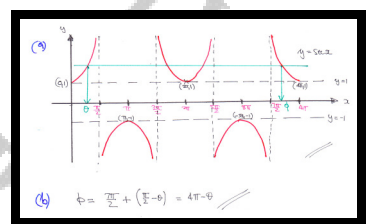
It is now given that

$$\sec \theta = \sec \varphi,$$

where  $0 < \theta < \frac{\pi}{2}$  and  $\frac{7\pi}{2} < \varphi < 4\pi$ .

- b) Express  $\varphi$  in terms of  $\theta$ .

$$\boxed{\phantom{000}}, \quad \boxed{\varphi = 4\pi - \theta}$$



## Question 118 (\*\*\*\*)

If  $\cos x = \frac{1}{3}$ , show by detailed workings that

$$\cos x \cos 2x \cos 4x = -\frac{119}{2187}.$$

proof

$$\begin{aligned} \cos x \cos 2x \cos 4x &= \cos x \cos 2x (2\cos^2 x - 1) \\ &= 2\cos x \cos 2x - \cos x \cos 2x \\ &= 2\cos x [2\cos^2 x - 1]^2 - \cos x [2\cos^2 x - 1] \\ &\quad \text{But } \cos x = \frac{1}{3} \\ &= 2 \times \frac{1}{3} \times [2(\frac{1}{3})^2 - 1]^2 - \frac{1}{3} [2 \times (\frac{1}{3})^2 - 1] \\ &= \frac{2}{3} (\frac{2}{9} - 1)^2 - \frac{1}{3} (\frac{2}{9} - 1) = -\frac{119}{2187} + \frac{7}{27} = -\frac{69 + 567}{2187} \\ &= -\frac{119}{2187} \end{aligned}$$

## Question 119 (\*\*\*\*)

$$4\sin\theta + \cos\theta = 2, \quad 0 \leq \theta < 360^\circ.$$

- a) Show that the above trigonometric equation can be written as

$$16\sin^2\theta = 4 - 4\cos\theta + \cos^2\theta.$$

- b) Show further that

$$\cos\theta = \frac{2 \pm 4\sqrt{13}}{17}.$$

- c) Hence, or otherwise, find the two values of  $\theta$  that satisfy the equation

$$4\sin\theta + \cos\theta = 2, \quad 0 \leq \theta < 360^\circ.$$

$$\theta \approx 15.0^\circ, 136.9^\circ$$

Handwritten solution for Question 119c:

(a)  $4\sin\theta + \cos\theta = 2$   
 $\Rightarrow 4\sin\theta = 2 - \cos\theta$   
 $\Rightarrow (4\sin\theta)^2 = (2 - \cos\theta)^2$   
 $\Rightarrow 16\sin^2\theta = 4 - 4\cos\theta + \cos^2\theta$   
 $\Rightarrow 16(1 - \cos^2\theta) = 4 - 4\cos\theta + \cos^2\theta$   
 $\Rightarrow 16 - 16\cos^2\theta = 4 - 4\cos\theta + \cos^2\theta$   
 $\Rightarrow 0 = 17\cos^2\theta - 4\cos\theta - 12$   
 By quadratic formula  
 $\cos\theta = \frac{4 \pm \sqrt{(-4)^2 - 4(17)(-12)}}{2 \times 17}$   
 $\cos\theta = \frac{4 \pm \sqrt{16 + 816}}{34}$   
 $\cos\theta = \frac{4 \pm \sqrt{832}}{34}$   
 $\cos\theta = \frac{2 \pm 4\sqrt{13}}{17}$   
 (4th Decimals)

(c)  $\cos\theta = \frac{2 \pm 4\sqrt{13}}{17}$   
 $\bullet \arccos\left(\frac{2 + 4\sqrt{13}}{17}\right) = 15.0^\circ$   
 $\bullet \arccos\left(\frac{2 - 4\sqrt{13}}{17}\right) = 136.9^\circ$   
 $\left(\alpha = 15^\circ \pm 360^\circ\right)$   
 $\left(\alpha = 344^\circ \pm 360^\circ\right)$   
 $\left(\alpha = 136.9^\circ \pm 360^\circ\right)$   
 $\left(\alpha = 223.1^\circ \pm 360^\circ\right)$   
 $\alpha = 15.0^\circ$   
 $\alpha = 136.9^\circ$   
 $\alpha = 223.1^\circ$   
 $\alpha = 344.0^\circ$   
 NOT SATISFY OR EQUAL

## Question 120 (\*\*\*\*)

It is given that

$$\frac{\sin 2x + \sin x}{\cos 2x + \cos x + 1} \equiv \tan x.$$

a) Prove the validity of the above trigonometric identity.

b) Hence, or otherwise, solve for  $0 \leq x < 360^\circ$ 

$$\frac{\sin 2x + \sin x}{\cos 2x + \cos x + 1} = 3 \cot x.$$

$$x = 60^\circ, 120^\circ, 180^\circ, 300^\circ$$

(a)  $\frac{\sin 2x + \sin x}{\cos 2x + \cos x + 1} = \frac{2 \sin \frac{3}{2}x \cos \frac{1}{2}x + \sin x}{2 \cos \frac{3}{2}x \cos \frac{1}{2}x + 1 + \cos x} = \frac{\sin(2 \cdot \frac{3}{2}x + \frac{1}{2}x)}{2 \cos \frac{3}{2}x \cos \frac{1}{2}x + 1 + \cos x} = \tan x$   
 (b)  $\frac{\sin 2x + \sin x}{\cos 2x + \cos x + 1} = 3 \cot x$   
 $\Rightarrow \tan x = 3 \cot x$   
 $\Rightarrow \tan^2 x = 3$   
 $\Rightarrow \tan x = \pm \sqrt{3}$   
 $\bullet \arctan(\sqrt{3}) = 60$   
 $\bullet \arctan(-\sqrt{3}) = -60$   
 $\therefore x = 60 \pm 180n$   
 $x = 60 \pm 180n$   
 $x = 60, 240, -60, -180$   
 $x = 60, 120, 240, 300$



## Question 121 (\*\*\*\*)

It is given that

$$\sin P + \sin Q \equiv 2 \sin\left(\frac{P+Q}{2}\right) \cos\left(\frac{P-Q}{2}\right).$$

- a) Prove the validity of the above trigonometric identity, by using the compound angle identities for  $\sin(A+B)$  and  $\sin(A-B)$ .
- b) Hence, or otherwise, solve the trigonometric equation

$$\sin \theta - \sin 3\theta + \sin 5\theta = 0, \quad 0 \leq \theta \leq 180^\circ.$$

$$\boxed{\phantom{000}}, \quad \boxed{\theta = 0^\circ, 30^\circ, 60^\circ, 120^\circ, 150^\circ, 180^\circ}$$

(a)  $\sin(A+B) = \sin A \cos B + \cos A \sin B$   
 $\sin(A-B) = \sin A \cos B - \cos A \sin B$  by addition  
 $\sin(A+B) + \sin(A-B) = 2 \sin A \cos B$  (4)  
 Let  $A+B = P$      $A+B = P$   
 $A-B = Q$      $A-B = Q$      $A+B = P+Q$      $A-B = P-Q$   
 Hence (4) becomes  
 $\sin P + \sin Q = 2 \sin\left(\frac{P+Q}{2}\right) \cos\left(\frac{P-Q}{2}\right)$  AS REQUIRED

(b)  $\sin \theta - \sin 3\theta + \sin 5\theta = 0$   
 $\Rightarrow \sin \theta + \sin 5\theta - \sin 3\theta = 0$   
 $\Rightarrow 2 \sin\left(\frac{\theta+5\theta}{2}\right) \cos\left(\frac{\theta-5\theta}{2}\right) - \sin 3\theta = 0$   
 $\Rightarrow 2 \sin 3\theta \cos(-2\theta) - \sin 3\theta = 0$      $\cos(-\theta) \equiv \cos \theta$   
 $\Rightarrow \sin 3\theta [2 \cos 2\theta - 1] = 0$

•  $\sin 3\theta = 0$   
 $\sin 3\theta = 0$   
 $3\theta = 0^\circ \pm 360^\circ$      $\theta = 0^\circ$   
 $3\theta = 180^\circ \pm 360^\circ$      $\theta = 60^\circ \pm 120^\circ$   
 $\theta = 0^\circ, 120^\circ, 60^\circ, 180^\circ, 150^\circ$

•  $\cos 2\theta = \frac{1}{2}$   
 $\cos 2\theta = \frac{1}{2}$   
 $2\theta = 60^\circ \pm 360^\circ$      $\theta = 30^\circ \pm 180^\circ$   
 $2\theta = 300^\circ \pm 360^\circ$      $\theta = 150^\circ \pm 180^\circ$   
 $\theta = 30^\circ, 150^\circ, 120^\circ, 180^\circ$

**Question 122** (\*\*\*\*)

It is given that

$$\frac{\cos 2x}{\sin x} + \frac{\sin 2x}{\cos x} \equiv \operatorname{cosec} x.$$

a) Prove the validity of the above trigonometric identity.

b) Hence, or otherwise, solve for  $0 \leq x < 180^\circ$ 

$$\frac{\cos 6x}{\sin 3x} + \frac{\sin 6x}{\cos 3x} = 2.$$

$$x = 10^\circ, 50^\circ, 130^\circ, 170^\circ$$

(a) LHS =  $\frac{\cos 2x}{\sin x} + \frac{\sin 2x}{\cos x} = \frac{\cos 2x \cos x + \sin 2x \sin x}{\sin x \cos x} = \frac{\cos(2x-x)}{\sin x \cos x}$   
 $= \frac{\cos x}{\sin x \cos x} = \frac{1}{\sin x} = \operatorname{cosec} x = \text{RHS}$   
 Alternative: LHS =  $\frac{\cos 2x}{\sin x} + \frac{\sin 2x}{\cos x} = \frac{1-2\sin^2 x}{\sin x} + \frac{2\sin x \cos x}{\cos x}$   
 $= \frac{1}{\sin x} - 2\sin x + 2\sin x = \frac{1}{\sin x} = \operatorname{cosec} x = \text{RHS}$   
 (b)  $\frac{\cos 6x}{\sin 3x} + \frac{\sin 6x}{\cos 3x} = 2$   
 $\Rightarrow \operatorname{cosec} 3x = 2 \quad (\text{from a})$   
 $\Rightarrow \sin 3x = \frac{1}{2}$   
 $\Rightarrow \operatorname{cosec} \left(\frac{1}{2}\right) = 3x$   
 $3x = 30^\circ \text{ or } 150^\circ$   
 $x = 10^\circ \text{ or } 50^\circ$   
 $x = 130^\circ \text{ or } 170^\circ$

**Question 123** (\*\*\*\*)

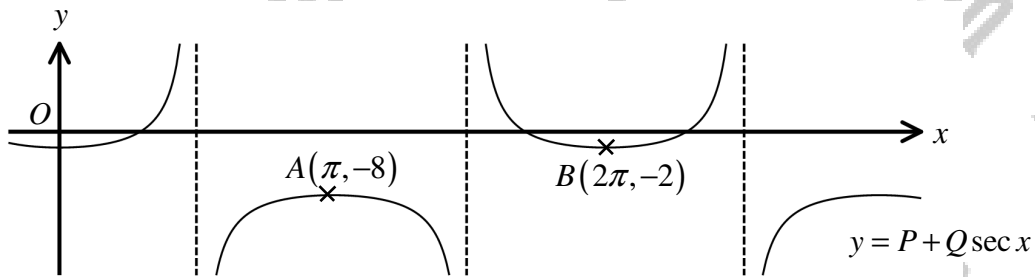
Solve the following trigonometric equation

$$\sin 4y = \sin 2y, \quad 0 \leq y < 180^\circ.$$

$$y = 0^\circ, 30^\circ, 90^\circ, 150^\circ$$

$\sin 4y = \sin 2y$   
 $\Rightarrow \sin(2 \times 2y) = \sin 2y$   
 $\Rightarrow 2\sin 2y \cos 2y = \sin 2y$   
 $\Rightarrow 2\sin 2y \cos 2y - \sin 2y = 0$   
 $\Rightarrow \sin 2y (2\cos 2y - 1) = 0$   
 $\Rightarrow \sin 2y = 0$   
 $2y = 0^\circ \text{ or } 180^\circ$   
 $y = 0^\circ \text{ or } 90^\circ$   
 $\cos 2y = \frac{1}{2}$   
 $2y = 60^\circ \text{ or } 300^\circ$   
 $y = 30^\circ \text{ or } 150^\circ$   
 $y = 0^\circ, 30^\circ, 150^\circ, 180^\circ$   
 Alternatively:  
 $\sin 4y = \sin 2y$   
 $\Rightarrow 4y = 2y \text{ or } 180^\circ$   
 $4y = (180-2y) + 360^\circ$   
 $\Rightarrow 4y = 180 - 2y + 360$   
 $6y = 540$   
 $y = 90^\circ$

## Question 124 (\*\*\*\*)



The figure above shows part of the curve with equation

$$y = P + Q \sec x,$$

where  $P$  and  $Q$  are non zero constants.

The curve has turning points at  $A(\pi, -8)$  and  $B(2\pi, -2)$ .

Determine the value of  $P$  and the value of  $Q$ .

$$\boxed{\phantom{00}}, \boxed{P = -5}, \boxed{Q = 3}$$

$y = P + Q \sec x$   
 $(\pi, -8) \Rightarrow -8 = P + Q \sec \pi \Rightarrow -8 = P - Q$   
 $(2\pi, -2) \Rightarrow -2 = P + Q \sec 2\pi \Rightarrow -2 = P + Q$   
 $\begin{cases} -8 = P - Q \\ -2 = P + Q \end{cases} \Rightarrow \begin{cases} -8 = P - Q \\ -2 = P + Q \end{cases}$   
 $\text{Add} \Rightarrow 2P = -10 \Rightarrow P = -5$   
 $\text{Subtract} \Rightarrow 6 = -2Q \Rightarrow Q = -3$   
ALTERNATIVE  
 $\bullet$  Sec x exist between  $-1$  &  $1$ , i.e. a gap of 2  
 $\bullet$  This graph has a gap of 6 (from  $-2$  to  $-8$ ), so it must have been stretched by factor of 3 in the y direction  
 $\bullet$  But this means it should have a gap between  $-3$  &  $3$   
 $\bullet$  But it has a gap between  $-5$  &  $-1$ , so it must have been translated by 5 units down  
 $\therefore y = -5 + 3 \sec x$

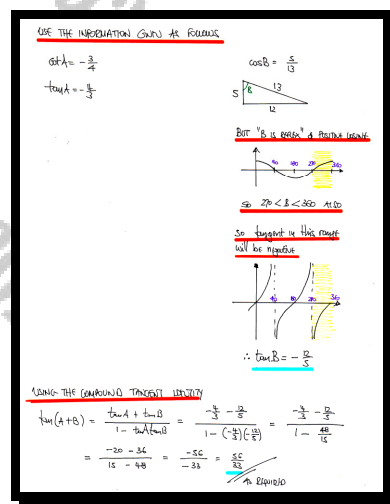
## Question 125 (\*\*\*\*)

$$\cot A = -\frac{3}{4} \quad \text{and} \quad \cos B = \frac{5}{13}.$$

If  $A$  is reflex and  $B$  is also reflex, show that

$$\tan(A+B) = \frac{56}{33}.$$

,  proof



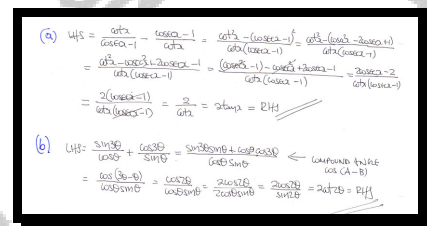
## Question 126 (\*\*\*\*)

Prove the validity of each of the following trigonometric identities.

$$\text{a) } \frac{\cot x}{\operatorname{cosec} x - 1} - \frac{\operatorname{cosec} x - 1}{\cot x} \equiv 2 \tan x.$$

$$\text{b) } \frac{\sin 3\theta}{\cos \theta} + \frac{\cos 3\theta}{\sin \theta} \equiv 2 \cot 2\theta.$$

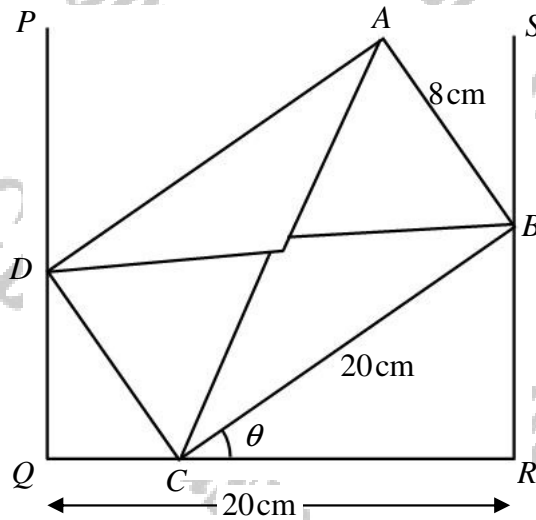
proof



(a) 
$$\begin{aligned} \text{LHS} &= \frac{\cot x}{\operatorname{cosec} x - 1} - \frac{\operatorname{cosec} x - 1}{\cot x} = \frac{\cot^2 x - (\operatorname{cosec} x - 1)^2}{(\cot x)(\operatorname{cosec} x - 1)} \\ &= \frac{\cot^2 x - \operatorname{cosec}^2 x + 2\operatorname{cosec} x - 1}{(\cot x)(\operatorname{cosec} x - 1)} = \frac{(\cot^2 x - 1) - \operatorname{cosec}^2 x + 2\operatorname{cosec} x - 1}{(\cot x)(\operatorname{cosec} x - 1)} \\ &= \frac{2(\operatorname{cosec} x - 1)}{(\cot x)(\operatorname{cosec} x - 1)} = \frac{2}{\cot x} = 2 \tan x = \text{RHS} \end{aligned}$$

(b) 
$$\begin{aligned} \text{LHS} &= \frac{\sin 3\theta}{\cos \theta} + \frac{\cos 3\theta}{\sin \theta} = \frac{\sin(3\theta - \theta) + \cos(3\theta - \theta)}{(\cos \theta)(\sin \theta)} \quad \leftarrow \text{Compound Angle for } (A-B) \\ &= \frac{\cos(3\theta - \theta)}{(\cos \theta)(\sin \theta)} = \frac{\cos 2\theta}{(\cos \theta)(\sin \theta)} = \frac{2 \cos 2\theta}{2 \cos \theta \sin \theta} = 2 \cot 2\theta = \text{RHS} \end{aligned}$$

## Question 127 (\*\*\*\*)



The figure above shows the cross section of a letter inside a filling slot.

The letter  $ABCD$  is modelled as a rectangle with  $|AB| = 8\text{ cm}$  and  $|BC| = 20\text{ cm}$ .

The width of the filling slot  $QR$  is also  $20\text{ cm}$  and the angle  $BCR$  is  $\theta$ .

a) Show clearly that

$$5\cos\theta + 2\sin\theta = 5.$$

b) Express  $5\cos\theta + 2\sin\theta$  in the form  $R\cos(\theta - \alpha)$ ,  $R > 0$ ,  $0 < \alpha < 90^\circ$ .

c) Hence, determine the value of  $\theta$ .

$$\boxed{\phantom{000}}, \quad 5\cos\theta + 2\sin\theta = \sqrt{29}\cos(\theta - 21.8^\circ), \quad \boxed{\theta \approx 43.6^\circ}$$

(a)

(b)  $2\sin\theta + 5\cos\theta = R\cos(\theta - \alpha)$   
 $\equiv R\cos\theta\cos\alpha + R\sin\theta\sin\alpha$   
 $\equiv (R\cos\alpha)\cos\theta + (R\sin\alpha)\sin\theta$   
 $\begin{cases} R\cos\alpha = 5 \\ R\sin\alpha = 2 \end{cases} \Rightarrow R = \sqrt{5^2 + 2^2} = \sqrt{29}$   
 $\tan\alpha = \frac{2}{5} \Rightarrow \alpha \approx 21.8^\circ$   
 $\therefore 2\sin\theta + 5\cos\theta = \sqrt{29}\cos(\theta - 21.8^\circ)$

(c)  $2\sin\theta + 5\cos\theta = 5$   
 $\sqrt{29}\cos(\theta - 21.8^\circ) = 5$   
 $\cos(\theta - 21.8^\circ) = \frac{5}{\sqrt{29}}$   
 $\theta - 21.8^\circ = \cos^{-1}\left(\frac{5}{\sqrt{29}}\right) = 21.8^\circ$   
 $\theta = 21.8^\circ + 21.8^\circ = 43.6^\circ$

## Question 128 (\*\*\*\*)

Solve the following trigonometric equation

$$\operatorname{cosec}^4 \theta - \cot^4 \theta = \frac{2}{3} + \sqrt{3} \cot \theta, \quad 0 \leq \theta < 2\pi.$$

$$\theta \approx 1.05^\circ, 1.29^\circ, 4.19^\circ, 4.43^\circ$$

Using the identity  $1 + \cot^2 \theta = \operatorname{cosec}^2 \theta \Leftrightarrow \operatorname{cosec}^2 \theta - \cot^2 \theta = 1$

$\Rightarrow (\operatorname{cosec}^2 \theta - \cot^2 \theta) = \frac{2}{3} + \sqrt{3} \cot \theta$   
 $\Rightarrow (\operatorname{cosec}^2 \theta - \cot^2 \theta) - 1 = \frac{2}{3} + \sqrt{3} \cot \theta - 1$  Difference of Squares  
 $\Rightarrow (\operatorname{cosec} \theta + \cot \theta)(\operatorname{cosec} \theta - \cot \theta) = \frac{2}{3} + \sqrt{3} \cot \theta - 1$   
 $\Rightarrow (1 + \cot^2 \theta) + \cot \theta = \frac{2}{3} + \sqrt{3} \cot \theta$   
 $\Rightarrow 2\cot^2 \theta + 1 = \frac{2}{3} + \sqrt{3} \cot \theta$   
 $\Rightarrow 6\cot^2 \theta + 3 = 2 + 3\sqrt{3} \cot \theta$   
 $\Rightarrow 6\cot^2 \theta - 3\sqrt{3} \cot \theta + 1 = 0$

Quadratic Formula yields

$$\cot \theta = \frac{3\sqrt{3} \pm \sqrt{27 - 4 \times 6 \times 1}}{2 \times 6} = \frac{3\sqrt{3} \pm 6}{12}$$

$\therefore \cot \theta < \frac{3\sqrt{3}}{12} = \frac{\sqrt{3}}{4}$

Working 3 terms + essential 4th term

- $\cot \theta = \frac{\sqrt{3}}{4}$
- $\theta = \frac{\pi}{3} \pm \pi$  No solution
- $\cot \theta = \frac{3\sqrt{3} + 6}{12}$
- $\theta = \pm 20.71^\circ \pm \pi$  No solution

$\theta_1 = 1.05^\circ$   
 $\theta_2 = 4.43^\circ$   
 $\theta_3 = 1.29^\circ$   
 $\theta_4 = 4.19^\circ$

**Question 129** (\*\*\*)

$$\frac{\cos \theta \cos 2\theta}{\cos \theta + \sin \theta} = \frac{1}{2}, \quad 0 \leq x < 2\pi.$$

Given that  $\cos \theta + \sin \theta \neq 0$ , find the solutions of the above trigonometric equation, giving the answers in radians in terms of  $\pi$ .

$$\boxed{\phantom{000}}, \quad x = \frac{\pi}{8}, \frac{5\pi}{8}, \frac{9\pi}{8}, \frac{13\pi}{8}$$

MULTIPLY ACROSS A TIDY

$$\Rightarrow 2(\cos 2\theta \sin 2\theta) = \cos 2\theta + \sin 2\theta$$

$$\Rightarrow 2\cos 2\theta (\sin 2\theta - \cos 2\theta) = \cos 2\theta + \sin 2\theta$$

$$\Rightarrow 2\cos 2\theta (\cos 2\theta - \sin 2\theta)(\sin 2\theta + \cos 2\theta) = \cos 2\theta + \sin 2\theta$$

} different 1. 2. same

If  $(\cos 2\theta + \sin 2\theta) \neq 0$ , WE MAY DIVIDE IT THROUGH

$$\Rightarrow 2(\cos 2\theta - \sin 2\theta)(\cos 2\theta + \sin 2\theta) = (\cos 2\theta + \sin 2\theta)$$

$$\Rightarrow 2\cos 2\theta (\cos 2\theta - \sin 2\theta) = 1$$

$$\Rightarrow 2\cos 2\theta = 2\cos 2\theta \sin 2\theta = 1$$

$$\Rightarrow 2(\frac{1}{2} + \frac{1}{2}\cos 2\theta) = \cos 2\theta + \sin 2\theta = 1$$

}  $2(\cos 2\theta - \sin 2\theta) = 1$

$$\Rightarrow \cancel{1} + \cos 2\theta = \sin 2\theta = \cancel{1}$$

$$\Rightarrow \cos 2\theta = \sin 2\theta$$

} Divide by  $\cos 2\theta$

$$\Rightarrow 1 = \tan 2\theta$$

•  $2\theta = \frac{\pi}{4} \pm n\pi$

•  $\theta = \frac{\pi}{8} \pm \frac{n\pi}{2}$

$\therefore \theta = \frac{\pi}{8}, \frac{5\pi}{8}, \frac{9\pi}{8}, \frac{13\pi}{8}$

**Question 130** (\*\*\*\*)

Solve in degrees the following trigonometric equation

$$\tan 2x + \tan x = 0, \quad 0 \leq x < 360.$$

$$x = 0^\circ, 60^\circ, 120^\circ, 180^\circ, 240^\circ, 300^\circ$$

$\tan 2\alpha + \tan \alpha = 0$   
 $\Rightarrow \frac{\sin 2\alpha}{1 - \tan^2 \alpha} + \tan \alpha = 0$   
 $\Rightarrow \frac{2T}{1 - T^2} + T = 0 \quad (T = \tan \alpha)$   
 $\Rightarrow 2T + T(1 - T^2) = 0$   
 $\Rightarrow 2T + T - T^3 = 0$   
 $\Rightarrow 3T - T^3 = 0$   
 $\Rightarrow T(3 - T^2) = 0$   
 $\Rightarrow \boxed{\tan \alpha = 0} \Leftrightarrow \boxed{\tan \alpha = 3}$   
 $\quad \quad \quad \tan \alpha = \pm \frac{3}{4}$   
 $\begin{cases} \alpha = 0 \pm 180^\circ \\ \alpha = 60 \pm 180^\circ \\ \alpha = -60 \pm 180^\circ \end{cases} \quad \begin{cases} \alpha = 0^\circ \\ \alpha = 60^\circ \\ \alpha = 120^\circ \end{cases}$   
 $\begin{cases} \alpha = 180^\circ \\ \alpha = 240^\circ \\ \alpha = 300^\circ \end{cases}$   
 $\alpha = 0^\circ, 60^\circ, 120^\circ, 180^\circ, 240^\circ, 300^\circ$



### Question 131 (\*\*\*)

It is given that

$$2 \cot 2x + \tan x \equiv \cot x.$$

- a)** Prove the validity of the above trigonometric identity.
- b)** Hence solve the following trigonometric equation

$$\cot x - \tan x = \frac{1}{2} \tan 2x, \quad 0 \leq x < 180^\circ.$$

$$x \approx 31.7^\circ, 58.3^\circ, 121.7^\circ, 148.3^\circ$$

(a)  $\text{LHS} = 2\cot 2\alpha + \tan \alpha = \frac{2\cos \alpha}{\sin 2\alpha} + \frac{\sin \alpha}{\cos 2\alpha} = \frac{2\cos \alpha \cos \alpha + \sin \alpha \sin 2\alpha}{2\sin \alpha \cos \alpha} = \frac{2\cos^2 \alpha + \sin 2\alpha \sin \alpha}{2\sin \alpha \cos \alpha}$   
 $= \frac{\cos \alpha}{\sin \alpha \cos \alpha} + \frac{\sin \alpha}{\sin \alpha \cos \alpha} = \cot \alpha + \tan \alpha = \text{RHS}$

(ii)  $\text{LHS} = 2\cot 2\alpha + \tan \alpha = \frac{2(1 - \tan^2 \alpha)}{2\tan \alpha} + \tan \alpha = \frac{1}{\tan \alpha} - \tan \alpha + \tan \alpha = \cot \alpha = \text{RHS}$

(b)  $\cot \alpha - \tan \alpha = \frac{1}{\tan \alpha} - \tan \alpha$   
 $\Rightarrow 2\cot 2\alpha = \frac{1}{\tan 2\alpha} \quad (\text{RHS})$   
 $\Rightarrow \frac{2}{\tan 2\alpha} = \frac{\tan 2\alpha}{2}$   
 $\Rightarrow \tan^2 2\alpha = 4$   
 $\Rightarrow \tan 2\alpha = 2$

$2\alpha = 63.4^\circ \text{ or } 116.6^\circ$   
 $2\alpha = -63.4^\circ \text{ or } 116.6^\circ$   
 $\alpha = 31.7^\circ \text{ or } 58.3^\circ$   
 $2 = -31.7^\circ, 58.3^\circ$   
 $2\alpha = 31.7^\circ, 121.7^\circ, 58.3^\circ, 148.3^\circ$

$\bullet \arctan(2) = 63.43^\circ$   
 $\bullet \arctan(-2) = -63.43^\circ$

**Question 132** (\*\*\*\*)

If  $\cot \theta = 2$ , use the tangent double angle identity to show

$$\tan \theta \cot 2\theta \tan 4\theta = -\frac{9}{7}.$$

You must show detailed workings in this question

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proof

$$\begin{aligned} \tan(0) \cot(20) \tan(60) &= \tan(0) \frac{1}{\tan(20)} \frac{2\sqrt{3}\tan(20)}{1-\tan^2(20)} & \tan(24) &= \frac{2\sqrt{3}\tan(24)}{1-\tan^2(24)} \\ &= \frac{2\sqrt{3}}{1-\tan^2(20)} = \frac{2\sqrt{3}}{1-\left(\frac{2\sqrt{3}}{1-\sqrt{3}}\right)^2} = \dots \text{ BUT } & \text{also } 2 &= \frac{2\sqrt{3}}{6\sqrt{3}-1} \\ &= \frac{2 \times \frac{1}{2}}{1-\left(\frac{2 \times \frac{1}{2}}{1-\frac{1}{2}}\right)^2} = \frac{1}{1-\left(\frac{1}{1-\frac{1}{2}}\right)^2} = \frac{1}{1-\frac{16}{9}} = \frac{9}{9-16} = -\frac{9}{7} \end{aligned}$$

**Question 133** (\*\*\*\*)

It is given that

$$(\sec \theta - \cos \theta)(\operatorname{cosec} \theta - \sin \theta) \equiv \sin \theta \cos \theta.$$

- a) Prove the validity of the above trigonometric identity.
- b) Hence solve the trigonometric equation

$$(\sec \theta - \cos \theta)(\operatorname{cosec} \theta - \sin \theta) = -\frac{1}{4}, \quad 0 \leq \theta < 360^\circ.$$

$$\theta = 105^\circ, 165^\circ, 285^\circ, 345^\circ$$

**Question 134** (\*\*\*\*)The three angles of a triangle are denoted by  $\alpha$ ,  $\beta$  and  $\gamma$ .

Show clearly that ...

a) ...  $\sin(\alpha + \beta) = \sin \gamma$ .

b) ...  $\sin\left(\frac{\alpha + \beta}{2}\right) = \cos\left(\frac{\gamma}{2}\right)$ .

proof

## Question 135 (\*\*\*\*)

$$f(x) = \sqrt{3} \sin x + \cos x, \quad x \in \mathbb{R}.$$

- a) Express  $f(x)$  in the form  $R \cos(x - \alpha)$ ,  $R > 0$ ,  $0 < \alpha < 90^\circ$ .
- b) State the maximum value of  $f(x)$  and find the smallest positive value of  $x$  for which this maximum occurs.

The temperature of the water  $T^\circ\text{C}$  in a tropical fish tank is modelled by the equation

$$T = 32 + \sqrt{3} \sin(15t)^\circ + \cos(15t)^\circ, \quad 0 \leq t < 24,$$

where  $t$  is the time in hours measured since midnight.

- c) State the maximum temperature of the water in the tank and the time when this maximum temperature occurs.
- d) Show that the temperature of the water in the tank reaches  $30.5^\circ\text{C}$  at 13:14 and at 18:46.

[You may not verify the answers in this part]

$$\boxed{\phantom{000}}, \quad \boxed{\sqrt{3} \sin x + \cos x \equiv 2 \cos(x - 60^\circ)}, \quad \boxed{\max = 2}, \quad \boxed{x = 60^\circ}, \quad \boxed{T_{\max} = 34},$$

04:00

**a) USING THE COMPOUND ANGLE IDENTITY FOR  $\cos(A-B)$**

$$\sqrt{3} \sin x + \cos x \equiv R \cos(x - \alpha)$$

$$\equiv R \cos x \cos \alpha + R \sin x \sin \alpha$$

$$\equiv (R \cos \alpha) \cos x + (R \sin \alpha) \sin x$$

$\bullet R \cos \alpha = \sqrt{3}$   
 $\bullet R \sin \alpha = 1$

SOUGHT  $\alpha$  AND  $R$   $R = \sqrt{(\sqrt{3})^2 + 1^2}$   
 $R = 2$

INVERTING SIDE BY SIDE:  $\tan \alpha = \frac{1}{\sqrt{3}}$   
 $\alpha = 30^\circ$

$\therefore f(x) = 2 \cos(x - 60^\circ)$

**ALTERNATIVE BY MANIPULATION**

$$\sqrt{3} \sin x + \cos x = 2 \left[ \frac{\sqrt{3}}{2} \sin x + \frac{1}{2} \cos x \right] = 2 \left[ \frac{1}{2} \cos x + \frac{\sqrt{3}}{2} \sin x \right]$$

$$= 2 \left[ \cos(60^\circ) \cos x + \sin(60^\circ) \sin x \right] = 2 \cos(x - 60^\circ)$$

**b) MAX VALUE OF  $f(x) = 2 \cos(x - 60^\circ)$  IS 2**  
 (As  $-1 \leq \cos(x - 60^\circ) \leq 1$ )  
 TO GET THIS MAX VALUE OF 2,  $\cos(x - 60^\circ) = 1$   
 $x - 60^\circ = 0$  (SIMPLE VALUE)  
 $x = 60^\circ$

**c) REVISIT THE SIMILARITY/ANALOGY TO PART (a) & (b)**

$$T(t) = 32 + 2 \cos\left(\frac{15t - 60}{60}\right)$$

$\therefore T_{\max} = 32 + 2$   
 $T_{\max} = 34^\circ\text{C}$

**d) SOLVING THE QUESTION  $T = 30.5$**

$$30.5 = 32 + 2 \cos\left(\frac{15t - 60}{60}\right)$$

$$\Rightarrow -1.5 = 2 \cos\left(\frac{15t - 60}{60}\right)$$

$$\Rightarrow \cos\left(\frac{15t - 60}{60}\right) = -0.75$$

OFFICAL  $\rightarrow 72 = 138.59$   
 $\Rightarrow (15t - 60) = 138.59 \pm 360n$   
 $15t - 60 = 298.59 \pm 360n$   
 $15t = 358.59 \pm 360n$   
 $t = 23.91 \pm 24n$   
 $t = 23.91$

$\therefore t_1 = 13.24 \quad \therefore 13:14$   
 $t_2 = 18.76 \quad \therefore 18:46$

$\leftarrow 0.14 \times 60 = 8.4$   
 $\leftarrow 0.76 \times 60 = 45.6$

**Question 136** (\*\*\*\*)

It is given that

$$\tan \theta (1 + \sec 2\theta) \equiv \tan 2\theta.$$

- a) Prove the validity of the above trigonometric identity.
- b) Hence, or otherwise, solve for  $0 \leq \theta < 180^\circ$

$$\tan \theta (1 + \sec 2\theta) = 4 \tan \theta.$$

$$\theta = 0^\circ \quad \theta \approx 35.3^\circ, 144.7^\circ$$

(a) LHS =  $\tan \theta (1 + \sec 2\theta) = \tan \theta \left(1 + \frac{1}{\cos 2\theta}\right) = \tan \theta \left(\frac{\cos 2\theta + 1}{\cos 2\theta}\right)$   
 $= \frac{\sin \theta}{\cos \theta} \times \frac{2 \cos^2 \theta + 1 - 1}{\cos 2\theta} = \frac{\sin \theta}{\cos \theta} \times \frac{2 \cos^2 \theta}{\cos 2\theta} = \frac{2 \sin \theta \cos^2 \theta}{\cos 2\theta}$   
 $= \frac{\sin 2\theta}{\cos 2\theta} = \tan 2\theta$

(b)  $\tan \theta (1 + \sec 2\theta) = 4 \tan \theta$   
 $\Rightarrow \tan 2\theta = 4 \tan \theta$   
 $\Rightarrow \frac{2 \tan \theta}{1 - \tan^2 \theta} = 4 \tan \theta$   
 $\Rightarrow 2 \tan \theta = 4 \tan \theta - 4 \tan^3 \theta$   
 $\Rightarrow 4 \tan^3 \theta - 2 \tan \theta = 0$   
 $\Rightarrow 2 \tan \theta (2 \tan^2 \theta - 1) = 0$   
 $\therefore \tan \theta = 0 \quad 2 \tan^2 \theta = 1$   
 $\tan \theta = 0 \quad \tan \theta = \pm \frac{1}{\sqrt{2}}$   
 $\theta = 0^\circ, 180^\circ$   
 $\theta = 35.3^\circ, 144.7^\circ$

**Question 137** (\*\*\*\*)

By twice applying the identity

$$\sin 2\theta = 2 \sin \theta \cos \theta,$$

solve the trigonometric equation

$$\sin x \cos x \cos 2x = \frac{1}{8}, \quad 0 \leq x < \pi.$$

$$x = \frac{\pi}{24}, \frac{5\pi}{24}, \frac{13\pi}{24}, \frac{17\pi}{24}$$

$\Rightarrow \sin x \cos x \cos 2x = \frac{1}{8}$   
 $\Rightarrow 2 \sin x \cos x \cos 2x = \frac{1}{4}$   
 $\Rightarrow \sin 2x \cos 2x = \frac{1}{4}$   
 $\Rightarrow 2 \sin 2x \cos 2x = \frac{1}{2}$   
 $\Rightarrow \sin 4x = \frac{1}{2}$   
 $\therefore 4x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$   
 $x = \frac{\pi}{24}, \frac{5\pi}{24}, \frac{7\pi}{24}, \frac{11\pi}{24}$

## Question 138 (\*\*\*\*)

$$f(x) = 2.5 \sin 2x + 6 \cos 2x, \quad 0 < x < 2\pi.$$

- a) Express  $f(x)$  in the form  $R \sin(2x + \alpha)$ ,  $R > 0$ ,  $0 < \alpha < \frac{\pi}{2}$ .
- b) Determine the value of the constant  $A$  so that

$$5 \sin x \cos x + 12 \cos^2 x \equiv f(x) + A.$$

- c) Hence, or otherwise, find the minimum and the maximum value of

$$5 \sin x \cos x + 12 \cos^2 x.$$

$$f(x) = 6.5 \sin(2x + 1.176^\circ), \quad A = 6, \quad \max = 12.5, \quad \min = -0.5$$

$f(x) = 2.5 \sin 2x + 6 \cos 2x \equiv R \sin(2x + \alpha)$   
 $\equiv R \sin 2x \cos \alpha + R \cos 2x \sin \alpha$   
 $\equiv (R \cos \alpha) \sin 2x + (R \sin \alpha) \cos 2x$   
 $R \cos \alpha = 2.5$   
 $R \sin \alpha = 6$   
 $\therefore R = \sqrt{2.5^2 + 6^2} = 6.5$   
 $\tan \alpha = \frac{6}{2.5} \Rightarrow \alpha = 1.176^\circ$   
 $\therefore f(x) = 6.5 \sin(2x + 1.176^\circ)$

$5 \sin x \cos x + 12 \cos^2 x = \frac{5}{2} \sin 2x + 6 \cos 2x + 6$   
 $= 2.5 \sin 2x + 6 \cos 2x + 6$   
 $= f(x) + 6$   
 $\therefore A = 6$

$5 \sin x \cos x + 12 \cos^2 x = f(x) + A$   
 $5 \sin x \cos x + 12 \cos^2 x = 2.5 \sin 2x + 6 \cos 2x + A$   
 $5 \sin x \cos x + 12 \cos^2 x = 2.5 (2 \sin x \cos x) + 6 (2 \cos^2 x - 1) + A$   
 $5 \sin x \cos x + 12 \cos^2 x = 5 \sin x \cos x + 12 \cos^2 x - 6 + A$   
 $\therefore A = 6$

$5 \sin x \cos x + 12 \cos^2 x = f(x) + 6$   
 $= 6.5 \sin(2x + 1.176^\circ) + 6$   
 $\therefore \text{MAX} = 6.5 + 6 = 12.5$   
 $\text{MIN} = -6.5 + 6 = -0.5$

**Question 139** (\*\*\*\*)Find the **two** solutions of the trigonometric equation

$$(1 + \sec y)(1 - \cos y) = \tan y, \quad 0 \leq y < 2\pi,$$

giving the answers in terms of  $\pi$ .

$$y = 0, \pi$$

Handwritten solution for Question 139:

$$(1 + \sec y)(1 - \cos y) = \tan y$$

$$\Rightarrow 1 - \cos y + \sec y - \sec y \cos y = \tan y$$

$$\Rightarrow 1 - \cos y + \sec y - 1 = \tan y$$

$$\Rightarrow \sec y - \cos y = \tan y$$

$$\Rightarrow \frac{1}{\cos y} - \cos y = \frac{\sin y}{\cos y}$$

$$\Rightarrow 1 - \cos^2 y = \sin y$$

$$\Rightarrow 1 - (1 - \sin^2 y) = \sin y$$

$$\Rightarrow \sin^2 y - \sin y = 0$$

$$\Rightarrow \sin y (\sin y - 1) = 0$$

$$\Rightarrow \sin y = 0$$

$$\Rightarrow y = 0, \pi$$

Check:

$$\sec y = 0 \Rightarrow \cos y = \frac{1}{0} \Rightarrow \text{undefined}$$

$$y = 0 \pm 2\pi \quad y = \pi \pm 2\pi$$

$$y = \frac{\pi}{2} \pm 2\pi \quad y = \frac{3\pi}{2} \pm 2\pi$$

$$\therefore y = 0, \pi$$

NOT 1 solution  
K sec y / tan y etc. instance

**Question 140** (\*\*\*\*)

Solve each of the following trigonometric equations.

i.  $\frac{1 - 2\operatorname{cosec}^2 y}{2\cot y} - 2 = \cot y, \quad 0 < y < 2\pi, \quad y \neq \pi.$

ii.  $\cos 2\theta + 6\cos \theta + 5 = 0, \quad 0 \leq \theta < 360^\circ.$

$$y \approx 2.03^\circ, 5.18^\circ, \quad \theta = 180^\circ$$

Handwritten solution for Question 140:

(i)  $\frac{1 - 2\operatorname{cosec}^2 y}{2\cot y} - 2 = \cot y$

$$\Rightarrow 1 - 2\operatorname{cosec}^2 y - 4\cot y = 2\cot^2 y$$

$$\Rightarrow 1 - 2(1 + \cot^2 y) - 4\cot y = 2\cot^2 y$$

$$\Rightarrow 1 - 2 - 2\cot^2 y - 4\cot y = 2\cot^2 y$$

$$\Rightarrow 0 = 4\cot^2 y + 4\cot y + 1$$

$$\Rightarrow (2\cot y + 1)^2 = 0$$

$$\Rightarrow 2\cot y = -1$$

$$\cot y = -\frac{1}{2}$$

$$\tan y = -2$$

$$\arctan(-2) = -1.107^\circ$$

$$\therefore y = -1.107^\circ \pm \pi \quad y = 1.107^\circ, 2.228^\circ$$

$$y = 2.03^\circ$$

$$y = 5.18^\circ$$

(ii)  $\cos 2\theta + 6\cos \theta + 5 = 0$

$$\Rightarrow (2\cos^2 \theta - 1) + 6\cos \theta + 5 = 0$$

$$\Rightarrow 2\cos^2 \theta + 6\cos \theta + 4 = 0$$

$$\Rightarrow \cos^2 \theta + 3\cos \theta + 2 = 0$$

$$\Rightarrow (\cos \theta + 1)(\cos \theta + 2) = 0$$

$$\cos \theta = -1$$

$$\arccos(-1) = 180^\circ$$

$$\theta = 180^\circ \pm 360^\circ \quad y = 0, 1, 2, \dots$$

$$(\theta = 180^\circ \pm 360^\circ)$$

$$\therefore \theta = 180^\circ$$

**Question 141** (\*\*\*\*)

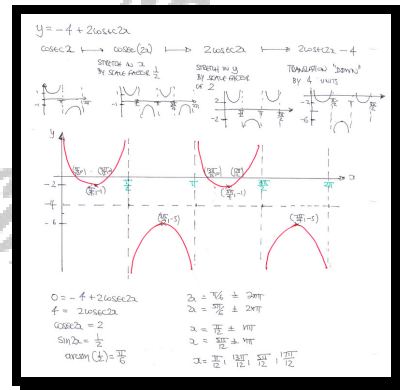
Sketch the graph of

$$y = -4 + 2\operatorname{cosec} 2x, \quad 0 \leq x \leq 2\pi.$$

The sketch must include

- the equations of any asymptotes to the curve
- the exact coordinates of any stationary points.
- the exact coordinates of any points where the curve crosses the coordinate axes.

graph



## Question 142 (\*\*\*\*)

Prove the validity of each of the trigonometric identities.

a)  $\operatorname{cosec} \theta - \cot \theta \equiv \tan \left( \frac{\theta}{2} \right).$

b)  $\frac{2 \tan 2x}{\tan 2x - \sin 2x} \equiv \operatorname{cosec}^2 x.$

proof

(a)  $\operatorname{LHS} = \operatorname{cosec} \theta - \cot \theta = \frac{1}{\sin \theta} - \frac{\cos \theta}{\sin \theta} = \frac{1 - \cos \theta}{\sin \theta} = \frac{1 - (-2\sin^2 \frac{\theta}{2})}{2\sin \frac{\theta}{2} \cos \frac{\theta}{2}}$   
 $= \frac{2\sin^2 \frac{\theta}{2}}{2\sin \frac{\theta}{2} \cos \frac{\theta}{2}} = \frac{\sin \frac{\theta}{2}}{\cos \frac{\theta}{2}} = \tan \frac{\theta}{2} = \operatorname{RHS}$

(b)  $\operatorname{LHS} = \frac{2 \tan 2x}{\tan 2x - \sin 2x} = \frac{2 \left( \frac{\sin 2x}{\cos 2x} \right)}{\frac{\sin 2x}{\cos 2x} - \sin 2x} = \dots$  Multiply top & bottom by  $\cos 2x$   
 $= \frac{2 \sin 2x}{\sin 2x - \sin 2x \cos 2x} = \frac{2}{1 - \cos 2x} = \frac{2}{1 - (1 - 2\sin^2 x)} = \frac{2}{2\sin^2 x} = \frac{1}{\sin^2 x} = \operatorname{cosec}^2 x = \operatorname{RHS}$



**Question 143** (\*\*\*\*)

$$f(x) \equiv 27x^3 - 9x - 2, \quad x \in \mathbb{R}.$$

- a)** Show that  $(3x+1)$  is a factor of  $f(x)$ .

It is further given that

$$36 \cos 2\theta \cos \theta + 9 \sin 2\theta \sin \theta = 4.$$

- b)** Find the possible values of  $\cos \theta$ .

$$\boxed{\frac{1}{3}}, \quad \boxed{\cos \theta = -\frac{1}{3}, \frac{2}{3}}$$

$$\begin{aligned} \textcircled{a} \quad f(x) &= 27x^3 - 9x - 2 \\ f\left(\frac{1}{3}\right) &= 27\left(\frac{1}{3}\right)^3 - 9\left(\frac{1}{3}\right) - 2 = 27\left(\frac{1}{27}\right) + 3 - 2 = 0 \\ \therefore (3x+1) &\text{ is a factor of } f(x) \\ \textcircled{b} \quad &36\omega^6\omega^6 + 9\omega^3\omega^3\omega^6 = 4 \\ &\Rightarrow 36\omega^6(2\omega^6 - 1) + 9\omega^3(2\omega^3\omega^6) = 4 \\ &\Rightarrow 72\omega^6 - 36\omega^6 + 18\omega^9\omega^6 = 4 \\ &\Rightarrow 72\omega^6 - 36\omega^6 + 18\omega^6(1 - \omega^6) = 4 \\ &\Rightarrow 72\omega^6 - 36\omega^6 + 18\omega^6 - 18\omega^6\omega^6 = 4 \\ &\Rightarrow 54\omega^6 - 18\omega^6 - 4 = 0 \\ &\Rightarrow 37\omega^6 - 9\omega^6 - 2 = 0 \\ &\quad \text{Let } x = \omega^6 \\ &\Rightarrow 27x^2 - 9x - 2 = 0 \\ &\Rightarrow (3x+1)(9x^2 - 3x - 2) = 0 \\ &\Rightarrow (3x+1)(3x+1)(3x-2) = 0 \\ &\Rightarrow 3x < -\frac{1}{3} \quad \therefore \omega^6 < -\frac{1}{3} \end{aligned}$$

## Question 144 (\*\*\*\*)

It is given that

$$\frac{1 - \tan^2 \theta}{1 + \tan^2 \theta} \equiv \cos 2\theta.$$

a) Prove the validity of the above trigonometric identity.

b) Given that  $\cos 36^\circ = \frac{1 + \sqrt{5}}{4}$ , show clearly that

$$\tan^2 18^\circ = \frac{5 - 2\sqrt{5}}{5}.$$

proof

Handwritten mathematical proof for part b of the question:

a) 
$$\frac{1 - \tan^2 \theta}{1 + \tan^2 \theta} = \frac{1 - \frac{\sin^2 \theta}{\cos^2 \theta}}{1 + \frac{\sin^2 \theta}{\cos^2 \theta}} = \frac{\frac{\cos^2 \theta - \sin^2 \theta}{\cos^2 \theta}}{\frac{\cos^2 \theta + \sin^2 \theta}{\cos^2 \theta}} = \frac{\cos^2 \theta - \sin^2 \theta}{\cos^2 \theta + \sin^2 \theta} = \frac{\cos 2\theta}{1} = \cos 2\theta$$

b) 
$$\frac{1 - \tan^2 18^\circ}{1 + \tan^2 18^\circ} = \cos 36^\circ$$
  

$$\frac{1 - \frac{\sin^2 18^\circ}{\cos^2 18^\circ}}{1 + \frac{\sin^2 18^\circ}{\cos^2 18^\circ}} = \cos 36^\circ$$
  

$$\Rightarrow \frac{1 - \frac{\sin^2 18^\circ}{\cos^2 18^\circ}}{1 + \frac{\sin^2 18^\circ}{\cos^2 18^\circ}} = \frac{1 + \sqrt{5}}{4}$$
  

$$\Rightarrow \frac{1 - \tan^2 18^\circ}{1 + \tan^2 18^\circ} = \frac{1 + \sqrt{5}}{4}$$
  

$$\Rightarrow 4 - 4\tan^2 18^\circ = (1 + \tan^2 18^\circ)(1 + \sqrt{5})$$
  

$$\Rightarrow 4 - 4T = 1 + \sqrt{5} + T + T\sqrt{5}$$
  

$$\Rightarrow 3 - 4\sqrt{5} = 5T + T\sqrt{5}$$
  

$$\Rightarrow T = \frac{3 - 4\sqrt{5}}{5 + \sqrt{5}}$$
  

$$\Rightarrow T = \frac{(3 - 4\sqrt{5})(5 - \sqrt{5})}{(5 + \sqrt{5})(5 - \sqrt{5})}$$
  

$$\Rightarrow T = \frac{15 - 3\sqrt{5} - 20\sqrt{5} + 20}{25 - 5}$$
  

$$\Rightarrow T = \frac{5 - 2\sqrt{5}}{5}$$
  
 as required

## Question 145 (\*\*\*\*)

$$f(\theta) \equiv 5\cos\theta - 12\sin\theta, \theta \in \mathbb{R}.$$

- a) Express  $f(\theta)$  in the form  $R\cos(\theta + \alpha)$ ,  $R > 0$ ,  $0 < \alpha < \frac{\pi}{2}$ .

Give the value of  $\alpha$  correct to 3 decimal places.

- b) State the maximum value of  $f(\theta)$  and find the smallest positive value of  $\theta$  for which this maximum occurs.

The pressure  $P$ , in suitable units, in a nuclear plant is modelled by the equation

$$P = 20 + 5\cos\left(\frac{4\pi t}{25}\right) - 12\sin\left(\frac{4\pi t}{25}\right), \quad 0 \leq t < 12,$$

where  $t$  is the time in hours measured from midnight.

- c) State the maximum pressure in the plant and the value of  $t$  when this maximum pressure occurs.
- d) Find the times, to the nearest minute, when  $P = 15$ .

$$\boxed{\phantom{000}}, \boxed{5\cos\theta - 12\sin\theta \equiv 13\cos(\theta + 1.176^\circ)}, \boxed{\max = 33}, \boxed{\theta = 5.107^\circ},$$

$$\boxed{P_{\max} = 33}, \boxed{t_{\max} = 10.16}, \boxed{01:34/06:15}$$

a) USING THE STANDARD METHOD

$$5\cos\theta - 12\sin\theta \equiv R\cos(\theta + \alpha)$$

$$5\cos\theta - 12\sin\theta \equiv R\cos\theta\cos\alpha - R\sin\theta\sin\alpha$$

$$5\cos\theta - 12\sin\theta \equiv (R\cos\alpha)\cos\theta - (R\sin\alpha)\sin\theta$$

COMPARING & SOLVING

$$R\cos\alpha = 5$$

$$R\sin\alpha = 12$$

SQUARING & ADDING YIELDS

$$R = \sqrt{5^2 + 12^2} = 13$$

DIVIDING THE EQUATIONS SIDE-BY-SIDE

$$\frac{R\sin\alpha}{R\cos\alpha} = \frac{12}{5} \Rightarrow \tan\alpha = \frac{12}{5}$$

$$\alpha \approx 1.176^\circ$$

$\therefore f(\theta) \equiv 13\cos(\theta + 1.176^\circ)$

b) FIRSTLY  $f(\theta)_{\max} = 13$

NEXT, TO GET + MAX OF 13

$$\cos(\theta + 1.176^\circ) = 1$$

$$\theta + 1.176^\circ = 0$$

$$\theta = -1.176^\circ$$

$$\theta = 360^\circ - 1.176^\circ = 358.824^\circ$$

c) USING PARTS (a) & (b)

$$P = 20 + 5\cos\left(\frac{4\pi t}{25}\right) - 12\sin\left(\frac{4\pi t}{25}\right)$$

$$P = 20 + 13\cos\left(\frac{4\pi t}{25} + 1.176^\circ\right)$$

$\therefore P_{\max} = 20 + 13 = 33$

AND FROM PART (b)  $\Rightarrow \theta = 5.107^\circ$

$$\Rightarrow \frac{4\pi t}{25} = 5.107^\circ$$

$$\Rightarrow t \approx 10.16$$

d) SOLVING THE EQUATION  $P = 15$

$$15 = 20 + 13\cos\left(\frac{4\pi t}{25} + 1.176^\circ\right)$$

$$\Rightarrow -5 = 13\cos\left(\frac{4\pi t}{25} + 1.176^\circ\right)$$

$$\Rightarrow \cos\left(\frac{4\pi t}{25} + 1.176^\circ\right) = -\frac{5}{13}$$

$\left[\arccos\left(-\frac{5}{13}\right) = 1.95587\dots\right]$

$$\Rightarrow \left(\frac{4\pi t}{25} + 1.176^\circ\right) = 1.95587 \dots \pm 2\pi n$$

$$\Rightarrow \frac{4\pi t}{25} = 0.77987 \dots \pm 2\pi n$$

$$\Rightarrow \frac{4\pi t}{25} = \dots \pm 2\pi n$$

$$\Rightarrow t = 1.5108 \dots \pm \frac{25}{4}n$$

$$\Rightarrow t = 6.25 \pm \frac{25}{4}n$$

$t = 1.5107$

TIME  $\begin{cases} 01:34 \\ 06:15 \end{cases}$

**Question 146** (\*\*\*\*)

It is given that

$$\frac{\tan 2x - \sin 2x}{\tan 2x} = 2\sin^2 x, \quad \tan 2x \neq 0.$$

- a) Prove the validity of the above trigonometric identity.
- b) Hence find, in terms of  $\pi$ , the solutions of the trigonometric equation

$$\frac{\tan 2x - \sin 2x}{\tan 2x} = 1, \quad 0 \leq x < 2\pi.$$

$$x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$$

**Question 147** (\*\*\*\*)If  $\sin x = \frac{12}{13}$  and  $x$  is obtuse, show clearly that

$$\cot 2x = \frac{119}{120}.$$

, proof

## Question 148 (\*\*\*\*)

It is given that

$$\cos x + \sin x \tan 2x \equiv \frac{\cos x}{\cos 2x}.$$

- a) Prove the validity of the above trigonometric identity.
- b) Hence find, in terms of  $\pi$ , the solutions of the trigonometric equation

$$\cos x + \sin x \tan 2x = 1, \quad 0 \leq x < 2\pi,$$

giving the answers in terms of  $\pi$ .

$$x = 0, \frac{2\pi}{3}, \frac{4\pi}{3}$$

Handwritten solution for Question 148:

(a)  $\cos x + \sin x \tan 2x = \cos x + \sin x \frac{\sin 2x}{\cos 2x}$   
 $= \frac{\cos x \cos 2x + \sin x \sin 2x}{\cos 2x} = \frac{\cos(2x - x)}{\cos 2x} = \frac{\cos x}{\cos 2x}$

(b)  $\cos x + \sin x \tan 2x = 1$   
 $\Rightarrow \cos x + \sin x \frac{\sin 2x}{\cos 2x} = 1$   
 $\Rightarrow \frac{\cos x \cos 2x + \sin x \sin 2x}{\cos 2x} = 1$   
 $\Rightarrow \frac{\cos(2x - x)}{\cos 2x} = 1$   
 $\Rightarrow \frac{\cos x}{\cos 2x} = 1$   
 $\Rightarrow \cos x = \cos 2x$   
 $\Rightarrow \cos x = 2\cos^2 x - 1$   
 $\Rightarrow 0 = 2\cos^2 x - \cos x - 1$   
 $\Rightarrow 0 = (2\cos x + 1)(\cos x - 1)$   
 $\Rightarrow \cos x = -\frac{1}{2} \text{ or } \cos x = 1$

$\cos x = 1$   
 $\Rightarrow x = 0 \text{ or } 2\pi$   
 $\Rightarrow x = 0$  (since  $0 \leq x < 2\pi$ )

$\cos x = -\frac{1}{2}$   
 $\Rightarrow x = \frac{2\pi}{3} \text{ or } \frac{4\pi}{3}$

$\therefore x = 0, \frac{2\pi}{3}, \frac{4\pi}{3}$

## Question 149 (\*\*\*\*)

It is given that

$$\frac{\operatorname{cosec} x}{1 + \operatorname{cosec} x} - \frac{\operatorname{cosec} x}{1 - \operatorname{cosec} x} \equiv 2 \sec^2 x, \quad \operatorname{cosec} x \neq \pm 1.$$

- a) Prove the validity of the above trigonometric identity.
- b) Hence find the solutions of the trigonometric equation

$$\frac{\operatorname{cosec} x}{1 + \operatorname{cosec} x} - \frac{\operatorname{cosec} x}{1 - \operatorname{cosec} x} = 5 \tan x, \quad 0 \leq x < 2\pi.$$

$$x \approx 0.464^\circ, 1.11^\circ, 3.61^\circ, 4.25^\circ$$

(a) 
$$\text{LHS} = \frac{\operatorname{cosec} x}{1 + \operatorname{cosec} x} - \frac{\operatorname{cosec} x}{1 - \operatorname{cosec} x} = \frac{\operatorname{cosec} x (1 - \operatorname{cosec} x) - \operatorname{cosec} x (1 + \operatorname{cosec} x)}{(1 + \operatorname{cosec} x)(1 - \operatorname{cosec} x)}$$

$$= \frac{\operatorname{cosec} x - \operatorname{cosec}^2 x - \operatorname{cosec} x - \operatorname{cosec}^2 x}{1 - \operatorname{cosec}^2 x} = \frac{-2 \operatorname{cosec}^2 x}{1 - \operatorname{cosec}^2 x}$$

$$= \frac{-2 \operatorname{cosec}^2 x}{-\tan^2 x} = \frac{2 \operatorname{cosec}^2 x}{\tan^2 x} = \frac{2 \frac{1}{\sin^2 x}}{\frac{\sin x}{\cos x}} = \frac{2 \cos x}{\sin^3 x}$$

$$= 2 \sec^2 x \quad \text{RHS}$$

(b) 
$$\frac{\operatorname{cosec} x}{1 + \operatorname{cosec} x} - \frac{\operatorname{cosec} x}{1 - \operatorname{cosec} x} = 5 \tan x$$

$$\Rightarrow 2 \sec^2 x = 5 \tan x$$

$$\Rightarrow 2(1 + \tan^2 x) = 5 \tan x$$

$$\Rightarrow 2 + 2 \tan^2 x = 5 \tan x$$

$$\Rightarrow 2 \tan^2 x - 5 \tan x + 2 = 0$$

$$\Rightarrow (2 \tan x - 1)(\tan x - 2) = 0$$

$$\tan x = \frac{1}{2} \quad \text{or} \quad \tan x = 2$$

$\bullet \arctan\left(\frac{1}{2}\right) = 0.464^\circ$   
 $\bullet \arctan(2) = 1.107^\circ$   
 $2 = 0.464^\circ + \pi$   
 $2 = 1.107^\circ + \pi$   
 $x_1 = 0.464^\circ$   
 $x_2 = 3.61^\circ$   
 $x_3 = 1.11^\circ$   
 $x_4 = 4.25^\circ$

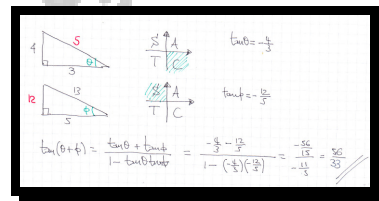
## Question 150 (\*\*\*\*)

$$\cot A = -\frac{3}{4} \quad \text{and} \quad \cos B = \frac{5}{13}.$$

If  $A$  is reflex and  $B$  is also reflex, show that

$$\tan(A+B) = \frac{56}{33}.$$

,  proof



## Question 151 (\*\*\*\*)

Given that

$$\cos\left(x - \frac{\pi}{3}\right) = 2 \sin\left(x + \frac{\pi}{3}\right),$$

show clearly that

$$\tan x = -4 - 3\sqrt{3}.$$

proof

**Question 152** (\*\*\*)

Make  $x$  the subject of the equation

$$\arctan(1+x) + \arctan(1-x) = y.$$

$$x = \pm \sqrt{\frac{2}{\tan y}}$$

$$\begin{aligned} & \arctan_{\theta}(1+2) + \arctan_{\phi}(1-2) = \frac{\pi}{4} \\ \Rightarrow & \tan[\arctan_{\theta}(1+2) + \arctan_{\phi}(1-2)] = \tan \frac{\pi}{4} \\ \Rightarrow & \frac{(1+2) + (1-2)}{1 - (1+2)(1-2)} = \tan y \\ \Rightarrow & \frac{2}{1 - (1-2)} = \tan y \\ \Rightarrow & \frac{2}{2} = \tan y \\ \Rightarrow & \frac{2}{\tan y} = 2^2 \\ \Rightarrow & 2 = \pm \sqrt{\frac{2}{\tan y}} \end{aligned}$$

**Question 153** (\*\*\*\*)

It is given that

$$\sin(x+y)\sin(x-y)\equiv\cos^2y-\cos^2x.$$

- a) Prove the validity of the above trigonometric identity.
- b) Hence, show that

$$\sin\left(\frac{7\pi}{12}\right)\sin\left(\frac{\pi}{12}\right)=\frac{1}{4}.$$

**proof**

(a)  $\begin{aligned} \text{LHS} &= \sin(x+y) \sin(x-y) \\ &= [\sin x \cos y + \cos x \sin y] [\sin x \cos y - \cos x \sin y] \\ &\quad \text{(a+b)(a-b) = a}^2\text{-b}^2 \\ &= \sin^2 x \cos^2 y - \cos^2 x \sin^2 y \\ &= (1 - \cos^2 x) \cos^2 y - \cos^2 x (1 - \cos^2 y) \\ &= \cos^2 y - \cos^2 x \cos^2 y - \cos^2 x + \cos^2 x \cos^2 y \\ &= \cos^2 y - \cos^2 x = \text{RHS} \end{aligned}$

(b)  $\begin{aligned} \text{Let } x &= \frac{\pi}{3} \quad y = \frac{\pi}{4} \\ \Rightarrow \sin\left(\frac{\pi}{3} + \frac{\pi}{4}\right) \sin\left(\frac{\pi}{4} - \frac{\pi}{4}\right) &= \cos^2 \frac{\pi}{4} - \sin^2 \frac{\pi}{3} \\ \Rightarrow \sin\left(\frac{7\pi}{12}\right) \sin\left(\frac{\pi}{12}\right) &= \frac{1}{4} - \frac{3}{4} \\ \Rightarrow \sin\left(\frac{7\pi}{12}\right) \sin\left(\frac{\pi}{12}\right) &= -\frac{1}{2} \end{aligned}$



## Question 154 (\*\*\*\*)

It is given that

$$\frac{1}{\cos x - \sin x} + \frac{1}{\cos x + \sin x} \equiv \frac{2 \sec x}{1 - \tan^2 x}$$

- a) Prove the validity of the above trigonometric identity.
- b) Hence solve the trigonometric equation

$$\frac{1}{\cos x - \sin x} + \frac{1}{\cos x + \sin x} = 2, \quad 0 < x < 2\pi,$$

giving the answers in terms of  $\pi$ .

$$x = \frac{2\pi}{3}, \frac{4\pi}{3}$$

(a) 
$$\frac{1}{\cos x - \sin x} + \frac{1}{\cos x + \sin x} = \frac{(\cos x + \sin x) + (\cos x - \sin x)}{(\cos x - \sin x)(\cos x + \sin x)} = \frac{2 \cos x}{\cos^2 x - \sin^2 x}$$
  

$$= \frac{2 \cos x}{\cos^2 x - \sin^2 x} = \frac{2 \cos x}{\cos^2 x (1 - \tan^2 x)} = \frac{2 \sec x}{1 - \tan^2 x}$$

(b) 
$$\frac{1}{\cos x - \sin x} + \frac{1}{\cos x + \sin x} = 2$$
  

$$\Rightarrow \frac{2 \sec x}{1 - \tan^2 x} = 2$$
  

$$\Rightarrow 2 \sec x = 2 - 2 \tan^2 x$$
  

$$\Rightarrow 2 \tan^2 x + 2 \sec x - 2 = 0$$
  

$$\Rightarrow \tan^2 x + \sec x - 1 = 0$$
  

$$\Rightarrow (\sec x - 1) + \sec x - 1 = 0$$
  

$$\Rightarrow \sec^2 x + \sec x - 2 = 0$$
  

$$\Rightarrow (\sec x - 1)(\sec x + 2) = 0$$
  

$$\Rightarrow \sec x = 1 \quad \text{or} \quad \sec x = -2$$
  

$$\Rightarrow \cos x = 1 \quad \text{or} \quad \cos x = -\frac{1}{2}$$
  

$$\Rightarrow x = 0 \pm 2\pi \quad \text{or} \quad x = \frac{2\pi}{3} \pm 2\pi \quad \text{or} \quad x = \frac{4\pi}{3} \pm 2\pi$$
  

$$\Rightarrow x = 0, 2\pi, \frac{2\pi}{3}, \frac{4\pi}{3}$$
  

$$\therefore x = \frac{2\pi}{3}, \frac{4\pi}{3}$$

## Question 155 (\*\*\*\*)

$$f(x) = \frac{6}{2\cos x + 2\sin x} \text{ for } 0 < x < \pi, x \neq \beta.$$

- a) Express  $2\cos x + 2\sin x$  in the form  $R\cos(x - \alpha)$ ,  $R > 0$ ,  $0 < \alpha < \frac{\pi}{2}$ .

The curve with equation  $y = f(x)$  has a vertical asymptote at  $x = \beta$ .

- b) Determine the value of  $\beta$ .

- c) Solve the equation

$$f(3x) - \sqrt{6} = 0,$$

giving the answers in terms of  $\pi$ .

$$\boxed{\phantom{000}}, \quad 2\cos x + 2\sin x \equiv 2\sqrt{2}\cos\left(x - \frac{\pi}{4}\right), \quad \beta = \frac{3\pi}{4}, \quad x = \frac{\pi}{36}, \frac{5\pi}{36}, \frac{25\pi}{36}, \frac{29\pi}{36}$$

(a) BY STANDARD TECHNIQUE ...  
 $2\cos x + 2\sin x = 2\sqrt{2}\left(\frac{1}{\sqrt{2}}\cos x + \frac{1}{\sqrt{2}}\sin x\right)$   
 $= 2\sqrt{2}\left(\cos\left(x - \frac{\pi}{4}\right)\right)$   
 $= 2\sqrt{2}\cos\left(x - \frac{\pi}{4}\right)$

(b) ASYMPTOTE (VERTICAL)  $\Rightarrow$  DENOMINATOR  $= 0$   
 $\Rightarrow \cos\left(x - \frac{\pi}{4}\right) = 0$   
 $\Rightarrow x - \frac{\pi}{4} = \frac{\pi}{2}$   
 $\Rightarrow x = \frac{3\pi}{4} \therefore \beta = \frac{3\pi}{4}$

(c)  $f(x) = \frac{6}{2\cos x + 2\sin x} = \frac{6}{2\sqrt{2}\cos\left(x - \frac{\pi}{4}\right)}$   
 $f(3x) - \sqrt{6} = 0$   
 $\Rightarrow \frac{6}{2\sqrt{2}\cos\left(3x - \frac{\pi}{4}\right)} - \sqrt{6} = 0$   
 $\Rightarrow \frac{6}{2\sqrt{2}\cos\left(3x - \frac{\pi}{4}\right)} = \sqrt{6}$   
 $\Rightarrow \frac{3}{\sqrt{2}\cos\left(3x - \frac{\pi}{4}\right)} = \sqrt{6}$   
 $\Rightarrow \frac{\sqrt{2}\cos\left(3x - \frac{\pi}{4}\right)}{3} = \frac{1}{\sqrt{6}}$   
 $\Rightarrow \cos\left(3x - \frac{\pi}{4}\right) = \frac{\sqrt{3}}{2}$   
 $\bullet \text{ since } \cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$   
 $3x - \frac{\pi}{4} = \frac{\pi}{6} \pm 2\pi n$   
 $3x = \frac{5\pi}{12} \pm 2\pi n$   
 $x = \frac{5\pi}{36} \pm \frac{2\pi n}{3}$   
 $\bullet \text{ since } \cos\left(\frac{5\pi}{6}\right) = -\frac{\sqrt{3}}{2}$   
 $3x - \frac{\pi}{4} = \frac{5\pi}{6} \pm 2\pi n$   
 $3x = \frac{11\pi}{12} \pm 2\pi n$   
 $x = \frac{11\pi}{36} \pm \frac{2\pi n}{3}$   
 $\therefore x = \frac{\pi}{36}, \frac{5\pi}{36}, \frac{25\pi}{36}, \frac{29\pi}{36}$

## Question 156 (\*\*\*\*)

It is given that

$$\tan(x+60^\circ)\tan(x-60^\circ) \equiv \frac{\tan^2 x - 3}{1 - 3\tan^2 x}$$

- a) Prove the validity of the above trigonometric identity.
- b) Hence, or otherwise, solve the trigonometric equation

$$\tan(x+60^\circ)\tan(x-60^\circ) + 11 = 0, \quad 0 \leq x < 360^\circ.$$

$$x \approx 26.6^\circ, 153.4^\circ, 186.6^\circ, 333.4^\circ$$

(a) LHS =  $\tan(x+60^\circ)\tan(x-60^\circ) = \frac{\tan x + \tan 60^\circ}{1 - \tan x \tan 60^\circ} \times \frac{\tan x - \tan 60^\circ}{1 + \tan x \tan 60^\circ}$   
 $= \frac{(\tan x + \sqrt{3})(\tan x - \sqrt{3})}{(1 - \sqrt{3}\tan x)(1 + \sqrt{3}\tan x)} = \frac{\tan^2 x - 3}{1 - 3\tan^2 x} = \text{RHS}$

(b)  $\tan(x+60^\circ)\tan(x-60^\circ) + 11 = 0$   
 $\Rightarrow \frac{\tan^2 x - 3}{1 - 3\tan^2 x} + 11 = 0$   
 $\Rightarrow \tan^2 x - 3 + 11(1 - 3\tan^2 x) = 0$   
 $\Rightarrow \tan^2 x - 3 + 11 - 33\tan^2 x = 0$   
 $\Rightarrow 8 = 32\tan^2 x$   
 $\Rightarrow \tan^2 x = \frac{1}{4}$   
 $\Rightarrow \tan x = \pm \frac{1}{2}$

•  $\arctan\left(\frac{1}{2}\right) = 26.6$   
 •  $\arctan\left(-\frac{1}{2}\right) = -26.6$   
 $\therefore x = 26.6^\circ \pm 180^\circ \quad (\text{use } 90^\circ, 270^\circ)$   
 $x_1 = 26.6^\circ$   
 $x_2 = 206.6^\circ$   
 $x_3 = 153.4^\circ$   
 $x_4 = 333.4^\circ$

**Question 157** (\*\*\*\*)

$$f(x) \equiv \cos x + \sqrt{3} \sin x, \quad x \in \mathbb{R}.$$

- a)** Express  $f(x)$  in the form  $R \cos(x - \alpha)$ , where  $R > 0$  and  $0 < \alpha < \frac{\pi}{2}$ .
- b)** Hence solve the equation

$$\cos 2\theta + \sqrt{3} \sin 2\theta = 2 \cos \theta, \quad 0 \leq \theta < 2\pi.$$

$$\boxed{\text{A}}, \quad \boxed{f(x) = 2 \cos\left(x - \frac{\pi}{3}\right)}, \quad \boxed{\theta = \frac{\pi}{3}, \frac{7\pi}{9}, \frac{13\pi}{9}}$$

$$\begin{aligned} \text{a) } \cos x + \sqrt{3} \sin x &\equiv R \cos(x - \alpha) \\ &\equiv R \cos \alpha \cos x + R \sin \alpha \sin x \\ &\equiv (R \cos \alpha) \cos x + (R \sin \alpha) \sin x \\ R \cos \alpha &= 1 \\ R \sin \alpha = \sqrt{3} \end{aligned} \Rightarrow \text{square of both } R = \sqrt{1^2 + 3^2} = 2$$
$$\text{Divide } \frac{\cos \alpha = \frac{1}{2}}{\sin \alpha = \frac{\sqrt{3}}{2}}$$
$$\therefore f(x) = 2 \cos(x - \frac{\pi}{6})$$
$$\text{b) } \cos 2\theta + \sqrt{3} \sin 2\theta = 2 \cos \theta$$
$$2 \cos \theta (\cos \theta - \frac{\sqrt{3}}{2}) = 2 \cos \theta$$
$$\cos \theta (\cos \theta - \frac{\sqrt{3}}{2}) = \cos \theta$$
$$\left. \begin{aligned} 2\theta - \frac{\pi}{6} &= \frac{\pi}{2} \neq 2\pi \\ 2\theta - \frac{\pi}{6} &= (2\pi - \frac{\pi}{6}) \neq 2\pi \end{aligned} \right\} \Rightarrow \theta = \frac{\pi}{4}, \frac{5\pi}{4}, \dots$$
$$\left. \begin{aligned} \theta &= \frac{\pi}{6} \pm 2\pi \\ \theta &= \frac{5\pi}{6} \pm 2\pi \end{aligned} \right\}$$
$$\left. \begin{aligned} \theta &= \frac{\pi}{3} \pm 2\pi \\ \theta &= \frac{5\pi}{3} \pm 2\pi \end{aligned} \right\}$$
$$\begin{aligned} \theta &= \frac{\pi}{6} \\ \theta &= \frac{5\pi}{6} \\ \theta &= \frac{7\pi}{6} \end{aligned}$$

## Question 158 (\*\*\*\*)

It is given that

$$\frac{\operatorname{cosec} x - \sin x}{\cot x \cos^2 x} \equiv \sec x.$$

- a) Prove the validity of the above trigonometric identity.
- b) Hence, or otherwise, solve the trigonometric equation

$$\tan^2 x - \sec x = \frac{\operatorname{cosec} x - \sin x}{2 \cot x \cos^2 x}, \quad 0 \leq x < 360^\circ.$$

$$x = 60^\circ, 300^\circ$$

Handwritten solution for Question 158b:

(a) LHS =  $\frac{\operatorname{cosec} x - \sin x}{\cot x \cos^2 x} = \frac{\frac{1}{\sin x} - \sin x}{\frac{\cos x}{\sin x} \times \cos^2 x} = \frac{\frac{1 - \sin^2 x}{\sin x}}{\frac{\cos^3 x}{\sin x}} = \frac{1 - \sin^2 x}{\cos^3 x} = \frac{\cos^2 x}{\cos^3 x} = \frac{1}{\cos x} = \sec x = \text{RHS}$

(b)  $\tan^2 x - \sec x = \frac{\operatorname{cosec} x - \sin x}{2 \cot x \cos^2 x}$

$\Rightarrow 2 \tan^2 x - 2 \sec x = \sec x$

$\Rightarrow 2(\sec^2 x - 1) - 2 \sec x = \sec x$

$\Rightarrow 2 \sec^2 x - 2 \sec x - 2 = 0$

$\Rightarrow (\sec x + 1)(\sec x - 2) = 0$

$\Rightarrow \sec x = -1$

$\Rightarrow \cos x = -\frac{1}{2}$

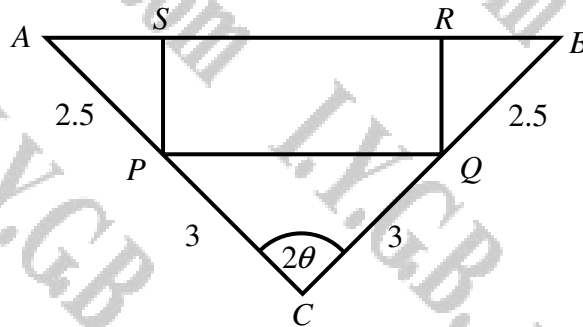
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$\begin{cases} x = 60 + 360n \\ x = 300 + 360n \end{cases}$  for  $n \in \mathbb{Z}$ .

$x_1 = 60$

$x_2 = 300$

## Question 159 (\*\*\*\*)



The figure above shows an isosceles triangle  $ABC$  where the angle  $ACB = 2\theta$ .

A rectangle  $PQRS$  is drawn inside  $ABC$ , so that  $S$  and  $R$  lie on  $AB$ ,  $P$  lies on  $AC$  and  $Q$  lies on  $BC$ .

It is further given that  $|AP| = |BQ| = 2.5$  and  $|PC| = |QC| = 3$ .

- a) Show clearly that the perimeter of  $PQRS$  is

$$5\cos\theta + 12\sin\theta.$$

- b) Express  $5\cos\theta + 12\sin\theta$  in the form  $R\sin(\theta + \alpha)$ ,  $R > 0$ ,  $0 < \alpha < 90^\circ$ .

- c) Find the value of  $\theta$ , given that the perimeter of  $PQRS$  is 10.

$$\boxed{\phantom{000}}, \quad \boxed{5\cos\theta + 12\sin\theta \approx 13\sin(\theta + 22.62^\circ)}, \quad \boxed{\theta \approx 27.7^\circ}$$

LOOKING AT THE TRIANGLES  $APC$  &  $BQC$

$|AP| = 2.5$   
 $|PC| = 3$   
 $|PQ| = 2.5\cos\theta$   
 $|QR| = 3\sin\theta$   
 $\Rightarrow |PQ| + |QR| = 2.5\cos\theta + 3\sin\theta$   
 $\Rightarrow \text{Perimeter} = 2 \times (2.5\cos\theta + 3\sin\theta)$   
 $\Rightarrow \text{Perimeter} = 5\cos\theta + 6\sin\theta$

USING THE COMPOUND ANGLE IDENTITY FOR  $\sin(A+B)$

$(2.5\cos\theta + 3\sin\theta) = R\sin(\theta + \alpha)$   
 $(2.5\cos\theta + 3\sin\theta) = R(\sin\theta\cos\alpha + \cos\theta\sin\alpha)$   
 $(2.5\cos\theta + 3\sin\theta) = (R\cos\alpha)\sin\theta + (R\sin\alpha)\cos\theta$

$R\cos\alpha = 2.5$   
 $R\sin\alpha = 3$   
 $\Rightarrow \tan\alpha = \frac{3}{2.5} = 1.2$   
 $\alpha = 50.1^\circ$   
 $R = \frac{2.5}{\cos\alpha} = \frac{2.5}{\cos 50.1^\circ} \approx 3.99$   
 $\Rightarrow 2.5\cos\theta + 3\sin\theta \approx 3.99\sin(\theta + 50.1^\circ)$

4) SETTING  $P=10$

$\Rightarrow 5\cos\theta + 12\sin\theta = 10$   
 $\Rightarrow 13.54(\sin\theta + 22.62^\circ) = 10$   
 $\Rightarrow \sin(\theta + 22.62^\circ) = \frac{10}{13.54}$   
 $\Rightarrow \theta + 22.62^\circ = \sin^{-1}\left(\frac{10}{13.54}\right) = 47.7^\circ$   
 $\Rightarrow \theta = 47.7^\circ - 22.62^\circ = 25.1^\circ$

## Question 160 (\*\*\*\*)

$$f(\theta) \equiv \sqrt{3} \sin \theta + \sin \theta, \quad 0 \leq \theta < 2\pi.$$

a) Express  $f(\theta)$  in the form  $R \cos(\theta - \alpha)$ ,  $R > 0$ ,  $0 < \alpha < \frac{\pi}{2}$ .

b) Find the solutions of the trigonometric equation

$$\sqrt{3} + \sin 2\theta = \sqrt{3} \cos 2\theta + 2 \sin \theta, \quad 0 \leq \theta < 2\pi,$$

giving the answers in radians in terms of  $\pi$ .

$$\theta = 0, \frac{2\pi}{3}, \pi$$

(a)  $\sqrt{3} \sin \theta + \cos \theta \equiv R \cos(\theta - \alpha)$   
 $\equiv R \cos \theta \cos \alpha + R \sin \theta \sin \alpha$   
 $\equiv (R \cos \alpha) \cos \theta + (R \sin \alpha) \sin \theta$   
 $R \cos \alpha = 1$   
 $R \sin \alpha = \sqrt{3}$   
 $\Rightarrow R = \sqrt{1^2 + 3^2} = 2$   
 $\tan \alpha = \sqrt{3} \quad \alpha = \frac{\pi}{3}$   
 $\therefore \sqrt{3} \sin \theta + \cos \theta = 2 \cos(\theta - \frac{\pi}{3})$

(b)  $\sqrt{3} + \sin 2\theta = \sqrt{3} \cos 2\theta + 2 \sin \theta$   
 $\sqrt{3} + 2 \sin \theta \cos \theta = \sqrt{3} (1 - 2 \sin^2 \theta) + 2 \sin \theta$   
 $\sqrt{3} + 2 \sin \theta \cos \theta = \sqrt{3} - 2\sqrt{3} \sin^2 \theta + 2 \sin \theta$   
 $2\sqrt{3} \sin^2 \theta + 2 \sin \theta \cos \theta - 2 \sin \theta = 0$   
 $2 \sin \theta [\sqrt{3} \sin \theta + \cos \theta - 1] = 0$   
 $2 \sin \theta [2 \cos(\theta - \frac{\pi}{3}) - 1] = 0$   
 $\bullet \sin \theta = 0$   
 $\theta = 0 \pm 2\pi$   
 $\theta = \pi \pm 2\pi$   
 $\theta = 0, \pi$   
 $\bullet \cos(\theta - \frac{\pi}{3}) = \frac{1}{2}$   
 $\theta - \frac{\pi}{3} = \frac{\pi}{3} \pm 2\pi n$   
 $\theta - \frac{\pi}{3} = \frac{5\pi}{3} \pm 2\pi n$   
 $\theta = \frac{4\pi}{3} \pm 2\pi n$   
 $\theta = \frac{2\pi}{3} \pm 2\pi n$   
 $\theta = 0, \frac{2\pi}{3}, \pi$

## Question 161 (\*\*\*\*)

It is given that

$$\cos P - \cos Q \equiv -2 \sin\left(\frac{P+Q}{2}\right) \sin\left(\frac{P-Q}{2}\right).$$

- a) Prove the validity of the above trigonometric identity by using the compound angle identities for  $\cos(A+B)$  and  $\cos(A-B)$ .
- b) Hence, or otherwise, solve the trigonometric equation

$$\cos 6x + \sin 4x = \cos 2x, \quad 0 \leq x \leq \frac{\pi}{2},$$

giving the answers in terms of  $\pi$ .

$$x = 0, \frac{\pi}{12}, \frac{\pi}{4}, \frac{5\pi}{12}, \frac{\pi}{2}$$

(a)  $\cos(A+B) = \cos A \cos B - \sin A \sin B$   
 $\cos(A-B) = \cos A \cos B + \sin A \sin B$   
 $\cos(A+B) - \cos(A-B) = -2 \sin A \sin B$  Subtract

Let  $\frac{A+B}{2} = P$  and  $\frac{A-B}{2} = Q$  Add  
 $\frac{A+B}{2} = P \Rightarrow A+B = 2P$   
 $\frac{A-B}{2} = Q \Rightarrow A-B = 2Q$   
 $\therefore \cos P - \cos Q = -2 \sin\left(\frac{P+Q}{2}\right) \sin\left(\frac{P-Q}{2}\right)$  ✓

(b)  $\cos 6x + \sin 4x = \cos 2x$   
 $\Rightarrow \cos 6x - \cos 2x = -\sin 4x$   
 $\Rightarrow -2 \sin\left(\frac{6x+2x}{2}\right) \sin\left(\frac{6x-2x}{2}\right) = -\sin 4x$   
 $\Rightarrow -2 \sin 4x \sin 2x = -\sin 4x$   
 $\Rightarrow \sin 4x - 2 \sin 4x \sin 2x = 0$   
 $\Rightarrow \sin 4x (1 - 2 \sin 2x) = 0$   
 $\Rightarrow \sin 4x = 0 \quad \sin 2x = \frac{1}{2}$

•  $\sin 4x = 0$   
 $\begin{cases} 4x = 0 \pm 2\pi \\ 4x = \pi \pm 2\pi \end{cases} \Rightarrow \begin{cases} x = 0 \pm \frac{\pi}{2} \\ x = \frac{\pi}{4} \pm \frac{\pi}{2} \end{cases}$

•  $\sin 2x = \frac{1}{2}$   
 $\begin{cases} 2x = \frac{\pi}{6} \pm 2\pi \\ 2x = \frac{5\pi}{6} \pm 2\pi \end{cases} \Rightarrow \begin{cases} x = \frac{\pi}{12} \pm \pi \\ x = \frac{5\pi}{12} \pm \pi \end{cases}$

$0, \frac{\pi}{12}, \frac{\pi}{4}, \frac{5\pi}{12}, \frac{\pi}{2}$  ✓



## Question 162 (\*\*\*\*)

Solve each of the following trigonometric equations.

i.  $\frac{\sec^2 x + 8}{4 - \tan x} = 3 \tan x, \quad 0 \leq x < 2\pi, \quad \tan x \neq 4.$

ii.  $\cos 2\theta = \sin \theta, \quad 0 \leq \theta < 360^\circ.$

,  $x \approx 0.983^\circ, 4.12^\circ$ ,  $\theta = 30^\circ, 150^\circ, 270^\circ$

$$\frac{\sec^2 x + 8}{4 - \tan x} = 3 \tan x$$

$$\Rightarrow \sec^2 x + 8 = 12 \tan x - 3 \tan^2 x$$

$$\Rightarrow (1 + \tan^2 x) + 8 = 12 \tan x - 3 \tan^2 x$$

$$\Rightarrow 4 \tan^2 x - 12 \tan x + 9 = 0$$

$$\Rightarrow (2 \tan x - 3)^2 = 0$$

$$\tan x = \frac{3}{2}$$

$$\arctan\left(\frac{3}{2}\right) = 0.983^\circ$$

$$\bullet 2 = 0.983^\circ + \pi$$

$$\bullet \bullet 2 = 0.983^\circ$$

$$\bullet 2 = 4.12^\circ$$

$$\cos 2\theta = \sin \theta$$

$$\Rightarrow 1 - 2 \sin^2 \theta = \sin \theta$$

$$\Rightarrow 0 = 2 \sin^2 \theta + \sin \theta - 1$$

$$\Rightarrow (2 \sin \theta - 1)(\sin \theta + 1) = 0$$

$$\Rightarrow \sin \theta = \frac{1}{2}$$

$$\bullet \arcsin\left(\frac{1}{2}\right) = 30^\circ$$

$$\left( \begin{array}{l} \theta = 30^\circ \text{ or } 300^\circ \\ \theta = 150^\circ \text{ or } 330^\circ \end{array} \right)$$

$$\bullet \arcsin(-1) = -90^\circ$$

$$\left( \begin{array}{l} \theta = -90^\circ \text{ or } 270^\circ \\ \theta = 270^\circ \text{ or } 360^\circ \end{array} \right)$$

$$\theta = 30^\circ, 150^\circ, 270^\circ$$

## Question 163 (\*\*\*\*)

Solve the following trigonometric equation

$$\tan(\arctan 3x - \arctan 2) + \tan(\arctan 3 - \arctan 2x) = \frac{3}{8}.$$

,  $x = \frac{1}{2}$

USING THE IDENTITY FOR  $\tan(A-B)$   

$$\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

$$\Rightarrow \tan(\arctan 3x - \arctan 2) + \tan(\arctan 3 - \arctan 2x) = \frac{3}{8}$$

$$\Rightarrow \frac{3x - 2}{1 + (3x)(2)} + \frac{3 - 2x}{1 + 3(2x)} = \frac{3}{8}$$

$$\Rightarrow \frac{3x - 2}{1 + 6x} + \frac{3 - 2x}{1 + 6x} = \frac{3}{8}$$

$$\Rightarrow \frac{3x - 2 + 3 - 2x}{1 + 6x} = \frac{3}{8}$$

$$\Rightarrow \frac{x + 1}{1 + 6x} = \frac{3}{8}$$

$$\Rightarrow 8x + 8 = 3 + 18x$$

$$\Rightarrow 5 = 10x$$

$$\Rightarrow x = \frac{1}{2}$$

**Question 164** (\*\*\*\*)

It is given that

$$\cos(A+B) \equiv \cos A \cos B - \sin A \sin B$$

$$\cos(A-B) \equiv \cos A \cos B + \sin A \sin B$$

a) Use the above trigonometric identities show that

$$\cos(A+B) + \cos(A-B) \equiv 2 \cos A \cos B.$$

b) Hence show further that

$$\cos P + \cos Q \equiv 2 \cos\left(\frac{P+Q}{2}\right) \cos\left(\frac{P-Q}{2}\right).$$

It is further given that

$$\sin P + \sin Q \equiv 2 \sin\left(\frac{P+Q}{2}\right) \cos\left(\frac{P-Q}{2}\right).$$

c) Show clearly that

$$\frac{\cos 2x + \cos 2y}{\sin 2x + \sin 2y} \equiv \cot(x+y).$$

d) Use the above results to show that

$$\cot(52.5^\circ) = \sqrt{6} + \sqrt{3} - \sqrt{2} - 2$$

proof

Handwritten proof for Question 164d:

(a)  $\cos(A+B) = \cos A \cos B - \sin A \sin B$   
 $\cos(A-B) = \cos A \cos B + \sin A \sin B$   
 $\cos(A+B) + \cos(A-B) = 2 \cos A \cos B$  ✓ All the equation

(b) Let  $P = A+B$      Add  $\Rightarrow \frac{P+Q}{2} = A$   
 $Q = A-B$      Subtract  $\Rightarrow \frac{P-Q}{2} = B$

Then  $\cos P + \cos Q = 2 \cos\left(\frac{P+Q}{2}\right) \cos\left(\frac{P-Q}{2}\right)$

(c) LHS =  $\frac{\cos 2x + \cos 2y}{\sin 2x + \sin 2y} = \frac{2 \cos\left(\frac{2x+2y}{2}\right) \cos\left(\frac{2x-2y}{2}\right)}{2 \sin\left(\frac{2x+2y}{2}\right) \cos\left(\frac{2x-2y}{2}\right)}$  (Note: The original image has a typo here, it should be  $\cos$  in the denominator for the sum-to-product formula)  
 $= \frac{\cos(x+y)}{\sin(x+y)} = \cot(x+y) = \text{RHS}$  ✓

(d)  $\frac{\cos 2x + \cos 2y}{\sin 2x + \sin 2y} = \cot(x+y)$       $\Rightarrow \cot(52.5^\circ) = \frac{1 + \sqrt{2}}{\sqrt{3} + \sqrt{2}}$   
 Let  $x = 30^\circ$       $y = 22.5^\circ$       $\Rightarrow \cot(52.5^\circ) = \frac{(1+\sqrt{2})(\sqrt{3}-\sqrt{2})}{(\sqrt{3}+\sqrt{2})(\sqrt{3}-\sqrt{2})}$   
 $\Rightarrow \frac{\cos 60^\circ + \cos 45^\circ}{\sin 60^\circ + \sin 45^\circ} = \cot(52.5^\circ)$       $\Rightarrow \cot(52.5^\circ) = \frac{\sqrt{2}-\sqrt{3}+\sqrt{2}-2}{1}$   
 $\Rightarrow \frac{1}{\sqrt{2}} + \frac{\sqrt{2}}{2} = \cot(52.5^\circ)$       $\Rightarrow \cot(52.5^\circ) = \sqrt{2} + \sqrt{3} - \sqrt{2} - 2$  ✓ (Proved)

## Question 165 (\*\*\*\*)

Solve the trigonometric equation

$$\ln(\operatorname{cosec} \theta) = \ln 4 - \ln(\sec \theta), \quad 0 < \theta < \frac{\pi}{2},$$

giving the answers in terms of  $\pi$ .

$$\theta = \frac{\pi}{12}, \frac{5\pi}{12}$$

Handwritten solution for Question 165:

$$\begin{aligned} \ln(\operatorname{cosec} \theta) &= \ln 4 - \ln(\sec \theta) \\ \ln(\operatorname{cosec} \theta) + \ln(\sec \theta) &= \ln 4 \\ \ln(\operatorname{cosec} \theta \sec \theta) &= \ln 4 \\ \ln\left(\frac{1}{\sin \theta \cos \theta}\right) &= \ln 4 \\ \ln\left(\frac{2}{2 \sin \theta \cos \theta}\right) &= \ln 4 \\ \ln\left(\frac{2}{\sin 2\theta}\right) &= \ln 4 \\ \frac{2}{\sin 2\theta} &= 4 \end{aligned}$$

Then:

$$\begin{aligned} \sin 2\theta &= \frac{1}{2} \\ \arcsin\left(\frac{1}{2}\right) &= \frac{\pi}{6} \\ 2\theta &= \frac{\pi}{6} + 2n\pi \quad \text{or} \quad \frac{5\pi}{6} + 2n\pi \\ \theta &= \frac{\pi}{12} + n\pi \quad \text{or} \quad \frac{5\pi}{12} + n\pi \\ \theta_1 &= \frac{\pi}{12} \\ \theta_2 &= \frac{5\pi}{12} \end{aligned}$$

## Question 166 (\*\*\*\*)

Solve the trigonometric equation

$$\sin \theta \cos \frac{\pi}{5} = \frac{1}{2} - \cos \theta \sin \frac{\pi}{5}, \quad 2\pi < \theta < 4\pi,$$

giving the answers in terms of  $\pi$ .

$$\theta = \frac{79\pi}{30}, \frac{119\pi}{30}$$

Handwritten solution for Question 166:

RECALLING ALSO A 'COMPOUND' ANGLE IDENTITY

$$\begin{aligned} \Rightarrow \sin \theta \cos \frac{\pi}{5} &= \frac{1}{2} - \cos \theta \sin \frac{\pi}{5} \\ \Rightarrow \sin \theta \cos \frac{\pi}{5} + \cos \theta \sin \frac{\pi}{5} &= \frac{1}{2} \\ \Rightarrow \sin\left(\theta + \frac{\pi}{5}\right) &= \frac{1}{2} \end{aligned}$$

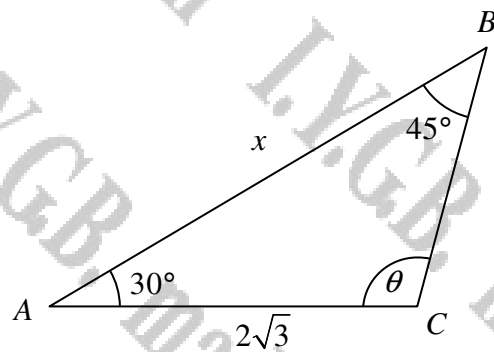
Then:

$$\begin{aligned} \arcsin\left(\frac{1}{2}\right) &= \frac{\pi}{6} \\ \theta + \frac{\pi}{5} &= \frac{\pi}{6} + 2n\pi \\ \theta + \frac{\pi}{5} &= \frac{5\pi}{6} + 2n\pi \quad \forall n \in \mathbb{Z} \\ \theta &= -\frac{\pi}{10} + 2n\pi \\ \theta &= \frac{19\pi}{10} + 2n\pi \end{aligned}$$

And for  $2\pi < \theta < 4\pi$

$$\begin{aligned} \theta_1 &= \frac{19\pi}{10} \\ \theta_2 &= \frac{29\pi}{10} \end{aligned}$$

## Question 167 (\*\*\*\*) non calculator



The figure above shows a triangle  $ABC$  where  $|AC| = 2\sqrt{3}$  and  $|AB| = x$ .

The angles  $ABC$ ,  $CAB$  and  $BCA$  are  $45^\circ$ ,  $30^\circ$  and  $\theta^\circ$ , respectively.

- a) By using a suitable compound angle identity show clearly that

$$\sin 105^\circ = \frac{\sqrt{6} + \sqrt{2}}{4}.$$

- b) Show without the use of a calculating aid that the exact length of  $AB$ , is

$$3 + \sqrt{3}.$$

proof

(a)  $\sin(105^\circ) = \sin(60^\circ + 45^\circ) = \sin 60^\circ \cos 45^\circ + \cos 60^\circ \sin 45^\circ$   
 $= \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{2}}{2} + \frac{1}{2} \cdot \frac{\sqrt{2}}{2}$   
 $= \frac{\sqrt{6}}{4} + \frac{\sqrt{2}}{4}$   
 $= \frac{\sqrt{6} + \sqrt{2}}{4}$   $\checkmark$   $\text{Rasp=0.966}$

(b)  $\frac{x}{\sin \theta} = \frac{2\sqrt{3}}{\sin 45^\circ}$  (by sine rule)  
 $x = \frac{2\sqrt{3} \sin \theta}{\sin 45^\circ} = \frac{2\sqrt{3}}{\frac{\sqrt{2}}{2}} \cdot \left( \frac{\sqrt{6} + \sqrt{2}}{4} \right) = \frac{4\sqrt{3}}{\sqrt{2}} \cdot \frac{\sqrt{6} + \sqrt{2}}{4}$   
 $= \frac{\sqrt{6} + \sqrt{2}}{\sqrt{2}} = \frac{3\sqrt{2} + \sqrt{2}}{\sqrt{2}} = 3 + \frac{\sqrt{2}}{\sqrt{2}} = 3 + 1 = 4$   $\checkmark$

## Question 168 (\*\*\*\*)

$$f(\theta) = 2\cos\theta + 3\sin\theta, \theta \in \mathbb{R}.$$

- a) Express  $f(\theta)$  in the form  $R\cos(\theta - \alpha)$ ,  $R > 0$ ,  $0 < \alpha < \frac{\pi}{2}$ .

Give the value of  $\alpha$  correct to 3 decimal places.

- b) State the maximum value of  $f(\theta)$  and find the smallest positive value of  $\theta$  for which this maximum occurs.

The temperature  $T$  °C in a warehouse is modelled by the equation

$$T = 16 + 2\cos\left(\frac{\pi t}{12}\right) + 3\sin\left(\frac{\pi t}{12}\right), \quad 0 \leq t < 24,$$

where  $t$  is the time in hours measured since midnight.

- c) State the maximum temperature in the warehouse and a value of  $t$  when this maximum temperature occurs.
- d) Find the times, to the nearest minute using 24 hour clock notation, when the temperature in the warehouse is 17 °C.

$$\boxed{\phantom{0000}}, \quad 2\cos\theta + 3\sin\theta \equiv \sqrt{13}\cos(\theta - 0.983^\circ), \quad \boxed{\max = \sqrt{13}}, \quad \boxed{\theta = 0.983^\circ},$$

$$\boxed{T_{\max} \approx 19.6}, \quad \boxed{t_{\max} = 3.75}, \quad \boxed{08:41/22:50}$$

**9) APPLICATION AS PER QUAL**

$2\cos\theta + 3\sin\theta \equiv R\cos(\theta - \alpha)$   
 $2\cos\theta + 3\sin\theta \equiv R\cos\theta\cos\alpha + R\sin\theta\sin\alpha$   
 $2\cos\theta + 3\sin\theta \equiv (R\cos\alpha)\cos\theta + (R\sin\alpha)\sin\theta$

**COMPARING**

$R\cos\alpha = 2$   
 $R\sin\alpha = 3$

**SQUARE AND ADD**

$R = \sqrt{2^2 + 3^2}$   
 $R = \sqrt{13}$  ( $R > 0$ )

**DIVIDE BY SINE**

$\frac{R\cos\alpha}{R\sin\alpha} = \frac{2}{3}$   
 $\cot\alpha = \frac{2}{3}$   
 $\alpha = 0.983^\circ$

$\therefore f(\theta) \equiv \sqrt{13}\cos(\theta - 0.983^\circ)$

**10) FIND THE MAX**

$f(\theta)_{\max} = \sqrt{13}$   
 IT OCCURS WHEN  $\cos(\theta - 0.983^\circ) = 1$   
 $\theta - 0.983^\circ = 0$   
 $\theta = 0.983^\circ$

**11) FIND THE MIN**

$T = 16 + 2\cos\left(\frac{\pi t}{12}\right) + 3\sin\left(\frac{\pi t}{12}\right)$   
 $T_{\max} = 16 + \sqrt{13} \approx 19.6^\circ$

**12) FIND THE TIMES**

$T = 17$   
 $17 = 16 + 2\cos\left(\frac{\pi t}{12}\right) + 3\sin\left(\frac{\pi t}{12}\right)$   
 $1 = \sqrt{13}\cos\left(\frac{\pi t}{12} - 0.983^\circ\right)$   
 $\cos\left(\frac{\pi t}{12} - 0.983^\circ\right) = \frac{1}{\sqrt{13}}$   
 $\frac{\pi t}{12} - 0.983^\circ = \pm \cos^{-1}\left(\frac{1}{\sqrt{13}}\right)$   
 $\frac{\pi t}{12} = 0.983^\circ \pm 0.983^\circ$   
 $\frac{\pi t}{12} = 0 \text{ or } 1.966^\circ$   
 $t = 0 \text{ or } 3.75$  (hours)

**13) FINALLY USE THE FORMULA**

$T = 17$   
 $17 = 16 + 2\cos\left(\frac{\pi t}{12}\right) + 3\sin\left(\frac{\pi t}{12}\right)$   
 $1 = \sqrt{13}\cos\left(\frac{\pi t}{12} - 0.983^\circ\right)$   
 $\cos\left(\frac{\pi t}{12} - 0.983^\circ\right) = \frac{1}{\sqrt{13}}$   
 $\frac{\pi t}{12} - 0.983^\circ = \pm \cos^{-1}\left(\frac{1}{\sqrt{13}}\right)$   
 $\frac{\pi t}{12} = 0.983^\circ \pm 0.983^\circ$   
 $\frac{\pi t}{12} = 0 \text{ or } 1.966^\circ$   
 $t = 0 \text{ or } 3.75$  (hours)

$\therefore t = 08:41 \text{ or } 20:50$

## Question 169 (\*\*\*\*)

It is given that

$$\sin P + \sin Q \equiv 2 \sin\left(\frac{P+Q}{2}\right) \cos\left(\frac{P-Q}{2}\right).$$

- a) Prove the validity of the above trigonometric identity by using the compound angle identities for  $\sin(A+B)$  and  $\sin(A-B)$ .
- b) Hence, or otherwise, solve the equation

$$\sin \theta + \sin 2\theta + \sin 3\theta = 0, \quad 0 \leq \theta < \pi,$$

giving the answers in terms of  $\pi$ .

$$\theta = 0, \frac{\pi}{2}, \frac{2\pi}{3}$$

a)  $\sin(A+B) = \sin A \cos B + \cos A \sin B$   
 $\sin(A-B) = \sin A \cos B - \cos A \sin B$   
 $\sin(A+B) + \sin(A-B) = 2 \sin A \cos B$  (by adding)

Let  $P = A+B$   
 $Q = A-B$  }  $\Rightarrow$  Add  $P+Q = 2A$  Subtract  $P-Q = 2B$   
 $A = \frac{P+Q}{2}$   $B = \frac{P-Q}{2}$

Use the previous unit done  
 $\sin P + \sin Q = 2 \sin\left(\frac{P+Q}{2}\right) \cos\left(\frac{P-Q}{2}\right)$

b)  $\sin \theta + \sin 2\theta + \sin 3\theta = 0$   
 $\Rightarrow \sin \theta + \sin \theta + \sin 2\theta = 0$   
 $\Rightarrow 2 \sin \theta + \sin 2\theta = 0$   
 $\Rightarrow 2 \sin \theta (1 + \cos \theta) = 0$   
 $\Rightarrow \sin \theta (2 \cos \theta + 1) = 0$

$\bullet \sin \theta = 0$   
 $\theta = 0$   
 $\theta = \pi$  (not in range)

$\bullet 2 \cos \theta + 1 = 0$   
 $\cos \theta = -\frac{1}{2}$   
 $\theta = \frac{2\pi}{3}$   
 $\theta = \frac{4\pi}{3}$  (not in range)

$\therefore \theta = 0, \frac{2\pi}{3}$

## Question 170 (\*\*\*\*)

Given that

$$64 \cos 2\theta \cos \theta + 32 \sin 2\theta \sin \theta = 27,$$

find the value of  $\cos \theta$ .

$$\boxed{\phantom{000}}, \cos \theta = \frac{3}{4}$$

37044. Solve the following arguments & find

$$\Rightarrow 64 \cos 2\theta \cos \theta + 32 \sin 2\theta \sin \theta = 27$$

$$\Rightarrow 64 (2 \cos^2 \theta - 1) \cos \theta + 32 (2 \sin \theta \cos \theta) \sin \theta = 27$$

$$\Rightarrow 128 \cos^3 \theta - 64 \cos \theta + 64 \sin^2 \theta \cos \theta = 27$$

$$\Rightarrow 128 \cos^3 \theta - 64 \cos \theta + 64 (1 - \cos^2 \theta) \cos \theta = 27$$

$$\Rightarrow 128 \cos^3 \theta - 64 \cos \theta + 64 \cos \theta - 64 \cos^3 \theta = 27$$

$$\Rightarrow 64 \cos^3 \theta = 27$$

$$\Rightarrow \cos^3 \theta = \frac{27}{64}$$

$$\Rightarrow \cos \theta = \frac{3}{4}$$

## Question 171 (\*\*\*\*)

$$f(x) = 2\sin x + 2\cos x, \quad x \in \mathbb{R}.$$

a) Express  $f(x)$  in the form  $R\sin(x+\alpha)$ ,  $R > 0$ ,  $0 < \alpha < \frac{\pi}{2}$ .

b) State the minimum and the maximum value of ...

i. ...  $y = f\left(x - \frac{\pi}{2}\right)$ .

ii. ...  $y = 2f(x) + 1$ .

iii. ...  $y = [f(x)]^2$ .

iv. ...  $y = \frac{10}{f(x) + 3\sqrt{2}}$ .

$$\boxed{\mathbb{R}}, \quad y = \sqrt{8} \sin\left(\theta + \frac{\pi}{4}\right), \quad \boxed{[-\sqrt{8}, \sqrt{8}]}, \quad \boxed{[-2\sqrt{8} + 1, 2\sqrt{8} + 1]}, \quad \boxed{[0, 8]},$$

$$\boxed{[\sqrt{2}, 5\sqrt{2}]}$$

(i)  $f(x) = 2\sin x + 2\cos x = R\sin(x+\alpha)$   
 $= R\sin x \cos \alpha + R\cos x \sin \alpha$   
 $= (R\cos \alpha)\sin x + (R\sin \alpha)\cos x$   
 $R\cos \alpha = 2 \Rightarrow R = \sqrt{2^2 + 2^2} = \sqrt{8}$   
 $R\sin \alpha = 2 \Rightarrow \sin \alpha = 1 \Rightarrow \alpha = \frac{\pi}{2}$   
 $\therefore f(x) = \sqrt{8} \sin\left(x + \frac{\pi}{4}\right)$   
 (ii) (i)  $\min = -\sqrt{8}$   $\max = \sqrt{8}$  (INDICATE! DON'T JUST COPY)  
 (ii)  $\min = -2\sqrt{8} + 1$   $\max = 2\sqrt{8} + 1$  (CHECK! IS IT BY FACTOR OF 2?)  
 (iii)  $\min = 0$   $\max = 8$  (OTHER TRANSFORMATION! UP BY 1 UNIT)  
 (iv)  $\min = \frac{10}{\sqrt{8} + 3\sqrt{2}}$   $\max = \frac{5\sqrt{2}}{-\sqrt{8} + 3\sqrt{2}}$



## Question 172 (\*\*\*\*)

It is given that  $\theta$  and  $\varphi$  are such so that

$$\tan \theta = t \quad \text{and} \quad \tan \varphi = t - 1,$$

where  $t$  is a constant.

It is further given that

$$\frac{1}{\cos^2 \theta} - \frac{1}{\cos^2 \varphi} = 3.$$

- a) Show clearly that  $t = 2$ .
- b) Determine the exact value of  $\tan(\theta + \varphi)$ , showing clearly all the steps in the workings.

$$\boxed{\phantom{000}}, \quad \tan(\theta + \varphi) = -3$$

a) working as follows

$$\begin{aligned} \Rightarrow \frac{1}{\cos^2 \theta} - \frac{1}{\cos^2 \varphi} &= 3 \\ \Rightarrow \sec^2 \theta - \sec^2 \varphi &= 3 \\ \Rightarrow (1 + \tan^2 \theta) - (1 + \tan^2 \varphi) &= 3 \\ \Rightarrow \tan^2 \theta - \tan^2 \varphi &= 3 \\ \Rightarrow t^2 - (t-1)^2 &= 3 \\ \Rightarrow t^2 - (t^2 - 2t + 1) &= 3 \\ \Rightarrow t^2 - t^2 + 2t - 1 &= 3 \\ \Rightarrow 2t - 1 &= 3 \\ \Rightarrow 2t &= 4 \\ \Rightarrow t &= 2 \end{aligned}$$

b) using the tangent compound identity

$$\begin{aligned} \Rightarrow \tan(\theta + \varphi) &= \frac{\tan \theta + \tan \varphi}{1 - \tan \theta \tan \varphi} = \frac{t + (t-1)}{1 - t(t-1)} \\ &= \frac{2t-1}{1-t^2+t} = \frac{2(2)-1}{1-2^2+2} = \frac{3}{-3} \\ &= -3 \end{aligned}$$

## Question 173 (\*\*\*\*)

$$\sin 2x + \cos 2x = 1 + \sin x, \quad 0 < x < \frac{\pi}{2}.$$

- a) Show that the above trigonometric equation can be written as

$$\cos x - \sin x = \frac{1}{2}.$$

- b) Express  $\cos\left(x + \frac{\pi}{4}\right)$  in the form  $R(A\cos x + B\sin x)$ , where  $R$ ,  $A$  and  $B$  are constants to be found.

- c) Use the results of part (a) and (b) to solve the trigonometric equation

$$\sin 2x + \cos 2x = 1 + \sin x, \quad 0 < x < \frac{\pi}{2}.$$

$$\boxed{R = \frac{\sqrt{2}}{2}}, \quad \boxed{A = 1}, \quad \boxed{B = -1}$$

(a)  $\sin 2x + \cos 2x = 1 + \sin x$   
 $\Rightarrow 2\sin x \cos x + (1 - 2\sin^2 x) = 1 + \sin x$   
 $\Rightarrow 2\sin x \cos x + 1 - 2\sin^2 x = 1 + \sin x$   
 $\Rightarrow 2\sin x \cos x - 2\sin^2 x = \sin x$   
 $\Rightarrow 2\sin x (\cos x - \sin x) = \sin x$   
 $\Rightarrow 2(\cos x - \sin x) = 1$   
 $\Rightarrow \cos x - \sin x = \frac{1}{2}$   
 $\Rightarrow \cos\left(x + \frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$   
 $\Rightarrow \cos\left(x + \frac{\pi}{4}\right) = \cos\left(\frac{\pi}{4}\right)$   
 $\Rightarrow x + \frac{\pi}{4} = \frac{\pi}{4} + 2n\pi$   
 $\Rightarrow x = 0 + 2n\pi$   
 $\Rightarrow x = 0$   
 $\Rightarrow x = 0.424$   
 $\Rightarrow x = 4.21$

(b)  $\cos\left(x + \frac{\pi}{4}\right) = \cos x \cos \frac{\pi}{4} - \sin x \sin \frac{\pi}{4}$   
 $\Rightarrow \cos\left(x + \frac{\pi}{4}\right) = \frac{\sqrt{2}}{2} \cos x - \frac{\sqrt{2}}{2} \sin x$   
 $\Rightarrow \cos\left(x + \frac{\pi}{4}\right) = \frac{\sqrt{2}}{2} (\cos x - \sin x)$   
 $\Rightarrow \cos\left(x + \frac{\pi}{4}\right) = \frac{\sqrt{2}}{2} \cdot \frac{1}{2}$   
 $\Rightarrow \cos\left(x + \frac{\pi}{4}\right) = \frac{1}{2}$   
 $\Rightarrow x + \frac{\pi}{4} = \frac{\pi}{3}$   
 $\Rightarrow x = \frac{\pi}{12}$   
 $\Rightarrow x = 0.424$   
 $\Rightarrow x = 4.21$

(c)  $\cos x - \sin x = \frac{1}{2}$   
 $\Rightarrow \cos\left(x + \frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$   
 $\Rightarrow \cos\left(x + \frac{\pi}{4}\right) = \cos\left(\frac{\pi}{4}\right)$   
 $\Rightarrow x + \frac{\pi}{4} = \frac{\pi}{4} + 2n\pi$   
 $\Rightarrow x = 0 + 2n\pi$   
 $\Rightarrow x = 0$   
 $\Rightarrow x = 0.424$   
 $\Rightarrow x = 4.21$

## Question 174 (\*\*\*\*)

$$f(x) = \sin 2x, \quad x \in \mathbb{R}$$

$$g(x) = f\left(x + \frac{\pi}{4}\right) - f\left(x - \frac{\pi}{4}\right), \quad x \in \mathbb{R}.$$

a) Show clearly that

$$g(x) = 2 \cos 2x.$$

b) Express  $g'(x)$  in terms of  $f(x)$ .

$$\boxed{\phantom{00000}}, \quad \boxed{g'(x) = -4f(x)}$$

a) Find the definition of  $g(x)$  and using  $\sin(A \pm B)$

$$\begin{aligned} \Rightarrow g(x) &= f\left(x + \frac{\pi}{4}\right) - f\left(x - \frac{\pi}{4}\right) \\ \Rightarrow g(x) &= \sin\left[2\left(x + \frac{\pi}{4}\right)\right] - \sin\left[2\left(x - \frac{\pi}{4}\right)\right] \\ \Rightarrow g(x) &= \sin\left[2x + \frac{\pi}{2}\right] - \sin\left[2x - \frac{\pi}{2}\right] \\ \Rightarrow g(x) &= \sin\left[2x + \frac{\pi}{2}\right] + \cos 2x \sin \frac{\pi}{2} - \left[\sin 2x \cos \frac{\pi}{2} - \cos 2x \sin \frac{\pi}{2}\right] \\ \Rightarrow g(x) &= 2 \cos 2x \sin \frac{\pi}{2} \quad \left(\sin \frac{\pi}{2} = 1\right) \\ \Rightarrow g(x) &= 2 \cos 2x \end{aligned}$$

b) Differentiating  $g(x)$

$$\begin{aligned} g(x) &= 2 \cos 2x \\ g'(x) &= -4 \sin 2x \\ g'(x) &= -4f(x) \end{aligned}$$

**Question 175** (\*\*\*\*)

Solve the trigonometric equation

$$\sec \theta - \cos \theta = 8(\operatorname{cosec} \theta - \sin \theta), \quad 0 \leq \theta < 360^\circ.$$

$$\theta \approx 63.4^\circ, 243.6^\circ$$

Handwritten solution for Question 175:

$$\begin{aligned} \sec \theta - \cos \theta &= 8(\operatorname{cosec} \theta - \sin \theta) \\ \Rightarrow \frac{1}{\cos \theta} - \cos \theta &= 8\left(\frac{1}{\sin \theta} - \sin \theta\right) \\ \Rightarrow \frac{1 - \cos^2 \theta}{\cos \theta} &= 8\left(\frac{1 - \sin^2 \theta}{\sin \theta}\right) \\ \Rightarrow \frac{\sin^2 \theta}{\cos \theta} &= 8 \times \frac{\cos^2 \theta}{\sin \theta} \\ \Rightarrow \frac{\sin^3 \theta}{\cos^3 \theta} &= 8 \\ \Rightarrow \tan^3 \theta &= 8 \\ \Rightarrow \tan \theta &= 2 \\ \Rightarrow \theta &= 63.4^\circ \pm 180^\circ \\ \theta_1 &= 63.4^\circ \\ \theta_2 &= 243.4^\circ \end{aligned}$$

**Question 176** (\*\*\*\*)

It is given that

$$\sin 3\theta \equiv 3\sin \theta - 4\sin^3 \theta.$$

- Prove the validity of the above trigonometric identity by writing  $3\theta$  as  $2\theta + \theta$ .
- Hence solve the trigonometric equation

$$12\sin^3 \theta - 9\sin \theta = 1.5, \quad 0 \leq \theta < 360^\circ.$$

$$\theta = 70^\circ, 110^\circ, 190^\circ, 230^\circ, 310^\circ, 350^\circ$$

Handwritten solution for Question 176:

(a)  $\sin 3\theta = \sin(2\theta + \theta) = \sin 2\theta \cos \theta + \cos 2\theta \sin \theta$   
 $= (2\sin \theta \cos \theta) \cos \theta + (1 - 2\sin^2 \theta) \sin \theta = 2\sin \theta \cos^2 \theta + \sin \theta - 2\sin^3 \theta$   
 $= 2\sin \theta (1 - \sin^2 \theta) + \sin \theta - 2\sin^3 \theta = 2\sin \theta - 2\sin^3 \theta + \sin \theta - 2\sin^3 \theta$   
 $= 3\sin \theta - 4\sin^3 \theta$

(b)  $12\sin^3 \theta - 9\sin \theta = 1.5$   
 $\Rightarrow 4\sin^3 \theta - 3\sin \theta = 0.5$   
 $\Rightarrow 3\sin \theta - 4\sin^3 \theta = -0.5$   
 $\Rightarrow \sin 3\theta = -\frac{1}{2}$   
 $\operatorname{arcsin}\left(-\frac{1}{2}\right) = -30^\circ$

$3\theta = -30^\circ \pm 360^\circ$   
 $3\theta = 210^\circ \pm 360^\circ$   
 $\theta = -10^\circ \pm 120^\circ$   
 $\theta = 70^\circ \pm 120^\circ$   
 $\theta = 110^\circ, 230^\circ, 310^\circ, 350^\circ$

**Question 177** (\*\*\*\*)

Solve the trigonometric equation

$$\frac{\cot \psi}{\operatorname{cosec} \psi - 1} - \frac{\cos \psi}{1 + \sin \psi} = 2, \quad 0 < \psi < 2\pi,$$

giving the answers in terms of  $\pi$ .

$$\psi = \frac{\pi}{4}, \frac{5\pi}{4}$$

**Question 178** (\*\*\*\*)

Solve each of the following trigonometric equations.

i.  $\frac{2\cot^2 x + 5}{\operatorname{cosec} x} + 2\operatorname{cosec} x = 13, \quad 0 \leq x < 2\pi.$

ii.  $2\cos 2\theta = 1 - 2\sin \theta, \quad 0 \leq \theta < 360^\circ.$

$$x \approx 0.340^\circ, 2.80^\circ, \quad \theta = 54^\circ, 126^\circ, 198^\circ, 342^\circ$$

## Question 179 (\*\*\*\*)

Solve the following trigonometric equation

$$\arctan\left(\frac{1}{x}\right) + \arctan\left(\frac{1}{x+1}\right) = \frac{\pi}{4}.$$

$$\boxed{\phantom{0}}, \boxed{x = -1, 2}$$

TRICK: "TAN" ON BOTH SIDES  $\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$

$$\Rightarrow \arctan\left(\frac{1}{x}\right) + \arctan\left(\frac{1}{x+1}\right) = \frac{\pi}{4}$$

$$\Rightarrow \tan\left[\arctan\left(\frac{1}{x}\right) + \arctan\left(\frac{1}{x+1}\right)\right] = \tan\left(\frac{\pi}{4}\right)$$

$$\Rightarrow \frac{\frac{1}{x} + \frac{1}{x+1}}{1 - \frac{1}{x(x+1)}} = 1$$

REWORKING ACROSS

$$\Rightarrow \frac{1}{x} + \frac{1}{x+1} = 1 - \frac{1}{x(x+1)} \quad \text{ } \times x(x+1)$$

$$\Rightarrow (x+1) + x = x(x+1) - 1$$

$$\Rightarrow x+1+x = x^2+x-1$$

$$\Rightarrow 0 = x^2 - x - 2$$

$$\Rightarrow (x-2)(x+1) = 0$$

$$\Rightarrow x = \frac{-1 \pm \sqrt{1+8}}{2}$$

Both are fine

- $x = -1$   
 $\arctan(-1) + \arctan(0) = -\frac{\pi}{4} + \frac{\pi}{4} = 0$
- $x = 2$   
 $\arctan\left(\frac{1}{2}\right) + \arctan\left(\frac{1}{3}\right) = \frac{\pi}{4}$

## Question 180 (\*\*\*\*)

Solve the following trigonometric equation

$$2 \arctan\left(\frac{3}{x}\right) = \arcsin\left(\frac{6x}{25}\right).$$

$$x = \pm 4$$

Proceder as follows

$$\Rightarrow 2 \arctan\left(\frac{3}{x}\right) = \arcsin\left(\frac{6x}{25}\right)$$

Let  $\theta = \arctan\left(\frac{3}{x}\right)$  then  $\tan \theta = \frac{3}{x}$

Draw a right-angled triangle with opposite side 3 and adjacent side x. The hypotenuse is  $\sqrt{x^2 + 9}$ .

Then  $\sin \theta = \frac{3}{\sqrt{x^2 + 9}}$  and  $\cos \theta = \frac{x}{\sqrt{x^2 + 9}}$ .

Since  $2\theta = \arcsin\left(\frac{6x}{25}\right)$ , then  $\sin 2\theta = \frac{6x}{25}$ .

Using the double angle formula for sine:

$$\sin 2\theta = 2 \sin \theta \cos \theta = 2 \left(\frac{3}{\sqrt{x^2 + 9}}\right) \left(\frac{x}{\sqrt{x^2 + 9}}\right) = \frac{6x}{x^2 + 9}$$

Equating the two expressions for  $\sin 2\theta$ :

$$\frac{6x}{x^2 + 9} = \frac{6x}{25}$$

Cancel 6x (assuming  $x \neq 0$ ):

$$\frac{1}{x^2 + 9} = \frac{1}{25}$$

Cross-multiply:

$$25 = x^2 + 9$$

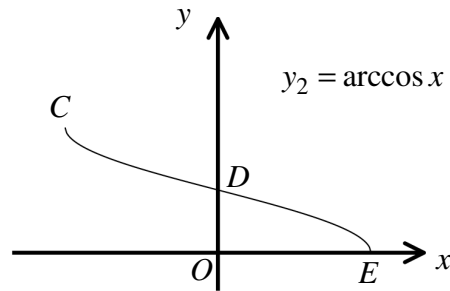
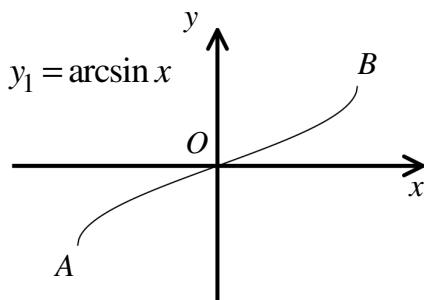
Rearrange:

$$x^2 = 16$$

Solve for x:

$$x = \pm 4$$

## Question 181 (\*\*\*\*)



The figures above show the graph of  $y_1 = \arcsin x$  and the graph of  $y_2 = \arccos x$ .

The graph of  $y_1$  has endpoints at  $A$  and  $B$ .

The graph of  $y_2$  has endpoints at  $C$  and  $E$ , and  $D$  is the point where the graph of  $y_2$  crosses the  $y$  axis.

- a) State the coordinates of  $A$ ,  $B$ ,  $C$ ,  $D$  and  $E$ .

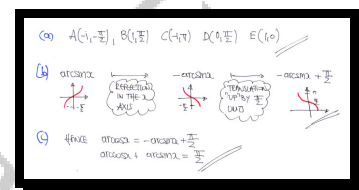
The graph of  $y_2$  can be obtained from the graph of  $y_1$  by a series of two geometric transformations which can be carried out in a specific order.

- b) Describe these two geometric transformations.  
c) Deduce using valid arguments that

$$\arcsin x + \arccos x = \text{constant},$$

stating the exact value of this constant.

$$\boxed{A\left(-1, \frac{\pi}{2}\right)}, \boxed{B\left(1, \frac{\pi}{2}\right)}, \boxed{C(-1, \pi)}, \boxed{D\left(0, \frac{\pi}{2}\right)}, \boxed{E(1, 0)}, \text{constant} = \frac{\pi}{2}$$





## Question 182 (\*\*\*\*)

It is given that

$$\sin(A+B) \equiv \sin A \cos B + \cos A \sin B.$$

- a) Use the above trigonometric identity to show that

$$\sin 3x \equiv 3 \sin x - 4 \sin^3 x.$$

- b) Hence find

$$\int \cos x (6 \sin x - 2 \sin 3x)^{\frac{2}{3}} dx.$$

$$\boxed{\phantom{000}}, \boxed{\frac{4}{3} \sin^3 x + C}$$

a) Prove it first

$$\begin{aligned} \sin 3x &= \sin(2x+x) = \sin 2x \cos x + \cos 2x \sin x \\ &= (2 \sin x \cos x) \cos x + (1 - 2 \sin^2 x) \sin x \\ &= 2 \sin x \cos^2 x + \sin x - 2 \sin^3 x \\ &= 2 \sin x (1 - \sin^2 x) + \sin x - 2 \sin^3 x \\ &= 2 \sin x - 2 \sin^3 x + \sin x - 2 \sin^3 x \\ &= 3 \sin x - 4 \sin^3 x \end{aligned}$$

no require

b) using the result of part (a)

$$\begin{aligned} &\int \cos x (6 \sin x - 2 \sin 3x)^{\frac{2}{3}} dx \\ &= \int \cos x [5 \sin x - 2(3 \sin x - 4 \sin^3 x)]^{\frac{2}{3}} dx \\ &= \int \cos x [5 \sin x - 6 \sin x + 8 \sin^3 x]^{\frac{2}{3}} dx \\ &= \int \cos x (8 \sin^3 x)^{\frac{2}{3}} dx \\ &= \int \cos x (4 \sin^2 x) dx \\ &= \int 4 \cos x \sin^2 x dx \end{aligned}$$

by inspection, or using the substitution  $u = \sin x$

$$= \frac{4}{3} \sin^3 x + C$$

## Question 183 (\*\*\*\*)

$$f(x) = A \sec 2x + B, \quad 0 \leq x < 2\pi.$$

The graph of  $f(x)$ , where  $A$  and  $B$  are non zero constants, passes through the points

$$\left(\frac{\pi}{2}, -7\right) \text{ and } (\pi, 1).$$

a) Determine the value of  $A$  and the value of  $B$ .

b) Solve the equation

$$f\left(x + \frac{3\pi}{2}\right) = 5.$$

$$\boxed{\phantom{000}}, \boxed{A=4}, \boxed{B=-3}, \boxed{x = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}}$$

a) USING THE GIVEN POINTS  $\left(\frac{\pi}{2}, -7\right)$  &  $(\pi, 1)$

$\left(\frac{\pi}{2}, -7\right)$	$(\pi, 1)$
$-7 = A \sec 2\left(\frac{\pi}{2}\right) + B$	$1 = A \sec 2\pi + B$
$-7 = -A + B$	$1 = A + B$

ADDING: GIVES

$$\frac{2B = -6}{B = -3} \quad \text{Hence } A = 4$$

b) SETTING UP THE EQUATION

$$\begin{aligned} \Rightarrow f\left(x + \frac{3\pi}{2}\right) &= 5 \\ \Rightarrow 4 \sec\left(2\left(x + \frac{3\pi}{2}\right)\right) - 3 &= 5 \\ \Rightarrow 4 \sec(2x + 3\pi) - 3 &= 5 \\ \Rightarrow \sec(2x + 3\pi) &= 2 \\ \Rightarrow \cos(2x + 3\pi) &= \frac{1}{2} \end{aligned}$$

$\cos(x) = \frac{1}{2}$

$$\begin{aligned} 2x + 3\pi &= \frac{\pi}{3} + 2\pi n \\ 2x + 3\pi &= \frac{5\pi}{3} + 2\pi n \quad n = 0, 1, 2, \dots \\ 2x &= -\frac{8\pi}{3} + 2\pi n \\ 2x &= -\frac{2\pi}{3} + 2\pi n \\ x &= -\frac{\pi}{3} + \pi n \end{aligned}$$

(range  $0 \leq x < 2\pi$ )

$$x = \frac{2\pi}{3}, \frac{5\pi}{3}, \frac{4\pi}{3}, \frac{\pi}{3}$$

## Question 184 (\*\*\*\*)

It is given that the angles  $\theta$ ,  $\frac{\pi}{4}$  and  $\varphi$  are in arithmetic progression.

Show that

$$(\sin \theta - \sin \varphi)^2 + (\cos \theta + \cos \varphi)^2 = k,$$

where  $k$  is a constant to be found.

$$\boxed{\phantom{000}}, \quad \boxed{k=2}$$

IF IN ARITHMETIC PROGRESSION, (10) THE ORDER GIVEN

$$\begin{aligned} \Rightarrow \frac{\pi}{4} - \theta &= \frac{\pi}{4} - \varphi \\ \Rightarrow \phi + \theta &= \frac{\pi}{2} \end{aligned}$$

NOW USE TRIG

$$\begin{aligned} &(\sin \theta - \sin \varphi)^2 + (\cos \theta + \cos \varphi)^2 \\ &= \sin^2 \theta - 2 \sin \theta \sin \varphi + \sin^2 \varphi + \cos^2 \theta + 2 \cos \theta \cos \varphi + \cos^2 \varphi \\ &= (\sin^2 \theta + \cos^2 \theta) + (\sin^2 \varphi + \cos^2 \varphi) + 2[\cos \theta \cos \varphi - \sin \theta \sin \varphi] \\ &= 2 + 2[\cos(\theta + \varphi)] \\ &= 2 + 2 \cos \frac{\pi}{2} \\ &= 2 \end{aligned}$$

∴  $k=2$

## Question 185 (\*\*\*\*)

It is given that

$$\cos x \cos \left( x + \frac{\pi}{4} \right) - \cos \left( 2x - \frac{\pi}{4} \right) = 0.$$

Given further that  $x \neq k\pi$ ,  $k \in \mathbb{Z}$ , show clearly that  $\tan x = 3$ 

proof

$$\begin{aligned} & \bullet \cos x \cos \left( x + \frac{\pi}{4} \right) - \cos \left( 2x - \frac{\pi}{4} \right) = 0 \\ & \Rightarrow \cos x \left[ \cos x \cos \frac{\pi}{4} - \sin x \sin \frac{\pi}{4} \right] - \left[ \cos 2x \cos \frac{\pi}{4} + \sin 2x \sin \frac{\pi}{4} \right] = 0 \\ & \Rightarrow \cos x \left[ \frac{1}{\sqrt{2}} \cos x - \frac{1}{\sqrt{2}} \sin x \right] - \left[ \frac{1}{\sqrt{2}} \cos 2x + \frac{1}{\sqrt{2}} \sin 2x \right] = 0 \\ & \Rightarrow \frac{1}{\sqrt{2}} \cos x - \frac{1}{\sqrt{2}} \cos x \sin x - \frac{1}{\sqrt{2}} \cos 2x - \frac{1}{\sqrt{2}} \sin 2x = 0 \\ & \Rightarrow \cos x - \cos x \sin x - \cos 2x - \sin 2x = 0 \\ & \Rightarrow \cos x - \cos x \sin x - (2\cos^2 x - 1) - 2\sin x \cos x = 0 \\ & \Rightarrow 1 - \cos x - 3\sin x \cos x = 0 \\ & \Rightarrow \sin x - 3\sin x \cos x = 0 \\ & \Rightarrow \sin x (\sin x - 3\cos x) = 0 \\ & \Rightarrow \sin x - 3\cos x = 0 \quad (\sin x \neq 0) \\ & \Rightarrow \sin x = 3\cos x \\ & \Rightarrow \tan x = 3 \end{aligned}$$

## Question 186 (\*\*\*\*)

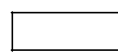
$$y = \arcsin x, \quad -1 \leq x \leq 1.$$

- a) By expressing  $\arccos x$  in terms of  $y$ , show that

$$\arcsin x + \arccos x = \frac{\pi}{2}.$$

- b) Hence, or otherwise, solve the equation

$$3\arcsin(x-1) = 2\arccos(x-1).$$



$$x = 1 + \sin\left(\frac{\pi}{5}\right) \approx 1.5878$$

4) MANIPULATE AS SUGGESTED

$\Rightarrow y = \arcsin x$   
 $\Rightarrow \sin y = x$   
 $\Rightarrow x = \sin y$

Take "Arccos" on BOTH SIDES

$\Rightarrow \arccos x = \arccos(\sin y)$   
 $\Rightarrow \arccos x = \arccos\left(\cos\left(\frac{\pi}{2} - y\right)\right) \leftarrow \sin A = \cos\left(\frac{\pi}{2} - A\right)$   
 $\Rightarrow \arccos x = \frac{\pi}{2} - y$   
 $\Rightarrow \arccos x = \frac{\pi}{2} - \arcsin x \leftarrow \text{As } y = \arcsin x$   
 $\Rightarrow \arccos x + \arcsin x = \frac{\pi}{2}$

5) USING PART (4), LET  $y = x-1$

$\Rightarrow \arcsin(x-1) = 2\arccos(x-1)$   
 $\Rightarrow \arcsin y = 2\arccos y$   
 $\Rightarrow \arcsin y = 2\left[\frac{\pi}{2} - \arcsin y\right]$   
 $\Rightarrow \arcsin y = \pi - 2\arcsin y$   
 $\Rightarrow 3\arcsin y = \pi$   
 $\Rightarrow \arcsin y = \frac{\pi}{3}$   
 $\Rightarrow y = \sin \frac{\pi}{3}$   
 $\Rightarrow x-1 = \sin \frac{\pi}{3}$   
 $\Rightarrow x = 1 + \sin \frac{\pi}{3} \approx 1.5878 \dots$

## Question 187 (\*\*\*\*)

Simplify, showing clearly all the workings, the trigonometric expression

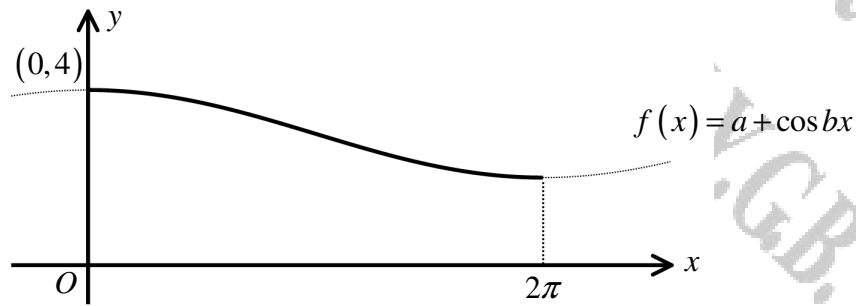
$$\cos^3 \theta \sin \theta - \sin^3 \theta \cos \theta,$$

giving the final answer in the form  $A \sin k\theta$ , where  $A$  and  $k$  are constants.

$$\boxed{\phantom{000}}, \boxed{\frac{1}{4} \sin 4\theta}$$

$$\begin{aligned} &\Rightarrow \cos^2 \theta \sin \theta - \sin^2 \theta \cos \theta \\ &= \cos^2 \theta \sin \theta - \sin^2 \theta \cos \theta \\ &= \cos^2 \theta \sin \theta \times \cos 2\theta \quad [\cos 2\theta = \cos^2 \theta - \sin^2 \theta] \\ &= \frac{1}{2} (2 \cos^2 \theta \sin \theta) \cos 2\theta \\ &= \frac{1}{2} \sin 2\theta \cos 2\theta \quad [\sin 2\theta = 2 \sin \theta \cos \theta] \\ &= \frac{1}{2} \times \frac{1}{2} \sin 4\theta \\ &= \frac{1}{4} \sin 4\theta \\ &\quad A = \frac{1}{4}, \quad k = 4 \end{aligned}$$

## Question 188 (\*\*\*\*)



The figure above shows the graph of the function

$$f(x) = a + \cos bx, \quad 0 \leq x \leq 2\pi,$$

where  $a$  and  $b$  are non zero constants.

The stationary points  $(0, 4)$  and  $(2\pi, 2)$  are the endpoints of the graph.

- State the range of  $f(x)$  and hence find the value of  $a$  and the value of  $b$ .
- Find an expression for  $f^{-1}(x)$ , the inverse function of  $f(x)$ .
- State the domain and range of  $f^{-1}(x)$ .
- Find the gradient at the point on  $f(x)$  with coordinates  $(\frac{4\pi}{3}, \frac{5}{2})$ .
- State the gradient at the point on  $f^{-1}(x)$  with coordinates  $(\frac{5}{2}, \frac{4\pi}{3})$ .

$$\boxed{\phantom{0 \leq f(x) \leq 4}}, \quad \boxed{2 \leq f(x) \leq 4}, \quad \boxed{a = 3, b = \frac{1}{2}}, \quad \boxed{f^{-1}(x) = 2 \arccos(x - 3)}, \quad \boxed{2 \leq x \leq 4},$$

$$\boxed{0 \leq f^{-1}(x) \leq 2\pi}, \quad \boxed{-\frac{\sqrt{3}}{4}}, \quad \boxed{-\frac{4}{\sqrt{3}}}$$

(a)  $\bullet$  RANGE:  $2 \leq f(x) \leq 4$   
 $-1 \leq \cos bx \leq 1$   
 $2 \leq a + \cos bx \leq 4$   
 $\therefore a = 3$

$\bullet$   $f(0) = 3 + \cos(0)$   
 $2 = 3 + \cos(2\pi b)$   
 $-1 = \cos(2\pi b)$   
 $\arccos(-1) = 2\pi b$   
 $2\pi b = \pi$   
 $2b = 1$   
 $b = \frac{1}{2}$

(b)  $y = 3 + \cos(\frac{1}{2}x)$   
 $y - 3 = \cos(\frac{1}{2}x)$   
 $\arccos(y - 3) = \frac{1}{2}x$   
 $x = 2 \arccos(y - 3)$   
 $f^{-1}(x) = 2 \arccos(x - 3)$

(c)  $\bullet$  Domain:  $2 \leq x \leq 4$   
 $\bullet$  Range:  $0 \leq f^{-1}(x) \leq 2\pi$

(d)  $\bullet$   $f(x) = 3 + \cos(\frac{1}{2}x)$   
 $f'(x) = -\frac{1}{2} \sin(\frac{1}{2}x)$   
 $f'(\frac{4\pi}{3}) = -\frac{1}{2} \sin(\frac{2\pi}{3}) = -\frac{\sqrt{3}}{4}$

(e)  $\bullet$   $f^{-1}(x) = 2 \arccos(x - 3)$   
 $f^{-1}(\frac{5}{2}) = 2 \arccos(\frac{5}{2} - 3) = 2 \arccos(-\frac{1}{2}) = \frac{4\pi}{3}$   
 $f^{-1}(\frac{5}{2}) = \frac{4\pi}{3}$

## Question 189 (\*\*\*\*)

Solve the following trigonometric equation.

$$\arctan 2x + \arctan x = \arctan 3, \quad x \in \mathbb{R}.$$

$$\boxed{\phantom{0}}, \quad x = \frac{1}{2}$$

$\arctan 2x + \arctan x = \arctan 3$   
 Using the compound angle formula for tangents  
 $\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$   
 $\Rightarrow \tan[\arctan 2x + \arctan x] = \tan(\arctan 3)$   
 $\Rightarrow \frac{\tan(\arctan 2x) + \tan(\arctan x)}{1 - \tan(\arctan 2x) \tan(\arctan x)} = 3$   
 $\Rightarrow \frac{2x + x}{1 - 2x^2} = 3$   
 $\Rightarrow 3x = 3(1 - 2x^2)$   
 $\Rightarrow x = 1 - 2x^2$   
 $\Rightarrow 2x^2 + x - 1 = 0$   
 $\Rightarrow (2x - 1)(x + 1) = 0$   
 $x = \frac{1}{2}$   
 As  $\arctan(-2) + \arctan(-1) < 0$   
 $\arctan 3 > 0$



## Question 190 (\*\*\*\*)

The function  $f$  is defined below.

$$f(x) \equiv 4\cos x - 3\sin\left(\frac{1}{2}x\right), \quad x \in \mathbb{R}.$$

Show that if  $\theta$  satisfies the equation

$$4\sin\left(\frac{1}{2}\theta\right) + \sqrt{3} = 0,$$

then  $f(\theta) = \frac{1}{4}(a + b\sqrt{3})$ , where  $a$  and  $b$  are integers to be found.

$$\boxed{-11}, \quad \boxed{\frac{1}{4}(10 + 3\sqrt{3})}$$

• START BY REARRANGING THE EQUATION  

$$4\sin\frac{\theta}{2} + \sqrt{3} = 0$$

$$4\sin\frac{\theta}{2} = -\sqrt{3}$$

$$\sin\frac{\theta}{2} = -\frac{\sqrt{3}}{4}$$

• WE NEED NO SOLUTION OVER AN ACTUAL RANGE, IF WE SIMPLY MANIPULATE THE FUNCTIONAL  

$$\Rightarrow f(\theta) = 4\cos\theta - 3\sin\frac{\theta}{2}$$

$$\Rightarrow f(\theta) = 4(1 - 2\sin^2\frac{\theta}{2}) - 3\sin\frac{\theta}{2}$$

$$\Rightarrow f(\theta) = 4 - 8\sin^2\frac{\theta}{2} - 3\sin\frac{\theta}{2}$$

$\cos 2A \equiv 1 - 2\sin^2 A$   
 $\cos 2B \equiv 1 - 2\sin^2 B$   
 $\cos A \equiv 1 - 2\sin^2 \frac{A}{2}$

• SUBSTITUTING THE ABOVE AS  $\sin\frac{\theta}{2} = -\frac{\sqrt{3}}{4}$   

$$\Rightarrow f(\theta) = 4 - 8\left(-\frac{\sqrt{3}}{4}\right)^2 - 3\left(-\frac{\sqrt{3}}{4}\right)$$

$$= 4 - 8\left(\frac{3}{16}\right) + \frac{3\sqrt{3}}{4}$$

$$= 4 - \frac{3}{2} + \frac{3\sqrt{3}}{4}$$

$$= \frac{8}{2} - \frac{3}{2} + \frac{3\sqrt{3}}{4}$$

$$= \frac{5}{2} + \frac{3\sqrt{3}}{4}$$

$$= \frac{1}{4}(10 + 3\sqrt{3})$$

## Question 191 (\*\*\*\*)


The obtuse angles  $A$  and  $B$ , satisfy the following relationships.


$$\cos 2A = \sin B = \frac{1}{3}.$$

Determine the exact value of  $\tan(A+B)$ .

$$\boxed{\phantom{000}}, \quad \tan(A+B) = -\sqrt{2}$$

• STARTING FROM  $\cos 2A = \frac{1}{3}$  & NOTING THAT  $A$  IS OBTUSE  
 $\Rightarrow \cos 2A = 2\cos^2 A - 1$   
 $\Rightarrow \frac{1}{3} = 2\cos^2 A - 1$   
 $\Rightarrow \frac{4}{3} = 2\cos^2 A$   
 $\Rightarrow \cos^2 A = \frac{2}{3}$   
 $\Rightarrow \cos A = -\sqrt{\frac{2}{3}}$  ( $A$  is obtuse)

• HENCE BY A STANDARD RIGHT-ANGLED TRIANGLE  

 $\therefore \tan A = -\frac{1}{\sqrt{2}}$

• SIMILARLY  $\sin B = \frac{1}{3}$  ( $B$  OBTUSE, SO BOTH  $\sin B$  &  $\tan B$  ARE NEGATIVE)  

 $\therefore \tan B = -\frac{1}{2\sqrt{2}}$

• FINALLY BY THE COMPOUND ANGLE IDENTITIES  

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B} = \frac{-\frac{1}{\sqrt{2}} - \frac{1}{4\sqrt{2}}}{1 - (-\frac{1}{2})(-\frac{1}{2\sqrt{2}})} = \frac{-\frac{5}{4\sqrt{2}}}{1 - \frac{1}{4}}$$

$$= -\frac{\frac{5}{4\sqrt{2}}}{\frac{3}{4}} = -\frac{5}{3\sqrt{2}} = -\frac{5\sqrt{2}}{6}$$

## Question 192 (\*\*\*\*)

Simplify  $(\tan x + \cot x) \sin 2x$  and hence prove that

$$\tan\left(\frac{1}{8}\pi\right) + \tan\left(\frac{5}{12}\pi\right) + \cot\left(\frac{1}{8}\pi\right) + \cot\left(\frac{5}{12}\pi\right) = 4 + 2\sqrt{2}.$$

$$\boxed{\phantom{000}}, \quad \text{proof}$$

START WITH THE SUGGESTED SIMPLIFICATION  

$$(\tan x + \cot x) \sin 2x = \frac{\sin x}{\cos x} + \frac{\cos x}{\sin x} \times 2 \sin x \cos x$$

$$= \frac{\sin^2 x + 2 \cos^2 x}{\sin x \cos x} = \frac{2 \cos^2 x + \sin^2 x}{\sin x \cos x}$$

$$= \frac{2 \cos^2 x + \sin^2 x}{\sin x \cos x} = 2 \left( \frac{\cos x}{\sin x} + \frac{\sin x}{\cos x} \right) = 2 \left( \cot x + \tan x \right)$$

$$\therefore (\tan x + \cot x) \sin 2x = 2(\tan x + \cot x)$$

NOW, SET THE EXPRESSIONS EQUAL, REMOVE  $\sin \frac{\pi}{8} = \cos \frac{\pi}{8}$   

$$\left( \tan \frac{\pi}{8} + \cot \frac{\pi}{8} \right) \sin \frac{\pi}{4} = 2 \left( \tan \frac{\pi}{8} + \cot \frac{\pi}{8} \right) \sin \frac{\pi}{8}$$

$$\left( \tan \frac{\pi}{8} + \cot \frac{\pi}{8} \right) \times \frac{1}{\sqrt{2}} = 2 \left( \tan \frac{\pi}{8} + \cot \frac{\pi}{8} \right) \sin \frac{\pi}{8}$$

$$\left( \tan \frac{\pi}{8} + \cot \frac{\pi}{8} \right) = 2\sqrt{2} \left( \tan \frac{\pi}{8} + \cot \frac{\pi}{8} \right) \sin \frac{\pi}{8}$$

$$\therefore \tan \frac{\pi}{8} + \cot \frac{\pi}{8} = 4$$

NOW, SET THE EXPRESSIONS EQUAL, REMOVE  $\sin \frac{5\pi}{12} = \cos \frac{\pi}{12}$   

$$\left( \tan \frac{5\pi}{12} + \cot \frac{5\pi}{12} \right) \sin \frac{5\pi}{12} = 2 \left( \tan \frac{5\pi}{12} + \cot \frac{5\pi}{12} \right) \sin \frac{5\pi}{12}$$

$$\left( \tan \frac{5\pi}{12} + \cot \frac{5\pi}{12} \right) \times \frac{1}{2} = 2 \left( \tan \frac{5\pi}{12} + \cot \frac{5\pi}{12} \right) \sin \frac{5\pi}{12}$$

$$\left( \tan \frac{5\pi}{12} + \cot \frac{5\pi}{12} \right) = 4$$

$$\therefore \tan \frac{\pi}{8} + \cot \frac{\pi}{8} + \tan \frac{5\pi}{12} + \cot \frac{5\pi}{12} = 4 + 2\sqrt{2}$$

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# HARD QUESTIONS

**Question 1** (\*\*\*\*+)

Find the solutions of the trigonometric equation

$$6 + 13 \sin(2\theta + \alpha)^\circ = 5 \cos 2\theta^\circ, \quad 0 \leq \theta < 360$$


where  $\tan \alpha^\circ = \frac{5}{12}, \quad 0 < \alpha < 90.$

,  $\theta = 105^\circ, 165^\circ, 285^\circ, 345^\circ$

THIS EQUATION IS NOT A COMMON ANGLE & NOT ABOUT DOUBLE ANGLES  
 BUT THE ALGEBRAIC ARE TWO THROUGHOUT

$$\begin{aligned} \Rightarrow 6 + 13 \sin(2\theta + \alpha) &= 5 \cos 2\theta \\ \Rightarrow 6 + 13 [\sin 2\theta \cos \alpha + \cos 2\theta \sin \alpha] &= 5 \cos 2\theta \\ \Rightarrow 6 + 13 \sin 2\theta \cos \alpha + 13 \cos 2\theta \sin \alpha &= 5 \cos 2\theta \end{aligned}$$

NOW  $\tan \alpha = \frac{5}{12}$  OR ACUTE



BY PYTHAGORAS  $\sin \alpha = \frac{5}{13}$   
 $\cos \alpha = \frac{12}{13}$

RETURNING TO THE MAIN LINE

$$\begin{aligned} \Rightarrow 6 + 13 \sin 2\theta \times \frac{12}{13} + 13 \cos 2\theta \times \frac{5}{13} &= 5 \cos 2\theta \\ \Rightarrow 6 + 12 \sin 2\theta + 5 \cos 2\theta &= 5 \cos 2\theta \\ \Rightarrow 12 \sin 2\theta &= -6 \\ \Rightarrow \sin 2\theta &= -\frac{1}{2} \end{aligned}$$

$\arcsin(-\frac{1}{2}) = -30^\circ$

$$\begin{aligned} 2\theta &= -30^\circ \pm 360^\circ \\ 2\theta &= 270^\circ \pm 360^\circ & \theta &= 135^\circ \pm 180^\circ \\ \theta &= 135^\circ \pm 180^\circ \end{aligned}$$

$\theta = 165^\circ, 345^\circ, 105^\circ, 285^\circ$

**Question 2** (\*\*\*\*+)

It is given that  $\sin 1^\circ \approx 0.8415$  and  $\cos 1^\circ \approx 0.5403$ .

Show that  $\sin(1.01^\circ) = 0.847$ , correct to three decimal places.

, proof

$$\begin{aligned} \sin(1.01^\circ) &= \sin(1^\circ + 0.01^\circ) = \sin 1^\circ \cos(0.01^\circ) + \cos 1^\circ \sin(0.01^\circ) \\ &= 0.8415 \times 1 + 0.5403 \times 0.01 = 0.8415 + 0.0054 \\ &= 0.847 \end{aligned}$$

(3 sf)

**Question 3** (\*\*\*\*+)

It is given that  $\theta$  satisfies the equation

$$4 \tan \theta + \cot \theta = 4.$$

Show clearly that

$$\cos 2\theta = \frac{3}{5}.$$

proof

$$\begin{aligned}
 4 \tan \theta + \frac{1}{\tan \theta} &= 4 \\
 \Rightarrow 4 \tan^2 \theta + 1 &= 4 \tan \theta \\
 \Rightarrow 4 \tan^2 \theta - 4 \tan \theta + 1 &= 0 \\
 \Rightarrow (2 \tan \theta - 1)^2 &= 0 \\
 \Rightarrow 2 \tan \theta - 1 &= 0 \\
 \Rightarrow \tan \theta &= \frac{1}{2} \\
 \Rightarrow \frac{\sin \theta}{\cos \theta} &= \frac{1}{2} \\
 \Rightarrow 2 \sin \theta &= \cos \theta
 \end{aligned}$$

$$\begin{aligned}
 \Rightarrow \sec^2 \theta &= \frac{5}{4} \\
 \Rightarrow \sec \theta &= \frac{\sqrt{5}}{2} \\
 \Rightarrow 2 \cos \theta &= \frac{2}{\sqrt{5}} \\
 \Rightarrow \cos \theta &= \frac{1}{\sqrt{5}}
 \end{aligned}$$

$$\begin{aligned}
 \Rightarrow \cos^2 \theta &= \frac{1}{5} \\
 \Rightarrow 1 - \sin^2 \theta &= \frac{1}{5} \\
 \Rightarrow \sin^2 \theta &= \frac{4}{5} \\
 \Rightarrow \sin \theta &= \frac{2}{\sqrt{5}}
 \end{aligned}$$

$$\begin{aligned}
 \Rightarrow \cos 2\theta &= \cos^2 \theta - \sin^2 \theta \\
 &= \frac{1}{5} - \frac{4}{5} = -\frac{3}{5}
 \end{aligned}$$

**Question 4** (\*\*\*\*+)

Find in radians, correct to two decimal places, the solutions of the trigonometric equation

$$\sec 2x - 3 \tan 2x = 2, \quad 0 \leq x < 2\pi.$$

$$x \approx 1.14^\circ, 2.99^\circ, 4.28^\circ, 6.13^\circ$$

$$\begin{aligned}
 \sec 2x - 3 \tan 2x &= 2 \\
 \Rightarrow \frac{1}{\cos 2x} - \frac{3 \sin 2x}{\cos 2x} &= 2 \\
 \Rightarrow 1 - 3 \sin 2x &= 2 \cos 2x \\
 \Rightarrow 1 &= 2 \cos 2x + 3 \sin 2x
 \end{aligned}$$

Now

$$\begin{aligned}
 2 \cos 2x + 3 \sin 2x &= R \sin(2x + \alpha) \\
 2 \cos 2x + 3 \sin 2x &= R \sin 2x \cos \alpha + R \cos 2x \sin \alpha \\
 &= (R \cos \alpha) \sin 2x + (R \sin \alpha) \cos 2x
 \end{aligned}$$

$$\begin{aligned}
 R \cos \alpha &= 3 \\
 R \sin \alpha &= 2 \Rightarrow R = \sqrt{3^2 + 2^2} = \sqrt{13} \\
 \tan \alpha &= \frac{2}{3} \Rightarrow \alpha = 0.588^\circ
 \end{aligned}$$

Thus

$$\begin{aligned}
 \Rightarrow \sqrt{13} \sin(2x + 0.588^\circ) &= 1 \\
 \Rightarrow \sin(2x + 0.588^\circ) &= \frac{1}{\sqrt{13}} \\
 \Rightarrow 2x + 0.588^\circ &= 0.285^\circ \pm 2\pi n \\
 2x + 0.588^\circ &= 2.851^\circ \pm 2\pi n \quad n=0,1,2,3 \\
 \Rightarrow 2x &= 0.193^\circ \pm 2\pi n \\
 \Rightarrow x &= 0.096^\circ \pm \pi n
 \end{aligned}$$

$$\therefore x = 1.14^\circ, 2.99^\circ, 4.28^\circ, 6.13^\circ$$

**Question 5** (\*\*\*\*+)

Solve in radians the trigonometric equation

$$\sin 8x = \sin 2x, \quad 0 \leq x < \frac{\pi}{2},$$

giving the answers in terms of  $\pi$ .

$$x = 0, \frac{\pi}{10}, \frac{3\pi}{10}, \frac{\pi}{3}, \frac{\pi}{2}$$

**Standard Method:**

$$\begin{aligned} \sin 8x &= \sin 2x \\ \sin 8x - \sin 2x &= 0 \\ \sin 8x - \sin 2x &= 2 \cos \frac{8x+2x}{2} \sin \frac{8x-2x}{2} \\ 2 \cos \left( \frac{8x+2x}{2} \right) \sin \left( \frac{8x-2x}{2} \right) &= 0 \\ 2 \cos 5x \sin 3x &= 0 \\ \cos 5x = 0 \quad \text{or} \quad \sin 3x &= 0 \\ \left( 5x = \frac{\pi}{2} \pm 2n\pi \right) \quad \text{or} \quad \left( 3x = 0 \pm 2n\pi \right) \\ \left( 5x = \frac{\pi}{2} \pm 2n\pi \right) \quad \text{or} \quad \left( 3x = 0 \pm 2n\pi \right) \\ x = \frac{\pi}{10} \pm \frac{2n\pi}{5}, \dots \quad \text{or} \quad x = \frac{2n\pi}{3}, \dots \\ x = \frac{\pi}{10}, \frac{3\pi}{10}, \frac{5\pi}{10}, \frac{7\pi}{10}, \dots \quad \text{or} \quad x = 0, \frac{2\pi}{3}, \frac{4\pi}{3}, \dots \end{aligned}$$

**Alternative Method:**

$$\begin{aligned} \sin 8x &= \sin 2x \\ \sin 8x &= \sin 2x \\ 8x &= 2x \pm 2n\pi \\ 8x &= (1-2n) \pm 2n\pi \\ 11x &= 2n\pi \pm 2n\pi \\ 11x &= 0 \pm 2n\pi \\ x &= 0 \pm \frac{2n\pi}{11} \\ x &= \frac{2n\pi}{11} \pm \frac{2n\pi}{11} \\ x &= 0, \frac{2\pi}{11}, \frac{4\pi}{11}, \frac{6\pi}{11}, \frac{8\pi}{11}, \frac{10\pi}{11}, \frac{12\pi}{11}, \dots \end{aligned}$$

**Question 6** (\*\*\*\*+)

Solve in degrees the trigonometric equation

$$\sin 5\theta + \sin 3\theta = 0, \quad 0^\circ \leq \theta < 180^\circ.$$

$$\theta = 0^\circ, 45^\circ, 90^\circ, 135^\circ$$

**Standard Method:**

$$\begin{aligned} \sin 5\theta + \sin 3\theta &= 0 \\ \sin 5\theta + \sin 3\theta &= 2 \sin \frac{5\theta+3\theta}{2} \cos \frac{5\theta-3\theta}{2} \\ 2 \sin \left( \frac{5\theta+3\theta}{2} \right) \cos \left( \frac{5\theta-3\theta}{2} \right) &= 0 \\ 2 \sin 4\theta \cos \theta &= 0 \\ \sin 4\theta = 0 \quad \text{or} \quad \cos \theta &= 0 \\ \left( 4\theta = 0 \pm 360^\circ \right) \quad \text{or} \quad \left( \theta = 90 \pm 360^\circ \right) \\ \left( 4\theta = 0 \pm 360^\circ \right) \quad \text{or} \quad \left( \theta = 90 \pm 360^\circ \right) \\ \theta = 0 \pm 90^\circ \quad \text{or} \quad \theta = 90 \pm 360^\circ, \dots \\ \theta = 0 \pm 90^\circ \quad \text{or} \quad \theta = 90 \pm 360^\circ, \dots \\ \therefore \theta = 0^\circ, 90^\circ, 135^\circ \end{aligned}$$

**Alternative Method:**

$$\begin{aligned} \sin 5\theta + \sin 3\theta &= 0 \\ \sin 5\theta &= -\sin 3\theta \\ \sin 5\theta &= \sin (-3\theta) \\ 5\theta &= -3\theta \pm 360^\circ n \\ 5\theta &= (10-3n) \pm 360^\circ n \\ 11\theta &= 360^\circ n \pm 360^\circ n \\ \theta &= 0 \pm 360^\circ n \\ \theta &= 0 \pm 360^\circ n \\ \theta &= 0 \pm 360^\circ n \\ \therefore \theta = 0^\circ, 90^\circ, 135^\circ \end{aligned}$$

## Question 7 (\*\*\*\*+)

Solve the following trigonometric equation

$$\frac{\cos 2x}{1 + \cos 2x} = 1 - 2 \tan x, \quad 0 \leq x < 2\pi,$$

giving the answers in terms of  $\pi$ .

$$\boxed{\phantom{0000}}, \quad x = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{13\pi}{12}, \frac{17\pi}{12}$$

START BY REDUCING THE DOUBLE ARGUMENTS

$$\Rightarrow \frac{\cos 2x}{1 + \cos 2x} = 1 - 2 \tan x$$

$$\Rightarrow \frac{\cos 2x - \sin^2 x}{1 + (2\cos^2 x - 1)} = 1 - 2 \tan x$$

$$\Rightarrow \frac{\cos 2x - \sin^2 x}{2\cos^2 x} = 1 - 2 \tan x$$

$$\Rightarrow \frac{\cos 2x - \sin^2 x}{\cos^2 x} = 2 - 4 \tan x$$

$$\Rightarrow 1 - \tan^2 x = 2 - 4 \tan x$$

$$\Rightarrow 0 = 1 - 4 \tan x + \tan^2 x$$

THE QUADRATIC DOES NOT FACTORISE NICELY, SO PROCEED BY THE QUADRATIC FORMULA OR BY COMPLETING THE SQUARE

$$\Rightarrow (\tan x - 2)^2 - 4 + 1 = 0$$

$$\Rightarrow (\tan x - 2)^2 = 3$$

$$\Rightarrow \tan x - 2 = \pm \sqrt{3}$$

$$\Rightarrow \tan x = \frac{2 + \sqrt{3}}{2 - \sqrt{3}} \Rightarrow \arctan\left(\frac{2 + \sqrt{3}}{2 - \sqrt{3}}\right) = \frac{\pi}{12}$$

SOLVING SEPARATELY IN RADIANS

- $x = \frac{\pi}{12} \pm \pi$
- $x = \frac{\pi}{12} \pm \pi$

$x = \frac{\pi}{12}, \frac{13\pi}{12}, \frac{5\pi}{12}, \frac{17\pi}{12}$

## Question 8 (\*\*\*\*+)

Solve in degrees the trigonometric equation

$$4 \tan(\theta + 60) \tan(\theta - 60) = \sec^2 \theta - 16, \quad 0^\circ \leq \theta < 180^\circ$$

$$\boxed{\phantom{000}}, \quad \theta = 15^\circ, 75^\circ, 105^\circ, 165^\circ$$

USING THE TANGENT ADDITION IDENTITY & THE PYTHAGOREAN IDENTITY

IDENTITY:  $\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$   
 $1 + \tan^2 A = \sec^2 A$

$\Rightarrow 4 \tan(\theta + 60) \tan(\theta - 60) = \sec^2 \theta - 16$

$\Rightarrow 4 \left( \frac{\tan \theta + \tan 60}{1 - \tan \theta \tan 60} \right) \left( \frac{\tan \theta - \tan 60}{1 + \tan \theta \tan 60} \right) = \sec^2 \theta - 16$

$\tan 60 = \sqrt{3}$

$\Rightarrow 4 \left( \frac{\tan \theta + \sqrt{3}}{1 - \sqrt{3} \tan \theta} \right) \left( \frac{\tan \theta - \sqrt{3}}{1 + \sqrt{3} \tan \theta} \right) = \sec^2 \theta - 16$

$\Rightarrow \frac{4(\tan^2 \theta - 3)}{1 - 3 \tan^2 \theta} = \sec^2 \theta - 16$

$\Rightarrow \frac{4(T - 3)}{1 - 3T} = T - 15$  where  $T = \tan \theta$

$\Rightarrow 4(T - 3) = (T - 15)(1 - 3T)$

$\Rightarrow 4T - 12 = T - 3T^2 - 15 + 45T$

$\Rightarrow 3T^2 - 42T + 3 = 0$

$\Rightarrow T^2 - 14T + 1 = 0$

$\Rightarrow (T - 7)^2 - 48 = 0$

$\Rightarrow (T - 7)^2 = 48$

$\Rightarrow T - 7 = \pm \sqrt{48}$

$\Rightarrow T = 7 \pm \sqrt{48}$

$\Rightarrow \tan \theta = 7 \pm \sqrt{48}$

$\Rightarrow \tan \theta = 7 \pm \sqrt{48}$

INVERTING ALL 4 RESULTS

$\theta = 15^\circ \pm 180^\circ$   
 $\theta = -15^\circ \pm 180^\circ$   
 $\theta = 75^\circ \pm 180^\circ$   
 $\theta = -75^\circ \pm 180^\circ$

$\therefore \theta = 15^\circ, 75^\circ, 105^\circ, 165^\circ$



## Question 9 (\*\*\*\*+)

Prove the validity of each of the following trigonometric identities.

a)  $(\tan \theta + \cot \theta)(\sin \theta + \csc \theta) \equiv \sec \theta + \operatorname{cosec} \theta.$

b)  $\tan\left(\theta + \frac{\pi}{4}\right) \equiv \sec 2\theta + \tan 2\theta.$

proof

(a) LHS =  $(\tan \theta + \cot \theta)(\sin \theta + \csc \theta)$   
 $= \left(\frac{\sin \theta}{\cos \theta} + \frac{\cos \theta}{\sin \theta}\right)(\sin \theta + \csc \theta) = \frac{\sin \theta + \cos \theta}{\sin \theta \cos \theta} \times (\sin \theta + \csc \theta)$   
 $= \frac{\sin \theta + \cos \theta}{\sin \theta \cos \theta} = \frac{\sin \theta}{\sin \theta \cos \theta} + \frac{\cos \theta}{\sin \theta \cos \theta}$   
 $= \frac{1}{\cos \theta} + \frac{1}{\sin \theta} = \sec \theta + \operatorname{cosec} \theta = \text{RHS} //$

(b) LHS =  $\tan\left(\theta + \frac{\pi}{4}\right) = \frac{\tan \theta + \tan \frac{\pi}{4}}{1 - \tan \theta \tan \frac{\pi}{4}} = \frac{\tan \theta + 1}{1 - \tan \theta}$   
 $= \frac{(\tan \theta + 1)(1 + \tan \theta)}{(-\tan \theta)(1 + \tan \theta)} = \frac{\tan^2 \theta + 1 + 2\tan \theta}{1 - \tan^2 \theta}$   
 $= \frac{\sec^2 \theta}{1 - \tan^2 \theta} + \frac{2\tan \theta}{1 - \tan^2 \theta} = \frac{1 + \tan^2 \theta}{1 - \tan^2 \theta} + \frac{2\tan \theta}{1 - \tan^2 \theta}$   
 $= \frac{1}{\cos^2 \theta - \sin^2 \theta} + \frac{2\tan \theta}{1 - \tan^2 \theta}$   
 $= \frac{1}{\cos 2\theta} + \tan 2\theta = \sec 2\theta + \tan 2\theta = \text{RHS} //$

## Question 10 (\*\*\*\*+)

$$2 \arctan \left[ \frac{1}{x-3} \right] + \arctan \left[ \frac{1}{x+2} \right] = \arctan \left[ \frac{31}{17} \right].$$

Show that  $x=5$  is one of the solutions of the above trigonometric equation and find, in exact surd form, the other two solutions.

$$x = \frac{10 \pm 5\sqrt{190}}{31}$$

Handwritten solution for Question 10:

$$2 \arctan \left( \frac{1}{x-3} \right) + \arctan \left( \frac{1}{x+2} \right) = \arctan \left( \frac{31}{17} \right)$$

$$\Rightarrow 2 \arctan \left( \frac{1}{x-3} \right) = \arctan \left( \frac{31}{17} \right) - \arctan \left( \frac{1}{x+2} \right)$$

• TAKING TANGENTS ON BOTH SIDES

$$\Rightarrow \frac{2 \left( \frac{1}{x-3} \right)}{1 - \left( \frac{1}{x-3} \right)^2} = \frac{\frac{31}{17} - \frac{1}{x+2}}{1 + \frac{31}{17} \times \frac{1}{x+2}}$$

• SIMPLIFYING

$$\Rightarrow \frac{2(x-3)}{(x-3)^2 - 1} = \frac{31(x+2) - 17}{17(x+2) + 31}$$

$$\Rightarrow \frac{2x-6}{x^2-6x+8} = \frac{31x+65}{17x+65}$$

$$\Rightarrow (31x+65)(x^2-6x+8) = (2x-6)(17x+65)$$

$$\Rightarrow 31x^3 - 186x^2 + 248x + 520x - 2730x + 360 = 34x^2 + 282x - 192x - 390$$

$$\Rightarrow 31x^3 - 104x^2 - 22x + 360 = 34x^2 + 282x - 390$$

$$\Rightarrow 31x^3 - 138x^2 - 302x + 750 = 0$$

• BY LONG DIVISION / OR MULTIPLICATION

$$\Rightarrow 31x^2(x-5) - 20x(x-5) - 150(x-5) = 0$$

$$\Rightarrow (x-5)(31x^2 - 20x - 150) = 0$$

either  $x=5$  or by quadratic formula  $x = \frac{20 \pm \sqrt{1900}}{2 \times 31}$

$$x = \frac{10 \pm 5\sqrt{190}}{31}$$

**Question 11** (\*\*\*\*+) **Non Calculator**

A triangle,  $ABC$  has  $|AB| = 2\sqrt{3}$  cm,  $\angle BAC = 45^\circ$  and  $\angle ACB = 60^\circ$ .

Determine, in exact simplified surd form, the area of this triangle

$$\boxed{\phantom{000}}, \text{ area} = 3 + \sqrt{3}$$

• Firstly  $\hat{ABC} = 75^\circ$   
 • BY THE SINE RULE WE HAVE  

$$\frac{|BC|}{\sin 45^\circ} = \frac{|AB|}{\sin 60^\circ}$$

$$|BC| = \frac{|AB| \sin 45^\circ}{\sin 60^\circ}$$

$$|BC| = \frac{2\sqrt{3} \times \frac{\sqrt{2}}{2}}{\frac{\sqrt{3}}{2}} = \frac{\sqrt{6}}{\frac{\sqrt{3}}{2}} = \frac{2\sqrt{6}}{\sqrt{3}} = 2\sqrt{\frac{6}{3}} = 2\sqrt{2}$$
 • NOW THE AREA CAN BE FOUND AS  

$$\text{Area} = \frac{1}{2} |AB| |BC| \sin 75^\circ$$

$$= \frac{1}{2} (2\sqrt{3}) (2\sqrt{2}) \sin (45^\circ + 30^\circ)$$

$$= 2\sqrt{6} \left[ \sin 45^\circ \cos 30^\circ + \cos 45^\circ \sin 30^\circ \right]$$

$$= 2\sqrt{6} \left[ \frac{\sqrt{2}}{2} \times \frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2} \times \frac{1}{2} \right]$$

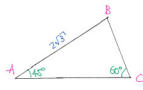
$$= 2\sqrt{6} \times \frac{\sqrt{6} + \sqrt{3}}{4}$$

$$= \frac{12 + 2\sqrt{18}}{4}$$

$$= \frac{12 + 2(3\sqrt{2})}{4}$$

$$= \frac{12 + 6\sqrt{2}}{4}$$

$$= 3 + \frac{3\sqrt{2}}{2}$$



## Question 12 (\*\*\*\*+)

It is given that

$$\sin^2\left(\theta + \frac{\pi}{4}\right) - \sin^2\left(\theta - \frac{\pi}{4}\right) \equiv \sin 2\theta.$$

a) Prove the validity of the above trigonometric identity.

b) Hence, or otherwise, show that ...

i. ...  $\sin\left(\theta + \frac{\pi}{4}\right)\cos\left(\theta + \frac{\pi}{4}\right) - \sin\left(\theta - \frac{\pi}{4}\right)\cos\left(\theta - \frac{\pi}{4}\right) \equiv \cos 2\theta.$

ii. ...  $\sin\left(\frac{5\pi}{12}\right)\cos\left(\frac{5\pi}{12}\right) + \sin\left(\frac{\pi}{12}\right)\cos\left(\frac{\pi}{12}\right) = \frac{1}{2}.$

proof

a) LHS =  $\sin^2\left(\theta + \frac{\pi}{4}\right) - \sin^2\left(\theta - \frac{\pi}{4}\right) = \left[\sin\left(\theta + \frac{\pi}{4}\right)\right]^2 - \left[\sin\left(\theta - \frac{\pi}{4}\right)\right]^2$   
 $= \left(\sin\theta\cos\frac{\pi}{4} + \cos\theta\sin\frac{\pi}{4}\right)^2 - \left(\sin\theta\cos\frac{\pi}{4} - \cos\theta\sin\frac{\pi}{4}\right)^2$   
 $= \left(\frac{\sqrt{2}}{2}\sin\theta + \frac{\sqrt{2}}{2}\cos\theta\right)^2 - \left(\frac{\sqrt{2}}{2}\sin\theta - \frac{\sqrt{2}}{2}\cos\theta\right)^2$   
 $= \left(\frac{1}{2}\sin^2\theta + \sin\theta\cos\theta + \frac{1}{2}\cos^2\theta\right) - \left(\frac{1}{2}\sin^2\theta - \sin\theta\cos\theta + \frac{1}{2}\cos^2\theta\right)$   
 $= 2\sin\theta\cos\theta = \sin 2\theta = \text{RHS}$

Alternatively:  
 LHS =  $\sin^2\left(\theta + \frac{\pi}{4}\right) - \sin^2\left(\theta - \frac{\pi}{4}\right) = \text{Difference of squares}$   
 $= \left[\sin\left(\theta + \frac{\pi}{4}\right) - \sin\left(\theta - \frac{\pi}{4}\right)\right]\left[\sin\left(\theta + \frac{\pi}{4}\right) + \sin\left(\theta - \frac{\pi}{4}\right)\right]$   
 $= \left[\sin\theta\cos\frac{\pi}{4} + \cos\theta\sin\frac{\pi}{4} - \sin\theta\cos\frac{\pi}{4} + \cos\theta\sin\frac{\pi}{4}\right]\left[\sin\theta\cos\frac{\pi}{4} + \cos\theta\sin\frac{\pi}{4} + \sin\theta\cos\frac{\pi}{4} - \cos\theta\sin\frac{\pi}{4}\right]$   
 $= (2\cos\theta\sin\frac{\pi}{4})(2\sin\theta\cos\frac{\pi}{4}) = (\sqrt{2}\cos\theta)(\sqrt{2}\sin\theta) = 2\sin\theta\cos\theta$   
 $= \sin 2\theta = \text{RHS}$

b)  $\sin 2\theta = \sin^2\left(\theta + \frac{\pi}{4}\right) - \sin^2\left(\theta - \frac{\pi}{4}\right)$   
 $\frac{d}{d\theta}(\sin 2\theta) = \frac{d}{d\theta}\left[\sin^2\left(\theta + \frac{\pi}{4}\right) - \sin^2\left(\theta - \frac{\pi}{4}\right)\right]$   
 $2\cos 2\theta = 2\sin\left(\theta + \frac{\pi}{4}\right)\cos\left(\theta + \frac{\pi}{4}\right) - 2\sin\left(\theta - \frac{\pi}{4}\right)\cos\left(\theta - \frac{\pi}{4}\right)$   
 Divide by 2  
 $\cos 2\theta = \sin\left(\theta + \frac{\pi}{4}\right)\cos\left(\theta + \frac{\pi}{4}\right) - \sin\left(\theta - \frac{\pi}{4}\right)\cos\left(\theta - \frac{\pi}{4}\right)$   
 Let  $\theta = \frac{\pi}{6}$   
 $\cos \frac{\pi}{3} = \sin\frac{\pi}{6}\cos\frac{\pi}{6} - \sin\left(\frac{\pi}{6} - \frac{\pi}{4}\right)\cos\left(\frac{\pi}{6} - \frac{\pi}{4}\right)$   
 $\frac{1}{2} = \sin\frac{\pi}{6}\cos\frac{\pi}{6} + \sin\frac{\pi}{12}\cos\frac{\pi}{12}$   
 $\frac{1}{2} = \sin\frac{\pi}{6}\cos\frac{\pi}{6} + \sin\frac{\pi}{12}\cos\frac{\pi}{12}$

**Question 13** (\*\*\*\*+)

Solve the trigonometric equation

$$(3\sin x + 5\cos x)^2 = 4\cos^2 x, \text{ for } 0 \leq x < 2\pi,$$

giving the answers correct to three significant figures.

$$x \approx 1.98^\circ, 2.36^\circ, 5.12^\circ, 5.50^\circ$$

Handwritten solution for Question 13:

$$(3\sin x + 5\cos x)^2 = 4\cos^2 x$$

$$3\sin x + 5\cos x = \pm 2\cos x$$

$$3\sin x = -3\cos x$$

$$\text{Divide to make tan}$$

$$\tan x = -1$$

$$x = -\frac{\pi}{4} + n\pi$$

$$\text{or } x = -1.465^\circ + n\pi$$

Hence

$$x = 236^\circ, 5.50^\circ, 198^\circ, 5.12^\circ$$

ALTERNATIVE: SQUARE, THEN R.T. TO DOUBLE ANGLES, THEN R.T. TO DOUBLE ANGLES

**Question 14** (\*\*\*\*+)

Solve the following trigonometric equation

$$\tan x + \cot x = 8\cos 2x, \quad 0 \leq x < \pi,$$

where  $x$  is measured in radians.

$$\boxed{\phantom{00000}}, \quad x = \frac{1}{24}\pi, \frac{5}{24}\pi, \frac{13}{24}\pi, \frac{17}{24}\pi$$

Handwritten solution for Question 14:

MANIPULATE JUST THE LHS OF THE EQUATION

$$\frac{\sin x + \cos x}{\cos x} = 8\cos 2x$$

$$\frac{\sin x + \cos x}{\cos x} = 8\cos 2x$$

$$\frac{\sin x + \cos x}{\cos x} = 8\cos 2x$$

$$\frac{1}{\cos x} = 8\cos 2x$$

$$\frac{1}{2\cos^2 x} = 4\cos 2x$$

$$\frac{1}{\sin^2 x} = 4\cos 2x$$

$$4\cos 2x \sin^2 x = 1$$

$$2\cos 2x \sin^2 x = \frac{1}{2}$$

$$\sin 4x = \frac{1}{2}$$

Use the arcsin rule

$$4x = \frac{\pi}{6} + 2n\pi$$

$$4x = \frac{5\pi}{6} + 2n\pi$$

$$x = \frac{\pi}{24} + \frac{n\pi}{2}$$

$$x = \frac{5\pi}{24} + \frac{n\pi}{2}$$

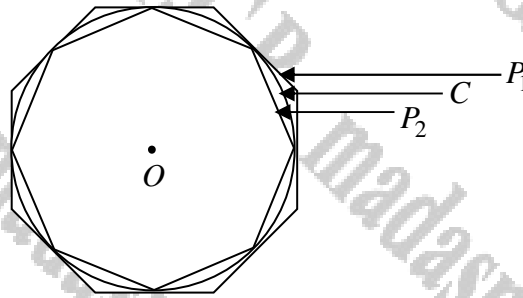
$$x = \frac{13\pi}{24} + \frac{n\pi}{2}$$

$$x = \frac{17\pi}{24} + \frac{n\pi}{2}$$

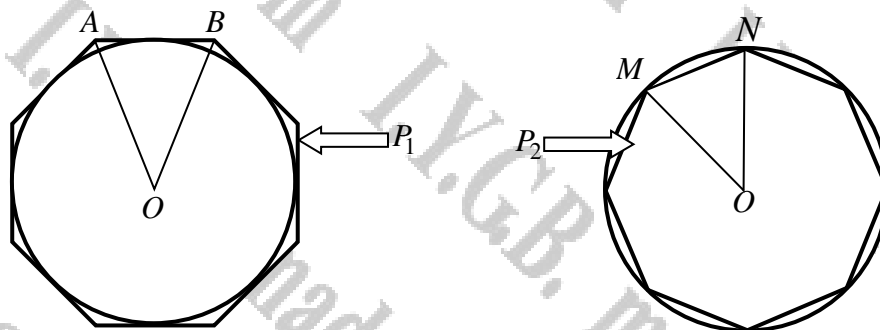
**Question 15** (\*\*\*\*+)

The figure below shows a regular octagon  $P_1$ . A circle  $C$  is inscribed inside  $P_1$  and another regular octagon  $P_2$  is inscribed inside the circle  $C$ .

The three objects have a common centre  $O$ .



The circle  $C$  has a radius of 1 unit. The points  $A$  and  $B$  are consecutive vertices of  $P_1$ , and the points  $M$  and  $N$  are consecutive vertices of  $P_2$ .



- By considering the triangle  $OAB$ , show that the perimeter of the octagon  $P_1$  is  $16 \tan \frac{\pi}{8}$ .
- Use the triangle  $OMN$  in a similar fashion to show that the perimeter of the octagon  $P_2$  is  $16 \sin \frac{\pi}{8}$ .

[continues overleaf]

c) Use a standard identity for  $\cos 2\theta$  to show that

**d)** Show further that

e) Deduce from the results obtained so far that

, proof

[illegible]

## Question 16 (\*\*\*\*+)

Solve the following trigonometric equation

$$\frac{\cot x}{\operatorname{cosec} x - 1} + \frac{\operatorname{cosec} x - 1}{\cot x} = 4, \quad 0^\circ \leq x < 360^\circ.$$

$$x = 60^\circ, 300^\circ$$

## Question 17 (\*\*\*\*+)

$$f(x) = \ln(1 + \sin x), \quad \sin x \neq \pm 1.$$

Show clearly that

$$f(x) - f(-x) = 2 \ln(\sec x + \tan x).$$

 ,  proof



**Question 18** (\*\*\*\*+)

It is given that

$$\cos 3x \equiv 4\cos^3 x - 3\cos x.$$

- a) Prove the validity of the above trigonometric identity.
- b) By differentiating both sides of the above identity with respect to  $x$ , show that

$$\sin 3x \equiv 3\sin x - 4\sin^3 x.$$

proof

**Question 19** (\*\*\*\*+)

Prove the validity of each of the following trigonometric identities.

- a)  $\frac{\cos 2x - \cos x + 1}{\sin 2x - \sin x} \equiv \cot x.$
- b)  $2\cos^4 \theta + \frac{1}{2}\sin^2 2\theta - 1 \equiv \cos 2\theta.$

proof

**Question 20** (\*\*\*\*+)

It is given that

$$\frac{\tan x}{\sec x - 1} - \frac{\sec x - 1}{\tan x} = 2 \cot x, \quad \theta \neq \frac{n\pi}{2}, \quad n \in \mathbb{Z}.$$

a) Prove the validity of the above trigonometric identity.

b) Hence solve the trigonometric equation

$$\frac{\tan 3\theta}{\sec 3\theta - 1} - \frac{\sec 3\theta - 1}{\tan 3\theta} = \frac{2}{\sqrt{3}}, \quad 0 \leq \theta < \pi,$$

giving the answers in terms of  $\pi$ .

$$x = \frac{\pi}{9}, \frac{4\pi}{9}, \frac{7\pi}{9}$$

(a) LHS =  $\frac{\tan x}{\sec x - 1} - \frac{\sec x - 1}{\tan x}$   
 $= \frac{\tan^2 x - (\sec x - 1)^2}{(\sec x - 1)\tan x}$   
 $= \frac{\tan^2 x - (\sec^2 x - 2\sec x + 1)}{(\sec x - 1)\tan x}$   
 $= \frac{\tan^2 x - \sec^2 x + 2\sec x - 1}{(\sec x - 1)\tan x}$   
 $= \frac{(\tan^2 x - \sec^2 x) + 2\sec x - 1}{(\sec x - 1)\tan x}$   
 $= \frac{-1 + 2\sec x - 1}{(\sec x - 1)\tan x}$   
 $= \frac{2\sec x - 2}{(\sec x - 1)\tan x}$   
 $= \frac{2(\sec x - 1)}{(\sec x - 1)\tan x}$   
 $= \frac{2}{\tan x} = 2 \cot x$

(b)  $\frac{\tan 3\theta}{\sec 3\theta - 1} - \frac{\sec 3\theta - 1}{\tan 3\theta} = \frac{2}{\sqrt{3}}$   
 Reu (a)  
 $\Rightarrow 2 \cot 3\theta = \frac{2}{\sqrt{3}}$   
 $\Rightarrow \cot 3\theta = \frac{1}{\sqrt{3}}$   
 $\Rightarrow \tan 3\theta = \sqrt{3}$   
 $\Rightarrow 3\theta = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$   
 $\therefore \theta = \frac{\pi}{9}, \frac{2\pi}{9}, \frac{4\pi}{9}, \frac{5\pi}{9}$

**Question 21** (\*\*\*\*+)

Prove the validity of the following trigonometric identity.

$$\frac{1 + \cos \theta}{1 - \cos \theta} = (\operatorname{cosec} \theta + \cot \theta)^2.$$

proof

LHS =  $\frac{1 + \cos \theta}{1 - \cos \theta}$   
 $= \frac{1 + \cos \theta}{1 - \cos \theta} \cdot \frac{1 + \cos \theta}{1 + \cos \theta} = \frac{(1 + \cos \theta)^2}{1 - \cos^2 \theta}$   
 $= \frac{1 + 2\cos \theta + \cos^2 \theta}{1 - \cos^2 \theta}$   
 $= \frac{1 + 2\cos \theta + \cos^2 \theta}{\sin^2 \theta}$   
 $= \frac{1}{\sin^2 \theta} + \frac{2\cos \theta}{\sin^2 \theta} + \frac{\cos^2 \theta}{\sin^2 \theta}$   
 $= \operatorname{cosec}^2 \theta + 2 \operatorname{cosec} \theta \cot \theta + \cot^2 \theta$   
 $= (\operatorname{cosec} \theta + \cot \theta)^2 = \text{RHS}$   
 (b)  $\frac{1 + \cos \theta}{1 - \cos \theta} = \frac{(1 + \cos \theta)^2}{1 - \cos^2 \theta} = \frac{(1 + \cos \theta)^2}{\sin^2 \theta} = (\operatorname{cosec} \theta + \cot \theta)^2 = \text{RHS}$

## Question 22 (\*\*\*\*+)

It is given that

$$\sin 3\theta \equiv 3\sin \theta - 4\sin^3 \theta.$$

- a) Prove the validity of the above trigonometric identity.
- b) State the maximum value of

$$f(x) = 6\sin(5x) - 8\sin^3(5x), \quad x \in \mathbb{R},$$

and determine, in degrees, the smallest positive value of  $x$  which produces this maximum value.

$$f_{\max}(x) = 2, \quad x = 6^\circ$$

$\sin 3\theta = \sin(2\theta + \theta) = \sin 2\theta \cos \theta + \cos 2\theta \sin \theta$   
 $= (2\sin \theta \cos \theta) \cos \theta + (1 - 2\sin^2 \theta) \sin \theta = 2\sin \theta \cos^2 \theta + \sin \theta - 2\sin^3 \theta$   
 $= 2\sin \theta (1 - \sin^2 \theta) + \sin \theta - 2\sin^3 \theta = 2\sin \theta - 2\sin^3 \theta + \sin \theta - 2\sin^3 \theta$   
 $= 3\sin \theta - 4\sin^3 \theta$

$f(x) = 6\sin 5x - 8\sin^3 5x$   
 $f'(x) = 2[3\sin 5x - 4\sin^3 5x]$   
 $f'(x) = 2\sin 5x$   
 $\therefore f'(x) = 0 \quad \text{if } \sin 5x = 0$   
 $5x = 0, \pi, 2\pi, \dots$   
 $x = 0, 18^\circ, 36^\circ, \dots$   
 $\therefore x = 18^\circ$

## Question 23 (\*\*\*\*+)

The obtuse angles  $\theta$  and  $\varphi$  satisfy the equation

$$\sin 6\theta^\circ + \cos 4\varphi^\circ = -2.$$

Find the possible values of  $\theta$  and  $\varphi$ .

$$\boxed{\phantom{000}}, (\theta, \varphi) = (105^\circ, 135^\circ) \cup (165^\circ, 135^\circ)$$

As the sine and cosine functions oscillate between -1 and 1, it is evident that  $\sin 6\theta + \cos 4\varphi = -2 \implies \begin{cases} \sin 6\theta = -1 \\ \cos 4\varphi = -1 \end{cases}$

APPLY THE RULE

- $\sin 6\theta = -1$ 
  - $6\theta = -90 \pm 360n$
  - $6\theta = 270 \pm 360n$
  - $\theta = -15 \pm 60n$
  - $\theta = 45 \pm 60n$
  - $\theta = 105^\circ$
  - $\theta = 165^\circ$
- $\cos 4\varphi = -1$ 
  - $4\varphi = 180 \pm 360n$
  - $4\varphi = 540 \pm 360n$
  - $\varphi = 45 \pm 90n$
  - $\varphi = 135^\circ$

$\therefore (\theta, \varphi) = (105^\circ, 135^\circ) \cup (165^\circ, 135^\circ)$

## Question 24 (\*\*\*\*+)

It is given that

$$\tan(A+B) \equiv \frac{\tan A + \tan B}{1 - \tan A \tan B}.$$

- a) Prove the validity of the above trigonometric identity by using the compound angle formulae for  $\sin(A+B)$  and  $\cos(A+B)$ .

- a) Deduce an exact simplified expression for  $\tan\left(\theta - \frac{\pi}{3}\right)$ , in terms of  $\tan \theta$ .

- b) Solve, for  $0 \leq \theta < 2\pi$ , the trigonometric equation

$$\tan \theta - \sqrt{3} = (1 + \sqrt{3} \tan \theta) \tan(2\pi - \theta),$$

giving the answers in terms of  $\pi$ .

$$\tan\left(\theta - \frac{\pi}{3}\right) = \frac{\tan \theta - \sqrt{3}}{1 + \sqrt{3} \tan \theta}, \quad \theta = \frac{\pi}{6}, \frac{2\pi}{3}, \frac{7\pi}{6}, \frac{5\pi}{3}$$

Handwritten solution for Question 24:

(a)  $\tan(A+B) = \frac{\sin(A+B)}{\cos(A+B)} = \frac{\sin A \cos B + \cos A \sin B}{\cos A \cos B - \sin A \sin B} = \frac{\tan A + \tan B}{1 - \tan A \tan B}$  (Divide by  $\cos A \cos B$ )

(b)  $\tan\left(\theta - \frac{\pi}{3}\right) = \frac{\tan \theta - \tan \frac{\pi}{3}}{1 + \tan \theta \tan \frac{\pi}{3}} = \frac{\tan \theta - \sqrt{3}}{1 + \sqrt{3} \tan \theta}$

Given:  $\tan \theta - \sqrt{3} = (1 + \sqrt{3} \tan \theta) \tan(2\pi - \theta)$

$\Rightarrow \tan \theta - \sqrt{3} = (1 + \sqrt{3} \tan \theta) \times \frac{\tan \theta - \sqrt{3}}{1 + \sqrt{3} \tan \theta}$

$\Rightarrow \tan \theta - \sqrt{3} = \tan \theta - \sqrt{3}$

$\Rightarrow \sqrt{3} \tan \theta + 2 \tan \theta - \sqrt{3} = 0$

$\Rightarrow \tan \theta + \frac{2}{\sqrt{3}} \tan \theta - 1 = 0$

$\Rightarrow \left(\tan \theta + \frac{2}{\sqrt{3}}\right)^2 - \left(\frac{1}{\sqrt{3}}\right)^2 = 0$

$\Rightarrow \left(\tan \theta + \frac{1}{\sqrt{3}}\right)^2 = \frac{4}{3}$

$\Rightarrow \tan \theta + \frac{1}{\sqrt{3}} = \pm \frac{2}{\sqrt{3}}$

$\Rightarrow \tan \theta = -\frac{1}{\sqrt{3}} \pm \frac{2}{\sqrt{3}}$

$\Rightarrow \tan \theta = \frac{1}{\sqrt{3}} \text{ or } -\frac{3}{\sqrt{3}}$

$\bullet \tan \theta = \frac{1}{\sqrt{3}} \Rightarrow \theta = \frac{\pi}{6} \text{ or } \frac{5\pi}{6}$

$\bullet \tan \theta = -\frac{3}{\sqrt{3}} \Rightarrow \theta = \frac{2\pi}{3} \text{ or } \frac{4\pi}{3}$

$\bullet \theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{2\pi}{3}, \frac{4\pi}{3}$

Alternative method:

$\tan \theta - \sqrt{3} = (1 + \sqrt{3} \tan \theta) \tan(2\pi - \theta)$

$\tan \theta - \sqrt{3} = \tan(2\pi - \theta)$

$\tan\left(\theta - \frac{\pi}{3}\right) = \tan(2\pi - \theta)$

$\theta - \frac{\pi}{3} = 2\pi - \theta + n\pi$

$2\theta = \frac{7\pi}{3} + n\pi$

$\theta = \frac{7\pi}{6} + \frac{n\pi}{2}$

$\theta = \frac{7\pi}{6}, \frac{5\pi}{3}, \frac{11\pi}{6}, \frac{2\pi}{3}$

## Question 25 (\*\*\*\*+)

Solve the following trigonometric equation

$$\tan 4x - \tan 2x = 0, \quad 0^\circ \leq x < 360^\circ.$$

$$\boxed{\phantom{000}}, \quad x = 0^\circ, 90^\circ, 180^\circ, 270^\circ$$

METHOD A

$$\begin{aligned} \Rightarrow \tan 4x - \tan 2x &= 0 \\ \Rightarrow \tan 4x &= \tan 2x \\ \Rightarrow 4x &= 2x \pm 180^\circ n \quad n=0,1,2,3,\dots \\ \Rightarrow 2x &= 0^\circ \pm 180^\circ n \\ \Rightarrow x &= 0^\circ \pm 90^\circ n \\ x &= 0^\circ, 90^\circ, 180^\circ, 270^\circ \end{aligned}$$

METHOD B

using  $\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$

$$\begin{aligned} \Rightarrow \tan(2 \times 2x) - \tan 2x &= 0 \\ \Rightarrow \frac{2\tan 2x}{1-\tan^2 2x} - \tan 2x &= 0 \\ \Rightarrow 2\tan 2x - \tan 2x(1-\tan^2 2x) &= 0 \\ \Rightarrow \tan 2x [2 - (1-\tan^2 2x)] &= 0 \\ \Rightarrow \tan 2x (1 + \tan^2 2x) &= 0 \\ \Rightarrow \tan 2x &= 0 \quad \text{NO SOLUTIONS} \end{aligned}$$

$$\begin{aligned} \Rightarrow 2x &= 0^\circ \pm 180^\circ n \quad n=0,1,2,3,\dots \\ \Rightarrow x &= 0^\circ \pm 90^\circ n \\ x &= 0^\circ, 90^\circ, 180^\circ, 270^\circ \end{aligned}$$

As before

## Question 26 (\*\*\*\*+)

It is given that

$$\cos 3\theta \equiv 4\cos^3 \theta - 3\cos \theta.$$

- a) Prove the validity of the above trigonometric identity.
- b) Hence solve in degrees the trigonometric equation

$$2 + \cos 6x \sec 2x = 0, \quad 0^\circ \leq x < 180^\circ.$$

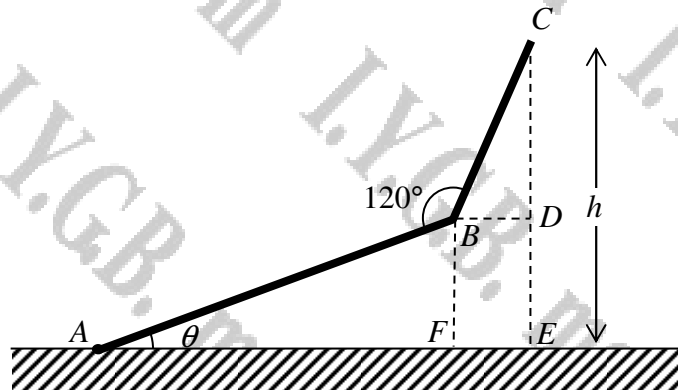
$$x = 30^\circ, 60^\circ, 120^\circ, 150^\circ$$

a)  $\cos 3\theta = \cos(2\theta + \theta) = \cos 2\theta \cos \theta - \sin 2\theta \sin \theta$   
 $= (2\cos^2 \theta - 1)\cos \theta - (2\sin \theta \cos \theta)\sin \theta$   
 $= 2\cos^3 \theta - \cos \theta - 2\sin^2 \theta \cos \theta$   
 $= 2\cos^3 \theta - \cos \theta - 2(1 - \cos^2 \theta)\cos \theta$   
 $= 2\cos^3 \theta - \cos \theta - 2\cos \theta + 2\cos^3 \theta$   
 $= 4\cos^3 \theta - 3\cos \theta$   
 Proved

b)  $\cos 2x \sec 2x + 2 = 0$   
 $\Rightarrow \cos(2 \times 2) \sec 2x = -2$   
 Let  $\theta = 2x$   
 $\Rightarrow \cos 2\theta \sec \theta = -2$   
 $\Rightarrow (4\cos^2 \theta - 3\cos \theta) \frac{1}{\cos \theta} = -2$   
 $\Rightarrow 4\cos \theta - 3 = -2$   
 $\Rightarrow 4\cos \theta = 1$   
 $\Rightarrow \cos \theta = \frac{1}{4}$   
 $\Rightarrow \cos \theta = \pm \frac{1}{4}$   
 $\Rightarrow \cos 2x = \pm \frac{1}{4}$

Thus  
 $2x = 60^\circ \pm 360^\circ$   
 $2x = 300^\circ \pm 360^\circ$   
 or  
 $2x = 120^\circ \pm 360^\circ$   
 $2x = 240^\circ \pm 360^\circ$   
 $x = 30^\circ, 150^\circ, 120^\circ, 210^\circ$   
 $2x = 30^\circ \pm 180^\circ$   
 $2x = 60^\circ \pm 180^\circ$   
 $2x = 120^\circ \pm 180^\circ$   
 $2x = 150^\circ \pm 180^\circ$   
 $x = 30^\circ, 60^\circ, 120^\circ, 150^\circ$

Question 27 (\*\*\*\*+)



The figure above shows a rigid rod  $ABC$  where  $AB$  is 6 metres,  $BC$  is 4 metres and the angle  $ABC$  is  $120^\circ$ . The rod is hinged at  $A$  so it can be rotated in a vertical plane forming an angle  $\theta^\circ$  with the horizontal ground.

Let  $h$  metres be the height of the point  $C$  from the horizontal ground.

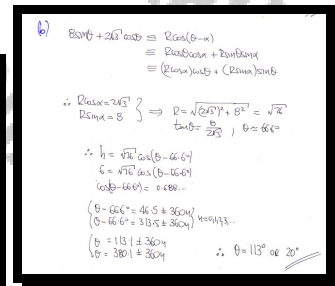
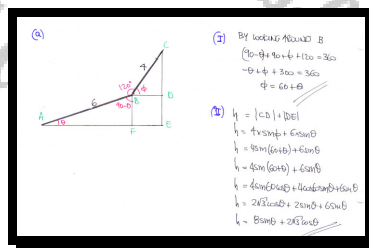
a) Show clearly that ...

i. ...  $\angle DBC = \theta^\circ + 60^\circ$ .

ii. ...  $h = 8\sin\theta + 2\sqrt{3}\cos\theta$ .

b) By expressing  $h$  in the form  $R\cos(\theta - \alpha)$ , where  $R > 0$  and  $0 < \alpha < 90^\circ$ , find to the nearest degree, the values of  $\theta$  when  $h = 6$ .

$\theta \approx 20^\circ, 113^\circ$





**Question 28** (\*\*\*\*+)

By considering the expansion of  $\tan(2A + A)$ , show clearly that

$$\tan 3A \equiv \frac{3 \tan A - \tan^3 A}{1 - 3 \tan^2 A}.$$

proof

Handwritten proof for Question 28:

$$\begin{aligned} \text{LHS} &= \tan(2A) = \tan(2A + A) = \frac{\tan 2A + \tan A}{1 - \tan 2A \tan A} \\ &= \frac{\frac{2 \tan A}{1 - \tan^2 A} + \tan A}{1 - \frac{2 \tan A}{1 - \tan^2 A} \tan A} = \dots \text{Multiply top/bottom by } (1 - \tan^2 A) \\ &= \frac{2 \tan A + \tan A(1 - \tan^2 A)}{(1 - \tan^2 A) - 2 \tan^2 A} = \frac{3 \tan A - \tan^3 A}{1 - 3 \tan^2 A} = \text{RHS} \end{aligned}$$

**Question 29** (\*\*\*\*+)

Show clearly that

$$\frac{\tan^2 x}{\tan^2 x + 2 + \cot^2 x} \equiv \sin^4 x.$$

proof

Handwritten proof for Question 29:

$$\begin{aligned} \text{LHS} &= \frac{\tan^2 x}{\tan^2 x + 2 + \cot^2 x} = \frac{\tan^2 x}{\tan^2 x + 2 \sec^2 x + \frac{1}{\tan^2 x}} = \frac{\tan^2 x}{\left(\tan^2 x + \frac{1}{\tan^2 x}\right)^2} = \frac{\tan^2 x}{\left(\frac{\tan^4 x + 1}{\tan^2 x}\right)^2} \\ &= \frac{\tan^2 x}{\frac{\tan^4 x + 1}{\tan^2 x}} = \frac{\tan^2 x \cdot \tan^2 x}{\tan^4 x + 1} = \frac{\tan^4 x}{\tan^4 x + 1} \\ &= \frac{\sin^4 x}{\cos^4 x + 1} = \text{RHS} \end{aligned}$$

Alternative method shown in the image:

$$\begin{aligned} \text{LHS} &= \frac{\tan^2 x}{\tan^2 x + 2 + \cot^2 x} = \frac{\frac{\sin^2 x}{\cos^2 x}}{\frac{\sin^2 x}{\cos^2 x} + 2 + \frac{\cos^2 x}{\sin^2 x}} = \frac{\frac{\sin^2 x}{\cos^2 x}}{\frac{\sin^2 x + 2 \cos^2 x + \cos^4 x}{\sin^2 x \cos^2 x}} = \frac{\sin^2 x}{\sin^2 x + 2 \cos^2 x + \cos^4 x} \\ &= \frac{\sin^2 x}{\sin^2 x + 2 \cos^2 x + \cos^4 x} = \frac{\sin^2 x}{\sin^2 x + 2 \cos^2 x + \cos^4 x} = \frac{\sin^2 x}{\sin^2 x + 2 \cos^2 x + \cos^4 x} = \sin^4 x = \text{RHS} \end{aligned}$$

**Question 30** (\*\*\*\*+)

It is given that

$$\cos 3\theta \equiv 4\cos^3 \theta - 3\cos \theta.$$

- a) Prove the validity of the above trigonometric identity.
- b) Hence, or otherwise, prove that

$$\cos 6\theta \equiv 32\cos^6 \theta - 48\cos^4 \theta + 18\cos^2 \theta - 1.$$

proof

(a)  $\cos 3\theta = \cos(2\theta + \theta) = \cos 2\theta \cos \theta - \sin 2\theta \sin \theta$   
 $= (2\cos^2 \theta - 1)\cos \theta - (2\sin \theta \cos \theta) \sin \theta$   
 $= 2\cos^3 \theta - \cos \theta - 2\cos \theta \sin^2 \theta$   
 $= 2\cos^3 \theta - \cos \theta - 2\cos \theta (1 - \cos^2 \theta)$   
 $= 2\cos^3 \theta - \cos \theta - 2\cos \theta + 2\cos^3 \theta$   
 $= 4\cos^3 \theta - 3\cos \theta$   $\checkmark$  As required

(b)  $\cos 6\theta = \cos(2 \times 3\theta) = 2\cos^2(3\theta) - 1$   
 $= 2(4\cos^3 \theta - 3\cos \theta)^2 - 1 = 2(16\cos^6 \theta - 24\cos^4 \theta + 9\cos^2 \theta) - 1$   
 $= 32\cos^6 \theta - 48\cos^4 \theta + 18\cos^2 \theta - 1$   $\checkmark$  As required

**Question 31** (\*\*\*\*+)

Prove the validity of each of the following trigonometric identities.

- a)  $\sin^2(x+y) - \sin^2(x-y) \equiv \sin 2x \sin 2y.$
- b)  $\frac{\cot 2\theta + \cos 2\theta}{\cot 2\theta} \equiv (\cos \theta + \sin \theta)^2.$

proof

(i) LHS  $= \sin^2(x+y) - \sin^2(x-y)$   
 $= [\sin(x+y) - \sin(x-y)][\sin(x+y) + \sin(x-y)]$   
 $= [\sin x \cos y + \cos x \sin y - (\sin x \cos y - \cos x \sin y)]$   
 $\times [\sin x \cos y + \cos x \sin y + \sin x \cos y - \cos x \sin y]$   
 $= (2\cos x \sin y)(2\sin x \cos y)$   
 $= (2\cos x \sin x)(2\sin y \cos y)$   
 $= \sin 2x \sin 2y$   
 $= \text{RHS}$   $\checkmark$

(ii) LHS  $= \frac{\cot 2\theta + \cos 2\theta}{\cot 2\theta} = \frac{\frac{\cos 2\theta}{\sin 2\theta} + \cos 2\theta}{\frac{\cos 2\theta}{\sin 2\theta}} = 1 + \cos 2\theta \tan 2\theta$   
 $= 1 + \cos 2\theta \times \frac{\sin 2\theta}{\cos 2\theta} = 1 + \sin 2\theta = 1 + 2\sin \theta \cos \theta$   
 $= \cos^2 \theta + 2\sin \theta \cos \theta + \sin^2 \theta = (\cos \theta + \sin \theta)^2 = \text{RHS}$   $\checkmark$

**Question 32** (\*\*\*\*+)

Show clearly that

$$\tan\left(\theta - \frac{\pi}{4}\right) \equiv \frac{\sin 2\theta - 1}{\cos 2\theta}.$$

proof

Handwritten proof for Question 32:

$$\begin{aligned} \text{LHS} = \tan\left(\theta - \frac{\pi}{4}\right) &= \frac{\tan\theta - \tan\frac{\pi}{4}}{1 + \tan\theta \tan\frac{\pi}{4}} = \frac{\tan\theta - 1}{1 + \tan\theta} = \frac{\frac{\sin\theta}{\cos\theta} - 1}{\frac{\cos\theta}{\cos\theta} + \frac{\sin\theta}{\cos\theta}} \\ &= \frac{\frac{\sin\theta - \cos\theta}{\cos\theta}}{\frac{\cos\theta + \sin\theta}{\cos\theta}} = \frac{\sin\theta - \cos\theta}{\cos\theta + \sin\theta} = \frac{(\sin\theta - \cos\theta)(\cos\theta - \sin\theta)}{(\cos\theta + \sin\theta)(\cos\theta - \sin\theta)} \\ &= \frac{-(\cos\theta - \sin\theta)^2}{\cos^2\theta - \sin^2\theta} = \frac{-(\cos^2\theta - 2\sin\theta\cos\theta + \sin^2\theta)}{\cos 2\theta} = \frac{-(\cos^2\theta + \sin^2\theta) + 2\sin\theta\cos\theta}{\cos 2\theta} \\ &= \frac{-1 + \sin 2\theta}{\cos 2\theta} = \frac{\sin 2\theta - 1}{\cos 2\theta} = \text{RHS} \end{aligned}$$

**Question 33** (\*\*\*\*+)

It is given that

$$\cos 3x \equiv 4\cos^3 x - 3\cos x.$$

a) Prove the validity of the above trigonometric identity.

b) Hence, or otherwise solve the trigonometric equation

$$2 + \cos 6\theta \sec 2\theta = 0, \quad 0^\circ \leq \theta < 360^\circ.$$

$$\boxed{\phantom{0000}}, \quad \theta = 30^\circ, 60^\circ, 120^\circ, 150^\circ, 210^\circ, 240^\circ, 300^\circ, 330^\circ$$

Handwritten solutions for Question 33:

a)  $\cos 3x = \cos(2x + x)$   
 $= \cos 2x \cos x - \sin 2x \sin x$   
 $= (\cos^2 x - \sin^2 x) \cos x - (2\sin x \cos x) \sin x$   
 $= 2\cos^3 x - \cos x - 2\sin^2 x \cos x$   
 $= 2\cos^3 x - \cos x - 2(1 - \cos^2 x) \cos x$   
 $= 2\cos^3 x - \cos x - 2\cos x + 2\cos^3 x$   
 $= 4\cos^3 x - 3\cos x$   
 As required

b) Proceed as follows  
 $\cos 2\theta \equiv 4\cos^2 \theta - 3\cos \theta$   
 $\cos 6\theta = \cos(3 \times 2\theta) = 4\cos^3 2\theta - 3\cos 2\theta$   
 Rearranging the equation  
 $\Rightarrow 2 + \cos 6\theta \sec 2\theta = 0$   
 $\Rightarrow 2 + [4\cos^3 2\theta - 3\cos 2\theta] \sec 2\theta = 0$   
 $\Rightarrow 2 + 4\cos^3 2\theta \sec 2\theta - 3\cos 2\theta \sec 2\theta = 0$   
 $\Rightarrow 2 + 4\cos^2 2\theta - 3 = 0$

$\Rightarrow 4\cos^2 2\theta = 1$   
 $\Rightarrow 4\left(\frac{1}{2} + \frac{1}{2}\cos 4\theta\right) = 1$   
 $\Rightarrow 2 + 2\cos 4\theta = 1$   
 $\Rightarrow 2\cos 4\theta = -1$   
 $\Rightarrow \cos 4\theta = -\frac{1}{2}$   
 $\text{Hence } \cos^{-1}\left(-\frac{1}{2}\right) = 120^\circ$   
 $\Rightarrow \begin{matrix} 4\theta = 120^\circ \pm 360^\circ \\ 4\theta = 240^\circ \pm 360^\circ \end{matrix} \quad \theta = 30^\circ, 60^\circ, \dots$   
 $\Rightarrow \begin{matrix} \theta = 30^\circ \pm 90^\circ \\ \theta = 60^\circ \pm 90^\circ \end{matrix}$   
 $\Rightarrow \theta = 30^\circ, 60^\circ, 120^\circ, 150^\circ, 210^\circ, 240^\circ, 300^\circ, 330^\circ$

Alternative from  $4\cos^2 2\theta = 1$   
 $\cos 2\theta = \pm \frac{1}{2}$   
 And solve from there

## Question 34 (\*\*\*\*+)

It is given that

$$(\cos x + \sin x)(1 - \sin x \cos x) \equiv \cos^3 x + \sin^3 x.$$

- a) Prove the validity of the above trigonometric identity.
- b) Hence find, in terms of  $\pi$ , the solutions of the trigonometric equation

$$\frac{\cos^3 x + \sin^3 x}{\cos x + \sin x} = \frac{3}{4}, \quad 0 \leq x < \pi,$$

giving the answers in terms of  $\pi$ .

$$x = \frac{\pi}{12}, \frac{5\pi}{12}$$

(a) 
$$\begin{aligned} \text{L.H.S.} &= (\cos x + \sin x)(1 - \sin x \cos x) \\ &= (\cos x + \sin x)(\cos^2 x + \sin^2 x - \sin x \cos x) \\ &= \cos^3 x + \sin^3 x - \sin x \cos^2 x - \sin^2 x \cos x \\ &= \cos^3 x + \sin^3 x - \sin x \cos^2 x - \sin^2 x \cos x \\ &= \cos^3 x + \sin^3 x \\ &= \text{R.H.S.} \end{aligned}$$

(b) 
$$\frac{\cos^3 x + \sin^3 x}{\cos x + \sin x} = \frac{3}{4}$$
  

$$\Rightarrow \frac{(\cos x + \sin x)(1 - \sin x \cos x)}{\cos x + \sin x} = \frac{3}{4}$$
  

$$\Rightarrow 1 - \sin x \cos x = \frac{3}{4}$$
  

$$\Rightarrow \frac{1}{4} = \sin x \cos x$$
  

$$\Rightarrow \frac{1}{4} = \frac{1}{2} \sin 2x$$
  

$$\Rightarrow \frac{1}{2} = \sin 2x$$

$\arcsin\left(\frac{1}{2}\right) = \frac{\pi}{6}$   
 $2x = \frac{\pi}{6} + 2n\pi$   
 $x = \frac{\pi}{12} + n\pi$   
 $2x = \frac{5\pi}{6} + 2n\pi$   
 $x = \frac{5\pi}{12} + n\pi$

## Question 35 (\*\*\*\*+)

It is given that

$$\cos(x+36)^\circ = \sin(x-54)^\circ.$$

- a) Show clearly without a calculating aid that the above trigonometric equation is equivalent to

$$\tan x^\circ = \tan 54^\circ.$$

- b) Hence solve the trigonometric equation

$$\cos(3y+36)^\circ = \sin(3y-54)^\circ, \quad 0 \leq y < 180.$$

$$y = 18, 78, 138$$

(a)  $\cos(x+36) = \sin(x-54)$   
 $\Rightarrow \cos x \cos 36 - \sin x \sin 36 = \sin x \cos 54 - \cos x \sin 54$   
 $\Rightarrow \cos x \cos 36 + \cos x \sin 54 = \sin x \cos 54 + \sin x \sin 36$   
 $\Rightarrow \cos x (\cos 36 + \sin 54) = \sin x (\cos 54 + \sin 36)$   
 $\Rightarrow \cos x = \sin x$  (dividing both sides by  $\cos 36 + \sin 54$ )  
 $\Rightarrow \tan x = 1$   
 $\Rightarrow x = 45^\circ$

(b)  $\tan 3y = \tan 54$   
 $3y = 54 + 180n \quad n = 0, 1, 2, \dots$   
 $y = 18 + 60n$   
 $\therefore y = 18, 78, 138$

**Question 36** (\*\*\*\*+)

Solve the trigonometric equation

$$\sin 3x = \cos 2x + \sin x, \quad \text{for } 0 \leq \theta < \pi,$$

giving the answers in terms of  $\pi$ .

$$x = \frac{\pi}{6}, \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{6}$$

$\sin 3\alpha = \cos 2\alpha + \sin \alpha$

$\Rightarrow \sin 3\alpha - \sin \alpha = \cos 2\alpha$

$\Rightarrow \sin 4\alpha - \sin 8\alpha = 2\sin \frac{4\alpha - 8\alpha}{2} \cos \frac{4\alpha + 8\alpha}{2}$

$\Rightarrow 2\cos 2\alpha \sin 2\alpha = \cos 2\alpha$

$\Rightarrow 2\cos 2\alpha \sin 2\alpha - \cos 2\alpha = 0$

$\Rightarrow \cos 2\alpha (2\sin 2\alpha - 1) = 0$

$\cos 2\alpha = 0 \quad \text{or} \quad \sin 2\alpha = \frac{1}{2}$

$2\alpha = \frac{\pi}{2} + 2n\pi$   
 $2\alpha = \frac{3\pi}{2} + 2n\pi$

$\alpha = \frac{\pi}{4} + n\pi$   
 $\alpha = \frac{3\pi}{4} + n\pi$

hence,  $\alpha = \frac{\pi}{4}, \frac{3\pi}{4}, \dots$

ALTERNATIVE

$\sin 3\alpha = \sin 2\alpha + \sin \alpha$

Then

$\Rightarrow \sin 3\alpha = \cos 2\alpha + \sin \alpha$

$\Rightarrow \sin \alpha - \sin 2\alpha = (-2\sin \frac{\alpha}{2} \cos \frac{3\alpha}{2})$

$\Rightarrow 0 = 4\sin \frac{\alpha}{2} \cdot 2\sin \frac{3\alpha}{2} - 2\sin \alpha + 1$

$\Rightarrow 0 = 4\sin^2 \frac{\alpha}{2} - 2\sin \frac{\alpha}{2} + 1$

$\Rightarrow 0 = (2\sin \frac{\alpha}{2} - 1)(2\sin \frac{\alpha}{2} + 1)$

$\Rightarrow y = \frac{1}{2}$   
 $y = \frac{1}{2}$

$\sin \alpha = \frac{1}{2}$

etc etc

**Question 37** (\*\*\*\*+)

Solve the trigonometric equation

$$2\sin 2x \tan x + 4\sec 2x + 5 = 0, \quad 0^\circ \leq x < 360^\circ.$$

$x = 60^\circ, 120^\circ, 240^\circ, 300^\circ$

$2\sin 2\alpha \cos 2\alpha + 4\sin^2 \alpha = 5 = 0$   
 $2(\sin 2\alpha \cos 2\alpha) + \frac{4}{\cos^2 2\alpha} + 5 = 0$   
 $4\sin^2 \alpha + \frac{4}{1-2\sin^2 \alpha} + 5 = 0$   
 $4\sin^2 \alpha (1-2\sin^2 \alpha) + 4 + 5(1-2\sin^2 \alpha) = 0$   
 $4\sin^2 \alpha - 8\sin^4 \alpha + 4 + 5 - 10\sin^2 \alpha = 0$   
 $0 = 8\sin^4 \alpha + 6\sin^2 \alpha - 9$   
 $(4\sin^2 \alpha - 3)(2\sin^2 \alpha + 3) = 0$   
 $\sin^2 \alpha = \begin{cases} \frac{3}{4} \\ \frac{3}{2} \end{cases} \quad \sin \alpha = \begin{cases} \pm \frac{\sqrt{3}}{2} \\ \pm \frac{\sqrt{6}}{2} \end{cases}$   
 $\begin{cases} \alpha = 60^\circ \pm 360^\circ \\ \alpha = 120^\circ \pm 360^\circ \end{cases} \quad \text{or } \begin{cases} \alpha = -60^\circ \pm 360^\circ \\ \alpha = 240^\circ \pm 360^\circ \end{cases}$   
 $\rightarrow \alpha = 60^\circ, 120^\circ, 300^\circ, 240^\circ$

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ACTIVITY 14.11 Use Little's Theorem  
 If  $\tan \alpha = t \quad \sin 2\alpha = \frac{2t}{1+t^2} \quad \& \quad \cos 2\alpha = \frac{1-t^2}{1+t^2}$   
 So  
 $\Rightarrow 2\sin 2\alpha \cos 2\alpha + \frac{4}{\cos^2 2\alpha} + 5 = 0$   
 $\Rightarrow 2\left(\frac{2t}{1+t^2}\right) + 4\left(\frac{1+t^2}{1-t^2}\right) + 5 = 0$   
 $\Rightarrow \frac{4t}{1+t^2} + \frac{4(1+t^2)}{1-t^2} + 5 = 0$   
 $\Rightarrow 4(t^2 - t^4) + 4(1+t^4) + 5(1-t^2-t^4) = 0$   
 $\Rightarrow 4t^4 - 4t^2 + 4 + 4t^4 + 4t^4 - 5t^4 - 5t^2 + 5 = 0$   
 $\Rightarrow 0 = 5t^4 - 12t^2 + 9 = 0$

$\Rightarrow (t^2 + 3)(t^2 - 3) = 0$   
 $t^2 = \begin{cases} 3 \\ -3 \end{cases}$   
 $\tan \alpha = \pm \sqrt{3}$   
 $2\alpha = 60^\circ, 180^\circ \quad \text{or } 120^\circ, 300^\circ$   
 $2\alpha = 60^\circ, 240^\circ, 300^\circ, 120^\circ$

## Question 38 (\*\*\*\*+)

It is given that

$$\tan \theta + \tan \varphi = 3,$$

$$\sin^2 x + 2 \sin x + \sin(\theta + \varphi) = 3 \cos \theta \cos \varphi - 1,$$

for  $x \in \mathbb{R}$ ,  $\theta \in \mathbb{R}$ ,  $\varphi \in \mathbb{R}$ .

Show that the above relationships imply that

$$\sin x = -1.$$

□, proof

STARTING FROM THE SECOND EQUATION

$$\begin{aligned} \Rightarrow \frac{\sin \theta}{\cos \theta} + \frac{\sin \varphi}{\cos \varphi} &= 3 \\ \Rightarrow \frac{\sin \theta}{\cos \theta} + \frac{\sin \varphi}{\cos \varphi} &= 3 \\ \Rightarrow \frac{\sin \theta \cos \varphi + \sin \varphi \cos \theta}{\cos \theta \cos \varphi} &= 3 \\ \Rightarrow \frac{\sin(\theta + \varphi)}{\cos \theta \cos \varphi} &= 3 \\ \Rightarrow \sin(\theta + \varphi) &= 3 \cos \theta \cos \varphi \end{aligned}$$

NOW THE FIRST EQUATION SIMPLIFIED

$$\begin{aligned} \Rightarrow \sin^2 x + 2 \sin x + \sin(\theta + \varphi) &= 3 \cos \theta \cos \varphi - 1 \\ \Rightarrow \sin^2 x + 2 \sin x &= -1 \\ \Rightarrow \sin^2 x + 2 \sin x + 1 &= 0 \\ \Rightarrow (\sin x + 1)^2 &= 0 \\ \Rightarrow \sin x &= -1 \end{aligned}$$

As required

## Question 39 (\*\*\*\*+)

A geometric progression has first term  $\sin \theta$  and common ratio  $\cos \theta$ .

- a) Given the value of  $\theta$  is such so that the progression converges, show that its sum to infinity is  $\cot \frac{\theta}{2}$ .

A different geometric progression has first term  $\cos \theta$  and common ratio  $\sin \theta$ .

- b) Given the value of  $\theta$  is such so that this progression also converges, show that its sum to infinity is  $\sec \theta + \tan \theta$ .

,  proof

The image shows a handwritten proof on grid paper. Part (a) uses the standard formula for the sum to infinity of a geometric series,  $S_{\infty} = \frac{a}{1-r}$ , where  $a = \sin \theta$  and  $r = \cos \theta$ . It then uses the double-angle identity  $\sin \theta = 2 \sin \frac{\theta}{2} \cos \frac{\theta}{2}$  and  $\cos \theta = 1 - 2 \sin^2 \frac{\theta}{2}$  to simplify the expression to  $\cot \frac{\theta}{2}$ . Part (b) uses the sum formula with  $a = \cos \theta$  and  $r = \sin \theta$ . It uses the identity  $\cos \theta = \cos^2 \frac{\theta}{2} - \sin^2 \frac{\theta}{2}$  and  $\sin \theta = 2 \sin \frac{\theta}{2} \cos \frac{\theta}{2}$  to simplify the expression to  $\sec \theta + \tan \theta$ .

a) USE THE STANDARD FORMULA FOR THE SUM TO INFINITY

$$S_{\infty} = \frac{a}{1-r} \quad -1 < r < 1$$

$$S_{\infty} = \frac{\sin \theta}{1 - \cos \theta} = \frac{2 \sin \frac{\theta}{2} \cos \frac{\theta}{2}}{1 - (1 - 2 \sin^2 \frac{\theta}{2})} = \frac{2 \sin \frac{\theta}{2} \cos \frac{\theta}{2}}{2 \sin^2 \frac{\theta}{2}} = \frac{\cos \frac{\theta}{2}}{\sin \frac{\theta}{2}} = \cot \frac{\theta}{2}$$

b) USE THE SUM FORMULA WITH  $\sin \theta$  &  $\cos \theta$  "TRIGONOMETRIC"

$$S_{\infty} = \frac{\cos \theta}{1 - \sin \theta} = \frac{\cos \theta (1 + \sin \theta)}{(1 - \sin \theta)(1 + \sin \theta)} = \frac{\cos \theta (1 + \sin \theta)}{1 - \sin^2 \theta}$$

$$= \frac{\cos \theta (1 + \sin \theta)}{\cos^2 \theta} = \frac{1 + \sin \theta}{\cos \theta} = \frac{1}{\cos \theta} + \frac{\sin \theta}{\cos \theta}$$

$$= \sec \theta + \tan \theta$$



**Question 40** (\*\*\*\*+)

Is given that

- $\cos^2 x + \sin^2 x \equiv 1.$
- $\operatorname{cosec} 15^\circ = \sqrt{6} + \sqrt{2}.$

Use these facts **only** to show that

- a)  $1 + \cot^2 x \equiv \operatorname{cosec}^2 x.$
- b)  $\cot 15^\circ = 2 + \sqrt{3}.$

proof

Handwritten proof for Question 40b:

a)  $\cos^2 x + \sin^2 x = 1$   
 $\Rightarrow \frac{\cos^2 x}{\sin^2 x} + \frac{\sin^2 x}{\sin^2 x} = \frac{1}{\sin^2 x}$   
 $\Rightarrow \cot^2 x + 1 = \operatorname{cosec}^2 x$

b) Let  $2 = 15^\circ$   
 $\Rightarrow \cot^2 15^\circ + 1 = \operatorname{cosec}^2 15^\circ$   
 $\Rightarrow \cot^2 15^\circ + 1 = (\sqrt{6} + \sqrt{2})^2$   
 $\Rightarrow \cot^2 15^\circ + 1 = 6 + 2\sqrt{12} + 2$   
 $\Rightarrow \cot^2 15^\circ = 7 + 4\sqrt{3}$   
 $\Rightarrow \cot 15^\circ = \sqrt{7 + 4\sqrt{3}}$   
 But  $15^\circ$  is acute, so  
 $\Rightarrow \cot 15^\circ = \sqrt{7 + 4\sqrt{3}}$   
 $\Rightarrow \cot 15^\circ = \sqrt{2^2 + 2 \times 2\sqrt{3} + (\sqrt{3})^2}$   
 $(a+b)^2 = a^2 + 2ab + b^2$   
 $\Rightarrow \cot 15^\circ = \sqrt{(2 + \sqrt{3})^2}$   
 $\Rightarrow \cot 15^\circ = 2 + \sqrt{3}$   
 As Required

**Question 41** (\*\*\*\*+)

Prove the validity of each of the following trigonometric identities.

- a)  $\sqrt{1 + \sin 2\theta} \equiv \cos \theta + \sin \theta.$
- b)  $8 \cos^4 \left( \frac{1}{2} \theta \right) = \cos 2\theta + 4 \cos \theta + 3.$

proof

Handwritten proof for Question 41a:

a) LHS =  $\sqrt{1 + \sin 2\theta} = \sqrt{\cos^2 \theta + \sin^2 \theta + 2 \sin \theta \cos \theta}$   
 $= \sqrt{(\cos \theta + \sin \theta)^2} = \cos \theta + \sin \theta = \text{RHS}$

b) Note:  $\cos 2A = 2 \cos^2 A - 1$  Hence  $\cos^2 \frac{\theta}{2} = \frac{1}{2} (1 + \cos \theta)$   
 $2 \cos^2 A = 1 + \cos 2A$  Hence  $\cos^2 \theta = \frac{1}{2} (1 + \cos 2\theta)$   
 $\cos^2 A = \frac{1}{2} (1 + \cos 2A)$

LHS =  $8 \cos^4 \left( \frac{\theta}{2} \right) = 8 \left( \cos^2 \frac{\theta}{2} \right)^2 = 8 \left( \frac{1}{2} (1 + \cos \theta) \right)^2$   
 $= 8 \times \left( \frac{1}{4} \right) (1 + \cos \theta)^2 = 2 (1 + \cos \theta)^2 = 2 (1 + 2 \cos \theta + \cos^2 \theta)$   
 $= 2 + 4 \cos \theta + 2 \cos^2 \theta = 2 + 4 \cos \theta + 2 \left( \frac{1}{2} (1 + \cos 2\theta) \right)$   
 $= 2 + 4 \cos \theta + 1 + \cos 2\theta = \cos 2\theta + 4 \cos \theta + 3 = \text{RHS}$

**Question 42** (\*\*\*\*+)

A relationship between  $x$  and  $y$  is given by the equations

$$x = \sin 2\theta, \quad 0 < \theta < \pi,$$

$$y = \cot \theta, \quad 0 < \theta < \pi.$$

Use trigonometric identities to show that

$$y(2-xy)=x.$$

proof

$x = \sin 2\theta$   
 $y = \cos 2\theta$

$\Rightarrow x = 2\sin\theta\cos\theta$   
 $x^2 = 4\sin\theta\cos\theta$   
 $x^2 = 4\sin\theta(1-\sin^2\theta)$

$\text{now } y^2 = \cos^2\theta = \cos^2\theta - 1$   
 $y^2 + 1 = \cos^2\theta$   
 $\sin^2\theta = \frac{y^2 + 1}{4}$

$\text{Thus } x^2 = 4 \times \frac{1}{y^2 + 1} \times \left(1 - \frac{y^2 + 1}{4}\right)$   
 $\Rightarrow x^2 = \frac{4}{y^2 + 1} \times \frac{y^2 + 1 - y^2 - 1}{4}$   
 $\Rightarrow x^2 = \frac{dy^2}{(y^2 + 1)^2}$   
 $\Rightarrow x = \frac{2y}{y^2 + 1}$

$\text{or } x^2 + y^2 = 2y$   
 $\Rightarrow 2y = 2y - 2y^2$   
 $= 2y(2 - y)$   
 $\therefore y(2 - y) = 2$

**Question 43** (\*\*\*\*+)

Show clearly that

$$\arctan x + \arctan \left( \frac{1-x}{1+x} \right) = \frac{\pi}{4}.$$

**proof**

$$\begin{aligned} \text{Let } \arctan x &+ \arctan \left( \frac{-x}{1-x^2} \right) = \psi \\ \text{if } x=1 \\ \tan \left[ \arctan 1 + \arctan \left( \frac{-1}{1-1^2} \right) \right] &= \frac{\tan(\arctan 1) + \tan(\arctan \frac{-1}{1-1^2})}{1 - \tan(\arctan 1) \tan(\arctan \frac{-1}{1-1^2})} \\ &= \frac{1 + \frac{-1}{1-1^2}}{1 - 1 \cdot \frac{-1}{1-1^2}} = \dots \text{undefined} / \text{Arctan by } (1/x) \dots \\ &= \frac{x(1+x) + (-1-x)}{(1+x) - x(1-x)} = \frac{x(1+x) + (-1-x)}{1+x-x+x^2} = \frac{x^2-1}{x^2+1} = 1 \\ \therefore \theta + \phi &= \psi \\ \Rightarrow \tan(\theta + \phi) &= \tan \psi \\ \Rightarrow 1 &= \tan \psi \\ \Rightarrow \psi &= \frac{\pi}{4} \end{aligned}$$

**Question 44** (\*\*\*\*+)

It is given that

$$\frac{2 \tan x}{\tan x + \sin x} \equiv \sec^2 \left( \frac{x}{2} \right), \quad x \neq n\pi, n \in \mathbb{Z}.$$

- a) Prove the validity of the above trigonometric identity.
- b) Hence solve the trigonometric equation

$$\frac{2 \tan x}{\tan x + \sin x} = 4, \quad 0 \leq x < 360^\circ.$$

$$x = 120^\circ, 240^\circ$$

(a) LHS =  $\frac{2 \tan x}{\tan x + \sin x} = \frac{2 \frac{\sin x}{\cos x}}{\frac{\sin x}{\cos x} + \sin x} = \frac{2 \sin x}{\sin x + \sin x \cos x} = \frac{2}{1 + \cos x} = \frac{2}{1 + (2 \cos^2 \frac{x}{2} - 1)} = \frac{2}{2 \cos^2 \frac{x}{2}} = \frac{1}{\cos^2 \frac{x}{2}} = \sec^2 \left( \frac{x}{2} \right) = \text{RHS}$

(b)  $\frac{2 \tan x}{\tan x + \sin x} = 4$   
 $\sec^2 \frac{x}{2} = 4$   
 $\sec \frac{x}{2} = \pm 2$   
 $\cos \frac{x}{2} = \pm \frac{1}{2}$   
 $\frac{x}{2} = 60^\circ \text{ or } 300^\circ$   
 $\frac{x}{2} = 120^\circ \text{ or } 240^\circ$   
 $x = 240^\circ \text{ or } 480^\circ$   
 $x = 120^\circ, 240^\circ$

**Question 45** (\*\*\*\*+)

Prove the validity of the following trigonometric identity

$$\frac{\sin x}{1 + \tan x} \equiv \frac{\cos x}{1 + \cot x}.$$

proof

LHS =  $\frac{\sin x}{1 + \tan x} = \frac{\sin x}{1 + \frac{\sin x}{\cos x}} = \frac{\sin x \cos x}{\cos x + \sin x} = \frac{\sin x \cos x}{\cos x + \sin x}$

RHS =  $\frac{\cos x}{1 + \cot x} = \frac{\cos x}{1 + \frac{\cos x}{\sin x}} = \frac{\cos x \sin x}{\sin x + \cos x} = \frac{\sin x \cos x}{\cos x + \sin x}$

LHS = RHS

**Question 46** (\*\*\*\*+)

Let  $t = \tan \frac{x}{2}$ .

a) Show clearly that ...

i. ...  $\sin x = \frac{2t}{1+t^2}$ ,

ii. ...  $\cos x = \frac{1-t^2}{1+t^2}$ .

b) Use these results to solve the trigonometric equation

$$5 \sin x + 4 \cos x = 3, \quad 0^\circ \leq x < 360^\circ.$$

$$x \approx 113.4^\circ, 349.3^\circ$$

$$\sin x = \frac{2 \tan \frac{x}{2}}{1 + \tan^2 \frac{x}{2}} = \frac{2 \tan \frac{x}{2}}{\sec^2 \frac{x}{2}} = 2 \tan \frac{x}{2} \cos^2 \frac{x}{2} = \left[ \frac{\sin \frac{x}{2}}{\cos \frac{x}{2}} \right] \cos^2 \frac{x}{2}$$

$$= 2 \sin \frac{x}{2} \cos \frac{x}{2} = \sin x$$

$$\bullet \frac{1-t}{1+t} = \frac{1 - \tan^2 \frac{x}{2}}{1 + \tan^2 \frac{x}{2}} = \frac{1 - \tan^2 \frac{x}{2}}{\sec^2 \frac{x}{2}} = [1 - \tan^2 \frac{x}{2}] \cos^2 \frac{x}{2}$$

$$= \cos^2 \frac{x}{2} - \tan^2 \frac{x}{2} \cos^2 \frac{x}{2} = \cos^2 \frac{x}{2} - \frac{\sin^2 \frac{x}{2}}{\cos^2 \frac{x}{2}} \cos^2 \frac{x}{2}$$

$$= \cos(2 \times \frac{x}{2}) = \cos x$$

$$\textcircled{b} \quad 5 \sin x + 4 \cos x = 3$$

$$\Rightarrow 5 \left( \frac{2t}{1+t^2} \right) + 4 \left( \frac{1-t^2}{1+t^2} \right) = 3$$

$$\Rightarrow 10t + 4(1-t^2) = 3(1+t^2)$$

$$\Rightarrow 10t + 4 - 4t^2 = 3 + 3t^2$$

$$\Rightarrow 0 = 7t^2 - 10t - 1$$

$$20 \pm \sqrt{100 + 28} = 192$$

$$t = \frac{10 \pm \sqrt{128}}{14} = \frac{10 \pm 11.31}{14}$$

$$\left. \begin{aligned} \arctan(1.122) &= 50.7^\circ \\ \arctan(-0.0714) &= -5^\circ \end{aligned} \right\} \begin{aligned} \frac{x}{2} &= 50.7^\circ \pm 180^\circ \\ \frac{x}{2} &= -5^\circ \pm 180^\circ \end{aligned} \quad \begin{aligned} x &= 101.4^\circ, \dots \\ x &= -10^\circ \pm 360^\circ \end{aligned}$$

$$\therefore x = 113.4^\circ, 349.3^\circ$$

## Question 47 (\*\*\*\*+)

It is given that for  $\theta \in \mathbb{R}$ ,  $\varphi \in \mathbb{R}$

$$3 \tan \theta = 4 \tan \varphi.$$

Show that the above relationship implies that

$$\tan(\theta - \varphi) = \frac{\sin 2\theta}{7 + \cos 2\theta}.$$

, proof

START BY APPLYING THE COMPOUND ANGLE IDENTITY FOR  $\tan(\theta - \varphi)$

$$\Rightarrow \tan(\theta - \varphi) = \frac{\tan \theta - \tan \varphi}{1 + \tan \theta \tan \varphi}$$

$$= \frac{4 \tan \theta - 4 \tan \varphi}{4 + 4 \tan \theta \tan \varphi}$$

BUT  $4 \tan \theta = 3 \tan \varphi$

$$= \frac{4 \tan \theta - (3 \tan \theta)}{4 + \tan \theta (3 \tan \theta)}$$

$$= \frac{4 \tan \theta - 3 \tan \theta}{4 + 3 \tan^2 \theta}$$

$$= \frac{\tan \theta}{4 + 3 \tan^2 \theta}$$

$$= \frac{\frac{\sin \theta}{\cos \theta}}{4 + \frac{3 \sin^2 \theta}{\cos^2 \theta}}$$

MULTIPLY "TOP & BOTTOM" OF THE FRACTION BY  $\cos^2 \theta$

$$= \frac{\sin \theta \cos \theta}{4 \cos^2 \theta + 3 \sin^2 \theta}$$

$$= \frac{2 \sin \theta \cos \theta}{5 \cos^2 \theta + 3 \sin^2 \theta}$$

$$= \frac{\sin 2\theta}{5 \cos^2 \theta + 3 \sin^2 \theta}$$

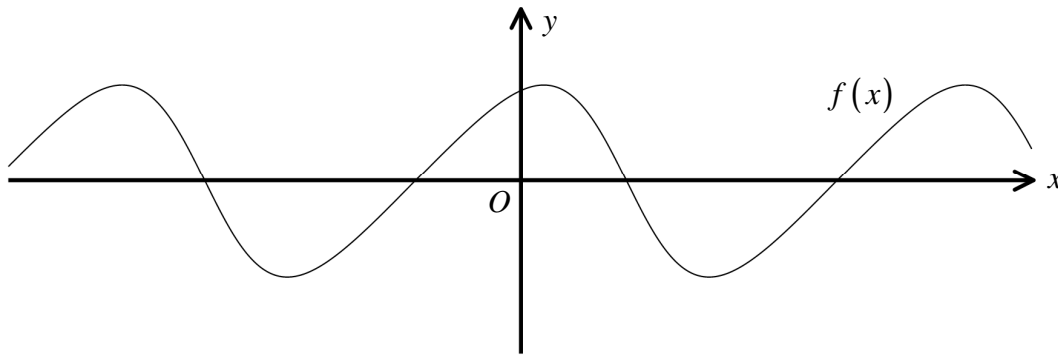
$\cos 2\theta = \cos^2 \theta - \sin^2 \theta$   
 $\cos 2\theta = (-\sin^2 \theta)$

$$= \frac{\sin 2\theta}{4 + 4 \cos 2\theta + 3 - 3 \cos 2\theta}$$

$$= \frac{\sin 2\theta}{7 + \cos 2\theta}$$

As Required

## Question 48 (\*\*\*\*+)



The figure above shows part of the graph of the curve with equation

$$f(x) = \frac{\cos x}{3 - \sin x}, \quad x \in \mathbb{R}.$$

Use differentiation to show that

$$-\frac{1}{4}\sqrt{2} \leq f(x) \leq \frac{1}{4}\sqrt{2}.$$

,  proof

DIFFERENTIATE VIA QUOTIENT RULE & TRY

$$f(x) = \frac{(3 - \sin x)(-\cos x) - (\cos x)(-\cos x)}{(3 - \sin x)^2}$$

$$= \frac{-3\cos x + \sin x \cos x + \cos^2 x}{(3 - \sin x)^2} = \frac{1 - 3\sin x}{(3 - \sin x)^2}$$

SEPARATE THE ZERO

$$1 - 3\sin x = 0$$

$$\sin x = \frac{1}{3} \leftarrow \text{"STATIONARY VALUE"}$$

USE THE SINE + COSINE IDENTITY

$$\Rightarrow \cos^2 x = 1 - \sin^2 x$$

$$\Rightarrow \cos x = \pm \sqrt{1 - \left(\frac{1}{3}\right)^2}$$

$$\Rightarrow \cos x = \pm \sqrt{\frac{8}{9}}$$

$$\Rightarrow \cos x = \pm \frac{2\sqrt{2}}{3}$$

FIND THE CORRESPONDING x WITH  $\cos x = \pm \frac{2\sqrt{2}}{3}$

$$\frac{\cos x}{3 - \sin x} = \frac{\frac{2\sqrt{2}}{3}}{3 - \frac{1}{3}} = \frac{2\sqrt{2}}{9 - 1} = \frac{1}{4}\sqrt{2}$$

$$-\frac{2\sqrt{2}}{3 - \frac{1}{3}} = \frac{-2\sqrt{2}}{9 - 1} = -\frac{1}{4}\sqrt{2}$$

$$\therefore -\frac{1}{4}\sqrt{2} \leq f(x) \leq \frac{1}{4}\sqrt{2}$$

**Question 49** (\*\*\*\*+)

It is given that

$$\tan 3x \equiv \frac{3 \tan x - \tan^3 x}{1 - 3 \tan^2 x}, \quad x \neq \frac{n\pi}{3}, \quad n = 0, 1, 2, 3, \dots$$

- a) Use the above identity to express  $\cot 3x$  in terms of  $\cot x$ .
- b) Show clearly that

$$\frac{\cos 2x - \cos x + 1}{\sin 2x - \sin x} \equiv \cot x, \quad \cos x \neq \frac{1}{2}.$$

- a) Hence, or otherwise, given that  $\cos 3x \neq \frac{1}{2}$  solve the trigonometric equation

$$\cos 6x + \sin 6x - \cos 3x - \sin 3x + 1 = 0,$$

for  $0 < x < \pi$ , giving the answers in terms of  $\pi$ .

$$\boxed{\phantom{000}}, \quad x = \frac{\pi}{4}, \frac{7\pi}{12}, \frac{11\pi}{12}$$

a) Using the sum identity

$$\begin{aligned} \tan 3x &= \frac{\sin 3x}{\cos 3x} = \frac{3 \sin x - 4 \sin^3 x}{1 - 3 \sin^2 x} \\ \frac{1}{\tan 3x} &= \frac{1 - 3 \sin^2 x}{3 \sin x - 4 \sin^3 x} \\ \cot 3x &= \frac{1 - 3 \sin^2 x}{3 \sin x - 4 \sin^3 x} \\ \cot 3x &= \frac{1 - 3 \sin^2 x}{3 \sin x (1 - \sin^2 x)} \\ &= \frac{1 - 3 \sin^2 x}{3 \sin x \cos^2 x} \end{aligned}$$

Wanted for a result in the R.H.S. by cot x

$$\cot 3x = \frac{\cot x - 3 \cot^3 x}{3 \cot^2 x - 1}$$

b) Using the double angle identities for  $\cos 2x$  &  $\sin 2x$

$$\begin{aligned} \text{L.H.S.} &= \frac{\cos 2x - \cos x + 1}{\sin 2x - \sin x} = \frac{(2 \cos^2 x - 1) - \cos x + 1}{2 \sin x \cos x - \sin x} \\ &= \frac{2 \cos^2 x - \cos x}{2 \sin x \cos x - \sin x} = \frac{\cos x (2 \cos x - 1)}{\sin x (2 \cos x - 1)} \quad (\cos x \neq \frac{1}{2}) \\ &= \frac{\cos x}{\sin x} = \cot x = \text{R.H.S.} \end{aligned}$$

As required

c) Using part (b) — Part (a) is not actually needed!

$$\begin{aligned} \Rightarrow \cos 6x + \sin 6x - \cos 3x - \sin 3x + 1 &= 0 \\ \Rightarrow \cos 6x - \cos 3x + 1 &= \sin 3x - \sin 6x \\ \Rightarrow \frac{\cos 6x - \cos 3x + 1}{\sin 6x - \sin 3x} &= -1 \end{aligned}$$

This is the result of part (b) with  $x \mapsto 3x$

$$\Rightarrow \cot 3x = -1$$

$$\Rightarrow \tan 3x = -1$$

$$\arctan(-1) = -\frac{\pi}{4}$$

$$\Rightarrow 3x = -\frac{\pi}{4} + n\pi \quad n = 0, 1, 2, \dots$$

$$\Rightarrow x = -\frac{\pi}{12} + \frac{n\pi}{3}$$

2.  $x = \frac{7\pi}{12}$   
3.  $x = \frac{11\pi}{12}$   
 All solutions are ok

## Question 50 (\*\*\*\*+)

$$f(x) = \frac{\sin 3x}{\cos x} + \frac{\cos 3x}{\sin x}, \quad x \in \mathbb{R}, \quad x \neq \frac{n\pi}{2}, \quad n \in \mathbb{Z}.$$

a) Show clearly that

$$f(x) \equiv 2 \cot 2x.$$

b) Solve the trigonometric equation

$$\frac{1}{4} f(x) + 1 = \tan x, \quad 0 \leq x < 2\pi.$$

$$\boxed{\phantom{0.785}}, \quad \boxed{x = 0.785^\circ, 2.94^\circ, 3.93^\circ, 6.09^\circ}$$

a) AVOID THE TRIPLE-ANGLE FORMULAS !!

$$f(x) = \frac{\sin 3x}{\cos x} + \frac{\cos 3x}{\sin x} = \frac{\sin 3x \sin x + \cos 3x \cos x}{\cos x \sin x}$$

$$\cos(A+B) = \cos A \cos B - \sin A \sin B$$

$$= \frac{\cos(3x-x)}{\cos x \sin x} = \frac{\cos 2x}{\cos x \sin x} = \frac{2 \cos 2x}{2 \cos x \sin x} = \frac{2 \cos 2x}{\sin 2x} = 2 \cot 2x$$

At Equival

b) FINISHED SOLVING THE EQUATION IN  $0 \leq x < 2\pi$

$$\Rightarrow \frac{1}{4} f(x) + 1 = \tan x$$

$$\Rightarrow \frac{1}{4} (2 \cot 2x) + 1 = \tan x$$

$$\Rightarrow \frac{1}{2} \cot 2x + 1 = \tan x$$

$$\Rightarrow \frac{1}{2} \cot 2x + 2 = 2 \tan x$$

$$\Rightarrow \frac{1}{2} \cot 2x = 2 \tan x - 2$$

$$\Rightarrow \frac{1 - \tan^2 x}{1 + \tan^2 x} = 4 \tan x - 4$$

$$\Rightarrow 1 - \tan^2 x + 4 \tan x - 4 = 4 \tan^2 x - 4$$

$$\Rightarrow 0 = 5 \tan^2 x - 4 \tan x + 3$$

$$\Rightarrow (5 \tan x + 3)(\tan x - 1) = 0$$

$$\Rightarrow \tan x = -\frac{3}{5} \quad \text{or} \quad \tan x = 1$$

$$\Rightarrow x = \arctan\left(-\frac{3}{5}\right) + k\pi \quad \text{or} \quad x = \arctan(1) + k\pi$$

$$\Rightarrow x = -0.577^\circ + k\pi \quad \text{or} \quad x = 45^\circ + k\pi$$

$$\Rightarrow x = 0.785^\circ \quad (74^\circ)$$

$$\Rightarrow x = 2.94^\circ \quad (94^\circ)$$

$$\Rightarrow x = 3.93^\circ$$

$$\Rightarrow x = 6.09^\circ$$



**Question 51** (\*\*\*\*+)

Solve the trigonometric equation

$$\sqrt{3}(\sec x - \tan x) = 1, \quad 0 \leq x \leq 2\pi,$$

giving the answers in terms of  $\pi$ .

$$x = \frac{\pi}{6}$$

Handwritten solution for Question 51:

Method 1 (Left):

$$\begin{aligned} \sqrt{3}(\sec x - \tan x) &= 1 \\ \Rightarrow \sec x - \tan x &= \frac{1}{\sqrt{3}} \\ \Rightarrow \frac{1}{\cos x} - \frac{\sin x}{\cos x} &= \frac{1}{\sqrt{3}} \\ \Rightarrow 1 - \sin x &= \frac{\sqrt{3}}{3} \\ \Rightarrow 3 - 3\sin x &= \sqrt{3} \\ \Rightarrow 3 - 3\sin x &= \sqrt{3} \\ \Rightarrow 3\sin x + \sqrt{3} &= 3 \\ \Rightarrow 3\sin x &= 3 - \sqrt{3} \\ \Rightarrow \sin x &= \frac{3 - \sqrt{3}}{3} \end{aligned}$$

Method 2 (Right):

$$\begin{aligned} \sec x - \tan x &= \frac{1}{\tan(x + \frac{\pi}{4})} \\ \Rightarrow \tan(x + \frac{\pi}{4}) &= \sqrt{3} \\ \Rightarrow x + \frac{\pi}{4} &= \frac{\pi}{3} + 2n\pi \\ \Rightarrow x &= \frac{\pi}{12} + 2n\pi \end{aligned}$$

**Question 52** (\*\*\*\*+)

Prove the validity of the following trigonometric identity

$$\cot^2 x - \tan^2 x \equiv 4 \cot 2x \operatorname{cosec} 2x.$$

proof

Handwritten proof for Question 52:

$$\begin{aligned} \text{LHS} &= \cot^2 x - \tan^2 x = \frac{\cos^2 x}{\sin^2 x} - \frac{\sin^2 x}{\cos^2 x} = \frac{\cos^2 x - \sin^2 x}{\sin^2 x \cos^2 x} \\ &= \frac{(\cos^2 x - \sin^2 x)^2}{\frac{1}{4} \times 4 \sin^2 x \cos^2 x} = \frac{(\cos^2 x - \sin^2 x)(\cos^2 x + \sin^2 x)}{\frac{1}{4} (2 \sin x \cos x)^2} \\ &= \frac{\cos 2x}{\frac{1}{4} (2 \sin x \cos x)^2} = \frac{4 \cos 2x}{\sin^2 2x} = \frac{4 \cos 2x}{\sin 2x} \times \frac{1}{\sin 2x} \\ &= 4 \cot 2x \operatorname{cosec} 2x = \text{RHS} \end{aligned}$$

## Question 53 (\*\*\*\*+)

Let  $t = \tan \frac{x}{2}$ .

a) Show clearly that ...

i. ...  $\sin x = \frac{2t}{1+t^2}$ ,

ii. ...  $\cos x = \frac{1-t^2}{1+t^2}$ .

b) Use these results to solve the trigonometric equation

$$2 \sin x + 3 \cos x = 1, \quad 0 \leq x < 2\pi.$$

$$x \approx 1.88^\circ, 5.58^\circ$$

(a)  $t = \tan \frac{x}{2}$

Diagram: A right-angled triangle with angle  $\frac{x}{2}$ . The side opposite the angle is  $t$ , the adjacent side is 1, and the hypotenuse is  $\sqrt{1+t^2}$ .

•  $\sin x = 2 \sin \frac{x}{2} \cos \frac{x}{2}$   
 $= 2 \left( \frac{t}{\sqrt{1+t^2}} \right) \left( \frac{1}{\sqrt{1+t^2}} \right)$   
 $= \frac{2t}{1+t^2}$

•  $\cos x = \cos^2 \frac{x}{2} - \sin^2 \frac{x}{2}$   
 $= \left( \frac{1}{\sqrt{1+t^2}} \right)^2 - \left( \frac{t}{\sqrt{1+t^2}} \right)^2$   
 $= \frac{1-t^2}{1+t^2}$

(b)  $2 \sin x + 3 \cos x = 1$   
 $\Rightarrow 2 \left( \frac{2t}{1+t^2} \right) + 3 \left( \frac{1-t^2}{1+t^2} \right) = 1$   
 $\Rightarrow 4t + 3(1-t^2) = 1+t^2$   
 $\Rightarrow 4t + 3 - 3t^2 = 1+t^2$   
 $\Rightarrow 0 = 4t^2 - 4t - 2$   
 $\Rightarrow 0 = t^2 - t - 1$   
 $\Rightarrow$  quadratic formula  
 $t = \frac{1 \pm \sqrt{5}}{2}$

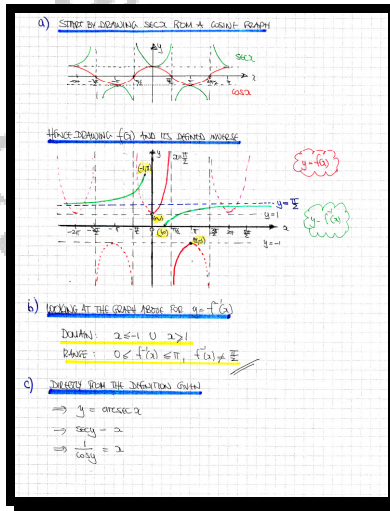
•  $\sin x = 2 \sin \left( \frac{1 \pm \sqrt{5}}{2} \right) \cos \left( \frac{1 \pm \sqrt{5}}{2} \right) = 0.989$   
 $\Rightarrow x = 0.989 \pm \pi$   
 $x = 1.8715 \pm 2\pi$   
 $\therefore x = 1.88^\circ, 5.58^\circ$

**Question 54** (\*\*\*\*+)

$$f(x) = \sec x, \quad 0 \leq x < \frac{\pi}{2} \cup \frac{\pi}{2} < x \leq \pi.$$

- Sketch in the same diagram the graphs of  $f(x)$  and  $f^{-1}(x) = \operatorname{arcsec} x$ .
- State the domain and range of  $f^{-1}(x) = \operatorname{arcsec} x$ .
- Show clearly that  $\operatorname{arcsec} x = \arccos\left(\frac{1}{x}\right)$ .
- Show further that  $\frac{d}{dx}(\operatorname{arcsec} x) = \frac{1}{\sqrt{x^4 - x^2}}$ .

$$\boxed{\mathbb{R}}, \text{ domain: } x \leq -1 \cup x \geq 1, \text{ range: } 0 \leq f^{-1}(x) \leq \pi, f^{-1}(x) \neq \frac{\pi}{2}$$



$\Rightarrow \cos y = \frac{1}{x}$   
 $\Rightarrow y = \arccos\left(\frac{1}{x}\right)$   
 $\therefore \underline{\arcsin 2 \equiv \arccos\left(\frac{1}{2}\right)}$

d) GRAPH THE STRIPPED RESULT & ANSWER OF PART C)  

$$\frac{d}{dx}(\arcsin x) = \frac{d}{dx}(\arccos(x)) = -\frac{1}{\sqrt{1-(\frac{1}{x})^2}} \times \frac{1}{x^2}(\frac{1}{x})$$

$$= -\frac{1}{\sqrt{1-\frac{1}{x^2}}} \times \left(\frac{1}{x^3}\right) = \frac{1}{\sqrt{x^2-1}} \times \frac{1}{x^3}$$

$$= \frac{1}{x\sqrt{x^2-1}} = \frac{1}{\sqrt{x^4-x^2}} = \underline{\frac{1}{\sqrt{x^2(x^2-1)}}}$$

DO BY THE INVERSE RULE

$\Rightarrow y = \arcsin x$   
 $\Rightarrow \sin y = x$   
 $\Rightarrow x = \sin y$   
 $\Rightarrow \frac{dx}{dy} = \sin y \cdot \tan y$   
 $\Rightarrow \left(\frac{dx}{dy}\right)^2 = \sin^2 y \cdot \tan^2 y$   
 $\Rightarrow \left(\frac{dx}{dy}\right)^2 = \sin^2 y (\sec^2 y - 1)$   
 $\Rightarrow \left(\frac{dx}{dy}\right)^2 = \sin^2 y - \sin^4 y$

$\Rightarrow \left(\frac{dy}{dx}\right)^2 = \frac{1}{\sin^2 y - \sin^4 y} = \frac{1}{\sin^2 y (1 - \sin^2 y)}$   
 $\Rightarrow \frac{dy}{dx} = \pm \frac{1}{\sin y \sqrt{1 - \sin^2 y}}$   
 $\Rightarrow \frac{dy}{dx} = \frac{1}{\sin y \cos y}$

PROBLEM OCCURRED IN THE  
 SAME EXAM (2004)

## Question 55 (\*\*\*\*+)

It is given that

$$\sin 3\theta \equiv 3\sin \theta - 4\sin^3 \theta.$$

- a) Prove the validity of the above trigonometric identity.
- b) By differentiating both sides of the above identity with respect to  $\theta$ , show that

$$\cos 3\theta \equiv 4\cos^3 \theta - 3\cos \theta.$$

- c) Hence show that

$$\tan 3\theta \equiv \frac{3\tan \theta \sec^2 \theta - 4\tan^3 \theta}{4 - 3\sec^2 \theta}.$$

- d) Deduce that

$$\tan 3\theta \equiv \frac{3\tan \theta - \tan^3 \theta}{1 - 3\tan^2 \theta}.$$

, proof

**a) STRONG AND THE L.H.S**

LHS =  $\sin 3\theta$   
 $= \sin(2\theta + \theta)$   
 $= \sin 2\theta \cos \theta + \cos 2\theta \sin \theta$   
 $= (2\sin \theta \cos \theta) \cos \theta + (1 - 2\sin^2 \theta) \sin \theta$   
 $= 2\sin \theta \cos^2 \theta + \sin \theta - 2\sin^3 \theta$   
 $= 2\sin \theta (1 - \sin^2 \theta) + \sin \theta - 2\sin^3 \theta$   
 $= 2\sin \theta - 2\sin^3 \theta + \sin \theta - 2\sin^3 \theta$   
 $= 3\sin \theta - 4\sin^3 \theta$   
 $= \text{RHS}$

**b) DIFFERENTIATING THE IDENTITY W.R.T  $\theta$**

$\frac{d}{d\theta} [\sin 3\theta] = \frac{d}{d\theta} [3\sin \theta - 4\sin^3 \theta]$   
 $3\cos 3\theta = 3\cos \theta - 12\sin^2 \theta \times \cos \theta$   
 $\cos 3\theta = \cos \theta - 4\cos \theta \sin^2 \theta$   
 $\cos 3\theta = \cos \theta - 4\cos \theta (1 - \cos^2 \theta)$   
 $\cos 3\theta = \cos \theta - 4\cos \theta + 4\cos^3 \theta$   
 $\cos 3\theta = 4\cos^3 \theta - 3\cos \theta$

**c) PROCEED AS BEFORE**

$\tan 3\theta = \frac{\sin 3\theta}{\cos 3\theta} = \frac{3\sin \theta - 4\sin^3 \theta}{4\cos^3 \theta - 3\cos \theta} = \frac{\frac{3\sin \theta}{\cos \theta} - \frac{4\sin^3 \theta}{\cos^3 \theta}}{\frac{4\cos^3 \theta}{\cos \theta} - \frac{3\cos \theta}{\cos \theta}}$   
 $= \frac{\frac{3\sin \theta}{\cos \theta} - \frac{4\sin^3 \theta}{\cos^3 \theta}}{\frac{4\cos^3 \theta}{\cos \theta} - \frac{3\cos \theta}{\cos \theta}} = \frac{3\tan \theta \sec^2 \theta - 4\tan^3 \theta}{4 - 3\sec^2 \theta}$

**d) USING THE IDENTITY  $1 + \tan^2 \theta = \sec^2 \theta$  WE HAVE**

$\tan 3\theta = \frac{3\tan \theta \sec^2 \theta - 4\tan^3 \theta}{4 - 3\sec^2 \theta} = \frac{3\tan \theta (1 + \tan^2 \theta) - 4\tan^3 \theta}{4 - 3(1 + \tan^2 \theta)}$   
 $= \frac{3\tan \theta + 3\tan^3 \theta - 4\tan^3 \theta}{4 - 3 - 3\tan^2 \theta} = \frac{3\tan \theta - \tan^3 \theta}{1 - 3\tan^2 \theta}$

**Question 56** (\*\*\*\*+)

Prove the validity of the trigonometric identity

$$\frac{1 + \sin \theta}{1 - \sin \theta} \equiv (\sec \theta + \tan \theta)^2.$$

proof

$$\begin{aligned} \text{LHS} &= \frac{1 + \sin \theta}{1 - \sin \theta} = \frac{(1 + \sin \theta)(1 + \sin \theta)}{(1 - \sin \theta)(1 + \sin \theta)} = \frac{1 + 2\sin \theta + \sin^2 \theta}{1 - \sin^2 \theta} \\ &= \frac{1 + 2\sin \theta + \sin^2 \theta}{\cos^2 \theta} = \frac{1}{\cos^2 \theta} + \frac{2\sin \theta}{\cos^2 \theta} + \frac{\sin^2 \theta}{\cos^2 \theta} \\ &= \sec^2 \theta + 2 \frac{\sin \theta}{\cos^2 \theta} + \tan^2 \theta = \sec^2 \theta + 2 \tan \theta \sec \theta + \tan^2 \theta \\ &= (\sec \theta + \tan \theta)^2 = \text{RHS} \end{aligned}$$

**Question 57** (\*\*\*\*+)

Solve the following trigonometric equation.

$$\arctan\left(\frac{x-5}{x-1}\right) + \arctan\left(\frac{x-4}{x-3}\right) = \frac{\pi}{4}, \quad x \in \mathbb{R}.$$

$$x=3 \cup x=6$$

$$\begin{aligned} \text{Let } \theta &= \arctan\left(\frac{x-5}{x-1}\right) \quad \text{and} \quad \phi = \arctan\left(\frac{x-4}{x-3}\right) \\ \Rightarrow \theta + \phi &= \frac{\pi}{4} \\ \Rightarrow \tan(\theta + \phi) &= \tan \frac{\pi}{4} \\ \Rightarrow \frac{\tan \theta + \tan \phi}{1 - \tan \theta \tan \phi} &= 1 \\ \Rightarrow \frac{\frac{x-5}{x-1} + \frac{x-4}{x-3}}{1 - \frac{x-5}{x-1} \cdot \frac{x-4}{x-3}} &= 1 \\ \Rightarrow \frac{\frac{x-5}{x-1} + \frac{x-4}{x-3}}{\frac{x-1}{x-1} - \frac{x-4}{x-3}} &= 1 - \frac{(x-5)(x-4)}{(x-1)(x-3)} \\ \text{Multiply through by } (x-1)(x-3) & \\ (x-5)(x-3) + (x-4)(x-1) &= (x-1)(x-3) - (x-5)(x-4) \\ x^2 - 8x + 15 + x^2 - 5x + 4 &= x^2 - 4x + 3 - (x^2 - 9x + 20) \\ 2x^2 - 13x + 19 &= 5x - 17 \\ 2x^2 - 18x + 36 &= 0 \\ x^2 - 9x + 18 &= 0 \\ (x-3)(x-6) &= 0 \\ x &= \frac{3}{2} \quad \text{or} \quad 6 \end{aligned}$$

Both are valid

$\arctan\left(\frac{3-5}{3-1}\right) + \arctan\left(\frac{3-4}{3-3}\right) = \frac{\pi}{4}$   
 $\arctan(-1) + \arctan(\infty) = -\frac{\pi}{4} + \frac{\pi}{2} = \frac{\pi}{4}$

**Question 58** (\*\*\*\*+)

Given that

$$\sec^2 x - (1 + \sqrt{3}) \tan x + \sqrt{3} = 1,$$

show that either  $\tan x = 1$  or  $\tan x = \sqrt{3}$ .*Detailed workings must be shown in this question.*,  proof

$\sec^2 x - (1 + \sqrt{3}) \tan x + \sqrt{3} = 1$   
 $\Rightarrow \sec^2 x - (1 + \sqrt{3}) \tan x + \sqrt{3} - 1 = 0$   
 $\Rightarrow \sec^2 x - (1 + \sqrt{3}) \tan x + \sqrt{3} - 1 = 0$   
 $\Rightarrow \sec^2 x - (1 + \sqrt{3}) \tan x + \sqrt{3} - 1 = 0$   
 BY THE QUADRATIC FORMULA  
 $\Rightarrow \tan x = \frac{(1 + \sqrt{3}) \pm \sqrt{(1 + \sqrt{3})^2 - 4 \times 1 \times (\sqrt{3} - 1)}}{2 \times 1}$   
 $\Rightarrow \tan x = \frac{(1 + \sqrt{3}) \pm \sqrt{1 + 2\sqrt{3} + 3 - 4\sqrt{3} + 4}}{2}$   
 $\Rightarrow \tan x = \frac{1 + \sqrt{3} \pm \sqrt{4 - 2\sqrt{3}}}{2}$   
 $\Rightarrow \tan x = \frac{1 + \sqrt{3} \pm \sqrt{(2 - \sqrt{3})^2}}{2}$   
 $\Rightarrow \tan x = \frac{1 + \sqrt{3} \pm (2 - \sqrt{3})}{2}$   
 $\Rightarrow \tan x = \frac{1 + \sqrt{3} + 2 - \sqrt{3}}{2}$   
 $\Rightarrow \tan x = \frac{3}{2}$   
 $\Rightarrow \tan x = \frac{3}{2}$

**Question 59** (\*\*\*\*+)

It is given that

$$u = \sin 2\theta, \quad v = \cot \theta.$$

Use trigonometric identities to find a simplified expression for  $u^2$  in terms of  $v$ .

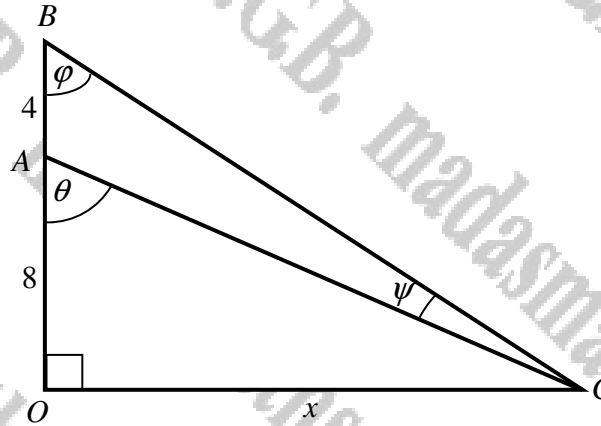
$$u^2 = \frac{2v}{v^2 + 1}$$

$u = \sin 2\theta$   
 $u = 2 \sin \theta \cos \theta$   
 $u^2 = 4 \sin^2 \theta \cos^2 \theta$   
 $u^2 = 4 \sin^2 \theta (1 - \sin^2 \theta)$   
 $u^2 = 4 \left( \frac{1}{1 + v^2} \right) \left( 1 - \frac{1}{1 + v^2} \right)$   
 $u^2 = 4 \frac{1}{1 + v^2} \times \frac{v^2}{1 + v^2}$   
 $u^2 = \frac{4v^2}{(1 + v^2)^2}$   
 $u^2 = \frac{2v}{v^2 + 1}$

**Question 60** (\*\*\*\*+)

The diagram below shows a right angled triangle  $OBC$  where  $|OC| = x$  and the point  $A$  on  $OB$  so that  $|OA| = 8$ ,  $|AB| = 4$ .

The angles  $OAC$ ,  $OBC$  and  $ACB$  are denoted by  $\theta$ ,  $\phi$  and  $\psi$  respectively.



By considering a relationship between the angles  $\theta$ ,  $\phi$  and  $\psi$ , show that

$$\tan \psi = \frac{4x}{96 + x^2}.$$

□, proof

LOOKING AT THE DIAGRAM

$\angle BAC = 180 - \phi$

LOOKING AT  $\triangle ABC$

$\phi + (180 - \phi) + \psi = 180$   
 $\phi - \phi + \psi = 0$   
 $\psi = 0 - \phi$

TAKING TANGENTS BOTH SIDES

$\tan \psi = \tan (0 - \phi)$   
 $\tan \psi = \frac{\tan 0 - \tan \phi}{1 - \tan 0 \tan \phi}$   
 $\tan \psi = \frac{0 - \frac{4}{12}}{1 - 0 \cdot \frac{4}{12}}$   
 $\tan \psi = \frac{-\frac{4}{12}}{1 - \frac{x^2}{96}}$   
 $\tan \psi = \frac{-\frac{4}{12} \cdot \frac{96}{96 - x^2}}{1}$   
 $\tan \psi = \frac{-4 \cdot 8}{96 - x^2}$   
 $\tan \psi = \frac{-32}{96 - x^2}$

$\therefore \tan \psi = \frac{32}{96 - x^2}$   
 At 12/04/20

**Question 61** (\*\*\*\*+)

Solve the trigonometric equation

$$\sin x \cos x \cos 2x \cos 4x = \frac{\sqrt{2}}{16}, \quad 0 \leq x \leq \frac{\pi}{2},$$

giving the answers in terms of  $\pi$ .

$$\boxed{\phantom{0000}}, \quad x = \frac{\pi}{32}, \frac{3\pi}{32}, \frac{9\pi}{32}, \frac{11\pi}{32}$$

Using the double angle identity for sine

$$\begin{aligned} \Rightarrow \sin 2x \cos 2x \cos 4x &= \frac{\sqrt{2}}{16} \quad \times 2 \\ \Rightarrow 2 \sin x \cos x \cos 2x \cos 4x &= \frac{\sqrt{2}}{8} \\ \Rightarrow \sin 2x \cos 2x \cos 4x &= \frac{\sqrt{2}}{16} \quad \times 2 \\ \Rightarrow 2 \sin 2x \cos 2x \cos 4x &= \frac{\sqrt{2}}{8} \\ \Rightarrow \sin 4x \cos 4x &= \frac{\sqrt{2}}{8} \quad \times 2 \\ \Rightarrow 2 \sin 4x \cos 4x &= \frac{\sqrt{2}}{4} \\ \Rightarrow \sin 8x &= \frac{\sqrt{2}}{4} \end{aligned}$$

$\begin{cases} 8x = 7\pi/4 + 2n\pi \\ 8x = 3\pi/4 + 2n\pi \end{cases} \quad n \in \mathbb{Z}, n = 0, 1, 2, \dots$   
 $\begin{cases} x = 7\pi/32 + n\pi/4 \\ x = 3\pi/32 + n\pi/4 \end{cases}$

For the range  $0 \leq x < \pi/2$

$$x = \frac{\pi}{32}, \frac{3\pi}{32}, \frac{9\pi}{32}, \frac{11\pi}{32}$$

**Question 62** (\*\*\*\*+)

Solve the trigonometric equation

$$\frac{4}{2 \sec \varphi - 2 \sin \varphi + 1} = \cot \varphi, \quad 0 < \varphi < 2\pi, \quad \varphi \neq \pi,$$

giving the answers in terms of  $\pi$ .

$$\boxed{\varphi = \frac{\pi}{6}, \frac{5\pi}{6}}$$

$$\begin{aligned} \frac{4}{2 \sec \varphi - 2 \sin \varphi + 1} &= \cot \varphi \\ \Rightarrow 4 &= \cot \varphi (2 \sec \varphi - 2 \sin \varphi + 1) \\ \Rightarrow 4 &= \frac{\cos \varphi}{\sin \varphi} (2 \sec \varphi - 2 \sin \varphi + 1) \\ \Rightarrow 4 &= \frac{2}{\sin \varphi} - 2 \cot \varphi + \frac{\cos \varphi}{\sin \varphi} \\ \Rightarrow 4 \sin \varphi &= 2 - 2 \cos \varphi + \cos \varphi \\ \Rightarrow 4 \sin \varphi + 2 \cos \varphi &= 2 - \cos \varphi = 0 \\ \Rightarrow 2 \sin \varphi (2 + \cos \varphi) - (2 + \cos \varphi) &= 0 \\ \Rightarrow (2 + \cos \varphi) (2 \sin \varphi - 1) &= 0 \end{aligned}$$

$\therefore \cos \varphi < -2$   
 $\sin \varphi = \frac{1}{2}$   
 $\cos \varphi = \frac{1}{2}$   
 $\begin{cases} \varphi = \frac{\pi}{6} + 2n\pi \\ \varphi = \frac{5\pi}{6} + 2n\pi \end{cases} \quad n \in \mathbb{Z}, n = 0, 1, 2, \dots$   
 $\therefore \varphi = \frac{\pi}{6}$   
 $\varphi = \frac{5\pi}{6}$



**Question 63** (\*\*\*\*+)

Solve the trigonometric equation

$$\tan \theta (1 + \cos 2\theta) = 2 \sin^2 2\theta, \quad 0 \leq \theta \leq 90^\circ.$$

$$\theta = 0^\circ, 15^\circ, 75^\circ, 90^\circ$$

Handwritten solution for Question 63:

$$\begin{aligned} \tan \theta (1 + \cos 2\theta) &= 2 \sin^2 2\theta \\ \tan \theta (1 + (2\cos^2 \theta - 1)) &= 2 \sin^2 2\theta \\ \tan \theta \cdot 2\cos^2 \theta &= 2 \sin^2 2\theta \\ \frac{\sin \theta}{\cos \theta} \cdot 2\cos^2 \theta &= 2 \sin^2 2\theta \\ 2\sin \theta \cos \theta &= 2 \sin^2 2\theta \\ \sin 2\theta &= 2 \sin^2 2\theta \\ 0 &= 2 \sin^2 2\theta - \sin 2\theta \\ 0 &= \sin 2\theta (2 \sin 2\theta - 1) \\ \sin 2\theta &= 0 \quad \text{or} \quad \frac{1}{2} \\ \sin 2\theta = 0 & \Rightarrow 2\theta = 0^\circ \text{ or } 180^\circ \\ \theta &= 0^\circ \text{ or } 90^\circ \\ \sin 2\theta = \frac{1}{2} & \Rightarrow 2\theta = 30^\circ \text{ or } 150^\circ \\ \theta &= 15^\circ \text{ or } 75^\circ \end{aligned}$$

**Question 64** (\*\*\*\*+)

Solve the trigonometric equation

$$\arcsin x + \arccos \frac{3}{5} = 2 \arctan \frac{3}{4}.$$

$$\boxed{\phantom{000}}, \quad x = \frac{44}{125}$$

Handwritten solution for Question 64:

Let  $\theta = \arcsin x$  and  $\phi = \arctan \frac{3}{4}$

From the triangles,  $\cos \theta = \frac{4}{5}$  and  $\cos \phi = \frac{4}{5}$ .

TRANSFORM THE EQUATION

$$\begin{aligned} \Rightarrow \arcsin x + \arccos \frac{4}{5} &= 2 \arctan \frac{3}{4} \\ \Rightarrow \arcsin x + \theta &= 2\phi \\ \Rightarrow \arcsin x &= 2\phi - \theta \\ \Rightarrow \sin(\arcsin x) &= \sin(2\phi - \theta) \\ \Rightarrow x &= \sin 2\phi \cos \theta - \cos 2\phi \sin \theta \end{aligned}$$

SINCE DOUBLE-ANGLE IDENTITIES:  $\sin 2\phi = 2 \sin \phi \cos \phi$  and  $\cos 2\phi = 2 \cos^2 \phi - 1$

$$\begin{aligned} \Rightarrow x &= 2 \sin \phi \cos \phi \cos \theta - (2 \cos^2 \phi - 1) \sin \theta \\ \Rightarrow x &= 2 \left( \frac{3}{5} \right) \left( \frac{4}{5} \right) \left( \frac{4}{5} \right) - \frac{1}{5} \left( 2 \left( \frac{4}{5} \right)^2 - 1 \right) \\ \Rightarrow x &= \frac{72}{125} - \frac{12}{125} \\ \Rightarrow x &= \frac{44}{125} \end{aligned}$$

## Question 65 (\*\*\*\*+)

It is given that

$$\frac{\cot x}{\operatorname{cosec} x - 1} - \frac{\cos x}{1 + \sin x} \equiv 2 \tan x, \quad x \neq 180^\circ n, n \in \mathbb{Z}.$$

a) Prove the validity of the above trigonometric identity.

b) Hence solve the trigonometric equation

$$\frac{\cot 3\theta}{\operatorname{cosec} 3\theta - 1} - \frac{\cos 3\theta}{1 + \sin 3\theta} = 2 \tan \theta, \quad 0 \leq \theta < 2\pi,$$

giving the answers in terms of  $\pi$ .

$$\theta = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$$

(a) LHS =  $\frac{\cot x}{\operatorname{cosec} x - 1} - \frac{\cos x}{1 + \sin x}$   
 $= \frac{\frac{\cos x}{\sin x}}{\frac{1}{\sin x} - 1} - \frac{\cos x}{1 + \sin x}$   
 $= \frac{\cos x}{1 - \sin x} - \frac{\cos x}{1 + \sin x}$   
 $= \cos x \left[ \frac{1 + \sin x}{(1 - \sin x)(1 + \sin x)} - \frac{1 - \sin x}{(1 - \sin x)(1 + \sin x)} \right]$   
 $= \cos x \left[ \frac{1 + \sin x - 1 + \sin x}{1 - \sin^2 x} \right] = \cos x \left[ \frac{2\sin x}{1 - \sin^2 x} \right]$   
 $= \cos x \frac{2\sin x}{\cos^2 x} = \frac{2\sin x}{\cos x} = 2 \tan x$   
 (b)  $\frac{\cot 3\theta}{\operatorname{cosec} 3\theta - 1} - \frac{\cos 3\theta}{1 + \sin 3\theta} = 2 \tan \theta$   
 $2 \tan 3\theta = 2 \tan \theta$   
 $\tan 3\theta = \tan \theta$   
 $3\theta = \theta + n\pi$   
 $2\theta = 0 + n\pi$   
 $\theta = 0 + \frac{n\pi}{2}$   
 $\therefore \theta = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$

## Question 66 (\*\*\*\*+)

Prove the validity of the following trigonometric identities.

i.  $\frac{\tan 2x}{\tan 2x + \sin 2x} \equiv \frac{1}{2} \sec^2 x.$

ii.  $\frac{\tan \phi}{(1 - \cos \phi)(1 + \sec \phi)} \equiv \operatorname{cosec} \phi.$

proof

Handwritten proof for Question 66(i) and (ii):

(i) LHS =  $\frac{\tan 2x}{\tan 2x + \sin 2x} = \frac{\frac{\sin 2x}{\cos 2x}}{\frac{\sin 2x}{\cos 2x} + \sin 2x}$  Divide top and bottom of the fraction by  $\sin 2x$   
 $= \frac{\frac{1}{\cos 2x}}{\frac{1}{\cos 2x} + 1}$  Multiply the fraction by  $\cos 2x$   
 $= \frac{1}{1 + (\cos 2x)}$   $\cos 2x = 2\cos^2 x - 1$   
 $= \frac{1}{1 + (2\cos^2 x - 1)} = \frac{1}{2\cos^2 x} = \frac{1}{2} \sec^2 x = \text{RHS}$

(ii) LHS =  $\frac{\tan \phi}{(1 - \cos \phi)(1 + \sec \phi)} = \frac{\frac{\sin \phi}{\cos \phi}}{(1 - \cos \phi)(1 + \frac{1}{\cos \phi})} = \frac{\frac{\sin \phi}{\cos \phi}}{(1 - \cos \phi)(\frac{\cos \phi + 1}{\cos \phi})} = \frac{\frac{\sin \phi}{\cos \phi}}{\frac{(1 - \cos \phi)(\cos \phi + 1)}{\cos \phi}}$  Multiply top and bottom by  $\cos \phi$   
 $= \frac{\sin \phi}{1 - \cos^2 \phi} = \frac{\sin \phi}{\sin^2 \phi} = \frac{1}{\sin \phi} = \operatorname{cosec} \phi = \text{RHS}$

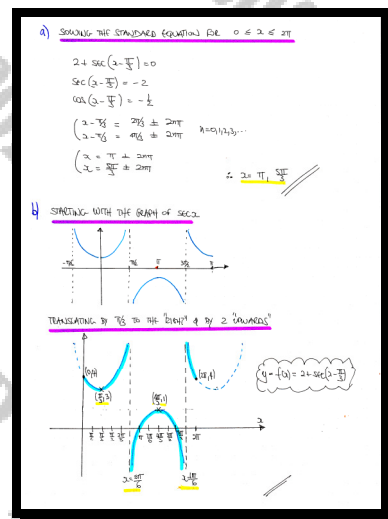
## Question 67 (\*\*\*\*+)

$$f(x) = 2 + \sec\left(x - \frac{\pi}{3}\right), \quad x \in \mathbb{R}, \quad 0 \leq x \leq 2\pi.$$

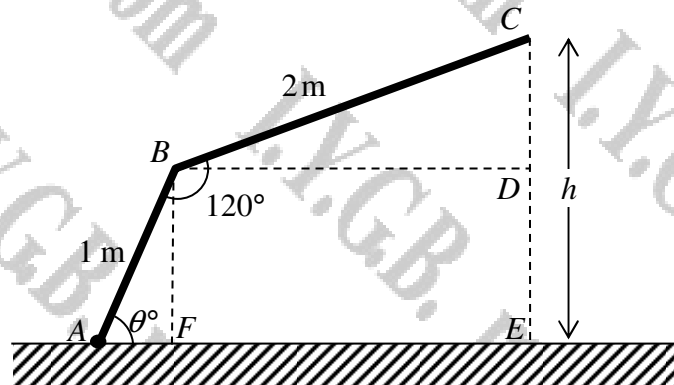
- a) Solve the equation  $f(x) = 0$ .
- b) Sketch the graph of  $f(x)$ .

The sketch must include the coordinates of any stationary points, the coordinates of any  $x$  or  $y$  intercepts and equations of the vertical asymptotes.

$$x = \pi, \quad \frac{5\pi}{3}$$



## Question 68 (\*\*\*\*+)



The figure above shows a rigid rod  $ABC$  where  $AB$  is 1 metre,  $BC$  is 2 metres and the angle  $ABC$  is  $120^\circ$ . The rod is hinged at  $A$  so it can be rotated in a vertical plane forming an angle  $\theta^\circ$  with the horizontal ground.

Let  $h$  metres be the height of the point  $C$  from the horizontal ground.

a) Show that ...

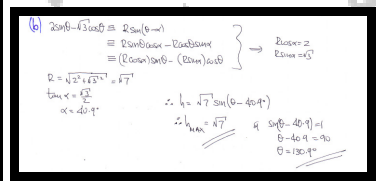
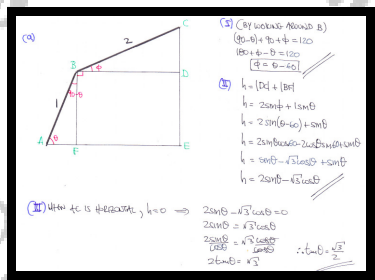
i. ...  $\angle DBC = \theta^\circ - 60^\circ$ .

ii. ...  $h = 2\sin\theta - \sqrt{3}\cos\theta$ .

iii. ... when  $AC$  is horizontal,  $\tan\theta = \frac{\sqrt{3}}{2}$ .

b) By expressing  $h$  in the form  $R\sin(\theta - \alpha)$ , where  $R > 0$  and  $0 < \alpha < 90^\circ$ , find the maximum value of  $h$  and the value of  $\theta$  when  $h$  takes this maximum value.

,  $h_{\max} = \sqrt{7}$  ,  $\theta \approx 130.9^\circ$



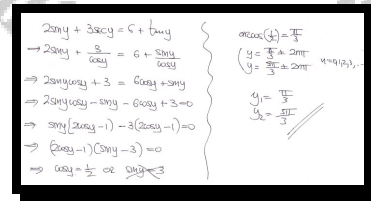
**Question 69** (\*\*\*\*+)

Solve the trigonometric equation

$$2\sin y + 3\sec y = 6 + \tan y, \quad 0 \leq y < 2\pi,$$

giving the answers in terms of  $\pi$ .

$$y = \frac{\pi}{3}, \frac{5\pi}{3}$$



Handwritten solution for Question 69:

$$\begin{aligned}
 2\sin y + 3\sec y &= 6 + \tan y \\
 \rightarrow 2\sin y + \frac{3}{\cos y} &= 6 + \frac{\sin y}{\cos y} \\
 \Rightarrow 2\sin y \cos y + 3 &= 6\cos y + \sin y \\
 \Rightarrow 2\sin y \cos y - \sin y - 6\cos y + 3 &= 0 \\
 \Rightarrow \sin y (2\cos y - 1) - 3(2\cos y - 1) &= 0 \\
 \Rightarrow (2\cos y - 1)(\sin y - 3) &= 0 \\
 \Rightarrow \cos y = \frac{1}{2} \text{ or } \sin y &= 3
 \end{aligned}$$

Since  $\sin y = 3$  is not possible, we solve  $\cos y = \frac{1}{2}$ :

$$\begin{aligned}
 \cos y &= \frac{1}{2} \\
 y &= \frac{\pi}{3}, \frac{5\pi}{3}
 \end{aligned}$$

## Question 70 (\*\*\*\*+)

$$f(x) = \sec^2 x, \quad x \in \mathbb{R}, \quad x \neq \frac{\pi}{2}(2n+1), \quad n \in \mathbb{N}.$$

Show that if

$$f(x) = \frac{1}{2} f\left(x + \frac{\pi}{4}\right),$$

then either  $\sin x = 0$  or  $\tan x = 2$ .

, proof

Write the equation explicitly

$$\begin{aligned} \Rightarrow f(x) &= \frac{1}{2} f\left(x + \frac{\pi}{4}\right) \\ \Rightarrow 2f(x) &= f\left(x + \frac{\pi}{4}\right) \\ \Rightarrow 2\sec^2 x &= \sec^2\left(x + \frac{\pi}{4}\right) \\ \Rightarrow \frac{2}{\cos^2 x} &= \frac{1}{\cos^2\left(x + \frac{\pi}{4}\right)} \\ \Rightarrow \frac{\cos^2 x}{2} &= \cos^2\left(x + \frac{\pi}{4}\right) \end{aligned}$$

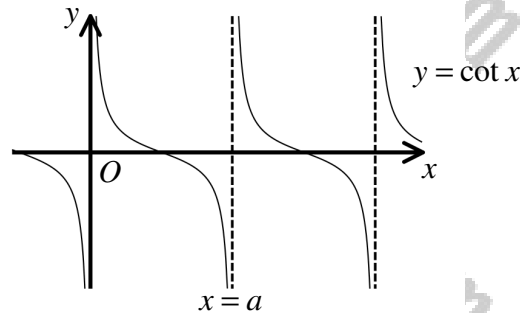
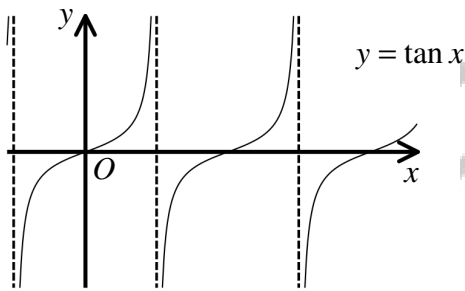
USE THE COMPOUND ANGLE IDENTITY,  $\cos(A+B)$

$$\begin{aligned} \Rightarrow \frac{1}{2} \cos^2 x &= \left[ \cos x \cos \frac{\pi}{4} - \sin x \sin \frac{\pi}{4} \right]^2 \\ \Rightarrow \frac{1}{2} \cos^2 x &= \left[ \frac{1}{\sqrt{2}} \cos x - \frac{1}{\sqrt{2}} \sin x \right]^2 \quad \text{Since } \cos \frac{\pi}{4} = \sin \frac{\pi}{4} = \frac{1}{\sqrt{2}} \\ \Rightarrow \frac{1}{2} \cos^2 x &= \left( \frac{1}{\sqrt{2}} \right)^2 (\cos x - \sin x)^2 \\ \Rightarrow \frac{1}{2} \cos^2 x &= \frac{1}{2} (\cos x - \sin x)^2 \end{aligned}$$

FINALLY TWO DIFFERENT APPROACHES TO FOLLOW

$\begin{aligned} \cos^2 x &= \cos^2 x + \sin^2 x - 2\sin x \cos x \\ 0 &= \sin^2 x - 2\sin x \cos x \\ 0 &= \sin x (\sin x - 2\cos x) \\ \frac{0}{\cos x} &= \frac{\sin x (\sin x - 2\cos x)}{\cos x} \\ 0 &= \sin x (\tan x - 2) \end{aligned}$ <p><math>\sin x = 0</math> or <math>\tan x = 2</math></p>	$\begin{aligned} \cos^2 x &= \cos^2 x + \sin^2 x \\ + \cos x + \cos x &= -\sin x \\ \pm \cos x - \cos x &= \pm \sin x \\ \cos x - \cos x &= \sin x \\ -\cos x - \cos x &= -\sin x \\ \sin x &= 0 \quad \text{or} \quad \sin x = 2\cos x \\ \tan x &= 2 \end{aligned}$
--	---

## Question 71 (\*\*\*\*+)



The diagrams above shows part of the graphs of  $y = \tan x$  and  $y = \cot x$ .

- a) Sketch the graph of  $y_1 = -\tan(-x)$  and hence write a simplified expression for  $y_1$  in terms of  $\tan x$ .

The graph of  $\cot x$  has vertical asymptotes and the equation of one of them is labelled in the diagram as  $x = a$ .

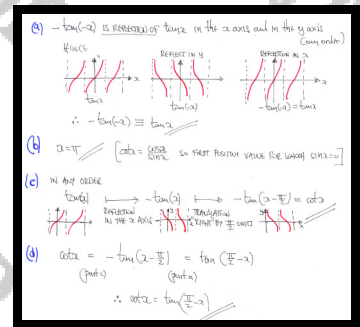
- b) State the value of  $a$ .

The graph of  $\cot x$  can be obtained from the graph of  $\tan x$  by a series of two geometric transformations which can be carried out in any order.

- c) Describe the two geometric transformations.  
d) Deduce using valid arguments that

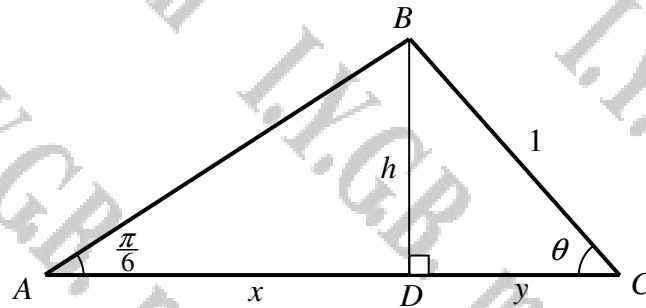
$$\cot x = \tan\left(\frac{\pi}{2} - x\right).$$

$a = \pi$ , reflection in the  $x$  axis/translation to the "right" by  $\frac{\pi}{2}$  units





## Question 72 (\*\*\*\*+)



The figure above shows a triangle  $ABC$ , where  $\angle BAC = \frac{\pi}{6}$ ,  $\angle BCA = \theta$  and  $|BC| = 1$ .

The straight line segment  $BD$ , labelled as  $h$ , is perpendicular to  $AC$ .

Let  $AD = x$  and  $DC = y$ .

- a) By expressing  $h$  in terms of  $\theta$ , and  $x$  in terms of  $h$ , show that

$$x + y = \sqrt{3} \sin \theta + \cos \theta,$$

and hence deduce that the area of the triangle  $ABC$  is given by

$$\sin \theta \sin \left( \theta + \frac{\pi}{6} \right).$$

- b) By using the trigonometric identities for

$$\cos \left[ \theta + \left( \theta + \frac{\pi}{6} \right) \right] \quad \text{and} \quad \cos \left[ \theta - \left( \theta + \frac{\pi}{6} \right) \right],$$

write a simplified expression for the area of the triangle  $ABC$ .

[continues overleaf]

[continued from overleaf]

The value of  $\theta$  can vary.

- c) By using part (b), deduce that the maximum value of the area of the triangle ABC is

$$\frac{1}{4}(2 + \sqrt{3})$$

and this maximum value occurs when  $\theta = \frac{5\pi}{12}$ .

$$\boxed{\phantom{000}}, \sin \theta \sin \left( \theta + \frac{\pi}{6} \right) = \frac{\sqrt{3}}{4} - \frac{1}{2} \cos \left( 2\theta + \frac{\pi}{6} \right)$$

q) LOOKING AT  $\triangle BDC$

- $h = 1 \times \sin \theta$
- $y = 1 \times \cos \theta$

LOOKING AT  $\triangle ABD$

$\Rightarrow \frac{1}{x} = \tan \frac{\pi}{6}$

$\Rightarrow \frac{\sin \theta}{x} = \frac{\sqrt{3}}{3}$

$\Rightarrow \sqrt{3}x = 3 \sin \theta$

$\Rightarrow x = \sqrt{3} \sin \theta$

$\therefore x+y = \cos \theta + \sqrt{3} \sin \theta$

$\therefore \text{Area} = \frac{1}{2} |AC| |BD|$

$= \frac{1}{2} (x+y) h$

$= \frac{1}{2} (\cos \theta + \sqrt{3} \sin \theta) \sin \theta$

$= \left( \frac{1}{2} \cos \theta + \frac{\sqrt{3}}{2} \sin \theta \right) \sin \theta$

$= \left( \sin \frac{\pi}{6} \cos \theta + \cos \frac{\pi}{6} \sin \theta \right) \sin \theta$

$= \sin \left( \theta + \frac{\pi}{6} \right) \sin \theta$

LOOKING AT THE COMPOUND ANGLE IDENTITIES FOR  $\cos(A \pm B)$

$\cos \left( \theta + \left( \theta + \frac{\pi}{6} \right) \right) \equiv \cos \theta \cos \left( \theta + \frac{\pi}{6} \right) - \sin \theta \sin \left( \theta + \frac{\pi}{6} \right)$

$\cos \left[ \theta - \left( \theta + \frac{\pi}{6} \right) \right] \equiv \cos \theta \cos \left( \theta + \frac{\pi}{6} \right) + \sin \theta \sin \left( \theta + \frac{\pi}{6} \right)$

SUBTRACTING "EQUATIONS"

$\Rightarrow \cos \left( -\frac{\pi}{6} \right) - \cos \left( 2\theta + \frac{\pi}{6} \right) \equiv 2 \sin \theta \sin \left( \theta + \frac{\pi}{6} \right)$

$\Rightarrow \frac{1}{2} \cos \left( -\frac{\pi}{6} \right) - \frac{1}{2} \cos \left( 2\theta + \frac{\pi}{6} \right) \equiv \sin \theta \sin \left( \theta + \frac{\pi}{6} \right)$

$\Rightarrow \sin \theta \sin \left( \theta + \frac{\pi}{6} \right) \equiv \frac{\sqrt{3}}{4} - \frac{1}{2} \cos \left( 2\theta + \frac{\pi}{6} \right)$

$\Rightarrow \text{Area} = \frac{1}{2} \left[ \sqrt{3} - 2 \cos \left( 2\theta + \frac{\pi}{6} \right) \right]$

c) MAXIMUM AREA OCCURS WHEN  $\cos \left( 2\theta + \frac{\pi}{6} \right) = -1$

$\text{Area}_{\text{max}} = \frac{1}{2} \left[ \sqrt{3} - 2(-1) \right]$

$\text{Area}_{\text{max}} = \frac{1}{2} (\sqrt{3} + 2)$

AND FINALLY

$\cos \left( 2\theta + \frac{\pi}{6} \right) = -1$

$2\theta + \frac{\pi}{6} = \pi$

$2\theta = \frac{5\pi}{6}$

$\theta = \frac{5\pi}{12}$

**Question 73** (\*\*\*\*+)

The height of tide,  $h$  meters, in a harbour on a certain day can be modelled by

$$h(t) = 10 + \sqrt{3} \sin(30t)^\circ + \cos(30t)^\circ, \quad 0 \leq t \leq 12,$$

where  $t$  is the time in hours since midnight.

- a) Find the time when the high tide and the low tide occur during the morning hours of that day and state the corresponding depth of water in the harbour at these times.

The depth of water in this harbour needs to be at least 8.5 metres for a boat to dock

A boat arrives outside the harbour at high tide and needs five hours to unload.

- b) Show that the boat has to wait until 09:23 to enter the harbour.

, high tide of 12 metres at 02:00 , low tide of 8 metres at 08:00

(a)  $h(t) = 10 + \sqrt{3} \sin(30t) + \cos(30t)$  or  $R \sin(30t + \alpha)$

$$\begin{aligned} \sqrt{3} \sin(30t) + \cos(30t) &= R \cos(30t - \alpha) \\ &= R \cos 30t \cos \alpha + R \sin 30t \sin \alpha \\ &= (R \cos \alpha) \cos 30t + (R \sin \alpha) \sin 30t \end{aligned}$$

$R \cos \alpha = \sqrt{3}$   $R \sin \alpha = 1$

$$R^2 \cos^2 \alpha + R^2 \sin^2 \alpha = 3 + 1 = 4$$

$$R = 2$$

$$\tan \alpha = \frac{1}{\sqrt{3}} \Rightarrow \alpha = 30^\circ$$

$$\therefore h(t) = 10 + 2 \sin(30t + 30)$$

High Tide is 12 metres  $\Rightarrow \cos(30t - 60) = 1$

$$30t - 60 = 0$$

$$30t = 60$$

$$t = 2$$

Low Tide is 8 metres  $\Rightarrow \cos(30t - 60) = -1$

$$30t - 60 = 180$$

$$30t = 240$$

$$t = 8$$

$\therefore$  High Tide 12 metres At 02:00  
Low Tide 8 metres At 08:00

(b)  $h = 8.5$

$$8.5 = 10 + 2 \sin(30t - 60)$$

$$\Rightarrow -1.5 = 2 \sin(30t - 60)$$

$$\Rightarrow \cos(30t - 60) = -0.75$$

$$\Rightarrow \left\{ \begin{array}{l} 30t - 60 = 136.5 \pm 360n \\ 30t - 60 = 223.41 \pm 360n \end{array} \right. \Rightarrow 9.55, \dots$$

$$\left\{ \begin{array}{l} 30t = 196.5 \pm 360n \\ 30t = 283.41 \pm 360n \end{array} \right.$$

$$\left\{ \begin{array}{l} t = 6.55 \pm 12n \\ t = 9.44 \pm 12n \end{array} \right.$$

$\therefore$  Ship Arrives At 02:00 + 5 hours = 07:00

IT OMAHUA GO IN BECAUSE WATER  $t = 6.42$  (06:57)

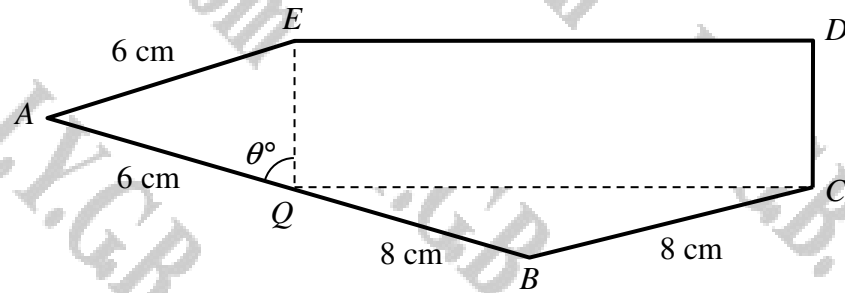
IT BLA AHEAD

$\therefore$  IT MUST WAIT UNTIL  $t = 9.38$

It 0:30 x 60 = 22.8

It 09:23

## Question 74 (\*\*\*\*+)



The figure above shows an irregular pentagon  $ABCDE$ . The lengths of  $AB$ ,  $BC$  and  $AE$  are 14 cm, 8 cm and 6 cm respectively.

The point  $Q$  lies on  $AB$  so that  $AQ$  is 6 cm and  $QB$  is 8 cm. The point  $D$  is then constructed so that  $QEDC$  is a rectangle.

Let the angle  $AQE$  be  $\theta^\circ$  and assume that  $\theta^\circ$  can vary.

- a) Given that  $P$  cm and  $R$  cm<sup>2</sup> are the perimeter and the area of the pentagon respectively, show that ...

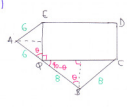
i. ...  $P = 28 + 12\cos\theta + 16\sin\theta$ .

ii. ...  $R = 146\sin 2\theta$ .

- b) Hence show that when the pentagon has a maximum area

$$P = 14(2 + \sqrt{2}) \text{ cm}^2.$$

 , proof

6) 

- $|EQ| = 2(4\cos\theta)$   
 $= 2 \times 4 \cos\theta$   
 $= 8\cos\theta$   
 $\therefore |EQ| = 8\cos\theta$
- $|QC| = 2(4\sin\theta)$   
 $= 2 \times 4 \sin\theta$   
 $= 8\sin\theta$   
 $\therefore |QC| = 8\sin\theta$

$\therefore \text{Perimeter} = C + E + B + 2QC + 2EQ$   
 $= 28 + 16\cos\theta + 16\sin\theta$   
 as required

6)  $\therefore \text{Area} = \frac{1}{2}|AQ||EQ|\sin\theta + \frac{1}{2}|QB||QC|\sin\theta + |EQ||QC|$   
 $= \frac{1}{2} \times 6 \times 8\cos\theta \sin\theta + \frac{1}{2} \times 8 \times 8 \sin\theta \cos\theta + 8\cos\theta \times 8\sin\theta$   
 $= 24\cos\theta \sin\theta + 32\sin\theta \cos\theta + 64\sin\theta \cos\theta$   
 $= 112\sin\theta \cos\theta = 56\sin 2\theta$   
 as required

6)  $\text{Max Area} = 112$  (occurs when  $\sin 2\theta = 1$ )  
 $2\theta = \frac{\pi}{2} \pm 2n\pi$   
 $\theta = \frac{\pi}{4} \pm n\pi$   
 $\therefore \theta = \frac{\pi}{4}$

$\therefore \text{Perimeter}_{\text{max}} = 28 + 16\cos\frac{\pi}{4} + 16\sin\frac{\pi}{4}$   
 $= 28 + 16\sqrt{2} + 16\sqrt{2}$   
 $= 28 + 32\sqrt{2}$   
 $= 14(2 + \sqrt{2})$   
 as required

## Question 75 (\*\*\*\*+)

Let  $t = \tan\left(\frac{x}{2}\right)$ .

a) Show that ...

i. ...  $\sin x = \frac{2t}{1+t^2}$ ,

ii. ...  $\cos x = \frac{1-t^2}{1+t^2}$ .

b) Use these results to solve the trigonometric equation

$$5 \sin x - 5 \cos x = 1, \quad 0 \leq x < 2\pi.$$

$$x = 3.79^\circ, 0.927^\circ$$

$$5 \sin x - 5 \cos x = 1$$

$$\Rightarrow \sin x - \cos x = \frac{1}{5}$$

$$\Rightarrow \frac{2t}{1+t^2} - \frac{1-t^2}{1+t^2} = \frac{1}{5}$$

$$\Rightarrow \frac{2t - (1-t^2)}{1+t^2} = \frac{1}{5}$$

$$\Rightarrow \frac{2t - 1 + t^2}{1+t^2} = \frac{1}{5}$$

$$\Rightarrow 5(2t - 1 + t^2) = 1 + t^2$$

$$\Rightarrow 10t - 5 + 5t^2 = 1 + t^2$$

$$\Rightarrow 4t^2 + 10t - 6 = 0$$

$$\Rightarrow 2t^2 + 5t - 3 = 0$$

$$\Rightarrow (2t-1)(t+3) = 0$$

$$\Rightarrow t = \frac{1}{2} \text{ or } t = -3$$

$$\therefore x_1 = 0.927^\circ$$

$$x_2 = 3.79^\circ$$

## Question 76 (\*\*\*\*+)

The functions  $f$  and  $g$  are defined by

$$f(x) \equiv 3 \sin x, \quad x \in \mathbb{R}, \quad -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$$

$$g(x) \equiv 6 - 3x^2, \quad x \in \mathbb{R}.$$

a) Find an expression for  $f^{-1}g(x)$ .

b) Determine the domain of  $f^{-1}g(x)$ .

$$\boxed{\phantom{000}}, \quad f^{-1}g(x) = \arcsin(2 - x^2), \quad -\sqrt{3} \leq x \leq -1 \quad \text{or} \quad 1 \leq x \leq \sqrt{3}$$

a)  $f(x) = 3 \sin x, \quad -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$   
 $g(x) = 6 - 3x^2, \quad x \in \mathbb{R}$

$\Rightarrow y = 3 \sin x$   
 $\Rightarrow \frac{y}{3} = \sin x$   
 $\Rightarrow x = \arcsin \frac{y}{3}$   
 $\therefore f^{-1}(y) = \arcsin \frac{y}{3}$

Now  $f^{-1}(g(x)) = f^{-1}(6 - 3x^2)$   
 $= \arcsin \left( \frac{6 - 3x^2}{3} \right)$   
 $= \arcsin(2 - x^2)$

b)  $f(x)$  has domain  $[-\frac{\pi}{2}, \frac{\pi}{2}]$  AND RANGE  $[-\frac{3}{2}, \frac{3}{2}]$   
 $g(x)$  has domain  $[-\infty, \infty]$  AND RANGE  $[-\infty, 6]$

IN  $(-\infty, \infty)$   $g(x)$  OUT  $g(x) \leq 6$   $\xrightarrow{\text{arcsin}}$   $f^{-1}(y)$  IN  $[-\frac{\pi}{2}, \frac{\pi}{2}]$  NOT

$\frac{f^{-1}(y)}{y}$   
 $\Rightarrow -\frac{3}{2} \leq g(x) \leq \frac{3}{2}$   
 $\Rightarrow -\frac{3}{2} \leq 6 - 3x^2 \leq \frac{3}{2}$   
 $\Rightarrow -9 \leq -3x^2 \leq -3$   
 $\Rightarrow 1 \leq x^2 \leq 3$

$x^2 \leq 3 \Rightarrow -\sqrt{3} \leq x \leq \sqrt{3}$   
 $x^2 \geq 1 \Rightarrow x \geq 1 \text{ or } x \leq -1$

OR

sketching  $\arcsin(2 - x^2)$   
 $-1 \leq 2 - x^2 \leq 1$   
 $-3 \leq -x^2 \leq -1$   
 $1 \leq x^2 \leq 3$  etc

$\therefore -\sqrt{3} \leq x \leq -1 \quad \text{OR} \quad 1 \leq x \leq \sqrt{3}$

## Question 77 (\*\*\*\*+)

Find the solution of the equation

$$\arctan\left(\frac{1-x}{1+x}\right) = \frac{1}{2} \arctan x.$$

$$x = \frac{\sqrt{3}}{3}$$

Handwritten solution for Question 77:

Left page:

$$\Rightarrow \arctan\left(\frac{1-x}{1+x}\right) = \frac{1}{2} \arctan x$$

$$\Rightarrow 2 \arctan\left(\frac{1-x}{1+x}\right) = \arctan x$$

Let  $\theta = \arctan\left(\frac{1-x}{1+x}\right) \Rightarrow \tan \theta = \frac{1-x}{1+x}$

$$\Rightarrow 2\theta = \arctan x$$

$$\Rightarrow \tan 2\theta = \tan(\arctan x)$$

$$\Rightarrow \frac{2 \tan \theta}{1 - \tan^2 \theta} = x$$

$$\Rightarrow \frac{2 \left(\frac{1-x}{1+x}\right)}{1 - \left(\frac{1-x}{1+x}\right)^2} = x$$

$$\Rightarrow \frac{2(1-x)}{1 - \frac{(1-x)^2}{(1+x)^2}} = x$$

MULTIPLY TOP AND BOTTOM OF THE FRACTION OF THE LHS BY  $(1+x)^2$

$$\Rightarrow \frac{2(1-x)(1+x)}{(1+x)^2 - (1-x)^2} = x$$

$$\Rightarrow \frac{2(1-x^2)}{(1+x)^2 - (1-x)^2} = x$$

$$\Rightarrow \frac{2(1-x^2)}{4x} = x$$

$$\Rightarrow \frac{1-x^2}{2x} = x$$

$$\Rightarrow 1-x^2 = 2x^2$$

Right page:

$$\Rightarrow 1 = 2x^2$$

$$\Rightarrow x^2 = \frac{1}{2}$$

$$\therefore x = \frac{\sqrt{2}}{2}$$

WHS > 0  
RHS < 0

## Question 78 (\*\*\*\*+)

$$\frac{1}{2} \sin^4 x + \frac{1}{3} \cos^4 x = \frac{1}{5}.$$

Show that the above trigonometric equation is equivalent to

$$\tan^2 x = \frac{2}{3}.$$

☐ , ☐ proof

STRICTLY MANIPULATING AS FOLLOWS

$$\Rightarrow \frac{1}{2} \sin^4 x + \frac{1}{3} \cos^4 x = \frac{1}{5}$$

$$\Rightarrow \frac{1}{2} (\sin^2 x)^2 + \frac{1}{3} (\cos^2 x)^2 = \frac{1}{5}$$

$$\Rightarrow \frac{1}{2} \left( \frac{1}{2} + \frac{1}{2} \cos 2x \right)^2 + \frac{1}{3} \left( \frac{1}{2} + \frac{1}{2} \cos 2x \right)^2 = \frac{1}{5}$$

$$\Rightarrow \frac{1}{8} - \frac{1}{4} \cos 2x + \frac{1}{8} \cos^2 2x + \frac{1}{12} + \frac{1}{6} \cos 2x + \frac{1}{12} \cos^2 2x = \frac{1}{5}$$

MULTIPLY THROUGH BY 120, AND TRY UP

$$\Rightarrow \left( \frac{15}{8} - 30 \cos 2x + 15 \cos^2 2x \right) + \left( \frac{10}{6} + 20 \cos 2x + 10 \cos^2 2x \right) = 24$$

$$\Rightarrow 25 \cos^2 2x - 10 \cos 2x + 1 = 0$$

$$\Rightarrow (5 \cos 2x - 1)^2 = 0$$

$$\Rightarrow \cos 2x = \frac{1}{5}$$

$$\Rightarrow 2 \cos^2 x - 1 = \frac{1}{5}$$

$$\Rightarrow 2 \cos^2 x = \frac{6}{5}$$

$$\Rightarrow \cos^2 x = \frac{3}{5}$$

$$\Rightarrow \sec^2 x = \frac{5}{3}$$

$$\Rightarrow \sec^2 x - 1 = \frac{5}{3} - 1$$

$$\Rightarrow \tan^2 x = \frac{2}{3}$$

As Required



**Question 79** (\*\*\*\*+)

A relationship is defined as

$$x = \sin^2 \theta, \quad 0 \leq \theta < \frac{\pi}{4}.$$

$$y = \tan 2\theta, \quad 0 \leq \theta < \frac{\pi}{4}.$$

Use trigonometric identities to show that

$$y^2 = \frac{4x(1-x)}{(1-2x)^2}.$$

proof

Handwritten proof for Question 79:

$$\begin{aligned} x &= \sin^2 \theta \\ y &= \tan 2\theta \\ \Rightarrow y &= \frac{2 \tan \theta}{1 - \tan^2 \theta} = \frac{2 \frac{x}{1-x}}{1 - \frac{x}{1-x}} = \frac{2x}{1-2x} \\ \Rightarrow y^2 &= \frac{4x^2}{(1-2x)^2} \\ \text{But } \sin^2 \theta &= x \text{ so } \frac{4x^2}{(1-2x)^2} = \frac{4x(1-x)}{(1-2x)^2} \\ \Rightarrow y^2 &= \frac{4x(1-x)}{(1-2x)^2} \end{aligned}$$

**Question 80** (\*\*\*\*+)

Find the solutions of the trigonometric equation

$$\cos \theta + \cos 3\theta + \cos 5\theta + \cos 7\theta = 0,$$

for which  $0 \leq \theta < 180^\circ$ .

$$\theta = 22.5^\circ, 45^\circ, 67.5^\circ, 90^\circ, 112.5^\circ, 135^\circ, 157.5^\circ$$

Handwritten solution for Question 80:

$$\begin{aligned} \cos \theta + \cos 3\theta + \cos 5\theta + \cos 7\theta &= 0 \\ \cos \theta + \cos 7\theta + \cos 3\theta + \cos 5\theta &= 0 \\ 2 \cos 4\theta \cos 2\theta + 2 \cos 4\theta \cos \theta &= 0 \\ 2 \cos 4\theta (\cos 2\theta + \cos \theta) &= 0 \\ 2 \cos 4\theta \times 2 \cos \frac{3\theta}{2} \cos \frac{\theta}{2} &= 0 \\ \cos 4\theta \cos \frac{3\theta}{2} \cos \frac{\theta}{2} &= 0 \\ \bullet \cos 4\theta &= 0 & \bullet \cos 2\theta &= 0 & \bullet \cos \frac{\theta}{2} &= 0 \\ \theta &= 90^\circ \pm 360^\circ & 2\theta &= 90^\circ \pm 360^\circ & \frac{\theta}{2} &= 90^\circ \pm 360^\circ \\ \theta &= 270^\circ \pm 360^\circ & \theta &= 45^\circ \pm 180^\circ & \theta &= 180^\circ \pm 720^\circ \\ \theta &= 22.5^\circ \pm 90^\circ & \theta &= 135^\circ \pm 180^\circ & \theta &= 180^\circ \pm 720^\circ \\ \text{Hence } \theta &= 22.5^\circ, 45^\circ, 135^\circ, 22.5^\circ, 112.5^\circ, 67.5^\circ, 157.5^\circ \end{aligned}$$

## Question 81 (\*\*\*\*+)

$$\sin 3\theta \equiv 3\sin \theta - 4\sin^3 \theta.$$

- a) Prove the validity of the above trigonometric identity.
- b) Hence or otherwise solve the trigonometric equation

$$\arcsin x = 3\arcsin\left(\frac{1}{3}\right).$$

$$\boxed{\phantom{000}}, \quad x = \frac{23}{27}$$

(a)  $\text{LHS} = \sin 3\theta = \sin(\theta + 2\theta) = \sin\theta\cos 2\theta + \cos\theta\sin 2\theta$   
 $= \sin\theta(\cos^2\theta - \sin^2\theta) + (\cos\theta)(2\sin\theta\cos\theta)$   
 $= \sin\theta\cos^2\theta - \sin^3\theta + 2\sin^2\theta\cos\theta$   
 $= \sin\theta(\cos^2\theta + 2\sin^2\theta) - \sin^3\theta$   
 $= \sin\theta(\cos^2\theta + 2(1 - \cos^2\theta)) - \sin^3\theta$   
 $= \sin\theta(\cos^2\theta + 2 - 2\cos^2\theta) - \sin^3\theta$   
 $= \sin\theta(2 - \cos^2\theta) - \sin^3\theta$   
 $= 2\sin\theta - \sin\theta\cos^2\theta - \sin^3\theta$   
 $= 2\sin\theta - \sin\theta(\cos^2\theta + \sin^2\theta)$   
 $= 2\sin\theta - \sin\theta(1) = \sin\theta$   
 $= \sin\theta$   
 $\therefore \text{LHS} = \text{RHS}$

(b)  $\arcsin x = 3\arcsin\left(\frac{1}{3}\right)$   
 $\Rightarrow \sin(\arcsin x) = \sin\left(3\arcsin\left(\frac{1}{3}\right)\right)$   
 $\Rightarrow x = \sin\left(3\arcsin\left(\frac{1}{3}\right)\right)$   
 $\bullet \text{ let } \theta = \arcsin\left(\frac{1}{3}\right) \Rightarrow \sin\theta = \frac{1}{3}$   
 $\Rightarrow x = \sin 3\theta$   
 $\Rightarrow x = 3\sin\theta - 4\sin^3\theta$   
 $\Rightarrow x = 3 \times \frac{1}{3} - 4 \times \left(\frac{1}{3}\right)^3$   
 $\Rightarrow x = 1 - \frac{4}{27}$   
 $\therefore x = \frac{23}{27}$

## Question 82 (\*\*\*\*+)

Solve the simultaneous equations

$$\arctan x + \arctan y = \arctan 8$$

$$x + y = 2.$$

$$\boxed{\phantom{000}}, \quad x = \frac{1}{2}, y = \frac{3}{2}, \text{ in either order}$$

$\arctan x + \arctan y = \arctan 8$      $x + y = 2$   
 $\tan(\arctan x + \arctan y) = \tan(\arctan 8)$   
 $\frac{x+y}{1-xy} = 8$      $x+y=2$   
 $\frac{2}{1-xy} = 8$      $\frac{1}{4}(y-2)$   
 $\Rightarrow 1-xy = \frac{1}{4}$   
 $\Rightarrow xy = \frac{3}{4}$   
 $\Rightarrow x(2-x) = \frac{3}{4}$   
 $\Rightarrow 2x - x^2 = \frac{3}{4}$   
 $\Rightarrow x^2 - 2x + \frac{3}{4} = 0$   
 $\Rightarrow (x - \frac{1}{2})(x - \frac{3}{2}) = 0$   
 $\therefore x = \frac{1}{2} \text{ or } \frac{3}{2}$   
 $\therefore y = \frac{3}{2} \text{ or } \frac{1}{2}$   
 $\therefore x = \frac{1}{2}, y = \frac{3}{2} \text{ or } x = \frac{3}{2}, y = \frac{1}{2}$

**Question 83** (\*\*\*\*+)

Solve the trigonometric equation

$$4 \tan 2\psi + 3 \cot \psi \sec^2 \psi = 0, \quad 0 \leq \psi < 2\pi,$$

giving the answers in terms of  $\pi$ .

$$\psi = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$$

**Question 84** (\*\*\*\*+)

Show clearly that

$$2 \arctan\left(\frac{3}{2}\right) + \arctan\left(\frac{12}{5}\right) = \pi.$$

**V**, **proof**

## Question 85 (\*\*\*\*+)

Find, in terms of  $\pi$ , the solutions of the trigonometric equation

$$\cos 2x + 3\cos x - 2\cos^2 x - \sqrt[3]{\cos x} = 1, \quad 0 \leq x < 4\pi.$$

$$x = 0, 2\pi$$

• SIMPLY BECAUSE THE ARGUMENT OF  $2x$   
 $\Rightarrow \cos 2x + 3\cos x - 2\cos^2 x - \sqrt[3]{\cos x} = 1$   
 $\Rightarrow (2\cos^2 x - 1) + 3\cos x - 2\cos^2 x - (\cos x)^{\frac{1}{3}} = 1$   
 $\Rightarrow 3\cos x - (\cos x)^{\frac{1}{3}} - 2 = 0$   
 • LET  $y = (\cos x)^{\frac{1}{3}}$   
 $\Rightarrow 3y^3 - y - 2 = 0$   
 • BY INSPECTION  $y = 1$  IS A ROOT  
 $\Rightarrow 3y^2(y-1) + 3y(y-1) + 2(y-1) = 0$   
 $\Rightarrow (3y^2 + 3y + 2)(y-1) = 0$   
 $\uparrow$   
 $6^2 - 4ac = 3^2 - 4(3)(2) < 0$   
 • ONLY SOLUTION  $y = (\cos x)^{\frac{1}{3}} = 1$   
 $\Rightarrow \cos x = 1$   
 $\Rightarrow \begin{cases} x = 0 + 2n\pi \\ x = 2\pi + 2n\pi \end{cases} \quad n \in \mathbb{Z}$   
 $\therefore x_1 = 0$   
 $x_2 = 2\pi$

## Question 86 (\*\*\*\*+)

Solve the trigonometric equation

$$\cos x + \cos 5x = 0, \quad 0 \leq \theta < \pi,$$

giving the answers in terms of  $\pi$ .

$$x = \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \frac{5\pi}{6}$$

$\cos x + \cos 5x = 0$   
 $\Rightarrow \cos 5x = -\cos x$   
 $\Rightarrow \cos 5x = \cos(\pi - x)$   
 $\Rightarrow \begin{cases} 5x = \pi - x + 2n\pi \\ 5x = x - \pi + 2n\pi \end{cases} \quad n \in \mathbb{Z}$   
 $\Rightarrow \begin{cases} 6x = \pi + 2n\pi \\ 4x = -\pi + 2n\pi \end{cases}$   
 $\Rightarrow \begin{cases} x = \frac{\pi}{6} + \frac{n\pi}{3} \\ x = -\frac{\pi}{4} + \frac{n\pi}{2} \end{cases}$   
 ATTENTION: USE  $\cos A + \cos B = 2\cos \frac{A+B}{2} \cos \frac{A-B}{2}$

## Question 87 (\*\*\*\*+)

Use the substitution  $t = \tan \frac{1}{2}x$  to show that if

$$6 \tan \frac{1}{2}x = 1 + 5 \sin x,$$

then the three possible values of  $\tan \frac{1}{2}x$  are

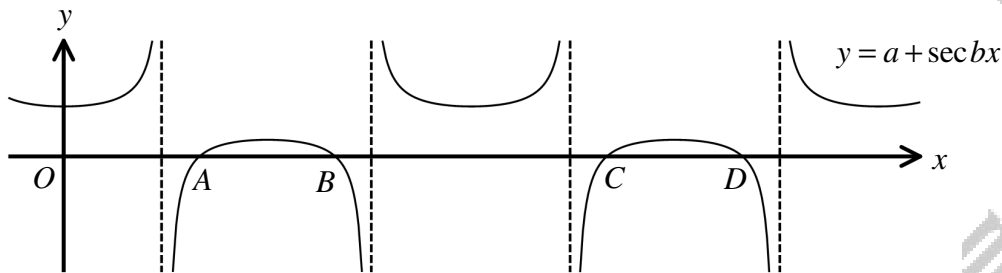
$$1, -\frac{1}{2} \text{ or } -\frac{1}{3}.$$

,  proof

$6 \tan \frac{1}{2}x = 1 + 5 \sin x$   
 By letting  $t = \tan \frac{1}{2}x$   
 $\sin x = \frac{2t}{1+t^2}$  where  $t = \tan \frac{1}{2}x$   
 Thus  
 $\Rightarrow 6t = 1 + 5 \left( \frac{2t}{1+t^2} \right)$   
 $\Rightarrow 6t = 1 + \frac{10t}{1+t^2}$   
 $\Rightarrow 6t(1+t^2) = 1+t^2 + 10t$   
 $\Rightarrow 6t^3 + 6t = t^2 + 10t + 1$   
 $\Rightarrow 6t^3 - t^2 - 4t - 1 = 0$

By inspection  $t=1$  is a solution, so  $t-1$  is a factor.  
 By long division or algebraic manipulations  
 $6t^3 - t^2 - 4t - 1 = (t-1)(6t^2 + 5t + 1) = 0$   
 $(t-1)(3t+1)(2t+1) = 0$   
 $t = 1, -\frac{1}{3}, -\frac{1}{2}$   
 $\therefore \tan \frac{1}{2}x = 1, -\frac{1}{3}, -\frac{1}{2}$   
 As required

## Question 88 (\*\*\*\*+)



The figure above shows part of the graph of

$$y = a + \sec bx,$$

where  $a$  and  $b$  are positive constants.

The points  $A$ ,  $B$ ,  $C$  and  $D$  are the  $x$  intercepts of the graph, with respective coordinates  $\left(\frac{\pi}{3}, 0\right)$ ,  $\left(\frac{2\pi}{3}, 0\right)$ ,  $\left(\frac{4\pi}{3}, 0\right)$  and  $\left(\frac{5\pi}{3}, 0\right)$ .

Determine the value of  $a$  and the value of  $b$ .

$$a = 2, \quad b = 2$$

$y = a + \sec bx$   
 • bounded at A & B (or C & D), they are the same  
 • the secant function has a period of  $2\pi$  (the cosine), so it has been stretched in  $x$  by scale factor of  $\frac{1}{2}$   
 $\therefore b = 2$   
 $y = a + \sec 2x$ , using A  $\left(\frac{\pi}{3}, 0\right)$   
 $\Rightarrow 0 = a + \sec\left(\frac{2\pi}{3}\right)$   
 $\Rightarrow 0 = a + \sec\left(\frac{2\pi}{3}\right)$   
 $\Rightarrow 0 = a + \frac{1}{\cos\left(\frac{2\pi}{3}\right)}$   
 $\Rightarrow 0 = a + \frac{1}{-\frac{1}{2}}$   
 $\Rightarrow 0 = a - 2$   
 $\Rightarrow a = 2$

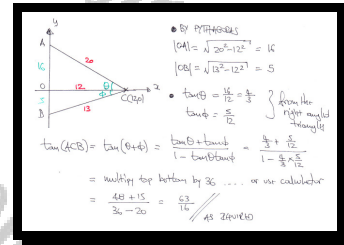
**Question 89** (\*\*\*\*+)

The point  $A$  lies on the  $y$  axis above the origin  $O$  and the point  $B$  lies on the  $y$  axis below the origin  $O$ .

The point  $C(12,0)$  is at a distance of 20 units from  $A$  and at a distance of 13 units from  $B$ .

By considering the tangent ratios of  $\angle OCA$  and  $\angle OCB$ , show that the tangent of the angle  $ACB$  is exactly  $\frac{63}{16}$ .

,  proof

**Question 90** (\*\*\*\*+)

By using the substitution  $t = \tan\left(\frac{1}{2}x\right)$  solve the trigonometric equation

$$3\cos x + 4\sin x = 3 - \tan\left(\frac{1}{2}x\right), \quad 0 \leq x < 360^\circ.$$

$x = 0^\circ, 143.1^\circ$

Handwritten solution for Question 90:

Equation:  $3\cos x + 4\sin x = 3 - \tan\left(\frac{1}{2}x\right)$

Substitution:  $t = \tan\left(\frac{1}{2}x\right)$

Using the double-angle formulas:

- $\cos x = \frac{1-t^2}{1+t^2}$
- $\sin x = \frac{2t}{1+t^2}$
- $\tan\left(\frac{1}{2}x\right) = t$

Substituting into the equation:

$$3\left(\frac{1-t^2}{1+t^2}\right) + 4\left(\frac{2t}{1+t^2}\right) = 3 - t$$

$$\Rightarrow 3(1-t^2) + 8t = (3-t)(1+t^2)$$

$$\Rightarrow 3 - 3t^2 + 8t = 3 + 3t^2 - t - t^3$$

$$\Rightarrow t^3 - 6t^2 + 9t = 0$$

$$\Rightarrow t(t^2 - 6t + 9) = 0$$

$$\Rightarrow t(t-3)^2 = 0$$

$$\Rightarrow t = 0 \text{ or } t = 3$$

When  $t = 0$ ,  $x = 0^\circ$

When  $t = 3$ ,  $x = 2 \times \tan^{-1}(3) = 143.1^\circ$

## Question 91 (\*\*\*\*+)

$$\sin x - \cos x = \sin 2x + \cos 2x - 1, \quad \cos x \neq 0,$$

Show that the above trigonometric equation is equivalent to

$$(\tan x - 1)(\sec x + 2 \tan x) = 0.$$

proof

Handwritten proof for Question 91:

$$\begin{aligned} \sin x - \cos x &= \sin 2x + \cos 2x - 1 \\ \sin x - \cos x &= 2 \sin x \cos x + 1 - 2 \cos^2 x - 1 \\ \sin x - \cos x &= 2 \sin x \cos x - 2 \cos^2 x \\ \sin x - \cos x + 2 \cos^2 x - 2 \sin x \cos x &= 0 \\ (\sin x - \cos x) + 2 \cos x (\cos x - \sin x) &= 0 \\ (\sin x - \cos x)(1 + 2 \cos x) &= 0 \\ \left( \frac{\sin x}{\cos x} - \frac{\cos x}{\cos x} \right) \left( \frac{1}{\cos x} + \frac{2 \cos x}{\cos x} \right) &= \frac{0}{\cos x} \\ (\tan x - 1)(\sec x + 2 \tan x) &= 0 \end{aligned}$$

da Equations

## Question 92 (\*\*\*\*+)

Solve the trigonometric equation

$$\frac{d}{dx}(\sqrt{1 - \cos 2x}) = 1, \quad 0 \leq x < 2\pi.$$

$$x = \frac{\pi}{4}, \frac{5\pi}{4}$$

Handwritten solution for Question 92:

$$\begin{aligned} \frac{d}{dx}[\sqrt{1 - \cos 2x}] &= 1 \\ \frac{d}{dx}[\sqrt{1 - (-2 \sin^2 x)}] &= 1 \\ \frac{d}{dx}[\sqrt{2 \sin^2 x}] &= 1 \\ \frac{d}{dx}[\sqrt{2} \sin x] &= 1 \\ \sqrt{2} \cos x &= 1 \\ \Rightarrow \cos x &= \frac{1}{\sqrt{2}} \\ \Rightarrow \arccos\left(\frac{1}{\sqrt{2}}\right) &= \frac{\pi}{4} \\ x &= \frac{\pi}{4} + 2n\pi \quad \text{or} \quad x = \frac{7\pi}{4} + 2n\pi \quad (0 \leq x < 2\pi) \\ x &= \frac{\pi}{4}, \frac{7\pi}{4} \end{aligned}$$



## Question 93 (\*\*\*\*+)

Given that

$$2 \sec^2 \left( \frac{x}{2} \right) - \frac{1 - \cos x}{\sin x} = 5,$$

find the **finite** value of  $\tan x$ .

$$\tan x = -\frac{12}{5}$$

## Question 94 (\*\*\*\*+)

$$2 \cot 2x + \tan x + 7 = \operatorname{cosec}^2 x, \quad 0 \leq x < \pi.$$

Given that  $x \neq \frac{\pi}{2}$ , find the solutions of the above trigonometric equation, giving the answers in radians correct to two decimal places.

$$x = 0.32^\circ, 2.68^\circ$$

## Question 95 (\*\*\*\*+)

$$\frac{1 + \cos x}{1 - \cos x} = 3 + \sqrt{8} \operatorname{cosec}\left(\frac{x}{2}\right), \quad 0 \leq x < 720.$$

Find the solutions of the above trigonometric equation, giving the answers in **degrees**.

$$x = 30^\circ, 330^\circ, 510^\circ, 570^\circ$$

Handwritten solution for Question 95:

$$\frac{1 + \cos x}{1 - \cos x} = 3 + \sqrt{8} \operatorname{cosec} \frac{x}{2}$$

$$\Rightarrow \frac{1 + (2\cos^2 \frac{x}{2} - 1)}{1 - (2\cos^2 \frac{x}{2} - 1)} = 3 + \sqrt{8} \operatorname{cosec} \frac{x}{2}$$

$$\Rightarrow \frac{2\cos^2 \frac{x}{2}}{2 - 2\cos^2 \frac{x}{2}} = 3 + \sqrt{8} \operatorname{cosec} \frac{x}{2}$$

$$\Rightarrow \frac{\cos^2 \frac{x}{2}}{\sin^2 \frac{x}{2}} = 3 + \sqrt{8} \operatorname{cosec} \frac{x}{2}$$

$$\Rightarrow \operatorname{cosec}^2 \frac{x}{2} = 3 + \sqrt{8} \operatorname{cosec} \frac{x}{2}$$

$$\Rightarrow \operatorname{cosec}^2 \frac{x}{2} - 1 = 3 + \sqrt{8} \operatorname{cosec} \frac{x}{2}$$

$$\Rightarrow \operatorname{cosec}^2 \frac{x}{2} - \sqrt{8} \operatorname{cosec} \frac{x}{2} - 4 = 0$$

$$\Rightarrow \operatorname{cosec}^2 \frac{x}{2} - 2\sqrt{2} \operatorname{cosec} \frac{x}{2} - 4 = 0$$

$$\Rightarrow (\operatorname{cosec} \frac{x}{2} - \sqrt{2})^2 - 2 - 4 = 0$$

$$\Rightarrow (\operatorname{cosec} \frac{x}{2} - \sqrt{2})^2 = 6$$

$$\Rightarrow \operatorname{cosec} \frac{x}{2} - \sqrt{2} = \pm \sqrt{6}$$

$$\Rightarrow \operatorname{cosec} \frac{x}{2} = \sqrt{2} \pm \sqrt{6}$$

$$\Rightarrow \sin \frac{x}{2} = \frac{1}{\sqrt{2} \pm \sqrt{6}}$$

$$\Rightarrow \sin \frac{x}{2} = \frac{\sqrt{2} \pm \sqrt{6}}{4}$$

Thus

$$\frac{x}{2} = 15^\circ \pm 30^\circ$$

$$\frac{x}{2} = 165^\circ \pm 30^\circ$$

$$\frac{x}{2} = -75^\circ \pm 30^\circ$$

$$\frac{x}{2} = 255^\circ \pm 30^\circ$$

$$\begin{cases} x = 30^\circ \pm 720^\circ \\ x = 330^\circ \pm 720^\circ \\ x = -150^\circ \pm 720^\circ \\ x = 510^\circ \pm 720^\circ \end{cases}$$

$$\therefore x = 30^\circ, 330^\circ, 510^\circ, 570^\circ$$

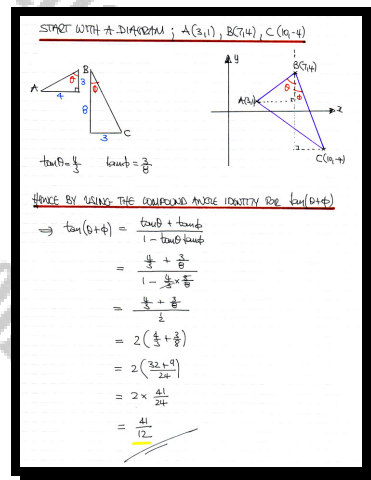
**Question 96** (\*\*\*\*+)

A triangle has vertices at the points with coordinates  $A(3,1)$ ,  $B(7,4)$  and  $C(10,-4)$ .

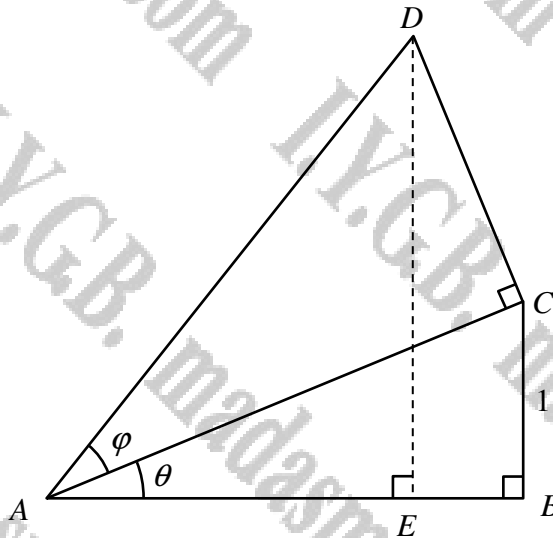
The acute angle  $\theta$  is defined as the angle formed between  $AB$  and the straight line which is parallel to the  $y$  axis and passes through  $B$ .

Find the value of  $\tan \theta$  and hence show that  $\tan(\angle ABC) = \frac{41}{12}$ .

$$\boxed{\phantom{000}}, \quad \tan \theta = \frac{4}{3}$$



## Question 97 (\*\*\*\*+)



The figure above shows two right angles triangles  $ABC$  and  $ACD$ . The angles  $CAB$  and  $DAC$  are denoted by  $\theta$  and  $\phi$ , respectively.

The length of  $BC$  is 1.

The point  $E$  lies on  $AB$  so that the angle  $AED$  is  $90^\circ$ .

Show clearly that the length of  $AE$  is  $\cot \theta - \tan \phi$ .

 , proof

LOOKING AT THE RIGHT ANGLED TRIANGLE ABC

$$\frac{|BC|}{|AC|} = \sin \theta$$

$$\frac{1}{|AC|} = \sin \theta$$

$$|AC| = \frac{1}{\sin \theta}$$

LOOKING AT THE RIGHT ANGLED TRIANGLE ACD

$$|AC| = \sec \phi$$

$$|AB| = \frac{|AC|}{\cos \phi}$$

$$|AB| = \frac{1}{\sin \theta \cos \phi}$$

FINALLY LOOKING AT THE RIGHT ANGLED TRIANGLE ADE

$$|AE| = |AB| \cos(\theta + \phi)$$

$$|AE| = \frac{\cos(\theta + \phi)}{\sin \theta \cos \phi}$$

$$|AE| = \frac{\cos \theta \cos \phi - \sin \theta \sin \phi}{\sin \theta \cos \phi}$$

$$|AE| = \frac{\cos \theta \cos \phi}{\sin \theta \cos \phi} - \frac{\sin \theta \sin \phi}{\sin \theta \cos \phi}$$

$$|AE| = \cot \theta - \tan \phi$$

A. BEVERIO

## Question 98 (\*\*\*\*+)

It is given that

$$\sin \varphi = k \sin \theta, \quad k \neq 0, \quad k \neq \pm 1.$$

Show, by a detailed method, that

$$1 + \left( \frac{d\varphi}{d\theta} \right)^2 = (k^2 - 1) \sec^2 \varphi$$

 ,  proof

$\sin \varphi = k \sin \theta \quad k \neq 0, k \neq \pm 1$   
 Differentiate w.r.t  $\theta$   
 $\Rightarrow \cos \varphi \frac{d\varphi}{d\theta} = k \cos \theta$   
 $\Rightarrow \cos \varphi \left( \frac{d\varphi}{d\theta} \right)^2 = k^2 \cos^2 \theta$   
 $\Rightarrow \cos \varphi \left( \frac{d\varphi}{d\theta} \right)^2 = k^2 (1 - \sin^2 \theta)$   
 $\Rightarrow \cos \varphi \left( \frac{d\varphi}{d\theta} \right)^2 = k^2 - k^2 \sin^2 \theta$   
 $\Rightarrow \cos \varphi \left( \frac{d\varphi}{d\theta} \right)^2 = k^2 - \sin^2 \varphi$   
 $\Rightarrow \left( \frac{d\varphi}{d\theta} \right)^2 = \frac{k^2 - \sin^2 \varphi}{\cos \varphi}$   
 $\Rightarrow \left( \frac{d\varphi}{d\theta} \right)^2 = k^2 \sec \varphi - \tan \varphi$   
 $\Rightarrow \left( \frac{d\varphi}{d\theta} \right)^2 = k^2 \sec \varphi - (\sec \varphi - 1)$   
 $\Rightarrow 1 + \left( \frac{d\varphi}{d\theta} \right)^2 = (k^2 - 1) \sec \varphi$

## Question 99 (\*\*\*\*+)

It is given that if  $x \in (0, \frac{1}{2}\pi)$ , then

$$\frac{\sin 3x}{\sin x} = \frac{1}{2}.$$

Determine the exact value of  $\frac{\cos 3x}{\cos x}$ .

$$\square, -\frac{9}{16}$$

START WITH THE TRIPLE IDENTITY FOR  $\sin 3x$  — USE COMPLEX NUMBERS AS WE WILL ALSO NEED  $\cos 3x$

$$\begin{aligned} \cos x + i \sin x &= e^{ix} \\ (\cos x + i \sin x)^3 &= (e^{ix})^3 \\ \cos 3x + i \sin 3x &= e^{3ix} = \cos 3x + i \sin 3x \end{aligned}$$

$\bullet \cos 3x = \cos^3 x - 3\cos x \sin^2 x$ $\cos 3x = \cos^3 x - 3\cos x(1 - \cos^2 x)$ $\cos 3x = \cos^3 x - 3\cos x + 3\cos^3 x$ $\cos 3x = 4\cos^3 x - 3\cos x$	$\bullet \sin 3x = 3\cos^2 x \sin x - \sin^3 x$ $\sin 3x = 3\cos^2 x \sin x - \sin^3 x$ $\sin 3x = 3\cos^2 x \sin x - \sin^3 x$ $\sin 3x = 3\cos^2 x \sin x - \sin^3 x$
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THIS WE KNOW FROM TRIPLE IDENTITY

$$\begin{aligned} \Rightarrow \frac{\sin 3x}{\sin x} &= \frac{1}{2} \\ \Rightarrow \frac{3\cos^2 x - \sin^2 x}{\sin x} &= \frac{1}{2} \\ \Rightarrow 3 - 4\sin^2 x &= \frac{1}{2} \\ \Rightarrow \sin^2 x &= \frac{3}{8} \end{aligned}$$

SIMILARLY NOW, WORK OUT  $\cos 3x$

$$\begin{aligned} \frac{\cos 3x}{\cos x} &= \frac{4\cos^3 x - 3\cos x}{\cos x} \\ &= 4\cos^2 x - 3 \quad \left(3 \times \frac{3}{8}\right) \\ &= 4\left(\frac{3}{8}\right) - 3 \\ &= \frac{3}{2} - 3 \\ &= -\frac{3}{2} \\ &= -\frac{9}{16} \end{aligned}$$

# 147

## ENRICHMENT QUESTIONS

## Question 1 (\*\*\*\*)

Use trigonometric algebra to find the solution of the following simultaneous equations, in the intervals  $0 \leq x < 2\pi$ ,  $0 \leq y < 2\pi$ .

$$4 \cos y = 3 - 2 \sin x \quad \text{and} \quad 4y - 2x = \pi.$$

$$\boxed{\phantom{000}}, \quad x = \frac{1}{6}\pi, \quad y = \frac{1}{3}\pi$$

• START BY REARRANGING THE SECOND EQUATION FOR  $x$

$$\begin{aligned} \Rightarrow 4y - 2x &= \pi \\ \Rightarrow 4y - \pi &= 2x \\ \Rightarrow x &= 2y - \frac{\pi}{2} \end{aligned}$$

• SUBSTITUTE INTO THE OTHER

$$\begin{aligned} \Rightarrow 4 \cos y &= 3 - 2 \sin x \\ \Rightarrow 4 \cos y &= 3 - 2 \sin \left( 2y - \frac{\pi}{2} \right) \\ \Rightarrow 4 \cos y &= 3 + 2 \sin \left( \frac{\pi}{2} - 2y \right) \\ \Rightarrow 4 \cos y &= 3 + 2 \cos(2y) \\ \Rightarrow 4 \cos y &= 3 + 2 \left[ 2 \cos^2 y - 1 \right] \\ \Rightarrow 4 \cos y &= 3 + 4 \cos^2 y - 2 \\ \Rightarrow 0 &= 4 \cos^2 y - 4 \cos y + 1 \\ \Rightarrow (2 \cos y - 1)^2 &= 0 \\ \Rightarrow \cos y &= \frac{1}{2} \end{aligned}$$

• HENCE WE HAVE

$$\begin{aligned} y &= \dots -\frac{\pi}{3}, \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}, \dots \\ x &= \dots -\frac{\pi}{6}, \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \dots \end{aligned}$$


∴ ONLY SOLUTION IS  $\left( \frac{\pi}{6}, \frac{\pi}{3} \right)$  //

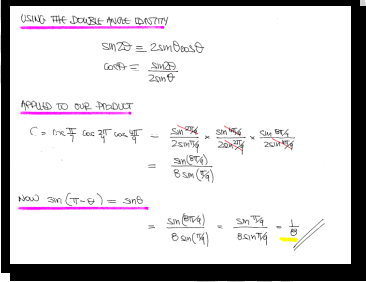


## Question 2 (\*\*\*\*)

$$C = \cos\left(\frac{1}{9}\pi\right) \cos\left(\frac{2}{9}\pi\right) \cos\left(\frac{4}{9}\pi\right)$$

Use the **sine double angle identity** for sine to show that  $C = \frac{1}{8}$ .

, **proof**



USING THE DOUBLE ANGLE IDENTITY  
 $\sin 2\theta = 2 \sin \theta \cos \theta$   
 $\cos \theta = \frac{\sin 2\theta}{2 \sin \theta}$   
 APPLY TO OUR PRODUCT  
 $C = \cos \frac{\pi}{9} \cos \frac{2\pi}{9} \cos \frac{4\pi}{9} = \frac{\sin \frac{2\pi}{9}}{2 \sin \frac{\pi}{9}} \times \frac{\sin \frac{4\pi}{9}}{2 \sin \frac{2\pi}{9}} \times \frac{\sin \frac{8\pi}{9}}{2 \sin \frac{4\pi}{9}}$   
 $= \frac{\sin \frac{8\pi}{9}}{8 \sin \frac{\pi}{9}}$   
 NOW  $\sin(\pi - \theta) = \sin \theta$   
 $= \frac{\sin \frac{8\pi}{9}}{8 \sin \frac{\pi}{9}} = \frac{\sin \frac{\pi}{9}}{8 \sin \frac{\pi}{9}} = \frac{1}{8}$

The acute angles  $x$  and  $y$ , satisfy the following relationships.

$$\boxed{\phantom{000}}, \quad \tan y = \begin{cases} 3 \\ \frac{19}{3} \end{cases}$$

$$\boxed{\phantom{000}}, \quad \tan y = \begin{cases} 3 \\ \frac{19}{3} \end{cases}$$

$$\boxed{\phantom{000}}, \quad \tan y = \begin{cases} 3 \\ \frac{19}{3} \end{cases}$$

**Question 4** (\*\*\*\*)

Two circles,  $C_1$  and  $C_2$ , have respective radii of 4 units and 1 unit and are touching each other externally at the point  $A$ .

The coordinates axes are tangents to  $C_1$ , whose centre  $P$  lies in the first quadrant.

The  $x$  axis is a tangent to  $C_2$ , whose centre  $Q$  also lies in the first quadrant.

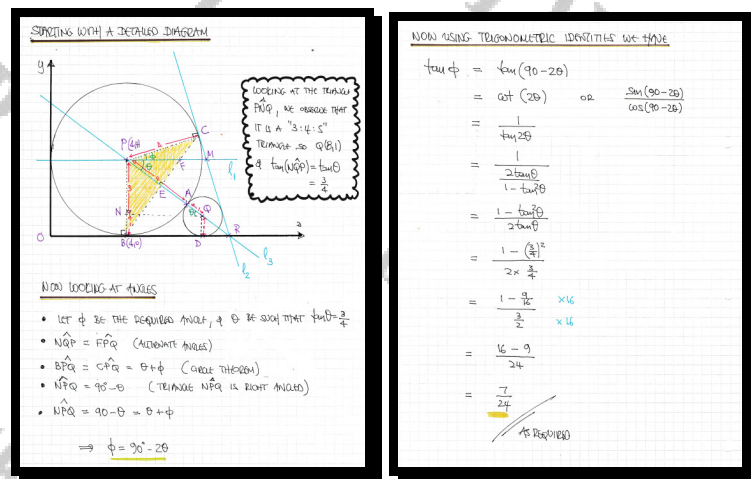
The straight line  $l_1$  is parallel to the  $x$  axis and passes through  $P$ .

The straight line  $l_2$  has negative gradient and is a common tangent to  $C_1$  and  $C_2$ , touching  $C_1$  at the point  $C$ .

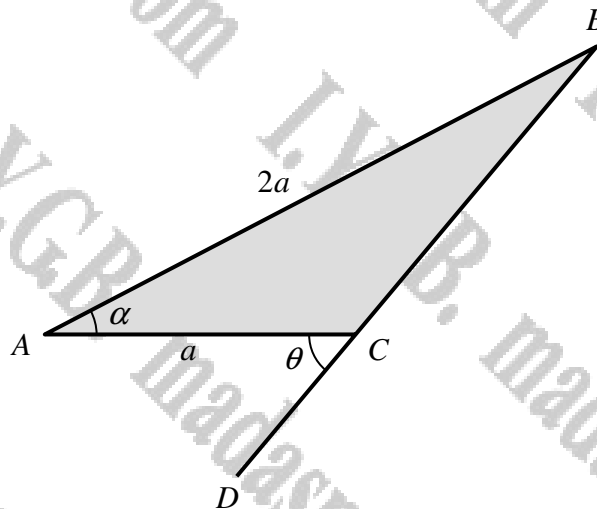
The acute angle formed by  $PC$  and  $l_1$  is denoted by  $\phi$ .

Show that  $\tan \phi = \frac{7}{24}$ .

,  proof



Question 5 (\*\*\*\*)



The figure above shows a triangle  $ABC$ , where  $|AB| = a$  and  $|AC| = 2a$ .

The angle  $BAC$  is  $\alpha$ , where  $\tan \alpha = \frac{3}{4}$ .

The side  $BC$  is extended to the point  $D$  so that the angle  $ACD$  is denoted by  $\theta$ .

Show clearly that  $\theta = \arctan 2$ .

☐ , ☐ proof

• START WITH THE DIAGONAL, LET  $BE = b$

• BY THE COSINE RULE ON  $\triangle ABC$

$$|BC|^2 = |AB|^2 + |AC|^2 - 2|AB||AC|\cos \alpha$$

$$\Rightarrow |BC|^2 = a^2 + 4a^2 - 4a^2 \cos \alpha$$

$$\Rightarrow |BC|^2 = 5a^2 - 4a^2 \times \frac{4}{5}$$

$$\Rightarrow |BC|^2 = 5a^2 - \frac{16a^2}{5}$$

$$\Rightarrow |BC|^2 = \frac{9}{5}a^2$$

$$\Rightarrow |BC| = \frac{3}{\sqrt{5}}a$$

• NEXT BY THE SINE RULE ON  $\triangle ABC$

$$\Rightarrow \frac{\sin \alpha}{\frac{3}{\sqrt{5}}a} = \frac{\sin b}{a}$$

$$\Rightarrow \sin b = \frac{3}{\sqrt{5}} \sin \alpha$$

$$\Rightarrow \sin b = \frac{3}{\sqrt{5}} \times \frac{3}{5}$$

$$\Rightarrow \sin b = \frac{9}{25}$$

• NEXT GET THE SINUS RATIO OF  $b$

$\sin b = \frac{3}{5}$   
 $\cos b = \frac{4}{5}$   
 $\tan b = \frac{3}{4}$

• FINALLY WE HAVE

$$\Rightarrow b = \alpha + \theta$$

$$\Rightarrow \tan b = \tan(\alpha + \theta)$$

$$\Rightarrow \tan b = \frac{\tan \alpha + \tan \theta}{1 - \tan \alpha \tan \theta}$$

$$\Rightarrow \tan b = \frac{\frac{3}{4} + \frac{1}{2}}{1 - \frac{3}{4} \times \frac{1}{2}}$$

$$\Rightarrow \tan b = \frac{\frac{3}{2}}{1 - \frac{3}{8}}$$

$$\Rightarrow \tan b = \frac{10}{5 - 3}$$

$$\Rightarrow \tan b = \frac{10}{2}$$

$$\Rightarrow \tan b = 2$$

$$\Rightarrow b = \arctan 2$$

## Question 6 (\*\*\*\*\*)

The acute angles  $\theta$ ,  $\psi$  and  $\alpha$  satisfy the following equations.

$$4 \tan \theta = \tan \alpha$$

$$(5 + 3 \cos 2\alpha) \tan \psi = 3 \sin 2\alpha.$$

Express  $\theta + \psi$ , in terms of  $\alpha$ .

$$\boxed{\phantom{000}}, \quad \theta + \psi = \alpha$$

**Method 1 (Left):**

- Given:  $4 \tan \theta = \tan \alpha$  and  $(5 + 3 \cos 2\alpha) \tan \psi = 3 \sin 2\alpha$
- Start from the second equation:
 
$$\Rightarrow \tan \psi = \frac{3 \sin 2\alpha}{5 + 3 \cos 2\alpha} = \frac{3(2 \sin \alpha \cos \alpha)}{5 + 3(1 - 2 \sin^2 \alpha)}$$

$$= \frac{6 \sin \alpha \cos \alpha}{8 - 6 \sin^2 \alpha}$$
- Divide top & bottom by  $\cos^2 \alpha$ :
 
$$= \frac{6 \tan \alpha}{\frac{8}{\cos^2 \alpha} - 6 \tan^2 \alpha} = \frac{6 \tan \alpha}{4 \sec^2 \alpha - 3 \tan^2 \alpha}$$

$$= \frac{3 \tan \alpha}{4(1 + \tan^2 \alpha) - 3 \tan^2 \alpha} = \frac{3 \tan \alpha}{4 + \tan^2 \alpha}$$
- Substituting  $T = \tan \alpha$  into both equations:
 
$$\tan \theta = \frac{1}{4} T$$

$$\tan \psi = \frac{3T}{4 + T^2}$$
- Hence by the compound angle identities:
 
$$\tan(\theta + \psi) = \frac{\tan \theta + \tan \psi}{1 - \tan \theta \tan \psi} = \frac{\frac{1}{4} T + \frac{3T}{4 + T^2}}{1 - \frac{1}{4} T \left( \frac{3T}{4 + T^2} \right)}$$

**Method 2 (Right):**

- Multiply "top & bottom" by  $4(4 + T^2)$ :
 
$$\dots = \frac{T(4 + T^2) + 12T}{4(4 + T^2) - 3T^2} = \frac{T^3 + 16T}{T^2 + 16}$$

$$= \frac{T(T^2 + 16)}{T^2 + 16} = T$$
- $\therefore \tan(\theta + \psi) = \tan \alpha$
- $\therefore \theta + \psi = \alpha$

## Question 7 (\*\*\*\*)

The acute angles  $\theta$  and  $\varphi$  satisfy the following equations

$$2 \cos \theta = \cos \varphi$$

$$2 \sin \theta = 3 \sin \varphi$$

Show clearly that

$$\theta + \varphi = \pi - \arctan \sqrt{15}$$

,  proof

**Method 1: Using the Addition Formulae**

Given:  $2 \cos \theta = \cos \varphi$  and  $2 \sin \theta = 3 \sin \varphi$

Divide the two equations to find  $\tan \varphi$ :

$$\frac{2 \sin \theta}{2 \cos \theta} = \frac{\cos \varphi}{3 \sin \varphi} \Rightarrow \tan \theta = \frac{\cos \varphi}{3 \sin \varphi}$$

From  $2 \cos \theta = \cos \varphi$ , we have  $\cos \theta = \frac{\cos \varphi}{2}$ . Using the identity  $\sin^2 \theta + \cos^2 \theta = 1$ :

$$\sin^2 \theta = 1 - \cos^2 \theta = 1 - \frac{\cos^2 \varphi}{4}$$

From  $2 \sin \theta = 3 \sin \varphi$ , we have  $\sin \theta = \frac{3 \sin \varphi}{2}$ . Squaring both sides:

$$\sin^2 \theta = \frac{9 \sin^2 \varphi}{4}$$

Equating the two expressions for  $\sin^2 \theta$ :

$$\frac{9 \sin^2 \varphi}{4} = 1 - \frac{\cos^2 \varphi}{4}$$

$$9 \sin^2 \varphi = 4 - \cos^2 \varphi$$

$$9(1 - \cos^2 \varphi) = 4 - \cos^2 \varphi$$

$$9 - 9 \cos^2 \varphi = 4 - \cos^2 \varphi$$

$$5 = 8 \cos^2 \varphi$$

$$\cos^2 \varphi = \frac{5}{8}$$

$$\cos \varphi = \frac{\sqrt{10}}{4}$$

Since  $\varphi$  is acute,  $\cos \varphi = \frac{\sqrt{10}}{4}$ . Then  $\sin \varphi = \frac{3}{4}$ .

Now find  $\tan \theta$ :

$$\tan \theta = \frac{\cos \varphi}{3 \sin \varphi} = \frac{\frac{\sqrt{10}}{4}}{3 \cdot \frac{3}{4}} = \frac{\sqrt{10}}{9}$$

Let  $\theta = \arctan \frac{\sqrt{10}}{9}$ . Then  $\varphi = \arccos \frac{\sqrt{10}}{4}$ .

**Method 2: Using the Compound Angle Formulae**

Let  $\theta + \varphi = \alpha$ . Then  $\alpha = \pi - \arctan \sqrt{15}$ .

Using the compound angle formulae for  $\cos(\theta + \varphi)$  and  $\sin(\theta + \varphi)$ :

$$\cos(\theta + \varphi) = \cos \theta \cos \varphi - \sin \theta \sin \varphi$$

$$\sin(\theta + \varphi) = \sin \theta \cos \varphi + \cos \theta \sin \varphi$$

From the given equations, we have  $\cos \varphi = 2 \cos \theta$  and  $\sin \varphi = \frac{2}{3} \sin \theta$ .

Substitute these into the compound angle formulae:

$$\cos(\theta + \varphi) = \cos \theta (2 \cos \theta) - \sin \theta \left( \frac{2}{3} \sin \theta \right) = 2 \cos^2 \theta - \frac{2}{3} \sin^2 \theta$$

$$\sin(\theta + \varphi) = \sin \theta (2 \cos \theta) + \cos \theta \left( \frac{2}{3} \sin \theta \right) = 2 \sin \theta \cos \theta + \frac{2}{3} \sin \theta \cos \theta = \frac{8}{3} \sin \theta \cos \theta$$

Divide the two equations to find  $\tan(\theta + \varphi)$ :

$$\tan(\theta + \varphi) = \frac{\sin(\theta + \varphi)}{\cos(\theta + \varphi)} = \frac{\frac{8}{3} \sin \theta \cos \theta}{2 \cos^2 \theta - \frac{2}{3} \sin^2 \theta}$$

Divide numerator and denominator by  $\cos^2 \theta$ :

$$\tan(\theta + \varphi) = \frac{\frac{8}{3} \tan \theta}{2 - \frac{2}{3} \tan^2 \theta}$$

Let  $t = \tan \theta$ . Then:

$$\tan(\theta + \varphi) = \frac{\frac{8}{3} t}{2 - \frac{2}{3} t^2}$$

From Method 1, we know  $\tan \theta = \frac{\sqrt{10}}{9}$ . Substitute this value:

$$\tan(\theta + \varphi) = \frac{\frac{8}{3} \cdot \frac{\sqrt{10}}{9}}{2 - \frac{2}{3} \left( \frac{\sqrt{10}}{9} \right)^2} = \frac{\frac{8\sqrt{10}}{27}}{2 - \frac{20}{27}} = \frac{\frac{8\sqrt{10}}{27}}{\frac{34}{27}} = \frac{8\sqrt{10}}{34} = \frac{4\sqrt{10}}{17}$$

Since  $\theta + \varphi$  is obtuse,  $\tan(\theta + \varphi) = -\frac{4\sqrt{10}}{17}$ .

Therefore,  $\theta + \varphi = \pi - \arctan \frac{4\sqrt{10}}{17}$ .

## Question 8 (\*\*\*\*)

Determine the range of the following function

$$f(\theta) \equiv \frac{1}{\sin^2 \theta + 3 \sin \theta \cos \theta + 5 \cos^2 \theta}, \theta \in \mathbb{R}$$

$$\boxed{\phantom{00}}, \boxed{\frac{2}{11} \leq f(\theta) \leq 2}$$

Handwritten solution for Question 8:

$$f(\theta) = \frac{1}{\sin^2 \theta + 3 \sin \theta \cos \theta + 5 \cos^2 \theta} \quad \theta \in \mathbb{R}$$

$$\Rightarrow f(\theta) = \frac{1}{\sin^2 \theta + \cos^2 \theta + 2 \sin \theta \cos \theta + 4 \cos^2 \theta}$$

$$\Rightarrow f(\theta) = \frac{1}{1 + 2 \sin \theta \cos \theta + 4 \cos^2 \theta}$$

$$\Rightarrow f(\theta) = \frac{1}{1 + \sin 2\theta + 2 + 2 \cos 2\theta}$$

$$\Rightarrow f(\theta) = \frac{1}{3 + \sin 2\theta + 2 \cos 2\theta}$$

"R TRANSFORMATION" / HARMONIC FORM

$$R = \sqrt{(\sin)^2 + (\cos)^2} = \sqrt{1^2 + 2^2} = \sqrt{5} = \sqrt{\frac{20}{4}} = \frac{\sqrt{20}}{2}$$

$$\Rightarrow f(\theta) = \frac{1}{3 + \frac{\sqrt{20}}{2} \cos(2\theta - \alpha)}$$

$$\therefore f(\theta)_{\min} = \frac{1}{3 + \frac{\sqrt{20}}{2}} = \frac{2}{6 + \sqrt{20}} = \frac{2}{6 + 2\sqrt{5}} = \frac{1}{3 + \sqrt{5}}$$

$$f(\theta)_{\max} = \frac{1}{3 - \frac{\sqrt{20}}{2}} = \frac{2}{6 - \sqrt{20}} = \frac{2}{6 - 2\sqrt{5}} = \frac{1}{3 - \sqrt{5}}$$

$$\therefore \text{RANGE: } \frac{2}{11} \leq f(\theta) \leq 2$$

It is given that

where  $a$  is a non zero constant.

$$\boxed{\phantom{000}}, \quad \boxed{\tan\left(\frac{1}{2}x\right) = \frac{1}{2}}, \quad \boxed{\tan\left(\frac{1}{2}x\right) = \frac{1}{a}}$$

● USING THE HALF-ANGLE IDENTITIES

$\sin x = \frac{2t}{1+t^2}$

$\cos x = \frac{1-t^2}{1+t^2}$

where  $t = \tan \frac{x}{2}$

$$\Rightarrow (x+2) \sin x + (2x-1) \cos x = 2x+1$$

$$\Rightarrow (x+2) \cdot \frac{2t}{1+t^2} + (2x-1) \cdot \frac{1-t^2}{1+t^2} = 2x+1$$

$$\Rightarrow (x+2) \cdot 2t + (2x-1) \cdot (1-t) = (2x+1)(1+t)$$

$$\Rightarrow 2(x+2)t + (2x-1) - (2x-1)t = (2x+1) + (2x+1)t$$

$$\Rightarrow 0 = (2x+1)t + (2x-1)t - 2(x+2)t + (2x+1) - (2x-1)$$

$$\Rightarrow 0 = 4t^2 - 2(x+2)t + 2$$

$$\Rightarrow 2t^2 - (x+2)t + 1 = 0$$

● BY PERMUTATION

$$\Rightarrow (x-1)(x-1) = 0$$

$$\Rightarrow t = \frac{1}{2}$$

$$\Rightarrow \tan \frac{x}{2} = \frac{1}{2}$$

OF THE QUADRATIC FORMULA

$$t = \frac{(x+2) \pm \sqrt{(x+2)^2 - 4 \cdot 2 \cdot 1}}{2 \cdot 2}$$

$$t = \frac{(x+2) \pm \sqrt{x^2 + 4x + 4 - 8}}{4}$$

$$t = \frac{(x+2) \pm \sqrt{x^2 + 4x - 4}}{4}$$

$$t = \frac{(x+2) \pm \sqrt{(x+2)^2 - 8}}{4}$$

$$t = \frac{x+2 \pm \sqrt{(x+2)^2 - 8}}{4}$$

$$t = \frac{x+2 \pm \sqrt{x^2 + 4x - 4}}{4}$$



## Question 10 (\*\*\*\*\*)

Solve the following trigonometric equation.

$$\cos 4x^\circ = \cos 40^\circ + \cos 80^\circ, \quad 0^\circ \leq x \leq 180^\circ.$$

$$\boxed{\phantom{000}}, \quad \boxed{x = 5^\circ, 85^\circ, 95^\circ, 175^\circ}$$

$\cos 4x = \cos 40^\circ + \cos 80^\circ \quad 0 \leq x < 180^\circ$

• START BY MANIPULATING THE RIGHT HAND SIDE

$$\cos(A+B) = \cos A \cos B - \sin A \sin B$$

$$\cos(A-B) = \cos A \cos B + \sin A \sin B$$

$$\cos(A+B) + \cos(A-B) = 2 \cos A \cos B$$

$$\cos(40^\circ + 2x) + \cos(40^\circ - 2x) = 2 \cos 40^\circ \cos 2x$$

• HENCE WE HAVE

$$\Rightarrow \cos 4x = 2 \cos 40^\circ \cos 2x$$

$$\Rightarrow \cos 4x = 2 \times \frac{1}{2} \times \cos 2x$$

$$\Rightarrow \cos 4x = \cos 2x$$

$$\begin{pmatrix} 4x = 2x \pm 360^\circ \\ 4x = 360^\circ \pm 360^\circ \end{pmatrix} \quad \Rightarrow x = 0, 180^\circ$$

$$\begin{pmatrix} x = 5^\circ \pm 90^\circ \\ x = 85^\circ \pm 90^\circ \end{pmatrix}$$

$\therefore x = 5^\circ, 95^\circ, 85^\circ, 175^\circ$

## Question 11 (\*\*\*\*\*)

A right circular cone, of radius  $r$  and semi-vertical angle  $\theta$ , lies with one of its generators in contact with a horizontal surface.

The cone is then rolled on the horizontal surface with its vertex at rest, so that the rolling circumference of its base completes a full circle on the surface, while the cone completes  $N$  revolutions about its own axis.

Show that  $N = \operatorname{cosec} \theta$ .

$$\boxed{\text{SP}}, \quad \boxed{\text{proof}}$$

• LET THE RADIUS OF THE CONE BE  $r$  AND ITS HEIGHT  $h$

• THEN LOOKING AT THE DIAGONAL, THE CIRCLE TO BE ROLLED HAS RADIUS  $\sqrt{r^2 + h^2}$

• THIS THE CIRCLE HAS CIRCUMFERENCE  $2\pi \sqrt{r^2 + h^2}$

• THE CIRCUMFERENCE OF THE BASE IS  $2\pi r$

• HENCE THE NUMBER OF TURNS IS GIVEN BY

$$N = \frac{2\pi \sqrt{r^2 + h^2}}{2\pi r} = \frac{\sqrt{r^2 + h^2}}{r} = \frac{h \sqrt{\frac{r^2}{h^2} + 1}}{r}$$

$$= \frac{\sqrt{\left(\frac{r}{h}\right)^2 + 1}}{\frac{r}{h}} = \frac{\sqrt{\tan^2 \theta + 1}}{\tan \theta} = \frac{\sqrt{\sec^2 \theta}}{\tan \theta}$$

$$= \frac{\sec \theta}{\tan \theta} = \sec \theta \times \cot \theta = \frac{1}{\sin \theta} \times \frac{\cos \theta}{\sin \theta} = \operatorname{cosec} \theta$$

AS REQUIRED

## Question 12 (\*\*\*\*)

Solve the following trigonometric equation.

$$2\sqrt{3}\sin\left(x + \frac{7\pi}{12}\right) = 3\operatorname{cosec}\left(x + \frac{5\pi}{12}\right), \quad 0 \leq x \leq 2\pi.$$

$$\boxed{\phantom{000}}, \quad x = \frac{\pi}{12}, \frac{11\pi}{12}, \frac{13\pi}{12}, \frac{23\pi}{12}$$

$2\sqrt{3}\sin\left(x + \frac{7\pi}{12}\right) = 3\operatorname{cosec}\left(x + \frac{5\pi}{12}\right), \quad 0 \leq x < 2\pi$   
 $\Rightarrow 2\sqrt{3}\sin\left(x + \frac{7\pi}{12}\right) = \frac{3}{\sin\left(x + \frac{5\pi}{12}\right)}$   
 $\Rightarrow 2\sin\left(x + \frac{7\pi}{12}\right)\sin\left(x + \frac{5\pi}{12}\right) = \frac{3}{\sqrt{3}}$   
 NOW INCLUDE AN IDENTITY BASED ON THE COMPOUND ANGLE IDENTITIES  
 $\cos(A+B) \equiv \cos A \cos B - \sin A \sin B$   
 $\cos(A-B) \equiv \cos A \cos B + \sin A \sin B$   
 $\cos(A-B) - \cos(A+B) \equiv 2\sin A \sin B$   
 $\Rightarrow \cos\left[\left(x + \frac{7\pi}{12}\right) - \left(x + \frac{5\pi}{12}\right)\right] - \cos\left[\left(x + \frac{7\pi}{12}\right) + \left(x + \frac{5\pi}{12}\right)\right] = \frac{3/\sqrt{3}}{2}$   
 $\Rightarrow \cos\frac{\pi}{6} - \cos(2x + \pi) = \sqrt{3}$   
 $\Rightarrow \frac{\sqrt{3}}{2} - [\cos 2x \cos \pi - \sin 2x \sin \pi] = \sqrt{3}$   
 $\Rightarrow \frac{\sqrt{3}}{2} + \cos 2x = \sqrt{3}$   
 $\Rightarrow \cos 2x = \frac{\sqrt{3}}{2}$   
 $\text{OR } \cos\left(\frac{2x}{2}\right) = \frac{\sqrt{3}}{2}$   
 $\left(\begin{array}{l} 2x = \frac{\pi}{6} \pm 2\pi n \\ 2x = \frac{11\pi}{6} \pm 2\pi n \end{array} \quad n=0,1,2,\dots\right)$   
 $\left(\begin{array}{l} x = \frac{\pi}{12} \pm \pi n \\ x = \frac{11\pi}{12} \pm \pi n \end{array}\right)$   
 $x = \frac{\pi}{12}, \frac{11\pi}{12}, \frac{13\pi}{12}, \frac{23\pi}{12}$

## Question 13 (\*\*\*\*\*)

Solve the following trigonometric equation.

$$\sin(2\theta + 58)^\circ + 2\sin^2(42^\circ) = 1, \quad 0 \leq \theta < 360.$$

$$\boxed{\phantom{000}}, \quad \theta = \{58, 154, 238, 334\}$$

$\sin(2\theta + 58^\circ) + 2\sin^2 42^\circ = 1 \quad 0 \leq \theta < 360^\circ$   
Solving the equation as follows  
 $\Rightarrow \sin(2\theta + 58^\circ) = 1 - 2\sin^2 42^\circ$   
 $\Rightarrow \sin(2\theta + 58^\circ) = \cos(2 \times 42^\circ)$   
 $\Rightarrow \sin(2\theta + 58^\circ) = \cos 84^\circ$   
 $\Rightarrow \sin(2\theta + 58^\circ) = \sin(180^\circ - 84^\circ)$   
 $\Rightarrow \sin(2\theta + 58^\circ) = \sin 96^\circ$   
Extracting a general solution  
 $\Rightarrow \begin{cases} 2\theta + 58^\circ = 96^\circ \pm 360^\circ \\ 2\theta + 58^\circ = 176^\circ \pm 360^\circ \end{cases} \quad n = 0, 1, 2, \dots$   
 $\Rightarrow \begin{cases} 2\theta = -52^\circ \pm 360^\circ \\ 2\theta = 116^\circ \pm 360^\circ \end{cases}$   
 $\Rightarrow \begin{cases} \theta = -26^\circ \pm 180^\circ \\ \theta = 58^\circ \pm 180^\circ \end{cases}$   
 $\Rightarrow \theta = 154^\circ, 334^\circ, 58^\circ, 238^\circ$   
 $\Rightarrow \theta = 58^\circ, 154^\circ, 238^\circ, 334^\circ$

## Question 14 (\*\*\*\*\*)

Solve the following trigonometric equation

$$\cos\left(\arcsin\frac{1}{4}\right) \sin(\arccos x) = \frac{1}{4}(4-x), \quad x \in \mathbb{R}.$$

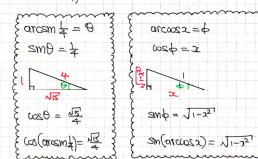
$$\boxed{\phantom{000}}, \quad \boxed{x = \frac{1}{4}}$$

METHOD AATTEMPT TO CREATE A SINE COMPOUND IDENTITY

$$\begin{aligned} \Rightarrow \cos(\arcsin \frac{1}{4}) \sin(\arccos x) &= \frac{1}{4}(4-x) \\ \Rightarrow \cos(\arcsin \frac{1}{4}) \sin(\arccos x) &= 1 - \frac{1}{4}x \\ \Rightarrow \frac{1}{4}x + \cos(\arcsin \frac{1}{4}) \sin(\arccos x) &= 1 \\ \text{Let } A = \arcsin \frac{1}{4} \quad B = \arccos x \\ \Rightarrow \sin(\arcsin \frac{1}{4}) \cos(\arccos x) + \cos(\arcsin \frac{1}{4}) \sin(\arccos x) &= 1 \\ \Rightarrow \sin(\arcsin \frac{1}{4} + \arccos x) &= 1 \\ \Rightarrow \arcsin \frac{1}{4} + \arccos x &= \frac{\pi}{2} + 2n\pi, \quad n = 0, 1, 2, \dots \\ \text{But } \arccos x \text{ can only obtain values between } 0 \text{ \& } \pi \\ \Rightarrow \arcsin \frac{1}{4} + \arccos x &= \frac{\pi}{2} \\ \Rightarrow \arccos x &= \frac{\pi}{2} - \arcsin \frac{1}{4} \\ \Rightarrow \cos(\arccos x) &= \cos(\frac{\pi}{2} - \arcsin \frac{1}{4}) \\ \Rightarrow x &= \sin(\arcsin \frac{1}{4}) \quad \left( \cos(\frac{\pi}{2} - b) = \sin b \right) \\ \Rightarrow x &= \frac{1}{4} \end{aligned}$$

METHOD B

$$\Rightarrow \cos(\arcsin \frac{1}{4}) \sin(\arccos x) = \frac{1}{4}(4-x)$$



$$\begin{aligned} \Rightarrow \frac{\sqrt{15}}{4} \sqrt{1-x^2} &= \frac{1}{4}(4-x) \\ \Rightarrow \sqrt{15(1-x^2)} &= 4-x \\ \Rightarrow 15(1-x^2) &= (4-x)^2 \\ \Rightarrow 15 - 15x^2 &= 16 - 8x + x^2 \\ \Rightarrow 0 &= 16x^2 - 8x + 1 \\ \Rightarrow (4x-1)^2 &= 0 \\ \Rightarrow x &= \frac{1}{4} \end{aligned}$$

SOLUTION CHECKED AGAINST THE QUESTION BEFORE SUBMITTING

## Question 15 (\*\*\*\*\*)

It is given that  $0 < x < \frac{1}{2}\pi$  and  $0 < y < \frac{1}{2}\pi$ .

It is further given that

$$\sin(x+y)\sin(x-y) = \frac{5}{36} \quad \text{and} \quad \cos x + \cos y = \frac{5}{6}.$$

Show that  $\cos(x-y) = \frac{1+\sqrt{n}}{n}$ , where  $n$  is a positive integer to be found.

$$\boxed{\phantom{000}}, \quad \boxed{n=6}$$

IDENTIFY THE EQUATIONS

$$\Rightarrow \sin(x+y)\sin(x-y) = \frac{5}{36}$$

$$\Rightarrow [\sin x \cos y + \cos x \sin y] [\sin x \cos y - \cos x \sin y] = \frac{5}{36}$$

$$\Rightarrow \sin^2 x \cos^2 y - \cos^2 x \sin^2 y = \frac{5}{36}$$

$$\Rightarrow (1 - \cos^2 x) \cos^2 y - \cos^2 x (1 - \cos^2 y) = \frac{5}{36}$$

$$\Rightarrow \cos^2 y - \cos^2 x \cos^2 y - \cos^2 x + \cos^2 x \cos^2 y = \frac{5}{36}$$

$$\Rightarrow \cos^2 y - \cos^2 x = \frac{5}{36}$$

$$\Rightarrow (\cos y - \cos x)(\cos y + \cos x) = \frac{5}{36}$$

BUT  $\cos x + \cos y = \frac{5}{6}$

$$\Rightarrow \frac{5}{6} (\cos y - \cos x) = \frac{5}{36} \Rightarrow \cos y - \cos x = \frac{1}{6}$$

SO WE HAVE TWO EQUATIONS

$$\begin{aligned} \cos y + \cos x &= \frac{5}{6} \\ \cos y - \cos x &= \frac{1}{6} \end{aligned} \quad \text{ADDING THEM} \quad \begin{aligned} 2\cos y &= 1 \\ \cos y &= \frac{1}{2} \end{aligned} \quad \text{AND} \quad \cos x = \frac{2}{3}$$

FINDING SINCE 2, 3, 4 ARE 30, 60, 90

- $\cos x = \frac{2}{3} = \sqrt{1 - \sin^2 x} = \sqrt{1 - (\frac{1}{3})^2} = \sqrt{\frac{8}{9}} = \frac{2\sqrt{2}}{3}$
- $\cos y = \frac{1}{2} = \sqrt{1 - \sin^2 y} = \sqrt{1 - (\frac{\sqrt{3}}{2})^2} = \sqrt{\frac{1}{4}} = \frac{1}{2}$

THESE ARE THE COMPOUND ANGLES IDENTITIES

$$\begin{aligned} \cos(x-y) &= \cos x \cos y + \sin x \sin y = \frac{2}{3} \cdot \frac{1}{2} + \frac{\sqrt{2}}{3} \cdot \frac{\sqrt{3}}{2} \\ &= \frac{1}{3} + \frac{\sqrt{6}}{6} \\ &= \frac{1}{3} (1 + \sqrt{6}) \end{aligned}$$

## Question 16 (\*\*\*\*\*)

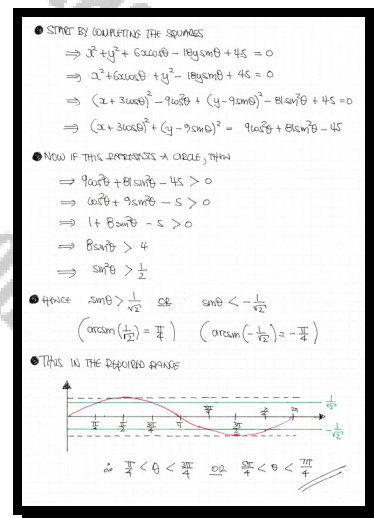
A curve in the  $x$ - $y$  plane has equation

$$x^2 + y^2 + 6x \cos \theta - 18y \sin \theta + 45 = 0,$$

where  $\theta$  is a parameter such that  $0 \leq \theta < 2\pi$ .

Given that curve represents a circle determine the range of possible values of  $\theta$ .

$$\boxed{\phantom{000}}, \left\{ \frac{1}{4}\pi < \theta < \frac{3}{4}\pi \right\} \cup \left\{ \frac{5}{4}\pi < \theta < \frac{7}{4}\pi \right\}$$



**Question 17** (\*\*\*\*)

Prove the validity of the following trigonometric identity.

$$\frac{1 + \tan \theta \tan 3\theta}{1 + \tan 2\theta \tan 3\theta} \equiv \frac{\cos^2 2\theta}{\cos^2 \theta}.$$

$\square$ ,  $\square$  proof

$$\begin{aligned} \text{Simplifying at the 4th} \\ L.H.S. &= \frac{1 + \tan 36^\circ \tan 36^\circ}{1 + \tan 36^\circ \tan 36^\circ} = \frac{\frac{1}{\tan 36^\circ} + \frac{\tan 36^\circ \tan 36^\circ}{\tan 36^\circ}}{\frac{1}{\tan 36^\circ} + \frac{\tan 36^\circ \tan 36^\circ}{\tan 36^\circ}} \\ &= \frac{\cot 36^\circ + \tan 36^\circ}{\cot 36^\circ + \tan 36^\circ} = \frac{\frac{\cos 36^\circ}{\sin 36^\circ} + \frac{\sin 36^\circ}{\cos 36^\circ}}{\frac{\cos 36^\circ}{\sin 36^\circ} + \frac{\sin 36^\circ}{\cos 36^\circ}} \\ &= \frac{\frac{\cos 36^\circ \cos 36^\circ + \sin 36^\circ \sin 36^\circ}{\sin 36^\circ \cos 36^\circ}}{\frac{\cos 36^\circ \cos 36^\circ + \sin 36^\circ \sin 36^\circ}{\sin 36^\circ \cos 36^\circ}} = \frac{\frac{\cos(36^\circ - 36^\circ)}{\sin 36^\circ \cos 36^\circ}}{\frac{\cos(36^\circ - 36^\circ)}{\sin 36^\circ \cos 36^\circ}} \\ &= \frac{\frac{\cos 0^\circ}{\sin 36^\circ \cos 36^\circ}}{\frac{\cos 0^\circ}{\sin 36^\circ \cos 36^\circ}} = \frac{\cos 0^\circ \sin 36^\circ}{\cos 0^\circ \sin 36^\circ} = \left( \frac{\sin 36^\circ}{\sin 36^\circ} \right)^2 = 12.H.S. \end{aligned}$$

**Question 18** (\*\*\*\*)

A surveyor views the top of a building, of height  $h$ , at an angle of elevation  $\alpha$ .

The surveyor walks a distance  $a$ , directly towards the building.

From this new position he views the top of the building at an angle of elevation  $\beta$ .

Show that

$$h = \frac{a \sin \alpha \sin \beta}{\sin(\beta - \alpha)}.$$

sp\_v, proof

[illegible]

## Question 19 (\*\*\*\*\*)

$$\sin^8 x - \cos^8 x = 1 - \frac{1}{2} \sin^2 2x.$$

Use trigonometric identities to show that the general solution of the above equation is  $x = k\pi$ ,  $k \in \mathbb{Z}$ .

proof

$$\begin{aligned} \sin^8 x - \cos^8 x &= 1 - \frac{1}{2} \sin^2 2x \\ \Rightarrow (\sin^2 x - \cos^2 x)(\sin^2 x + \cos^2 x) &= 1 - \frac{1}{2} (2 \sin x \cos x)^2 \\ \Rightarrow (\sin^2 x - \cos^2 x)(\sin^2 x + \cos^2 x) &= 1 - 2 \sin^2 x \cos^2 x \\ \Rightarrow \cos 2x (\sin^2 x + \cos^2 x) &= 1^2 - 2 \sin^2 x \cos^2 x \\ \Rightarrow \cos 2x (\sin^2 x + \cos^2 x) &= (\sin^2 x + \cos^2 x)^2 - 2 \sin^2 x \cos^2 x \\ \Rightarrow \cos 2x (\sin^2 x + \cos^2 x) &= \cos^2 x + 2 \sin^2 x \cos^2 x + \sin^2 x - 2 \sin^2 x \cos^2 x \\ \Rightarrow \cos 2x (\sin^2 x + \cos^2 x) &= \cos^2 x + \sin^2 x \\ \Rightarrow \cos 2x &= 1 \\ \begin{cases} 2x = 0 \pm 2\pi n \\ 2x = \dots \end{cases} & \quad n = 0, 1, 2, 3, \dots \\ x &= 0 \pm \pi n \\ x &= k\pi \quad k \in \mathbb{Z} \end{aligned}$$

## Question 20 (\*\*\*\*\*)

It is given that  $x$  is a solution of the following equation.

$$\sec x + \tan x = \frac{5}{7}.$$

Without solving the above equation for  $x$ , find the value of  $\operatorname{cosec} x + \cot x$ .

V, , 

$$\begin{aligned} \sec x + \tan x &= \frac{5}{7} \\ \Rightarrow \sec x + \tan x &= \frac{5}{7} \\ \Rightarrow (\sec x + \tan x)(\sec x - \tan x) &= \frac{5}{7} (\sec x - \tan x) \\ \Rightarrow \sec^2 x - \tan^2 x &= \frac{5}{7} (\sec x - \tan x) \\ \Rightarrow 1 &= \frac{5}{7} (\sec x - \tan x) \\ \Rightarrow \sec x - \tan x &= \frac{7}{5} \end{aligned}$$


Then we have:

$$\begin{aligned} \sec x + \tan x &= \frac{5}{7} \\ \sec x - \tan x &= \frac{7}{5} \end{aligned} \quad \begin{cases} \text{Adding} \\ \text{Subtracting} \end{cases}$$

$$\begin{aligned} 2 \sec x &= \frac{5}{7} + \frac{7}{5} = \frac{25 + 49}{35} = \frac{74}{35} \\ \sec x &= \frac{37}{35} \\ \tan x &= \frac{5}{7} - \frac{37}{35} = \frac{25 - 59}{35} = -\frac{34}{35} \end{aligned}$$

As we have positive values, we can use the "Sine Rule" triangle.

OR, AS EASY — AS YOU STILL USE THE "SINE RULE" TRIANGLE



$$\begin{aligned} \Rightarrow \operatorname{cosec} x + \cot x &= \frac{1}{\sin x} + \frac{1}{\tan x} \\ &= \frac{1}{\frac{34}{35}} + \frac{1}{-\frac{34}{35}} \\ &= \frac{35}{34} - \frac{35}{34} \\ &= 0 \end{aligned}$$



**Question 21** (\*\*\*\*)

The positive solution of the quadratic equation  $x^2 - x - 1 = 0$  is denoted by  $\phi$ , and is commonly known as the golden section or golden number.

Solve the equation

$$x \cos\left(\frac{1}{2} \arctan 2\right) = \sqrt{\phi}, \quad x \in \mathbb{R}.$$

Give the answer in the form  $\sqrt[n]{m}$ , where  $m$  and  $n$  are positive integers.

**V**, ,  $x = \sqrt[4]{5}$

[illegible]

## Question 22 (\*\*\*\*\*)

Prove that

$$\arctan \left[ \sqrt{\frac{1-x}{1+x}} \right] = \frac{1}{2} \arccos x.$$

V, ☐ SP, ☐ proof

Write in terms

Let  $\cos \theta = x = \arccos \sqrt{\frac{1-x}{1+x}}$

Maximum in terms

$$\Rightarrow \tan \theta = \sqrt{\frac{1-x}{1+x}}$$

$$\Rightarrow \tan \theta = \frac{1-x}{1+x}$$

$$\Rightarrow 1 + \tan^2 \theta = \frac{1-x^2}{1+x^2} + 1$$

$$\Rightarrow \sec^2 \theta = \frac{1-x^2+1+x^2}{1+x^2}$$

$$\Rightarrow \sec^2 \theta = \frac{2}{1+x^2}$$

$$\Rightarrow \cos^2 \theta = \frac{1+x^2}{2}$$

Now using  $\cos 2\theta = 2\cos^2 \theta - 1$

$$\Rightarrow 2\cos^2 \theta = x+1$$

$$\Rightarrow 2\cos^2 \theta - 1 = x$$

$$\Rightarrow \cos 2\theta = x$$

$$\Rightarrow 2\theta = \arccos x$$

$$\Rightarrow \theta = \frac{1}{2} \arccos x$$

$$\Rightarrow \arctan \sqrt{\frac{1-x}{1+x}} = \frac{1}{2} \arccos x$$

As required

**Question 23** (\*\*\*\*)

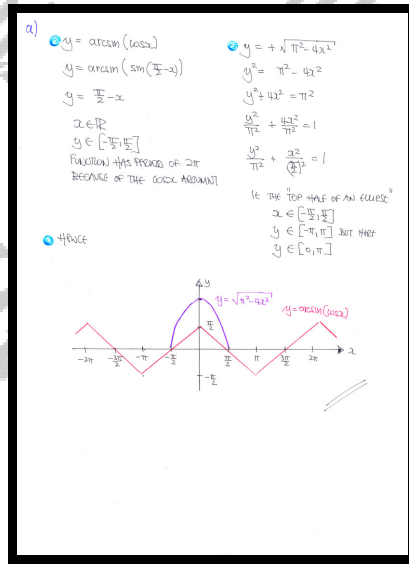
The functions  $f$  and  $g$  are defined in the largest possible domain by the equations

$$f(x) = \arcsin(\cos x) \quad \text{and} \quad g(x) = \sqrt{\pi^2 - 4x^2}.$$

- Sketch the graphs of  $f$  and  $g$  on the same set of axes.
- Use an algebraic method to solve the equation

$$\arcsin(\cos x) = \sqrt{\pi^2 - 4x^2}.$$

$$x = \pm \frac{\pi}{2}$$



b)

$\arcsin(\cos x) = \sqrt{\pi^2 - 4x^2}$   
 $\cos x = \sin \sqrt{\pi^2 - 4x^2}$   
 $\sin(\frac{\pi}{2} - x) = \sin \sqrt{\pi^2 - 4x^2}$   
 $\Rightarrow \frac{\pi}{2} - x = \sqrt{\pi^2 - 4x^2}$   
 $\Rightarrow \frac{\pi^2}{4} - \pi x + x^2 = \pi^2 - 4x^2$   
 $\Rightarrow 5x^2 - \pi x - \frac{3\pi^2}{4} = 0$   
 $\Rightarrow 20x^2 - 4\pi x - 3\pi^2 = 0$   
 $\Rightarrow (2x - \pi)(10x + 3\pi) = 0$   
 $x = \frac{\pi}{2} \quad \text{or} \quad x = -\frac{3\pi}{10}$

NOTE THAT BOTH SIDES ARE GIVEN SO POTENTIALLY  $\pm \frac{\pi}{2} \pm \frac{3\pi}{10}$   
 COULD BE SOLUTIONS BUT DUE TO SQUARING WE MUST CHECK

$\arcsin(\cos(\frac{\pi}{2})) = \arcsin(0) = 0$   
 $\sqrt{\pi^2 - 4(\frac{\pi}{2})^2} = \sqrt{\pi^2 - \pi^2} = 0 \quad \therefore x = \frac{\pi}{2}$

NOW WE KNOW FROM THE GRAPH THERE ARE NO MORE SOLUTIONS

$\sqrt{\pi^2 - 4(\frac{3\pi}{10})^2} = \sqrt{\pi^2 - \frac{36\pi^2}{25}} = \sqrt{\frac{64\pi^2}{25}} = \frac{8\pi}{5}$   
 $\arcsin(\cos(\frac{3\pi}{10})) = \arcsin(\cos(\frac{\pi}{5})) = \arcsin(\sin[\frac{\pi}{2} - \frac{\pi}{5}])$   
 $= \arcsin(\sin \frac{4\pi}{5}) = \frac{4\pi}{5}$   
 $\frac{4\pi}{5} \neq \frac{3\pi}{10}$  SO ONLY SOLUTION  
 $x = \frac{\pi}{2}$

**Question 24 (\*\*\*\*\*) non calculator**

It is given that

$$2\cos\theta + \sin\theta = 1.$$

Determine the possible values of

$$7\cos\theta + 6\sin\theta.$$

$$\boxed{\phantom{000}}, \quad 7\cos\theta + 6\sin\theta = 2, \quad 7\cos\theta + 6\sin\theta = 6$$


**METHOD A - BY SQUARING**

- START BY SQUARING THE GIVEN EQUATION
 
$$\begin{aligned} 2\cos\theta + \sin\theta &= 1 \\ 2\cos\theta &= 1 - \sin\theta \\ 4\cos^2\theta &= (1 - \sin\theta)^2 \\ 4(1 - \sin^2\theta) &= 1 - 2\sin\theta + \sin^2\theta \\ 4 - 4\sin^2\theta &= 1 - 2\sin\theta + \sin^2\theta \\ 0 &= 5\sin^2\theta - 2\sin\theta - 3 \\ (5\sin\theta + 3)(\sin\theta - 1) &= 0 \\ \sin\theta &= \frac{-3}{5} \end{aligned}$$
- OBTAIN THE CORRESPONDING SOLUTIONS FOR  $\cos\theta$ 

$$\begin{aligned} \cos^2\theta + \sin^2\theta &= 1 & \cos^2\theta + \sin^2\theta &= 1 \\ \cos^2\theta + 1 &= 1 & \cos^2\theta + \frac{9}{25} &= 1 \\ \cos^2\theta &= 0 & \cos^2\theta &= \frac{16}{25} \\ \cos\theta &= 0 & \cos\theta &= \pm \frac{4}{5} \end{aligned}$$
- CHECK AGAINST THE ORIGINAL (ONE OF THE SQUARING)
 
$$\begin{aligned} \sin\theta &= 1, \cos\theta = 0 & \text{is ok} & \quad (2\cos\theta + 1 = 1) \\ \sin\theta = -\frac{3}{5}, \cos\theta = \frac{4}{5} & \text{is ALSO ok} & \quad (2\cos\theta - \frac{3}{5} = \frac{8}{5} - \frac{3}{5} = 1) \\ \sin\theta = -\frac{3}{5}, \cos\theta = -\frac{4}{5} & \text{is NOT ok} & \quad (2\cos\theta - \frac{3}{5} = -\frac{8}{5} - \frac{3}{5} \neq 1) \end{aligned}$$
- THIS WE OBTAIN
 
$$7\cos\theta + 6\sin\theta = \begin{cases} 7(0) + 6(1) = 6 \\ 7(\frac{4}{5}) + 6(-\frac{3}{5}) = \frac{28}{5} - \frac{18}{5} = 2 \end{cases}$$

**METHOD B - BY SINE/COS FORM**


- WRITE  $2\cos\theta + \sin\theta$  IN "HARMONIC FORM"
 
$$\begin{aligned} 2\cos\theta + \sin\theta &= \sqrt{5}\cos(\theta - \alpha) \\ &\text{BY IDENTIFYING} \\ &= \sqrt{5}\cos\theta\cos\alpha + \sqrt{5}\sin\theta\sin\alpha \\ &= (\sqrt{5}\cos\alpha)\cos\theta + (\sqrt{5}\sin\alpha)\sin\theta \end{aligned}$$

$$\begin{aligned} \frac{\sqrt{5}\sin\alpha}{\sqrt{5}\cos\alpha} &= \frac{1}{2} \\ \tan\alpha &= \frac{1}{2} \\ \alpha &= \arctan\frac{1}{2} \end{aligned}$$

- RETURNING TO THE GIVEN EQUATION TO SOLVE IT FOR  $\theta$ 

$$\begin{aligned} 2\cos\theta + \sin\theta &= 1 \\ \sqrt{5}\cos(\theta - \alpha) &= 1 \\ \cos(\theta - \alpha) &= \frac{1}{\sqrt{5}} \end{aligned}$$

$$\begin{aligned} \theta - \alpha &= \arccos\frac{1}{\sqrt{5}} \pm 2\pi n & \theta - \alpha &= \arccos\frac{1}{\sqrt{5}} \pm 2\pi n \\ \theta - \alpha &= -\arccos\frac{1}{\sqrt{5}} \pm 2\pi n & \theta - \alpha &= -\arccos\frac{1}{\sqrt{5}} \pm 2\pi n \\ \theta &= \alpha + \arccos\frac{1}{\sqrt{5}} \pm 2\pi n & \theta &= \alpha - \arccos\frac{1}{\sqrt{5}} \pm 2\pi n \\ \theta &= \alpha + \arccos\frac{1}{\sqrt{5}} \pm 2\pi n & \theta &= \alpha - \arccos\frac{1}{\sqrt{5}} \pm 2\pi n \end{aligned}$$

- NOW SUBSTITUTE EACH SOLUTION INTO  $7\cos\theta + 6\sin\theta$ 

$$\begin{aligned} 7\cos\theta + 6\sin\theta &= \frac{1}{\sqrt{5}}[2\cos\theta + \sin\theta] + \frac{3}{\sqrt{5}}\sin\theta \\ &= \frac{1}{\sqrt{5}} \times 1 + \frac{3}{\sqrt{5}}\sin\theta \\ &= \frac{1}{\sqrt{5}} + \frac{3}{\sqrt{5}}\sin\theta \end{aligned}$$
- NOW LET  $A = \arccos\frac{1}{\sqrt{5}}$  & WORKING AT THE TRIANGLE
 
$$A + B = \frac{\pi}{2}$$

- THIS WE OBTAIN
 
$$\begin{aligned} \dots \frac{1}{\sqrt{5}} + \frac{3}{\sqrt{5}}\sin\theta &= \frac{1}{\sqrt{5}} + \frac{3}{\sqrt{5}}\sin(A+B) \\ &= \frac{1}{\sqrt{5}} + \frac{3}{\sqrt{5}}\sin A \cos B \\ \text{OR} \dots \frac{1}{\sqrt{5}} + \frac{3}{\sqrt{5}}\sin\theta &= \frac{1}{\sqrt{5}} + \frac{3}{\sqrt{5}}(\sin A \cos B - \cos A \sin B) \\ &= \frac{1}{\sqrt{5}} + \frac{3}{\sqrt{5}}\left[\frac{1}{\sqrt{5}}\cos\frac{\pi}{2} - \frac{2}{\sqrt{5}}\sin\frac{\pi}{2}\right] \\ &= \frac{1}{\sqrt{5}} + \frac{3}{\sqrt{5}}\left[\frac{1}{\sqrt{5}} - \frac{2}{\sqrt{5}}\right] \\ &= 2 \end{aligned}$$

**Question 25** (\*\*\*\*\*)

The point  $M$  is the midpoint of  $AB$ , on a triangle  $ABC$ .

Given further that

$$\tan[\angle CAM] = \frac{2}{5} \quad \text{and} \quad \tan[\angle CBM] = \frac{2}{3},$$

Use trigonometric identities to find the value of  $\tan[\angle CMB]$ .

$$\boxed{2}, \quad \tan[\angle CMB] = 2$$

**WORKING AT ACM**

LOOKING AT ACM

$$\Rightarrow \frac{\sin \alpha}{\sin \theta} = \frac{\sin(\theta - \alpha)}{y}$$

$$\Rightarrow \frac{y}{\sin \alpha} = \frac{\sin(\theta - \alpha)}{\sin \theta}$$

$$\Rightarrow \frac{y}{\sin \alpha} = \frac{\sin \theta \cos \alpha - \cos \theta \sin \alpha}{\sin \theta}$$

$$\Rightarrow \frac{y}{\sin \alpha} = \sin \theta \cot \alpha - \cos \theta$$

$$\Rightarrow \frac{y}{\sin \alpha} = \frac{2}{5} \sin \theta - \cos \theta$$

**LOOKING AT BCM**

$$\Rightarrow \frac{\sin \theta}{\sin \beta} = \frac{\sin(\theta - \beta)}{y}$$

$$\Rightarrow \frac{\sin \theta}{\sin \beta} = \frac{\sin(\theta - \beta)}{y}$$

$$\Rightarrow \frac{y}{\sin \beta} = \frac{\sin(\theta - \beta)}{\sin \theta}$$

$$\Rightarrow \frac{y}{\sin \beta} = \frac{\sin \theta \cos \beta - \cos \theta \sin \beta}{\sin \theta}$$

**WORKING AT CMB**

$$\Rightarrow \frac{y}{\sin \theta} = \cos \beta + \cot \theta \sin \beta$$

$$\Rightarrow \frac{y}{\sin \theta} = \cos \beta + \frac{2}{3} \sin \beta$$

SPINNING EXPRESSIONS FOR  $\frac{y}{\sin \theta}$  MENUS

$$\Rightarrow \frac{2}{5} \sin \theta - \cos \theta = \cos \beta + \frac{2}{3} \sin \beta$$

$$\Rightarrow \sin \theta = 2 \cos \beta$$

$$\Rightarrow \tan \theta = 2$$

## Question 26 (\*\*\*\*)

Show, with detailed workings, that if  $\sin 2x = \frac{2}{3}$  then

$$\cos^6 x + \sin^6 x$$

also equals to  $\frac{2}{3}$ .

**V**, ☐, **proof**

THIS REQUIRES IDENTITIES - BUT THE STUDENT IS NOW TELECOMMUNICATING

$$A^3 \pm B^3 \equiv (A+B)(A^2 \pm AB + B^2)$$

$$\Rightarrow \cos^6 x + \sin^6 x = (\cos^2 x)^3 + (\sin^2 x)^3$$

$$= (\cos^2 x + \sin^2 x)(\cos^4 x - \cos^2 x \sin^2 x + \sin^4 x)$$

$$= 1 \times (\cos^4 x - \cos^2 x \sin^2 x + \sin^4 x)$$

$$= (\cos^4 x) - (\cos^2 x)(\sin^2 x) + (\sin^4 x)$$

NOW DECKATE THE IDENTITY  $(A+B)^2 \equiv A^2 + 2AB + B^2$

$$= [(\cos^2 x)^2 + 2(\cos^2 x)(\sin^2 x) + (\sin^2 x)^2] - 3(\cos^2 x)(\sin^2 x)$$

$$= [\cos^4 x + \sin^4 x]^2 - 3(\cos^2 x \sin^2 x)^2$$

$$= (1 - \frac{2}{3}(\sin 2x))^2$$

$$= 1 - \frac{2}{3}(\frac{2}{3})^2$$

$$= 1 - \frac{2}{3} \times \frac{4}{9}$$

$$= 1 - \frac{8}{27}$$

$$= \frac{2}{3}$$

**As Required**

**Question 27** (\*\*\*\*\*)

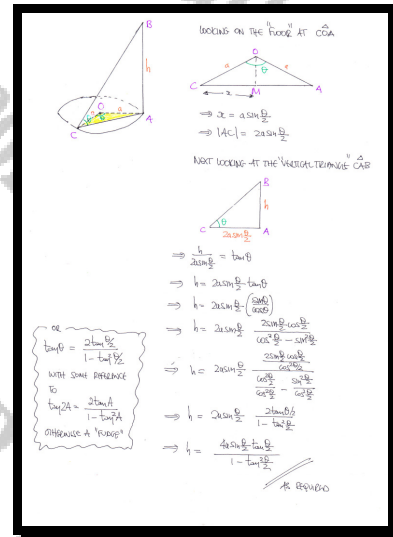
A pole  $AB$ , of height  $h$ , is standing vertically on level horizontal ground with  $A$  on the circumference of a circle of radius  $a$ , centred at the point  $O$ .

The point  $C$  is another point on the circumference of this circle so that  $\angle COA = \theta$  and  $\angle ACB = \theta$ .

Use a detailed method to show that

$$h = \frac{4a \sin\left(\frac{1}{2}\theta\right) \tan\left(\frac{1}{2}\theta\right)}{1 - \tan^2\left(\frac{1}{2}\theta\right)}.$$

 , proof



**Question 28** (\*\*\*\*\*)

A triangle has angles of  $36^\circ$ ,  $72^\circ$  and  $72^\circ$ .

By suitably partitioning this triangle and using similar triangles, show that

$$\sin 18^\circ + \cos 36^\circ = \frac{1}{2}\sqrt{5} \quad \text{and} \quad \sin^2 36^\circ + \cos^2 18^\circ = \frac{5}{4}$$

$$\boxed{\phantom{000}}, \quad \boxed{-\frac{9}{16}}$$

SIMILAR WITH THE TRIANGLE SUGGESTING, ABC WHERE BAC = 36°

- LET  $|AB| = |AC| = 1$
- LET  $BC = x$
- SELECT THE ANGLE  $\angle ABC$ , WITH BD

THEN TRANSFER ALL THE INFORMATION INTO 3 SEPARATE FIGURES

APPLYING THE TRIANGLE ANGLES NEXT TO EACH OTHER (20) = 2, THEN THE 3RD FIGURE

AND THIS IMPLES  $|AD| = x$  FROM THE 2ND FIGURE - SIMILAR TRIANGLES

BY SIMILAR TRIANGLES

$$\begin{aligned} \Rightarrow \frac{x}{1-x} &= \frac{1}{x} \\ \Rightarrow x^2 &= 1-x \\ \Rightarrow x^2 + x &= 1 \\ \Rightarrow (x+1)^2 &= 5 \\ \Rightarrow x+1 &= \pm\sqrt{5} \\ \Rightarrow x &= \frac{-1+\sqrt{5}}{2} \end{aligned}$$

REVISIT THE ORIGINAL TRIANGLE

- $\sin 18^\circ = \frac{|BC|}{|AC|} = \frac{1+\sqrt{5}}{4}$
- $\cos 36^\circ = 1 - 2\sin^2 18^\circ$   
 $\cos 36^\circ = 1 - 2\left(\frac{1+\sqrt{5}}{4}\right)^2$   
 $= 1 - 2\left(\frac{1+2\sqrt{5}+5}{16}\right)$   
 $= 1 - \frac{6+2\sqrt{5}}{8}$   
 $= 1 - \frac{3+\sqrt{5}}{4}$   
 $= \frac{4-3-\sqrt{5}}{4}$   
 $= \frac{1-\sqrt{5}}{4}$

$\therefore \sin 18^\circ + \cos 36^\circ = \frac{1+\sqrt{5}}{4} + \frac{1-\sqrt{5}}{4} = \frac{2}{4} = \frac{1}{2}$

ALSO USE HYPERBOLIC COSINE:  $\cos^2 \theta + \sin^2 \theta = 1$

- $\cos^2 18^\circ = 1 - \sin^2 18^\circ = 1 - \left(\frac{1+\sqrt{5}}{4}\right)^2 = 1 - \frac{1+2\sqrt{5}+5}{16} = \frac{15-1-2\sqrt{5}}{16} = \frac{14-2\sqrt{5}}{16} = \frac{7-\sqrt{5}}{8}$
- $\sin^2 36^\circ = 1 - \cos^2 36^\circ = 1 - \left(\frac{1-\sqrt{5}}{4}\right)^2 = 1 - \frac{1-2\sqrt{5}+5}{16} = \frac{15-1+2\sqrt{5}}{16} = \frac{14+2\sqrt{5}}{16} = \frac{7+\sqrt{5}}{8}$

$\therefore \cos^2 18^\circ + \sin^2 36^\circ = \frac{7-\sqrt{5}}{8} + \frac{7+\sqrt{5}}{8} = \frac{14}{8} = \frac{7}{4}$

$\cos^2 \theta + \sin^2 \theta = 1$

**Question 329** (\*\*\*\*\*) **non calculator**

Solve the trigonometric equation

$$\sin(y-30) = \sin(y-45), \quad 0 \leq y < 360^\circ.$$

$$\boxed{y = 82.5^\circ, 262.5^\circ}$$

$$\begin{aligned} \sin(y-30) &= \sin(y+45) \\ \Rightarrow (y-30) &= (y+45) + 360n \\ y-30 &= 180-(y+45) + 360n \\ \Rightarrow (y-30) &= 135-y + 360n \\ \Rightarrow 2y &= 165 + 360n \\ \Rightarrow y &= 82.5 + 180n \\ \therefore y &= 82.5^\circ, 262.5^\circ \end{aligned}$$



## Question 30 (\*\*\*\*\*)

$$\cot^2 x - \tan^2 x = 8 \cot 2x, \quad 0 \leq x < 180.$$

Find the solutions of the above trigonometric equation, giving the answers in **degrees**.

$$x = 15^\circ, 45^\circ, 75^\circ, 135^\circ$$

Handwritten solution for Question 30:

$$\begin{aligned} \cot^2 x - \tan^2 x &= 8 \cot 2x \\ \Rightarrow \cot^2 x - \tan^2 x &= \frac{8}{\tan 2x} \\ \Rightarrow \frac{1}{\tan^2 x} - \tan^2 x &= \frac{8(1 - \tan^2 x)}{2 \tan 2x} \\ \Rightarrow \frac{1}{T^2} - T^2 &= \frac{8(1 - T^2)}{2T} \quad \text{where } T = \tan x \\ \Rightarrow 1 - T^4 &= 4T(1 - T^2) \\ \Rightarrow (1 - T^2)(1 + T^2) &= 4T(1 - T^2) \\ \Rightarrow (1 - T^2)[(1 + T^2) - 4T] &= 0 \\ \Rightarrow (1 - T^2)(T^2 - 4T + 1) &= 0 \\ \Rightarrow (1 - T)(1 + T)(T^2 - 4T + 1) &= 0 \\ \Rightarrow (1 - T)(1 + T)(T - 2 - \sqrt{3})(T - 2 + \sqrt{3}) &= 0 \\ \Rightarrow \tan x &= \begin{cases} 1 \\ 2 + \sqrt{3} \\ 2 - \sqrt{3} \end{cases} \\ \Rightarrow \begin{cases} x = 45^\circ + 180n \\ x = 75^\circ + 180n \\ x = 15^\circ + 180n \end{cases} \quad \text{for } n \in \mathbb{Z} \end{aligned}$$

For  $0 \leq x < 180$ :

$$\begin{aligned} x_1 &= 45^\circ \\ x_2 &= 75^\circ \\ x_3 &= 15^\circ \end{aligned}$$

**Question 31** (\*\*\*\*\*)Solve the following trigonometric equation for  $0 \leq x < 360^\circ$ 

$$\tan x \sec x + \frac{1}{1 + \sin x} = \frac{4}{3}$$

$$x = 30^\circ, 150^\circ, 210^\circ, 330^\circ$$

Handwritten solution for Question 31:

$\tan x \sec x + \frac{1}{1 + \sin x} = \frac{4}{3}$   
 $\Rightarrow \frac{\sin x}{\cos x} \cdot \frac{1}{\cos x} + \frac{1}{1 + \sin x} = \frac{4}{3}$   
 $\Rightarrow \frac{\sin x}{\cos^2 x} + \frac{1}{1 + \sin x} = \frac{4}{3}$   
 $\Rightarrow \frac{\sin x}{1 - \sin^2 x} + \frac{1}{1 + \sin x} = \frac{4}{3}$   
 $\Rightarrow \frac{\sin x}{(1 - \sin x)(1 + \sin x)} + \frac{1}{1 + \sin x} = \frac{4}{3}$   
 $\Rightarrow \frac{\sin x + 1 - \sin x}{(1 - \sin x)(1 + \sin x)} = \frac{4}{3}$   
 $\Rightarrow \frac{1}{1 - \sin x} = \frac{4}{3}$   
 $\Rightarrow 3 = 4(1 - \sin x)$   
 $\Rightarrow 3 = 4 - 4\sin x$   
 $\Rightarrow 4\sin x = 1$   
 $\Rightarrow \sin x = \frac{1}{4}$   
 $\Rightarrow x = \arcsin\left(\frac{1}{4}\right)$   
 $\Rightarrow x = 14.5^\circ, 175.5^\circ$   
 $\Rightarrow x = 30^\circ, 150^\circ, 210^\circ, 330^\circ$

**Question 32** (\*\*\*\*\*) non calculatorGiven that  $\alpha = \arctan \frac{1}{2}$  and  $\beta = \arctan \frac{9}{13}$ , find the value of  $\tan(3\alpha - \beta)$ .

$$\tan(3\alpha - \beta) = 1$$

Handwritten solution for Question 32:

$\tan 3\alpha = \tan(2\alpha + \alpha) = \frac{\tan 2\alpha + \tan \alpha}{1 - \tan 2\alpha \tan \alpha} = \frac{\frac{2 \tan \alpha}{1 - \tan^2 \alpha} + \tan \alpha}{1 - \frac{2 \tan \alpha}{1 - \tan^2 \alpha} \tan \alpha}$   
 $\Rightarrow \frac{2 \tan \alpha + \tan \alpha (1 - \tan^2 \alpha)}{1 - \tan^2 \alpha - 2 \tan^2 \alpha} = \frac{2 \tan \alpha + \tan \alpha - \tan^3 \alpha}{1 - 3 \tan^2 \alpha}$   
 $\Rightarrow \frac{3 \tan \alpha - \tan^3 \alpha}{1 - 3 \tan^2 \alpha}$   
 $\Rightarrow \frac{3 \cdot \frac{1}{2} - \left(\frac{1}{2}\right)^3}{1 - 3 \left(\frac{1}{2}\right)^2} = \frac{\frac{3}{2} - \frac{1}{8}}{1 - \frac{3}{4}} = \frac{\frac{12}{8} - \frac{1}{8}}{\frac{1}{4}} = \frac{\frac{11}{8}}{\frac{1}{4}} = \frac{11}{2}$   
 $\Rightarrow \tan 3\alpha = \frac{11}{2}$   
 $\tan(3\alpha - \beta) = \frac{\tan 3\alpha - \tan \beta}{1 + \tan 3\alpha \tan \beta} = \frac{\frac{11}{2} - \frac{9}{13}}{1 + \frac{11}{2} \cdot \frac{9}{13}} = \frac{\frac{143 - 18}{26}}{1 + \frac{99}{26}} = \frac{\frac{125}{26}}{\frac{125}{26}} = 1$

**Question 33** (\*\*\*\*)

It is given that for some value of the constant  $k$

$$\cos 4x \equiv 1 + k \sin^2 x \cos^2 x.$$

- a) Determine the value of  $k$ .
- b) Hence, or otherwise, show clearly that for  $x \in \mathbb{R}$

$$-\frac{3}{4} \leq 4 \cos^2 2x - 3 \sin^2 x \cos^2 x \leq 4.$$

$$\boxed{\phantom{000}}, \quad k = -8$$

(a)  $\cos 4x = \cos(2 \cdot 2x)$   
 $= 1 - 2 \sin^2 2x$   
 $= 1 - 2(\sin 2x)^2$   
 $= 1 - 2(2 \sin x \cos x)^2$   
 $= 1 - 2(4 \sin^2 x \cos^2 x)$   
 $= 1 - 8 \sin^2 x \cos^2 x$   
 $\therefore k = -8$

(b)  $f(x) = 4 \cos^2 2x - 3 \sin^2 x \cos^2 x$   
 $f(x) = 2 + 2 \cos 4x - \frac{3}{4} + \frac{3}{4} \cos 4x$   
 $f(x) = \frac{13}{4} + \frac{13}{4} \cos 4x$   
 $\text{Since } -1 \leq \cos 4x \leq 1$   
 $f(x)_{\max} = \frac{13}{4} + \frac{13}{4} \times 1 = 4$   
 $f(x)_{\min} = \frac{13}{4} + \frac{13}{4} \times (-1) = -\frac{3}{4}$   
 $\therefore -\frac{3}{4} \leq 4 \cos^2 2x - 3 \sin^2 x \cos^2 x \leq 4$

**Question 34** (\*\*\*\*) non calculator

Solve the following trigonometric equation

$$\cos(\psi - 60) = \cos(\psi - 45), \quad 0 \leq \psi < 360^\circ.$$

$$\boxed{\psi = 52.5^\circ, 232.5^\circ}$$

$\cos(\psi - 60) = \cos(\psi - 45)$   
 $\Rightarrow \begin{cases} \psi - 60 = \psi - 45 + 360n \\ \psi - 60 = 45 - \psi + 360n \end{cases} \quad n = 0, 1, 2, \dots$   
 $\Rightarrow \begin{cases} \text{impossible} \\ 2\psi = 105 + 360n \end{cases}$   
 $\Rightarrow \psi = 52.5^\circ + 180n$   
 $\therefore \psi = 52.5^\circ, 232.5^\circ$

## Question 35 (\*\*\*\*\*)

$$S = \sin(10^\circ) \sin(30^\circ) \sin(50^\circ) \sin(70^\circ)$$

Use the **sine double angle identity** for sine to show that  $S = \frac{1}{16}$

**V**, , **proof**

SINCE WE'RE TRYING TO GET  $\sin(40^\circ)$

$$\Rightarrow S = \sin(10^\circ) \sin(30^\circ) \sin(50^\circ) \sin(70^\circ)$$

$$\Rightarrow S = \cos(80^\circ) \cos(40^\circ) \cos(20^\circ)$$

NOW USING THE SINE DOUBLE ANGLE

- $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$
- $\cos(\theta) = \frac{\sin(2\theta)}{2\sin(\theta)}$

$$\Rightarrow S = \frac{\sin(20^\circ)}{2\sin(10^\circ)} \times \frac{\sin(40^\circ)}{2\sin(20^\circ)} \times \frac{\sin(80^\circ)}{2\sin(40^\circ)}$$

$$\Rightarrow S = \frac{\sin(20^\circ) \sin(40^\circ) \sin(80^\circ)}{16 \sin(10^\circ) \sin(20^\circ) \sin(40^\circ)}$$

BUT  $\sin(\theta) = \sin(180^\circ - \theta)$

$$\Rightarrow S = \frac{\sin(20^\circ) \sin(40^\circ) \sin(80^\circ)}{16 \sin(10^\circ) \sin(20^\circ) \sin(40^\circ)}$$

$$\Rightarrow S = \frac{1}{16}$$

## Question 36 (\*\*\*\*\*)

Solve the trigonometric equation

$$(\cos 4x + \cos x)^2 + (\sin 4x + \sin x)^2 = 2\sqrt{3} \sin 3x, \quad 0 \leq x < \pi,$$

giving the answers in terms of  $\pi$ .

$$\boxed{\phantom{000}}, \quad \frac{\pi}{9}, \frac{\pi}{3}, \frac{7\pi}{9}$$

START BY EXPANDING THE LHS

$$\Rightarrow (\cos 4x + \cos x)^2 + (\sin 4x + \sin x)^2 = 2\sqrt{3} \sin 3x$$

$$\Rightarrow \{ \cos^2 4x + 2\cos 4x \cos x + \cos^2 x \} + \{ \sin^2 4x + 2\sin 4x \sin x + \sin^2 x \} = 2\sqrt{3} \sin 3x$$

$$\Rightarrow 1 + 2[\cos 4x \cos x + \sin 4x \sin x] + 1 = 2\sqrt{3} \sin 3x$$

$$\Rightarrow 1 + 2 \cos(4x - x) + 1 = 2\sqrt{3} \sin 3x$$

$$\Rightarrow 2 + 2 \cos 3x = 2\sqrt{3} \sin 3x$$

$$\Rightarrow 1 + \cos 3x = \sqrt{3} \sin 3x$$

$$\Rightarrow \sqrt{3} \sin 3x - \cos 3x = 1$$

NOW WE MAY PROCEED BY AN "R TRANSFORMATION" (OR AS YOU LIKE)

$$\Rightarrow \sqrt{3} \sin 3x - \frac{1}{2} \cos 3x = \frac{1}{2}$$

$$\Rightarrow \cos \frac{\pi}{6} \sin 3x - \sin \frac{\pi}{6} \cos 3x = \frac{1}{2}$$

$$\Rightarrow \sin(3x - \frac{\pi}{6}) = \frac{1}{2}$$

$$\arcsin(\frac{1}{2}) = \frac{\pi}{6}$$

$$\Rightarrow \left( 3x - \frac{\pi}{6} = \frac{\pi}{6} + 2n\pi \right) \quad n=0,1,2,\dots$$

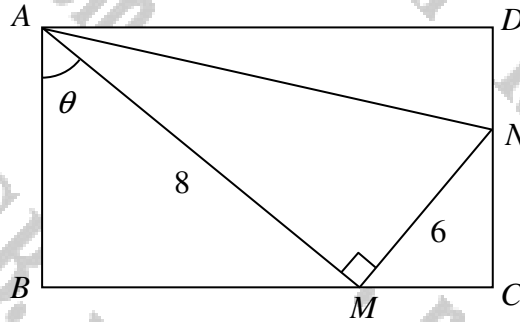
$$\Rightarrow 3x - \frac{\pi}{6} = \frac{\pi}{6} + 2n\pi$$

$$\Rightarrow 3x = \frac{\pi}{3} + 2n\pi$$

$$\Rightarrow x = \frac{\pi}{9} + \frac{2n\pi}{3}$$

$\alpha = 20^\circ \approx 120^\circ$   
 $\alpha = 60^\circ \approx 120^\circ$   
 $\alpha = 20, 140, 60$   
 $\therefore x = \frac{\pi}{9}, \frac{\pi}{3}, \frac{7\pi}{9}$

## Question 37 (\*\*\*\*)



The figure above shows a rectangle  $ABCD$ .

The points  $M$  and  $N$  lie on  $BC$  and  $CD$  respectively.

The angle  $AMN$  is  $90^\circ$ ,  $|AM| = 8$  and  $|MN| = 6$ . The angle  $BAM$  is denoted by  $\theta$ .

- Given that the perimeter of the rectangle  $ABCD$  is fixed at 24 units, determine the possible value(s) of  $\theta$ .
- Given instead that the perimeter of the rectangle  $ABCD$  can vary, determine the largest possible area of the triangle  $ADN$ .

$$\theta \approx 71.7^\circ, \text{ area}_{\max} = 25$$

$|BC| = |AM| + |MC| = 8 \sin \theta + 6 \cos \theta$   
 $|AB| = 6 \cos \theta$   
 $|AB| + |BC| = (8 \sin \theta + 6 \cos \theta) + (6 \cos \theta)$   
 $= 8 \sin \theta + 12 \cos \theta$   
 $\therefore \text{PERIMETER} = 16 \sin \theta + 24 \cos \theta$   
 $24 = 16 \sin \theta + 24 \cos \theta$   
 $16 \sin \theta + 24 \cos \theta = 24$

BY TRANSFORMATION  
 $16 \sin \theta + 24 \cos \theta = R \sin(\theta + \alpha)$   
 $= R \sin \theta \cos \alpha + R \cos \theta \sin \alpha$   
 $= (R \cos \alpha) \sin \theta + (R \sin \alpha) \cos \theta$   
 $\therefore R \cos \alpha = 16$   
 $R \sin \alpha = 24$   
 $R = \sqrt{16^2 + 24^2} = \sqrt{65}$   
 $\tan \alpha = \frac{24}{16} = \frac{3}{2} \therefore \alpha \approx 60.25^\circ$   
 $\therefore \sqrt{65} \sin(\theta + 60.25^\circ) = 24$   
 $\sin(\theta + 60.25^\circ) = \frac{24}{\sqrt{65}} \approx 0.744$   
 $(\theta + 60.25^\circ) = 48.01^\circ \dots \pm 360^\circ$   
 $(\theta + 60.25^\circ) = 131.99^\circ \dots \pm 360^\circ$   
 $(\theta = -12.24^\circ \pm 360^\circ)$   
 $(\theta = 71.75^\circ \pm 360^\circ)$   
 $\therefore \theta = 71.7^\circ$

(b) Area of  $\triangle ADN = \frac{1}{2} |AD| |DN|$   
 $= \frac{1}{2} [AB - MC] [BC - CN]$   
 $= \frac{1}{2} [6 \cos \theta - 6 \cos \theta] [8 \sin \theta + 6 \cos \theta]$   
 $= \frac{1}{2} [6 \cos \theta - 6 \cos \theta] [8 \sin \theta + 6 \cos \theta]$   
 $= \frac{1}{2} [24 \cos \theta \sin \theta + 48 \cos^2 \theta - 48 \cos^2 \theta - 36 \cos \theta \sin \theta]$   
 $= \frac{1}{2} [14 \sin 2\theta + 48 \cos 2\theta]$   
 $= 7 \sin 2\theta + 24 \cos 2\theta$   
 $\therefore R = \sqrt{7^2 + 24^2} = 25$   
 $\therefore \text{MAX AREA} = 25$

Question 38 (\*\*\*\*\*) non calculator

$$\tan 2x^\circ + \tan 2x^\circ \tan 25^\circ = 1 - \tan 25^\circ, \quad 0 \leq x < 360.$$

Find the solutions of the above trigonometric equation, giving the answers in **degrees**.

$$x = 10^\circ, 100^\circ, 190^\circ, 280^\circ$$

Handwritten solution for Question 38:

Method 1 (Left):

$$\begin{aligned} \tan 2x + \tan 2x \tan 25 &= 1 - \tan 25 \\ \Rightarrow \tan 2x + \tan 25 &= 1 - \tan 2x \tan 25 \\ \Rightarrow \frac{\tan 2x + \tan 25}{1 - \tan 2x \tan 25} &= 1 \\ \Rightarrow \tan(2x + 25) &= 1 \\ \arctan 1 &= 45 \\ \Rightarrow 2x + 25 &= 45 \pm 180n \\ n &= 0, 1, 2, 3, \dots \\ \Rightarrow 2x &= 20 \pm 180n \\ \Rightarrow x &= 10 \pm 90n \\ \therefore x &= 10^\circ, 100^\circ, 190^\circ, 280^\circ \end{aligned}$$

Method 2 (Right - Alternative):

$$\begin{aligned} \tan 2x + \tan 2x \tan 25 &= 1 - \tan 25 \\ \Rightarrow \tan 2x(1 + \tan 25) &= 1 - \tan 25 \\ \Rightarrow \tan 2x &= \frac{1 - \tan 25}{1 + \tan 25} \\ &= \frac{\tan 45 - \tan 25}{1 + \tan 45 \tan 25} \\ &= \tan(45 - 25) \\ \text{Since } \tan 45 &= 1 \\ \Rightarrow \tan 2x &= \tan(45 - 25) \\ \Rightarrow \tan 2x &= \tan 20 \\ 2x &= 20 \pm 180n \\ n &= 0, 1, 2, 3, \dots \\ x &= 10 \pm 90n \\ \therefore x &= 10^\circ, 100^\circ, 190^\circ, 280^\circ \end{aligned}$$

Question 39 (\*\*\*\*\*)

$$2 \tan x - \sin 2x = \sin^2 x, \quad 0 \leq x < 360.$$

Find the solutions of the above trigonometric equation, giving the answers in **degrees**.

$$x = 0^\circ, 26.6^\circ, 180^\circ, 206.6^\circ$$

Handwritten solution for Question 39:

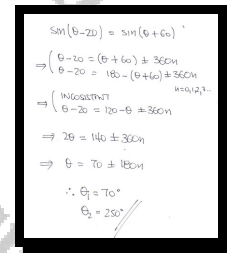
$$\begin{aligned} 2 \tan x - \sin 2x &= \sin^2 x \\ \frac{2 \sin x}{\cos x} - 2 \sin x \cos x &= \sin^2 x \\ 2 \sin x - 2 \sin x \cos^2 x &= \sin^2 x \\ 0 &= \sin^2 x \cos^2 x + 2 \sin x \cos^2 x - 2 \sin x \\ \sin x (\sin^2 x \cos^2 x + 2 \cos^2 x - 2) &= 0 \\ \sin x \left( \frac{1}{2} \sin^2 x + (1 + \cos^2 x) - 2 \right) &= 0 \\ \sin x \left( \frac{1}{2} \sin^2 x + \cos^2 x - 1 \right) &= 0 \\ \sin x (\sin^2 x + 2 \cos^2 x - 2) &= 0 \\ \text{Either } \sin x &= 0 \\ \text{OR } \sin^2 x + 2 \cos^2 x - 2 &= 0 \\ \sin^2 x + 2 \cos^2 x &= 2 \\ \sin^2 x + 2 \cos^2 x &\equiv R \sin(2x + \alpha) \\ &\equiv R \sin 2x \cos \alpha + R \cos 2x \sin \alpha \\ &\equiv (R \cos \alpha) \sin 2x + (R \sin \alpha) \cos 2x \\ R \cos \alpha &= 1 \\ R \sin \alpha &= 2 \\ \Rightarrow R &= \sqrt{1^2 + 2^2} = \sqrt{5} \\ \tan \alpha &= 2, \quad \alpha = 63.43^\circ \\ \sqrt{5} \sin(2x + 63.43^\circ) &= 2 \\ \sin(2x + 63.43^\circ) &= \frac{2}{\sqrt{5}} \\ 2x + 63.43^\circ &= 63.43^\circ + 360n \\ 2x + 63.43^\circ &= 116.57^\circ + 360n \\ 2x &= 0 \pm 180n \\ x &= 0^\circ \pm 90n \\ (n &= 0, 1, 2, 3, \dots) \\ \text{OR } 2x &= 180 \pm 360n \\ x &= 90 \pm 180n \\ \therefore x &= 0^\circ, 26.6^\circ, 180^\circ, 206.6^\circ \end{aligned}$$

**Question 40 (\*\*\*\*\*) non calculator**

Solve the following trigonometric equation

$$\sin(\theta - 20) = \sin(\theta + 60), \quad 0 \leq \theta < 360^\circ.$$

$$\theta = 70^\circ, 250^\circ$$



Handwritten solution for Question 40:

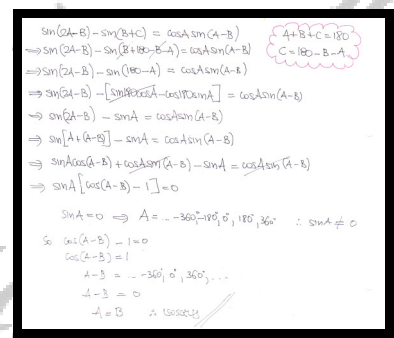
$$\begin{aligned} \sin(\theta - 20) &= \sin(\theta + 60) \\ \Rightarrow \theta - 20 &= (\theta + 60) \pm 360n \\ \theta - 20 &= 180 - (\theta + 60) \pm 360n \quad \text{using } \sin(A) = \sin(180 - A) \\ \Rightarrow \theta - 20 &= 120 - \theta \pm 360n \\ \Rightarrow 2\theta &= 140 \pm 360n \\ \Rightarrow \theta &= 70 \pm 180n \\ \therefore \theta_1 &= 70^\circ \\ \theta_2 &= 250^\circ \end{aligned}$$

**Question 41 (\*\*\*\*\*)**The three angles in a triangle  $ABC$  satisfy the relationship

$$\sin(2A - B) - \sin(B + C) = \cos A \sin(A - B).$$

Show that the triangle  $ABC$  is isosceles.

proof



Handwritten proof for Question 41:

$$\begin{aligned} \sin(2A - B) - \sin(B + C) &= \cos A \sin(A - B) \\ \Rightarrow \sin(2A - B) - \sin(180 - B - A) &= \cos A \sin(A - B) \\ \Rightarrow \sin(2A - B) - \sin(180 - A) &= \cos A \sin(A - B) \\ \Rightarrow \sin(2A - B) - [\sin(180 - A) - \cos A \sin A] &= \cos A \sin(A - B) \\ \Rightarrow \sin(2A - B) - \sin A &= \cos A \sin(A - B) \\ \Rightarrow \sin[A + (A - B)] - \sin A &= \cos A \sin(A - B) \\ \Rightarrow \sin A \cos(A - B) + \cos A \sin(A - B) - \sin A &= \cos A \sin(A - B) \\ \Rightarrow \sin A [\cos(A - B) - 1] &= 0 \\ \sin A = 0 &\Rightarrow A = -360^\circ, -180^\circ, 0^\circ, 180^\circ, 360^\circ \quad \therefore \sin A \neq 0 \\ \text{So } \cos(A - B) - 1 &= 0 \\ \cos(A - B) &= 1 \\ A - B &= -360^\circ, 0^\circ, 360^\circ, \dots \\ A - B &= 0 \\ A &= B \quad \therefore \text{isosceles} \end{aligned}$$



## Question 42 (\*\*\*\*\*)

It is given that

$$\sin(A+B) \equiv \sin A \cos B + \cos A \sin B.$$

Use the above trigonometric identity to show that

$$\sin 3x \equiv 3\sin x - 4\sin^3 x,$$

and hence find

$$\int \sqrt[3]{3\sin 2x - 2\sin 3x \cos x} \, dx.$$

$$-\frac{3}{2} \sin^{\frac{4}{3}} x + C$$

$$\begin{aligned} \sin 3x &= \sin(2x+x) = \sin 2x \cos x + \cos 2x \sin x \\ &= (2\sin x \cos x) \cos x + (1-2\sin^2 x) \sin x \\ &= 2\sin x \cos^2 x + \sin x - 2\sin^3 x \\ &= 2\sin x (1-\sin^2 x) + \sin x - 2\sin^3 x \\ &= 2\sin x - 2\sin^3 x + \sin x - 2\sin^3 x \\ &= 3\sin x - 4\sin^3 x \end{aligned}$$

$$\begin{aligned} &\int (3\sin x - 4\sin^3 x) \cos x \, dx \quad \text{since } \sin x = \cos(-x) \\ &= \int (3\sin x \cos x - 4\sin^3 x \cos x) \, dx \\ &= \int (3\sin x \cos x - 4\sin^2 x \sin x \cos x) \, dx \\ &= \int 3\sin x \cos x \, dx \\ &= \frac{3}{2} (\cos x)^2 + C \\ &= -\frac{3}{2} (\cos x)^2 + C = -\frac{3}{2} \cos^2 x + C \end{aligned}$$

## Question 43 (\*\*\*\*)

It is given that  $a$ ,  $b$  and  $c$  are consecutive terms of an arithmetic progression.

It is further given that

$$a \cos^2 \frac{x}{2} - (2a + c) \sin^2 \frac{x}{2} = a \cos x - b(1 + \sin x), \quad x \in \mathbb{R}.$$

Show clearly that

$$\tan x = -1.$$

 , proof

• IF  $a, b, c$  ARE CONSECUTIVE TERMS OF AN ARITHMETIC PROGRESSION

$$\Rightarrow b - a = c - b$$

$$\Rightarrow 2b = a + c$$

$$\Rightarrow 2b + a = 2a + c$$

• HENCE, WE NOW HAVE

$$\Rightarrow a \cos^2 \frac{x}{2} - (2a + c) \sin^2 \frac{x}{2} = a \cos x - b(1 + \sin x)$$

$$\Rightarrow a \cos^2 \frac{x}{2} - (2b + a) \sin^2 \frac{x}{2} = a \cos x - b(1 + \sin x)$$

$$\Rightarrow a \cos^2 \frac{x}{2} - a \sin^2 \frac{x}{2} - 2b \sin^2 \frac{x}{2} = a \cos x - b(1 + \sin x)$$

$$\Rightarrow a(\cos^2 \frac{x}{2} - \sin^2 \frac{x}{2}) - 2b \sin^2 \frac{x}{2} = a \cos x - b(1 + \sin x)$$

$$\Rightarrow a \cos x - 2b \sin^2 \frac{x}{2} = a \cos x - b(1 + \sin x)$$

$$\Rightarrow 2 \sin^2 \frac{x}{2} = 1 + \sin x$$

$\cos 2\theta = 1 - 2\sin^2 \theta$   
 $2\sin^2 \theta = 1 - \cos 2\theta$

$$\Rightarrow 1 - \cos x = 1 + \sin x$$

$$\Rightarrow \sin x = -\cos x$$

$$\Rightarrow \tan x = -1$$

-14/11/20

**Question 44** (\*\*\*\*\*)

Prove the validity of each of the following trigonometric identities.

$$\text{a) } \tan 3x \equiv \frac{3 \tan x - \tan^3 x}{1 - 3 \tan^2 x}$$

$$\text{b) } \frac{2 \sec^2 \theta - \cos 2\theta - 1}{2 \tan \theta + \sin 2\theta} \equiv \tan \theta.$$

proof

$$\text{a) } \tan 3x = \tan(2x+x) = \frac{\tan 2x + \tan x}{1 - \tan 2x \tan x} = \frac{\frac{2 \tan x}{1 - \tan^2 x} + \tan x}{1 - \frac{2 \tan x}{1 - \tan^2 x} \tan x}$$

Multiply top & bottom of the double fraction by  $(1 - \tan^2 x)$

$$= \frac{2 \tan x + \tan x(1 - \tan^2 x)}{1 - \tan^2 x - 2 \tan^2 x} = \frac{2 \tan x + \tan x - \tan^3 x}{1 - 3 \tan^2 x} = \frac{3 \tan x - \tan^3 x}{1 - 3 \tan^2 x} = \tan 3x$$

$$\text{b) } \frac{2 \sec^2 \theta - \cos 2\theta - 1}{2 \tan \theta + \sin 2\theta} = \frac{2 \sec^2 \theta - (\cos^2 \theta - \sin^2 \theta) - 1}{2 \tan \theta + 2 \sin \theta \cos \theta} = \frac{2 \sec^2 \theta - \cos^2 \theta + \sin^2 \theta - 1}{2 \tan \theta + 2 \sin \theta \cos \theta}$$

Multiply top & bottom of the double fraction by  $\cos^2 \theta$

$$= \frac{2 - \cos^2 \theta}{2 \sin \theta \cos \theta + 2 \sin \theta \cos \theta} = \frac{2 - \cos^2 \theta}{4 \sin \theta \cos \theta} = \frac{1 - \cos^2 \theta}{2 \sin \theta \cos \theta} = \frac{\sin^2 \theta}{2 \sin \theta \cos \theta} = \frac{\sin \theta}{2 \cos \theta} = \tan \theta = \text{RHS}$$

**Question 45** (\*\*\*\*\*) non calculator

Solve the trigonometric equation

$$\cos(\psi - 36) = \cos(\psi - 72), \quad 0 \leq \psi < 360^\circ.$$

$$\psi = 54^\circ, 234^\circ$$

$$\begin{aligned}
 \cos(\psi - 36) &= \cos(\psi - 72) \\
 \Rightarrow \psi - 36 &= \psi - 72 \pm 360^\circ n \\
 \Rightarrow \psi - 36 &= 72 - \psi \pm 360^\circ n \\
 2\psi &= 108 \pm 360^\circ n \\
 \psi &= 54 \pm 180^\circ n \\
 \therefore \psi_1 &= 54^\circ \\
 \psi_2 &= 234^\circ
 \end{aligned}$$

## Question 46 (\*\*\*\*\*)

Solve the trigonometric equation

$$\sin x + \sin 2x + \sin 3x + \sin 4x = 0, \quad 0 \leq \theta \leq \pi,$$

giving the answers in terms of  $\pi$ .

$$x = 0, \frac{2\pi}{5}, \frac{\pi}{2}, \frac{4\pi}{5}, \pi$$

Handwritten solution for the trigonometric equation  $\sin x + \sin 2x + \sin 3x + \sin 4x = 0$ .

$\sin 2x + \sin 4x + \sin 3x + \sin x = 0$   
 $\Rightarrow (\sin 2x + \sin 4x) + (\sin 3x + \sin x) = 0$   
 Using  $\sin A + \sin B = 2 \sin \left( \frac{A+B}{2} \right) \cos \left( \frac{A-B}{2} \right)$   
 $\Rightarrow 2 \sin \left( \frac{2x+4x}{2} \right) \cos \left( \frac{2x-4x}{2} \right) + 2 \sin \left( \frac{3x+x}{2} \right) \cos \left( \frac{3x-x}{2} \right) = 0$   
 $\Rightarrow \sin \left( \frac{3x}{2} \right) \cos \left( \frac{3x}{2} \right) + \sin \left( \frac{2x}{2} \right) \cos \left( \frac{2x}{2} \right) = 0$  (Note:  $\cos(-\theta) = \cos \theta$ )  
 $\Rightarrow \sin \left( \frac{3x}{2} \right) \left[ \cos \frac{3x}{2} + \cos \frac{2x}{2} \right] = 0$   
 Using  $\cos A + \cos B = 2 \cos \left( \frac{A+B}{2} \right) \cos \left( \frac{A-B}{2} \right)$   
 $\Rightarrow \sin \left( \frac{3x}{2} \right) \times 2 \cos \left( \frac{3x+2x}{2} \right) \cos \left( \frac{3x-2x}{2} \right) = 0$   
 $\Rightarrow \cos 2x \cos \frac{5x}{2} \sin \frac{3x}{2} = 0$   
 Hence  $\cos \frac{3x}{2} = 0$  or  $\sin \frac{3x}{2} = 0$  or  $\cos 2x = 0$   
 Case 1:  $\cos \frac{3x}{2} = 0 \Rightarrow \frac{3x}{2} = \frac{\pi}{2} + 2n\pi$   
 $x = \frac{\pi}{3} + \frac{4n\pi}{3}$   
 Case 2:  $\sin \frac{3x}{2} = 0 \Rightarrow \frac{3x}{2} = 0 + 2n\pi$   
 $x = 0 + \frac{4n\pi}{3}$   
 Case 3:  $\cos 2x = 0 \Rightarrow 2x = \frac{\pi}{2} + 2n\pi$   
 $x = \frac{\pi}{4} + n\pi$   
 For  $n=0, 1, 2, \dots$   
 $\therefore x = \frac{\pi}{3}, 0, \frac{4\pi}{3}, \frac{2\pi}{3}, \pi$

**Question 47** (\*\*\*\*\*)Use the substitution  $t = \tan x$  to show that if

$$10 \sin 2x = 3 + 10 \tan x,$$

$$\text{then either } \tan x = \frac{1}{2} \text{ or } \tan x = \frac{-2 \pm \sqrt{19}}{5}.$$

proof

$10 \sin 2x = 10 \tan x + 3$   
 By letting  $t = \tan x$  if  $t = \tan x$   $\sin 2x = \frac{2t}{1+t^2}$   
 $\Rightarrow 10 \left( \frac{2t}{1+t^2} \right) = 10t + 3$   
 $\Rightarrow \frac{20t}{1+t^2} = 10t + 3$   
 $\Rightarrow 20t = (10t+3)(1+t^2)$   
 $\Rightarrow 20t = 10t + 10t^3 + 3 + 3t^2$   
 $\Rightarrow 0 = 10t^3 + 3t^2 - 10t + 3$   
 Now  $t = \frac{1}{2}$  is a solution  
 so  $(2t-1)$  is a factor  
 By long divide or manipulate  
 $\Rightarrow (2t-1)(5t^2+4t-3) = 0$   
 $\Rightarrow (2t-1)(5t^2+4t-3) = 0$   
 By quadratic formula  
 $t = \frac{-4 \pm \sqrt{16 - 4(5)(-3)}}{2 \times 5}$   
 $t = \frac{-4 \pm \sqrt{64}}{10}$   
 $t = \frac{-4 \pm 8}{10}$   
 $\therefore \tan x = \frac{1}{2}$  or  $\frac{-2 \pm \sqrt{19}}{5}$

**Question 48** (\*\*\*\*\*)

Solve the following simultaneous equations.

$$x + y = \frac{1}{5}\pi, \quad \cos x + \cos y = 0, \quad 0 \leq x < 2\pi.$$

$$\boxed{\phantom{000}}, \quad (x, y) = \left( \frac{3}{5}\pi, -\frac{2}{5}\pi \right) = \left( \frac{8}{5}\pi, -\frac{7}{5}\pi \right)$$

$x + y = \frac{\pi}{5}$   
 $\cos x + \cos y = 0 \Rightarrow y = \frac{\pi}{5} - x$   
 • Substitute into the other equation  
 $\cos x + \cos\left(\frac{\pi}{5} - x\right) = 0$   
 $\cos x = -\cos\left(\frac{\pi}{5} - x\right)$   
 • Use the minus of the R.H.S inside using  $\cos(\pi - \theta) = -\cos \theta$   
 $\cos x = \cos\left[\pi - \left(\frac{\pi}{5} - x\right)\right]$   
 $\cos x = \cos\left[\frac{4\pi}{5} + x\right]$   
 • Hence we have  
 $x = \pm \left(\frac{4\pi}{5} + x\right) + 2n\pi \quad n = 0, 1, 2, \dots$   
 $2x = -\frac{4\pi}{5} + 2n\pi$   
 $x = -\frac{2\pi}{5} + n\pi$   
 $\therefore x_1 = \frac{3\pi}{5} \quad \text{and} \quad x_2 = \frac{8\pi}{5}$   
 • Find  $y$  using  $y = \frac{\pi}{5} - x$   
 $y = -\frac{2\pi}{5} \quad \text{and} \quad y_2 = -\frac{7\pi}{5}$   
 $\therefore \left(\frac{3\pi}{5}, -\frac{2\pi}{5}\right) \quad \text{and} \quad \left(\frac{8\pi}{5}, -\frac{7\pi}{5}\right)$

## Question 49 (\*\*\*\*)

$$f(x) \equiv \frac{1 - \sin 2x}{\sin x - \cos x}, \quad x \in \mathbb{R}, \sin x \neq \cos x.$$

a) Show clearly that

$$f(x) \equiv \sin x - \cos x.$$

b) Solve the equation

$$f(x)f(-x) = \cos\left(x - \frac{2\pi}{3}\right), \quad 0 \leq x < 2\pi,$$

giving the answers in terms of  $\pi$ .

$$x = \frac{2\pi}{9}, \frac{8\pi}{9}, \frac{4\pi}{3}, \frac{14\pi}{9}$$

a)  $f(x) = \frac{1 - \sin 2x}{\sin x - \cos x} = \frac{\cos^2 x + \sin^2 x - \sin 2x}{\sin x - \cos x} = \frac{\sin^2 x - 2\sin x \cos x + \cos^2 x}{\sin x - \cos x}$   
 $= \frac{(\sin x - \cos x)^2}{\sin x - \cos x} = \sin x - \cos x$   
 b) Now  $f(x)f(-x) = \cos(x - \frac{2\pi}{3})$   
 $\Rightarrow (\sin x - \cos x)(\sin(-x) - \cos(-x)) = \cos(x - \frac{2\pi}{3})$   
 $\Rightarrow (\sin x - \cos x)(-\sin x - \cos x) = \cos(x - \frac{2\pi}{3})$   
 $\Rightarrow -\sin^2 x - \sin x \cos x + \cos x \sin x + \cos^2 x = \cos(x - \frac{2\pi}{3})$   
 $\Rightarrow \cos^2 x - \sin^2 x = \cos(x - \frac{2\pi}{3})$   
 $\Rightarrow \cos 2x = \cos(x - \frac{2\pi}{3})$   
 $\Rightarrow \begin{cases} 2x = (x - \frac{2\pi}{3}) + 2k\pi \\ 2x = (\frac{2\pi}{3} - x) + 2k\pi \end{cases} \quad \eta = 0, 1, 2, 3, \dots$   
 $\Rightarrow \begin{cases} x = -\frac{2\pi}{3} + 2k\pi \\ x = \frac{2\pi}{3} + 2k\pi \end{cases}$   
 $\Rightarrow \begin{cases} x = \frac{4\pi}{3} + 2k\pi \\ x = \frac{2\pi}{3} + 2k\pi \end{cases} \quad \therefore x = \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{8\pi}{9}, \frac{14\pi}{9}$

**Question 50 (\*\*\*\*\*)**

Find the solutions of the trigonometric equation

$$\sin(x+15) + 5\sin x + \sin(x-15) = [\cos(x+15) + 5\cos x + \cos(x-15)] \tan(2x+15),$$

in the range  $0^\circ \leq x < 360^\circ$ .

$$x = 165^\circ, 345^\circ$$

Handwritten solution for Question 50:

$$\begin{aligned} \Rightarrow \sin(x+15) + 5\sin x + \sin(x-15) &= \tan(2x+15) [\cos(x+15) + 5\cos x + \cos(x-15)] \\ \Rightarrow \frac{\sin(x+15) + 5\sin x + \sin(x-15)}{\cos(x+15) + 5\cos x + \cos(x-15)} &= \tan(2x+15) \\ \Rightarrow \frac{\sin x \cos 15 + \cos x \sin 15 + 5\sin x + \sin x \cos 15 - \cos x \sin 15}{\cos x \cos 15 - \sin x \sin 15 + 5\cos x + \cos x \cos 15 + \sin x \sin 15} &= \tan(2x+15) \\ \Rightarrow \frac{2\sin x \cos 15 + 5\sin x}{2\cos x \cos 15 + 5\cos x} &= \tan(2x+15) \\ \Rightarrow \frac{\sin x (2\cos 15 + 5)}{\cos x (2\cos 15 + 5)} &= \tan(2x+15) \\ \Rightarrow \tan x &= \tan(2x+15) \\ \therefore x &= 2x+15 \pmod{180} \quad \text{or } x = 2x+15 \pmod{180} \\ -x &= 15 \pmod{180} \\ x &= -15 \pmod{180} \\ \therefore x_1 &= 165^\circ \\ x_2 &= 345^\circ \end{aligned}$$

**Question 51 (\*\*\*\*\*)**

Solve the trigonometric equation

$$4\cos(x+30) = \sec(x-60), \quad 0 \leq x < 360^\circ.$$

$$x = 45^\circ, 165^\circ, 225^\circ, 345^\circ$$

Handwritten solution for Question 51 (Method 1):

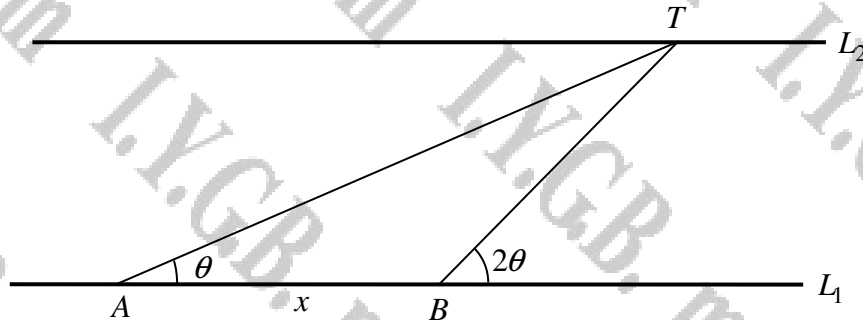
$$\begin{aligned} 4\cos(x+30) &= \sec(x-60) \\ \Rightarrow 4\cos(x+30) &= \frac{1}{\cos(x-60)} \\ \Rightarrow 4\cos(x+30)\cos(x-60) &= 1 \\ \Rightarrow 2[2\cos(x+30)\cos(x-60)] &= 1 \\ \Rightarrow 2[\cos(x+30+x-60) + \cos(x+30-x+60)] &= 1 \\ \Rightarrow 2[\cos(2x-30) + \cos(90)] &= 1 \\ \Rightarrow 2[\cos(2x-30) + 0] &= 1 \\ \Rightarrow 2\cos(2x-30) &= 1 \\ \Rightarrow \cos(2x-30) &= \frac{1}{2} \\ \text{where } \cos\left(\frac{\pi}{3}\right) &= \frac{1}{2} \\ \therefore 2x-30 &= 60 \pmod{360} \quad \text{or } 2x-30 = 300 \pmod{360} \\ \therefore 2x &= 90 \pmod{360} \quad \text{or } 2x = 330 \pmod{360} \\ \therefore x &= 45 \pmod{180} \quad \text{or } x = 165 \pmod{180} \\ \therefore x_1 &= 45^\circ, x_2 = 165^\circ, x_3 = 225^\circ, x_4 = 345^\circ \end{aligned}$$

Handwritten solution for Question 51 (Method 2):

$$\begin{aligned} \text{ALTERNATIVE} \\ 4\cos(x+30) &= \sec(x-60) \\ \Rightarrow 4\cos(x+30) &= \frac{1}{\cos(x-60)} \\ \Rightarrow 4\cos(x+30)\cos(x-60) &= 1 \\ \Rightarrow 2[2\cos(x+30)\cos(x-60)] &= 1 \\ \Rightarrow 2[\cos(x+30+x-60) + \cos(x+30-x+60)] &= 1 \\ \Rightarrow 2[\cos(2x-30) + \cos(90)] &= 1 \\ \Rightarrow 2[\cos(2x-30) + 0] &= 1 \\ \Rightarrow 2\cos(2x-30) &= 1 \\ \Rightarrow \cos(2x-30) &= \frac{1}{2} \\ \therefore 2x-30 &= 60 \pmod{360} \quad \text{or } 2x-30 = 300 \pmod{360} \\ \therefore 2x &= 90 \pmod{360} \quad \text{or } 2x = 330 \pmod{360} \\ \therefore x &= 45 \pmod{180} \quad \text{or } x = 165 \pmod{180} \\ \therefore x_1 &= 45^\circ, x_2 = 165^\circ, x_3 = 225^\circ, x_4 = 345^\circ \end{aligned}$$

## Question 52 (\*\*\*\*)

In the following question you may not use the sine or the cosine rule.



The figure above shows the plan of a river whose banks are modelled as straight parallel lines  $L_1$  and  $L_2$ .

The points  $A$  and  $B$  lie on  $L_1$ , so that  $|AB| = x$ .

A tree is positioned at the point  $T$  on  $L_2$ , so that  $AT$  and  $BT$  subtend angles of  $\theta$  and  $2\theta$ , respectively.

The tree located at  $T$  has height  $h$ . The angle of elevation of the top of the tree as viewed from  $A$  is  $\theta$ .

Show that

$$h = 2x \sin \theta.$$

,  proof

Let the width be  $y$  if  $|BC| = w$

- $\frac{y}{w} = \tan 2\theta$      $\frac{y}{x+w} = \tan \theta$
- $\frac{y}{w} = \frac{2 \tan \theta}{1 - \tan^2 \theta}$      $\frac{x+w}{y} = \frac{1}{\tan \theta}$
- $\frac{xy}{y} = \frac{1 - \tan^2 \theta}{2 \tan \theta}$      $\frac{x}{y} + \frac{w}{y} = \cot \theta$

$\Rightarrow \frac{x}{y} + \frac{1 - \tan^2 \theta}{2 \tan \theta} = \cot \theta$   
 $\Rightarrow \frac{x}{y} = \cot \theta - \frac{1 - \tan^2 \theta}{2 \tan \theta}$   
 $\Rightarrow \frac{x}{y} = \frac{2 \cot \theta \tan \theta - 1 + \tan^2 \theta}{2 \tan \theta}$   
 $\Rightarrow \frac{x}{y} = \frac{1 + \tan^2 \theta}{2 \tan \theta}$   
 $\Rightarrow \frac{x}{y} = \frac{1 + \tan^2 \theta}{2 \tan \theta}$   
 $\Rightarrow y = \frac{2x \tan \theta}{1 + \tan^2 \theta}$

Looking at  $\triangle ATC$

$$\frac{h}{d} = \sin \theta$$

$$d = \frac{h}{\sin \theta}$$

$$d = \frac{2x \tan \theta}{1 + \tan^2 \theta} \times \frac{1}{\sin \theta}$$

Now find of elevation, looking at the diagram below

$\Rightarrow \frac{h}{d} = \sin \theta$   
 $\Rightarrow h = d \sin \theta$   
 $\Rightarrow h = \frac{2x \tan \theta}{1 + \tan^2 \theta} \times \frac{1}{\sin \theta}$   
 $\Rightarrow h = \frac{2x \tan \theta}{1 + \tan^2 \theta} \times \frac{1}{\sin \theta}$   
 $\Rightarrow h = \frac{2x \sin \theta \cos \theta}{1 + \tan^2 \theta} \times \frac{1}{\sin \theta}$   
 $\Rightarrow h = 2x \sin \theta$   
 as required



**Question 53** (\*\*\*\*)

By using the trigonometric identity for  $\sin 2\theta$ , or otherwise, show clearly that

$$\cos \frac{\pi}{7} \cos \frac{2\pi}{7} \cos \frac{4\pi}{7} = -\frac{1}{8}.$$

**proof**

$$\begin{aligned} \sin 2\alpha &= 2 \sin \alpha \cos \alpha \rightarrow \cos \alpha = \frac{\sin 2\alpha}{2 \sin \alpha} \\ \cos \frac{\pi}{5} \cos \left( \frac{\pi}{5} + \frac{\pi}{5} \right) &= \frac{\sin \frac{\pi}{5}}{2 \sin \frac{\pi}{5}} \times \sin \frac{2\pi}{5} = \frac{\sin \frac{\pi}{5}}{2 \sin \frac{\pi}{5}} \\ &= \frac{\sin \left( \frac{\pi}{5} + \frac{\pi}{5} \right)}{2 \sin \frac{\pi}{5}} = \frac{\sin \frac{2\pi}{5} \cos \frac{\pi}{5} + \cos \frac{2\pi}{5} \sin \frac{\pi}{5}}{2 \sin \frac{\pi}{5}} \\ &= \frac{-\sin \frac{\pi}{5}}{2 \sin \frac{\pi}{5}} = -\frac{1}{2} \end{aligned}$$

**Question 54** (\*\*\*\*)

Show clearly that

$$\tan \frac{3\pi}{8} - \tan \frac{\pi}{8} - \tan \frac{3\pi}{8} \tan \frac{\pi}{8} = 1.$$

$\square$ , proof

• SKETCHING BY THE L.H.S.

$$\begin{aligned} & \tan \frac{\pi}{8} - \tan \frac{\pi}{4} - \tan \frac{\pi}{8} \tan \frac{\pi}{4} \\ &= \tan \left( \frac{\pi}{8} - \frac{\pi}{4} \right) - \tan \frac{\pi}{4} \left[ \tan \left( \frac{\pi}{8} - \frac{\pi}{4} \right) \right] \\ &= \tan \left( \frac{\pi}{8} - \frac{\pi}{4} \right) - \tan \frac{\pi}{4} \tan \left( \frac{\pi}{8} - \frac{\pi}{4} \right) - \tan \frac{\pi}{4} \\ &= \tan \left( \frac{\pi}{8} - \frac{\pi}{4} \right) \left[ 1 - \tan \frac{\pi}{4} \right] - \tan \frac{\pi}{4} \\ &= \frac{\tan \frac{\pi}{8} - \tan \frac{\pi}{4}}{1 - \tan \frac{\pi}{8} \tan \frac{\pi}{4}} \left[ 1 - \tan \frac{\pi}{4} \right] - \tan \frac{\pi}{4} \end{aligned}$$

CRASHING ABOVE THEORY  
FOR  $\tan(210)$

• LET  $\tan \frac{\pi}{4} = T$  & MORE  $\tan \frac{\pi}{4} = 1$

$$\begin{aligned} &= \frac{1+T}{1-T} (1-T) - T \\ &= 1+T-T \\ &= 1 \end{aligned}$$

ALTERNATIVE METHOD

• USING THE IDENTITY  $\tan \left( \frac{\pi}{2} - \theta \right) = \cot \theta = \frac{1}{\tan \theta}$  (WE CERTAIN)

$$\begin{aligned} & \tan \frac{\pi}{8} - \tan \frac{\pi}{4} - \tan \frac{\pi}{8} \tan \frac{\pi}{4} \\ &= \tan \left( \frac{\pi}{8} - \frac{\pi}{4} \right) - \tan \left( \frac{\pi}{8} - \frac{\pi}{4} \right) \tan \frac{\pi}{4} \end{aligned}$$

$$\begin{aligned}
 &= \frac{1}{\tan \frac{\theta}{2}} - \tan \frac{\theta}{2} - \frac{1}{\tan \frac{\theta}{2}} \times \tan \frac{\theta}{2} \\
 \bullet \text{ If } T &= \tan \frac{\theta}{2} \text{ then} \\
 &= \frac{1}{T} - T - 1 \\
 &= \frac{1-T^2}{T} - 1 \\
 &= 2 \left( \frac{1-T^2}{2T} \right) - 1 \\
 &= 2 \left( \frac{1 - \tan^2 \frac{\theta}{2}}{2 \tan \frac{\theta}{2}} \right) - 1 \quad \leftarrow \tan 2\theta = \frac{2 \tan \frac{\theta}{2}}{1 - \tan^2 \frac{\theta}{2}} \\
 &= 2 \times \left[ \frac{1}{\tan(2\theta)} \right] - 1 \\
 &= 2 \times \frac{1}{\tan 2\theta} - 1 \\
 &= 2 \times 1 - 1 \\
 &= 1
 \end{aligned}$$

**Question 55 (\*\*\*\*\*) non calculator**

Solve the trigonometric equation

$$\sin(\varphi + 30) = \cos(\varphi - 45), \quad 0 \leq \varphi < 360^\circ.$$

$$\varphi = 52.5^\circ, 232.5^\circ$$

$\sin(\varphi + 30) = \cos(\varphi - 45)$   
 $\Rightarrow \sin[90 - (\varphi + 30)] = \cos(\varphi - 45)$   $\sin A = \cos(90 - A)$   
 $\Rightarrow \cos(60 - \varphi) = \cos(\varphi - 45)$   
 $\Rightarrow \begin{cases} 60 - \varphi = \varphi - 45 \pm 360n \\ 60 - \varphi = 45 - \varphi \pm 360n \end{cases} \quad n=0, 1, 2, \dots$   
 $\Rightarrow \begin{cases} -2\varphi = -105 \pm 360n \\ \text{Inconsistent} \end{cases}$   
 $\Rightarrow 2\varphi = 105 \pm 360n$   
 $\Rightarrow \varphi = 52.5 \pm 180n$   
 $\therefore \varphi = 52.5^\circ, 232.5^\circ //$

**Question 56** (\*\*\*\*)

Eliminate  $\theta$  from the following pair of equation.

$$\tan \theta + \cot \theta = x^3$$

$$\sec \theta - \cos \theta = y^3$$

Write the answer in the form

$$f(x, y) = 1.$$

$$\boxed{\text{SPM}}, \quad \boxed{x^4 y^2 - y^4 x^2 = 1}$$

- $$\frac{1}{\tan \theta} + \frac{1}{\cot \theta} = 2$$

$$\frac{\sec \theta}{\tan \theta} - \frac{\sec \theta}{\cot \theta} = 2$$

● Simplify both equations now since a constant

$$\frac{\sec \theta}{\tan \theta} = \frac{\sec \theta}{\frac{\sin \theta}{\cos \theta}} = 2$$

$$\frac{\sec \theta \cdot \cos \theta}{\sin \theta} = 2$$

$$\frac{1}{\sin \theta} = 2$$

$$\frac{1}{\sin \theta} = 2$$

$$\frac{1}{\sin \theta} = 2$$
- Multiply the two expressions side by side

$$\frac{1}{\tan \theta} \times \frac{\sec \theta}{\cot \theta} = 2^2$$

$$\frac{1}{\tan \theta} = 2^2$$

$$\frac{1}{\tan \theta} = \frac{1}{2^2}$$

$$\tan \theta = \frac{1}{2^2}$$
- Substitute into the second equation

$$\sec \theta - \sec \theta = 2^2$$

$$\frac{1}{\tan \theta} - \frac{1}{\tan \theta} = 2^2$$

$$\frac{1}{\tan^2 \theta} - 1 = 2^2$$

$$\frac{1}{\tan^2 \theta} - 2^2 = 1$$

## Question 57 (\*\*\*\*\*)

It is given that

$$\tan^2\left(\frac{\pi}{4} - \frac{\theta}{2}\right) \equiv \frac{1 - \sin \theta}{1 + \sin \theta}, \quad x \neq \frac{\pi}{2}(4n+3), \quad n \in \mathbb{Z}.$$

- a) Prove the validity of the above trigonometric identity
- b) Hence solve the equation

$$\tan\left(\frac{\pi}{4} - 2x\right) = \sqrt{7+4\sqrt{3}}, \quad 0 \leq x < \pi.$$

$$x = \frac{5\pi}{12}, \frac{11\pi}{6}$$

(a) LHS -  $\tan^2\left(\frac{\pi}{4} - \frac{\theta}{2}\right) = \left[\frac{\tan\frac{\pi}{4} - \tan\frac{\theta}{2}}{1 + \tan\frac{\pi}{4}\tan\frac{\theta}{2}}\right]^2 = \left[\frac{1 - \tan\frac{\theta}{2}}{1 + \tan\frac{\theta}{2}}\right]^2 = \left[\frac{1 - \frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}}{1 + \frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}}\right]^2$   
 $= \frac{(\cos\frac{\theta}{2} - \sin\frac{\theta}{2})^2}{(\cos\frac{\theta}{2} + \sin\frac{\theta}{2})^2} = \frac{\cos^2\frac{\theta}{2} - 2\cos\frac{\theta}{2}\sin\frac{\theta}{2} + \sin^2\frac{\theta}{2}}{\cos^2\frac{\theta}{2} + 2\cos\frac{\theta}{2}\sin\frac{\theta}{2} + \sin^2\frac{\theta}{2}} = \frac{1 - 2\cos\frac{\theta}{2}\sin\frac{\theta}{2}}{1 + 2\cos\frac{\theta}{2}\sin\frac{\theta}{2}}$   
 $= \frac{1 - \sin\theta}{1 + \sin\theta} = \text{RHS}$

(b)  $\tan\left(\frac{\pi}{4} - 2x\right) = \sqrt{7+4\sqrt{3}} \Rightarrow 1 + \sin 4x = 1 - \frac{\sqrt{3}}{2}$   
 $\Rightarrow \tan\left(\frac{\pi}{4} - 2x\right) = 7 + 4\sqrt{3}$   
 $\Rightarrow \frac{1 - \sin 4x}{1 + \sin 4x} = 7 + 4\sqrt{3}$   
 $\Rightarrow \frac{2 - (1 + \sin 4x)}{1 + \sin 4x} = 7 + 4\sqrt{3}$   
 $\Rightarrow \frac{2}{1 + \sin 4x} - 1 = 7 + 4\sqrt{3}$   
 $\Rightarrow \frac{2}{1 + \sin 4x} = 8 + 4\sqrt{3}$   
 $\Rightarrow \frac{1}{1 + \sin 4x} = 4 + 2\sqrt{3}$   
 $\Rightarrow 1 + \sin 4x = \frac{1}{4 + 2\sqrt{3}}$   
 $\Rightarrow 1 + \sin 4x = \frac{4 - 2\sqrt{3}}{16 - 12}$

$\Rightarrow \sin 4x = -\frac{\sqrt{3}}{2}$   
 $4x = \frac{4\pi}{3} \pm 2n\pi$   
 $x = \frac{\pi}{3} \pm \frac{n\pi}{2}$   
 $\therefore x_1 = \frac{\pi}{3}$   
 $x_2 = \frac{5\pi}{6}$   
 $x_3 = \frac{4\pi}{3}$   
 $x_4 = \frac{3\pi}{2}$   
 $x_5 = \frac{7\pi}{6}$   
 $x_6 = \frac{2\pi}{3}$   
 $x_7 = \frac{\pi}{6}$   
 $x_8 = 0$   
 $x_9 = \frac{11\pi}{6}$   
 $x_{10} = \frac{5\pi}{3}$   
 $x_{11} = \frac{7\pi}{3}$   
 $x_{12} = \frac{8\pi}{3}$   
 $x_{13} = \frac{9\pi}{3} = \pi$   
 $x_{14} = \frac{10\pi}{3}$   
 $x_{15} = \frac{11\pi}{3}$   
 $x_{16} = \frac{12\pi}{3} = 4\pi$

## Question 58 (\*\*\*\*\*)

It is given that

$$p = \sin^2 \theta, \quad q = \tan 2\theta.$$

Use trigonometric identities to find a simplified expression for  $q^2$  in terms of  $p$ .

$$q^2 = \frac{4p(1-p)}{(1-2p)^2}$$

Handwritten solution for Question 58:

Given  $p = \sin^2 \theta$ , then  $\frac{1}{p} = \csc^2 \theta$ .

Using the identity  $\frac{1}{p} - 1 = \csc^2 \theta - 1 = \cot^2 \theta$ , we have  $\cot^2 \theta = \frac{1}{p} - 1 = \frac{1-p}{p}$ .

Since  $\tan^2 \theta = \frac{1}{\cot^2 \theta}$ , we have  $\tan^2 \theta = \frac{p}{1-p}$ .

Using the double angle identity  $\tan 2\theta = \frac{2 \tan \theta}{1 - \tan^2 \theta}$ , we have  $q = \frac{2 \tan \theta}{1 - \tan^2 \theta}$ .

Squaring both sides, we get  $q^2 = \frac{4 \tan^2 \theta}{(1 - \tan^2 \theta)^2}$ .

Substituting  $\tan^2 \theta = \frac{p}{1-p}$ , we have  $q^2 = \frac{4 \left( \frac{p}{1-p} \right)}{\left( 1 - \frac{p}{1-p} \right)^2}$ .

Simplifying the denominator, we have  $1 - \frac{p}{1-p} = \frac{1-p-p}{1-p} = \frac{1-2p}{1-p}$ .

Therefore,  $q^2 = \frac{4 \left( \frac{p}{1-p} \right)}{\left( \frac{1-2p}{1-p} \right)^2} = \frac{4p(1-p)}{(1-2p)^2}$ .

## Question 659 (\*\*\*\*)

$$\sin 2x - \sqrt{3} \cos 2x = \tan x, \quad 0 \leq x < 2\pi.$$

Find the solutions of the above trigonometric equation, giving the answers terms of  $\pi$ .

$$x = \frac{\pi}{4}, \frac{\pi}{3}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{4\pi}{3}, \frac{7\pi}{4}$$

$\sin 2x - \sqrt{3} \cos 2x = \tan x$   
 $\Rightarrow \sin 2x - \tan x = \sqrt{3} \cos 2x$   
 $\Rightarrow \frac{2 \sin x \cos x - \sin x}{\cos x} = \sqrt{3} \cos 2x$   
 $\Rightarrow \frac{\sin x (2 \cos x - 1)}{\cos x} = \sqrt{3} \cos 2x$   
 $\Rightarrow \tan x (2 \cos x - 1) = \sqrt{3} \cos 2x$   
 $\Rightarrow \tan x \cos 2x - \sqrt{3} \cos 2x = 0$   
 $\Rightarrow \cos 2x (\tan x - \sqrt{3}) = 0$   
 $\Rightarrow \cos 2x = 0 \quad \text{or} \quad \tan x = \sqrt{3}$   
 $\Rightarrow 2x = \frac{\pi}{2} + k\pi \quad \text{or} \quad x = \frac{\pi}{3} + k\pi$   
 $\Rightarrow x = \frac{\pi}{4} + \frac{k\pi}{2} \quad \text{or} \quad x = \frac{\pi}{3} + k\pi$   
 $\Rightarrow x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4} \quad \text{or} \quad x = \frac{\pi}{3}, \frac{4\pi}{3}$

$\Rightarrow \sin 2x - \sqrt{3} \cos 2x = \tan x$   
 $\Rightarrow \frac{2t}{1+t^2} - \sqrt{3} \frac{1-t^2}{1+t^2} = \frac{t}{1+t^2}$   
 $\Rightarrow 2t - \sqrt{3}(1-t^2) = t$   
 $\Rightarrow 2t - \sqrt{3} + \sqrt{3}t^2 = t$   
 $\Rightarrow 0 = t^2 - \sqrt{3}t - (t - \sqrt{3})$   
 $\Rightarrow 0 = (t - \sqrt{3})(t + 1)$   
 $\Rightarrow (t - \sqrt{3})(t + 1) = 0$   
 $\Rightarrow t = \sqrt{3} \quad \text{or} \quad t = -1$   
 $\Rightarrow x = \frac{\pi}{3} + k\pi \quad \text{or} \quad x = \frac{3\pi}{4} + k\pi$   
 $\Rightarrow x = \frac{\pi}{3}, \frac{4\pi}{3} \quad \text{or} \quad x = \frac{3\pi}{4}, \frac{7\pi}{4}$

## Question 60 (\*\*\*\*)

$$\sin\left(3x - \frac{\pi}{3}\right) + \sin\left(x - \frac{\pi}{3}\right) = \sin x, \quad 0 \leq x < \pi.$$

Determine the solutions of the above trigonometric equation, giving the answers in terms of  $\pi$ .

$$\boxed{\phantom{0}}, \quad x = \frac{\pi}{4}, \frac{\pi}{3}, \frac{3\pi}{4}$$

**METHOD 1**

• SIMILAR BY DERIVING AN IDENTITY TO COMBINE THE SINS IN THE LHS

$$\begin{aligned} \sin(A+B) &= \sin A \cos B + \cos A \sin B \\ \sin(A-B) &= \sin A \cos B - \cos A \sin B \\ \hline \sin(A+B) + \sin(A-B) &= 2\sin A \cos B \end{aligned}$$

Let  $A = 4x$  and  $B = \frac{\pi}{3}$

$$\sin(4x + \frac{\pi}{3}) + \sin(4x - \frac{\pi}{3}) = 2\sin 4x \cos \frac{\pi}{3}$$

So  $\sin(4x + \frac{\pi}{3}) + \sin(4x - \frac{\pi}{3}) = 2\sin 4x \cos \frac{\pi}{3}$

• THIS USES ADDING FORMULAE

$$\begin{aligned} \Rightarrow \sin\left(4x - \frac{\pi}{3}\right) + \sin\left(4x - \frac{\pi}{3}\right) &= \sin 2x \\ \Rightarrow 2\sin\left(\frac{4x - \frac{\pi}{3}}{2}\right) \cos\left(\frac{(4x - \frac{\pi}{3}) - (4x - \frac{\pi}{3})}{2}\right) &= \sin 2x \\ \Rightarrow 2\sin\left(\frac{4x - \frac{\pi}{3}}{2}\right) \cos\left(\frac{0}{2}\right) &= \sin 2x \\ \Rightarrow 2\sin\left(2x - \frac{\pi}{6}\right) \cos 0 &= \sin 2x \\ \Rightarrow 2\sin\left(2x - \frac{\pi}{6}\right) \cdot 1 &= \sin 2x \\ \Rightarrow 2\sin\left(2x - \frac{\pi}{6}\right) &= \sin 2x \\ \Rightarrow 2\sin\left(2x - \frac{\pi}{6}\right) - \sin 2x &= 0 \end{aligned}$$

• NOW PROCEED AS BEFORE

$$\begin{aligned} \Rightarrow \sin 2x - \sin 2x &= \sqrt{3} \cos 2x \\ \Rightarrow 2\sin 2x \cos 2x - \sin 2x &= \sqrt{3} \cos 2x \\ \Rightarrow \frac{2\sin 2x \cos 2x - \sin 2x}{\cos 2x} &= \sqrt{3} \cos 2x \\ \Rightarrow \frac{\sin 2x (2\cos 2x - 1)}{\cos 2x} &= \sqrt{3} \cos 2x \\ \Rightarrow \tan 2x \cos 2x &= \sqrt{3} \cos 2x \\ \Rightarrow \tan 2x \cos 2x - \sqrt{3} \cos 2x &= 0 \\ \Rightarrow \cos 2x (\tan 2x - \sqrt{3}) &= 0 \end{aligned}$$

• HENCE WE HAVE

$$\begin{aligned} \cos 2x &= 0 & \tan 2x &= \sqrt{3} \\ 2x &= \frac{\pi}{2} \pm n\pi & 2x &= \frac{\pi}{3} \pm n\pi \\ 2x &= \frac{5\pi}{2} \pm 2n\pi & 2x &= \frac{4\pi}{3} \pm 2n\pi \\ x &= \frac{5\pi}{4} \pm n\pi & x &= \frac{2\pi}{3} \pm n\pi \end{aligned}$$

• HENCE WE HAVE

$$\begin{aligned} \Rightarrow \sin 2x - \sqrt{3} \cos 2x &= \sin 2x \\ \Rightarrow \sin 2x - \sqrt{3} \cos 2x - \sin 2x &= 0 \\ \Rightarrow -\sqrt{3} \cos 2x &= 0 \end{aligned}$$

**METHOD 2**

• THIS REQUIRES KNOWLEDGE OF THE "LITTLE t" IDENTITIES

STARTING FROM THE "YOUNG" BOXED EXPRESSION OF METHOD 1

$$\begin{aligned} \Rightarrow \sin 2x - \sqrt{3} \cos 2x &= \sin 2x \\ \Rightarrow \frac{2t}{1+t^2} - \sqrt{3} \frac{1-t^2}{1+t^2} &= \frac{2t}{1+t^2} \\ \Rightarrow 2t - \sqrt{3}(1-t^2) &= 2t \\ \Rightarrow 2t - \sqrt{3} + \sqrt{3}t^2 &= 2t \\ \Rightarrow 0 &= \sqrt{3}t^2 - \sqrt{3} \\ \Rightarrow 0 &= t^2 - 1 \\ \Rightarrow (t-1)(t+1) &= 0 \end{aligned}$$

• FACTORISE IN PARS

$$\begin{aligned} \Rightarrow 0 &= t^2(t-1) - (t-1) \\ \Rightarrow 0 &= (t-1)(t^2-1) \\ \Rightarrow (t-1)(t-1)(t+1) &= 0 \end{aligned}$$

THIS USES THE

$$\frac{1}{\tan 2x} = \frac{\sqrt{3}}{1} \Rightarrow \begin{cases} 2x = \frac{\pi}{3} \pm n\pi \\ 2x = \frac{2\pi}{3} \pm n\pi \\ 2x = \frac{4\pi}{3} \pm n\pi \end{cases}$$

• HENCE WE HAVE

$$x = \frac{\pi}{6}, \frac{\pi}{3}, \frac{2\pi}{3}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{4\pi}{3}, \frac{3\pi}{2}, \frac{5\pi}{3}$$

• HENCE WE HAVE

$$x = \frac{\pi}{6}, \frac{\pi}{3}, \frac{2\pi}{3}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{4\pi}{3}, \frac{3\pi}{2}, \frac{5\pi}{3}$$

## Question 61 (\*\*\*\*\*)

The function  $f$  is defined as

$$f(x) = \sin\left(x + \frac{7\pi}{12}\right) \sin\left(x + \frac{\pi}{12}\right), \quad 0 \leq x < 2\pi$$

Solve the equation

$$f(x) + f(-x) = f\left(\frac{\pi}{4} - x\right).$$

$$\boxed{\phantom{00000}}, \quad x = \frac{\pi}{24}, \frac{13\pi}{24}, \frac{25\pi}{24}, \frac{37\pi}{24}$$

• START BY SIMPLIFYING THE FUNCTION FIRST, BY RECALLING THE IDENTITY

$$\begin{aligned} \cos(A+B) &= \cos A \cos B - \sin A \sin B \\ \cos(A-B) &= \cos A \cos B + \sin A \sin B \end{aligned}$$

SUBTRACTING "MINUSES" GIVES  
 $2\cos A \cos B = \cos(A+B) + \cos(A-B)$

• APPLY THE ABOVE

$$\begin{aligned} f(x) &= \sin\left(x + \frac{7\pi}{12}\right) \sin\left(x + \frac{\pi}{12}\right) \\ &= \frac{1}{2} \left[ \cos\left(x + \frac{7\pi}{12} - x - \frac{\pi}{12}\right) - \cos\left(x + \frac{7\pi}{12} + x + \frac{\pi}{12}\right) \right] \\ &= \frac{1}{2} \left[ \cos\left(\frac{6\pi}{12}\right) - \cos\left(2x + \frac{8\pi}{12}\right) \right] \\ &= \frac{1}{2} \left[ \cos\left(\frac{\pi}{2}\right) - \cos\left(2x + \frac{2\pi}{3}\right) \right] \\ &= -\frac{1}{2} \cos\left(2x + \frac{2\pi}{3}\right) \end{aligned}$$

• NEXT THE EQUATION BECOMES

$$\begin{aligned} \Rightarrow f(x) + f(-x) &= f\left(\frac{\pi}{4} - x\right) \\ \Rightarrow -\frac{1}{2} \cos\left(2x + \frac{2\pi}{3}\right) - \frac{1}{2} \cos\left(-2x + \frac{2\pi}{3}\right) &= -\frac{1}{2} \cos\left(\frac{\pi}{4} - x - \frac{2\pi}{3}\right) \\ \Rightarrow \cos\left(2x + \frac{2\pi}{3}\right) + \cos\left(\frac{2\pi}{3} - 2x\right) &= \cos\left(\frac{\pi}{4} + \frac{2\pi}{3} - x\right) \\ \Rightarrow \cos\left(2x + \frac{2\pi}{3}\right) + \cos\left(\frac{2\pi}{3} - 2x\right) &= \cos\left(\frac{11\pi}{12} - x\right) \\ \Rightarrow \cos\left(2x + \frac{2\pi}{3}\right) + \cos\left(2x - \frac{2\pi}{3}\right) &= \cos\left(2x - \frac{\pi}{12}\right) \\ \Rightarrow \cos 2x \cos \frac{2\pi}{3} - \sin 2x \sin \frac{2\pi}{3} &= \cos 2x \cos \frac{\pi}{12} + \sin 2x \sin \frac{\pi}{12} \\ \Rightarrow \cos 2x \cos \frac{2\pi}{3} + \sin 2x \sin \frac{2\pi}{3} &= \cos 2x \cos \frac{\pi}{12} + \sin 2x \sin \frac{\pi}{12} \end{aligned}$$

• DIVIDE THE EQUATION BY  $\cos 2x$

$$\begin{aligned} \Rightarrow 2 \cos \frac{2\pi}{3} &= \cos \frac{\pi}{12} + \tan 2x \sin \frac{\pi}{12} \\ \Rightarrow 2\left(-\frac{1}{2}\right) &= -\frac{\sqrt{3}}{2} + \tan 2x \times \left(\frac{1}{2}\right) \\ \Rightarrow -1 &= -\frac{\sqrt{3}}{2} - \frac{1}{2} \tan 2x \\ \Rightarrow -2 &= -\sqrt{3} - \tan 2x \\ \Rightarrow \tan 2x &= 2 - \sqrt{3} \\ \Rightarrow \arctan(2 - \sqrt{3}) &= \frac{\pi}{12} \end{aligned}$$

• FINALLY WE OBTAIN

$$\begin{aligned} 2x &= \frac{\pi}{12} \pm n\pi \quad n = 0, 1, 2, 3, \dots \\ x &= \frac{\pi}{24} \pm \frac{n\pi}{2} \end{aligned}$$

$\therefore x = \frac{\pi}{24}, \frac{13\pi}{24}, \frac{25\pi}{24}, \frac{37\pi}{24}$



**Question 62** (\*\*\*\*\*)

It is given that the three angles of a triangle  $\alpha$ ,  $\beta$  and  $\gamma$  satisfy the relationship

$$\tan \frac{\alpha}{2} = \left(1 + \tan^2 \frac{\alpha}{2}\right) \sin(\beta - \gamma).$$

Assuming that the triangle is not right angled, show that

$$3 \tan \gamma = \tan \beta.$$

,  proof

**START WITH IDENTITIES - AS FOLLOWS**

$$\Rightarrow \tan \frac{\alpha}{2} = \left[1 + \tan^2 \frac{\alpha}{2}\right] \sin(\beta - \gamma)$$

$$\Rightarrow \tan \frac{\alpha}{2} = \sec^2 \frac{\alpha}{2} \sin(\beta - \gamma)$$

$$\Rightarrow \frac{\tan \frac{\alpha}{2}}{\sec^2 \frac{\alpha}{2}} = \sin(\beta - \gamma)$$

$$\Rightarrow \tan \frac{\alpha}{2} \cos^2 \frac{\alpha}{2} = \sin(\beta - \gamma)$$

$$\Rightarrow \frac{\sin \frac{\alpha}{2}}{\cos \frac{\alpha}{2}} \times \cos^2 \frac{\alpha}{2} = \sin(\beta - \gamma)$$

$$\Rightarrow \sin \frac{\alpha}{2} \cos \frac{\alpha}{2} = \sin(\beta - \gamma)$$

$$\Rightarrow 2 \sin \frac{\alpha}{2} \cos \frac{\alpha}{2} = 2 \sin(\beta - \gamma)$$

$$\Rightarrow \sin \alpha = 2 \sin(\beta - \gamma)$$

**NOW  $\alpha + \beta + \gamma = \pi$**   
 $\alpha = \pi - (\beta + \gamma)$

$$\Rightarrow \sin(\pi - (\beta + \gamma)) = 2 \sin(\beta - \gamma)$$

$$\Rightarrow \sin \pi \cos(\beta + \gamma) - \cos \pi \sin(\beta + \gamma) = 2 \sin \beta \cos \gamma - 2 \cos \beta \sin \gamma$$

$$\Rightarrow \sin(\beta + \gamma) = 2 \sin \beta \cos \gamma - 2 \cos \beta \sin \gamma$$

$$\Rightarrow \sin \beta \cos \gamma + \cos \beta \sin \gamma = 2 \sin \beta \cos \gamma - 2 \cos \beta \sin \gamma$$

**3 cos β sin γ = sin β cos γ**  
 $\Rightarrow \frac{3 \cos \beta \sin \gamma}{\sin \beta \cos \gamma} = \frac{\sin \beta \cos \gamma}{\sin \beta \cos \gamma}$  cos β ≠ 0  
 $\Rightarrow 3 \tan \gamma = \tan \beta$  As required

**Question 63** (\*\*\*\*\*) non calculator

Solve the trigonometric equation

$$\sin(y - 48) = \cos(y + 12), \quad 0 \leq y < 360^\circ.$$

$$y = 63^\circ, 243^\circ$$

$$\sin(y - 48) = \cos(y + 12)$$

$$\Rightarrow \cos[90 - (y - 48)] = \cos(y + 12)$$

$$\Rightarrow \cos(138 - y) = \cos(y + 12)$$

Thus

$$\Rightarrow \begin{cases} 138 - y = y + 12 + 360n \\ 138 - y = -y - 12 + 360n \end{cases} \quad n = 0, 1, 2, 3, \dots$$

$$\Rightarrow \begin{cases} -2y = -126 + 360n \\ \text{no constraint} \end{cases}$$

$$\Rightarrow y = 63 \pm 180n$$

$\therefore y_1 = 63^\circ$   
 $y_2 = 243^\circ$

**Question 64** (\*\*\*\*)

The 2<sup>nd</sup>, 3<sup>rd</sup> and 4<sup>th</sup> terms of a geometric series are  $\cos \theta$ ,  $\sqrt{2} \sin \theta$  and  $\sqrt{3} \tan \theta$ , respectively, where  $0 < \theta < \frac{\pi}{2}$ .

Show clearly that the sum of the first 6 terms of the series is

$$\frac{43}{12}(6 + \sqrt{6}).$$

proof

$u_2 = \cos \theta$   
 $u_3 = \sqrt{2} \sin \theta$   
 $u_4 = \sqrt{2} \tan \theta$   
 $\text{GP} \Rightarrow \frac{u_2}{u_3} = \frac{u_3}{u_4}$   
 $\Rightarrow \frac{\cos \theta}{\sqrt{2} \sin \theta} = \frac{\sqrt{2} \sin \theta}{\sqrt{2} \tan \theta}$   
 $\Rightarrow 2 \sin^2 \theta = \sqrt{3} \tan^2 \theta$   
 $\Rightarrow 2 \sin^2 \theta = \sqrt{3} \frac{\sin^2 \theta}{\cos^2 \theta}$   
 $\Rightarrow 2 \cos^2 \theta = \sqrt{3} \sin^2 \theta$   
 $\Rightarrow \tan^2 \theta = \frac{2}{\sqrt{3}}$   
 $\Rightarrow \tan \theta = \sqrt{\frac{2}{\sqrt{3}}}$   
 $\Rightarrow 2 \sin \theta = \sqrt{2}$   
 $\Rightarrow \sin \theta = \frac{1}{\sqrt{2}}$   
 $\theta = \sin^{-1} \frac{1}{\sqrt{2}}$   
 $\theta = 45^\circ$   
 only valid in PASE

## Question 65 (\*\*\*\*)

It is given that  $\theta$  and  $\varphi$  satisfy the relationship

$$\tan \theta = \frac{3 \sin \varphi \cos \varphi}{1 - 3 \sin^2 \varphi}.$$

Show clearly that

$$\tan(\theta - \varphi) = 2 \tan \varphi.$$

 , proof

● START BY EXPRESSING  $\tan \theta$  IN TERMS OF  $\tan \varphi$  AS GIVEN.

$$\tan \theta = \frac{3 \sin \varphi \cos \varphi}{1 - 3 \sin^2 \varphi} = \frac{\frac{3 \sin \varphi \cos \varphi}{\cos^2 \varphi}}{\frac{1}{\cos^2 \varphi} - \frac{3 \sin^2 \varphi}{\cos^2 \varphi}} = \frac{3 \tan \varphi}{\sec^2 \varphi - 3 \tan^2 \varphi}$$

$$= \frac{3 \tan \varphi}{(1 + \tan^2 \varphi) - 3 \tan^2 \varphi} = \frac{3 \tan \varphi}{1 - 2 \tan^2 \varphi}$$

● FINALLY USING THE COMPOUND ANGLE IDENTITY FOR  $\tan(A - B)$

$$\tan(\theta - \varphi) = \frac{\tan \theta - \tan \varphi}{1 + \tan \theta \tan \varphi}$$

$$= \frac{\frac{3 \tan \varphi}{1 - 2 \tan^2 \varphi} - \tan \varphi}{1 + \left( \frac{3 \tan \varphi}{1 - 2 \tan^2 \varphi} \right) \tan \varphi}$$

MISTERY "TRICK" OF THE FRACTIONS BY (1 - 2tan^2)

$$= \frac{3 \tan \varphi - \tan \varphi (1 - 2 \tan^2 \varphi)}{(1 - 2 \tan^2 \varphi) + 3 \tan^2 \varphi}$$

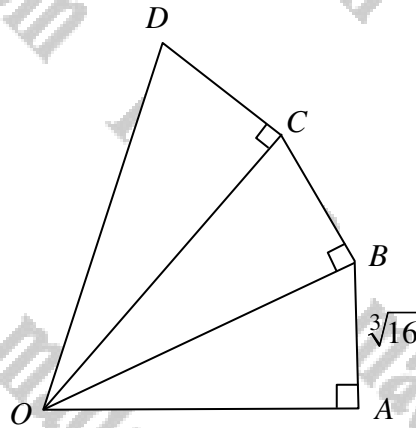
$$= \frac{3 \tan \varphi - \tan \varphi + 2 \tan^3 \varphi}{1 - 2 \tan^2 \varphi + 3 \tan^2 \varphi}$$

$$= \frac{2 \tan \varphi + 2 \tan^3 \varphi}{1 + \tan^2 \varphi}$$

$$= \frac{2 \tan \varphi (1 + \tan^2 \varphi)}{1 + \tan^2 \varphi}$$

$$= 2 \tan \varphi \quad \text{As required}$$

## Question 66 (\*\*\*\*\*)



The figure above shows three right angle triangles,  $OAB$ ,  $OBC$  and  $OCD$ .

It is given that  $\angle AOB = \angle BOC = \angle COD$  and  $|OD| = 2|OA|$ .

Given further that the length of  $AB$  is  $\sqrt[3]{16}$ , determine the length of  $DC$ .

$$\boxed{\phantom{000}}, \quad |DC| = 4$$

Firstly by similar triangles/trigonometry

$$\rightarrow \frac{y}{x} = \frac{AB}{OA} = \frac{\sqrt[3]{16}}{OA} = \cos \theta$$

$$\rightarrow y = \cos \theta \cdot x \quad \& \quad \frac{y}{x} = \cos \theta$$

$$\rightarrow \frac{y}{x} = \cos \theta \quad \& \quad \frac{y}{x} = \cos \theta$$

$$\Rightarrow \frac{y}{x} = \cos \theta$$

Now use HAT by similar triangles

$$\rightarrow \frac{y}{x} = \cos \theta \quad \& \quad \frac{y}{x} = \cos \theta$$

$$\rightarrow \frac{y}{x} = \cos \theta \quad \& \quad \frac{y}{x} = \cos \theta$$

$$\rightarrow \frac{y}{x} = \cos \theta \quad \& \quad \frac{y}{x} = \cos \theta$$

$$\Rightarrow \cos \theta = \left(\frac{1}{3}\right)^{\frac{1}{3}}$$

$$\Rightarrow \cos \theta = 2^{-\frac{1}{3}}$$

Now use the scale factor

$$\frac{y}{x} = \cos \theta$$

$$y = \frac{x}{2^{\frac{1}{3}}}$$

$$y = \frac{x}{2^{\frac{1}{3}}}$$

$$y = 2^{\frac{1}{3}}x$$

Scale factor OA : OD

$$\frac{OA}{OD} = \frac{OA}{2OA} = \frac{1}{2}$$

$$\therefore |DC| = 2^{\frac{1}{3}} |AB|$$

$$= 2^{\frac{1}{3}} \times \sqrt[3]{16}$$

$$= 2^{\frac{1}{3}} \times 2^{\frac{4}{3}} = 2^{\frac{5}{3}} = 4$$

Question 67 (\*\*\*\*\*)

Solve the trigonometric equation

$$8 \cos 2x \cos 4x \cos 8x + 1 = 0, \quad 0 < x < \frac{\pi}{2}.$$

$$x = \frac{\pi}{14}, \frac{\pi}{9}, \frac{3\pi}{14}, \frac{2\pi}{9}, \frac{\pi}{3}, \frac{5\pi}{14}, \frac{4\pi}{9}$$

$8 \cos 2x \cos 4x \cos 8x + 1 = 0 \quad 0 < x < \frac{\pi}{2}$

• SPLIT BY RECOGNISING THE SINH-DOUBLE-ANGLE FORMULA  
 $\sin 2A \equiv 2 \sin A \cos A$   
 $\cos A = \frac{\sin 2A}{2 \sin A}$

• Thus we may write the equation as follows  
 $\Rightarrow 8 \cos 2x \cos 4x \cos 8x = -1$   
 $\Rightarrow 8 \times \frac{\sin 4x}{2 \sin 2x} \times \frac{\sin 8x}{2 \sin 4x} \times \frac{\sin 16x}{2 \sin 8x} = -1$   
 $\Rightarrow \frac{\sin 16x}{\sin 2x} = -1$   
 $\Rightarrow \sin 16x = -\sin 2x$   
 $\Rightarrow \sin 16x = \sin(-2x)$   
 $\Rightarrow \begin{cases} 16x = -2x + 2n\pi \\ 16x = (\pi + 2x) + 2n\pi \end{cases} \quad n \in \mathbb{Z}$   
 $\Rightarrow \begin{cases} 18x = 0 + 2n\pi \\ 14x = \pi + 2n\pi \end{cases}$   
 $\Rightarrow \begin{cases} x = 0 + \frac{n\pi}{9} \\ x = \frac{\pi}{14} + \frac{n\pi}{7} \end{cases} \quad \left( n = \frac{1}{14}, \frac{3}{14}, \frac{5}{14}, \frac{7}{14}, \frac{9}{14}, \frac{11}{14}, \frac{13}{14} \right)$

collecting the solutions  $x = \frac{\pi}{14}, \frac{\pi}{9}, \frac{3\pi}{14}, \frac{2\pi}{9}, \frac{\pi}{3}, \frac{5\pi}{14}, \frac{4\pi}{9}$

Question 68 (\*\*\*\*\*)

Find, in terms of  $\pi$ , the solutions of the equation

$$\sqrt{x} \frac{d}{dx} (\sqrt{x} + 2 \cos \sqrt{x}) = 1, \quad 0 \leq x < 4\pi^2.$$

$$x = \frac{49\pi^2}{36}, \frac{121\pi^2}{36}$$

$x^{\frac{1}{2}} \frac{d}{dx} \left[ x^{\frac{1}{2}} + 2 \cos x^{\frac{1}{2}} \right] = 1$   
 $x^{\frac{1}{2}} \left[ \frac{1}{2} x^{-\frac{1}{2}} + 2 \times \frac{1}{2} x^{-\frac{1}{2}} (-\sin x^{\frac{1}{2}}) \right] = 1$   
 $x^{\frac{1}{2}} \left[ \frac{1}{2} x^{-\frac{1}{2}} - x^{\frac{1}{2}} \sin x^{\frac{1}{2}} \right] = 1$   
 $\frac{1}{2} - \sin x^{\frac{1}{2}} = \frac{1}{x}$   
 $\sin x^{\frac{1}{2}} = -\frac{1}{2}$   
 $\sin \theta = -\frac{1}{2}$   
 $\arcsin\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$

$\theta = -\frac{\pi}{6} + 2n\pi \quad n \in \mathbb{Z}$   
 $\theta = \frac{5\pi}{6} + 2n\pi$   
 $\theta = -\frac{\pi}{6} - 2\pi, -\frac{\pi}{6} - 4\pi, \dots$   
 $x^{\frac{1}{2}} = -\frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}, \dots$   
 $x = \frac{49\pi^2}{36}, \frac{121\pi^2}{36}$

**Question 69** (\*\*\*\*)Find in terms of  $\pi$  the solutions of the trigonometric equation

$$\cot\left(\frac{\pi}{2}\cos x\right) = 1, \quad 0 \leq x < 2\pi.$$

$$x = \frac{\pi}{3}, \frac{5\pi}{3}$$

Handwritten solution for Question 69:

$$\begin{aligned} \cot\left(\frac{\pi}{2}\cos x\right) &= 1 \\ \Rightarrow \tan\left(\frac{\pi}{2}\cos x\right) &= 1 \\ \Rightarrow \frac{\pi}{2}\cos x &= \frac{\pi}{4} + n\pi \\ \Rightarrow \cos x &= \frac{1}{2} + 2n \\ \Rightarrow \cos x &= \frac{1}{2} \pm 2n \end{aligned}$$

Then:

$$\begin{aligned} \cos x &= \dots -2, -\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \dots \\ \cos x &= \frac{1}{2} \\ x &= \frac{\pi}{3} \pm 2n\pi \\ x &= \frac{5\pi}{3} \pm 2n\pi \end{aligned}$$

$\therefore x = \frac{\pi}{3}, \frac{5\pi}{3}$

**Question 70** (\*\*\*\*)

Solve the trigonometric equation

$$8\sin\left(\frac{\pi}{18}\right)\cos\left(\frac{\pi}{18}\right)\cos\left(\frac{\pi}{9}\right)\cos\left(\frac{2\pi}{9}\right) = \cos 3x, \quad 0 \leq x < \frac{\pi}{2},$$

giving the answer in terms of  $\pi$ .

$$\boxed{\phantom{000}}, x = \frac{\pi}{54}$$

Handwritten solution for Question 70:

$$8\sin\frac{\pi}{18}\cos\frac{\pi}{18}\cos\frac{\pi}{9}\cos\frac{2\pi}{9} = \cos 3x$$

● SIMPLIFY BY SUCCESSIVE APPLICATIONS OF  $\sin 2A \equiv 2\sin A \cos A$

$$\begin{aligned} \Rightarrow 4(2\sin\frac{\pi}{18}\cos\frac{\pi}{18})\cos\frac{\pi}{9}\cos\frac{2\pi}{9} &= \cos 3x \\ \Rightarrow 4\sin\frac{\pi}{9}\cos\frac{\pi}{9}\cos\frac{2\pi}{9} &= \cos 3x \\ \Rightarrow 2(2\sin\frac{\pi}{9}\cos\frac{\pi}{9})\cos\frac{2\pi}{9} &= \cos 3x \\ \Rightarrow 2\sin\frac{2\pi}{9}\cos\frac{2\pi}{9} &= \cos 3x \\ \Rightarrow \sin\frac{4\pi}{9} &= \cos 3x \end{aligned}$$

● SWITCH INTO ALL SINES OR ALL COSINES (QUANT) BY  $\sin A = \cos(\frac{\pi}{2}-A)$   
 $\cos A = \sin(\frac{\pi}{2}-A)$

$$\begin{aligned} \Rightarrow \cos\left(\frac{\pi}{2}-\frac{4\pi}{9}\right) &= \cos 3x \\ \Rightarrow \cos 3x &= \cos\left(\frac{\pi}{2}-\frac{4\pi}{9}\right) \\ \Rightarrow 3x &= \cos\frac{\pi}{2} \end{aligned}$$

● SET UP A GENERAL SOLUTION

$$\begin{aligned} 3x &= -\frac{\pi}{2} \pm 2n\pi \\ 3x &= \frac{\pi}{2} \pm 2n\pi \end{aligned} \quad n=0,1,2,\dots$$

$$\begin{aligned} x &= -\frac{\pi}{6} \pm \frac{2n\pi}{3} \\ x &= \frac{\pi}{6} \pm \frac{2n\pi}{3} \end{aligned}$$

ONLY SOLUTION IN RANGE IS  $\frac{\pi}{6}$

## Question 71 (\*\*\*\*\*)

Solve the trigonometric equation

$$\sin(2\theta + 20^\circ) + \cos(3\theta + 50^\circ) = 0, \quad 0 \leq \theta \leq 360^\circ.$$

$$\theta = 40^\circ, 60^\circ, 112^\circ, 184^\circ, 256^\circ, 328^\circ$$

Handwritten solution for Question 71:

$$\begin{aligned} \Rightarrow \sin(2\theta + 20^\circ) + \cos(3\theta + 50^\circ) &= 0 \\ \Rightarrow \cos(3\theta + 50^\circ) &= -\sin(2\theta + 20^\circ) \\ \Rightarrow \cos(3\theta + 50^\circ) &= \sin(2\theta + 20^\circ) \\ \Rightarrow \cos(3\theta + 50^\circ) &= \cos(90^\circ - (2\theta + 20^\circ)) \\ \Rightarrow \cos(3\theta + 50^\circ) &= \cos(70^\circ - 2\theta) \end{aligned}$$

Using the identity  $\sin(-A) \equiv -\sin A$  and  $\cos(A) = \cos(360^\circ - A)$ :

$$\begin{aligned} 3\theta + 50^\circ &= 70^\circ - 2\theta \quad \text{or} \quad 3\theta + 50^\circ = 360^\circ - (70^\circ - 2\theta) \\ 3\theta + 50^\circ &= 70^\circ - 2\theta \quad \text{or} \quad 3\theta + 50^\circ = 360^\circ - 70^\circ + 2\theta \\ 3\theta + 50^\circ &= 70^\circ - 2\theta \quad \text{or} \quad 3\theta + 50^\circ = 290^\circ + 2\theta \\ 5\theta &= 20^\circ \quad \text{or} \quad \theta = 4^\circ \\ 5\theta &= 240^\circ \quad \text{or} \quad \theta = 48^\circ \end{aligned}$$

General solutions:

$$\begin{aligned} \theta &= 4^\circ + 360^\circ k \quad \text{or} \quad \theta = 48^\circ + 360^\circ k \\ \theta &= 4^\circ + 360^\circ k \quad \text{or} \quad \theta = 48^\circ + 360^\circ k \end{aligned}$$

Final solutions in the range  $0 \leq \theta < 360^\circ$ :

$$\theta = 4^\circ, 48^\circ, 404^\circ, 408^\circ$$

## Question 72 (\*\*\*\*\*)

A curve has equation

$$y = \ln \left[ \tan \left( x + \frac{\pi}{4} \right) \right], \quad -\frac{\pi}{4} < x < \frac{\pi}{4}.$$

Show clearly that

$$\frac{dy}{dx} = 2 \sec 2x.$$

proof

Handwritten proof for Question 72:

$$\begin{aligned} y &= \ln \left( \tan \left( x + \frac{\pi}{4} \right) \right) = \ln \left( \frac{\tan x + \tan \frac{\pi}{4}}{1 - \tan x \tan \frac{\pi}{4}} \right) = \ln \left( \frac{\tan x + 1}{1 - \tan x} \right) \\ \therefore y &= \ln(1 + \tan x) - \ln(1 - \tan x) \\ \frac{dy}{dx} &= \frac{1}{1 + \tan x} \times \sec^2 x - \frac{1}{1 - \tan x} \times (-\sec^2 x) \\ &= \sec^2 x \left[ \frac{1}{1 + \tan x} + \frac{1}{1 - \tan x} \right] = \sec^2 x \left[ \frac{1 - \tan x + 1 + \tan x}{(1 + \tan x)(1 - \tan x)} \right] \\ &= \frac{2 \sec^2 x}{1 - \tan^2 x} = \frac{2 \sec^2 x}{\cos^2 x - \sin^2 x} = \frac{2}{\cos^2 x - \sin^2 x} = \frac{2}{\cos 2x} = 2 \sec 2x \end{aligned}$$

Alternative method:

$$\begin{aligned} y &= \ln \left[ \tan \left( x + \frac{\pi}{4} \right) \right] \\ \frac{dy}{dx} &= \frac{1}{\tan \left( x + \frac{\pi}{4} \right)} \times \sec^2 \left( x + \frac{\pi}{4} \right) = \frac{1}{\frac{\sin \left( x + \frac{\pi}{4} \right)}{\cos \left( x + \frac{\pi}{4} \right)}} \times \frac{1}{\cos^2 \left( x + \frac{\pi}{4} \right)} \\ &= \frac{\cos \left( x + \frac{\pi}{4} \right)}{\sin \left( x + \frac{\pi}{4} \right)} \times \frac{1}{\cos^2 \left( x + \frac{\pi}{4} \right)} = \frac{1}{\sin \left( x + \frac{\pi}{4} \right) \cos \left( x + \frac{\pi}{4} \right)} = \frac{2}{2 \sin \left( x + \frac{\pi}{4} \right) \cos \left( x + \frac{\pi}{4} \right)} \\ &= \frac{2}{\sin \left[ 2 \left( x + \frac{\pi}{4} \right) \right]} = \frac{2}{\sin \left( 2x + \frac{\pi}{2} \right)} = \frac{2}{\cos 2x} = 2 \sec 2x \end{aligned}$$

## Question 73 (\*\*\*\*\*)

Prove the validity of each of the following trigonometric identities.

i.  $\sin^2 \theta \tan \theta + \cos^2 \theta \cot \theta + 2 \sin \theta \cos \theta \equiv \tan \theta + \cot \theta.$

ii.  $\sin \theta (1 + \tan \theta) + \cos \theta (1 + \cot \theta) \equiv \sec \theta + \operatorname{cosec} \theta.$

, proof

i) L.H.S. =  $\sin^2 \theta \tan \theta + \cos^2 \theta \cot \theta + 2 \sin \theta \cos \theta$   
 $= \sin^2 \theta \left( \frac{\sin \theta}{\cos \theta} \right) + \cos^2 \theta \left( \frac{\cos \theta}{\sin \theta} \right) + 2 \sin \theta \cos \theta$   
 $= \frac{\sin^3 \theta}{\cos \theta} + 2 \sin \theta \cos \theta + \frac{\cos^3 \theta}{\sin \theta}$   
 $= \frac{\sin^3 \theta + 2 \sin^2 \theta \cos \theta + \cos^3 \theta}{\cos \theta \sin \theta}$   
 $= \frac{(\sin \theta + \cos \theta)^2}{\cos \theta \sin \theta} = \frac{1}{\cos \theta \sin \theta}$   
 $= \frac{\cos^2 \theta + \sin^2 \theta}{\cos \theta \sin \theta} = \frac{\cos^2 \theta}{\cos \theta \sin \theta} + \frac{\sin^2 \theta}{\cos \theta \sin \theta}$   
 $= \frac{\cos \theta}{\sin \theta} + \frac{\sin \theta}{\cos \theta} = \cot \theta + \tan \theta = \text{R.H.S.} //$

ii) L.H.S. =  $\sin \theta (1 + \tan \theta) + \cos \theta (1 + \cot \theta)$   
 $= \sin \theta + \sin \theta \tan \theta + \cos \theta + \cos \theta \cot \theta$   
 $= \sin \theta + \sin \theta \left( \frac{\sin \theta}{\cos \theta} \right) + \cos \theta + \cos \theta \left( \frac{\cos \theta}{\sin \theta} \right)$   
 $= \sin \theta + \frac{\sin^2 \theta}{\cos \theta} + \cos \theta + \frac{\cos^2 \theta}{\sin \theta}$   
 $= \frac{\sin^2 \theta \cos \theta + \sin^3 \theta + \cos^2 \theta \sin \theta + \cos^3 \theta}{\cos \theta \sin \theta}$   
 $= \frac{\sin^2 \theta (\cos \theta + \sin \theta) + \cos^2 \theta (\cos \theta + \sin \theta)}{\cos \theta \sin \theta}$   
 $= \frac{(\cos \theta + \sin \theta)(\sin^2 \theta + \cos^2 \theta)}{\cos \theta \sin \theta} = \frac{\cos \theta + \sin \theta}{\cos \theta \sin \theta}$   
 $= \frac{\cos \theta}{\cos \theta \sin \theta} + \frac{\sin \theta}{\cos \theta \sin \theta} = \sec \theta + \operatorname{cosec} \theta = \text{R.H.S.} //$



## Question 74 (\*\*\*\*)

Given the simultaneous equations

$$3 \tan \theta + 4 \tan \varphi = 8$$

$$\theta + \varphi = \frac{\pi}{2},$$

find the possible values of  $\tan \theta$  and  $\tan \varphi$ .

$$[\tan \theta, \tan \varphi] = \left[2, \frac{1}{2}\right] = \left[\frac{2}{3}, \frac{3}{2}\right]$$

$3 \tan \theta + 4 \tan \varphi = 8$   
 $\theta + \varphi = \frac{\pi}{2}$

$\Rightarrow \begin{cases} 3 \tan \theta + \frac{4}{\tan \theta} = 8 \\ 3 \tan^2 \theta + 4 = 8 \tan \theta \\ 3 \tan^2 \theta - 8 \tan \theta + 4 = 0 \\ (3 \tan \theta - 2)(\tan \theta - 2) = 0 \\ \tan \theta = \frac{2}{3} \text{ or } 2 \\ \text{if } \tan \theta = \frac{2}{3} \Rightarrow \tan \varphi = \frac{3}{2} \\ \text{if } \tan \theta = 2 \Rightarrow \tan \varphi = \frac{1}{2} \end{cases}$

Hence either  $\tan \theta = 2, \tan \varphi = \frac{1}{2}$   
 or  $\tan \theta = \frac{2}{3}, \tan \varphi = \frac{3}{2}$

**Question 75** (\*\*\*\*)

A relationship between  $x$  and  $y$  is given by the equations

$$x = \operatorname{cosec} \theta - \sin \theta, \quad 0 < \theta < \frac{\pi}{2}$$

$$y = \sec \theta - \cos \theta, \quad 0 < \theta < \frac{\pi}{2}.$$

Use trigonometric identities to show that

$$y^2 x^2 \left( x^{\frac{2}{3}} + y^{\frac{2}{3}} \right)^3 = 1.$$

proof

$$\begin{aligned}
 \left. \begin{aligned} 2 &= \cos 2\theta - \sin 2\theta \\ y &= \sec 2\theta - \tan 2\theta \end{aligned} \right\} &= \frac{1}{\sin \theta} - \sin \theta = \frac{1 - \sin^2 \theta}{\sin \theta} = \frac{\cos^2 \theta}{\sin \theta} \\
 &= \frac{1}{\cos \theta} - \cos \theta = \frac{1 - \cos^2 \theta}{\cos \theta} = \frac{\sin^2 \theta}{\cos \theta} = \tan \theta \sec \theta
 \end{aligned}$$

$$\frac{y}{x} = \frac{\frac{\cos^2 \theta}{\sin \theta}}{\frac{\sin^2 \theta}{\cos \theta}} = \tan^2 \theta$$

$$\tan \theta = \left( \frac{y}{x} \right)^{\frac{1}{2}}$$

$$\begin{aligned}
 y &= \tan^2 \theta \sec \theta \\
 y^2 &= \tan^4 \theta \sec^2 \theta \\
 y^2 &= \tan^4 \theta (1 + \sec^2 \theta) \\
 y^2 &= \tan^4 \theta \left( 1 + \frac{1}{\sec^2 \theta} \right) \\
 y^2 &= \tan^4 \theta \left( 1 + \frac{1}{1 + \tan^2 \theta} \right) \\
 y^2 &= \tan^4 \theta \left( \frac{1 + 1 + \tan^2 \theta}{1 + \tan^2 \theta} \right) \\
 y^2 &= \frac{\tan^4 \theta}{1 + \tan^2 \theta}
 \end{aligned}$$

$$\begin{aligned}
 y^2 &= \frac{\left( \frac{y}{x} \right)^4}{1 + \left( \frac{y}{x} \right)^2} \\
 \Rightarrow y^2 &= \frac{y^4}{x^2 + y^2} \\
 \Rightarrow y^2 &= \frac{y^4}{x^2 + y^2} \\
 \Rightarrow y^2 x^2 (x^2 + y^2) &= y^4 \\
 \Rightarrow y^2 x^2 (x^2 + y^2) &= y^4 \\
 \Rightarrow y^2 x^2 (x^2 + y^2) &= y^4
 \end{aligned}$$

MISTAKE  
 = 1 + \frac{y^2}{x^2} \neq 1 + \frac{y^2}{x^2}

**Question 76** (\*\*\*\*)

Solve the trigonometric equation

$$\sec x + \sqrt{3} \operatorname{cosec} x = 4, \quad 0 \leq \theta < \pi,$$

giving the answers in terms of  $\pi$ .

$$\theta = \frac{2\pi}{9}, \frac{\pi}{3}, \frac{8\pi}{9}$$

$\sin 2\alpha + \sqrt{3} \cos 2\alpha = 4$   
 $\Rightarrow \frac{1}{\cos \alpha} + \frac{\sqrt{3}}{\sin \alpha} = 4$   
 $\Rightarrow \sin 2\alpha + \sqrt{3} \cos 2\alpha = 4 \cos 2\alpha \sin 2\alpha$   
 $\Rightarrow \sin 2\alpha + \sqrt{3} \cos 2\alpha = 2 \sin 2\alpha$   
 $\Rightarrow 2 \left[ \sin \frac{2\alpha}{2} + \sqrt{3} \cos \frac{2\alpha}{2} \right] = 2 \sin 2\alpha$   
 $\Rightarrow 2 \left[ \sin \frac{\alpha}{2} + \sqrt{3} \cos \frac{\alpha}{2} \right] = 2 \sin 2\alpha$   
 $\Rightarrow 2 \sin \left( 2\alpha + \frac{\pi}{6} \right) = 2 \sin 2\alpha$   
 $\Rightarrow \sin \left( 2\alpha + \frac{\pi}{6} \right) = \sin 2\alpha$   
 $\Rightarrow \left( 2\alpha + \frac{\pi}{6} \right) = 2\alpha + 2\pi n$  for  $n \in \mathbb{Z}$   
 $\Rightarrow 2\alpha + \frac{\pi}{6} = 2\alpha + 2\pi n$   
 $\Rightarrow 2\alpha = \frac{\pi}{6} + 2\pi n$   
 $\Rightarrow \alpha = \frac{\pi}{12} + \pi n$   
 $\Rightarrow \alpha = \frac{\pi}{12}, \frac{13\pi}{12}$

**Question 77** (\*\*\*\*)

It is given that

$$\cos\left(\frac{\pi}{12}\right) \equiv \cos\left(\frac{\pi}{4}\right) [\cos(A\pi) + \cos(B\pi)],$$

where  $A$  and  $B$  are constants such that  $0 < A < 1$  and  $0 < B < 1$ .

Determine the value of  $A$  and the value of  $B$ .

$$A = \frac{1}{6}, \quad B = \frac{2}{3}$$

$$\begin{aligned} 5 \sin \frac{\pi}{12} &= 5 \sin \left( \frac{\pi}{4} - \frac{\pi}{6} \right) = 5 \sin \frac{\pi}{4} \cos \frac{\pi}{6} - 5 \cos \frac{\pi}{4} \sin \frac{\pi}{6} \\ &= 5 \sin \frac{\pi}{4} \cos \frac{\pi}{6} - 5 \cos \frac{\pi}{4} \sin \frac{\pi}{6} \\ &= 5 \sin \frac{\pi}{4} \left[ \cos \frac{\pi}{6} - \cos \frac{\pi}{4} \right] \\ &= 5 \sin \frac{\pi}{4} \left[ \cos \frac{\pi}{6} + \sin \left( \frac{\pi}{4} - \frac{\pi}{6} \right) \right] \\ &= 5 \sin \frac{\pi}{4} \left[ \cos \frac{\pi}{6} + \cos \left( \frac{\pi}{4} - \frac{\pi}{6} \right) \right] \\ &= 5 \sin \frac{\pi}{4} \left[ \cos \frac{\pi}{6} + \cos \frac{\pi}{12} \right] \end{aligned}$$

**Question 78** (\*\*\*\*)

Find the only finite solution of the trigonometric equation

$$\arcsin\left(\frac{x}{x-1}\right) + 2\arctan\left(\frac{1}{x+1}\right) = \frac{\pi}{2}.$$

$$\boxed{x=0}$$

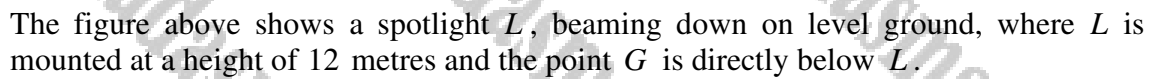
$\arcsin\left(\frac{x}{x-1}\right) + 2\arctan\left(\frac{1}{x+1}\right) = \frac{\pi}{2}$   
 $\Rightarrow 2\arctan\left(\frac{1}{x+1}\right) = \frac{\pi}{2} - \arcsin\left(\frac{x}{x-1}\right)$   
 $2\theta = \frac{\pi}{2} - \phi$   
 $\Rightarrow \cos(2\theta) = \cos\left(\frac{\pi}{2} - \phi\right)$   
 $\Rightarrow 1 - 2\sin^2\theta = \cos\left(\frac{\pi}{2} - \phi\right) + \sin\frac{\pi}{2}\sin\phi$   
 $\Rightarrow 1 - 2\left(\frac{1}{\sqrt{1+(x+1)^2}}\right)^2 = \sin\phi$   
 $\Rightarrow 1 - \frac{2}{x^2+2x+2} = \frac{x}{x-1}$   
 $\Rightarrow (x^2+2x+2)(x-1) - 2(x-1) = x(x^2+2x+2)$   
 $\Rightarrow x^3-2x^2+2x-2 = x^3+2x^2+2x$   
 $\Rightarrow x^3-2x^2-2x-2 = x^3+2x^2+2x$   
 $\Rightarrow x^3-2x^2-2x-2 = x^3+2x^2+2x$   
 $\Rightarrow -4x^2-4x-4 = 0$   
 $\Rightarrow x^2+x+1 = 0$   
 $\Rightarrow x = \frac{-1 \pm \sqrt{1-4}}{2}$   
 $\Rightarrow x = \frac{-1 \pm \sqrt{-3}}{2}$   
 $\Rightarrow x = \frac{-1 \pm i\sqrt{3}}{2}$   
 $\Rightarrow x = 0$

**Question 79** (\*\*\*\*)

Prove the validity of the following trigonometric identity

$$\sin^4\theta + \cos^4\theta \equiv \frac{1}{2}(2 - \sin^2 2\theta).$$

$LHS = \sin^4\theta + \cos^4\theta = (\sin^2\theta)^2 + (\cos^2\theta)^2$   
 $= \left(\frac{1 - \cos(2\theta)}{2}\right)^2 + \left(\frac{1 + \cos(2\theta)}{2}\right)^2$   
 $= \frac{1}{4} - \frac{1}{2}\cos(2\theta) + \frac{1}{4}\cos^2(2\theta) + \frac{1}{4} + \frac{1}{2}\cos(2\theta) + \frac{1}{4}\cos^2(2\theta)$   
 $= \frac{1}{2} + \frac{1}{2}\cos^2(2\theta)$   
 $= \frac{1}{2} + \frac{1}{2}(1 - \sin^2(2\theta)) = \frac{1}{2} + \frac{1}{2} - \frac{1}{2}\sin^2(2\theta) = 1 - \frac{1}{2}\sin^2(2\theta)$   
 $= \frac{1}{2}(2 - \sin^2(2\theta)) = RHS$   
 $\therefore LHS = RHS$



The beam is  $\varphi^\circ$  wide all the way round the axis of symmetry of the light cone.

**a)** Show that the length of  $AB$  is

$$|AB| = \frac{12 \sec^2 \theta}{\cot \varphi + \tan \theta}.$$

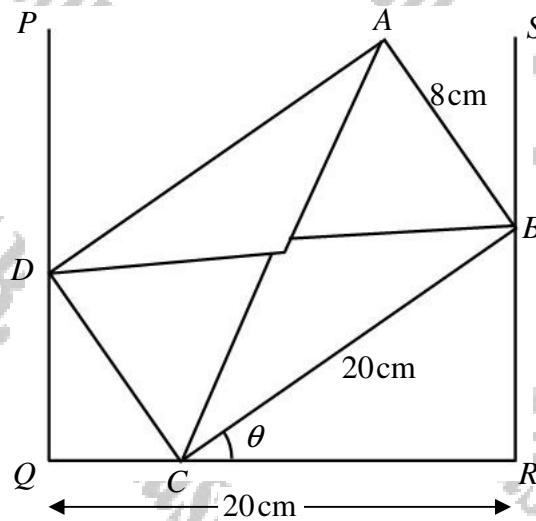
The lengths of  $LG$  and  $AB$  are  $8\sqrt{3}$  metres and 8 metres, respectively.

**b)** Show further that  $\tan \varphi = \frac{1}{11}(6 + \sqrt{3})$ .

~~SPR~~, proof

[illegible]

## Question 81 (\*\*\*\*\*)



The figure above shows the cross section of a letter inside a filling slot.

The letter  $ABCD$  is modelled as a rectangle with  $|AB| = 8\text{ cm}$  and  $|BC| = 20\text{ cm}$ .

The width of the filling slot  $QR$  is also  $20\text{ cm}$  and the angle  $BCR$  is  $\theta$ .

Determine the value of  $\theta$ .

,  $\theta \approx 43.6^\circ$

● START WITH A DIAGRAM TO WRITE EQUATIONS

$|QC| + |CR| = |QR|$   
 $|20\cos\theta| + |8\sin\theta| = 20$   
 $20\cos\theta + 8\sin\theta = 20$   
 $20\cos\theta + 8\sin\theta = 5$

● PROCEED WITH AN "R-TRANSFORMATION" IN SINUS

$20\sin\theta + 8\cos\theta \Rightarrow R\sin(\theta + \alpha)$   
 $20\sin\theta + 8\cos\theta \Rightarrow R\sin\theta\cos\alpha + R\cos\theta\sin\alpha$   
 $R\cos\alpha = 20$   
 $R\sin\alpha = 8 \Rightarrow R = \sqrt{20^2 + 8^2} = \sqrt{424}$   
 $\tan\alpha = \frac{8}{20} \Rightarrow \alpha \approx 21.8^\circ$

● SOLVING THE EQUATION, FOR  $0 < \theta < 90$

$\Rightarrow 20\sin\theta + 8\cos\theta = 5$   
 $\Rightarrow \sqrt{424} \sin(\theta + 21.8^\circ) = 5$   
 $\Rightarrow \sin(\theta + 21.8^\circ) = \frac{5}{\sqrt{424}}$   
 $\Rightarrow \theta + 21.8^\circ = 68.2^\circ \pm 360^\circ$   
 $\Rightarrow \theta = 46.4^\circ \pm 360^\circ$   
 $\therefore \text{ONLY PHYSICAL ANSWER IS } 46.4^\circ$

## Question 82 (\*\*\*\*\*)

Solve the following trigonometric equation

$$\sqrt{1 + \cos 2\theta} - \sqrt{1 - \cos 2\theta} = 1, \quad 0 \leq \theta < 2\pi.$$

$$\boxed{\phantom{000}}, \quad \theta = \frac{\pi}{12}, \frac{11\pi}{12}, \frac{13\pi}{12}, \frac{23\pi}{12}$$

$\sqrt{1 + \cos 2\theta} - \sqrt{1 - \cos 2\theta} = 1 \quad 0 \leq \theta < 2\pi$

1. FIRSTLY REMOVE THE SQUARE ROOTS BY THE CONJUGATE ROOT METHOD

$$\Rightarrow \sqrt{1 + (2\cos 2\theta - 1)} - \sqrt{1 - (2\cos 2\theta - 1)} = 1$$

$$\Rightarrow \sqrt{2\cos 2\theta} - \sqrt{2\sin 2\theta} = 1$$

$$\Rightarrow \sqrt{2}|\cos \theta| - \sqrt{2}|\sin \theta| = 1$$

2. NOW IN EACH QUADRANT WE HAVE

$0 \leq \theta < \frac{\pi}{2}$	$\sqrt{2}\cos \theta - \sqrt{2}\sin \theta = 1$
$\frac{\pi}{2} < \theta < \pi$	$-\sqrt{2}\cos \theta - \sqrt{2}\sin \theta = 1$
$\pi < \theta < \frac{3\pi}{2}$	$-\sqrt{2}\cos \theta + \sqrt{2}\sin \theta = 1$
$\frac{3\pi}{2} < \theta < 2\pi$	$\sqrt{2}\cos \theta + \sqrt{2}\sin \theta = 1$

3. SOLVING SEPARATELY IN EACH QUADRANT, STARTING WITH  $0 \leq \theta < \frac{\pi}{2}$

$$\Rightarrow \sqrt{2}\cos \theta - \sqrt{2}\sin \theta = 1$$

$$\Rightarrow \frac{\sqrt{2}}{2}\cos \theta - \frac{\sqrt{2}}{2}\sin \theta = \frac{1}{2}$$

$$\Rightarrow \cos \theta - \sin \theta = \frac{1}{\sqrt{2}}$$

$$\Rightarrow \cos(\theta + \frac{\pi}{4}) = \frac{1}{\sqrt{2}}$$

$$\Rightarrow \theta + \frac{\pi}{4} = \frac{\pi}{4} \pm 2n\pi \quad n=0,1,2,3,\dots$$

$$\Rightarrow \theta + \frac{\pi}{4} = \frac{7\pi}{4} \pm 2n\pi$$

$$\Rightarrow \theta = \frac{3\pi}{2} \pm 2n\pi$$

$$\Rightarrow \theta = \frac{3\pi}{2} \pm 2n\pi$$

FOR  $\frac{\pi}{2} < \theta < \pi$

$$\Rightarrow -\sqrt{2}\cos \theta - \sqrt{2}\sin \theta = 1$$

$$\Rightarrow \frac{\sqrt{2}}{2}\cos \theta + \frac{\sqrt{2}}{2}\sin \theta = -\frac{1}{2}$$

$$\Rightarrow \cos \theta + \sin \theta = -\frac{1}{\sqrt{2}}$$

$$\Rightarrow \cos(\theta - \frac{\pi}{4}) = -\frac{1}{\sqrt{2}}$$

$$\Rightarrow \theta - \frac{\pi}{4} = \frac{3\pi}{4} \pm 2n\pi \quad n=0,1,2,3,\dots$$

$$\Rightarrow \theta - \frac{\pi}{4} = \frac{5\pi}{4} \pm 2n\pi$$

$$\Rightarrow \theta = \frac{3\pi}{2} \pm 2n\pi$$

FOR  $\pi < \theta < \frac{3\pi}{2}$

$$\Rightarrow -\sqrt{2}\cos \theta + \sqrt{2}\sin \theta = 1$$

$$\Rightarrow \frac{\sqrt{2}}{2}\cos \theta - \frac{\sqrt{2}}{2}\sin \theta = \frac{1}{2}$$

$$\Rightarrow \cos \theta - \sin \theta = \frac{1}{\sqrt{2}}$$

$$\Rightarrow \cos(\theta + \frac{\pi}{4}) = \frac{1}{\sqrt{2}}$$

$$\Rightarrow \theta + \frac{\pi}{4} = \frac{\pi}{4} \pm 2n\pi \quad n=0,1,2,3,\dots$$

$$\Rightarrow \theta + \frac{\pi}{4} = \frac{7\pi}{4} \pm 2n\pi$$

$$\Rightarrow \theta = \frac{3\pi}{2} \pm 2n\pi$$

$$\Rightarrow \theta = \frac{3\pi}{2} \pm 2n\pi$$

FOR  $\frac{3\pi}{2} < \theta < 2\pi$

$$\Rightarrow \sqrt{2}\cos \theta + \sqrt{2}\sin \theta = 1$$

$$\Rightarrow \frac{\sqrt{2}}{2}\cos \theta + \frac{\sqrt{2}}{2}\sin \theta = \frac{1}{2}$$

$$\Rightarrow \cos \theta + \sin \theta = \frac{1}{\sqrt{2}}$$

$$\Rightarrow \cos(\theta - \frac{\pi}{4}) = \frac{1}{\sqrt{2}}$$

$$\Rightarrow \theta - \frac{\pi}{4} = \frac{\pi}{4} \pm 2n\pi \quad n=0,1,2,3,\dots$$

$$\Rightarrow \theta - \frac{\pi}{4} = \frac{7\pi}{4} \pm 2n\pi$$

$$\Rightarrow \theta = \frac{3\pi}{2} \pm 2n\pi$$

$$\Rightarrow \theta = \frac{3\pi}{2} \pm 2n\pi$$

$\therefore \theta = \frac{\pi}{12}, \frac{11\pi}{12}, \frac{13\pi}{12}, \frac{23\pi}{12}$

## Question 83 (\*\*\*\*\*)

Prove the validity of the following trigonometric identity.

$$\frac{\sin 4\theta - 8\sin^3 \theta \cos \theta}{\sin \theta} \equiv 4 \cos 3\theta, \quad \theta \neq n\pi.$$

☐ , proof

● STARTING ON THE LEFT HAND SIDE

$$\begin{aligned} \frac{\sin 4\theta - 8\sin^3 \theta \cos \theta}{\sin \theta} &= \frac{2\sin 2\theta \cos 2\theta - 8\sin^3 \theta \cos \theta}{\sin \theta} \\ &= \frac{2(2\sin \theta \cos \theta)(\cos^2 \theta - \sin^2 \theta) - 8\sin^3 \theta \cos \theta}{\sin \theta} \\ &= \frac{4\sin \theta \cos \theta (\cos^2 \theta - \sin^2 \theta) - 8\sin^3 \theta \cos \theta}{\sin \theta} \\ &= 4\cos \theta \cos 2\theta - 8\sin^2 \theta \cos \theta \\ &= 4\cos \theta \cos 2\theta - 4(2\sin^2 \theta) \cos \theta \\ &= 4\cos \theta \cos 2\theta - 4\sin 2\theta \sin \theta \\ &= 4[\cos 2\theta \cos \theta - \sin 2\theta \sin \theta] \\ &= 4 \cos(2\theta + \theta) \\ &= 4 \cos 3\theta \\ &= \text{R.H.S.} \end{aligned}$$

● ALTERNATIVE VERIFICATION STARTING WITH THE LEFT HAND SIDE AGAIN

$$\begin{aligned} \frac{\sin 4\theta - 8\sin^3 \theta \cos \theta}{\sin \theta} &= \frac{2\sin 2\theta \cos 2\theta - 8\sin^3 \theta \cos \theta}{\sin \theta} \\ &= \frac{2(2\sin \theta \cos \theta)(\cos^2 \theta - \sin^2 \theta) - 8\sin^3 \theta \cos \theta}{\sin \theta} \end{aligned}$$

$$\begin{aligned} &= 4\cos \theta \cos 2\theta - 8\sin^2 \theta \cos \theta \\ &= 4\cos \theta (2\cos^2 \theta - 1) - 8(1 - \cos^2 \theta) \cos \theta \\ &= 8\cos^3 \theta - 4\cos \theta - 8\cos \theta + 8\cos^3 \theta \\ &= 16\cos^3 \theta - 12\cos \theta \\ &= 4[4\cos^3 \theta - 3\cos \theta] \\ &= 4 \cos 3\theta \\ &= \text{R.H.S.} \end{aligned}$$

$$\begin{aligned} \cos 3\theta &\equiv \cos(2\theta + \theta) \\ &\equiv \cos 2\theta \cos \theta - \sin 2\theta \sin \theta \\ &\equiv (2\cos^2 \theta - 1) \cos \theta - (2\sin \theta \cos \theta) \sin \theta \\ &\equiv 2\cos^3 \theta - \cos \theta - 2\sin^2 \theta \cos \theta \\ &\equiv 2\cos^3 \theta - \cos \theta - 2(1 - \cos^2 \theta) \cos \theta \\ &\equiv 2\cos^3 \theta - \cos \theta - 2\cos \theta + 2\cos^3 \theta \\ &\equiv 4\cos^3 \theta - 3\cos \theta \\ &< \text{As given above} \end{aligned}$$



**Question 84** (\*\*\*\*\*)

The three angles of a triangle are denoted by  $\alpha$ ,  $\beta$  and  $\gamma$ .

a) Show clearly that ...

i. ...  $\sin(\alpha + \beta) = \sin \gamma$ .

ii. ...  $\sin\left(\frac{\alpha + \beta}{2}\right) = \cos\left(\frac{\gamma}{2}\right)$ .

iii. ...  $\sin \alpha + \sin \beta + \sin(\alpha + \beta) = 2 \sin\left(\frac{\alpha + \beta}{2}\right) \left[ \cos\left(\frac{\alpha + \beta}{2}\right) + \cos\left(\frac{\alpha - \beta}{2}\right) \right]$ .

iv. ...  $\sin \alpha + \sin \beta + \sin \gamma = 4 \cos \frac{\alpha}{2} \cos \frac{\beta}{2} \cos \frac{\gamma}{2}$ .

b) By using (iv) with suitable values for  $\alpha$ ,  $\beta$  and  $\gamma$ , show that

$$\cos\left(\frac{\pi}{12}\right) = \frac{\sqrt{6} + \sqrt{2}}{4}.$$

 , proof

**i)**  $\alpha + \beta + \gamma = \pi$   
 $\Rightarrow \alpha + \beta = \pi - \gamma$   
 $\Rightarrow \sin(\alpha + \beta) = \sin(\pi - \gamma)$   
 $\Rightarrow \sin(\alpha + \beta) = \sin \gamma$  (As  $\sin(\pi - x) = \sin x$ )

**ii)**  $\alpha + \beta + \gamma = \pi$   
 $\Rightarrow \frac{\alpha + \beta}{2} + \frac{\gamma}{2} = \frac{\pi}{2}$   
 $\Rightarrow \frac{\alpha + \beta}{2} = \frac{\pi}{2} - \frac{\gamma}{2}$   
 $\Rightarrow \sin\left(\frac{\alpha + \beta}{2}\right) = \sin\left(\frac{\pi}{2} - \frac{\gamma}{2}\right)$   
 $\Rightarrow \sin\left(\frac{\alpha + \beta}{2}\right) = \cos\left(\frac{\gamma}{2}\right)$  (As  $\sin(\frac{\pi}{2} - x) = \cos x$ )

**iii)**  $\sin \alpha + \sin \beta + \sin(\alpha + \beta)$   
 $\stackrel{\text{Sum-to-Product}}{=} 2 \sin \frac{\alpha + \beta}{2} \cos \frac{\alpha - \beta}{2} + \sin(\alpha + \beta)$   
 $\stackrel{\text{ii)}}{=} 2 \sin\left(\frac{\alpha + \beta}{2}\right) \cos\left(\frac{\alpha - \beta}{2}\right) + 2 \sin\left(\frac{\alpha + \beta}{2}\right) \cos\left(\frac{\alpha - \beta}{2}\right)$   
 $\stackrel{\text{ii)}}{=} 4 \sin\left(\frac{\alpha + \beta}{2}\right) \cos\left(\frac{\alpha - \beta}{2}\right)$  (As  $\sin(\frac{\alpha + \beta}{2}) = \cos(\frac{\gamma}{2})$ )

**(iv)** COMBINING THE CASE 3 RESULTS

$\sin \alpha + \sin \beta + \sin \gamma = \sin \alpha + \sin \beta + \sin(\alpha + \beta)$   
 $= 2 \sin\left(\frac{\alpha + \beta}{2}\right) \cos\left(\frac{\alpha - \beta}{2}\right) + \sin(\alpha + \beta)$   
 $= 2 \cos\left(\frac{\gamma}{2}\right) \left[ \cos\left(\frac{\alpha - \beta}{2}\right) + \cos\left(\frac{\alpha + \beta}{2}\right) \right]$   
 $\stackrel{\text{Sum-to-Product}}{=} 2 \cos\left(\frac{\gamma}{2}\right) \left[ 2 \cos\left(\frac{\alpha}{2}\right) \cos\left(\frac{\beta}{2}\right) \right]$   
 $= 4 \cos\left(\frac{\gamma}{2}\right) \cos\left(\frac{\alpha}{2}\right) \cos\left(\frac{\beta}{2}\right)$  (As  $\cos(\frac{\alpha}{2}) \cos(\frac{\beta}{2}) = \frac{\cos(\frac{\alpha + \beta}{2}) + \cos(\frac{\alpha - \beta}{2})}{2}$ )

**b)** USING  $\sin \alpha + \sin \beta + \sin \gamma = 4 \cos \frac{\alpha}{2} \cos \frac{\beta}{2} \cos \frac{\gamma}{2}$

With  $\alpha = \frac{\pi}{6}$   
 $\beta = \frac{\pi}{6}$   
 $\gamma = \frac{2\pi}{3}$

$\sin \frac{\pi}{6} + \sin \frac{\pi}{6} + \sin \frac{2\pi}{3} = 4 \cos \frac{\pi}{12} \cos \frac{\pi}{12} \cos \frac{\pi}{6}$   
 $1 + \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} = 4 \cos \frac{\pi}{12} \cos \frac{\pi}{12} \times \frac{\sqrt{3}}{2}$   
 $2 + \sqrt{3} = 2\sqrt{3} \cos^2 \frac{\pi}{12}$   
 $\cos^2 \frac{\pi}{12} = \frac{2 + \sqrt{3}}{2\sqrt{3}}$   
 $\cos \frac{\pi}{12} = \frac{\sqrt{2 + \sqrt{3}}}{\sqrt{2}}$   
 $\cos \frac{\pi}{12} = \frac{\sqrt{2} + \sqrt{3}}{2}$

## Question 85 (\*\*\*\*\*)

Simplify, showing all steps in the calculation, the expression

$$\arctan \frac{4}{3} + \arctan 2 - \arctan 3,$$

giving the answer in terms of  $\pi$ .

$$\frac{\pi}{4}$$

$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$       $\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$   
 $\therefore \tan(A+B-C) = \frac{\tan(A+B) - \tan C}{1 + \tan(A+B)\tan C}$   
 $\tan(A+B-C) = \frac{\frac{\tan A + \tan B}{1 - \tan A \tan B} - \tan C}{1 + \left(\frac{\tan A + \tan B}{1 - \tan A \tan B}\right)\tan C}$   
 $\tan(A+B-C) = \frac{\tan A + \tan B - \tan C(1 - \tan A \tan B)}{1 - \tan A \tan B + \tan A \tan B + \tan A \tan B \tan C}$   
 $\tan(A+B-C) = \frac{\tan A + \tan B - \tan C + \tan A \tan B \tan C}{1 - \tan A \tan B + \tan A \tan B + \tan A \tan B \tan C}$   
 $\text{let } A = \arctan \frac{4}{3} \Rightarrow \tan A = \frac{4}{3}$   
 $B = \arctan 2 \Rightarrow \tan B = 2$   
 $C = \arctan 3 \Rightarrow \tan C = 3$   
 $\therefore \tan(A+B-C) = \frac{\frac{4}{3} + 2 - 3 + \frac{4}{3} \times 2 \times 3}{1 - \frac{4}{3} \times 2 + \frac{4}{3} \times 2 \times 3 + \frac{4}{3} \times 2 \times 3} = \frac{\frac{34}{3}}{\frac{25}{3}} = 1$   
 $\therefore A+B-C = \arctan 1$   
 $A+B-C = \frac{\pi}{4}$   
 $\therefore \arctan \frac{4}{3} + \arctan 2 - \arctan 3 = \frac{\pi}{4}$

$\frac{(3+4i)(1+2i)}{1+3i} = \frac{3+4i+4i+8i^2}{1+3i} = \frac{-5+12i}{1+3i} = \frac{(-5+12i)(1-3i)}{(1+3i)(1-3i)}$   
 $= \frac{-5+15i+12i-36i^2}{1+9i^2} = \frac{25+27i}{10} = \frac{25}{10} + \frac{27}{10}i = \frac{5}{2} + \frac{27}{10}i$   
 $\therefore \arg\left(\frac{(3+4i)(1+2i)}{1+3i}\right) = \arg\left(\frac{5}{2} + \frac{27}{10}i\right)$   
 $\arg(3+4i) + \arg(1+2i) - \arg(1+3i) = \arg\left(\frac{5}{2} + \frac{27}{10}i\right)$   
 $\arctan \frac{4}{3} + \arctan 2 - \arctan 3 = \arctan 1$   
 $\arctan \frac{4}{3} + \arctan 2 - \arctan 3 = \frac{\pi}{4}$

## Question 86 (\*\*\*\*\*)

Solve the trigonometric equation

$$\sec x + \operatorname{cosec} x = 2\sqrt{2}, \quad 0 \leq x < 2\pi,$$

giving the answers in terms of  $\pi$ .

$$\boxed{\phantom{000}}, \quad x = \frac{\pi}{4}, \frac{11\pi}{12}, \frac{19\pi}{12}$$

**Method 1: Algebraic Manipulation**

$$\sec x + \operatorname{cosec} x = 2\sqrt{2} \quad 0 \leq x < 2\pi$$

$$\Rightarrow \frac{1}{\cos x} + \frac{1}{\sin x} = 2\sqrt{2}$$

$$\Rightarrow \sin x + \cos x = 2\sqrt{2} \sin x \cos x$$

$$\Rightarrow (\sin x + \cos x)^2 = (2\sqrt{2} \sin x \cos x)^2$$

$$\Rightarrow \sin^2 x + 2\sin x \cos x + \cos^2 x = 8\sin^2 x \cos^2 x$$

$$\Rightarrow 1 + 2\sin x \cos x = 2(2\sin x \cos x)^2$$

$$\Rightarrow 1 + \sin 2x = 2\sin^2 2x$$

$$\Rightarrow 2\sin^2 2x - \sin 2x - 1 = 0$$

$$\Rightarrow (2\sin 2x + 1)(\sin 2x - 1) = 0$$

$$\Rightarrow \sin 2x = -\frac{1}{2} \quad \text{(Check in degrees, because it would be easier to check the solutions)}$$

$$2x = 90^\circ \pm 360^\circ \quad \text{or} \quad 2x = 270^\circ \pm 360^\circ$$

$$x = 45^\circ \pm 180^\circ \quad \text{or} \quad x = 135^\circ \pm 180^\circ$$

$$x = 45^\circ, 225^\circ, 135^\circ, 315^\circ$$

**Method 2: Graphical Verification**

Check the best:  $\sec 15^\circ = \frac{1}{\cos 15^\circ} = \frac{1}{\cos(45^\circ - 30^\circ)} = \frac{1}{\cos 45^\circ \cos 30^\circ + \sin 45^\circ \sin 30^\circ} = \frac{1}{\frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2} \cdot \frac{1}{2}} = \frac{1}{\frac{\sqrt{2}(\sqrt{3}+1)}{4}} = \frac{4}{\sqrt{2}(\sqrt{3}+1)} = \sqrt{2}(\sqrt{3}-1)$

$\operatorname{cosec} 15^\circ = \frac{1}{\sin 15^\circ} = \frac{1}{\sin(45^\circ - 30^\circ)} = \frac{1}{\sin 45^\circ \cos 30^\circ - \cos 45^\circ \sin 30^\circ} = \frac{1}{\frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2} \cdot \frac{1}{2}} = \frac{1}{\frac{\sqrt{2}(\sqrt{3}-1)}{4}} = \frac{4}{\sqrt{2}(\sqrt{3}-1)} = \sqrt{2}(\sqrt{3}+1)$

Thus  $\sec 15^\circ + \operatorname{cosec} 15^\circ = \sqrt{2}(\sqrt{3}-1) + \sqrt{2}(\sqrt{3}+1) = 2\sqrt{2}$

Similarly,  $\sec 75^\circ + \operatorname{cosec} 75^\circ = 2\sqrt{2}$

Therefore we have 3 valid solutions in the interval  $0 \leq x < 2\pi$ :  $x = 45^\circ, 135^\circ, 225^\circ$

## Question 87 (\*\*\*\*\*)

Prove the validity of the trigonometric identity

$$\tan^2\left(\frac{3\pi}{4} - 2x\right) \equiv \frac{1 + \sin 4x}{1 - \sin 4x}, \quad x \neq \frac{\pi}{3}(4n+1), \quad n \in \mathbb{Z}$$

proof

$$\begin{aligned} \text{LHS} &= \tan^2\left(\frac{3\pi}{4} - 2x\right) = \left[\tan\left(\frac{3\pi}{4} - 2x\right)\right]^2 = \left[\frac{\tan \frac{3\pi}{4} - \tan 2x}{1 + \tan \frac{3\pi}{4} \tan 2x}\right]^2 \\ &= \left[\frac{-1 - \tan 2x}{1 + \tan 2x}\right]^2 = \left[\frac{-1 - \tan 2x}{1 + \tan 2x}\right]^2 = \left[\frac{1 + \tan 2x}{-1 - \tan 2x}\right]^2 \\ &= \frac{(1 + \tan 2x)^2}{(1 + \tan 2x)^2} = \frac{1 + 2\tan 2x + \tan^2 2x}{1 + 2\tan 2x + \tan^2 2x} \\ &= \frac{1 + \tan 2x}{1 - \tan 2x} = \frac{1 + \sin 4x}{1 - \sin 4x} \end{aligned}$$

## Question 88 (\*\*\*\*\*)

Use algebra to find, in terms of  $\pi$ , the solution of the trigonometric equation

$$x^2 - 8\pi x + 2 - 2\cos x + 16\pi^2 = 0, \quad x \in \mathbb{R}.$$

$$\boxed{\phantom{000}}, \quad \boxed{x = 4\pi}$$

$x^2 - 8\pi x + 2 - 2\cos x + 16\pi^2 = 0$   
 $\Rightarrow x^2 - 8\pi x + 2 - 2(1 - 2\sin^2 \frac{x}{2}) + 16\pi^2 = 0$   
 $\Rightarrow x^2 - 8\pi x + 4\sin^2 \frac{x}{2} = 0$   
 $\Rightarrow (x - 4\pi)^2 + 4\sin^2 \frac{x}{2} = 0$   
 • BOTH TERMS MUST BE ZERO  
 • THE FIRST TERM CAN ONLY BE ZERO IF  $x = 4\pi$   
 •  $x = 4\pi$  MAKES THE SECOND TERM ZERO  
 THEREFORE THE ONLY REAL SOLUTION IS  $x = 4\pi$

**Question 89** (\*\*\*\*\*)The piecewise continuous function  $f$  is given below

$$f(x) \equiv \begin{cases} \sin x^\circ & 0 \leq x < 360 \\ \sin 2x^\circ & 360 \leq x < 720 \\ \sin 3x^\circ & 720 \leq x < 1080 \end{cases}$$

a) Sketch the graph of  $f(x)$ .

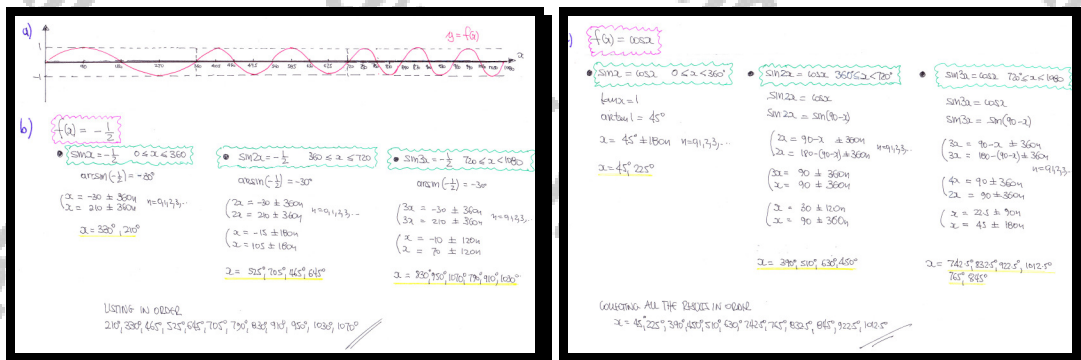
b) Solve the equation ...

i. ...  $f(x) = -\frac{1}{2}$ .

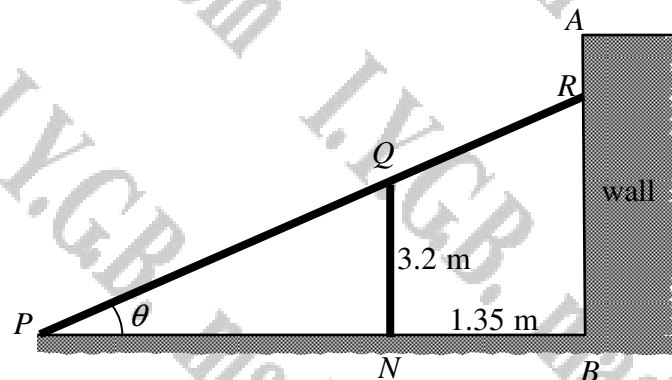
ii. ...  $f(x) = \cos x$ .

$$\boxed{\phantom{0000}}, \boxed{x = 210, 330, 465, 525, 645, 705, 790, 830, 910, 950, 1030, 1070},$$

$$\boxed{x = 45, 225, 390, 450, 510, 630, 735, 742.5, 832.5, 915, 922.5, 1012.5}$$



## Question 90 (\*\*\*\*)



The figure above shows the wall  $AB$  of a certain structure, which is supported by a straight rigid beam  $PR$ , where  $P$  is on level ground and  $R$  is at some point on the wall.

In order to increase the rigidity of the support, the beam is rested on a steady pole  $NQ$ , of height 3.2 metres.

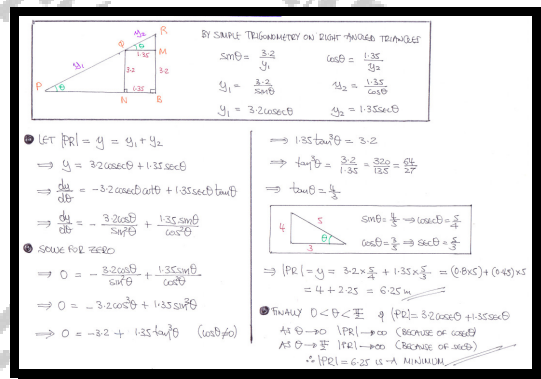
The pole is placed at a distance of 1.35 metres from the bottom of the wall  $B$ .

The beam  $PR$  is forming an acute angle  $\theta$  with the horizontal ground  $PNB$ .

The angle  $\theta$  is chosen so that the length of the beam  $PR$ , is least.

Determine the least value for the length of the beam  $PR$ , assuming that  $R$  lies on the wall, fully justifying that this is indeed the minimum value.

,



## Question 91 (\*\*\*\*)

The functions  $f$  and  $g$  are defined by

$$f(x) \equiv \cos x, \quad x \in \mathbb{R}, \quad 0 \leq x \leq \pi$$

$$g(x) \equiv 1 - x^2, \quad x \in \mathbb{R}.$$

a) Solve the equation

$$fg(x) = \frac{1}{2}.$$

b) Determine the values of  $x$  for which  $f^{-1}g(x)$  is **not** defined.

$$\boxed{\text{no}}, \quad \boxed{x = \pm \sqrt{1 - \frac{\pi}{6}}}, \quad \boxed{x < -\sqrt{2} \text{ or } x > \sqrt{2}}$$

**a)**  $f(x) = \cos x, 0 \leq x \leq \pi$   $g(x) = 1 - x^2, x \in \mathbb{R}$

$\rightarrow f(g(x)) = f(1 - x^2) = \cos(1 - x^2)$

$\rightarrow \cos(1 - x^2) = \frac{1}{2}$

$\arccos\left(\frac{1}{2}\right) = \frac{\pi}{6}$

$\rightarrow \begin{cases} 1 - x^2 = \frac{\pi}{6} \pm 2n\pi \\ 1 - x^2 = \frac{5\pi}{6} \pm 2n\pi \end{cases} \quad n = 0, 1, 2, \dots$

$\rightarrow \begin{cases} x^2 = 1 - \frac{\pi}{6} \pm 2n\pi \\ x^2 = 1 - \frac{5\pi}{6} \pm 2n\pi \end{cases}$

$x = \pm \sqrt{1 - \frac{\pi}{6}} \pm 2n\pi$  gives a solution if  $1 - \frac{\pi}{6} \geq 0$   $x = \pm \sqrt{1 - \frac{\pi}{6}}$

$x^2 = 1 - \frac{5\pi}{6} \pm 2n\pi$  give no solution in  $\mathbb{R}$  as  $1 - \frac{5\pi}{6} < 0$   $n=1$  gives  $x^2 > 0$

$\therefore x = \pm \sqrt{1 - \frac{\pi}{6}}$

**b)** Firstly if  $f^{-1}(x) = \cos^{-1} x$   $0 \leq x \leq \pi$

Then  $f^{-1}(x) = \arccos x$   $-1 \leq x \leq 1$

[We do not really need to work out  $f^{-1}(g(x))$ ]

$\begin{matrix} \text{if } x \in \mathbb{R} & \xrightarrow{g(x)} & \text{if } y \in \mathbb{R} & \xrightarrow{f^{-1}(y)} & \text{if } x \in \mathbb{R} \end{matrix}$

$\begin{matrix} \text{if } x \in \mathbb{R} & \xrightarrow{g(x)} & y \leq 1 & \xrightarrow{f^{-1}(y)} & \text{if } x \in \mathbb{R} \end{matrix}$

(Composition will be valid if)

$-1 \leq g(x) \leq 1$

$-1 \leq 1 - x^2 \leq 1$

$-2 \leq -x^2 \leq 0$

$0 \leq x^2 \leq 2$

$-\sqrt{2} \leq x \leq \sqrt{2}$

$\therefore$  It will not be defined if

$x < -\sqrt{2}$  or  $x > \sqrt{2}$

## Question 92 (\*\*\*\*\*)

$$f(x) \equiv \cos^2 x + \sin^2 x, \quad x \in \mathbb{R}.$$

- a) Determine an expression for  $f'(x)$  and find the value of  $f\left(\frac{1}{2}\pi\right)$ .
- b) By using the results of part (a) only, show that

$$\cos^2 x + \sin^2 x \equiv 1.$$

$$f'(x) = 0, \quad f\left(\frac{1}{2}\pi\right) = 1$$

(a)  $f(x) = \cos^2 x + \sin^2 x$   
 $f'(x) = -2\cos x \sin x + 2\sin x \cos x = 0$   
 $f\left(\frac{1}{2}\pi\right) = \cos^2\left(\frac{1}{2}\pi\right) + \sin^2\left(\frac{1}{2}\pi\right) = 1$   
 (b) Since  $f'(x) = 0, \forall x \in \mathbb{R} \Rightarrow f(x) = \text{constant} = c$   
 Since  $f\left(\frac{1}{2}\pi\right) = 1 \Rightarrow c = 1$   
 $\therefore f(x) = 1, \forall x \in \mathbb{R}$   
 $\cos^2 x + \sin^2 x = 1, \forall x \in \mathbb{R}$

## Question 93 (\*\*\*\*\*)

Prove the validity of each of the following trigonometric identities.

a)  $\frac{\sqrt{2-2\cos x}}{\sin x} \equiv \sec \frac{x}{2}.$

b)  $\sin^2 \theta \tan \theta + \cos^2 \theta \cot \theta + 2 \sin \theta \cos \theta \equiv \tan \theta + \cot \theta.$

proof

(a) LHS =  $\frac{\sqrt{2-2\cos x}}{\sin x}$   
 $= \frac{\sqrt{2(1-\cos x)}}{\sin x}$   
 $= \frac{\sqrt{4\sin^2 \frac{x}{2}}}{2\sin \frac{x}{2} \cos \frac{x}{2}} = \frac{2\sin \frac{x}{2}}{2\sin \frac{x}{2} \cos \frac{x}{2}} = \frac{1}{\cos \frac{x}{2}} = \sec \frac{x}{2} = \text{RHS}$   
 (b) LHS =  $\sin^2 \theta \tan \theta + \cos^2 \theta \cot \theta + 2 \sin \theta \cos \theta$   
 $= \frac{\sin^2 \theta \sin \theta}{\cos \theta} + \frac{\cos^2 \theta \cos \theta}{\sin \theta} + 2 \sin \theta \cos \theta$   
 $= \frac{\sin^3 \theta + \cos^3 \theta}{\sin \theta \cos \theta} + 2 \sin \theta \cos \theta$   
 $= \frac{(\sin \theta + \cos \theta)(\sin^2 \theta - \sin \theta \cos \theta + \cos^2 \theta)}{\sin \theta \cos \theta} + 2 \sin \theta \cos \theta$   
 $= \frac{(\sin \theta + \cos \theta)(1 - \sin \theta \cos \theta)}{\sin \theta \cos \theta} + 2 \sin \theta \cos \theta$   
 $= \frac{\sin \theta + \cos \theta}{\sin \theta \cos \theta} + 2 \sin \theta \cos \theta = \frac{\sin \theta}{\sin \theta \cos \theta} + \frac{\cos \theta}{\sin \theta \cos \theta} + 2 \sin \theta \cos \theta$   
 $= \frac{1}{\cos \theta} + \frac{1}{\sin \theta} + 2 \sin \theta \cos \theta = \sec \theta + \csc \theta + 2 \sin \theta \cos \theta$



## Question 94 (\*\*\*\*\*)

Solve the trigonometric equation

$$2 \arctan(x-2) + \arcsin\left(\frac{1-x}{1+x}\right) = \frac{\pi}{2}.$$

$$\boxed{x=4}$$

$2 \arctan(x-2) + \arcsin\left(\frac{1-x}{1+x}\right) = \frac{\pi}{2}$

• FIRSTLY REWRITE THE INVERSE TRIGONOMETRIC FUNCTIONS AS ANGLES

$\theta = \arctan(x-2)$   
 $\tan \theta = x-2$

$\phi = \arcsin\left(\frac{1-x}{1+x}\right)$   
 $\sin \phi = \frac{1-x}{1+x}$

BY PYTHAGORAS THE HYPOTENUSE WILL BE  $\sqrt{(x-2)^2 + 1} = \sqrt{x^2 - 4x + 5}$

BY PYTHAGORAS THE ADJACENT WILL BE  $\sqrt{(x^2 - 4x + 5) - (1-x)^2} = \sqrt{4x}$

• THEREFORE WE MAY REWRITE THE EQUATION AS FOLLOWS

$$\Rightarrow 2\theta + \phi = \frac{\pi}{2}$$

$$\Rightarrow 2\theta = \frac{\pi}{2} - \phi$$

• TAKE THE COSINE OF THE EQUATION, BECAUSE OF THE R.H.S

$$\Rightarrow \cos(2\theta) = \cos\left(\frac{\pi}{2} - \phi\right)$$

$$\Rightarrow 2\cos^2\theta - 1 = \sin\phi$$

$$\Rightarrow 2\left(\frac{1-x^2}{x^2-4x+5}\right) - 1 = \frac{1-x}{1+x}$$

$$\Rightarrow \frac{2-x^2-4x+5}{x^2-4x+5} = \frac{1-x}{1+x}$$

$$\Rightarrow \frac{2-x^2-4x+5}{x^2-4x+5} = \frac{1-x}{1+x}$$

• CHECKING THE SOLUTIONS AGAINST THE ORIGINAL

IF  $x=4$

$$\Rightarrow 2 \arctan(2) + \arcsin\left(\frac{1-4}{1+4}\right) = \pi$$

$$\Rightarrow 2 \arctan(2) - \arcsin\frac{3}{5} = \pi$$

$$\Rightarrow 2\theta - \phi = \pi$$

$$\Rightarrow \cos(2\theta - \phi) = \cos \pi$$

$$\Rightarrow \cos(2\theta)\cos\phi + \sin(2\theta)\sin\phi = \cos \pi$$

$$\Rightarrow (2\cos^2\theta - 1)\cos\phi + 2\sin\theta\cos\theta\sin\phi = \cos \pi$$

IF  $x=4$

$$\Rightarrow 2 \arctan(2) + \arcsin\left(\frac{1-4}{1+4}\right) = \pi$$

$$\Rightarrow 2 \arctan(2) - \arcsin\frac{3}{5} = \pi$$

$$\Rightarrow 2\theta - \phi = \pi$$

$$\Rightarrow \cos(2\theta - \phi) = \cos \pi$$

$$\Rightarrow \cos(2\theta)\cos\phi + \sin(2\theta)\sin\phi = \cos \pi$$

$$\Rightarrow (2\cos^2\theta - 1)\cos\phi + 2\sin\theta\cos\theta\sin\phi = \cos \pi$$

Question 95 (\*\*\*\*)

By considering the solution of trigonometric equation

$$\sin(x-30)^\circ = \cos(x-45)^\circ,$$

find, in degrees, the exact value of  $\arctan\left[\frac{1+\sqrt{2}}{\sqrt{3}-\sqrt{2}}\right]$ .

, 82.5°

The image shows two handwritten pages of a solution for the trigonometric equation  $\sin(x-30)^\circ = \cos(x-45)^\circ$ .

**Left Page:**

- Method 1: Compound Angle Formula**
  - Starts with  $\sin(x-30) = \cos(x-45)$ .
  - Expands both sides:  $\sin x \cos 30 - \cos x \sin 30 = \cos x \cos 45 + \sin x \sin 45$ .
  - Substitutes values:  $\frac{\sqrt{3}}{2} \sin x - \frac{1}{2} \cos x = \frac{\sqrt{2}}{2} \cos x + \frac{\sqrt{2}}{2} \sin x$ .
  - Multiplies by 2:  $\sqrt{3} \sin x - \cos x = \sqrt{2} \cos x + \sqrt{2} \sin x$ .
  - Rearranges:  $\sqrt{3} \sin x - \sqrt{2} \sin x = \sqrt{2} \cos x + \cos x$ .
  - Factorizes:  $(\sqrt{3}-\sqrt{2}) \sin x = (\sqrt{2}+1) \cos x$ .
  - Divides by  $\cos x$ :  $\tan x = \frac{\sqrt{2}+1}{\sqrt{3}-\sqrt{2}}$ .
  - Writes the principal value:  $x = \arctan\left(\frac{1+\sqrt{2}}{\sqrt{3}-\sqrt{2}}\right)$  (Principal Value).
- Method 2: Auxiliary Equation**
  - Starts with  $\sin(x-30) = \cos(x-45)$ .
  - Converts  $\cos$  to  $\sin$ :  $\sin(x-30) = \sin(90-(x-45))$ .
  - Simplifies:  $\sin(x-30) = \sin(135-x)$ .
  - Considers the general solution:  $x-30 = 135-x + 360n$  or  $x-30 = 180-(135-x) + 360n$ .
  - Solves for  $x$ :  $2x = 165 + 360n$  or  $2x = 45 + 360n$ .
  - Writes the general solution:  $x = 82.5 + 180n$  or  $x = 22.5 + 180n$ .

**Right Page:**

- Method 1: Principal Value**
  - Starts with  $\sin(x-30) = \cos(x-45)$ .
  - Converts  $\cos$  to  $\sin$ :  $\sin(x-30) = \sin(90-(x-45))$ .
  - Simplifies:  $\sin(x-30) = \sin(135-x)$ .
  - Considers the general solution:  $x-30 = 135-x + 360n$  or  $x-30 = 180-(135-x) + 360n$ .
  - Solves for  $x$ :  $2x = 165 + 360n$  or  $2x = 45 + 360n$ .
  - Writes the general solution:  $x = 82.5 + 180n$  or  $x = 22.5 + 180n$ .
- Method 2: Auxiliary Equation**
  - Starts with  $\sin(x-30) = \cos(x-45)$ .
  - Converts  $\cos$  to  $\sin$ :  $\sin(x-30) = \sin(90-(x-45))$ .
  - Simplifies:  $\sin(x-30) = \sin(135-x)$ .
  - Considers the general solution:  $x-30 = 135-x + 360n$  or  $x-30 = 180-(135-x) + 360n$ .
  - Solves for  $x$ :  $2x = 165 + 360n$  or  $2x = 45 + 360n$ .
  - Writes the general solution:  $x = 82.5 + 180n$  or  $x = 22.5 + 180n$ .

## Question 96 (\*\*\*\*\*)

Simplify, showing all steps in the calculation, the expression

$$\arctan 8 + \arctan 2 + \arctan \frac{2}{3},$$

giving the answer in terms of  $\pi$ .

$$\boxed{\phantom{000}}, \pi$$

• STARTING WITH THE FOUR-POINT TANGENT IDENTITY

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

• EXTEND THE IDENTITY

$$\tan(A+B+C) = \frac{\tan(A+B) + \tan C}{1 - \tan(A+B)\tan C}$$

$$= \frac{\frac{\tan A + \tan B}{1 - \tan A \tan B} + \tan C}{1 - \frac{\tan A + \tan B}{1 - \tan A \tan B} \tan C}$$

Multiply "top & bottom" of the fraction by  $1 - \tan A \tan B$

$$= \frac{\tan A + \tan B + \tan C(1 - \tan A \tan B)}{1 - \tan A \tan B - (\tan A + \tan B)\tan C}$$

$$\tan(A+B+C) = \frac{\tan A + \tan B + \tan C - \tan A \tan B \tan C}{1 - \tan A \tan B - \tan B \tan C - \tan C \tan A}$$

• NOW LET  $A = \arctan 8 \Rightarrow \tan A = 8$   
 $B = \arctan 2 \Rightarrow \tan B = 2$   
 $C = \arctan \frac{2}{3} \Rightarrow \tan C = \frac{2}{3}$

$$\Rightarrow \tan(A+B+C) = \frac{8 + 2 + \frac{2}{3} - 8 \times 2 \times \frac{2}{3}}{1 - (8 \times 2) - (8 \times \frac{2}{3}) - (2 \times \frac{2}{3})}$$

$$= \frac{10 + \frac{2}{3} - \frac{32}{3}}{1 - 16 - \frac{16}{3} - \frac{4}{3}}$$

$$= \frac{30 + 2 - 32}{3 - 48 - 16 - 4} = 0$$

• ALTERNATIVE:  $A+B+C = 0 \pm n\pi \quad n=0,1,2,3,\dots$   
 $A+B+C = -2\pi, -\pi, 0, \pi, 2\pi, \dots$   
 $\arctan 8 + \arctan 2 + \arctan \frac{2}{3} = \dots -2\pi, -\pi, 0, \pi, 2\pi, \dots$   
 $\forall A, B, C \text{ ARE ALL ACUTE } 0 < A+B+C < \frac{3\pi}{2}$   
 $\therefore \arctan 8 + \arctan 2 + \arctan \frac{2}{3} = \pi$

ALTERNATIVE BY COMPLEX NUMBERS

CONSIDER THE FOLLOWING

$$Z = (1+8i)(1+2i)(3+2i) = (1+8i)(3+2i+6i-4)$$

$$Z = (1+8i)(-1+8i)$$

$$Z = -1+8i-8i-64$$

$$Z = -65$$

TAKING ARGUMENT IN THE FOLLOWING EXPRESSION

$$\Rightarrow (1+8i)(1+2i)(3+2i) = -65$$

$$\Rightarrow \arg[(1+8i)(1+2i)(3+2i)] = \arg(-65)$$

$$\Rightarrow \arg(1+8i) + \arg(1+2i) + \arg(3+2i) = \arg(-65)$$

$$\Rightarrow \arctan\left(\frac{8}{1}\right) + \arctan\left(\frac{2}{1}\right) + \arctan\left(\frac{2}{3}\right) = \pi$$

$$\therefore \arctan 8 + \arctan 2 + \arctan \frac{2}{3} = \pi$$

**Question 97** (\*\*\*\*\*)

Given that  $0 < \theta < \frac{1}{2}\pi$ ,  $0 < \varphi < \frac{1}{2}\pi$ , solve the following simultaneous equations.

$$5 \cos \theta + 2 \tan \varphi = 5 \quad \text{and} \quad 5 \sin \theta + \cot \varphi = 5.$$

Give the answers in exact form in terms of inverse trigonometric functions.

$$\boxed{\theta, \varphi} = \left[ \arccos \frac{3}{5}, \arctan 1 \right], \quad \boxed{\theta, \varphi} = \left[ \arccos \frac{4}{5}, \arctan \frac{1}{2} \right],$$

$$\boxed{\theta, \varphi} = \left[ \arccos \left[ \frac{1}{10} (3 + \sqrt{41}) \right], \arctan \left[ \frac{1}{4} (7 + \sqrt{41}) \right] \right]$$

$$\begin{aligned} 5 \cos \theta + 2 \tan \varphi &= 5 \\ 5 \sin \theta + \cot \varphi &= 5 \end{aligned} \Rightarrow \begin{aligned} 5 \cos \theta &= 5 - 2 \tan \varphi \\ 5 \sin \theta &= 5 - \cot \varphi \end{aligned}$$

SQUARING & ADDING GIVES  

$$\begin{aligned} 25 \cos^2 \theta &= (5 - 2 \tan \varphi)^2 \\ 25 \sin^2 \theta &= (5 - \cot \varphi)^2 \end{aligned}$$

$$\Rightarrow 25 (\cos^2 \theta + \sin^2 \theta) = (25 - 20 \tan \varphi + 4 \tan^2 \varphi) + (25 - 10 \cot \varphi + \cot^2 \varphi)$$

$$\Rightarrow 25 = 25 - 20 \tan \varphi + 4 \tan^2 \varphi + 25 - 10 \cot \varphi + \cot^2 \varphi$$

LET  $T = \tan \varphi$   

$$\Rightarrow 0 = -20T + 4T^2 + 25 - \frac{10}{T} + \frac{1}{T^2}$$

$$\Rightarrow 0 = -20T^3 + 4T^4 + 25T^2 - 10T + 1$$

$$\Rightarrow 4T^4 - 20T^3 + 25T^2 - 10T + 1 = 0$$

BY INSPECTION  $T=1$  IS A SOLUTION — DIVIDE IT OUT  

$$\Rightarrow 4T^3(T-1) - 16T^2(T-1) + 9T(T-1) - (T-1) = 0$$

$$\Rightarrow (T-1)(4T^3 - 16T^2 + 9T - 1) = 0$$

LOOK FOR ONE MORE SOLUTION  $T = \frac{1}{2}$  OR  $T = \frac{1}{4}$   

$$4\left(\frac{1}{2}\right)^3 - 16\left(\frac{1}{2}\right)^2 + 9\left(\frac{1}{2}\right) - 1 = \frac{1}{2} - 4 + \frac{9}{2} - 1 = 0$$

IF  $(2T-1)$  IS A FACTOR  

$$\Rightarrow (T-1)[2T(2T-1) - 7T(2T-1) + (2T-1)] = 0$$

$$\Rightarrow (T-1)(2T-1)(-7T+1) = 0$$

USE BY THE QUADRATIC FORMULA  

$$T = \frac{7 \pm \sqrt{49 - 4 \times 2 \times 1}}{2 \times 2}$$

$$T = \frac{7 \pm \sqrt{41}}{4}$$

$$\Rightarrow \tan \varphi = \begin{cases} 1 \\ \frac{1}{2} \\ \frac{7 \pm \sqrt{41}}{4} \end{cases}$$

LOOKING AT THE FIRST EQUATION  

$$5 \cos \theta = 5 - 2 \tan \varphi$$

$$\cos \theta = 1 - \frac{2}{5} \tan \varphi$$

IF  $\tan \varphi = 1$       IF  $\tan \varphi = \frac{1}{2}$   

$$\cos \theta = \frac{3}{5}$$
      
$$\cos \theta = \frac{4}{5}$$

IF  $\tan \varphi = \frac{7 \pm \sqrt{41}}{4}$   

$$\cos \theta = 1 - \frac{2}{5} \left( \frac{7 \pm \sqrt{41}}{4} \right)$$

$$\cos \theta = 1 - \frac{1}{10} (7 \pm \sqrt{41})$$

$$\cos \theta = \frac{1}{10} (3 \pm \sqrt{41}) < 0$$

IF  $\tan \varphi = \frac{7 \pm \sqrt{41}}{4}$   

$$\cos \theta = 1 - \frac{1}{10} (7 \pm \sqrt{41})$$

$$\cos \theta = \frac{3}{10} \pm \frac{1}{10} \sqrt{41}$$

$$\cos \theta = \frac{1}{10} (3 \pm \sqrt{41})$$

THERE ARE 3 CASES  

$$(\theta, \varphi) = \begin{cases} \arccos \frac{3}{5}, \arctan 1 \\ \arccos \frac{4}{5}, \arctan \frac{1}{2} \\ \arccos \left[ \frac{1}{10} (3 + \sqrt{41}) \right], \arctan \left[ \frac{1}{4} (7 + \sqrt{41}) \right] \end{cases}$$

## Question 98 (\*\*\*\*)

Solve the trigonometric equation

$$\sqrt{3} \cos\left(x + \frac{\pi}{5}\right) = \sin\left(x + \frac{\pi}{5}\right), \quad 0 \leq \theta < 2\pi,$$

giving the answers in terms of  $\pi$ .

$$\boxed{\phantom{00}}, \quad x = \frac{2\pi}{15}, \frac{17\pi}{15}$$

Handwritten solution for Question 98:

$$\begin{aligned} \sqrt{3} \cos\left(x + \frac{\pi}{5}\right) &= \sin\left(x + \frac{\pi}{5}\right) \\ \Rightarrow \sqrt{3} \cos\left(x + \frac{\pi}{5}\right) - \sin\left(x + \frac{\pi}{5}\right) &= 0 \\ \Rightarrow \frac{\sqrt{3}}{2} \cos\left(x + \frac{\pi}{5}\right) - \frac{1}{2} \sin\left(x + \frac{\pi}{5}\right) &= 0 \\ \Rightarrow \cos\left(x + \frac{\pi}{5}\right) - \frac{1}{\sqrt{3}} \sin\left(x + \frac{\pi}{5}\right) &= 0 \\ \Rightarrow \cos\left(x + \frac{\pi}{5} + \frac{\pi}{6}\right) &= 0 \\ \Rightarrow \cos\left(x + \frac{11\pi}{30}\right) &= 0 \\ \Rightarrow x + \frac{11\pi}{30} &= \frac{\pi}{2} \pm 2n\pi \quad \text{for } n \in \mathbb{Z}, \dots \\ \Rightarrow x + \frac{11\pi}{30} &= \frac{3\pi}{2} \pm 2n\pi \\ \Rightarrow x &= \frac{2\pi}{15} \pm 2n\pi \quad \left\{ \begin{array}{l} \frac{\pi}{2} - \frac{11\pi}{30} = \frac{5\pi}{30} - \frac{11\pi}{30} = -\frac{6\pi}{30} = -\frac{\pi}{5} \\ \frac{3\pi}{2} - \frac{11\pi}{30} = \frac{45\pi}{30} - \frac{11\pi}{30} = \frac{34\pi}{30} = \frac{17\pi}{15} \end{array} \right\} \\ \Rightarrow x &= \frac{17\pi}{15} \pm 2n\pi \\ \therefore x_1 &= \frac{2\pi}{15} \\ x_2 &= \frac{17\pi}{15} \end{aligned}$$

## Question 99 (\*\*\*\*\*)

Solve the following trigonometric equation

$$\arcsin 2x + \arccos x = \frac{5\pi}{6}.$$

$$\boxed{x = \frac{1}{2}}$$

Handwritten solution for Question 99:

Let  $\theta = \arcsin 2x$  and  $\phi = \arccos x$ .  
 $\sin \theta = 2x$  and  $\cos \phi = x$ .

For  $\theta$ :  $\cos \theta = \sqrt{1 - 4x^2}$ .  
 For  $\phi$ :  $\sin \phi = \sqrt{1 - x^2}$ .

Hence the above equation becomes:  
 $\theta + \phi = \frac{5\pi}{6}$   
 $\Rightarrow \sin(\theta + \phi) = \sin \frac{5\pi}{6} = \frac{1}{2}$   
 $\Rightarrow \sin \theta \cos \phi + \cos \theta \sin \phi = \frac{1}{2}$   
 $\Rightarrow (2x)(x) + (\sqrt{1 - 4x^2})(\sqrt{1 - x^2}) = \frac{1}{2}$   
 $\Rightarrow 2x^2 + \sqrt{(1 - 4x^2)(1 - x^2)} = \frac{1}{2}$   
 $\Rightarrow 2x^2 + \sqrt{1 - 5x^2 + 4x^4} = \frac{1}{2}$   
 $\Rightarrow 2\sqrt{4x^4 - 5x^2 + 1} = \frac{1}{2} - 4x^2$   
 $\Rightarrow 4(4x^4 - 5x^2 + 1) = \left(\frac{1}{2} - 4x^2\right)^2$

On the right page, the quadratic equation is solved:  
 $16x^4 - 20x^2 + 4 = 16x^4 - 8x^2 + 1$   
 $\Rightarrow -20x^2 + 4 = -8x^2 + 1$   
 $\Rightarrow 3 = 12x^2$   
 $\Rightarrow x^2 = \frac{1}{4}$   
 $\Rightarrow x = \pm \frac{1}{2}$   
 Since  $\arcsin 2x$  is defined for  $-1/2 \leq x \leq 1/2$ , the solution is  $x = \frac{1}{2}$ .

**Question 100** (\*\*\*\*\*) non calculator

Find, in degrees, the solutions of the trigonometric equation

$$2 \cos(x+10)^\circ = \frac{\cos(x+22)^\circ}{\sin(x+10)^\circ}, \quad 0^\circ \leq x \leq 360^\circ.$$

$$\boxed{\phantom{000}}, \quad \boxed{x = 16^\circ, 92^\circ, 136^\circ, 256^\circ}$$

Handwritten solution for the trigonometric equation:

$$2 \cos(x+10) = \frac{\cos(x+22)}{\sin(x+10)}$$

$$\Rightarrow 2 \cos(x+10) \sin(x+10) = \cos(x+22)$$

$$\Rightarrow \sin(2(x+10)) = \cos(x+22)$$

$$\Rightarrow \sin(2x+20) = \cos(x+22)$$

$$\Rightarrow \sin(2x+20) = \sin(90 - (x+22))$$

$$\Rightarrow \sin(2x+20) = \sin(68-x)$$

Solving (i.e. now cosine)

$$2x+20 = 68-x \pm 360n$$

$$3x = 48 \pm 360n$$

$$x = 16 \pm 120n$$

$$2x+20 = 180 - (68-x) \pm 360n$$

$$2x+20 = 112-x \pm 360n$$

$$3x = 92 \pm 360n$$

$$x = 30.66 \pm 120n$$

$x = 16^\circ, 136^\circ, 256^\circ, 92^\circ$

## Question 101 (\*\*\*\*)

It is given that for  $\theta \neq (4k+1)\frac{\pi}{2}$ ,  $k \in \mathbb{Z}$ ,

$$\tan\left(\frac{\theta}{2} + \frac{\pi}{4}\right) \equiv \tan \theta + \sec \theta.$$

a) Prove the validity of the above trigonometric identity.

b) Hence find a similar expression for  $\tan\left(\frac{\theta}{2} - \frac{\pi}{4}\right)$ .

You are now given the equation

$$\tan x - \tan(x - \alpha) = 2 \tan x,$$

where  $\alpha$  is a constant.

c) Express  $\tan x$  in terms of trigonometric functions involving  $\alpha$  only.

d) Hence solve the trigonometric equation

$$\tan x - \tan\left(x - \frac{3\pi}{5}\right) = 2 \tan \frac{3\pi}{5}, \quad 0 \leq x < 2\pi,$$

giving the answers in terms of  $\pi$ .

$$\boxed{\phantom{0000}}, \quad \tan x = \tan \alpha \pm \sec \alpha, \quad x = \frac{\pi}{20}, \frac{11\pi}{20}, \frac{21\pi}{20}, \frac{31\pi}{20}$$

a) LHS =  $\tan\left(\frac{\theta}{2} + \frac{\pi}{4}\right) = \frac{\tan\frac{\theta}{2} + \tan\frac{\pi}{4}}{1 - \tan\frac{\theta}{2}\tan\frac{\pi}{4}} = \frac{\tan\frac{\theta}{2} + 1}{1 - \tan\frac{\theta}{2}}$   
 $= \frac{\frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}} + 1}{1 - \frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}} = \dots$  multiply top & bottom of the fraction by  $\cos\frac{\theta}{2}$   
 $= \frac{\sin\frac{\theta}{2} + \cos\frac{\theta}{2}}{\cos\frac{\theta}{2} - \sin\frac{\theta}{2}} = \frac{(\sin\frac{\theta}{2} + \cos\frac{\theta}{2})(\cos\frac{\theta}{2} + \sin\frac{\theta}{2})}{(\cos\frac{\theta}{2} - \sin\frac{\theta}{2})(\cos\frac{\theta}{2} + \sin\frac{\theta}{2})}$   
 $= \frac{\sin^2\frac{\theta}{2} + 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} + \cos^2\frac{\theta}{2}}{\cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2}} = \frac{1 + 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{\cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2}}$   
 $= \frac{1 + \sin\theta}{\cos\theta} = \frac{1}{\cos\theta} + \frac{\sin\theta}{\cos\theta} = \sec\theta + \tan\theta = \text{RHS}$   
b) EITHER RECALL AN IDENTITY  
OR  
REPLACE  $\theta$  BY  $\theta - \pi$   
 $\tan\left(\frac{\theta - \pi}{2} + \frac{\pi}{4}\right) = \sec(\theta - \pi) + \tan(\theta - \pi)$   
 $\tan\left(\frac{\theta}{2} - \frac{\pi}{4}\right) = -\sec\theta + \tan\theta$   
c)  $\tan x - \tan(x - \alpha) = 2 \tan \alpha$   
 $\Rightarrow \tan x - \frac{\tan x - \tan \alpha}{1 + \tan x \tan \alpha} = 2 \tan \alpha$   
 $\Rightarrow \tan x + \frac{\tan x \tan \alpha - \tan \alpha}{1 + \tan x \tan \alpha} = 2 \tan \alpha + \frac{2 \tan \alpha \tan x}{1 + \tan x \tan \alpha}$   
 $\Rightarrow \tan x \tan \alpha - 2 \tan \alpha \tan x - \tan \alpha = 0$   
 $\Rightarrow \tan x = 2 \tan \alpha \tan x - 1 = 0$   
d)  $\tan x - \tan\left(x - \frac{3\pi}{5}\right) = 2 \tan \frac{3\pi}{5}$   
 $\tan x < \frac{\tan \frac{3\pi}{5} + \sec \frac{3\pi}{5}}{1 + \tan x \tan \frac{3\pi}{5}} = \frac{\tan\left(\frac{3\pi}{5} + \frac{\pi}{4}\right)}{1 + \tan x \tan \frac{3\pi}{5}} = \tan\left(\frac{3\pi}{5} + \frac{\pi}{4}\right)$   
Correct c) Correct d)  
THIS SINKING  
 $\left(\tan x = \tan \frac{3\pi}{5} \Rightarrow x = \frac{3\pi}{5} + n\pi, n = 0, 1, 2, \dots\right)$   
 $\tan x = \tan \frac{3\pi}{5} \Rightarrow x = \frac{3\pi}{5} + n\pi, n = 0, 1, 2, \dots$   
 $\therefore x = \frac{3\pi}{5}, \frac{8\pi}{5}, \frac{13\pi}{5}, \frac{18\pi}{5}$



## Question 102 (\*\*\*\*)

It is given that  $\theta$  and  $\varphi$  satisfy the simultaneous equations

$$\frac{\sin 2\theta}{1 + \sin \theta} = 1 - \sin \varphi,$$

$$\theta + \varphi = \pi$$

where  $0 < \theta < \pi$ ,  $\theta \neq \frac{\pi}{2}$ ,  $0 < \varphi < \pi$ .

a) Determine the value of  $\tan \theta$ .

b) Show clearly that

$$\tan(3\theta + 5\varphi) = -\frac{4}{3}.$$

$$\boxed{\phantom{00}}, \quad \boxed{\tan \theta = \frac{1}{2}}$$

$$\begin{aligned} \text{a) } \frac{\sin 2\theta}{1 + \sin \theta} &= 1 - \sin \varphi \\ \theta + \varphi &= \pi \end{aligned} \quad \Rightarrow \quad \begin{aligned} \frac{\sin 2\theta}{1 + \sin \theta} &= 1 - \sin(\pi - \theta) \\ &= 1 - [\sin \pi \cos \theta - \cos \pi \sin \theta] \\ &\Rightarrow \frac{\sin 2\theta}{1 + \sin \theta} = 1 - \sin \theta \\ \Rightarrow \sin 2\theta &= (1 - \sin \theta)(1 + \sin \theta) \\ \Rightarrow 2 \sin \theta \cos \theta &= 1 - \sin^2 \theta \\ \Rightarrow 2 \sin \theta \cos \theta &= \cos^2 \theta \\ \Rightarrow 2 \sin \theta &= \cos \theta \quad (\cos \theta \neq 0, \theta \neq \frac{\pi}{2}) \\ \Rightarrow \tan \theta &= \frac{1}{2} \end{aligned}$$

$$\begin{aligned} \text{b) } \tan(3\theta + 5\varphi) &= \tan(3\theta + 5(\pi - \theta)) = \tan(3\theta + 5\pi - 5\theta) \\ &= \tan(-2\theta + 5\pi) = \tan(-2\theta) \quad \text{tan has period of } \pi \\ &= -\tan 2\theta \quad \text{tan is odd} \\ &= \frac{-2 \tan \theta}{1 - \tan^2 \theta} \\ &= \frac{-2 \times \frac{1}{2}}{1 - \frac{1}{4}} = \frac{-1}{\frac{3}{4}} = -\frac{4}{3} \quad \text{As required} \end{aligned}$$

## Question 103 (\*\*\*\*)

The angle  $\theta$  satisfies the equation

$$\tan \theta \tan 2\theta = \sum_{r=0}^{\infty} 2 \cos^r 2\theta.$$

Given that  $0 < \theta < \frac{\pi}{2}$ , find the exact value of  $\tan \theta$ .

$$\boxed{\phantom{000}}, \quad \tan \theta = 3^{-\frac{1}{4}}$$

The image shows two handwritten solutions for the problem. The left page shows a method using the geometric series sum formula, and the right page shows a method using the identity  $2T^2 = T - T^3 + 1 - T$ .

**Left Page Solution:**

$$\begin{aligned} \tan \theta \tan 2\theta &= \sum_{r=0}^{\infty} 2 \cos^r 2\theta \\ \Rightarrow \tan \theta \tan 2\theta &= 2 \left[ 1 + \cos 2\theta + \cos^2 2\theta + \cos^3 2\theta + \dots \right] \\ \text{As } 0 < \theta < \frac{\pi}{2}, \quad 1 < \cos 2\theta < 1 \\ \text{Take the G.P. on the R.H.S. according to } S_{\infty} &= \frac{a}{1-r} \\ \Rightarrow \tan \theta \tan 2\theta &= 2 \times \frac{1}{1 - \cos 2\theta} \\ \Rightarrow \tan \theta \left( \frac{2 \tan \theta}{1 - \tan^2 \theta} \right) &= \frac{2}{1 - (1 - 2 \sin^2 \theta)} \\ \Rightarrow \frac{2 \tan^2 \theta}{1 - \tan^2 \theta} &= \frac{2}{2 \sin^2 \theta} \\ \Rightarrow \frac{2 \tan^2 \theta}{1 - \tan^2 \theta} &= \sec^2 \theta \\ \Rightarrow \frac{2 \tan^2 \theta}{1 - \tan^2 \theta} &= 1 + \cot^2 \theta \\ \Rightarrow \frac{2 \tan^2 \theta}{1 - \tan^2 \theta} &= 1 + \frac{1}{\tan^2 \theta} \\ \text{Let } T &= \tan^2 \theta \\ \Rightarrow \frac{2T}{1-T} &= 1 + \frac{1}{T} \\ \Rightarrow 2T^2 &= T(1-T) + (1-T) \end{aligned}$$

**Right Page Solution:**

$$\begin{aligned} \Rightarrow 2T^2 &= T - T^3 + 1 - T \\ \Rightarrow 3T^2 &= 1 \\ \Rightarrow T^2 &= \frac{1}{3} \\ \Rightarrow (\tan^2 \theta)^2 &= \frac{1}{3} \\ \Rightarrow \tan^4 \theta &= \frac{1}{3} \\ \Rightarrow \tan^2 \theta &= \frac{1}{\sqrt{3}} \\ \Rightarrow \tan \theta &= 3^{-\frac{1}{4}} \quad (\tan \theta > 0 \text{ as } 0 < \theta < \frac{\pi}{2}) \end{aligned}$$

## Question 104 (\*\*\*\*)

Use trigonometric algebra to fully simplify

$$\arctan \left[ \frac{\sqrt{1+\sin x} - \sqrt{1-\sin x}}{\sqrt{1+\sin x} + \sqrt{1-\sin x}} \right], \quad 0 < x < \frac{\pi}{4},$$

giving the final answer in terms of  $x$ .

$$\boxed{\phantom{000}}, \quad \boxed{\frac{1}{2}x}$$

Handwritten solution for Question 104:

Given:  $\arctan \left[ \frac{\sqrt{1+\sin x} - \sqrt{1-\sin x}}{\sqrt{1+\sin x} + \sqrt{1-\sin x}} \right], \quad 0 < x < \frac{\pi}{4}$

• START BY CONJUGATING THE DENOMINATOR.

$$\begin{aligned} &= \arctan \left[ \frac{(\sqrt{1+\sin x} - \sqrt{1-\sin x})(\sqrt{1+\sin x} - \sqrt{1-\sin x})}{(\sqrt{1+\sin x} + \sqrt{1-\sin x})(\sqrt{1+\sin x} - \sqrt{1-\sin x})} \right] \\ &= \arctan \left[ \frac{(1+\sin x) - 2\sqrt{1-\sin^2 x} + (1-\sin x)}{(1+\sin x) - (1-\sin x)} \right] \\ &= \arctan \left[ \frac{2 - 2\sqrt{1-\sin^2 x}}{2\sin x} \right] \\ &= \arctan \left[ \frac{2 - 2\sqrt{\cos^2 x}}{2\sin x} \right] \\ &= \arctan \left[ \frac{1 - \cos x}{\sin x} \right] \end{aligned}$$

• THE COULD POSSIBLY PRODUCE AN ARGUMENT IN TANGENTS IF WE USE THE DOUBLE ANGLE FORMULAE

$$\begin{aligned} &= \arctan \left[ \frac{1 - (1 - 2\sin^2 \frac{x}{2})}{2\sin \frac{x}{2} \cos \frac{x}{2}} \right] \\ &= \arctan \left[ \frac{2\sin^2 \frac{x}{2}}{2\sin \frac{x}{2} \cos \frac{x}{2}} \right] \\ &= \arctan \left[ \frac{\sin \frac{x}{2}}{\cos \frac{x}{2}} \right] \\ &= \arctan \left( \tan \frac{x}{2} \right) \\ &= \frac{x}{2} \end{aligned}$$

Trigonometric identities used:

- $\cos 2\theta = 1 - 2\sin^2 \theta$
- $\sin 2\theta = 2\sin \theta \cos \theta$

## Question 105 (\*\*\*\*)

Solve the following trigonometric equation, for  $0 \leq x < 2\pi$ .

$$2\cos x \sin^2 x - 2\cos^2 x \sin x + \cos^2 x - 4\sin^2 x + 3\cos x \sin x + 2\sin x - 2\cos x = 0.$$

$$\boxed{\phantom{000}}, \quad x = \frac{\pi}{6}, \frac{\pi}{4}, \frac{5\pi}{6}, \frac{5\pi}{4}$$

2cosxsin^2x - 2cos^2xsinx + cos^2x - 4sin^2x + 3cosxsinx + 2sinx - 2cosx = 0

- LOOKING AT THE COEFFICIENTS OF THIS EQUATION IN TERMS OF POWER
  - 2cosxsin^2x - 2cos^2xsinx (CUBIC IN COS/SINX)
  - 1cos^2x - 4sin^2x + 3cosxsinx (QUADRATIC IN COS/SINX)
  - 2sinx - 2cosx (LINEAR IN COS/SINX)
- BY INSPECTION IF COSX = SINX THEN THE EQUATION IS SATISFIED AS WE FOUND A SOLUTION & THEREFORE A FACTOR (COSX - SINX)
- RESOLVE THE LHS COMPACTLY
 
$$2C^2S^2 - 2C^2S + C^2 - 4S^2 + 3CS + 2S - 2C = 0$$
- FACTORISE IN PAIRS BY INSPECTION
 
$$\Rightarrow 2CS(S-C) + C^2 - 4S^2 + 3CS + 2(S-C) = 0$$

$$\Rightarrow 2CS(S-C) + C^2 + 3CS - 4S^2 + 2(S-C) = 0$$

$$\Rightarrow 2CS(S-C) + (C-S)(C+4S) + 2(S-C) = 0$$

$$\Rightarrow 2CS(S-C) - (S-C)(C+4S) + 2(S-C) = 0$$

$$\Rightarrow (S-C)[2CS - (C+4S) + 2] = 0$$

$$\Rightarrow (S-C)(2CS - C - 4S + 2) = 0$$
- FACTORISE IN PAIRS AGAIN (RATIO 2:1 -1 & -4:2)
 
$$\Rightarrow (S-C)[C(2S-1) - 2(2S-1)] = 0$$

$\Rightarrow (S-C)(2S-1)(C-2) = 0$

$\Rightarrow$  EITHER  $SINX - COSX = 0$   
 OR  $SINX = \frac{1}{2}$   
 OR  $COSX = 2$

HENCE WE CAN NOW FIND SOLUTIONS

$SINX - COSX = 0$        $SINX = \frac{1}{2}$   
 $SINX = COSX$        $\Rightarrow X = \frac{\pi}{6}, \frac{5\pi}{6}$   
 $COSX = 1$        $\Rightarrow X = 0, 2\pi$

$2 = \frac{\pi}{4}, \frac{7\pi}{4}$

$\Rightarrow X = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{\pi}{4}, \frac{7\pi}{4}$

## Question 106 (\*\*\*\*\*)

Use trigonometric algebra to solve the equation

$$\arctan x + 2 \operatorname{arccot} x = \frac{2\pi}{3}.$$

You may assume that  $\operatorname{arccot} x$  is the inverse function for the part of  $\cot x$  for which  $0 \leq x \leq \pi$ .

$$\boxed{\phantom{000}}, \boxed{x = \sqrt{3}}$$

$\arctan x + 2 \operatorname{arccot} x = \frac{2\pi}{3}$

• USING THE TRIGONOMETRIC IDENTITY  $\arctan\left(\frac{1}{x}\right) \equiv \operatorname{arccot}\left(\frac{1}{x}\right)$  WHICH IS A CONSEQUENCE OF THE DEFINITIONS, EASILY VERIFIABLE BY A RIGHT ANGLED TRIANGLE

$\tan \phi = \frac{1}{x} \Rightarrow \phi = \arctan \frac{1}{x}$   
 $\tan \phi = \frac{1}{x} \Rightarrow \phi = \operatorname{arccot} x$   
 $\cot \phi = x \Rightarrow \phi = \operatorname{arccot} x$

$\Rightarrow \arctan x + 2 \operatorname{arccot} \frac{1}{x} = \frac{2\pi}{3}$   
 $\Rightarrow 0 + 2\phi = \frac{2\pi}{3}$

• TAKE TANGENTS ON BOTH SIDES OF THE EQUATION

$\Rightarrow \tan \left[ \arctan x + 2 \operatorname{arccot} \frac{1}{x} \right] = \tan \frac{2\pi}{3}$

$\Rightarrow \frac{\tan(\arctan x) + \tan(2 \operatorname{arccot} \frac{1}{x})}{1 - \tan(\arctan x) \tan(2 \operatorname{arccot} \frac{1}{x})} = -\sqrt{3}$

$\Rightarrow \frac{x + \tan(2 \operatorname{arccot} \frac{1}{x})}{1 - x \tan(2 \operatorname{arccot} \frac{1}{x})} = -\sqrt{3}$

• APPLY THE TANGENT DOUBLE ANGLE IDENTITY  $\tan 2\phi = \frac{2 \tan \phi}{1 - \tan^2 \phi}$

$\Rightarrow \frac{x + \frac{2 \tan(\operatorname{arccot} \frac{1}{x})}{1 - \tan^2(\operatorname{arccot} \frac{1}{x})}}{1 - x \left[ \frac{2 \tan(\operatorname{arccot} \frac{1}{x})}{1 - \tan^2(\operatorname{arccot} \frac{1}{x})} \right]} = -\sqrt{3}$

$\Rightarrow \frac{x + \frac{2 \cdot \frac{1}{x}}{1 - \frac{1}{x^2}}}{1 - x \left[ \frac{2}{1 - \frac{1}{x^2}} \right]} = -\sqrt{3}$

$\Rightarrow \frac{x + \frac{2}{x}}{1 - \frac{2x}{1 - \frac{1}{x^2}}} = -\sqrt{3}$

$\Rightarrow \frac{x + \frac{2}{x}}{1 - \frac{2x}{1 - \frac{1}{x^2}}} = -\sqrt{3}$

• MULTIPLY TOP & BOTTOM OF THE DOUBLE FRACTION BY  $1 - \frac{1}{x^2}$

$\Rightarrow \frac{x(1 - \frac{1}{x^2}) + \frac{2}{x}}{(1 - \frac{1}{x^2}) - 2} = -\sqrt{3}$

$\Rightarrow \frac{x - \frac{1}{x} + \frac{2}{x}}{1 - \frac{1}{x^2} - 2} = -\sqrt{3}$

$\Rightarrow \frac{x + \frac{1}{x}}{-1 - \frac{1}{x^2}} = -\sqrt{3}$

• MULTIPLY TOP & BOTTOM OF THE DOUBLE FRACTION BY  $x^2$

$\Rightarrow \frac{x^3 + x}{-x^2 - 1} = -\sqrt{3}$

$\Rightarrow -\frac{x(x^2 + 1)}{x^2 + 1} = -\sqrt{3}$

$\Rightarrow x = \sqrt{3}$   
 $(x+1) \neq 0$

**Question 107** (\*\*\*\*)

The distinct acute angles  $\theta$  and  $\varphi$ ,  $\theta > \varphi$  satisfy the equation

$$f(\theta, \varphi) = g(\theta, \varphi) \tan \varphi,$$

where the functions  $f$  and  $g$  are defined as

$$f(\theta, \varphi) \equiv \sin(\theta - \varphi) \quad \text{and} \quad g(\theta, \varphi) \equiv \cos(\theta - \varphi) - 2 \tan \varphi \sin(\theta - \varphi).$$

Use trigonometric identities to show that

$$\tan \theta = 2 \tan \varphi .$$

M2, proof

$f(\theta, \phi) = \sin(\theta - \phi)$   
 $g(\theta, \phi) = \cos(\theta - \phi) - 2 \tan \phi \sin(\theta - \phi)$

WE ARE GIVEN THAT  $f(\theta, \phi) = g(\theta, \phi) \tan \phi$

$$\Rightarrow \sin(\theta - \phi) = [\cos(\theta - \phi) - 2 \tan \phi \sin(\theta - \phi)] \tan \phi$$

$$\Rightarrow \frac{\sin(\theta - \phi)}{\cos(\theta - \phi)} = \left[ \frac{\cos(\theta - \phi)}{\cos(\theta - \phi)} - 2 \tan \phi \frac{\sin(\theta - \phi)}{\cos(\theta - \phi)} \right] \tan \phi$$

$$\Rightarrow \tan(\theta - \phi) = [1 - 2 \tan \phi \tan(\theta - \phi)] \tan \phi$$

$$\Rightarrow \tan(\theta - \phi) = \tan \phi - 2 \tan \phi \tan(\theta - \phi)$$

$$\Rightarrow \tan(\theta - \phi) + 2 \tan \phi \tan(\theta - \phi) = \tan \phi$$

$$\Rightarrow \tan(\theta - \phi) [1 + 2 \tan \phi] = \tan \phi$$

$$\Rightarrow \left( \frac{\tan \theta - \tan \phi}{1 + \tan \theta \tan \phi} \right) (1 + 2 \tan \phi) = \tan \phi$$

$$\Rightarrow (\tan \theta - \tan \phi) (1 + 2 \tan \phi) = \tan \phi (1 + \tan \theta \tan \phi)$$

$$\Rightarrow \tan \theta + 2 \tan \phi \tan \theta - \tan \phi - 2 \tan^2 \phi = \tan \phi + \tan \phi \tan \theta$$

$$\Rightarrow \tan \theta + \tan \phi \tan \theta = 2 \tan \phi + 2 \tan^2 \phi$$

$$\Rightarrow \tan \theta (1 + \tan \phi) = 2 \tan \phi (1 + \tan \phi) \quad (1 + \tan \phi \neq 0)$$

$$\Rightarrow \tan \theta = 2 \tan \phi$$

As required

ADDITIONALITY BY CHANGING INTO SINES & COSINES

$$\begin{aligned} \Rightarrow \sin(\theta - \phi) &= [\cos(\theta - \phi) - 2\sin\theta \sin(\theta - \phi)] \tan\phi \\ \Rightarrow \sin(\theta - \phi) &= [\cos(\theta - \phi) - 2\sin\theta \sin(\theta - \phi)] \tan\phi \\ \Rightarrow \sin(\theta - \phi) &= \frac{1}{\cos\phi} [\cos\phi \cos(\theta - \phi) - 2\sin\theta \sin(\theta - \phi)] \tan\phi \\ \Rightarrow \sin(\theta - \phi) &= \frac{\tan\phi}{\cos\phi} [\cos\phi \cos(\theta - \phi) - 2\sin\theta \sin(\theta - \phi) - \sin\theta \sin(\theta - \phi)] \\ \Rightarrow \sin(\theta - \phi) &= \frac{\sin\phi}{\cos\phi} [\cos\phi \cos(\theta - \phi) - 3\sin\theta \sin(\theta - \phi)] \\ \Rightarrow \sin(\theta - \phi) &= \frac{\sin\phi}{\cos\phi} [\cos\phi - 3\sin\theta \sin(\theta - \phi)] \\ \Rightarrow \cos^2\theta \sin(\theta - \phi) &= \sin\phi \cos\phi - 3\sin^2\theta \sin(\theta - \phi) \\ \Rightarrow \sin(\theta - \phi) \cos^2\theta &+ \sin(\theta - \phi) \sin^2\theta = \sin\phi \cos\phi \\ \Rightarrow \sin(\theta - \phi) [\cos^2\theta + \sin^2\theta] &= \sin\phi \cos\phi \\ \Rightarrow \sin(\theta - \phi) \cos\phi &= \cos\theta \sin\phi = \sin\phi \cos\theta \\ \Rightarrow \sin\theta \cos\phi &= 2\sin\theta \cos\theta \\ \Rightarrow \frac{\sin\theta}{\cos\theta} &= \frac{2\sin\theta}{\cos\phi} \\ \Rightarrow -\tan\theta &= 2 \tan\phi \end{aligned}$$

-A. K. SINGH

## Question 108 (\*\*\*\*)

Solve the following trigonometric equation, for  $0 \leq \theta < 2\pi$ .

$$3\cos^2 \theta - \sin^2 \theta - \sqrt{3} \cos \theta - \sin \theta = 0.$$

$$\boxed{\phantom{000}}, \theta = \frac{\pi}{6}, \frac{2\pi}{3}, \frac{3\pi}{2}, \frac{5\pi}{3}$$

$3\cos^2 \theta - \sin^2 \theta - \sqrt{3} \cos \theta - \sin \theta = 0$   
 • SOLVE BY FACTORISING IN PHASE  
 $\Rightarrow (\sqrt{3} \cos \theta)^2 - (\sin \theta)^2 - [\sqrt{3} \cos \theta + \sin \theta] = 0$   
 $\Rightarrow (\sqrt{3} \cos \theta - \sin \theta)(\sqrt{3} \cos \theta + \sin \theta) - (\sqrt{3} \cos \theta + \sin \theta) = 0$   
 $\Rightarrow (\sqrt{3} \cos \theta + \sin \theta) [(\sqrt{3} \cos \theta - \sin \theta) - 1] = 0$   
 • EITHER  
 $\sqrt{3} \cos \theta + \sin \theta = 0$   
 $\Rightarrow \sin \theta = -\sqrt{3} \cos \theta$   
 $\Rightarrow \tan \theta = -\sqrt{3}$   
 $\Rightarrow \theta = -\frac{\pi}{3} \pm n\pi$   
 • OR  
 $\sqrt{3} \cos \theta - \sin \theta = 1$   
 $\Rightarrow \frac{\sqrt{3}}{2} \cos \theta - \frac{1}{2} \sin \theta = \frac{1}{2}$   
 $\Rightarrow \cos(\theta + \frac{\pi}{6}) = \frac{1}{2}$   
 $\Rightarrow \theta + \frac{\pi}{6} = \frac{\pi}{3} \pm 2n\pi$   
 $\Rightarrow \theta = \frac{\pi}{6} \pm 2n\pi$   
 • COLLECTING ALL THE SOLUTIONS FOR  $0 \leq \theta < 2\pi$   
 $\theta = \frac{\pi}{6}, \frac{2\pi}{3}, \frac{3\pi}{2}, \frac{5\pi}{3}$

## Question 109 (\*\*\*\*\*)

Solve the following trigonometric equation, for  $0 < x < 2\pi$ .

$$\sin x \sin 2x + \sin 2x \sin 3x + \sin 3x \sin 4x = 0.$$

$$\boxed{\phantom{000}}, x = \frac{\pi}{2}, \pi, \frac{3\pi}{2}$$

$\sin x \sin 2x + \sin 2x \sin 3x + \sin 3x \sin 4x = 0 \quad 0 < x < 2\pi$

• LOOKING AT THE ABOVE WE MAY ATTEMPT TO FACTORISE  $\sin 2x$  OUT OF THE GIVEN EXPRESSION, FOLLOWED BY  $\sin x$ , AFTER RECOGNISING  $\sin 3x$  OUT OF THE LAST TWO TERMS & WRITE  $\sin 4x$  IN TERMS OF  $\sin 2x$

$$\Rightarrow \sin x \sin 2x + \sin 2x [\sin 2x + \sin 4x] = 0$$

$$\Rightarrow \sin x \sin 2x + \sin 2x [\sin 2x + 2\sin 2x \cos 2x] = 0$$

$$\Rightarrow \sin 2x [\sin x + \sin 2x (1 + 2\cos 2x)] = 0$$

• NEXT WE REQUIRE THE TRIPLE ANGLE IDENTITY, WHICH IF WE DO NOT REMEMBER WE CAN QUICKLY

$$\sin 3x = \sin(2x+x) = \sin 2x \cos x + \cos 2x \sin x$$

$$= (2\sin x \cos x) \cos x + (1-2\sin^2 x) \sin x$$

$$= 2\sin x \cos^2 x + \sin x - 2\sin^3 x$$

$$= 2\sin x (1-\sin^2 x) + \sin x - 2\sin^3 x$$

$$= 2\sin x - 2\sin^3 x + \sin x - 2\sin^3 x$$

$$= 3\sin x - 4\sin^3 x$$

• HENCE WE CAN QUICKLY FACTORISE  $\sin 2x$  OUT

$$\Rightarrow \sin 2x [\sin x + (3\sin x - 4\sin^3 x)(1+2\cos 2x)] = 0$$

$$\Rightarrow \sin 2x \sin x [1 + (3-4\sin^2 x)(1+2\cos 2x)] = 0$$

$$\Rightarrow \sin 2x \sin x [1 + (3-4\sin^2 x)(1+2(1-2\sin^2 x))] = 0$$

$\Rightarrow \sin 2x \sin x [1 + (3-4\sin^2 x)(1+2-4\sin^2 x)] = 0$

$\Rightarrow \sin 2x \sin x [1 + (3-4\sin^2 x)(3-4\sin^2 x)] = 0$

$\Rightarrow \sin 2x \sin x [1 + (3-4\sin^2 x)^2] = 0$

• EITHER  $\sin 2x = 0$  OR  $\sin x = 0$  HOWEVER  $1 + (3-4\sin^2 x)^2 \neq 0$  (As  $\pi$  IS AT  $x = \pi$ )

• AS ALL THE SOLUTIONS OF  $\sin 2x = 0$  ARE INCLUDED IN THE SOLUTION OF  $\sin x = 0$ , WE FINALLY OBTAIN

$$\Rightarrow \sin x = 0$$

$$\Rightarrow x = k\pi, \quad k \in \mathbb{Z}$$

$$\Rightarrow x = \frac{k\pi}{1}$$

$\therefore x = \frac{\pi}{2}, \pi, \frac{3\pi}{2} \text{ FOR } 0 < x < 2\pi //$



## Question 110 (\*\*\*\*)

Solve the following trigonometric equation, for  $0 < x < \frac{1}{2}\pi$ .

$$4\cos x \cos 2x \cos 5x + 1 = 0.$$

$$\boxed{\phantom{000}}, \quad \boxed{x = \frac{1}{8}\pi, \frac{1}{5}\pi, \frac{3}{8}\pi, \frac{2}{5}\pi}$$

Handwritten solution for Question 110:

**Method 1 (Left Panel):**

- Equation:  $4\cos x \cos 2x \cos 5x + 1 = 0$ ,  $0 < x < \frac{\pi}{2}$
- $\Rightarrow 4\cos x \cos 2x \cos 5x = -1$
- Firstly  $2 = \pi$  is not a solution of the equation (i.e.  $\sin 2 = 0$ ) so if we multiply both sides by  $\sin x$  we introduce this solution
- $\Rightarrow 4\sin x \cos x \cos 2x \cos 5x = -\sin x$
- $\Rightarrow 2\sin 2x \cos 2x \cos 5x = -\sin x$
- $\Rightarrow \sin 2x \cos 5x = -\sin x$
- Now looking at the LHS
- $\sin(4x + 5x) = \sin 4x \cos 5x + \cos 4x \sin 5x$
- $\sin(4x - 5x) = \sin 4x \cos 5x - \cos 4x \sin 5x$
- $\sin 2x + \sin(-x) = 2\sin 4x \cos 5x$
- $\therefore \sin 4x \cos 5x = \frac{1}{2}\sin 2x - \frac{1}{2}\sin x$
- Replacing in the equation
- $\frac{1}{2}\sin 2x - \frac{1}{2}\sin x = -\sin x$
- $\frac{1}{2}\sin 2x = -\frac{1}{2}\sin x$
- $\sin 2x = -\sin x$
- $\sin 2x = \sin(-x)$

**Method 2 (Right Panel):**

- $\Rightarrow \begin{cases} 4x = -x \pm 2\pi n \\ 4x = \pi + x \pm 2\pi n \end{cases} \quad n = 0, 1, 2, \dots$
- $\Rightarrow \begin{cases} 5x = 0 \pm 2\pi n \\ 3x = \pi \pm 2\pi n \end{cases}$
- $\Rightarrow \begin{cases} x = 0 \pm \frac{2\pi n}{5} \\ x = \frac{\pi}{3} \pm \frac{2\pi n}{3} \end{cases}$
- $\therefore x = 0, \frac{\pi}{5}, \frac{2\pi}{5}, \frac{\pi}{3}, \frac{2\pi}{3}$
- But 0 is not a solution (introduced)
- $\therefore x = \frac{\pi}{5}, \frac{2\pi}{5}, \frac{\pi}{3}, \frac{2\pi}{3}$

## Question 111 (\*\*\*\*)

Solve the following trigonometric equation.

$$2 \cot 2x = \sec x \sec\left(\frac{2\pi}{15}\right) + 2 \tan\left(\frac{2\pi}{15}\right), \quad 0 \leq x < 2\pi.$$

Give the answers in the form  $k\pi$ , where  $k$  is rational.

$$\boxed{\phantom{0}}, \quad k = \frac{1}{2}, \frac{3}{2}, \frac{11}{90}, \frac{71}{90}, \frac{131}{90}, \frac{41}{30}$$

Handwritten solution for Question 111:

Left page:

$$2 \cot 2x = \sec x \sec\left(\frac{2\pi}{15}\right) + 2 \tan\left(\frac{2\pi}{15}\right) \quad 0 \leq x < 2\pi$$

• Start by simplifying everything into sines & cosines

$$\frac{2 \cos 2x}{\sin 2x} = \frac{1}{\cos x} \frac{1}{\cos\left(\frac{2\pi}{15}\right)} + \frac{2 \sin\left(\frac{2\pi}{15}\right)}{\cos\left(\frac{2\pi}{15}\right)}$$

• Multiply through by  $\sin 2x \cos\left(\frac{2\pi}{15}\right)$

$$\Rightarrow 2 \cos 2x \cos\left(\frac{2\pi}{15}\right) = \sin 2x + 2 \sin x \sin 2x \tan\left(\frac{2\pi}{15}\right)$$

$$\Rightarrow 2 \cos 2x \cos\left(\frac{2\pi}{15}\right) = 2 \sin x \cos\left(\frac{2\pi}{15}\right) + 2 \sin 2x \tan\left(\frac{2\pi}{15}\right)$$

$$\Rightarrow \cos 2x \cos\left(\frac{2\pi}{15}\right) = \sin x \cos\left(\frac{2\pi}{15}\right) + \sin 2x \tan\left(\frac{2\pi}{15}\right)$$

• Everyting looks like a section (divide it all and account for it at the end)

$$\Rightarrow \cos 2x \cos\left(\frac{2\pi}{15}\right) = \sin x + \sin 2x \tan\left(\frac{2\pi}{15}\right)$$

$$\Rightarrow \cos 2x \cos\left(\frac{2\pi}{15}\right) - \sin 2x \tan\left(\frac{2\pi}{15}\right) = \sin x$$

$$\Rightarrow \cos\left(2x + \frac{2\pi}{15}\right) = \sin x$$

$$\Rightarrow \cos\left(2x + \frac{2\pi}{15}\right) = \cos\left(\frac{\pi}{2} - x\right) \quad \text{SMA} \equiv \cos\left(\frac{\pi}{2} - A\right)$$

• Setting up a general solution

$$\Rightarrow \left(2x + \frac{2\pi}{15} = \frac{\pi}{2} - x + 2n\pi\right) \quad \text{or} \quad \left(2x + \frac{2\pi}{15} = x - \frac{\pi}{2} + 2n\pi\right) \quad n = 0, 1, 2, \dots$$

• Check to make sure in degrees & switch to radians at the end

$$\Rightarrow \left(2x + 24 = 90 - x + 360n\right) \quad \text{or} \quad \left(2x + 24 = x - 90 + 360n\right)$$

Right page:

$$\Rightarrow \begin{cases} 3x = 66 + 360n \\ x = -114 + 360n \end{cases}$$

$$\Rightarrow \begin{cases} 2x = 22 + 120n \\ x = 246 + 360n \end{cases}$$

• Solutions in degrees:  $22^\circ, 142^\circ, 262^\circ, 246^\circ$

• To that we have to add the solutions of  $\cos x = 0$  i.e.  $\frac{\pi}{2}$  &  $\frac{3\pi}{2}$

•  $k = \frac{1}{2}, \frac{3}{2}, \frac{11}{90}, \frac{71}{90}, \frac{131}{90}, \frac{41}{30}$

## Question 112 (\*\*\*\*)

Use trigonometric algebra to fully simplify

$$2 \arctan\left(\frac{1}{5}\right) + \arccos\left(\frac{7}{5\sqrt{2}}\right) + \arctan\left(\frac{1}{8}\right),$$

giving the final answer in terms of  $\pi$ .

$$\boxed{\phantom{000}}, \boxed{\frac{\pi}{4}}$$

$2 \arctan\left(\frac{1}{5}\right) + \arccos\left(\frac{7}{5\sqrt{2}}\right) + 2 \arctan\left(\frac{1}{8}\right) = \psi$   
 $2\theta + \alpha + 2\phi = \psi$

$\tan \theta = \frac{1}{5}$        $\cos \alpha = \frac{7}{5\sqrt{2}}$        $\tan \phi = \frac{1}{8}$

WORKING WITH TRIANGLES AS FOLLOWS:  
 $\Rightarrow 2\theta + \alpha + 2\phi = \psi$   
 $\Rightarrow 2\theta + 2\phi = \psi - \alpha$   
 $\Rightarrow \tan(2\theta + 2\phi) = \tan(\psi - \alpha)$   
 $\Rightarrow \frac{\tan 2\theta + \tan 2\phi}{1 - \tan 2\theta \tan 2\phi} = \frac{\tan \psi - \tan \alpha}{1 - \tan \psi \tan \alpha}$   
 $\Rightarrow \frac{\frac{2 \tan \theta}{1 - \tan^2 \theta} + \frac{2 \tan \phi}{1 - \tan^2 \phi}}{1 - \frac{2 \tan \theta}{1 - \tan^2 \theta} \times \frac{2 \tan \phi}{1 - \tan^2 \phi}} = \frac{\tan \psi - \tan \alpha}{1 - \tan \psi \tan \alpha}$

SUBSTITUTING VALUES IN:  
 $\frac{\frac{2}{1 - \frac{1}{25}} + \frac{2}{1 - \frac{1}{64}}}{1 - \frac{2}{1 - \frac{1}{25}} \times \frac{2}{1 - \frac{1}{64}}} = \frac{\tan \psi - \frac{1}{5}}{1 - \tan \psi \times \frac{1}{5}}$

$\Rightarrow \frac{\frac{10}{24} + \frac{16}{64}}{1 - \frac{10}{24} \times \frac{16}{64}} = \frac{\tan \psi - \frac{1}{5}}{1 - \frac{1}{5} \tan \psi}$   
 $\Rightarrow \frac{\frac{5}{12} + \frac{16}{64}}{1 - \frac{5}{12} \times \frac{16}{64}} = \frac{\tan \psi - \frac{1}{5}}{1 - \frac{1}{5} \tan \psi}$   
 $\Rightarrow \frac{315 + 192}{750 - 80} = \frac{\tan \psi - \frac{1}{5}}{1 - \frac{1}{5} \tan \psi}$   
 $\Rightarrow \frac{3}{4} = \frac{\tan \psi - \frac{1}{5}}{1 - \frac{1}{5} \tan \psi}$   
 $\Rightarrow 21 + 3 \tan \psi = 28 \tan \psi - 4$   
 $\Rightarrow 25 \tan \psi = 25$   
 $\Rightarrow \tan \psi = 1$   
 $\Rightarrow \psi = \frac{\pi}{4}$

ALTERNATIVE BY COMPLEX NUMBERS  
 CONSIDER THE EXPRESSION  
 $(5+i)^2 (7+i) (8+i)^2$   
 $= (25 + 10i - 1)(7+i)(64 + 16i - 1)$   
 $= (24 + 10i)(7+i)(63 + 16i)$   
 $= 2(12 + 5i)(7+i)(63 + 16i)$

$= 2(84 + 12i + 35i - 5)(63 + 16i)$   
 $= 2(79 + 47i)(63 + 16i)$   
 $= 2(4977 + 1264i + 2961i - 752)$   
 $= 2(4225 + 4225i)$   
 $= 8450(1+i)$

THIS  
 $\arg[(5+i)^2 (7+i) (8+i)^2] = \arg[8450(1+i)]$   
 $\arg(5+i)^2 + \arg(7+i) + \arg(8+i)^2 = \arg 8450 + \arg(1+i)$   
 $2 \arg(5+i) + \arg(7+i) + 2 \arg(8+i) = \arg 8450 + \arg(1+i)$   
 $2 \arctan\left(\frac{1}{5}\right) + \arctan\left(\frac{1}{7}\right) + 2 \arctan\left(\frac{1}{8}\right) = 0 + \arctan 1$   
 $2 \arctan \frac{1}{5} + \arctan \frac{1}{7} + 2 \arctan \frac{1}{8} = \frac{\pi}{4}$

## Question 113 (\*\*\*\*\*)

It given that

$$\arctan x + \arctan y + \arctan z = \frac{\pi}{2}.$$

Show that  $x$ ,  $y$  and  $z$  satisfy the relationship

$$xy + yz + zx = 1.$$

,  proof

• WORK IN STAGES

$\Rightarrow \arctan x + \arctan y = \psi$   
 $\Rightarrow \psi + \phi = \frac{\pi}{2}$   
 $\Rightarrow \tan(\psi + \phi) = \tan \frac{\pi}{2}$   
 $\Rightarrow \frac{\tan \psi + \tan \phi}{1 - \tan \psi \tan \phi} = \tan \frac{\pi}{2}$   
 $\Rightarrow \frac{x + y}{1 - xy} = \tan \frac{\pi}{2}$   
 $\Rightarrow \psi = \arctan \left( \frac{x + y}{1 - xy} \right)$   
 THIS  
 $\arctan x + \arctan y = \arctan \left( \frac{x + y}{1 - xy} \right)$

• NOW USE THE EQUATION IN THE BOX WITH THE ARCTAN

$\Rightarrow \arctan x + \arctan y + \arctan z = \frac{\pi}{2}$   
 $\Rightarrow \arctan \left( \frac{x + y}{1 - xy} \right) + \arctan z = \frac{\pi}{2}$   
 $\Rightarrow \arctan \left[ \frac{\left( \frac{x + y}{1 - xy} \right) + z}{1 - \left( \frac{x + y}{1 - xy} \right) z} \right] = \frac{\pi}{2}$   
 TAKING TANGENTS ON BOTH SIDES  
 $\Rightarrow \frac{\frac{x + y}{1 - xy} + z}{1 - z \left( \frac{x + y}{1 - xy} \right)} = \infty$

• AS THE FRACTION IS INFINITE, THE DENOMINATOR MUST BE ZERO

$\Rightarrow 1 - z \left( \frac{x + y}{1 - xy} \right) = 0$   
 $\Rightarrow 1 - \frac{zx + yz}{1 - xy} = 0$   
 $\Rightarrow 1 - xy - (zx + yz) = 0$   
 $\Rightarrow 1 - xy - zx - yz = 0$   
 $\Rightarrow xy + yz + zx = 1$   
 Q.E.D.

## Question 114 (\*\*\*\*)

Use a fully detailed method to show that

$$\arctan[\sqrt{6} + \sqrt{3} - \sqrt{2} - 2] = 37.5^\circ.$$

 , proof

**Method A - By constructing a suitable equation**

• Start from a general equation for a "tan" equation

$$\Rightarrow 2x = 75^\circ \pm 360n$$

$$\Rightarrow (x - 45^\circ) = (36 - 2) \pm 360n$$

• This could have been a sine or cosine equation (general equation)

$$\Rightarrow \cos(x - 45^\circ) = \cos(36 - 2)$$

• Expand by using the addition/subtraction formulae

$$\Rightarrow \cos(x) \cos(45^\circ) + \sin(x) \sin(45^\circ) = \cos(36) \cos(2) + \sin(36) \sin(2)$$

$$\Rightarrow \frac{\sqrt{2}}{2} \cos(x) + \frac{\sqrt{2}}{2} \sin(x) = \cos(36) \cos(2) + \sin(36) \sin(2)$$

$$\Rightarrow \sqrt{2} \cos(x) + \sqrt{2} \sin(x) = \cos(36) \cos(2) + \sin(36) \sin(2)$$

$$\Rightarrow \sqrt{2} (\cos(x) + \sin(x)) = \cos(36) \cos(2) + \sin(36) \sin(2)$$

$$\Rightarrow \sqrt{2} (\cos(x) + \sin(x)) = \cos(36 - 2)$$

$$\Rightarrow \cos(x) + \sin(x) = \frac{\cos(36 - 2)}{\sqrt{2}}$$

$$\Rightarrow \cos(x) = \frac{\cos(36 - 2) - \sin(x)}{\sqrt{2}}$$

$$\Rightarrow x = \arctan(\sqrt{2} \cos(36 - 2) - \sin(x)) \pm 180n$$

$\therefore \arctan(\sqrt{2} \cos(36 - 2) - \sin(x)) = 37.5^\circ$

**Method B - By constructing the auxiliary line**

**THE CORRESPONDING ANGLE OF THE AUXILIARY TRIANGLE**

$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$        $\tan(2A) = \frac{2 \tan A}{1 - \tan^2 A}$

•  $\tan 75^\circ = \tan(45^\circ + 30^\circ) = \frac{\tan 45^\circ + \tan 30^\circ}{1 - \tan 45^\circ \tan 30^\circ} = \frac{1 + \frac{1}{\sqrt{3}}}{1 - \frac{1}{\sqrt{3}}}$

$$= \frac{\sqrt{3} + 1}{\sqrt{3} - 1} = \frac{(\sqrt{3} + 1)^2}{(\sqrt{3} - 1)(\sqrt{3} + 1)} = \frac{3 + 2\sqrt{3} + 1}{3 - 1} = \frac{4 + 2\sqrt{3}}{2} = 2 + \sqrt{3}$$

•  $\tan 75^\circ = \frac{2 \tan 37.5^\circ}{1 - \tan^2 37.5^\circ}$

$$\Rightarrow 2 + \sqrt{3} = \frac{2T}{1 - T^2}$$

$$\Rightarrow 1 - T^2 = \frac{2T}{2 + \sqrt{3}}$$

$$\Rightarrow T^2 - 1 = -\frac{2T}{2 + \sqrt{3}}$$

$$\Rightarrow T^2 + \frac{2T}{2 + \sqrt{3}} - 1 = 0$$

$$\Rightarrow \left(T + \frac{1}{2 + \sqrt{3}}\right)^2 - \left(\frac{1}{2 + \sqrt{3}}\right)^2 - 1 = 0$$

$$\Rightarrow (T + 2 - \sqrt{3})^2 - (2 - \sqrt{3})^2 - 1 = 0$$

$\frac{1}{2 + \sqrt{3}} = \frac{2 - \sqrt{3}}{(2 + \sqrt{3})(2 - \sqrt{3})} = \frac{2 - \sqrt{3}}{4 - 3} = 2 - \sqrt{3}$

$\Rightarrow (T + 2 - \sqrt{3})^2 = (2 - \sqrt{3})^2 + 1$

$$\Rightarrow (T + 2 - \sqrt{3})^2 = 4 - 4\sqrt{3} + 3 + 1$$

$$\Rightarrow (T + 2 - \sqrt{3})^2 = 8 - 4\sqrt{3}$$

$$\Rightarrow (T + 2 - \sqrt{3})^2 = 2(4 - 2\sqrt{3})$$

$$\Rightarrow (T + 2 - \sqrt{3})^2 = 2(1^2 - 2 \times 1 \times \sqrt{3} + (\sqrt{3})^2)$$

$$\Rightarrow (T + 2 - \sqrt{3})^2 = 2(1 - \sqrt{3})^2$$

$$\Rightarrow T + 2 - \sqrt{3} = \sqrt{2(1 - \sqrt{3})^2} = \sqrt{2} (1 - \sqrt{3})$$

$$\Rightarrow T = \sqrt{2} + \sqrt{3} - \sqrt{2} - 2 > 0$$

$$\Rightarrow T = \sqrt{2} + \sqrt{3} - \sqrt{2} - 2 < 0$$

$\therefore \tan 37.5^\circ = \sqrt{2} + \sqrt{3} - \sqrt{2} - 2$

$$\Rightarrow \arctan(\sqrt{2} + \sqrt{3} - \sqrt{2} - 2) = 37.5^\circ$$

**UNIQUE TO METHOD B**

• Use  $\tan 45^\circ = \frac{2 \tan 22.5^\circ}{1 - \tan^2 22.5^\circ}$  to find  $\tan 22.5^\circ = \sqrt{2} - 1$

• Use  $\tan 30^\circ = \frac{2 \tan 15^\circ}{1 - \tan^2 15^\circ}$  to find  $\tan 15^\circ = 2 - \sqrt{3}$

$\therefore \tan 37.5^\circ = \tan(15^\circ + 22.5^\circ)$  & use compound

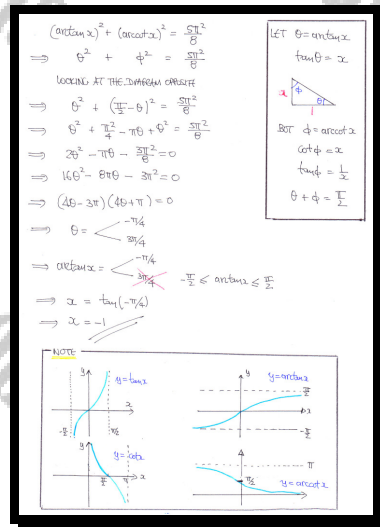
## Question 115 (\*\*\*\*)

Use a trigonometric algebra to solve the following equation

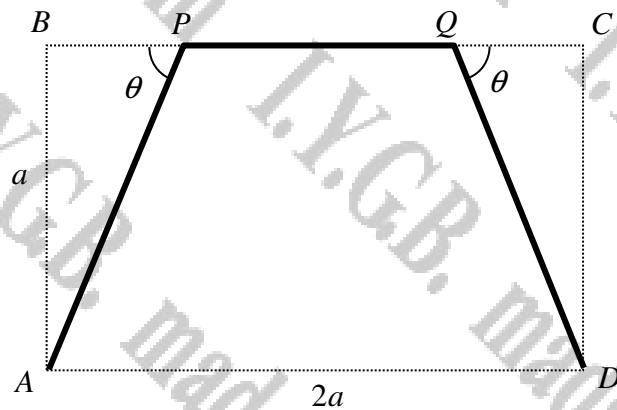
$$(\arctan x)^2 + (\operatorname{arccot} x)^2 = \frac{5\pi^2}{8}.$$

You may assume that  $y = \operatorname{arccot} x$  is the inverse function of  $y = \cot x$ ,  $0 \leq x \leq \pi$ 

$$\boxed{\phantom{000}}, \quad \boxed{x = -1}$$



## Question 116 (\*\*\*\*)



The figure above shows a network  $APQD$  inside a rectangle  $ABCD$ , where  $|AB| = a$  and  $|BC| = 2a$ . The endpoints of the network  $A$  and  $D$  are fixed. The points  $P$  and  $Q$  are variable so that they always lie on  $BC$  with  $|AP| = |QD|$ . The angles  $BPA$  and  $CQD$  are both equal to  $\theta$ . A particle travels with constant speed  $v$  on the sections  $AP$  and  $QD$ , and with constant speed  $\frac{5}{3}v$  on the section  $PQ$ .

Let  $T$  be the total time for the journey  $APQD$ .

Given that the positions of the points  $P$  and  $Q$  can be varied as appropriate, show that the minimum value of  $T$  is  $\frac{14a}{5v}$ , fully justifying that this is the minimum value.

,  proof

**Diagram and Variables:**

**Scale Triangle:**

$$\frac{a}{d_1} = \sin \theta \Rightarrow d_1 = \frac{a}{\sin \theta}$$

$$\frac{a}{d_2} = \cos \theta \Rightarrow d_2 = \frac{a}{\cos \theta}$$

$$\frac{a}{d_3} = \sin \theta \Rightarrow d_3 = \frac{a}{\sin \theta}$$

**Now we can express  $d_3$  also in terms of  $a$  &  $\theta$ :**

$$d_3 = 2a - d_1 = 2a - \frac{a}{\sin \theta} = 2a(1 - \frac{1}{\sin \theta})$$

**Steps:**  $\frac{\text{Distance}}{\text{Time}} \Leftrightarrow \text{Time} = \frac{\text{Distance}}{\text{Speed}}$

**Time for AP and QD:**

$$t_1 = \frac{d_1}{v} = \frac{a}{v \sin \theta}$$

$$t_3 = \frac{d_3}{v} = \frac{2a(1 - \frac{1}{\sin \theta})}{v}$$

**Time for PQ:**

$$t_2 = \frac{d_2}{\frac{5}{3}v} = \frac{3a}{5v \cos \theta}$$

**Total Time  $T = t_1 + t_2 + t_3$ :**

$$T = \frac{a}{v \sin \theta} + \frac{3a}{5v \cos \theta} + \frac{2a(1 - \frac{1}{\sin \theta})}{v}$$

$$T = \frac{a}{v} \left[ \frac{1}{\sin \theta} + \frac{3}{5 \cos \theta} + 2 - \frac{2}{\sin \theta} \right]$$

$$T = \frac{a}{v} \left[ 2 - \frac{1}{\sin \theta} + \frac{3}{5 \cos \theta} \right]$$

**Differentiate and set to zero:**

$$\frac{dT}{d\theta} = \frac{a}{v} \left[ \frac{1}{\sin^2 \theta} - \frac{3}{5 \cos^2 \theta} \right]$$

$$0 = \frac{a}{v} \left[ \frac{1}{\sin^2 \theta} - \frac{3}{5 \cos^2 \theta} \right]$$

$$0 = \frac{a}{v} \left[ \frac{5 \cos^2 \theta - 3 \sin^2 \theta}{\sin^2 \theta \cos^2 \theta} \right]$$

$$0 = 5 \cos^2 \theta - 3 \sin^2 \theta$$

**Check the nature of this stationary value:**

$$\frac{d^2T}{d\theta^2} = \frac{2a}{v} \left[ \frac{2 \cos \theta}{\sin^3 \theta} + \frac{6 \sin \theta}{5 \cos^3 \theta} \right]$$

**Now:**

At  $\theta = \frac{\pi}{4}$ :  $\sin \theta = \frac{\sqrt{2}}{2}$ ,  $\cos \theta = \frac{\sqrt{2}}{2}$

$$\frac{d^2T}{d\theta^2} = \frac{2a}{v} \left[ \frac{2 \times \frac{\sqrt{2}}{2}}{(\frac{\sqrt{2}}{2})^3} + \frac{6 \times \frac{\sqrt{2}}{2}}{5 (\frac{\sqrt{2}}{2})^3} \right]$$

$$= \frac{2a}{v} \left[ \frac{4 \sqrt{2}}{2 \sqrt{2}} + \frac{6 \sqrt{2}}{5 \sqrt{2}} \right] = \frac{2a}{v} \left[ 2 + \frac{6}{5} \right] = \frac{2a}{v} \left[ \frac{16}{5} \right] > 0$$

$\therefore A$  (LOCAL) MINIMUM

**Final Answer:**

$$T_{\min} = \frac{2a}{v} \left[ 2 - \frac{1}{\sin \frac{\pi}{4}} + \frac{3}{5 \cos \frac{\pi}{4}} \right]$$

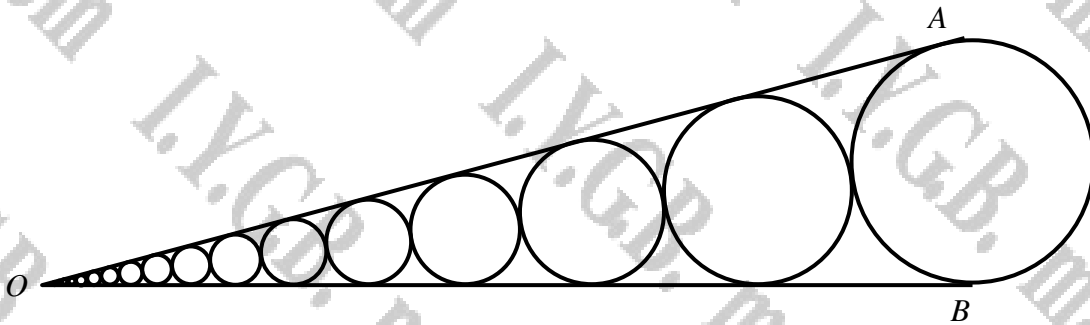
$$= \frac{2a}{v} \left[ 2 - \frac{1}{\frac{\sqrt{2}}{2}} + \frac{3}{5 \times \frac{\sqrt{2}}{2}} \right]$$

$$= \frac{2a}{v} \left[ 2 - \frac{2}{\sqrt{2}} + \frac{3}{5 \sqrt{2}} \right]$$

$$= \frac{14a}{5v}$$

As required

## Question 117 (\*\*\*\*)



The figure above shows a infinite sequence of circles of decreasing radius, the radius of the larger circle being  $\frac{4}{3}$ .

The centres of these circles lie on a straight line. The straight lines  $OA$  and  $OB$  are tangents to every circle in the sequence, the angle  $AOB$  denoted by  $2\theta$ .

Given that the total area of these circles is  $2\pi$ , determine the value of  $\theta$ .

$$\theta = \frac{1}{6}\pi$$

• START WITH A GOOD DIAGRAM

• LOOKING AT THE FIGURE ABOVE WE GET

$$\sin \theta = \frac{r_n}{2 + r_n} \quad \text{AND} \quad \sin \theta = \frac{r_n}{2 + r_n + r_{n+1}}$$

• ELIMINATE  $\sin \theta$

$$2 = \frac{r_n}{\sin \theta} \Rightarrow \sin \theta = \frac{r_n}{2 + r_n}$$

$$\Rightarrow \frac{r_n}{2 + r_n} = \frac{r_{n+1}}{2 + r_n + r_{n+1}}$$

$$\Rightarrow \frac{r_n}{r_{n+1}} = \frac{2 + r_n}{2 + r_n + r_{n+1}}$$

$$\Rightarrow \frac{r_n}{r_{n+1}} = \frac{2 + r_n}{2 + r_n + r_{n+1}}$$

• LET  $R_n$  DENOTE  $\frac{r_n}{r_{n+1}}$

$$\Rightarrow R + \sin \theta + R \sin \theta = 1$$

$$\Rightarrow R + R \sin \theta = 1 - \sin \theta$$

$$\Rightarrow R(1 + \sin \theta) = 1 - \sin \theta$$

$$\Rightarrow R = \frac{1 - \sin \theta}{1 + \sin \theta}$$

$$\Rightarrow \frac{r_n}{r_{n+1}} = \frac{1 - \sin \theta}{1 + \sin \theta} = \text{CONSTANT}$$

• A GEOMETRIC PROGRESSION WHICH CONVERGES

• AS THE RADIUS GOES TO 0, THE CIRCLES APPROXIMATE TO A POINT

SO WE CAN USE THE FORMULA FOR THE SUM OF AN INFINITE GEOMETRIC PROGRESSION

$$\Rightarrow \text{AREA} = \pi r_1^2 + \pi r_2^2 + \pi r_3^2 + \dots$$

$$\Rightarrow \text{AREA} = \pi [r_1^2 + (Rr_1)^2 + (R^2r_1)^2 + (R^3r_1)^2 + \dots]$$

$$\Rightarrow \text{AREA} = \pi r_1^2 [1 + R^2 + R^4 + R^6 + \dots]$$

$$\Rightarrow \text{AREA} = \pi r_1^2 \left[ \frac{1}{1 - R^2} \right]$$

$$\Rightarrow 2\pi = \pi \left( \frac{4}{3} \right)^2 \left[ \frac{1}{1 - R^2} \right]$$

$$\Rightarrow 2 = \frac{16}{9} \left[ \frac{1}{1 - R^2} \right]$$

$$\Rightarrow 9 = 8 \left[ \frac{1}{1 - R^2} \right]$$

$$\Rightarrow 9(1 - R^2) = 8$$

$$\Rightarrow 9 \left( 1 - \left( \frac{1 - \sin \theta}{1 + \sin \theta} \right)^2 \right) = 8$$

$$\Rightarrow 9 \left[ \frac{(1 + \sin \theta)^2 - (1 - \sin \theta)^2}{(1 + \sin \theta)^2} \right] = 8$$

$$\Rightarrow 9 \left[ \frac{4 \sin \theta}{(1 + \sin \theta)^2} \right] = 8$$

$$\Rightarrow \frac{9 \sin \theta}{(1 + \sin \theta)^2} = \frac{8}{9}$$

$$\Rightarrow 81 \sin \theta = 8(1 + \sin \theta)^2$$

$$\Rightarrow 81 \sin \theta = 8(1 + 2 \sin \theta + \sin^2 \theta)$$

$$\Rightarrow 81 \sin \theta = 8 + 16 \sin \theta + 8 \sin^2 \theta$$

$$\Rightarrow 73 \sin \theta - 8 \sin^2 \theta = 8$$

$$\Rightarrow 8 \sin^2 \theta - 73 \sin \theta + 8 = 0$$

$$\Rightarrow \sin \theta = \frac{73 \pm \sqrt{73^2 - 4 \cdot 8 \cdot 8}}{2 \cdot 8}$$

$$\Rightarrow \sin \theta = \frac{73 \pm \sqrt{5329 - 256}}{16}$$

$$\Rightarrow \sin \theta = \frac{73 \pm \sqrt{5073}}{16}$$

$$\Rightarrow \sin \theta = \frac{73 \pm 71.3}{16}$$

$$\Rightarrow \sin \theta = \frac{2.7}{16} \text{ OR } \frac{144.3}{16}$$

$$\Rightarrow \sin \theta = \frac{2.7}{16}$$

$$\Rightarrow \theta = \arcsin \left( \frac{2.7}{16} \right)$$

$$\Rightarrow \theta = \frac{\pi}{6}$$



## Question 118 (\*\*\*\*)

$$\theta = \arctan \left[ \frac{\cos 66^\circ - \sin 48^\circ}{\cos 48^\circ - \sin 66^\circ} \right]$$

Show by detailed working that  $\theta = -12^\circ$ .

, proof

**METHOD 1**

$$\arctan \left[ \frac{\cos 66^\circ - \sin 48^\circ}{\cos 48^\circ - \sin 66^\circ} \right] = \arctan \left[ \frac{\sin(90^\circ - 66^\circ) - \sin 48^\circ}{\sin(90^\circ - 48^\circ) + \sin 66^\circ} \right]$$

$$= \arctan \left[ \frac{\sin 24^\circ - \sin 48^\circ}{\sin 42^\circ + \sin 66^\circ} \right] = \dots$$

NEW FROM THE COMPOUND ANGLE IDENTITIES

$$\sin(A+B) = \sin A \cos B + \cos A \sin B$$

$$\sin(A-B) = \sin A \cos B - \cos A \sin B$$

ADDING & SUBTRACTING

$$\sin(A+B) + \sin(A-B) = 2 \sin A \cos B$$

$$\sin(A+B) - \sin(A-B) = 2 \cos A \sin B$$

LETTING  $P=A+B$  &  $Q=A-B$

$$\sin P + \sin Q = 2 \sin \frac{P+Q}{2} \cos \frac{P-Q}{2}$$

$$\sin P - \sin Q = 2 \cos \frac{P+Q}{2} \sin \frac{P-Q}{2}$$

RETURNING TO THE "MAIN LINE"

$$\dots \arctan \left[ \frac{2 \cos \frac{24^\circ + 42^\circ}{2} \sin \frac{24^\circ - 42^\circ}{2}}{2 \cos \frac{42^\circ + 66^\circ}{2} \cos \frac{42^\circ - 66^\circ}{2}} \right] = \arctan \left[ \frac{\cos 30^\circ \sin(-12^\circ)}{\cos 54^\circ \cos(-12^\circ)} \right]$$

Now  $\sin(-A) = -\sin A$  &  $\cos(-A) \equiv \cos A$

**METHOD 2**

$$\text{LET } \theta = \arctan \left[ \frac{\cos 66^\circ - \sin 48^\circ}{\cos 48^\circ - \sin 66^\circ} \right]; -90^\circ \leq \theta \leq 90^\circ$$

$$\Rightarrow \tan \theta = \frac{\cos 66^\circ - \sin 48^\circ}{\cos 48^\circ - \sin 66^\circ}$$

$$\Rightarrow \frac{\sin \theta}{\cos \theta} = \frac{\cos 66^\circ - \sin 48^\circ}{\cos 48^\circ - \sin 66^\circ}$$

$$\Rightarrow \sin \theta \cos 48^\circ + \sin \theta \sin 66^\circ = \cos \theta \cos 66^\circ - \cos \theta \sin 48^\circ$$

$$\Rightarrow \sin \theta \cos 48^\circ + \cos \theta \sin 48^\circ = \cos \theta \cos 66^\circ - \sin \theta \sin 66^\circ$$

$$\Rightarrow \sin(\theta + 48^\circ) = \cos(\theta + 66^\circ)$$

$$\Rightarrow \cos[90^\circ - (\theta + 48^\circ)] = \cos(\theta + 66^\circ)$$

$\sin A \equiv \cos(90^\circ - A)$

$\Rightarrow \cos(42^\circ - \theta) = \cos(\theta + 66^\circ)$

$\cos A \equiv \cos A$  OR  $\cos(A) \equiv \cos A$

$$\Rightarrow \cos(\theta - 42^\circ) = \cos(\theta + 66^\circ)$$

SCALING THE EQUATION IN RADIANS FORM

$$\Rightarrow \left( \theta - 42^\circ = \theta + 66^\circ \pm 360^\circ \right.$$

$$\left. \theta - 42^\circ = -\theta - 66^\circ \pm 360^\circ \right)$$

THE FIRST EQUATION YIELDS NOTHING, BUT THE SECOND EQUATION CAN PROVIDE A VALUABLE SOLUTION

$$\Rightarrow 2\theta = -24^\circ \pm 360^\circ$$

$$\Rightarrow \theta = -12^\circ \pm 180^\circ$$

$$\Rightarrow \arctan \left[ \frac{\cos 66^\circ - \sin 48^\circ}{\cos 48^\circ - \sin 66^\circ} \right] = \dots$$

ONLY ONE IN RANGE  $-90^\circ \leq \theta \leq 90^\circ$

## Question 119 (\*\*\*\*)

$$f(x) = \arctan(3x) + \arcsin\left(\frac{1}{\sqrt{9x^2+1}}\right), \quad x \in \mathbb{R}.$$

Show clearly that ...

a) ...  $f'(x) = 0$ .

b) ...  $\arctan(3x) + \arcsin\left(\frac{1}{\sqrt{9x^2+1}}\right) \equiv k\pi$ , stating the value of the constant  $k$ .

$$k = \frac{1}{2}$$

(a) Let  $f(x) = \arctan(3x) + \arcsin\left(\frac{1}{\sqrt{9x^2+1}}\right)$   
 $f'(x) = \frac{3}{1+9x^2} + \frac{1}{\sqrt{1-(\frac{1}{\sqrt{9x^2+1}})^2}} \times \left(-\frac{1}{2}\right) \left(\frac{1}{\sqrt{9x^2+1}}\right)^{\frac{3}{2}}$   
 $f'(x) = \frac{3}{1+9x^2} - \frac{9x}{\sqrt{1-\frac{1}{9x^2+1}}} \times \frac{1}{(9x^2+1)^{\frac{3}{2}}}$   
 $f'(x) = \frac{3}{1+9x^2} - \frac{9x}{\sqrt{\frac{9x^2+1-1}{9x^2+1}}} \times \frac{1}{(9x^2+1)^{\frac{3}{2}}}$   
 $f'(x) = \frac{3}{1+9x^2} - \frac{9x}{\sqrt{\frac{9x^2}{9x^2+1}}} \times \frac{1}{(9x^2+1)^{\frac{3}{2}}}$   
 $f'(x) = \frac{3}{1+9x^2} - \frac{9x}{3x\sqrt{9x^2+1}} \times \frac{1}{(9x^2+1)^{\frac{3}{2}}}$   
 $f'(x) = \frac{3}{1+9x^2} - \frac{3}{9x^2+1}$   
 $f'(x) = 0$

(b)  $f(x) = \text{constant}$   
 $f(0) = \arctan(0) + \arcsin(1) = \frac{\pi}{2}$   
 $\therefore \arctan(3x) + \arcsin\left(\frac{1}{\sqrt{9x^2+1}}\right) = \frac{\pi}{2}$

## Question 120 (\*\*\*\*)

It is given that

$$\cot x - 2 \cot 2x \equiv \tan x.$$

a) Prove the validity of the above trigonometric identity.

b) Hence, or otherwise, show that

$$\sum_{r=1}^{10} \frac{1}{2^{r-1}} \tan\left(\frac{x}{2^r}\right) = \frac{1}{512} \cot\left(\frac{x}{1024}\right) - 2 \cot x.$$

□, proof

a) STARTING WITH THE LEFT HAND SIDE

$$\begin{aligned} \text{L.H.S.} &= \cot x - 2 \cot 2x = \cot x - \frac{2}{\tan 2x} = \cot x - \frac{2}{\frac{2 \tan x}{1 - \tan^2 x}} \\ &= \cot x - \frac{2(1 - \tan^2 x)}{2 \tan x} = \frac{\cot x \tan x - (1 - \tan^2 x)}{\tan x} \\ &= \frac{1 - 1 + \tan^2 x}{\tan x} = \tan x = \text{R.H.S.} \end{aligned}$$

b) NOW USING THE IDENTITY WE OBTAIN

$$\begin{aligned} \cot x - 2 \cot 2x &= \tan x \\ \cot \frac{x}{2} - 2 \cot x &= \tan \frac{x}{2} \\ \cot \frac{x}{4} - 2 \cot \frac{x}{2} &= \tan \frac{x}{4} \\ \vdots &\vdots \\ \cot \frac{x}{2^{10}} - 2 \cot \frac{x}{2^{9}} &= \tan \frac{x}{2^{10}} \end{aligned}$$

ADDING THE GET

$$\frac{1}{2^{10}} \cot \frac{x}{2^{10}} - 2 \cot x = \sum_{r=1}^{10} \left[ \frac{1}{2^{r-1}} \tan \left( \frac{x}{2^r} \right) \right]$$

WHEN WE REARRANGE IF  $n=10$

$$\begin{aligned} \sum_{r=1}^{10} \left[ \frac{1}{2^{r-1}} \tan \left( \frac{x}{2^r} \right) \right] &= \frac{1}{2^{10}} \cot \left( \frac{x}{2^{10}} \right) - 2 \cot x \\ &= \frac{1}{512} \cot \left( \frac{x}{1024} \right) - 2 \cot x \end{aligned}$$

As required

## Question 121 (\*\*\*\*)

Given the trigonometric equation

$$\frac{\sin(x - \alpha)}{\cos(x - \alpha) - 2 \tan \alpha \sin(x - \alpha)} = \tan \alpha,$$

show clearly that

$$\tan x = 2 \tan \alpha.$$

,  proof

Handwritten solution for Question 121:

$$\frac{\sin(x - \alpha)}{\cos(x - \alpha) - 2 \tan \alpha \sin(x - \alpha)} = \tan \alpha$$

Divide through by  $\cos(x - \alpha)$

$$\Rightarrow \tan(x - \alpha) = \tan \alpha \cos(x - \alpha) - 2 \tan \alpha \sin(x - \alpha)$$

$$\Rightarrow \tan(x - \alpha) = \tan \alpha - 2 \tan \alpha \tan(x - \alpha)$$

$$\Rightarrow \tan(x - \alpha) + 2 \tan \alpha \tan(x - \alpha) = \tan \alpha$$

$$\Rightarrow \tan(x - \alpha) [1 + 2 \tan \alpha] = \tan \alpha$$

$$\Rightarrow \frac{\tan(x - \alpha)}{1 + 2 \tan \alpha} = \frac{\tan \alpha}{1 + 2 \tan \alpha}$$

$$\Rightarrow (\tan x - \tan \alpha)(1 + 2 \tan \alpha) = \tan \alpha (1 + 2 \tan \alpha)$$

$$\Rightarrow \tan x + 2 \tan \alpha \tan x - \tan \alpha - 2 \tan^2 \alpha = \tan \alpha + 2 \tan^2 \alpha$$

$$\Rightarrow \tan \alpha \tan x - 2 \tan^2 \alpha + \tan x - 2 \tan \alpha = 0$$

$$\Rightarrow \tan \alpha (\tan x - 2 \tan \alpha) + (\tan x - 2 \tan \alpha) = 0$$

$$\Rightarrow (\tan x - 2 \tan \alpha)(\tan \alpha + 1) = 0$$

$$\Rightarrow \tan x - 2 \tan \alpha = 0 \quad \boxed{\tan \alpha + 1 \neq 0}$$

$$\Rightarrow \tan x = 2 \tan \alpha$$

As required

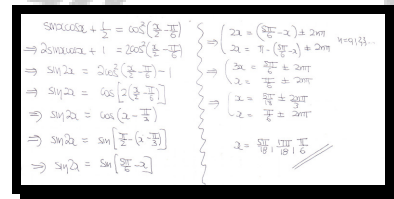
## Question 122 (\*\*\*\*)

Solve the trigonometric equation

$$\sin x \cos x + \frac{1}{2} = \cos^2 \left( \frac{x}{2} - \frac{\pi}{6} \right), \quad 0 \leq x < \pi,$$

giving the answers in terms of  $\pi$ .

$$\boxed{\phantom{000}}, \quad x = \frac{\pi}{6}, \frac{5\pi}{18}, \frac{17\pi}{18}$$



Handwritten solution showing the steps to solve the equation:

$$\begin{aligned} \sin x \cos x + \frac{1}{2} &= \cos^2 \left( \frac{x}{2} - \frac{\pi}{6} \right) \\ \Rightarrow 2 \sin x \cos x + 1 &= 2 \cos^2 \left( \frac{x}{2} - \frac{\pi}{6} \right) - 1 \\ \Rightarrow \sin 2x &= 2 \cos^2 \left( \frac{x}{2} - \frac{\pi}{6} \right) - 1 \\ \Rightarrow \sin 2x &= \cos \left[ 2 \left( \frac{x}{2} - \frac{\pi}{6} \right) \right] \\ \Rightarrow \sin 2x &= \cos \left( x - \frac{\pi}{3} \right) \\ \Rightarrow \sin 2x &= \sin \left[ \frac{\pi}{2} - \left( x - \frac{\pi}{3} \right) \right] \\ \Rightarrow \sin 2x &= \sin \left[ \frac{5\pi}{6} - x \right] \end{aligned}$$

Then solving for  $2x$ :

$$\begin{aligned} 2x &= \left( \frac{5\pi}{6} - x \right) + 2n\pi \\ 2x &= \frac{5\pi}{6} - x + 2n\pi \\ 3x &= \frac{5\pi}{6} + 2n\pi \\ x &= \frac{5\pi}{18} + \frac{2n\pi}{3} \end{aligned}$$

For  $n=0$ ,  $x = \frac{5\pi}{18}$ .  
For  $n=1$ ,  $x = \frac{5\pi}{18} + \frac{2\pi}{3} = \frac{17\pi}{18}$ .  
For  $n=-1$ ,  $x = \frac{5\pi}{18} - \frac{2\pi}{3} = -\frac{7\pi}{18}$  (not in the domain).

Also,  $\sin 2x = \cos \left( x - \frac{\pi}{3} \right) = \sin \left( \frac{\pi}{2} - \left( x - \frac{\pi}{3} \right) \right) = \sin \left( \frac{5\pi}{6} - x \right)$ .  
So  $2x = \frac{5\pi}{6} - x + 2n\pi$  or  $2x = \frac{\pi}{2} - \left( \frac{5\pi}{6} - x \right) + 2n\pi$ .  
Solving  $2x = \frac{\pi}{2} - \left( \frac{5\pi}{6} - x \right) + 2n\pi$  gives  $x = \frac{\pi}{6} + 2n\pi$ .  
For  $n=0$ ,  $x = \frac{\pi}{6}$ .

Final solutions:  $x = \frac{\pi}{6}, \frac{5\pi}{18}, \frac{17\pi}{18}$ .

## Question 123 (\*\*\*\*)

Prove the validity of the following trigonometric identities.

a)  $\cos^4 \theta + \sin^4 \theta \equiv \frac{1}{4}(3 + 4\cos 4\theta).$

b)  $32\sin^2 x \cos^4 x \equiv 2 + \cos 2x - 2\cos 4x - \cos 6x.$

proof

$$\begin{aligned} \text{LHS} &= \frac{1}{4}(3 + 4\cos 4\theta) = \frac{1}{4}(3 + 2\cos 2\theta - 1) \\ &= \frac{1}{4}(2 + 2\cos 2\theta) = \frac{1}{2}(1 + \cos 2\theta) \\ &= \frac{1}{2}(1 + (2\cos^2 \theta - 1)) = \frac{1}{2}[1 + 4\cos^2 \theta - 4\cos^2 \theta + 1] \\ &= \frac{1}{2}[4\cos^2 \theta - 4\cos^2 \theta + 2] = 2\cos^2 \theta - 2\cos^2 \theta + 1 \\ &= \cos^2 \theta + (\cos^2 \theta - 2\cos^2 \theta + 1) = \cos^2 \theta + (1 - \cos^2 \theta)^2 \\ &= \cos^2 \theta + (\sin^2 \theta)^2 = \cos^2 \theta + \sin^4 \theta = \text{RHS} \end{aligned}$$

$$\begin{aligned} \text{LHS} &= \cos^2 \theta + \sin^4 \theta = (\cos^2 \theta + 2\sin^2 \theta \cos^2 \theta + \sin^4 \theta) - 2\sin^2 \theta \cos^2 \theta \\ &= (\cos^2 \theta + \sin^2 \theta)^2 - 2\sin^2 \theta \cos^2 \theta = 1^2 - \frac{1}{2}(2\sin^2 \theta \cos^2 \theta)^2 \\ &= 1 - \frac{1}{2}\sin^2 2\theta \\ &= 1 - \frac{1}{2}\left(\frac{1}{2} - \frac{1}{2}\cos 4\theta\right) \\ &= 1 - \frac{1}{4} + \frac{1}{4}\cos 4\theta \\ &= \frac{3}{4} + \frac{1}{4}\cos 4\theta = \text{RHS} \end{aligned}$$

$$\begin{aligned} \text{LHS} &= 32\sin^2 x \cos^4 x = 8(4\sin^2 x \cos^4 x) = 8(2\sin x \cos x)^2 \cos^2 x \\ &= 8(\sin 2x)^2 \cos^2 x = 8\sin^2 2x \cos^2 x \\ &= 8 \times \left(\frac{1}{2} - \frac{1}{2}\cos 4x\right) \left(\frac{1}{2} + \frac{1}{2}\cos 2x\right) \\ &= 2(1 - \cos 4x)(1 + \cos 2x) \\ &= 2 + 2\cos 2x - 2\cos 4x - 2\cos 2x \cos 4x \\ &= 2 + 2\cos 2x - 2\cos 4x - (\cos 6x + \cos 2x) \\ &= 2 + \cos 2x - 2\cos 4x - \cos 6x \\ &= \text{RHS} \end{aligned}$$

## Question 124 (\*\*\*\*)

Show clearly that

$$4 \operatorname{arccot} 2 + \arctan\left(\frac{24}{7}\right) = \pi.$$

□, proof

$4 \operatorname{arccot} 2 + \arctan \frac{24}{7} = \psi$   
 let  $\psi = 4\theta + \phi = \psi$   
 $\Rightarrow \cos \psi = \cos(4\theta + \phi)$   
 $\Rightarrow \cos \psi = \cos 4\theta \cos \phi - \sin 4\theta \sin \phi$   
 $\Rightarrow \cos \psi = (2\cos 2\theta - 1)\cos \phi - (2\sin 2\theta \cos 2\theta)\sin \phi$   
 $\Rightarrow \cos \psi = [2(2\cos^2 \theta - 1) - 1]\cos \phi - [4\cos^2 \theta \sin 2\theta - \sin 2\theta]\sin \phi$   
 $\Rightarrow \cos \psi = -\frac{24}{25} - \frac{52}{25}\sin \phi$   
 $\Rightarrow \cos \psi = -1$   
 $\psi = \dots, \pi, 3\pi, \dots$   
 But  $\theta$  &  $\phi$  are acute  $\Rightarrow 0 < 4\theta + \phi < 5\pi/2$   
 $\therefore \psi = \pi$   
 $\therefore 4 \operatorname{arccot} 2 + \arctan \frac{24}{7} = \pi$

Alternatively by complex numbers  
 $4 \operatorname{arccot} 2 + \arctan \frac{24}{7} = 4 \operatorname{arctan} \frac{1}{2} + \arctan \frac{24}{7}$   
 Consider  $(2+i)^4 (7+24i) = (4+4i-1)^2 (7+24i) = (3+4i)^2 (7+24i)$   
 $= (9+24i-16)(7+24i) = (-7+24i)(7+24i)$   
 $= -49 - 168i + 168i - 576 = -625$   
 Thus  $\arg[(2+i)^4 (7+24i)] = \arg(-625)$   
 $\arg(2+i)^4 + \arg(7+24i) = \pi$   
 $4 \arg(2+i) + \arg(7+24i) = \pi$   
 $4 \operatorname{arctan} \frac{1}{2} + \arctan \frac{24}{7} = \pi$   
 $4 \operatorname{arccot} 2 + \arctan \frac{24}{7} = \pi$

## Question 125 (\*\*\*\*)

It is given that if  $x \neq y$ ,  $x \neq 0$ ,  $y \neq 0$ ,

$$\tan(x+y) = 2 \tan(x-y).$$

Show clearly that

$$\frac{\sin 2x}{\sin 2y} = 3.$$

☐ , ☐ proof

Handwritten proof for Question 125:

$$\begin{aligned} \tan(x+y) &= 2 \tan(x-y) \\ \Rightarrow \frac{\tan x + \tan y}{1 - \tan x \tan y} &= \frac{2(\tan x - \tan y)}{1 + \tan x \tan y} \\ \Rightarrow (\tan x + \tan y)(1 + \tan x \tan y) &= 2(\tan x - \tan y)(1 - \tan x \tan y) \\ \Rightarrow \tan x + \tan x \tan y + \tan y + \tan x \tan y &= 2 \tan x - 2 \tan x \tan y - 2 \tan y + 2 \tan x \tan y \\ \Rightarrow 0 &= \tan x - 3 \tan y - 3 \tan x \tan y + \tan x \tan y \\ \Rightarrow \tan x + \tan x \tan y - 3 \tan y - 3 \tan x \tan y &= 0 \\ \Rightarrow \tan x(1 + \tan y) - 3 \tan y(1 + \tan x) &= 0 \\ \Rightarrow \tan x \sec y - 3 \tan y \sec x &= 0 \\ \Rightarrow \frac{\tan x \sec y}{\tan y \sec x} - 3 &= 0 \\ \Rightarrow \frac{\tan x}{\tan y} \times \frac{\sec y}{\sec x} &= 3 \\ \Rightarrow \frac{\frac{\sin x}{\cos x}}{\frac{\sin y}{\cos y}} \times \frac{\frac{1}{\cos y}}{\frac{1}{\cos x}} &= 3 \\ \Rightarrow \frac{\sin x \cos x}{\cos x \sin y} &= 3 \\ \Rightarrow \frac{\sin x \cos x}{\sin y \cos y} &= 3 \end{aligned}$$

Alternative derivation shown in a box:

$$\begin{aligned} \Rightarrow \frac{2 \tan x \sec x}{2 \tan y \sec y} &= 3 \\ \Rightarrow \frac{\sin 2x}{\sin 2y} &= 3 \end{aligned}$$

✓ QED



Question 126 (\*\*\*\*)

A triangle  $ABC$  is such so that  $\angle BAC = \frac{1}{6}\pi$  and  $|BC| = 1$ .

Show that the maximum value of the area of the triangle  $ABC$  is

$$\frac{1}{4}(2 + \sqrt{3}).$$

,  proof

STARTING WITH A DIAGRAM

LOOKING AT  $\triangle BDC$

- $h = |x \sin \theta| \Rightarrow \boxed{h = \sin \theta}$
- $y = |x \cos \theta| \Rightarrow \boxed{y = \cos \theta}$

LOOKING AT  $\triangle ABD$

- $\frac{h}{x} = \sin \frac{\pi}{6} \Rightarrow \frac{h}{x} = \frac{1}{2} \Rightarrow \boxed{x = 2h}$
- $\frac{y}{x} = \cos \frac{\pi}{6} \Rightarrow \frac{y}{x} = \frac{\sqrt{3}}{2} \Rightarrow \boxed{y = \sqrt{3}h}$

HENCE THE AREA OF THE TRIANGLE  $ABC$

- $\Rightarrow \text{Area} = \frac{1}{2} |AC| |BD|$
- $\Rightarrow \text{Area} = \frac{1}{2} (x+y) h$
- $\Rightarrow \text{Area} = \frac{1}{2} (\sqrt{3}h + h) \sin \theta$
- $\Rightarrow \text{Area} = \left( \frac{\sqrt{3}}{2} \sin \theta + \frac{1}{2} \sin \theta \right) \sin \theta$
- $\Rightarrow \text{Area} = \left( \cos \frac{\pi}{6} \sin \theta + \sin \frac{\pi}{6} \sin \theta \right) \sin \theta$
- $\Rightarrow \text{Area} = \sin \left( \theta + \frac{\pi}{6} \right) \sin \theta$

BY DIFFERENTIATION

- $f(\theta) = \sin \theta \sin \left( \theta + \frac{\pi}{6} \right)$
- $f'(\theta) = \cos \theta \sin \left( \theta + \frac{\pi}{6} \right) + \sin \theta \cos \left( \theta + \frac{\pi}{6} \right)$
- $f'(\theta) = \sin \left( 2\theta + \frac{\pi}{6} \right)$
- $f''(\theta) = \sin \left( 2\theta + \frac{\pi}{6} \right)$

SCALING FOR ZERO, ONLY LOOKING FOR  $\theta > 0$

- $2\theta + \frac{\pi}{6} = 0, \pi, 2\pi, \dots$
- $2\theta = -\frac{\pi}{6}, \frac{5\pi}{6}, \dots$
- $\theta = -\frac{\pi}{12}, \frac{5\pi}{12}, \dots$

HENCE  $\theta_{\text{max}}$

- $\theta_{\text{max}} = \frac{5\pi}{12}$
- $\Rightarrow \sin \frac{5\pi}{12} \sin \left( \frac{5\pi}{12} + \frac{\pi}{6} \right)$
- $= \sin \frac{5\pi}{12} \sin \frac{7\pi}{12}$
- $= \sin \frac{5\pi}{12} \sin \frac{\pi}{12}$
- $= \sin^2 \frac{\pi}{12}$
- $= \frac{1}{4} - \frac{1}{4} \cos \left( 2 \times \frac{\pi}{12} \right)$
- $= \frac{1}{4} - \frac{1}{4} \cos \frac{\pi}{6}$
- $= \frac{1}{4} - \frac{1}{4} \left( \frac{\sqrt{3}}{2} \right)$
- $= \frac{1}{4} + \frac{\sqrt{3}}{8}$
- $= \frac{1}{4} (2 + \sqrt{3})$

ALTERNATIVE WITHOUT DIFFERENTIATION

Area =  $\sin \theta \sin \left( \theta + \frac{\pi}{6} \right)$

LOOKING AT THE COMPOUND ANGLES IDENTITIES FOR  $\cos(A+B)$

- $\cos \left( \theta + \left( \theta + \frac{\pi}{6} \right) \right) = \cos \theta \cos \left( \theta + \frac{\pi}{6} \right) - \sin \theta \sin \left( \theta + \frac{\pi}{6} \right)$
- $\cos \left( 2\theta + \frac{\pi}{6} \right) = \cos \theta \cos \left( \theta + \frac{\pi}{6} \right) - \sin \theta \sin \left( \theta + \frac{\pi}{6} \right)$

REARRANGING

- $\Rightarrow \cos \left( 2\theta + \frac{\pi}{6} \right) = \cos \theta \cos \left( \theta + \frac{\pi}{6} \right) - \sin \theta \sin \left( \theta + \frac{\pi}{6} \right)$
- $\Rightarrow \frac{1}{2} \cos \left( 2\theta + \frac{\pi}{6} \right) = \frac{1}{2} \cos \theta \cos \left( \theta + \frac{\pi}{6} \right) - \frac{1}{2} \sin \theta \sin \left( \theta + \frac{\pi}{6} \right)$
- $\Rightarrow \text{Area} = \frac{1}{4} \cos \left( 2\theta + \frac{\pi}{6} \right)$
- $\Rightarrow \text{Area}_{\text{max}} = \frac{1}{4} \cos \left( 2\theta + \frac{\pi}{6} \right)$
- $\Rightarrow \text{Area}_{\text{max}} = \frac{1}{4} \cos \left( 2\theta + \frac{\pi}{6} \right)$
- $\Rightarrow \text{Area}_{\text{max}} = \frac{1}{4} (2 + \sqrt{3})$

## Question 127 (\*\*\*\*)

It is given that the angles  $A$ ,  $B$  and  $C$  are the three angles of a triangle  $ABC$  with  $B \neq 90^\circ$ .

Given further that

$$\sin A - \sin(B - C) = \frac{\cos(B - C)}{\tan B},$$

show that the triangle  $ABC$  is right angled.

 , proof

Handwritten solution for Question 127:

$$\begin{aligned} \sin A - \sin(B - C) &= \frac{\cos(B - C)}{\tan B} & A + B + C &= 180^\circ \\ \Rightarrow \tan B [\sin A - \sin(B - C)] &= \cos(B - C) \\ \sin P - \sin Q &= 2 \cos \frac{P+Q}{2} \sin \frac{P-Q}{2} \\ \Rightarrow \tan B \left[ 2 \cos \frac{A+B+C}{2} \sin \frac{A-(B-C)}{2} \right] &= \cos(B - C) \\ \Rightarrow \tan B \left[ 2 \cos \frac{A+B+C-2C}{2} \sin \frac{A+B+C-2B}{2} \right] &= \cos(B - C) \\ \Rightarrow 2 \tan B \left[ \cos \frac{180-2C}{2} \sin \frac{180-2B}{2} \right] &= \cos(B - C) \\ \Rightarrow 2 \tan B \cos(90-C) \sin(90-B) &= \cos(B - C) \\ \Rightarrow 2 \frac{\sin B}{\cos B} \times \sin C \times \cos B &= \cos(B - C) & \cos(90-C) &= \sin C \\ \Rightarrow 2 \sin B \sin C &= \cos(B - C) & \sin(90-B) &= \cos B \\ \Rightarrow 2 \sin B \sin C &= \cos B \cos C + \sin B \sin C \\ \Rightarrow 0 &= \cos B \cos C - \sin B \sin C \\ \Rightarrow \cos(B + C) &= 0 \\ \Rightarrow B + C &= 90^\circ \\ \therefore A &= 90^\circ \end{aligned}$$

THE TRIANGLE IS RIGHT ANGLED AT A

## Question 128 (\*\*\*\*)

Solve the following trigonometric equation

$$\arctan\left[x \cos\left(2 \arcsin \frac{1}{x}\right)\right] = \frac{1}{4}\pi.$$

$$\boxed{\phantom{00}}, \quad x = -1, \quad x = 2$$

$\arctan\left[x \cos\left(2 \arcsin \frac{1}{x}\right)\right] = \frac{\pi}{4}$

• TAKING TANGENT ON BOTH SIDES OF THE EQUATION

$$\Rightarrow x \cos\left(2 \arcsin \frac{1}{x}\right) = 1$$

$$\Rightarrow \cos\left(2 \arcsin \frac{1}{x}\right) = \frac{1}{x}$$

$$\Rightarrow 2 \arcsin \frac{1}{x} = \pm \arccos \frac{1}{x} + 2n\pi \quad n=0,1,2,\dots$$

$$\Rightarrow 2 \arcsin \frac{1}{x} = \pm \left(\frac{\pi}{2} - \arcsin \frac{1}{x}\right) + 2n\pi$$

$\uparrow$   
arcsin + arccos =  $\frac{\pi}{2}$

$$\Rightarrow 2 \arcsin \frac{1}{x} = \begin{cases} \frac{\pi}{2} - \arcsin \frac{1}{x} + 2n\pi \\ -\frac{\pi}{2} + \arcsin \frac{1}{x} + 2n\pi \end{cases}$$

• DEAL WITH EACH POSSIBILITY SEPARATELY

$$\Rightarrow \begin{cases} 2 \arcsin \frac{1}{x} = \frac{\pi}{2} + 2n\pi \\ \arcsin \frac{1}{x} = -\frac{\pi}{2} + 2n\pi \end{cases}$$

$$\Rightarrow \begin{cases} \arcsin \frac{1}{x} = \frac{\pi}{4} + n\pi \\ \arcsin \frac{1}{x} = -\frac{\pi}{2} + 2n\pi \end{cases}$$

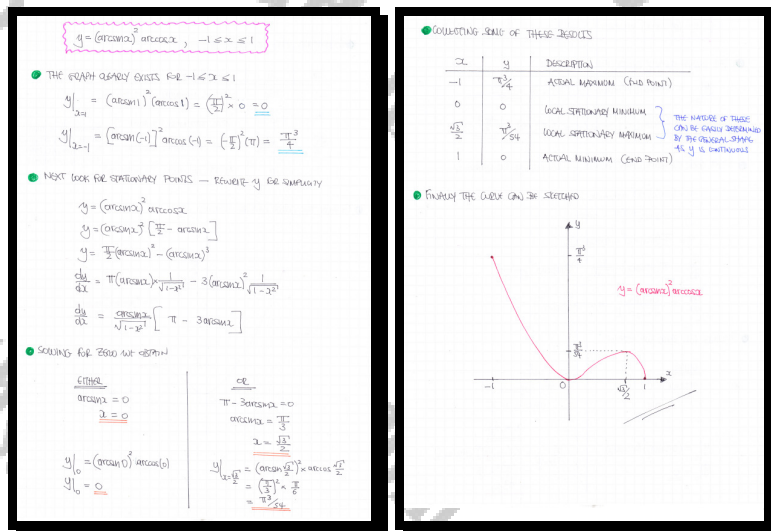
• BUT THE ARCSINE FUNCTION IS BOUNDED, i.e.  $-\frac{\pi}{2} \leq \arcsin \frac{1}{x} \leq \frac{\pi}{2}$   
therefore we obtain

$\arcsin \frac{1}{x} = \frac{\pi}{4}$ $\frac{1}{x} = \sin \frac{\pi}{4}$ $\frac{1}{x} = \frac{1}{2}$ $x = 2$	or	$\arcsin \frac{1}{x} = -\frac{\pi}{2}$ $\frac{1}{x} = \sin\left(-\frac{\pi}{2}\right)$ $\frac{1}{x} = -1$ $x = -1$
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## Question 129 (\*\*\*\*)

Sketch the graph of

$$y = (\arcsin x)^2 \arccos x, \quad -1 \leq x \leq 1$$

 , graph


## Question 130 (\*\*\*\*)

Solve the following trigonometric equation

$$\sin[\operatorname{arccot}(x+1)] = \cos(\arctan x).$$

You may assume that  $y = \operatorname{arccot} x$  is the inverse function for  $y = \cot x$ ,  $0 \leq x \leq \pi$ .

$$\boxed{\phantom{0}}, \quad x = -\frac{1}{2}$$

$\sin(\operatorname{arccot}(x+1)) = \cos(\arctan(x+1))$   
 • USING THE IDENTITY  $\cos A = \sin(\frac{\pi}{2} - A)$   
 $\Rightarrow \sin(\operatorname{arccot}(x+1)) = \sin(\frac{\pi}{2} - \arctan(x+1))$   
 • NOW THERE ARE TWO POSIBILITIES  
 $\Rightarrow \operatorname{arccot}(x+1) = \frac{\pi}{2} - \arctan(x+1)$  OR  $\Rightarrow \operatorname{arccot}(x+1) = \frac{\pi}{2} - \arctan(x+1) + \pi$   
 $\Rightarrow \operatorname{arccot}(x+1) + \arctan(x+1) = \frac{\pi}{2}$  OR  $\Rightarrow \operatorname{arccot}(x+1) + \arctan(x+1) = \frac{3\pi}{2}$   
 • NOW USING THE IDENTITY  $\arctan A = \arctan(\frac{1}{A})$   
 $\Rightarrow \arctan(\frac{1}{x+1}) + \arctan(x+1) = \frac{\pi}{2}$  OR  $\Rightarrow \arctan(\frac{1}{x+1}) + \arctan(x+1) = \frac{3\pi}{2}$   
 • TAKING TANGENTS ON BOTH SIDES IN EACH OF THE TWO EQUATIONS  
 $\Rightarrow \tan[\arctan(\frac{1}{x+1}) + \arctan(x+1)] = \tan(\frac{\pi}{2})$  OR  $\Rightarrow \tan[\arctan(\frac{1}{x+1}) + \arctan(x+1)] = \tan(\frac{3\pi}{2})$   
 $\Rightarrow \frac{\frac{1}{x+1} + x}{1 - \frac{1}{x+1} \cdot x} = \infty$  OR  $\Rightarrow \frac{\frac{1}{x+1} + x}{1 - \frac{1}{x+1} \cdot x} = \infty$   
 $\Rightarrow \frac{1+x(x+1)}{x+1-x} = \infty$  OR  $\Rightarrow \frac{1-x(x+1)}{x+1-x} = \infty$   
 $\Rightarrow \frac{1+x^2+x}{x+1-x} = \infty$  OR  $\Rightarrow \frac{1-x^2-x}{x+1-x} = \infty$   
 $\Rightarrow x^2+x+1 = \infty$  OR  $\Rightarrow x^2-x+1 = \infty$   
 $\Rightarrow x^2+x+1 = 0$  OR  $\Rightarrow x^2-x+1 = 0$   
 $\Rightarrow x = -\frac{1}{2}$

**Question 131** (\*\*\*\*)

The function  $f$  is defined in the largest possible real domain, contained in the interval  $(-2\pi, 2\pi)$ , and its equation is

$$f(x) \equiv \ln \left[ \tan \left( \frac{1}{8} \pi - \frac{1}{2} x \right) \right].$$

- a)** Find the domain of  $f$ .

- b)** Show that  $f'(x) \equiv \frac{k}{\sqrt{1 - \sin 2x}}$ , for some constant  $k$ .

$$\boxed{\text{DFA}}, \quad \left[ (-2\pi, -\frac{7}{4}\pi) \cup (-\frac{3}{4}\pi, \frac{1}{4}\pi) \cup (\frac{5}{4}\pi, 2\pi) \right]$$

a) For the function to be defined, the argument of the arctan must be non-negative - look at the graph or table

$$\tan(x) > 0 \Rightarrow 0 < x < \frac{\pi}{2} \quad -\pi < x < -\frac{\pi}{2}$$

$$\tan(x) < 0 \Rightarrow \frac{\pi}{2} < x < \pi \quad -\frac{\pi}{2} < x < 0$$

using transformations

$$x \mapsto x + \pi \quad x \mapsto \frac{x}{2} \quad x \mapsto -x$$

$$-\pi < x < -\frac{\pi}{2} \Rightarrow -\frac{\pi}{2} < x < -\frac{3\pi}{2} \Rightarrow -\frac{3\pi}{2} < x < -\frac{5\pi}{2}$$

$$-\frac{\pi}{2} < x < \frac{\pi}{2} \Rightarrow -\frac{3\pi}{2} < x < -\frac{\pi}{2}$$

$$0 < x < \frac{\pi}{2} \Rightarrow 0 < x < \frac{\pi}{2}$$

$$\frac{\pi}{2} < x < \frac{3\pi}{2} \Rightarrow \frac{\pi}{2} < x < \frac{3\pi}{2}$$

$$\frac{3\pi}{2} < x < \frac{5\pi}{2} \Rightarrow \frac{3\pi}{2} < x < \frac{5\pi}{2}$$

$$\tan(x) > 0 \quad \tan(x) < 0 \quad \tan(x) < 0$$

simplifying these gives

$$(-\frac{3\pi}{2}, -\frac{5\pi}{2}) \cup (-\frac{\pi}{2}, \frac{\pi}{2}) \cup (\frac{3\pi}{2}, \frac{5\pi}{2})$$

b) Only for the differentiation

$$\frac{d}{dx} \left[ \ln \left( \tan \left( \frac{x}{2} - \frac{\pi}{2} \right) \right) \right] = \frac{1}{\tan \left( \frac{x}{2} - \frac{\pi}{2} \right)} \times \frac{1}{2} \times \sec^2 \left( \frac{x}{2} - \frac{\pi}{2} \right)$$

$$= \frac{\sec \left( \frac{x}{2} - \frac{\pi}{2} \right)}{2 \tan \left( \frac{x}{2} - \frac{\pi}{2} \right)} = \frac{1}{\tan^2 \left( \frac{x}{2} - \frac{\pi}{2} \right)}$$

## Question 132 (\*\*\*\*\*)

The positive solution of the quadratic equation  $x^2 - x - 1 = 0$  is denoted by  $\phi$ , and is commonly known as the golden section or golden number.

It can be shown, and you **may assume in this question**, that  $\cos\left(\frac{1}{5}\pi\right) = \frac{1}{2}\phi$ .

Use trigonometric identities to show that

$$\tan\left(\frac{1}{5}\pi\right)\tan\left(\frac{2}{5}\pi\right)\tan\left(\frac{3}{5}\pi\right)\tan\left(\frac{4}{5}\pi\right) = 5.$$

*You may not use complex numbers in this question.*

 , proof

PROCEED AS FOLLOWS

$$\begin{aligned} & \tan\left(\frac{1}{5}\pi\right)\tan\left(\frac{2}{5}\pi\right)\tan\left(\frac{3}{5}\pi\right)\tan\left(\frac{4}{5}\pi\right) \\ &= [\tan\left(\frac{1}{5}\pi\right)\tan\left(\frac{2}{5}\pi\right) - 1][\tan\left(\frac{3}{5}\pi\right)\tan\left(\frac{4}{5}\pi\right) - 1] \\ &= \tan\left(\frac{1}{5}\pi\right)\tan\left(\frac{2}{5}\pi\right) - 1 \\ &= \frac{\sin\left(\frac{1}{5}\pi\right)\sin\left(\frac{2}{5}\pi\right)}{\cos\left(\frac{1}{5}\pi\right)\cos\left(\frac{2}{5}\pi\right)} - 1 \end{aligned}$$

SKETCH TO SHOW ALL CORNERS

USE THE DOUBLE ANGLE FORMULA i.e.  $\sin 2A = 2\sin A \cos A$

$$\begin{aligned} \sin 2A &= 2\sin A \cos A \\ \sin 2A &= 2\sin A \cos A \\ \cos 2A &= 2\cos^2 A - 1 \end{aligned}$$

$$\begin{aligned} &= \frac{\sin\left(\frac{1}{5}\pi\right)\sin\left(\frac{2}{5}\pi\right)}{\cos\left(\frac{1}{5}\pi\right)\cos\left(\frac{2}{5}\pi\right)} - 1 \\ &= \frac{4\left[1 - \cos\left(\frac{4}{5}\pi\right)\right]}{[2\cos^2\left(\frac{1}{5}\pi\right) - 1]^2} = \frac{4\left[1 - \cos\left(\frac{4}{5}\pi\right)\right]}{[2\cos^2\left(\frac{1}{5}\pi\right) - 1]^2} = \frac{4\left[1 - \cos\left(\frac{4}{5}\pi\right)\right]}{[2\cos^2\left(\frac{1}{5}\pi\right) - 1]^2} \end{aligned}$$

BUT  $\phi$  HAS THE PROPERTY OF  $\phi^2 = \phi + 1$  (CONSIDER  $x^2 = x + 1$ )

$$\begin{aligned} &= \frac{(4 - \phi - 1)^2}{(\phi - 1)^2} = \frac{(3 - \phi)^2}{(\phi - 1)^2} = \frac{9 - 6\phi + 1}{\phi^2 - 2\phi + 1} \\ &= \frac{10 - 5\phi}{\phi^2 - 2\phi + 1} = \frac{10 - 5\phi}{\phi^2 - 2\phi + 1} = 5 \end{aligned}$$

As required

## Question 133 (\*\*\*\*)

It is given that  $\theta$ ,  $\alpha$  and  $\beta$  are distinct real numbers which satisfy.

$$\tan(\theta - \alpha) + \tan(\theta - \beta) = x$$

$$\cot(\theta - \alpha) + \cot(\theta - \beta) = y.$$

Find, in exact simplified form, an expression for  $\tan(\alpha - \beta)$ , in terms of  $x$  and  $y$ .

$$\boxed{\phantom{000000}}, \tan(\alpha - \beta) = \pm \frac{\sqrt{x^2 y^2 - 4xy}}{x + y}$$

Handwritten solution for Question 133:

Given:  $\tan(\theta - \alpha) + \tan(\theta - \beta) = x$   
 $\cot(\theta - \alpha) + \cot(\theta - \beta) = y$

**Method 1: SIMPLY BY SETTING RELATIONSHIPS** — Let  $A = \tan(\theta - \alpha)$   
 $B = \tan(\theta - \beta)$

$A + B = x$   
 $\frac{1}{A} + \frac{1}{B} = y \Rightarrow \frac{A+B}{AB} = y$  or  $AB = \frac{x}{y}$

**Method 2: NEXT WE MAY PROCEED AS FOLLOWS**

$\tan(\alpha - \beta) = \tan(\alpha - \theta + \theta - \beta)$   
 $= \tan[(\theta - \beta) - (\theta - \alpha)]$   
 $= \frac{\tan(\theta - \beta) - \tan(\theta - \alpha)}{1 + \tan(\theta - \beta)\tan(\theta - \alpha)}$   
 $= \frac{B - A}{1 + AB}$   
 $= \frac{B - A}{1 + \frac{x}{y}}$   
 $= \frac{y(B - A)}{y + x}$   
 $= \frac{y(B^2 - 2AB + A^2)}{y + x}$   
 $= \frac{y(A^2 - 2AB + B^2) + 4AB}{y + x}$   
 $= \frac{y(A^2 + 2AB + B^2) - 4AB}{y + x}$

From Method 1, we have  $A + B = x$  and  $AB = \frac{x}{y}$ .  
 $(A + B)^2 = x^2 \Rightarrow A^2 + 2AB + B^2 = x^2$   
 $A^2 + B^2 = x^2 - 2AB = x^2 - \frac{2x}{y}$

From Method 2, we have  $\frac{y(A^2 + 2AB + B^2) - 4AB}{y + x}$   
 $= \frac{y(x^2 - \frac{2x}{y}) - 4AB}{y + x}$   
 $= \frac{yx^2 - 2x - 4AB}{y + x}$   
 $= \frac{yx^2 - 2x - 4 \cdot \frac{x}{y}}{y + x}$   
 $= \frac{yx^2 - 2x - \frac{4x}{y}}{y + x}$   
 $= \frac{y^2 x^2 - 2xy - 4x}{y(y + x)}$   
 $= \frac{x(y^2 x - 2y - 4)}{y(y + x)}$



## Question 134 (\*\*\*\*)

By considering the trigonometric identity for  $\tan(A-B)$ , with  $A = \arctan(n+1)$  and  $B = \arctan(n)$ , sum the following series

$$\sum_{n=1}^{\infty} \arctan\left(\frac{1}{n^2 + n + 1}\right).$$

You may assume the series converges.

$$\boxed{\phantom{000}}, \quad \boxed{\frac{\pi}{4}}$$

**Handwritten Solution:**

Consider the compound angle identity for  $\tan(A-B)$

$$\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

$$\tan[\arctan(n+1) - \arctan(n)] = \frac{\tan[\arctan(n+1)] - \tan[\arctan(n)]}{1 + \tan[\arctan(n+1)] \tan[\arctan(n)]}$$

$$\tan[\arctan(n+1) - \arctan(n)] = \frac{(n+1) - n}{1 + (n+1)n}$$

$$\tan[\arctan(n+1) - \arctan(n)] = \frac{1}{n^2 + n + 1}$$

$$\arctan\left[\tan[\arctan(n+1) - \arctan(n)]\right] = \arctan\left(\frac{1}{n^2 + n + 1}\right)$$

Hence the summation now becomes

$$\sum_{n=1}^{\infty} \arctan\left(\frac{1}{n^2 + n + 1}\right) = \sum_{n=1}^{\infty} [\arctan(n+1) - \arctan(n)]$$

$$= \sum_{k=1}^{\infty} \arctan(k+1) - \sum_{k=1}^{\infty} \arctan(k)$$

What now gives in a limiting sense

$$\lim_{k \rightarrow \infty} \left[ \sum_{n=1}^k \arctan(n+1) - \sum_{n=1}^k \arctan(n) \right]$$

Telescoping sum:

$$= \lim_{k \rightarrow \infty} [\arctan(k+1) - \arctan(1)]$$

$$= \frac{\pi}{2} - \frac{\pi}{4}$$

$$= \frac{\pi}{4}$$

## Question 135 (\*\*\*\*)

It is given that

$$(\arcsin x)^3 + (\arccos x)^3 = k\pi^3, \quad |x| \leq 1,$$

for some constant  $k$ .

- a) Show that a necessary but not sufficient condition for the above equation to have solutions is that

$$k \geq \frac{1}{32}.$$

- b) Solve the equation given that it only has one solution.
- c) Given instead that that  $k = \frac{7}{96}$ , find the two solutions of the equation, giving the answers in the form  $x = \sin(a\pi)$ , where  $a \in \mathbb{Q}$ .

$$\boxed{\phantom{000}}, \quad \boxed{x = \frac{\sqrt{2}}{2}}, \quad \boxed{x = \sin\left(\frac{\pi}{12}\right)}, \quad \boxed{x = \sin\left(\frac{5\pi}{12}\right)}$$

a)  $(\arcsin x)^3 + (\arccos x)^3 = k\pi^3$

● USING THE IDENTITY  $\arcsin x + \arccos x = \frac{\pi}{2}$

$$\Rightarrow (\arcsin x)^3 + \left(\frac{\pi}{2} - \arcsin x\right)^3 = k\pi^3$$

$$\Rightarrow (\arcsin x)^3 + \frac{\pi^3}{8} - \frac{3\pi^2}{2}(\arcsin x) + \frac{3\pi}{2}(\arcsin x)^2 - (\arcsin x)^3 = k\pi^3$$

$$\Rightarrow \frac{3\pi}{2}(\arcsin x)^2 - \frac{3\pi^2}{2}(\arcsin x) + \frac{\pi^3}{8} - k\pi^3 = 0$$

$$\Rightarrow \frac{3}{2}(\arcsin x)^2 - \frac{3}{2}\pi(\arcsin x) + \frac{\pi^2}{8} - k\pi^2 = 0$$

$$\Rightarrow (\arcsin x)^2 - \pi(\arcsin x) + \frac{\pi^2}{12}(1 - 8k) = 0$$

● FOR REAL SOLUTIONS  $b^2 - 4ac \geq 0$

$$\Rightarrow \pi^2 - 4 \times \frac{\pi^2}{12}(1 - 8k) \geq 0$$

$$\Rightarrow \frac{1}{3} - \frac{1}{3}(1 - 8k) \geq 0$$

$$\Rightarrow \frac{1}{3} - \frac{1}{3} + \frac{8k}{3} \geq 0$$

$$\Rightarrow 3 - 1 + 8k \geq 0$$

$$\Rightarrow 8k \geq 2$$

$$\Rightarrow k \geq \frac{1}{4}$$

● THIS CONDITION IS NECESSARY BUT NOT SUFFICIENT AS  $k \geq \frac{1}{4}$   
MAY PRODUCE SOLUTIONS SUCH AS  $|\arcsin x| > 1$  WHICH  
DO NOT EXIST DUE TO THE DEFINITION

b) IF THERE IS ONLY 1 SOLUTION  $\Rightarrow k = \frac{1}{32}$

$$\Rightarrow (\arcsin x)^2 - \pi(\arcsin x) + \frac{\pi^2}{12}\left(1 - 8 \times \frac{1}{32}\right) = 0$$

$$\Rightarrow (\arcsin x)^2 - \pi(\arcsin x) + \frac{\pi^2}{12} \times \frac{3}{4} = 0$$

$$\Rightarrow (\arcsin x)^2 - \pi(\arcsin x) + \frac{\pi^2}{16} = 0$$

$$\Rightarrow (\arcsin x - \frac{\pi}{4})^2 = 0$$

$$\Rightarrow \arcsin x = \frac{\pi}{4}$$

$$\Rightarrow x = \sin \frac{\pi}{4} = \frac{\sqrt{2}}{2}$$

c) FINALLY IF  $k = \frac{7}{96}$

$$\Rightarrow (\arcsin x)^2 - \pi(\arcsin x) + \frac{\pi^2}{12}\left(1 - 8 \times \frac{7}{96}\right) = 0$$

$$\Rightarrow (\arcsin x)^2 - \pi(\arcsin x) + \frac{\pi^2}{12}\left(1 - \frac{7}{12}\right) = 0$$

$$\Rightarrow (\arcsin x)^2 - \pi(\arcsin x) + \frac{\pi^2}{144} = 0$$

$$\Rightarrow \left[\arcsin x - \frac{\pi}{12}\right]^2 - \frac{\pi^2}{16} + \frac{\pi^2}{144} = 0$$

$$\Rightarrow \left[\arcsin x - \frac{\pi}{12}\right]^2 - \frac{4\pi^2}{144} + \frac{\pi^2}{144} = 0$$

$$\Rightarrow \left[\arcsin x - \frac{\pi}{12}\right]^2 = \frac{3\pi^2}{36}$$

$$\Rightarrow \arcsin x - \frac{\pi}{12} = \pm \frac{\pi}{6}$$

$$\Rightarrow \arcsin x = \frac{\pi}{12} + \frac{\pi}{6} = \frac{\pi}{4} \quad \text{OR} \quad \arcsin x = \frac{\pi}{12} - \frac{\pi}{6} = -\frac{\pi}{12}$$

$$\Rightarrow x = \sin \frac{\pi}{4} = \frac{\sqrt{2}}{2} \quad \text{OR} \quad x = \sin \left(-\frac{\pi}{12}\right) = -\sin \frac{\pi}{12}$$

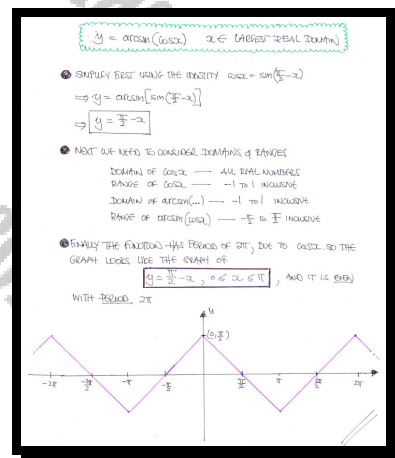
## Question 136 (\*\*\*\*)

Sketch the graph of

$$f(x) = \arcsin(\cos x),$$

in the largest domain that the function is defined.

Indicate the coordinates of any intersections with the axes, and the coordinates of the cusps of the curve.

, graph


## Question 137 (\*\*\*\*)

It is given that

$$\arctan 2 + \arctan A + \arctan B = \pi.$$

It is further given that  $A$  and  $B$  are distinct positive real numbers other than unity.Determine a pair of possible values for  $A$  and  $B$ .

$$\boxed{\phantom{00}}, \boxed{5} \text{ \& } \boxed{\frac{7}{9}}$$

• AS THE PROBLEM IS NOT UNIQUE, LET  $A=5$   
 $\arctan 2 + \arctan 5 + \arctan B = \pi$

• USING COMPLEX NUMBERS  
 $(1+2i)(1+5i) = 1 + 5i + 2i + 10i^2 = -9 + 7i$

• WE EXPECT  
 $(1+2i)(1+5i)z = \text{NEGATIVE REAL NUMBER (SAY } -1 \text{ AT THE ONCE)}$   
 $\Rightarrow (1+2i)(1+5i)z = -1$   
 $(-9+7i)z = -1$   
 $(9-7i)z = 1$   
 $z = \frac{1}{9-7i}$   
 $z = \frac{9+7i}{81+49}$   
 $z = \frac{9+7i}{130}$

• SO FOR OUR CASE  
 $\Rightarrow (1+2i)(1+5i)\left(\frac{9+7i}{130}\right) = -1$   
 $\Rightarrow (1+2i)(1+5i)(9+7i) = -130$   
 $\Rightarrow \arg[(1+2i)(1+5i)(9+7i)] = -150$   
 $\Rightarrow \arg(1+2i) + \arg(1+5i) + \arg(9+7i) = \arg(-130)$   
 $\Rightarrow \arctan 2 + \arctan 5 + \arctan \frac{7}{9} = \pi$

## Question 138 (\*\*\*\*)

Find the value of

$$\sum_{r=0}^{\infty} \left[ \frac{\sin^4(\pi \times 2^{r-2})}{4^r} \right].$$

Hint: Express  $\sin^4 \theta$  in terms of  $\sin^2 \theta$  and  $\sin^2 2\theta$  only.

$$\boxed{\phantom{000}}, \boxed{\frac{1}{2}}$$

• STARTING BY MANIPULATING THE SINE TO THE POWER 4  

$$\sin^4 \theta = (\sin^2 \theta)^2 = \left( \frac{1}{2} - \frac{1}{2} \cos 2\theta \right)^2 = \frac{1}{4} - \frac{1}{2} \cos 2\theta + \frac{1}{4} \cos^2 2\theta$$

$$= \frac{1}{4} - \frac{1}{2} (1 - 2\sin^2 \theta) + \frac{1}{4} (1 - \sin^2 2\theta)$$

$$= \frac{1}{4} - \frac{1}{2} + \sin^2 \theta + \frac{1}{4} - \frac{1}{4} \sin^2 2\theta$$

$$= \sin^2 \theta - \frac{1}{4} \sin^2 2\theta$$

• NOW WE THINK BY CONSIDERING THE SUM OF THE FIRST N TERMS  

$$\sum_{r=0}^n \frac{\sin^4(\pi \times 2^{r-2})}{4^r} = \sum_{r=0}^n \left[ \frac{1}{4^r} \left( \sin^2(\pi \times 2^{r-2}) - \frac{1}{4} \sin^2(\pi \times 2^{r-1}) \right) \right]$$

$$= \sum_{r=0}^n \left[ \frac{1}{4^r} \sin^2(\pi \times 2^{r-2}) - \frac{1}{4^{r+1}} \sin^2(\pi \times 2^{r-1}) \right]$$

$$= \frac{\sin^2 \frac{\pi}{2}}{4^0} - \frac{1}{4^1} \sin^2 \frac{\pi}{2} \quad \leftarrow r=0$$

$$+ \frac{1}{4^1} \sin^2 \pi - \frac{1}{4^2} \sin^2 2\pi \quad \leftarrow r=1$$

$$+ \frac{1}{4^2} \sin^2 2\pi - \frac{1}{4^3} \sin^2 4\pi \quad \leftarrow r=2$$

$$\vdots$$

$$+ \frac{1}{4^n} \sin^2(\pi \times 2^{n-2}) - \frac{1}{4^{n+1}} \sin^2(\pi \times 2^{n-1}) \quad \leftarrow r=n$$

$$= \sin^2 \frac{\pi}{2} - \frac{1}{4^{n+1}} \sin^2(\pi \times 2^{n-1})$$

• THEN WE HAVE  

$$\sum_{r=0}^{\infty} \frac{\sin^4(\pi \times 2^{r-2})}{4^r} = \sin^2 \frac{\pi}{2} = \left( \frac{1}{\sqrt{2}} \right)^2 = \frac{1}{2}$$

## Question 139 (\*\*\*\*\*)

The triangle  $ABC$  is isosceles with  $|AB| = |AC|$  and  $\angle BAC = 36^\circ$ .

The angle bisector of  $\angle ABC$  meets  $AC$  at the point  $D$ .

By using trigonometry in the above construction, or otherwise, show that

$$\cos 36^\circ = \frac{1}{2}(1 + \sqrt{5}).$$

,  proof

**METHOD 1: USING THE COSINE DOUBLE-ANGLE FORMULA (IF FINALLY OBTAIN)**

$$\begin{aligned} \Rightarrow 1 &= 2\cos 36^\circ - 2[2\cos^2 36^\circ - 1] \\ \Rightarrow 1 &= 2\cos 36^\circ - 4\cos^2 36^\circ + 2 \\ \Rightarrow 4\cos^2 36^\circ - 2\cos 36^\circ - 1 &= 0 \\ \Rightarrow \cos 36^\circ &= \frac{2 \pm \sqrt{(-2)^2 - 4 \times 4 \times (-1)}}{2 \times 4} \\ \Rightarrow \cos 36^\circ &= \frac{2 \pm \sqrt{20}}{8} \\ \Rightarrow \cos 36^\circ &= \frac{2 \pm 2\sqrt{5}}{8} \\ \Rightarrow \cos 36^\circ &= \frac{1 \pm \sqrt{5}}{4} \\ \Rightarrow \cos 36^\circ &= \frac{1 + \sqrt{5}}{4} \quad \left( \frac{1 - \sqrt{5}}{4} < 0 \right) \\ \cos 36^\circ &> 0 \end{aligned}$$

**METHOD 2: SIMILAR TRIANGLES (ANALYZING ALL THE ANGLES)**

Let  $|AD| = x$  &  $|DC| = y$   
 Then  $|AB| = |AC| = x + y$   
 $|BD| = |AD| = x$   
 $|BC| = |BC| = x$

**● NOW LOOKING AT THE TRIANGLE ABD**

$$\begin{aligned} |AB| &= 2|BD|\cos 36^\circ \\ x + y &= 2x\cos 36^\circ \end{aligned}$$

**● NOW LOOKING AT THE TRIANGLE BDC**

$$\begin{aligned} |BC| &= 2|DC|\cos 72^\circ \\ x &= 2y\cos 72^\circ \end{aligned}$$

**● SUBSTITUTING THE LAST TWO EQUATIONS AND TRYING**

$$\begin{aligned} x + y &= 2x\cos 36^\circ \\ y &= 2x\cos 72^\circ \\ \Rightarrow x &= 2x\cos 36^\circ - 2x\cos 72^\circ \\ \Rightarrow 1 &= 2\cos 36^\circ - 2\cos 72^\circ \end{aligned}$$

## Question 140 (\*\*\*\*)

$$f(x) = \frac{\cos 2x}{\sqrt{1 + \sin 2x}}, \quad x \in \mathbb{R}, \sin 2x \neq -1.$$

a) Express  $f(x)$  in the form

$$f(x) = \frac{g(x)g(-x)}{|g(x)|},$$

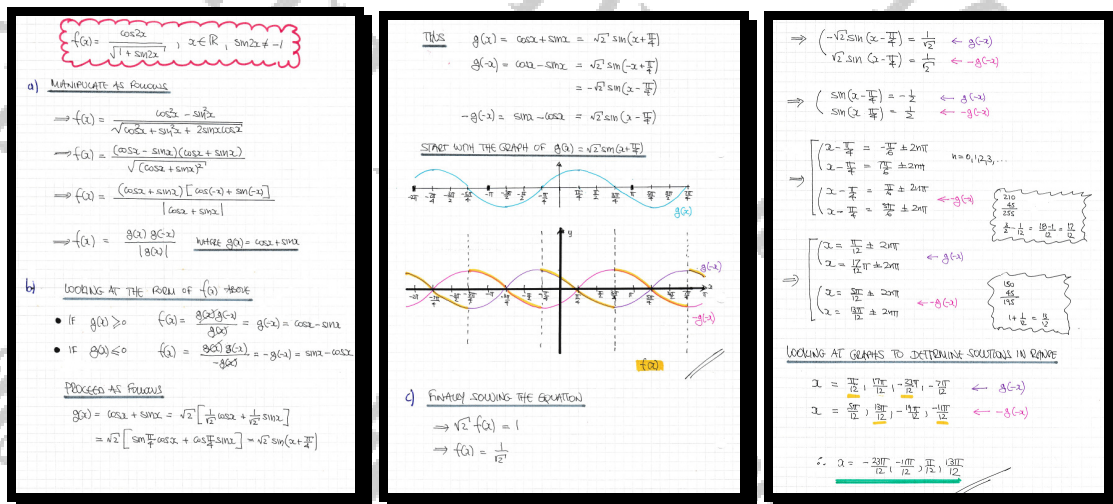
where  $g(x)$  is a function to be found.

b) Sketch the graph of  $f(x)$  for  $-2\pi \leq x \leq 2\pi$ .

c) Hence solve the trigonometric equation

$$\sqrt{2}f(x) = 1, \quad -2\pi \leq x \leq 2\pi.$$

$$\boxed{\phantom{000}}, \quad \boxed{g(x) = \cos x + \sin x}, \quad \boxed{x = -\frac{23}{12}\pi, -\frac{11}{12}\pi, \frac{1}{12}\pi, \frac{13}{12}\pi}$$



## Question 141 (\*\*\*\*)

Prove that for all  $x$  such that  $-1 \leq x \leq 1$ 

$$\arccos x + \arccos \left[ \frac{1}{2} \left( x + \sqrt{3-3x^2} \right) \right] = \frac{\pi}{3}.$$

□, proof

$\arccos x + \arccos \left( \frac{x + \sqrt{3-3x^2}}{2} \right) = \frac{\pi}{3}$

• LET  $\theta = \arccos x$   
 $\cos \theta = x$   
 $\sin \theta = \sqrt{1-x^2}$

• LET  $\phi = \arccos \left( \frac{x + \sqrt{3-3x^2}}{2} \right)$   
 $\cos \phi = \frac{x + \sqrt{3-3x^2}}{2}$

• WE NEED TO FIND THE EXACT VALUE OF  $\sin \phi$  SO WE USE PYTHAGORAS IN THE "SECOND" TRIANGLE TO FIND  $y$   
 $\Rightarrow y = \sqrt{4 - (x + \sqrt{3-3x^2})^2}$   
 $\Rightarrow y = \sqrt{4 - (x^2 + 2x\sqrt{3-3x^2} + 3-3x^2)}$   
 $\Rightarrow y = \sqrt{4 - x^2 - 2x\sqrt{3-3x^2} - 3 + 3x^2}$   
 $\Rightarrow y = \sqrt{1 + 2x^2 - 2x\sqrt{3-3x^2}}$

• ATTEMPTING TO SQUARE ROOT THE ARGUMENT OF THE R.H.S. BY INSPECTION  
 $\Rightarrow 1 + 2x^2 - 2\sqrt{3}x\sqrt{1-x^2} \equiv (Ax + \sqrt{1-x^2})^2$   
 EXPAND AND COMPARE COEFFICIENTS  
 $\Rightarrow 1 + 2x^2 - 2\sqrt{3}x\sqrt{1-x^2} \equiv A^2x^2 + 2Ax\sqrt{1-x^2} + 1 - x^2$   
 $\Rightarrow 1 + 2x^2 - 2\sqrt{3}x\sqrt{1-x^2} \equiv (A^2-1)x^2 + 2Ax\sqrt{1-x^2} + 1$

• THIS GENUINELY WORKS IF  $-A = \sqrt{3}$   
 $\Rightarrow y = \sqrt{3x + \sqrt{1-x^2}}$   
 $\Rightarrow \sin \phi = \frac{y}{2}$   
 $\Rightarrow \sin \phi = \frac{\sqrt{3x + \sqrt{1-x^2}}}{2}$

• RETURNING TO THE ORIGINAL EXPRESSION AND REWRITE-ING FINALLY  
 $\arccos x + \arccos \left( \frac{x + \sqrt{3-3x^2}}{2} \right) = \psi$   
 $\Rightarrow \theta + \phi = \psi$   
 $\Rightarrow \cos(\theta + \phi) = \cos \psi$   
 $\Rightarrow \cos \theta \cos \phi - \sin \theta \sin \phi = \cos \psi$   
 $\Rightarrow x \left[ \frac{x + \sqrt{3-3x^2}}{2} \right] - \sqrt{1-x^2} \left[ \frac{\sqrt{3x + \sqrt{1-x^2}}}{2} \right] = \cos \psi$   
 $\Rightarrow \frac{x^2 + \sqrt{3}x\sqrt{1-x^2} - \sqrt{3}x\sqrt{1-x^2} + (1-x^2)}{2} = \cos \psi$   
 $\Rightarrow \frac{x^2 + 1 - x^2}{2} = \cos \psi$   
 $\Rightarrow \cos \psi = \frac{1}{2}$   
 $\Rightarrow \psi = \frac{\pi}{3}$   
 $\therefore \arccos x + \arccos \left( \frac{x + \sqrt{3-3x^2}}{2} \right) = \frac{\pi}{3} //$



The function  $f$  is defined as

The function  $f$  is defined as

$$f(x) \equiv \sin x + \cos x + \tan x + \cot x + \sec x + \operatorname{cosec} x, \quad x \in \left(0, \frac{1}{2}\pi\right).$$

Determine with full justification the range of  $f$ .

$$\boxed{\phantom{000}}, \quad f(x) \in [2+3\sqrt{2}, \infty)$$

REWRITE THE FUNCTION IN SINES & COSINES

$$f(x) = \sin x + \cos x + \tan x + \cot x + \sec x + \csc x$$

$$f(x) = \sin x + \cos x + \frac{\sin x}{\cos x} + \frac{\cos x}{\sin x} + \frac{1}{\cos x} + \frac{1}{\sin x}$$

$$f(x) = \sin x + \cos x + \frac{\sin^2 x + \cos^2 x}{\cos x \sin x} + \frac{\sin x + \cos x}{\cos x \sin x}$$

$$f(x) = \sin x + \cos x + \frac{2(\sin x + \cos x)}{\sin x \cos x} + \frac{\sin x + \cos x}{\sin x \cos x}$$

$$f(x) = \sin x + \cos x + \frac{2}{\sin 2x} + \frac{1}{\sin 2x} \frac{2(\sin x + \cos x)}{\sin 2x}$$

NOW  $\sin x + \cos x$  &  $\sin 2x$  ARE RELATED AS FOLLOWS

If  $g(x) = \sin x + \cos x$

$$[g(x)]^2 = (\sin x + \cos x)^2$$

$$g^2 = \sin^2 x + 2\sin x \cos x + \cos^2 x$$

$$g^2 = 1 + \sin 2x$$

$$\sin 2x = g^2 - 1$$

REWRITE THE FUNCTION  $f(x)$  IN TERMS OF  $g(x)$

$$f(x) = g + \frac{2}{g^2 - 1} + \frac{2g}{g^2 - 1}$$

$$f(x) = g + \frac{2(g+1)}{g^2 - 1}$$

$$f(x) = g + \frac{2(g+1)}{(g-1)(g+1)}$$

$$f(x) = g(x) + \frac{2}{g(x)-1}$$

$g \neq 1$   
 $\sin x + \cos x \neq 1 \Rightarrow \sin(x/2) \neq 1$

## Question 143 (\*\*\*\*\*)

It is given that

- the angles  $A$ ,  $B$  and  $C$  are the three angles of a triangle  $ABC$ .
- the angles  $A$ ,  $B$  and  $C$  are in an increasing arithmetic progression, in that order.
- The lengths of the triangle  $ABC$ , opposite each of the angles  $A$ ,  $B$  and  $C$  are denoted by  $a$ ,  $b$  and  $c$ .

Show that

$$\frac{a}{c} \sin 2C + \frac{c}{a} \sin 2A = \sqrt{3}.$$

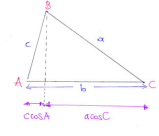
□, proof

✓ **SINCE BY A DIAGRAM**  
 $A+B+C = 180^\circ$   
 ANGLES ARE IN ARITHMETIC PROGRESSION IMPLIES  
 $B-A = C-B$

✓ **SOLVING THESE TWO EQUATIONS**  
 $A+B+C = 180^\circ$   
 $2B = C+A \rightarrow$   
 $2A+2B+2C = 360^\circ$   
 $2B = C+A \rightarrow$   
 $\Rightarrow 2A + C + A + 2C = 360$   
 $\Rightarrow 3A + 3C = 360$   
 $\Rightarrow A + C = 120$   
 $\Rightarrow 2B = 120$   
 $\Rightarrow B = 60$

✓ **USE THE SINE RULE**  
 $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} = k$  (SOME CONSTANT)

✓  $\frac{a}{c} \sin 2C + \frac{c}{a} \sin 2A$   
 $= \frac{a}{c} [2 \sin C \cos C] + \frac{c}{a} [2 \sin A \cos A]$   
 $= 2 \frac{\sin C}{\sin C} [\cos C] + 2 \frac{\sin A}{\sin A} [\cos A]$   
 $= 2k \cos C + 2k \cos A$   
 $= 2k [\cos C + \cos A]$  (NOT USE AT DIAGRAM)  
 $= 2k \cdot b = 2 \left( \frac{\sin B}{b} \right) b = 2 \sin B$   
 $= 2 \sin 60^\circ = 2 \times \frac{\sqrt{3}}{2} = \sqrt{3}$



## Question 144 (\*\*\*\*)

Find, in exact surd form, the only real solution of the following trigonometric equation

$$\arcsin(2x-1) - \arccos x = \frac{\pi}{6}$$

The rejection of any additional solutions must be fully justified.

$$\boxed{\phantom{000}}, x = \frac{1}{2} - \frac{1}{6}\sqrt{6}$$

$\arcsin(2x-1) - \arccos x = \frac{\pi}{6}$   
 $\Rightarrow \arcsin(2x-1) = \frac{\pi}{6} + \arccos x$   
 $\Rightarrow \sin[\arcsin(2x-1)] = \sin[\frac{\pi}{6} + \arccos x]$   
 $\Rightarrow 2x-1 = \sin \frac{\pi}{6} \cos(\arccos x) + \cos \frac{\pi}{6} \sin(\arccos x)$   
 $\Rightarrow 2x-1 = \frac{1}{2}x + \frac{\sqrt{3}}{2} \sin(\arccos x)$   
 $\Rightarrow 4x-2 = x + \sqrt{3} \sin(\arccos x)$   
 $\Rightarrow 3x-2 = \sqrt{3} \sin(\arccos x)$

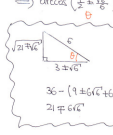
Let  $\theta = \arccos x$   
 $\cos \theta = x$   
 $\cos^2 \theta = x^2$   
 $1 - \cos^2 \theta = 1 - x^2$   
 $\sin^2 \theta = 1 - x^2$   
 $\sin(\arccos x) = \sqrt{1-x^2}$

$\Rightarrow (3x-2)^2 = 3 \sin^2(\arccos x)$   
 $\Rightarrow 9x^2 - 12x + 4 = 3(1-x^2)$   
 $\Rightarrow 9x^2 - 12x + 4 = 3 - 3x^2$   
 $\Rightarrow 12x^2 - 12x + 1 = 0$   
 $\Rightarrow 4x^2 - 4x + \frac{1}{3} = 0$

$\Rightarrow 4x^2 - 4x + \frac{1}{3} = 0$   
 $\Rightarrow (2x-1)^2 = \frac{4}{3}$   
 $\Rightarrow 2x-1 = \pm \sqrt{\frac{4}{3}}$   
 $\Rightarrow 2x = 1 \pm \frac{\sqrt{3}}{3}$   
 $\Rightarrow x = \frac{1}{2} \pm \frac{\sqrt{3}}{6}$   
 $\Rightarrow x = \frac{1}{2} + \frac{\sqrt{3}}{6}$  or  $\frac{1}{2} - \frac{\sqrt{3}}{6}$

NOW WE NEED TO CHECK THESE SOLUTIONS (GUT TO SQUARING)

$\Rightarrow \arcsin(2(\frac{1}{2} + \frac{\sqrt{3}}{6}) - 1) - \arccos(\frac{1}{2} + \frac{\sqrt{3}}{6}) = \pi$   
 $\Rightarrow \arcsin(\pm \frac{\sqrt{3}}{3}) + \arccos(\frac{1}{2} + \frac{\sqrt{3}}{6}) = \pi$   
 $\Rightarrow \arccos(\frac{1}{2} + \frac{\sqrt{3}}{6}) + \arccos(\frac{1}{2} + \frac{\sqrt{3}}{6}) = \pi$


 $\Rightarrow \arccos(\frac{1}{2} + \frac{\sqrt{3}}{6}) + \arccos(\frac{1}{2} + \frac{\sqrt{3}}{6}) = \pi$   
 $\Rightarrow \arccos(\frac{1}{2} + \frac{\sqrt{3}}{6}) = \frac{\pi}{2}$   
 $\Rightarrow \frac{1}{2} + \frac{\sqrt{3}}{6} = \cos \frac{\pi}{2} = 0$   
 $\Rightarrow \frac{1}{2} + \frac{\sqrt{3}}{6} = 0$   
 $\Rightarrow \frac{1}{2} + \frac{\sqrt{3}}{6} = 0$

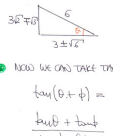
• HAD THE SQUARE ROOT OF  $21 \pm 6\sqrt{2}$  BEFORE WE TAKE TRIGONOMETRIC SIMPLIFY

$21 \pm 6\sqrt{2} = (a\sqrt{2})^2 + 2 \times a\sqrt{2} \times b + (b\sqrt{2})^2$   
 $a^2 \pm 2 \times a \times b + b^2 = 21 \pm 6\sqrt{2}$

• BY INSPECTION (IT WORKS IN THE FIRST CASE IF  $\frac{a}{b} = 1$ )

$21 \pm 6\sqrt{2} = (\sqrt{3})^2 + 2 \times \sqrt{3} \times 3\sqrt{2} + (3\sqrt{2})^2$   
 $\Rightarrow \sqrt{3} \pm 3\sqrt{2}$

• AS BOTH WILL BE POSITIVE  
 $21 \pm 6\sqrt{2} = (3\sqrt{2} \pm \sqrt{3})^2$   
 $\Rightarrow \sqrt{21 \pm 6\sqrt{2}} = 3\sqrt{2} \pm \sqrt{3}$



• NOW WE CAN TAKE TANGENTS IN EACH OF THE TWO CASES

$\tan(\theta + \phi) = \tan \psi$   
 $\frac{\tan \theta + \tan \phi}{1 - \tan \theta \tan \phi} = \tan \psi$   
 $\frac{\frac{3\sqrt{2}-\sqrt{3}}{3+\sqrt{2}} + \sqrt{2}}{1 - \frac{3\sqrt{2}-\sqrt{3}}{3+\sqrt{2}} \times \sqrt{2}} = \tan \psi$   
 $\frac{3\sqrt{2}-\sqrt{3} + \sqrt{2}(3+\sqrt{2})}{3+\sqrt{2} - (3\sqrt{2}-\sqrt{3})} = \tan \psi$   
 $\frac{3\sqrt{2}-\sqrt{3} + 3\sqrt{2} + 2}{3+\sqrt{2} - 3\sqrt{2} + \sqrt{3}} = \tan \psi$   
 $\frac{6\sqrt{2}-\sqrt{3} + 2}{3-2\sqrt{2} + \sqrt{3}} = \tan \psi$

$\tan(\theta - \phi) = \tan \psi$   
 $\frac{\tan \theta - \tan \phi}{1 + \tan \theta \tan \phi} = \tan \psi$   
 $\frac{\frac{3\sqrt{2}-\sqrt{3}}{3+\sqrt{2}} - \sqrt{2}}{1 + \frac{3\sqrt{2}-\sqrt{3}}{3+\sqrt{2}} \times \sqrt{2}} = \tan \psi$   
 $\frac{3\sqrt{2}-\sqrt{3} - \sqrt{2}(3+\sqrt{2})}{3+\sqrt{2} + (3\sqrt{2}-\sqrt{3})} = \tan \psi$   
 $\frac{3\sqrt{2}-\sqrt{3} - 3\sqrt{2} - 2}{3+\sqrt{2} + 3\sqrt{2} - \sqrt{3}} = \tan \psi$   
 $\frac{-\sqrt{3} - 2}{6+2\sqrt{2} - \sqrt{3}} = \tan \psi$

$\tan \psi = \frac{6\sqrt{2}-\sqrt{3}+\sqrt{2}}{3+\sqrt{2}+\sqrt{3}-6}$   
 $\tan \psi = \frac{6\sqrt{2}-2\sqrt{3}}{2\sqrt{2}-3}$   
 $\therefore \psi \neq \frac{\pi}{6}$   
 $\therefore x = \frac{1}{2} + \frac{\sqrt{3}}{6}$   
 IS NOT A SOLUTION

$\tan \psi = \frac{3\sqrt{2}-\sqrt{3}-3\sqrt{2}-2}{3-2\sqrt{2}+\sqrt{3}}$   
 $\tan \psi = \frac{-\sqrt{3}-2}{6+2\sqrt{2}-\sqrt{3}}$   
 $\tan \psi = \frac{\sqrt{3}}{3}$   
 $\psi = \frac{\pi}{6}$   
 $\therefore$  ONLY SOLUTION IS  $\frac{1}{2} - \frac{\sqrt{3}}{6}$

Given that  $n$  is an integer such that  $n > 3$ , use a detailed method to solve the following trigonometric equation.

$$\frac{1}{\sin\left[\frac{2\pi}{n}\right]} + \frac{1}{\sin\left[\frac{3\pi}{n}\right]} = \frac{1}{\sin\left[\frac{\pi}{n}\right]}.$$

$$\boxed{\phantom{00}}, n=7$$

[illegible]

## Question 146 (\*\*\*\*\*)

Find, in terms of  $\pi$ , the general solution of the equation

$$(x+y)^2 + 4(x+y)\cos(x-y) + 4 = 0.$$

$$\boxed{\phantom{000}}, \left( x, y \right) = \left( -1 + \frac{k\pi}{2}, -1 - \frac{k\pi}{2} \right), k = \text{even} \quad , \quad \left( x, y \right) = \left( 1 + \frac{k\pi}{2}, 1 - \frac{k\pi}{2} \right), k = \text{odd}$$

$(x+y)^2 + 4(x+y)\cos(x-y) + 4 = 0$

• START BY COMPLETING THE SQUARE IN  $(x+y)$

$$\Rightarrow [(x+y) + 2\cos(x-y)]^2 - [2\cos(x-y)]^2 + 4 = 0$$

$$\Rightarrow [(x+y) + 2\cos(x-y)]^2 - 4\cos^2(x-y) + 4 = 0$$

$$\Rightarrow [(x+y) + 2\cos(x-y)]^2 + 4[1 - \cos^2(x-y)] = 0$$

$$\Rightarrow [(x+y) + 2\cos(x-y)]^2 + 4\sin^2(x-y) = 0$$

$$\Rightarrow [(x+y) + 2\cos(x-y)]^2 + [2\sin(x-y)]^2 = 0$$

• SINCE THE BRACKETS ARE BOTH QUANTITIES ONLY WE CAN SAY IF BOTH BRACKETS ARE ZERO

$(x+y) + 2\cos(x-y) = 0$   
 $x+y = -2$   
 $x-y = \pi, 3\pi, 5\pi, \dots$

$2\sin(x-y) = 0$   
 $x-y = 0, \pi, 2\pi, 3\pi, \dots$

• NOW THERE ARE TWO CASES TO CONSIDER

• IF  $\eta = \text{EVEN}$

$x-y = 2\eta\pi, 4\eta\pi, \dots$

$\cos(x-y) = 1$

$\therefore x+y + 2 = 0$

$x+y = -2$

• IF  $\eta = \text{ODD}$

$x-y = (2\eta+1)\pi, 3\pi, 5\pi, \dots$

$\cos(x-y) = -1$

$\therefore x+y - 2 = 0$

$x+y = 2$

• OBSERVING THE RESULTS

•  $\eta = \text{EVEN}$

$\Rightarrow \begin{cases} x+y = -2 \\ x-y = \eta\pi \end{cases}$

ADDING

$2x = -2 + \eta\pi$

$x = -1 + \frac{\eta\pi}{2}$

SUBTRACTING

$2y = -2 - \eta\pi$

$y = -1 - \frac{\eta\pi}{2}$

•  $\eta = \text{ODD}$

$\Rightarrow \begin{cases} x+y = 2 \\ x-y = \eta\pi \end{cases}$

ADDING

$2x = 2 + \eta\pi$

$x = 1 + \frac{\eta\pi}{2}$

SUBTRACTING

$2y = 2 - \eta\pi$

$y = 1 - \frac{\eta\pi}{2}$

• THENCE THE GENERAL SOLUTION OF THE EQUATION IS GIVEN BY

$x = -1 + \frac{\eta\pi}{2}$

$y = -1 - \frac{\eta\pi}{2}$

$\eta = 0, 2, 4, \dots$

IF  $\eta$  IS AN EVEN INTEGER

$x = 1 + \frac{\eta\pi}{2}$

$y = 1 - \frac{\eta\pi}{2}$

$\eta = 1, 3, 5, \dots$

IF  $\eta$  IS AN ODD INTEGER

## Question 147 (\*\*\*\*)

Find the general solution of the following equation

$$\frac{d}{dx} \left[ \int_{\frac{1}{6}\pi}^{\sqrt{2x}} \sin(t^2) + \cos(2t^2) dx \right] = -\sqrt{\frac{2}{x}}, \quad x \in \mathbb{R}.$$

$$\boxed{\phantom{000}}, \quad x = \frac{1}{4}\pi(4k-1) \quad k \in \mathbb{Z}$$

PROCEED BY U-SUBSTITUTION RULE & DIFF.  $\frac{d}{dx} \left( \int \right) = 0$

$$\frac{d}{dx} \int_{\frac{1}{6}\pi}^{\sqrt{2x}} \sin(t^2) + \cos(2t^2) dt = -\sqrt{\frac{2}{x}}$$

$$\Rightarrow \sin(\sqrt{2x}) \times \frac{d}{dx}(\sqrt{2x}) + \cos(2\sqrt{2x}) \times \frac{d}{dx}(2\sqrt{2x}) = -\sqrt{\frac{2}{x}}$$

$$\Rightarrow [\sin 2x + \cos 4x] \times \frac{d}{dx}(\sqrt{2}x^{\frac{1}{2}}) = -\sqrt{\frac{2}{x}}$$

$$\Rightarrow (\sin 2x + \cos 4x) \times \frac{1}{2}\sqrt{2}x^{-\frac{1}{2}} = -\sqrt{\frac{2}{x}}$$

$$\Rightarrow \sin 2x + \cos 4x = -2$$

NO IDENTITIES NEEDED HERE - JUST NEED A COMMON SOLUTION

- $\sin 2x = -1$   
 $2x = -\frac{\pi}{2} + 2n\pi \quad n = 0, 1, 2, \dots$   
 $2x = -\frac{\pi}{2} [1 \pm 4n]$   
 $x = \frac{1}{4}\pi(1 \pm 4n)$
- $\cos 4x = -1$   
 $4x = \pi \pm 2n\pi \quad n = 0, 1, 2, \dots$   
 $4x = \pi(1 \pm 2n)$   
 $x = \frac{1}{4}\pi(1 \pm 2n)$

THE COMMON SOLUTIONS ARE THOSE OF THE SINE

$$\therefore x = \frac{1}{4}\pi(4k-1) \quad k \in \mathbb{Z}$$

