TRIGONOMETRY

MINOR TRIGONOMETRIC RATIOS

Question 1

Simplify the following trigonometric expressions.

The final answer must not contain trigonometric fractions.

a) $\dfrac{\operatorname{cosec} x}{\sin^3 x}$

b) $\dfrac{\cos\theta}{\sec^2\theta}$

c) $\dfrac{2\sin^2 x}{\operatorname{cosec} x}$

d) $\dfrac{\sec^2\theta}{2\cos^2\theta}$

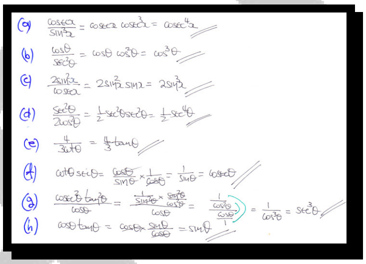
e) $\dfrac{4}{3\cot x}$

f) $\cot\theta\sec\theta$

g) $\dfrac{\operatorname{cosec}^2\theta\tan^2\theta}{\cos\theta}$

h) $\cos\theta\tan\theta$

$\boxed{\operatorname{cosec}^4 x}$, $\boxed{\cos^3\theta}$, $\boxed{2\sin^3 x}$, $\boxed{\frac{1}{2}\sec^4\theta}$, $\boxed{\frac{4}{3}\tan x}$, $\boxed{\operatorname{cosec}\theta}$, $\boxed{\sec^3\theta}$, $\boxed{\sin\theta}$


(a) $\dfrac{\operatorname{cosec}x}{\sin^3 x} = \operatorname{cosec}x \operatorname{cosec}^3 x = \operatorname{cosec}^4 x$
(b) $\dfrac{\cos\theta}{\sec^2\theta} = \cos\theta\cos^2\theta = \cos^3\theta$
(c) $\dfrac{2\sin^2 x}{\operatorname{cosec}x} = 2\sin^2 x\sin x = 2\sin^3 x$
(d) $\dfrac{\sec^2\theta}{2\cos^2\theta} = \frac{1}{2}\sec^2\theta\sec^2\theta = \frac{1}{2}\sec^4\theta$
(e) $\dfrac{4}{3\cot\theta} = \frac{4}{3}\tan\theta$
(f) $\cot\theta\sec\theta = \dfrac{\cancel{\cos\theta}}{\sin\theta}\times\dfrac{1}{\cancel{\cos\theta}} = \dfrac{1}{\sin\theta} = \operatorname{cosec}\theta$
(g) $\dfrac{\operatorname{cosec}^2\theta\tan^2\theta}{\cos\theta} = \dfrac{\frac{1}{\cancel{\sin^2\theta}}\times\frac{\cancel{\sin^2\theta}}{\cos^2\theta}}{\cos\theta} = \dfrac{\frac{1}{\cos^2\theta}}{\frac{\cos\theta}{1}} = \dfrac{1}{\cos^3\theta} = \sec^3\theta$
(h) $\cos\theta\tan\theta = \cancel{\cos\theta}\times\dfrac{\sin\theta}{\cancel{\cos\theta}} = \sin\theta$

Question 2

Simplify the following trigonometric expressions.

The final answer must not contain trigonometric fractions.

a) $\dfrac{1-\sin^2 x}{\sin^2 x}$

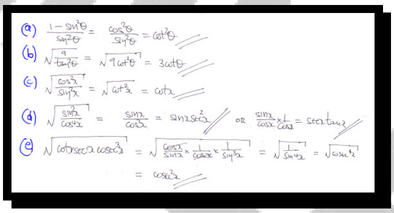
b) $\sqrt{\dfrac{9}{\tan^2\theta}}$

c) $\sqrt{\dfrac{\cos^2 x}{\sin^2 x}}$

d) $\sqrt{\dfrac{\sin^2 x}{\cos^4 x}}$

e) $\sqrt{\cot x \sec x \operatorname{cosec}^3 x}$

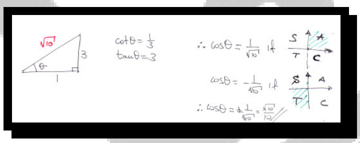
$\boxed{\cot^2 x}$, $\boxed{3\cot\theta}$, $\boxed{\cot x}$, $\boxed{\sec x \tan x}$, $\boxed{\operatorname{cosec}^2 x}$



Question 3

If $\cot\theta = \dfrac{1}{3}$, show that $\cos\theta = \pm\dfrac{\sqrt{10}}{10}$.

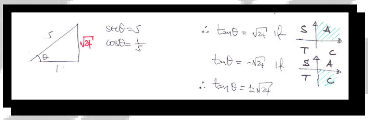
$\boxed{\text{proof}}$



Question 4

If $\sec\theta = 5$, show that $\tan\theta = \pm\sqrt{24}$.

$\boxed{\text{proof}}$



Question 5

Solve each of the following trigonometric equations.

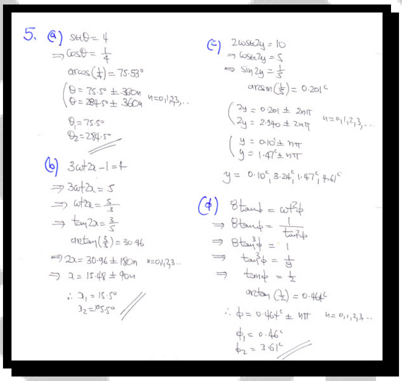
a) $\sec\theta = 4$, $\quad 0 \le \theta < 360°$

b) $3\cot 2x - 1 = 4$, $\quad 0 \le x < 180°$

c) $2\operatorname{cosec} 2y = 10$, $\quad 0 \le y < 2\pi$

d) $8\tan\varphi = \cot^2\varphi$, $\quad 0 \le \varphi < 2\pi$

$\boxed{\theta \approx 75.5°,\ 284.5°}$, $\boxed{x \approx 15.5°,\ 105.5°}$, $\boxed{y \approx 0.10^c,\ 1.47^c,\ 3.24^c,\ 4.61^c}$, $\boxed{\varphi \approx 0.46^c,\ 3.61^c}$


5. (a) $\sec\theta = 4$

$\Rightarrow \cos\theta = \frac{1}{4}$

$\arccos\left(\frac{1}{4}\right) = 75.53°$

$\theta = 75.5° \pm 360n$
$\theta = 284.5° \pm 360n$ $\quad n = 0,1,2,3,\ldots$

$\theta_1 = 75.5°$
$\theta_2 = 284.5°$

(b) $3\cot 2x - 1 = 4$

$\Rightarrow 3\cot 2x = 5$

$\Rightarrow \cot 2x = \frac{5}{3}$

$\Rightarrow \tan 2x = \frac{3}{5}$

$\arctan\left(\frac{3}{5}\right) = 30.96$

$\Rightarrow 2x = 30.96 \pm 180n$ $\quad n = 0,1,2,3\ldots$

$\Rightarrow x = 15.48 \pm 90n$

$\therefore x_1 = 15.5°$
$x_2 = 105.5°$

(c) $2\operatorname{cosec} 2y = 10$

$\Rightarrow \operatorname{cosec} 2y = 5$

$\Rightarrow \sin 2y = \frac{1}{5}$

$\arcsin\left(\frac{1}{5}\right) = 0.201^c$

$2y = 0.201 \pm 2n\pi$
$2y = 2.940 \pm 2n\pi$ $\quad n = 0,1,2,3,\ldots$

$y = 0.10^c \pm n\pi$
$y = 1.47^c \pm n\pi$

$y = 0.10^c, 3.24^c, 1.47^c, 4.61^c$

(d) $8\tan\phi = \cot^2\phi$

$\Rightarrow 8\tan\phi = \frac{1}{\tan^2\phi}$

$\Rightarrow 8\tan^3\phi = 1$

$\Rightarrow \tan^3\phi = \frac{1}{8}$

$\Rightarrow \tan\phi = \frac{1}{2}$

$\arctan\left(\frac{1}{2}\right) = 0.464^c$

$\therefore \phi = 0.464^c \pm n\pi$ $\quad n = 0,1,2,3\ldots$

$\phi_1 = 0.46^c$
$\phi_2 = 3.61^c$

Question 6

Solve each of the following trigonometric equations.

a) $2\sec\theta = 3, \quad 0 \le \theta < 360°$

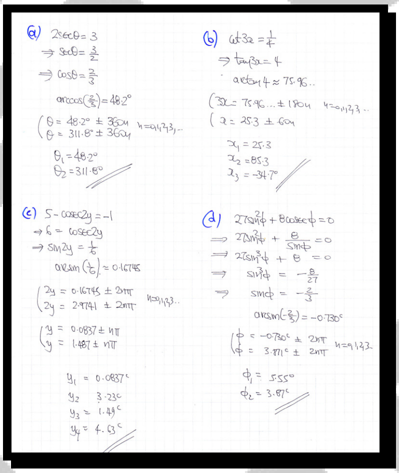
b) $\cot 3x = \frac{1}{4}, \quad -90° \le x < 90°$

c) $5 - \operatorname{cosec} 2y = -1, \quad 0 \le y < 2\pi$

d) $27\sin^2\varphi + 8\operatorname{cosec}\varphi = 0, \quad 0 \le \varphi < 2\pi$

$\boxed{\theta \approx 48.2°,\ 311.8°}$, $\boxed{x \approx -34.7°,\ 25.3°,\ 85.3°}$,

$\boxed{y \approx 0.0837^{c},\ 1.49^{c},\ 3.23^{c},\ 4.63^{c}}$, $\boxed{\varphi \approx 3.87^{c},\ 5.55^{c}}$



(a) $2\sec\theta = 3$
$\Rightarrow \sec\theta = \frac{3}{2}$
$\Rightarrow \cos\theta = \frac{2}{3}$
$\arccos\left(\frac{2}{3}\right) = 48.2°$
$\theta = 48.2° \pm 360n$
$\theta = 311.8° \pm 360n \quad n = 0,1,2,3,\ldots$
$\theta_1 = 48.2°$
$\theta_2 = 311.8°$

(b) $\cot 3x = \frac{1}{4}$
$\Rightarrow \tan 3x = 4$
$\arctan 4 \approx 75.96\ldots$
$3x = 75.96\ldots \pm 180n \quad n = 0,1,2,3\ldots$
$x = 25.3 \pm 60n$
$x_1 = 25.3$
$x_2 = 85.3$
$x_3 = -34.7°$

(c) $5 - \operatorname{cosec}2y = -1$
$\Rightarrow 6 = \operatorname{cosec}2y$
$\Rightarrow \sin 2y = \frac{1}{6}$
$\arcsin\left(\frac{1}{6}\right) \approx 0.16745$
$2y = 0.16745 \pm 2n\pi$
$2y = 2.9741 \pm 2n\pi \quad n = 0,1,2,3\ldots$
$y = 0.0837 \pm n\pi$
$y = 1.487 \pm n\pi$
$y_1 = 0.0837^{c}$
$y_2 \ 3.23^{c}$
$y_3 = 1.49^{c}$
$y_4 = 4.63^{c}$

(d) $27\sin^2\phi + 8\operatorname{cosec}\phi = 0$
$\Rightarrow 27\sin^2\phi + \frac{8}{\sin\phi} = 0$
$\Rightarrow 27\sin^3\phi + 8 = 0$
$\Rightarrow \sin^3\phi = -\frac{8}{27}$
$\Rightarrow \sin\phi = -\frac{2}{3}$
$\arcsin\left(-\frac{2}{3}\right) = -0.730^{c}$
$\phi = -0.730^{c} \pm 2n\pi$
$\phi = 3.871^{c} \pm 2n\pi \quad n = 0,1,2,3\ldots$
$\phi_1 = 5.55°$
$\phi_2 = 3.87^{c}$

Question 7

Solve each of the following trigonometric equations.

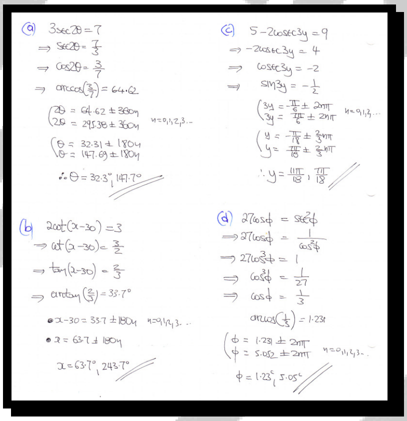
a) $3\sec 2\theta = 7$, $\quad 0 \le \theta < 180°$

b) $2\cot\left(x - 30°\right) = 3$, $\quad 0 \le x < 360°$

c) $5 - 2\operatorname{cosec} 3y = 9$, $\quad 0 \le y < \pi$

d) $27\cos\varphi = \sec^2\varphi$, $\quad 0 \le \varphi < 2\pi$

$\boxed{\theta \approx 32.3°,\ 147.7°}$, $\boxed{x \approx 63.7°,\ 243.7°}$, $\boxed{y = \frac{7\pi}{18},\ \frac{11\pi}{18}}$, $\boxed{\varphi \approx 1.23^{c},\ 5.05^{c}}$



(a) $3\sec 2\theta = 7$

$\Rightarrow \sec 2\theta = \frac{7}{3}$

$\Rightarrow \cos 2\theta = \frac{3}{7}$

$\Rightarrow \arccos\left(\frac{3}{7}\right) = 64.62$

$\begin{cases} 2\theta = 64.62 \pm 360n \\ 2\theta = 295.38 \pm 360n \end{cases}$ $n = 0, 1, 2, 3 \ldots$

$\begin{cases} \theta = 32.31 \pm 180n \\ \theta = 147.69 \pm 180n \end{cases}$

$\therefore \theta = 32.3°, 147.7°$

(b) $2\cot(x - 30) = 3$

$\Rightarrow \cot(x - 30) = \frac{3}{2}$

$\Rightarrow \tan(x - 30) = \frac{2}{3}$

$\Rightarrow \arctan\left(\frac{2}{3}\right) = 33.7°$

- $x - 30 = 33.7 \pm 180n$ $n = 0, 1, 2, 3 \ldots$
- $x = 63.7 \pm 180n$

$x = 63.7°, 243.7°$

(c) $5 - 2\operatorname{cosec} 3y = 9$

$\Rightarrow -2\operatorname{cosec} 3y = 4$

$\Rightarrow \operatorname{cosec} 3y = -2$

$\Rightarrow \sin 3y = -\frac{1}{2}$

$\begin{cases} 3y = -\frac{\pi}{6} \pm 2n\pi \\ 3y = \frac{7\pi}{6} \pm 2n\pi \end{cases}$ $n = 0, 1, 2 \ldots$

$\begin{cases} y = -\frac{\pi}{18} \pm \frac{2}{3}n\pi \\ y = \frac{7\pi}{18} \pm \frac{2}{3}n\pi \end{cases}$

$\therefore y = \frac{11\pi}{18}, \frac{7\pi}{18}$

(d) $27\cos\phi = \sec^2\phi$

$\Rightarrow 27\cos\phi = \frac{1}{\cos^2\phi}$

$\Rightarrow 27\cos^3\phi = 1$

$\Rightarrow \cos^3\phi = \frac{1}{27}$

$\Rightarrow \cos\phi = \frac{1}{3}$

$\arccos\left(\frac{1}{3}\right) = 1.231$

$\begin{cases} \phi = 1.231 \pm 2n\pi \\ \phi = 5.052 \pm 2n\pi \end{cases}$ $n = 0, 1, 2, 3 \ldots$

$\phi = 1.23^{c}, 5.05^{c}$

Question 8

Solve each of the following trigonometric equations.

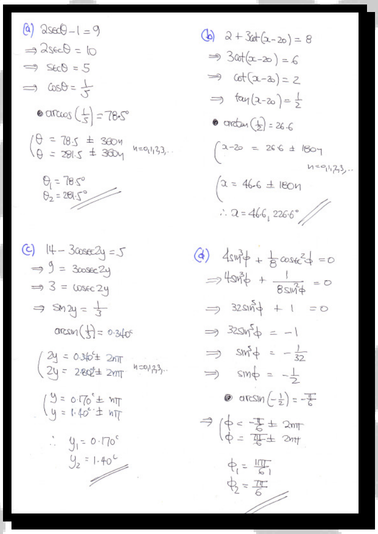
a) $2\sec\theta - 1 = 9$, $\quad 0 \le \theta < 360°$

b) $2 + 3\cot\left(x - 20°\right) = 8$, $\quad 0 \le x < 360°$

c) $14 - 3\operatorname{cosec} 2y = 5$, $\quad 0 \le y < \pi$

d) $4\sin^3\varphi + \frac{1}{8}\operatorname{cosec}^2\varphi = 0$, $\quad 0 \le \varphi < 2\pi$

$\boxed{\theta \approx 78.5°,\ 281.5°}$, $\boxed{x \approx 46.6°,\ 226.6°}$, $\boxed{y \approx 0.170^{c},\ 1.40^{c}}$, $\boxed{\varphi = \frac{7\pi}{6},\ \frac{11\pi}{6}}$



(a) $2\sec\theta - 1 = 9$

$\Rightarrow 2\sec\theta = 10$

$\Rightarrow \sec\theta = 5$

$\Rightarrow \cos\theta = \frac{1}{5}$

• $\arccos\left(\frac{1}{5}\right) = 78.5°$

$\begin{cases} \theta = 78.5 \pm 360n \\ \theta = 281.5 \pm 360n \end{cases}$ $n = 0, 1, 2, 3, \ldots$

$\theta_1 = 78.5°$

$\theta_2 = 281.5°$

(b) $2 + 3\cot(x - 20) = 8$

$\Rightarrow 3\cot(x - 20) = 6$

$\Rightarrow \cot(x - 20) = 2$

$\Rightarrow \tan(x - 20) = \frac{1}{2}$

• $\arctan\left(\frac{1}{2}\right) = 26.6$

$(x - 20 = 26.6 \pm 180n$ $\quad n = 0, 1, 2, 3, \ldots$

$(x = 46.6 \pm 180n$

$\therefore x = 46.6, 226.6°$

(c) $14 - 3\operatorname{cosec} 2y = 5$

$\Rightarrow 9 = 3\operatorname{cosec} 2y$

$\Rightarrow 3 = \operatorname{cosec} 2y$

$\Rightarrow \sin 2y = \frac{1}{3}$

$\arcsin\left(\frac{1}{3}\right) = 0.340^{c}$

$\begin{cases} 2y = 0.340^{c} \pm 2n\pi \\ 2y = 2.802^{c} \pm 2n\pi \end{cases}$ $n = 0, 1, 2, 3, \ldots$

$\begin{cases} y = 0.170^{c} \pm n\pi \\ y = 1.40^{c} \pm n\pi \end{cases}$

$\therefore y_1 = 0.170^{c}$

$y_2 = 1.40^{c}$

(d) $4\sin^3\phi + \frac{1}{8}\operatorname{cosec}^2\phi = 0$

$\Rightarrow 4\sin^3\phi + \frac{1}{8\sin^2\phi} = 0$

$\Rightarrow 32\sin^5\phi + 1 = 0$

$\Rightarrow 32\sin^5\phi = -1$

$\Rightarrow \sin^5\phi = -\frac{1}{32}$

$\Rightarrow \sin\phi = -\frac{1}{2}$

• $\arcsin\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$

$\Rightarrow \begin{cases} \phi = -\frac{\pi}{6} \pm 2n\pi \\ \phi = \frac{7\pi}{6} \pm 2n\pi \end{cases}$

$\phi_1 = \frac{11\pi}{6}$,

$\phi_2 = \frac{7\pi}{6}$

Question 9

Solve each of the following trigonometric equations.

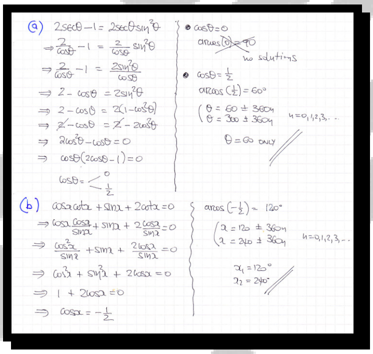
a) $2\sec\theta - 1 = 2\sec\theta\sin^2\theta$, $\quad 0 \le \theta < 180°$, $\theta \ne 90°$

b) $\cos x\cot x + \sin x + 2\cot x = 0$, $\quad 0 < x < 360°$, $\quad x \ne 180°$

c) $(\text{cosec}\, y - \sin y)\sec^2 y = 2$, $\quad 0 \le y < \pi$, $\quad y \ne \frac{\pi}{2}$

d) $\text{cosec}\,\varphi - \sin\varphi + 2\cos^2\varphi\cot\varphi = 0$, $\quad 0 < \varphi < 2\pi$, $\quad \varphi \ne \pi$

$\boxed{\theta = 60°}$, $\boxed{x = 120°, 240°}$, $\boxed{y = \frac{\pi}{6}, \frac{5\pi}{6}}$, $\boxed{\varphi = \frac{\pi}{2}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{3\pi}{2}}$



(a) $2\sec\theta - 1 = 2\sec\theta\sin^2\theta$

$\Rightarrow \frac{2}{\cos\theta} - 1 = \frac{2}{\cos\theta}\sin^2\theta$

$\Rightarrow \frac{2}{\cos\theta} - 1 = \frac{2\sin^2\theta}{\cos\theta}$

$\Rightarrow 2 - \cos\theta = 2\sin^2\theta$

$\Rightarrow 2 - \cos\theta = 2(1 - \cos^2\theta)$

$\Rightarrow \cancel{2} - \cos\theta = \cancel{2} - 2\cos^2\theta$

$\Rightarrow 2\cos^2\theta - \cos\theta = 0$

$\Rightarrow \cos\theta(2\cos\theta - 1) = 0$

$\cos\theta = \begin{cases} 0 \\ \frac{1}{2} \end{cases}$

• $\cos\theta = 0$

$\cancel{\arccos(0) = 90}$ no solutions

• $\cos\theta = \frac{1}{2}$

$\arccos\left(\frac{1}{2}\right) = 60°$

$\begin{cases} \theta = 60 \pm 360n \\ \theta = 300 \pm 360n \end{cases}$ $n = 0, 1, 2, 3, \ldots$

$\theta = 60$ ONLY

(b) $\cos x\cot x + \sin x + 2\cot x = 0$

$\Rightarrow \cos x\frac{\cos x}{\sin x} + \sin x + 2\frac{\cos x}{\sin x} = 0$

$\Rightarrow \frac{\cos^2 x}{\sin x} + \sin x + \frac{2\cos x}{\sin x} = 0$

$\Rightarrow \cos^2 x + \sin^2 x + 2\cos x = 0$

$\Rightarrow 1 + 2\cos x = 0$

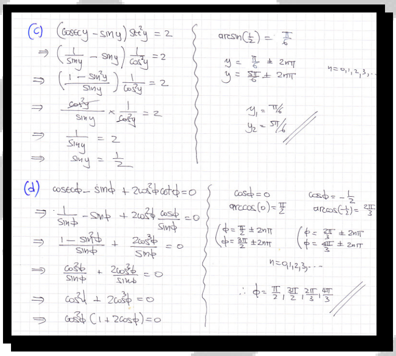
$\Rightarrow \cos x = -\frac{1}{2}$

$\arccos\left(-\frac{1}{2}\right) = 120°$

$\begin{cases} x = 120 \pm 360n \\ x = 240 \pm 360n \end{cases}$ $n = 0, 1, 2, 3, \ldots$

$x_1 = 120°$

$x_2 = 240°$



(c) $(\text{cosec}\, y - \sin y)\sec^2 y = 2$

$\Rightarrow \left(\frac{1}{\sin y} - \sin y\right)\frac{1}{\cos^2 y} = 2$

$\Rightarrow \left(\frac{1 - \sin^2 y}{\sin y}\right)\frac{1}{\cos^2 y} = 2$

$\Rightarrow \frac{\cancel{\cos^2 y}}{\sin y} \times \frac{1}{\cancel{\cos^2 y}} = 2$

$\Rightarrow \frac{1}{\sin y} = 2$

$\Rightarrow \sin y = \frac{1}{2}$

$\arcsin\left(\frac{1}{2}\right) = \frac{\pi}{6}$

$y = \frac{\pi}{6} \pm 2n\pi$

$y = \frac{5\pi}{6} \pm 2n\pi$ $\quad n = 0, 1, 2, 3, \ldots$

$y_1 = \pi/6$

$y_2 = 5\pi/6$

(d) $\text{cosec}\,\phi - \sin\phi + 2\cos^2\phi\cot\phi = 0$

$\Rightarrow \frac{1}{\sin\phi} - \sin\phi + 2\cos^2\phi\frac{\cos\phi}{\sin\phi} = 0$

$\Rightarrow \frac{1 - \sin^2\phi}{\sin\phi} + \frac{2\cos^3\phi}{\sin\phi} = 0$

$\Rightarrow \frac{\cos^2\phi}{\sin\phi} + \frac{2\cos^3\phi}{\sin\phi} = 0$

$\Rightarrow \cos^2\phi + 2\cos^3\phi = 0$

$\Rightarrow \cos^2\phi(1 + 2\cos\phi) = 0$

$\cos\phi = 0$

$\arccos(0) = \frac{\pi}{2}$

$\begin{cases} \phi = \frac{\pi}{2} \pm 2n\pi \\ \phi = \frac{3\pi}{2} \pm 2n\pi \end{cases}$

$\cos\phi = -\frac{1}{2}$

$\arccos\left(-\frac{1}{2}\right) = \frac{2\pi}{3}$

$\begin{cases} \phi = \frac{2\pi}{3} \pm 2n\pi \\ \phi = \frac{4\pi}{3} \pm 2n\pi \end{cases}$

$n = 0, 1, 2, 3, \ldots$

$\therefore \phi = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{2\pi}{3}, \frac{4\pi}{3}$

Question 10

Solve each of the following trigonometric equations.

a) $\sec\theta + \cos\theta = \frac{5}{2}$, $\quad 0 \le \theta < 360°$, $\quad \theta \ne 90°, 270°$

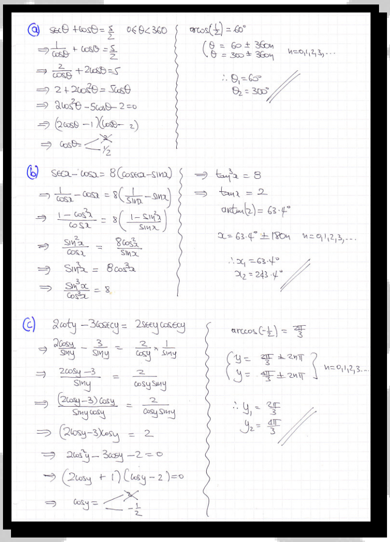
b) $\sec x - \cos x = 8(\operatorname{cosec} x - \sin x)$, $\quad 0 \le x < 360°$, $\quad x \ne 90°$

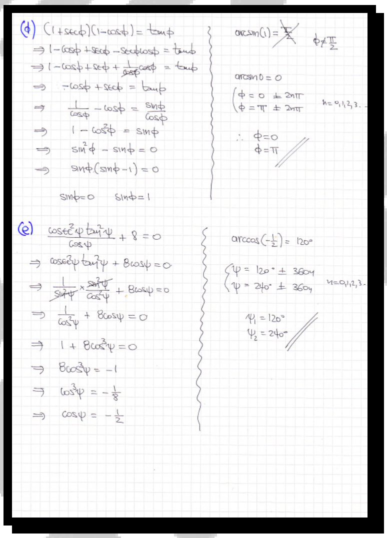
c) $2\cot y - 3\operatorname{cosec} y = 2\sec y \operatorname{cosec} y$, $\quad 0 < y < 2\pi$, $\quad y \ne \frac{k\pi}{2}, k \in \mathbb{Z}$

d) $(1+\sec\varphi)(1-\cos\varphi) = \tan\varphi$, $\quad 0 \le \varphi < 2\pi$, $\quad \varphi \ne \frac{\pi}{2}, \frac{3\pi}{2}$

e) $\frac{\operatorname{cosec}^2\psi \tan^2\psi}{\cos\psi} + 8 = 0$, $\quad 0 \le \psi < 360°$, $\quad \psi \ne 90°, 270°$

$\boxed{\theta = 60°, 300°}$, $\boxed{x = 63.4°, 243.6°}$, $\boxed{y = \frac{2\pi}{3}, \frac{4\pi}{3}}$, $\boxed{\varphi = 0, \pi}$, $\boxed{\psi = 120°, 240°}$





Question 11 (hard questions)

Solve each of the following trigonometric equations.

a) $2\sin\theta + 3\sec\theta = 6 + \tan\theta$, $\quad 0 \le \theta < 2\pi$, $\theta \ne \dfrac{\pi}{2}, \dfrac{3\pi}{2}$

b) $\sin^2 x \tan x + \cos^2 x \cot x + 2\sin x \cos x = 2$, $\quad 0 < x < 360°$, $x \ne 90°, 180°, 270°$

you may use in this part the fact that $\boxed{2\sin x \cos x \equiv \sin 2x}$

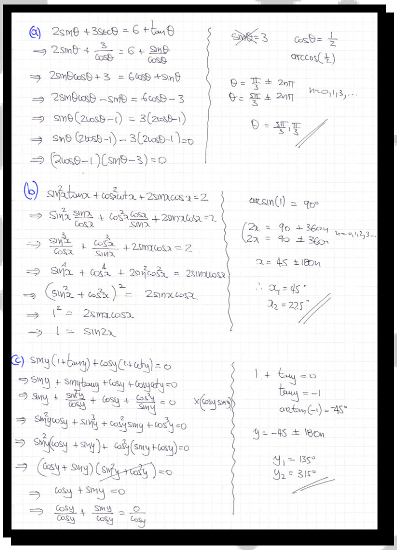
c) $\sin y\left(1 + \tan y\right) + \cos y\left(1 + \cot y\right) = 0$, $\quad 0 < y < 360°$, $y \ne 90°, 180°, 270°$

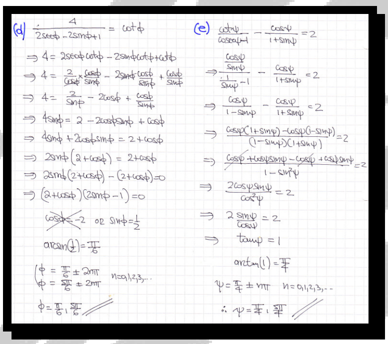
d) $\dfrac{4}{2\sec\varphi - 2\sin\varphi + 1} = \cot\varphi$, $\quad 0 < \varphi < 2\pi$, $\varphi \ne \pi$

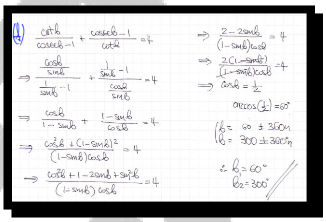
e) $\dfrac{\cot\psi}{\operatorname{cosec}\psi - 1} - \dfrac{\cos\psi}{1 + \sin\psi} = 2$, $\quad 0 < \psi < 2\pi$, $\psi \ne \dfrac{\pi}{2}, \pi, \dfrac{3\pi}{2}$

f) $\dfrac{\cot\beta}{\operatorname{cosec}\beta - 1} + \dfrac{\operatorname{cosec}\beta - 1}{\cot\beta} = 4$, $\quad 0 \le x < 360°$, $x \ne 90°, 180°, 270°$

$\boxed{\varphi = \dfrac{\pi}{3}, \dfrac{5\pi}{3}}$, $\boxed{x = 45°, 225°}$, $\boxed{y = 135°, 315°}$, $\boxed{\varphi = \dfrac{\pi}{6}, \dfrac{5\pi}{6}}$, $\boxed{\psi = \dfrac{\pi}{4}, \dfrac{5\pi}{4}}$, $\boxed{\beta = 60°, 300°}$







Question 12

Prove the validity of each of the following trigonometric identities.

a) $(2\cos x + \sin x)^2 + (\cos x - 2\sin x)^2 \equiv 5$

b) $\cos x \sin x (\cot x + \tan x) \equiv 1$

c) $\cot x + \tan x \equiv \sec x \operatorname{cosec} x$

d) $\sec\theta - \sec\theta \sin^2\theta \equiv \cos\theta$

e) $(1-\sin\theta)(1+\operatorname{cosec}\theta) \equiv \cos\theta \cot\theta$

(a) $\text{LHS} = (2\cos x + \sin x)^2 + (\cos x - 2\sin x)^2$
$= 4\cos^2 x + \cancel{4\cos x\sin x} + \sin^2 x + \cos^2 x - \cancel{4\cos x\sin x} + 4\sin^2 x$
$= 5\cos^2 x + 5\sin^2 x = 5(\cos^2 x + \sin^2 x) = 5 \times 1 = 5 = \text{RHS}$

(b) $\text{LHS} = \cos x\sin x(\cot x + \tan x) = \cos x \sin x \cot x + \cos x \sin x \tan x$
$= \cos x\cancel{\sin x} \times \frac{\cos x}{\cancel{\sin x}} + \sin x\cancel{\cos x} \times \frac{\sin x}{\cancel{\cos x}} = \cos^2 x + \sin^2 x = 1 = \text{RHS}$

(c) $\text{LHS} = \cot x + \tan x = \frac{\cos x}{\sin x} + \frac{\sin x}{\cos x} = \frac{\cos^2 x + \sin^2 x}{\sin x\cos x} = \frac{1}{\sin x \cos x}$
$= \frac{1}{\sin x} \times \frac{1}{\cos x} = \operatorname{cosec} x \sec x = \text{RHS}$

(d) $\text{LHS} = \sec\theta - \sec\theta\sin^2\theta = \sec\theta(1-\sin^2\theta) = \sec\theta\cos^2\theta = \frac{1}{\cancel{\cos\theta}} \times \cos^2\theta$
$= \cos\theta = \text{RHS}$

(e) $\text{LHS} = (1-\sin\theta)(1+\operatorname{cosec}\theta) = 1 + \operatorname{cosec}\theta - \sin\theta - \sin\theta\operatorname{cosec}\theta$
$= 1 + \operatorname{cosec}\theta - \sin\theta - \cancel{\sin\theta} \times \frac{1}{\cancel{\sin\theta}} = \cancel{1} + \operatorname{cosec}\theta - \sin\theta - \cancel{1} = \frac{1}{\sin\theta} - \sin\theta$
$= \frac{1-\sin^2\theta}{\sin\theta} = \frac{\cos^2\theta}{\sin\theta} = \cos\theta \times \frac{\cos\theta}{\sin\theta} = \cos\theta\cot\theta = \text{RHS}$

Question 13

Prove the validity of each of the following trigonometric identities.

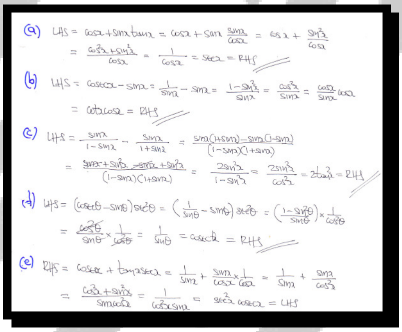
a) $\cos x + \sin x \tan x \equiv \sec x$

b) $\operatorname{cosec} x - \sin x \equiv \cos x \cot x$

c) $\dfrac{\sin x}{1-\sin x} - \dfrac{\sin x}{1+\sin x} \equiv 2\tan^2 x$

d) $\left(\operatorname{cosec}\theta - \sin\theta\right)\sec^2\theta \equiv \operatorname{cosec}\theta$

e) $\operatorname{cosec} x \sec^2 x \equiv \operatorname{cosec} x + \tan x \sec x$



(a) $\text{LHS} = \cos x + \sin x \tan x = \cos x + \sin x \dfrac{\sin x}{\cos x} = \cos x + \dfrac{\sin^2 x}{\cos x}$
$= \dfrac{\cos^2 x + \sin^2 x}{\cos x} = \dfrac{1}{\cos x} = \sec x = \text{RHS}$

(b) $\text{LHS} = \operatorname{cosec} x - \sin x = \dfrac{1}{\sin x} - \sin x = \dfrac{1-\sin^2 x}{\sin x} = \dfrac{\cos^2 x}{\sin x} = \dfrac{\cos x}{\sin x}\cos x$
$= \cot x \cos x = \text{RHS}$

(c) $\text{LHS} = \dfrac{\sin x}{1-\sin x} - \dfrac{\sin x}{1+\sin x} = \dfrac{\sin x(1+\sin x) - \sin x(1-\sin x)}{(1-\sin x)(1+\sin x)}$
$= \dfrac{\cancel{\sin x} + \sin^2 x \cancel{- \sin x} + \sin^2 x}{(1-\sin x)(1+\sin x)} = \dfrac{2\sin^2 x}{1-\sin^2 x} = \dfrac{2\sin^2 x}{\cos^2 x} = 2\tan^2 x = \text{RHS}$

(d) $\text{LHS} = (\operatorname{cosec}\theta - \sin\theta)\sec^2\theta = \left(\dfrac{1}{\sin\theta} - \sin\theta\right)\sec^2\theta = \left(\dfrac{1-\sin^2\theta}{\sin\theta}\right) \times \dfrac{1}{\cos^2\theta}$
$= \dfrac{\cancel{\cos^2\theta}}{\sin\theta} \times \dfrac{1}{\cancel{\cos^2\theta}} = \dfrac{1}{\sin\theta} = \operatorname{cosec}\theta = \text{RHS}$

(e) $\text{RHS} = \operatorname{cosec} x + \tan x \sec x = \dfrac{1}{\sin x} + \dfrac{\sin x}{\cos x} \times \dfrac{1}{\cos x} = \dfrac{1}{\sin x} + \dfrac{\sin x}{\cos^2 x}$
$= \dfrac{\cos^2 x + \sin^2 x}{\sin x \cos^2 x} = \dfrac{1}{\cos^2 x \sin x} = \sec^2 x \operatorname{cosec} x = \text{LHS}$

Question 14

Prove the validity of each of the following trigonometric identities.

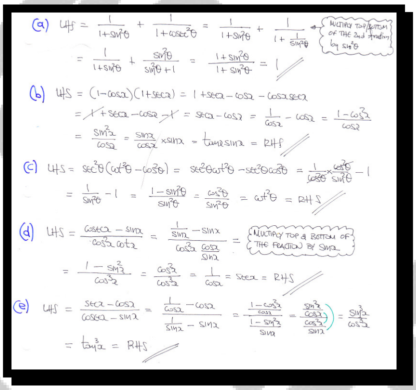
a) $\dfrac{1}{1+\sin^2\theta}+\dfrac{1}{1+\text{cosec}^2\theta}\equiv 1$

b) $(1-\cos x)(1+\sec x)\equiv \sin x\tan x$

c) $\sec^2\theta\left(\cot^2\theta-\cos^2\theta\right)\equiv \cot^2\theta$

d) $\dfrac{\text{cosec}\,x-\sin x}{\cos^2 x\cot x}\equiv \sec x$

e) $\dfrac{\sec x-\cos x}{\text{cosec}\,x-\sin x}\equiv \tan^3 x$



(a) $\text{LHS} = \dfrac{1}{1+\sin^2\theta}+\dfrac{1}{1+\text{cosec}^2\theta} = \dfrac{1}{1+\sin^2\theta}+\dfrac{1}{1+\frac{1}{\sin^2\theta}}$ ← MULTIPLY TOP/BOTTOM OF THE 2nd fraction by $\sin^2\theta$

$= \dfrac{1}{1+\sin^2\theta}+\dfrac{\sin^2\theta}{\sin^2\theta+1} = \dfrac{1+\sin^2\theta}{1+\sin^2\theta} = 1$

(b) $\text{LHS} = (1-\cos x)(1+\sec x) = 1+\sec x-\cos x-\cos x\sec x$

$= \cancel{1}+\sec x-\cos x-\cancel{1} = \sec x-\cos x = \dfrac{1}{\cos x}-\cos x = \dfrac{1-\cos^2 x}{\cos x}$

$= \dfrac{\sin^2 x}{\cos x} = \dfrac{\sin x}{\cos x}\times\sin x = \tan x\sin x = \text{RHS}$

(c) $\text{LHS} = \sec^2\theta(\cot^2\theta-\cos^2\theta) = \sec^2\theta\cot^2\theta-\sec^2\theta\cos^2\theta = \dfrac{1}{\cancel{\cos^2\theta}}\times\dfrac{\cancel{\cos^2\theta}}{\sin^2\theta}-1$

$= \dfrac{1}{\sin^2\theta}-1 = \dfrac{1-\sin^2\theta}{\sin^2\theta} = \dfrac{\cos^2\theta}{\sin^2\theta} = \cot^2\theta = \text{RHS}$

(d) $\text{LHS} = \dfrac{\text{cosec}\,x-\sin x}{\cos^2 x\cot x} = \dfrac{\frac{1}{\sin x}-\sin x}{\cos^2 x\,\frac{\cos x}{\sin x}} =$ MULTIPLY TOP & BOTTOM OF THE FRACTION BY $\sin x$

$= \dfrac{1-\sin^2 x}{\cos^3 x} = \dfrac{\cos^2 x}{\cos^3 x} = \dfrac{1}{\cos x} = \sec x = \text{RHS}$

(e) $\text{LHS} = \dfrac{\sec x-\cos x}{\text{cosec}\,x-\sin x} = \dfrac{\frac{1}{\cos x}-\cos x}{\frac{1}{\sin x}-\sin x} = \dfrac{\frac{1-\cos^2 x}{\cos x}}{\frac{1-\sin^2 x}{\sin x}} = \dfrac{\frac{\sin^2 x}{\cos x}}{\frac{\cos^2 x}{\sin x}} = \dfrac{\sin^3 x}{\cos^3 x}$

$= \tan^3 x = \text{RHS}$

Question 15

Prove the validity of each of the following trigonometric identities.

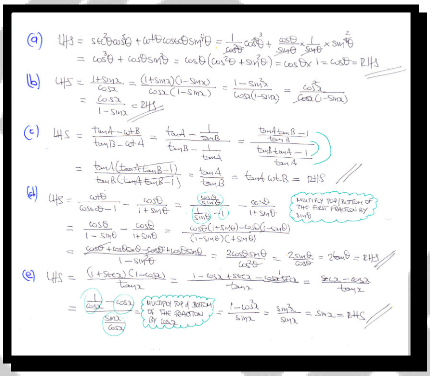
a) $\sec^2\theta\cos^5\theta + \cot\theta\,\text{cosec}\,\theta\sin^4\theta \equiv \cos\theta$

b) $\dfrac{1+\sin x}{\cos x} \equiv \dfrac{\cos x}{1-\sin x}$

c) $\dfrac{\tan A - \cot B}{\tan B - \cot A} \equiv \tan A\cot B$

d) $\dfrac{\cot\theta}{\text{cosec}\,\theta - 1} - \dfrac{\cos\theta}{1+\sin\theta} \equiv 2\tan\theta$

e) $\dfrac{(1+\sec x)(1-\cos x)}{\tan x} \equiv \sin x$



Question 16

Prove the validity of each of the following trigonometric identities.

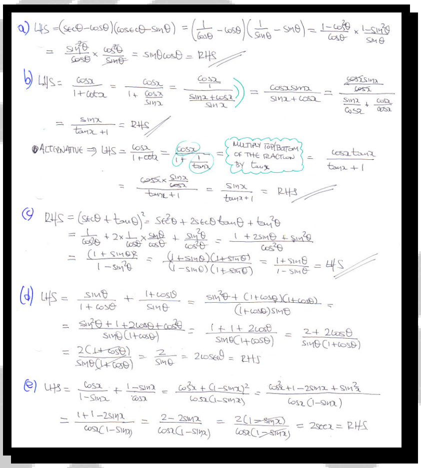
a) $(\sec\theta - \cos\theta)(\operatorname{cosec}\theta - \sin\theta) \equiv \sin\theta\cos\theta$

b) $\dfrac{\cos x}{1+\cot x} \equiv \dfrac{\sin x}{1+\tan x}$

c) $\dfrac{1+\sin\theta}{1-\sin\theta} \equiv (\sec\theta + \tan\theta)^2$

d) $\dfrac{\sin\theta}{1+\cos\theta} + \dfrac{1+\cos\theta}{\sin\theta} \equiv 2\operatorname{cosec}\theta$

e) $\dfrac{\cos x}{1-\sin x} + \dfrac{1-\sin x}{\cos x} \equiv 2\sec x$



a) $\text{LHS} = (\sec\theta - \cos\theta)(\operatorname{cosec}\theta - \sin\theta) = \left(\frac{1}{\cos\theta} - \cos\theta\right)\left(\frac{1}{\sin\theta} - \sin\theta\right) = \frac{1-\cos^2\theta}{\cos\theta} \times \frac{1-\sin^2\theta}{\sin\theta}$

$= \frac{\sin^2\theta}{\cos\theta} \times \frac{\cos^2\theta}{\sin\theta} = \sin\theta\cos\theta = \text{RHS}$

b) $\text{LHS} = \frac{\cos x}{1+\cot x} = \frac{\cos x}{1+\frac{\cos x}{\sin x}} = \frac{\frac{\cos x}{1}}{\frac{\sin x + \cos x}{\sin x}} = \frac{\cos x \sin x}{\sin x + \cos x} = \frac{\frac{\cos x \sin x}{\cos x}}{\frac{\sin x}{\cos x} + \frac{\cos x}{\cos x}}$

$= \frac{\sin x}{\tan x + 1} = \text{RHS}$

ALTERNATIVE $\Rightarrow \text{LHS} = \frac{\cos x}{1+\cot x} = \frac{\cos x}{1+\frac{1}{\tan x}} =$ MULTIPLY TOP/BOTTOM OF THE FRACTION BY $\tan x$ $= \frac{\cos x \tan x}{\tan x + 1}$

$= \frac{\cos x \times \frac{\sin x}{\cos x}}{\tan x + 1} = \frac{\sin x}{\tan x + 1} = \text{RHS}$

c) $\text{RHS} = (\sec\theta + \tan\theta)^2 = \sec^2\theta + 2\sec\theta\tan\theta + \tan^2\theta$

$= \frac{1}{\cos^2\theta} + 2\times\frac{1}{\cos\theta}\times\frac{\sin\theta}{\cos\theta} + \frac{\sin^2\theta}{\cos^2\theta} = \frac{1+2\sin\theta+\sin^2\theta}{\cos^2\theta}$

$= \frac{(1+\sin\theta)^2}{1-\sin^2\theta} = \frac{(1+\sin\theta)\cancel{(1+\sin\theta)}}{(1-\sin\theta)\cancel{(1+\sin\theta)}} = \frac{1+\sin\theta}{1-\sin\theta} = \text{LHS}$

d) $\text{LHS} = \frac{\sin\theta}{1+\cos\theta} + \frac{1+\cos\theta}{\sin\theta} = \frac{\sin^2\theta + (1+\cos\theta)(1+\cos\theta)}{(1+\cos\theta)\sin\theta} =$

$= \frac{\sin^2\theta + 1 + 2\cos\theta + \cos^2\theta}{\sin\theta(1+\cos\theta)} = \frac{1+1+2\cos\theta}{\sin\theta(1+\cos\theta)} = \frac{2+2\cos\theta}{\sin\theta(1+\cos\theta)}$

$= \frac{2\cancel{(1+\cos\theta)}}{\sin\theta\cancel{(1+\cos\theta)}} = \frac{2}{\sin\theta} = 2\operatorname{cosec}\theta = \text{RHS}$

e) $\text{LHS} = \frac{\cos x}{1-\sin x} + \frac{1-\sin x}{\cos x} = \frac{\cos^2 x + (1-\sin x)^2}{\cos x(1-\sin x)} = \frac{\cos^2 x + 1 - 2\sin x + \sin^2 x}{\cos x(1-\sin x)}$

$= \frac{1+1-2\sin x}{\cos x(1-\sin x)} = \frac{2-2\sin x}{\cos x(1-\sin x)} = \frac{2\cancel{(1-\sin x)}}{\cos x\cancel{(1-\sin x)}} = 2\sec x = \text{RHS}$

Question 17

Prove the validity of each of the following trigonometric identities.

a) $(\tan\theta+\cot\theta)(\sin\theta+\cos\theta)\equiv\sec\theta+\operatorname{cosec}\theta$

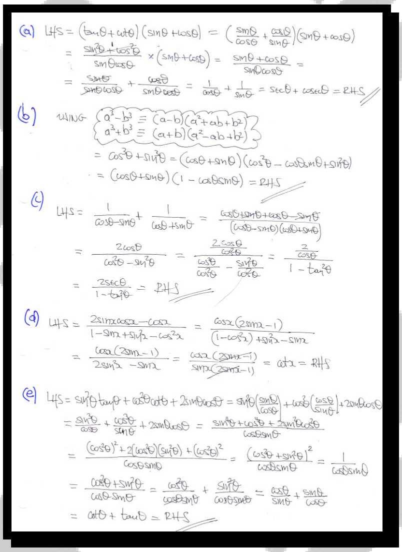
b) $\cos^3\theta+\sin^3\theta\equiv(\cos\theta+\sin\theta)(1-\sin\theta\cos\theta)$

c) $\dfrac{1}{\cos\theta-\sin\theta}+\dfrac{1}{\cos\theta+\sin\theta}\equiv\dfrac{2\sec\theta}{1-\tan^2\theta}$

d) $\dfrac{2\sin x\cos x-\cos x}{1-\sin x+\sin^2 x-\cos^2 x}\equiv\cot x$

e) $\sin^2\theta\tan\theta+\cos^2\theta\cot\theta+2\sin\theta\cos\theta\equiv\tan\theta+\cot\theta$

f) $\sin\theta(1+\tan\theta)+\cos\theta(1+\cot\theta)\equiv\sec\theta+\operatorname{cosec}\theta$



(a) $\text{LHS}=(\tan\theta+\cot\theta)(\sin\theta+\cos\theta)=\left(\frac{\sin\theta}{\cos\theta}+\frac{\cos\theta}{\sin\theta}\right)(\sin\theta+\cos\theta)$

$=\frac{\sin^2\theta+\cos^2\theta}{\sin\theta\cos\theta}\times(\sin\theta+\cos\theta)=\frac{\sin\theta+\cos\theta}{\sin\theta\cos\theta}=$

$=\frac{\sin\theta}{\sin\theta\cos\theta}+\frac{\cos\theta}{\sin\theta\cos\theta}=\frac{1}{\cos\theta}+\frac{1}{\sin\theta}=\sec\theta+\operatorname{cosec}\theta=\text{RHS}$

(b) USING $a^3-b^3\equiv(a-b)(a^2+ab+b^2)$, $a^3+b^3\equiv(a+b)(a^2-ab+b^2)$

$=\cos^3\theta+\sin^3\theta=(\cos\theta+\sin\theta)(\cos^2\theta-\cos\theta\sin\theta+\sin^2\theta)$

$=(\cos\theta+\sin\theta)(1-\cos\theta\sin\theta)=\text{RHS}$

(c) $\text{LHS}=\frac{1}{\cos\theta-\sin\theta}+\frac{1}{\cos\theta+\sin\theta}=\frac{\cos\theta+\sin\theta+\cos\theta-\sin\theta}{(\cos\theta-\sin\theta)(\cos\theta+\sin\theta)}$

$=\frac{2\cos\theta}{\cos^2\theta-\sin^2\theta}=\frac{\frac{2\cos\theta}{\cos^2\theta}}{\frac{\cos^2\theta}{\cos^2\theta}-\frac{\sin^2\theta}{\cos^2\theta}}=\frac{\frac{2}{\cos\theta}}{1-\tan^2\theta}$

$=\frac{2\sec\theta}{1-\tan^2\theta}=\text{RHS}$

(d) $\text{LHS}=\frac{2\sin x\cos x-\cos x}{1-\sin x+\sin^2 x-\cos^2 x}=\frac{\cos x(2\sin x-1)}{(1-\cos^2 x)+\sin^2 x-\sin x}$

$=\frac{\cos x(2\sin x-1)}{2\sin^2 x-\sin x}=\frac{\cos x(2\sin x-1)}{\sin x(2\sin x-1)}=\cot x=\text{RHS}$

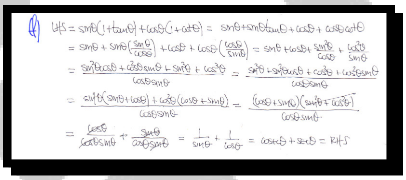
(e) $\text{LHS}=\sin^2\theta\tan\theta+\cos^2\theta\cot\theta+2\sin\theta\cos\theta=\sin^2\theta\left(\frac{\sin\theta}{\cos\theta}\right)+\cos^2\theta\left(\frac{\cos\theta}{\sin\theta}\right)+2\sin\theta\cos\theta$

$=\frac{\sin^3\theta}{\cos\theta}+\frac{\cos^3\theta}{\sin\theta}+2\sin\theta\cos\theta=\frac{\sin^4\theta+\cos^4\theta+2\sin^2\theta\cos^2\theta}{\cos\theta\sin\theta}$

$=\frac{(\cos^2\theta)^2+2(\cos^2\theta)(\sin^2\theta)+(\cos^2\theta)^2}{\cos\theta\sin\theta}=\frac{(\cos^2\theta+\sin^2\theta)^2}{\cos\theta\sin\theta}=\frac{1}{\cos\theta\sin\theta}$

$=\frac{\cos^2\theta+\sin^2\theta}{\cos\theta\sin\theta}=\frac{\cos^2\theta}{\cos\theta\sin\theta}+\frac{\sin^2\theta}{\cos\theta\sin\theta}=\frac{\cos\theta}{\sin\theta}+\frac{\sin\theta}{\cos\theta}$

$=\cot\theta+\tan\theta=\text{RHS}$



(f) $\text{LHS}=\sin\theta(1+\tan\theta)+\cos\theta(1+\cot\theta)=\sin\theta+\sin\theta\tan\theta+\cos\theta+\cos\theta\cot\theta$

$=\sin\theta+\sin\theta\left(\frac{\sin\theta}{\cos\theta}\right)+\cos\theta+\cos\theta\left(\frac{\cos\theta}{\sin\theta}\right)=\sin\theta+\cos\theta+\frac{\sin^2\theta}{\cos\theta}+\frac{\cos^2\theta}{\sin\theta}$

$=\frac{\sin^2\theta\cos\theta+\cos^2\theta\sin\theta+\sin^3\theta+\cos^3\theta}{\cos\theta\sin\theta}=\frac{\sin^3\theta+\sin^2\theta\cos\theta+\cos^3\theta+\cos^2\theta\sin\theta}{\cos\theta\sin\theta}$

$=\frac{\sin^2\theta(\sin\theta+\cos\theta)+\cos^2\theta(\cos\theta+\sin\theta)}{\cos\theta\sin\theta}=\frac{(\cos\theta+\sin\theta)(\sin^2\theta+\cos^2\theta)}{\cos\theta\sin\theta}$

$=\frac{\cos\theta}{\cos\theta\sin\theta}+\frac{\sin\theta}{\cos\theta\sin\theta}=\frac{1}{\sin\theta}+\frac{1}{\cos\theta}=\operatorname{cosec}\theta+\sec\theta=\text{RHS}$