FUNCTION PRACTICE

FUNCTION INTRODUCTION

Question 1

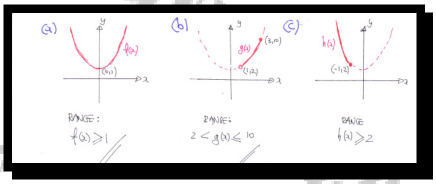
Find the range for each of the following functions.

a) $f(x) = x^2 + 1, \quad x \in \mathbb{R}$.

b) $g(x) = x^2 + 1, \quad x \in \mathbb{R}, \ 1 < x \le 3$.

c) $h(x) = x^2 + 1, \quad x \in \mathbb{R}, \ x \le -1$.

$\boxed{f(x) \in \mathbb{R}, \ f(x) \ge 1}, \ \boxed{g(x) \in \mathbb{R}, \ 2 < g(x) \le 10}, \ \boxed{h(x) \in \mathbb{R}, \ h(x) \ge 2}$



Question 2

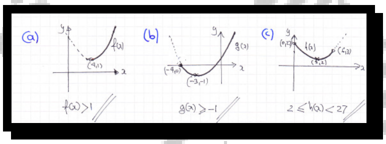
Find the range for each of the following functions.

a) $f(x) = (x-4)^2 + 1, \quad x \in \mathbb{R}, \ x > 4$.

b) $g(x) = (x+3)^2 - 1, \quad x \in \mathbb{R}, \ x \ge -4$.

c) $h(x) = (x-5)^2 + 2, \quad x \in \mathbb{R}, \ 0 < x < 6$.

$\boxed{f(x) \in \mathbb{R}, \ f(x) > 1}, \ \boxed{g(x) \in \mathbb{R}, \ g(x) \ge -1}, \ \boxed{h(x) \in \mathbb{R}, \ 2 \le h(x) < 27}$



Question 3

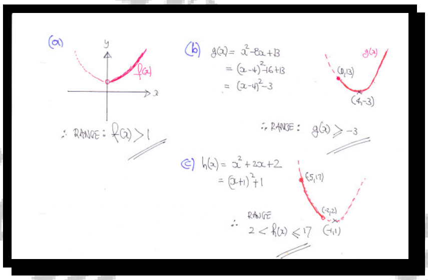
Find the range for each of the following functions.

a) $f(x) = x^2 + 1, \quad x \in \mathbb{R}, \; x > 0$.

b) $g(x) = x^2 - 8x + 13, \quad x \in \mathbb{R}, \; x \ge 0$.

c) $h(x) = x^2 + 2x + 2, \quad x \in \mathbb{R}, \; -5 \le x < -2$.

$\boxed{f(x) \in \mathbb{R}, \; f(x) > 1}$, $\boxed{g(x) \in \mathbb{R}, \; g(x) \ge -3}$, $\boxed{h(x) \in \mathbb{R}, \quad 2 < h(x) \le 17}$



Question 4

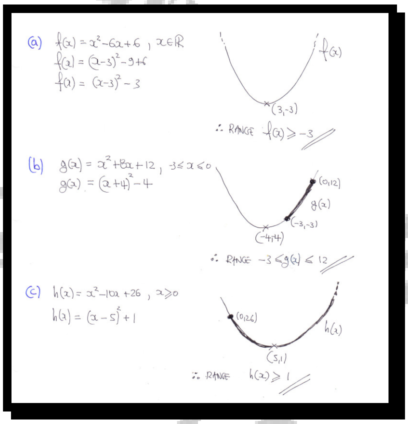
Find the range for each of the following functions.

a) $f(x) = x^2 - 6x + 6, \quad x \in \mathbb{R}$.

b) $g(x) = x^2 + 8x + 12, \quad x \in \mathbb{R}, -3 \le x \le 0$.

c) $h(x) = x^2 - 10x + 26, \quad x \in \mathbb{R}, x \ge 0$.

$\boxed{f(x) \in \mathbb{R},\ f(x) \ge -3}$, $\boxed{g(x) \in \mathbb{R},\ -3 \le g(x) \le 12}$, $\boxed{h(x) \in \mathbb{R},\ h(x) \ge 1}$



Question 5

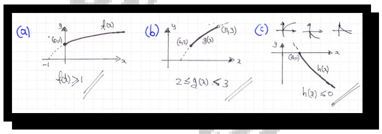
Find the range for each of the following functions.

a) $f(x) = \sqrt{x+1},\ \ x \in \mathbb{R},\ \ x \geq 0$.

b) $g(x) = \sqrt{x-2},\ \ x \in \mathbb{R},\ \ 6 \leq x < 11$.

c) $h(x) = 2 - \sqrt{x},\ \ x \in \mathbb{R},\ \ x \geq 4$.

$\boxed{f(x) \in \mathbb{R},\ f(x) \geq 1}$, $\boxed{g(x) \in \mathbb{R},\ 2 \leq g(x) < 3}$, $\boxed{h(x) \in \mathbb{R},\ h(x) \leq 0}$



Question 6

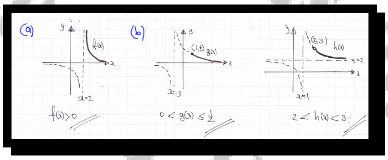
Find the range for each of the following functions.

a) $f(x) = \dfrac{1}{x-2},\ \ x \in \mathbb{R},\ \ x > 2$.

b) $g(x) = \dfrac{2}{x+3},\ \ x \in \mathbb{R},\ \ x \geq 1$.

c) $h(x) = \dfrac{1}{x-1} + 2,\ \ x \in \mathbb{R},\ \ x > 2$.

$\boxed{f(x) \in \mathbb{R},\ f(x) > 0}$, $\boxed{g(x) \in \mathbb{R},\ 0 < g(x) \leq \frac{1}{2}}$, $\boxed{h(x) \in \mathbb{R},\ 2 < h(x) < 3}$



Question 7

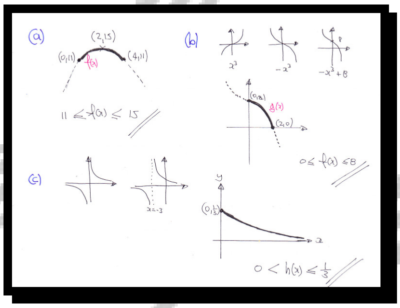
Find the range for each of the following functions.

a) $f(x) = 15 - (x-2)^2$, $x \in \mathbb{R}$, $0 \le x \le 4$.

b) $g(x) = 8 - x^3$, $x \in \mathbb{R}$, $0 \le x \le 2$.

c) $h(x) = \dfrac{1}{x+3}$, $x \in \mathbb{R}$, $x \ge 0$.

$\boxed{f(x) \in \mathbb{R},\, 11 \le f(x) \le 15}$, $\boxed{g(x) \in \mathbb{R},\, 0 \le g(x) \le 8}$, $\boxed{h(x) \in \mathbb{R},\, 0 < h(x) \le \frac{1}{3}}$



Question 8

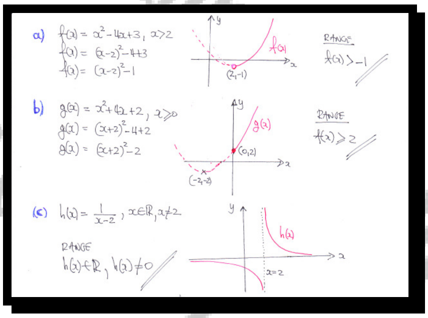
Find the range for each of the following functions.

a) $f(x) = x^2 - 4x + 3, \ \ x \in \mathbb{R}, \ x > 2$.

b) $g(x) = x^2 + 4x + 2, \ \ x \in \mathbb{R}, \ x \geq 0$.

c) $h(x) = \dfrac{1}{x-2}, \ \ x \in \mathbb{R}, \ x \neq 2$.

$\boxed{f(x) \in \mathbb{R}, \ f(x) > -1}$, $\boxed{g(x) \in \mathbb{R}, \ g(x) \geq 2}$, $\boxed{h(x) \in \mathbb{R}, \ h(x) \neq 0}$



Question 9

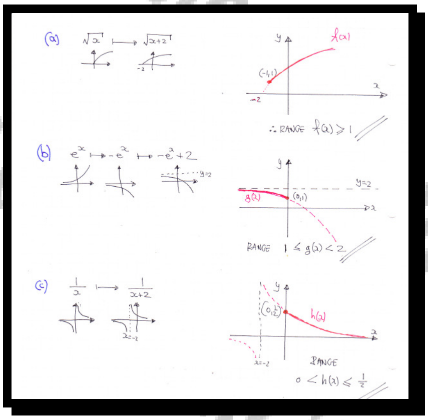
Find the range for each of the following functions.

a) $f(x) = \sqrt{x+2}, \quad x \in \mathbb{R}, \ x \geq -1$.

b) $g(x) = 2 - e^{x}, \quad x \in \mathbb{R}, \ x \leq 0$.

c) $h(x) = \dfrac{1}{x+2}, \quad x \in \mathbb{R}, \ x \geq 0$.

$\boxed{f(x) \in \mathbb{R}, \ f(x) \geq 1}$, $\boxed{g(x) \in \mathbb{R}, \ 1 \leq g(x) < 2}$, $\boxed{h(x) \in \mathbb{R}, \ 0 < h(x) \leq \frac{1}{2}}$



Question 10

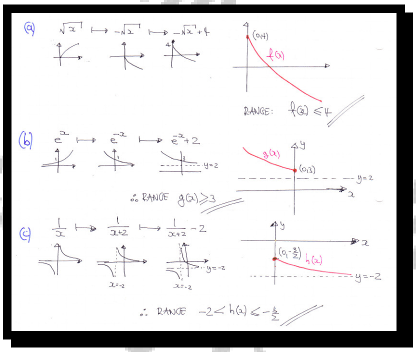
Find the range for each of the following functions.

a) $f(x) = 4 - \sqrt{x}, \quad x \in \mathbb{R}, \ x \geq 0$.

b) $g(x) = 2 + \mathrm{e}^{-x}, \quad x \in \mathbb{R}, \ x \leq 0$.

c) $h(x) = \dfrac{1}{x+2} - 2, \quad x \in \mathbb{R}, \ x \geq 0$.

$\boxed{f(x) \in \mathbb{R}, \ f(x) \leq 4}$, $\boxed{g(x) \in \mathbb{R}, \ \ g(x) \geq 3}$, $\boxed{h(x) \in \mathbb{R}, \ -2 < h(x) \leq -\frac{3}{2}}$



Question 11

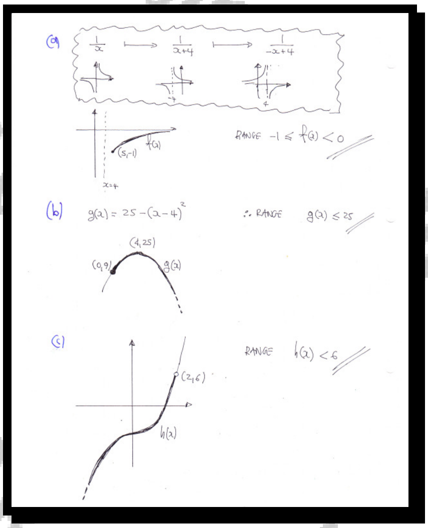
Find the range for each of the following functions.

a) $f(x) = \dfrac{1}{4-x}, \quad x \in \mathbb{R},\ x \geq 5.$

b) $g(x) = 25 - (x-4)^2, \quad x \in \mathbb{R}, \quad x \geq 0.$

c) $h(x) = x^3 - 2, \quad x \in \mathbb{R}, \quad x < 2.$

$\boxed{f(x) \in \mathbb{R},\ -1 \leq f(x) < 0}, \quad \boxed{g(x) \in \mathbb{R}, \quad g(x) \leq 25}, \quad \boxed{h(x) \in \mathbb{R}, \quad h(x) < 6}$



Question 12

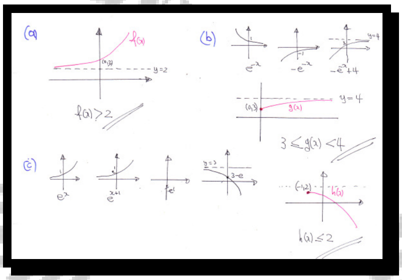
Find the range for each of the following functions.

a) $f(x) = \mathrm{e}^x + 2, \quad x \in \mathbb{R}$.

b) $g(x) = 4 - \mathrm{e}^{-x}, \quad x \in \mathbb{R}, \quad x \geq 0$.

c) $h(x) = 3 - \mathrm{e}^{x+1}, \quad x \in \mathbb{R}, \quad x \geq -1$.

$\boxed{f(x) \in \mathbb{R},\ f(x) > 2}$, $\boxed{g(x) \in \mathbb{R},\ \ g(x) \leq 25}$, $\boxed{h(x) \in \mathbb{R},\ \ h(x) \leq 2}$



Question 13

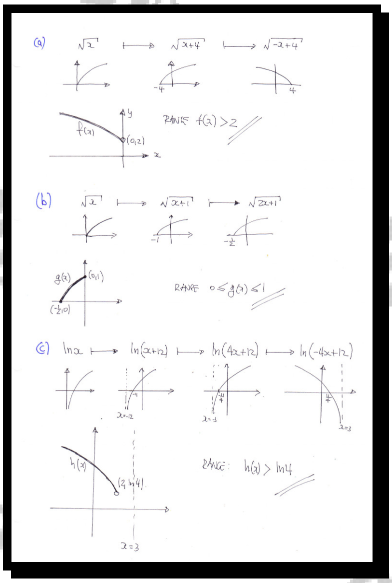
Find the range for each of the following functions.

a) $f(x) = \sqrt{4-x}, \ \ x \in \mathbb{R}, \ \ x < 0$.

b) $g(x) = \sqrt{2x+1}, \ \ x \in \mathbb{R}, \ \ -\frac{1}{2} \le x \le 0$.

c) $h(x) = \ln(12-4x), \ \ x \in \mathbb{R}, \ \ x < 2$.

$\boxed{f(x) \in \mathbb{R}, \ f(x) > 2}$, $\boxed{g(x) \in \mathbb{R}, \ 0 \le g(x) \le 1}$, $\boxed{h(x) \in \mathbb{R}, \ \ h(x) > \ln 4}$



FUNCTION COMPOSITIONS

Question 1

Find $fg(x)$ and $gf(x)$ if

$$f(x) = 2x + 1, \quad x \in \mathbb{R}$$

$$g(x) = x^2 - 1, \quad x \in \mathbb{R}.$$

Simplify the answers as much as possible.

$\boxed{fg(x) = 2x^2 - 1}$, $\boxed{gf(x) = 4x^2 + 4x}$

$fg(x) = f(g(x)) = f(x^2-1) = 2(x^2-1)+1 = 2x^2-1$

$gf(x) = g(f(x)) = g(2x+1) = (2x+1)^2-1 = 4x^2+4x$

Question 2

Find $fg(x)$ and $gf(x)$ if

$$f(x) = 4 - 3x, \quad x \in \mathbb{R}$$

$$g(x) = \sqrt{x}, \quad x \in \mathbb{R}, \quad x \geq 0.$$

Simplify the answers as much as possible.

$\boxed{fg(x) = 4 - 3\sqrt{x}}$, $\boxed{gf(x) = \sqrt{4 - 3x}}$

$fg(x) = f(g(x)) = f(\sqrt{x}) = 4-3\sqrt{x}$

$gf(x) = g(f(x)) = g(4-3x) = \sqrt{4-3x}$

Question 3

Find $fg(x)$ and $gf(x)$ if

$$f(x) = 3x - 8, \quad x \in \mathbb{R}$$

$$g(x) = \frac{1}{x}, \quad x \in \mathbb{R}, \quad x \neq 0.$$

Simplify the answers as much as possible.

$$\boxed{fg(x) = \frac{3}{x} - 8}, \quad \boxed{gf(x) = \frac{1}{3x-8}}$$

- $fg(x) = f(g(x)) = f\left(\frac{1}{x}\right) = 3\left(\frac{1}{x}\right) - 8 = \frac{3}{x} - 8$
- $gf(x) = g(f(x)) = g(3x-8) = \frac{1}{3x-8}$

Question 4

Find $fg(x)$ and $gf(x)$ if

$$f(x) = 4x - 1, \quad x \in \mathbb{R}$$

$$g(x) = \frac{x}{x+1}, \quad x \in \mathbb{R}, \quad x \neq -1.$$

Simplify the answers as much as possible.

$$\boxed{fg(x) = \frac{3x-1}{x+1}}, \quad \boxed{gf(x) = \frac{4x-1}{4x}}$$

- $fg(x) = f(g(x)) = f\left(\frac{x}{x+1}\right) = 4\left(\frac{x}{x+1}\right) - 1 = \frac{4x}{x+1} - 1$
  $= \frac{4x-(x+1)}{x+1} = \frac{3x-1}{x+1}$
- $gf(x) = g(f(x)) = g(4x-1) = \frac{4x-1}{(4x-1)+1} = \frac{4x-1}{4x}$

Question 5

Find $fg(x)$ and $gf(x)$ if

$$f(x) = 2x^2 + 1, \quad x \in \mathbb{R}$$

$$g(x) = \sqrt{x}, \quad x \in \mathbb{R}, \quad x \geq 0 .$$

Simplify the answers as much as possible.

$\boxed{fg(x) = 2x + 1}$, $\boxed{gf(x) = \sqrt{2x^2 + 1}}$

- $fg(x) = f(g(x)) = f(\sqrt{x}) = 2(\sqrt{x})^2 + 1 = 2x + 1$ //
- $gf(x) = g(f(x)) = g(2x^2+1) = \sqrt{2x^2+1}$ //

Question 6

Find $fg(x)$ and $gf(x)$ if

$$f(x) = (x+3)^2, \quad x \in \mathbb{R}$$

$$g(x) = 2x, \quad x \in \mathbb{R} .$$

Simplify the answers as much as possible.

$\boxed{fg(x) = (2x+3)^2}$, $\boxed{gf(x) = 2(x+3)^2}$

(a) $fg(x) = f(g(x)) = f(2x) = (2x+3)^2$ //

(b) $gf(x) = g(f(x)) = g((x+3)^2) = 2(x+3)^2$ //

Question 7

Find $fg(x)$ and $gf(x)$ if

$$f(x) = 2x - 1, \quad x \in \mathbb{R}$$

$$g(x) = \sqrt{x+3}, \quad x \in \mathbb{R}, \quad x \geq -3.$$

Simplify the answers as much as possible.

$$\boxed{fg(x) = 2\sqrt{x+3} - 1}, \quad \boxed{gf(x) = \sqrt{2x+2}}$$

$fg(x) = f(g(x)) = f(\sqrt{x+3}) = 2\sqrt{x+3} - 1$

$gf(x) = g(f(x)) = g(2x-1) = \sqrt{(2x-1)+3} = \sqrt{2x+2}$

Question 8

Find $fg(x)$ and $gf(x)$ if

$$f(x) = \sqrt{x}, \quad x \in \mathbb{R}, \quad x \geq 0$$

$$g(x) = \frac{2x^2}{x^2 - 1}, \quad x \in \mathbb{R}, \ x \neq \pm 1.$$

Simplify the answers as much as possible.

$$\boxed{fg(x) = \sqrt{\frac{2x^2}{x^2 - 1}}}, \quad \boxed{gf(x) = \frac{2x}{x-1}}$$

(a) $fg(x) = f(g(x)) = f\left(\frac{2x^2}{x^2-1}\right) = \sqrt{\frac{2x^2}{x^2-1}}$

(b) $gf(x) = g(f(x)) = g(\sqrt{x}) = \frac{2(\sqrt{x})^2}{(\sqrt{x})^2 - 1} = \frac{2x}{x-1}$

Question 9

Find $fg(x)$ and $gf(x)$ if

$$f(x) = 2x - 3, \quad x \in \mathbb{R}$$

$$g(x) = x - \frac{1}{x}, \quad x \in \mathbb{R}, \ x \neq 0.$$

Simplify the answer as much as possible.

$$\boxed{fg(x) = \frac{2x^2 - 3x - 2}{x}}, \quad \boxed{gf(x) = \frac{4x^2 - 12x + 8}{2x - 3}}$$

(a) $fg(x) = f(g(x)) = f\left(x - \frac{1}{x}\right) = 2\left(x - \frac{1}{x}\right) - 3 = 2x - \frac{2}{x} - 3$ $\left(\text{or } \frac{2x^2 - 3x - 2}{x}\right)$

(b) $gf(x) = g(f(x)) = g(2x-3) = (2x-3) - \frac{1}{2x-3}$

$= \frac{(2x-3)^2 - 1}{2x-3} = \frac{4x^2 - 12x + 8}{2x-3}$

Question 10

Find $fg(x)$ and $gf(x)$ if

$$f(x) = x^3 - 1, \quad x \in \mathbb{R}$$

$$g(x) = \frac{1}{\sqrt{x}}, \quad x \in \mathbb{R}, \ x > 0.$$

Simplify the answers as much as possible.

$$\boxed{fg(x) = \frac{1}{x\sqrt{x}} - 1}, \quad \boxed{gf(x) = \frac{1}{\sqrt{x^3 - 1}}}$$

- $fg(x) = f(g(x)) = f\left(\frac{1}{\sqrt{x}}\right) = \left(\frac{1}{\sqrt{x}}\right)^3 - 1 = \frac{1}{x\sqrt{x}} - 1$
- $gf(x) = g(f(x)) = g(x^3 - 1) = \frac{1}{\sqrt{x^3 - 1}}$

Question 11

Find $fg(x)$ and $gf(x)$ if

$$f(x) = 6 - x^2, \quad x \in \mathbb{R}$$

$$g(x) = \frac{x+1}{x}, \quad x \in \mathbb{R}, \ x \neq 0.$$

Simplify the answers as much as possible.

$$\boxed{fg(x) = \frac{5x^2 - 2x - 1}{x^2}}, \quad \boxed{gf(x) = \frac{7 - x^2}{6 - x^2}}$$

- $fg(x) = f(g(x)) = f\left(\frac{x+1}{x}\right) = 6 - \left(\frac{x+1}{x}\right)^2 = 6 - \frac{x^2+2x+1}{x^2}$
  $= \frac{6x^2 - (x^2+2x+1)}{x^2} = \frac{6x^2 - x^2 - 2x - 1}{x^2} = \frac{5x^2 - 2x - 1}{x^2}$
- $gf(x) = g(f(x)) = g(6 - x^2) = \frac{6 - x^2 + 1}{6 - x^2} = \frac{7 - x^2}{6 - x^2}$

Question 12

The following functions are defined by

$$f(x) = 2x+3, \quad x \in \mathbb{R}.$$

$$g(x) = 1-x^2, \quad x \in \mathbb{R}.$$

$$h(x) = \frac{1}{x}, \quad x \in \mathbb{R}, \; x \neq 0.$$

Find all six possible two-fold compositions for the above functions, simplifying the final answers as much as possible.

$$\boxed{\begin{array}{l} fg(x) = 5-2x^2 \\ gf(x) = -4x^2-12x-8 \end{array}}, \quad \boxed{\begin{array}{l} fh(x) = \dfrac{2}{x}+3 \\ hf(x) = \dfrac{1}{2x+3} \end{array}}, \quad \boxed{\begin{array}{l} gh(x) = 1-\dfrac{1}{x^2} \\ hg(x) = \dfrac{1}{1-x^2} \end{array}}$$

- $f(g(x)) = f(1-x^2) = 2(1-x^2)+3 = 5-2x^2$
- $g(f(x)) = g(2x+3) = 1-(2x+3)^2 = 1-(4x^2-12x+9) = -4x^2+12x-8$
- $f(h(x)) = f(\frac{1}{x}) = 2(\frac{1}{x})+3 = \frac{2}{x}+3$
- $h(f(x)) = h(2x+3) = \frac{1}{2x+3}$
- $g(h(x)) = g(\frac{1}{x}) = 1-\frac{1}{x^2} = \frac{x^2-1}{x^2}$
- $h(g(x)) = h(1-x^2) = \frac{1}{1-x^2}$

Question 13

The following functions are defined by

$$f(x) = 2x+1, \quad x \in \mathbb{R}.$$

$$g(x) = \mathrm{e}^{x}, \quad x \in \mathbb{R}.$$

$$h(x) = \sin x, \quad x \in \mathbb{R}.$$

Find all six possible two-fold compositions for the above functions simplifying the final answers as much as possible.

$$\boxed{\begin{aligned} fg(x) &= 2\mathrm{e}^{x}+1 \\ gf(x) &= \mathrm{e}^{2x+1} \end{aligned}}, \quad \boxed{\begin{aligned} fh(x) &= 2\sin x+1 \\ hf(x) &= \sin(2x+1) \end{aligned}}, \quad \boxed{\begin{aligned} gh(x) &= \mathrm{e}^{\sin x} \\ hg(x) &= \sin\left(\mathrm{e}^{x}\right) \end{aligned}}$$

(a) $fg(2) = f(g(2)) = f(2^2-5) = f(-1) = 1-(-1) = 2$

(b) $gf(4) = g(f(4)) = g(1-4) = g(3) = 3^2-5 = 4$

(c) $fh(9) = f(h(9)) = f(\sqrt{9}) = f(3) = 1-3 = -2$

(d) $hf(-15) = h(f(-15)) = h(1+15) = h(16) = \sqrt{16} = 4$

(e) $gh(4) = g(h(4)) = g(\sqrt{4}) = g(2) = 2^2-5 = -1$

(f) $hg(3) = h(g(3)) = h(3^2-5) = h(4) = \sqrt{4} = 2$

Question 14

The following functions are defined by

$$f(x) = 1 - 2x, \quad x \in \mathbb{R} .$$

$$g(x) = \mathrm{e}^{x}, \quad x \in \mathbb{R} .$$

$$h(x) = \sqrt{x}\ , \quad x \in \mathbb{R},\ x \geq 0 .$$

Find all six possible two-fold compositions for the above functions simplifying the final answers as much as possible.

$$\boxed{\begin{array}{l} fg(x) = 1 - 2\mathrm{e}^{x} \\ gf(x) = \mathrm{e}^{1-2x} \end{array}}, \quad \boxed{\begin{array}{l} fh(x) = 1 - 2\sqrt{x} \\ hf(x) = \sqrt{1-2x} \end{array}}, \quad \boxed{\begin{array}{l} gh(x) = \mathrm{e}^{\sqrt{x}} \\ hg(x) = \mathrm{e}^{\frac{1}{2}x} \end{array}}$$

$$fg(x) = f(g(x)) = f(e^{x}) = 1 - 2e^{x}$$
$$gf(x) = g(f(x)) = g(1-2x) = e^{1-2x}$$
$$fh(x) = f(h(x)) = f(\sqrt{x}) = 1 - 2\sqrt{x}$$
$$hf(x) = h(f(x)) = h(1-2x) = \sqrt{1-2x}$$
$$gh(x) = g(h(x)) = g(\sqrt{x}) = e^{\sqrt{x}}$$
$$hg(x) = h(g(x)) = h(e^{x}) = \sqrt{e^{x}} = (e^{x})^{\frac{1}{2}} = e^{\frac{1}{2}x}$$

Question 15

The following functions are defined by

$$f(x) = 1 - x, \quad x \in \mathbb{R}.$$

$$g(x) = x^2 - 5, \quad x \in \mathbb{R}.$$

$$h(x) = \sqrt{x}, \quad x \in \mathbb{R}, \; x \geq 0.$$

Evaluate the following function compositions.

a) $fg(2)$

b) $gf(4)$

c) $fh(9)$

d) $hf(-15)$

e) $gh(4)$

f) $hg(3)$

$\boxed{fg(2) = 2}$, $\boxed{gf(4) = 4}$, $\boxed{fh(9) = -2}$, $\boxed{hf(-15) = 4}$, $\boxed{gh(4) = -1}$, $\boxed{hg(3) = 2}$

(a) $fg(2) = f(g(2)) = f(2^2-5) = f(-1) = 1-(-1) = 2$

(b) $gf(4) = g(f(4)) = g(1-4) = g(-3) = (-3)^2 - 5 = 9 - 5 = 4$

(c) $fh(9) = f(h(9)) = f(\sqrt{9}) = f(3) = 1 - 3 = -2$

(d) $hf(-15) = h(f(-15)) = h(1-(-15)) = h(16) = \sqrt{16} = 4$

(e) $gh(4) = g(h(4)) = g(\sqrt{4}) = g(2) = 2^2 - 5 = 4 - 5 = -1$

(f) $hg(3) = h(g(3)) = h(3^2-5) = h(4) = \sqrt{4} = 2$

Question 16

The following functions are defined by

$$f(x) = 2x+5, \quad x \in \mathbb{R}$$

$$g(x) = \frac{4}{x}, \quad x \in \mathbb{R}, \; x \neq 0$$

$$h(x) = \sqrt{x+2}, \quad x \in \mathbb{R}, \; x \geq -2.$$

Evaluate the following function compositions.

a) $fg\left(\frac{1}{2}\right)$

b) $gf(-2)$

c) $hf(1)$

d) $fh(2)$

e) $hg\left(\frac{2}{7}\right)$

f) $gh\left(-\frac{7}{4}\right)$

g) $gfh(-1)$

h) $fgf(-2)$

i) $fff\left(\frac{1}{4}\right)$

$\boxed{fg\left(\frac{1}{2}\right)=21}$, $\boxed{gf(-2)=4}$, $\boxed{hf(1)=3}$, $\boxed{fh(2)=9}$, $\boxed{hg\left(\frac{2}{7}\right)=4}$, $\boxed{gh\left(-\frac{7}{4}\right)=8}$,
$\boxed{gfh(1)=\frac{4}{7}}$, $\boxed{fgf(-2)=13}$, $\boxed{fff\left(\frac{1}{4}\right)=37}$

(a) $f\left(g\left(\frac{1}{2}\right)\right) = f\left(\frac{4}{\frac{1}{2}}\right) = f(8) = 2\times 8+5 = 21$

(b) $g(f(-2)) = g(2(-2)+5) = g(1) = \frac{4}{1} = 4$

(c) $h(f(1)) = h(2\times 1+5) = h(7) = \sqrt{7+2} = \sqrt{9} = 3$

(d) $f(h(2)) = f\left(\sqrt{2+2}\right) = f(2) = 2\times 2+5 = 9$

(e) $h\left(g\left(\frac{2}{7}\right)\right) = h\left(\frac{4}{\frac{2}{7}}\right) = h(14) = \sqrt{14+2} = 4$

(f) $g\left(h\left(-\frac{7}{4}\right)\right) = g\left(\sqrt{-\frac{7}{4}+2}\right) = g\left(\sqrt{\frac{1}{4}}\right) = g\left(\frac{1}{2}\right) = \frac{4}{\frac{1}{2}} = 8$

(g) $g(f(h(-1))) = g\left(f\left(\sqrt{-1+2}\right)\right) = g(f(1)) = g(2\times 1+5) = g(7) = \frac{4}{7}$

(h) $f(g(f(-2))) = f(g(2(-2)+5)) = f(g(1)) = f\left(\frac{4}{1}\right) = 2\times 4+5 = 13$

(I) $f\left(f\left(f\left(\frac{1}{4}\right)\right)\right) = f\left(f\left(2\times\frac{1}{4}+5\right)\right) = f\left(f\left(\frac{11}{2}\right)\right) = f\left(2\times\frac{11}{2}+5\right) = f(16) = 37$

Question 17

The following functions are defined by

$$f(x) = x - 2, \quad x \in \mathbb{R}$$

$$g(x) = \ln x, \quad x \in \mathbb{R}, \ x > 0$$

$$h(x) = \mathrm{e}^{2x}, \quad x \in \mathbb{R}.$$

Find simplified expressions the following function compositions, stating in each case the domain and range.

a) $fg(x)$

b) $gf(x)$

c) $fh(x)$

d) $hf(x)$

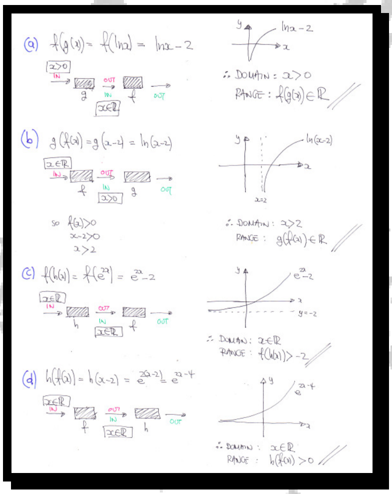
e) $gh(x)$

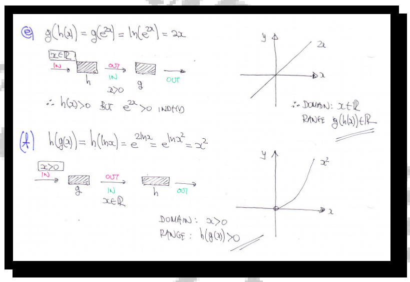
f) $hg(x)$

$\boxed{fg(x) = \ln x - 2, x > 0, fg(x) \in \mathbb{R}}$, $\boxed{gf(x) = \ln(x-2),\ x > 2,\ gf(x) \in \mathbb{R}}$,

$\boxed{fh(x) = \mathrm{e}^{2x} - 2,\ x \in \mathbb{R},\ fh(x) > -2}$, $\boxed{hf(x) = \mathrm{e}^{2x-4},\ x \in \mathbb{R},\ hf(x) \in \mathbb{R}}$,

$\boxed{gh(x) = 2x,\ x \in \mathbb{R},\ gh(x) \in \mathbb{R}}$, $\boxed{hg(x) = x^2,\ x > 0,\ hg(x) > 0}$





Question 17

The following functions are defined by

$$f(x) = 2x - 1, \quad x \in \mathbb{R}, \quad x \le 18$$

$$g(x) = x^2 + 2, \quad x \in \mathbb{R}, \quad x \ge 1$$

$$h(x) = \sqrt{x}, \quad x \in \mathbb{R}, \quad x \ge 0 .$$

Find simplified expressions for each of the following function compositions, stating in each case the domain and range.

a) $fg(x)$ **d)** $hf(x)$

b) $gf(x)$ **e)** $gh(x)$

c) $fh(x)$ **f)** $hg(x)$

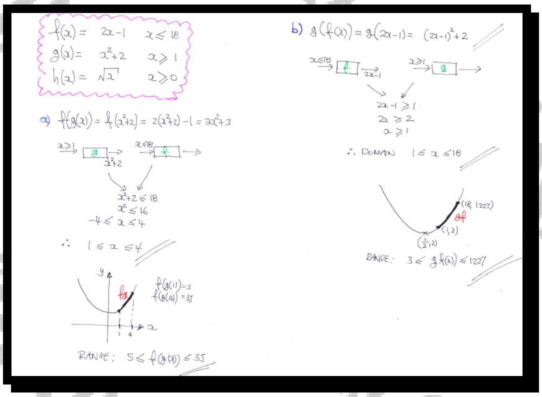
$\boxed{fg(x) = 2x^2 + 3, \quad 1 \le x \le 4, \quad 5 \le fg(x) \le 35}$,

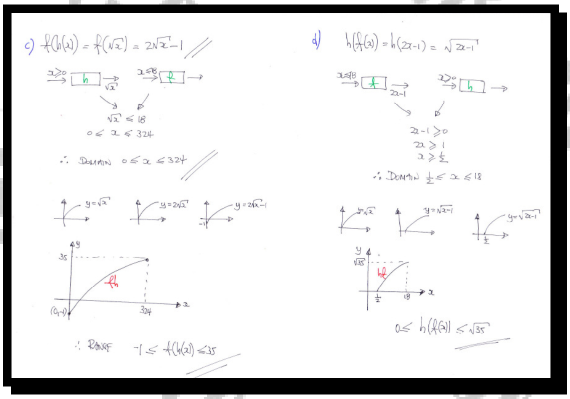
$\boxed{gf(x) = (2x-1)^2 + 2, \quad 1 \le x \le 18, \quad 3 \le gf(x) \le 1227}$,

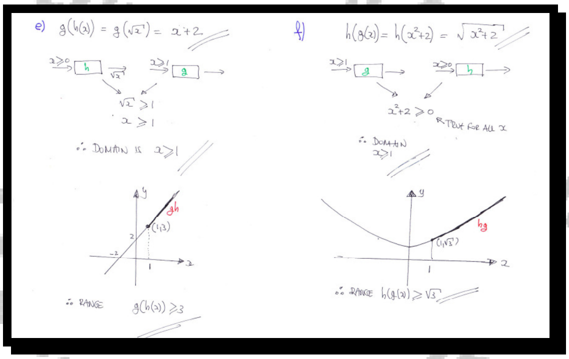
$\boxed{fh(x) = 2\sqrt{x} - 1, \quad 0 \le x \le 324, \quad -1 \le fh(x) \le 35}$,

$\boxed{hf(x) = \sqrt{2x-1}, \quad \frac{1}{2} \le x \le 18, \quad 0 \le hf(x) \le \sqrt{35}}$, $\boxed{gh(x) = x + 2, \quad x \ge 1, \quad gh(x) \ge 3}$,

$\boxed{hg(x) = \sqrt{x^2+2}, \quad x \ge 1, \quad hg(x) \ge \sqrt{3}}$







FUNCTION INVERSES

Question 1

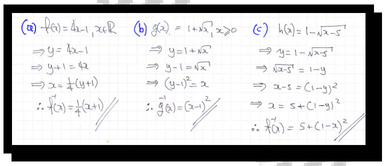
For each of the following functions find an expression for its inverse.

a) $f(x) = 4x - 1, \quad x \in \mathbb{R}$.

b) $g(x) = 1 + \sqrt{x}, \quad x \in \mathbb{R},\ x \geq 0$.

c) $h(x) = 1 - \sqrt{x-5}, \quad x \in \mathbb{R},\ x \geq 5$.

$$\boxed{f^{-1}(x) = \frac{x+1}{4}},\ \boxed{g^{-1}(x) = (x-1)^2},\ \boxed{h^{-1}(x) = 5 + (1-x)^2}$$



(a) $f(x) = 4x-1,\ x \in \mathbb{R}$

$\Rightarrow y = 4x - 1$

$\Rightarrow y + 1 = 4x$

$\Rightarrow x = \frac{1}{4}(y+1)$

$\therefore f^{-1}(x) = \frac{1}{4}(x+1)$

(b) $g(x) = 1 + \sqrt{x},\ x \geqslant 0$

$\Rightarrow y = 1 + \sqrt{x}$

$\Rightarrow y - 1 = \sqrt{x}$

$\Rightarrow (y-1)^2 = x$

$\therefore g^{-1}(x) = (x-1)^2$

(c) $h(x) = 1 - \sqrt{x-5}$

$\Rightarrow y = 1 - \sqrt{x-5}$

$\Rightarrow \sqrt{x-5} = 1 - y$

$\Rightarrow x - 5 = (1-y)^2$

$\Rightarrow x = 5 + (1-y)^2$

$\therefore h^{-1}(x) = 5 + (1-x)^2$

Question 2

For each of the following functions find an expression for its inverse.

a) $f(x) = 5 - 2x, \quad x \in \mathbb{R}$.

b) $g(x) = \dfrac{3}{x} - 2, \quad x \in \mathbb{R},\ x \neq 0$.

c) $h(x) = \sqrt{\dfrac{x}{2} - 1}, \quad x \in \mathbb{R},\ x \geq 2$.

$$\boxed{f^{-1}(x) = \frac{5-x}{2}},\ \boxed{g^{-1}(x) = \frac{3}{x+2}},\ \boxed{h^{-1}(x) = 2x^2 + 2}$$

(a) $f(x) = 5 - 2x,\ x \in \mathbb{R}$

- $y = 5 - 2x$
- $2x = 5 - y$

  $x = \frac{1}{2}(5 - y)$
- $f^{-1}(x) = \frac{1}{2}(5 - x)$

(b) $g(x) = \frac{3}{x} - 2,\ x \in \mathbb{R},\ x \neq 0$

- $y = \frac{3}{x} - 2$
- $y + 2 = \frac{3}{x}$

  $\frac{1}{y+2} = \frac{x}{3}$

  $\frac{3}{y+2} = x$
- $g^{-1}(x) = \frac{3}{x+2}$

(c) $h(x) = \sqrt{\frac{x}{2} - 1}\ \ x \in \mathbb{R},\ x \geqslant 2$

- $y = \sqrt{\frac{x}{2} - 1}$
- $y^2 = \frac{x}{2} - 1$

  $y^2 + 1 = \frac{x}{2}$

  $2(y^2 + 1) = x$
- $h^{-1}(x) = 2(x^2 + 1)$

Question 3

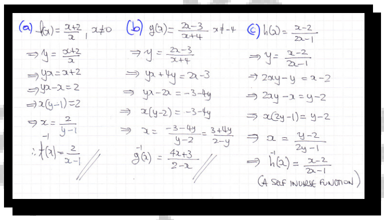
For each of the following functions find an expression for its inverse.

a) $f(x) = \dfrac{x+2}{x}, \quad x \in \mathbb{R},\ x \neq 0.$

b) $g(x) = \dfrac{2x-3}{x+4}, \quad x \in \mathbb{R},\ x \neq -4.$

c) $h(x) = \dfrac{x-2}{2x-1}, \quad x \in \mathbb{R},\ x \neq \dfrac{1}{2}.$

$$\boxed{f^{-1}(x) = \frac{2}{x-1}}, \boxed{g^{-1}(x) = \frac{4x+3}{2-x}}, \boxed{h^{-1}(x) = \frac{x-2}{2x-1}}$$



(a) $f(x) = \frac{x+2}{x}$, $x \neq 0$

$\Rightarrow y = \frac{x+2}{x}$

$\Rightarrow yx = x+2$

$\Rightarrow yx - x = 2$

$\Rightarrow x(y-1) = 2$

$\Rightarrow x = \frac{2}{y-1}$

$\therefore f^{-1}(x) = \frac{2}{x-1}$

(b) $g(x) = \frac{2x-3}{x+4}$ $x \neq -4$

$\Rightarrow y = \frac{2x-3}{x+4}$

$\Rightarrow yx + 4y = 2x - 3$

$\Rightarrow yx - 2x = -3 - 4y$

$\Rightarrow x(y-2) = -3-4y$

$\Rightarrow x = \frac{-3-4y}{y-2} = \frac{3+4y}{2-y}$

$g^{-1}(x) = \frac{4x+3}{2-x}$

(c) $h(x) = \frac{x-2}{2x-1}$

$\Rightarrow y = \frac{x-2}{2x-1}$

$\Rightarrow 2xy - y = x - 2$

$\Rightarrow 2xy - x = y - 2$

$\Rightarrow x(2y-1) = y - 2$

$\Rightarrow x = \frac{y-2}{2y-1}$

$\Rightarrow h^{-1}(x) = \frac{x-2}{2x-1}$

(A SELF INVERSE FUNCTION)

Question 4

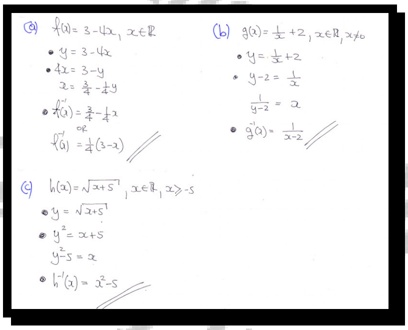
For each of the following functions find an expression for its inverse.

a) $f(x) = 3 - 4x, \quad x \in \mathbb{R}$.

b) $g(x) = \dfrac{1}{x} + 2, \quad x \in \mathbb{R},\ x \neq 0$.

c) $h(x) = \sqrt{x+5}, \quad x \in \mathbb{R},\ x \geq -5$.

$$\boxed{f^{-1}(x) = \frac{3-x}{4}}, \boxed{g^{-1}(x) = \frac{1}{x-2}}, \boxed{h^{-1}(x) = x^2 - 5}$$



(a) $f(x) = 3 - 4x,\ x \in \mathbb{R}$

- $y = 3 - 4x$
- $4x = 3 - y$

$x = \frac{3}{4} - \frac{1}{4}y$

- $f^{-1}(x) = \frac{3}{4} - \frac{1}{4}x$

OR

$f^{-1}(x) = \frac{1}{4}(3 - x)$

(b) $g(x) = \frac{1}{x} + 2,\ x \in \mathbb{R},\ x \neq 0$

- $y = \frac{1}{x} + 2$
- $y - 2 = \frac{1}{x}$

$\frac{1}{y-2} = x$

- $g^{-1}(x) = \frac{1}{x-2}$

(c) $h(x) = \sqrt{x+5},\ x \in \mathbb{R},\ x \geq -5$

- $y = \sqrt{x+5}$
- $y^2 = x + 5$

$y^2 - 5 = x$

- $h^{-1}(x) = x^2 - 5$

Question 5

For each of the following functions find an expression for its inverse.

a) $f(x) = 20 - 4x, \quad x \in \mathbb{R}$.

b) $g(x) = 5 - \dfrac{2}{x}, \quad x \in \mathbb{R},\ x \neq 0$.

c) $h(x) = \sqrt{x} - 2, \quad x \in \mathbb{R},\ x \geq 0$.

$$\boxed{f^{-1}(x) = 5 - \frac{1}{4}x}, \boxed{g^{-1}(x) = \frac{2}{5-x}}, \boxed{h^{-1}(x) = (x+2)^2}$$

(a) $f(x) = 20 - 4x$

$y = 20 - 4x$

$4x = 20 - y$

$x = 5 - \frac{1}{4}y$

$\therefore f^{-1}(x) = 5 - \frac{1}{4}x$

(b) $g(x) = 5 - \frac{2}{x}$

$y = 5 - \frac{2}{x}$

$\frac{2}{x} = 5 - y$

$\frac{x}{2} = \frac{1}{5-y}$

$x = \frac{2}{5-y}$

$\therefore g^{-1}(x) = \frac{2}{5-x}$

(c) $h(x) = \sqrt{x} - 2$

$y = \sqrt{x} - 2$

$y + 2 = \sqrt{x}$

$(y+2)^2 = x$

$\therefore h^{-1}(x) = (x+2)^2$

Question 6

For each of the following functions find an expression for its inverse.

a) $f(x) = \dfrac{4}{x+1}, \quad x \in \mathbb{R},\ x \neq -1.$

b) $g(x) = \dfrac{2x}{x+1}, \quad x \in \mathbb{R},\ x \neq -1.$

c) $h(x) = \dfrac{x+2}{x-4}, \quad x \in \mathbb{R},\ x \neq 4.$

$$\boxed{f^{-1}(x) = \frac{4}{x} - 1},\ \boxed{g^{-1}(x) = \frac{x}{2-x}},\ \boxed{h^{-1}(x) = \frac{4x+2}{x-1}}$$

(a) $f(x) = \frac{4}{x+1} \quad x \in \mathbb{R}, x \neq -1$

- $y = \frac{4}{x+1}$
- $\frac{1}{y} = \frac{x+1}{4}$

  $\frac{4}{y} = x+1$

  $\frac{4}{y} - 1 = x$
- $f^{-1}(x) = \frac{4}{x} - 1$

(b) $g(x) = \frac{2x}{x+1}, \ x \in \mathbb{R}, x \neq -1$

- $y = \frac{2x}{x+1}$
- $y(x+1) = 2x$

  $yx + y = 2x$

  $y = 2x - yx$

  $y = x(2-y)$

  $\frac{y}{2-y} = x$
- $g^{-1}(x) = \frac{x}{2-x}$

(c) $h(x) = \frac{x+2}{x-4}, \ x \in \mathbb{R}, x \neq 4$

- $y = \frac{x+2}{x-4}$
- $y(x-4) = x+2$

  $yx - 4y = x+2$

  $yx - x = 4y+2$

  $x(y-1) = 4y+2$

  $x = \frac{4y+2}{y-1}$
- $\therefore h^{-1}(x) = \frac{4x+2}{x-1}$

Question 7

For each of the following functions find an expression for its inverse.

a) $f(x) = \dfrac{x}{x-1}, \quad x \in \mathbb{R},\ x \neq 1.$

b) $g(x) = \dfrac{1}{2}\mathrm{e}^{2x}, \quad x \in \mathbb{R}.$

c) $h(x) = \ln(5-x), \quad x \in \mathbb{R},\ x > 5.$

$\boxed{f^{-1}(x) = \dfrac{x}{x-1}}$, $\boxed{g^{-1}(x) = \dfrac{1}{2}\ln 2x}$, $\boxed{h^{-1}(x) = 5 - \mathrm{e}^{x}}$

(a) $f(x) = \frac{x}{x-1},\ x \in \mathbb{R},\ x \neq 1$

- $y = \frac{x}{x-1}$
- $yx - y = x$

  $yx - x = y$

  $x(y-1) = x$

  $x = \frac{y}{y-1}$
- $f^{-1}(x) = \frac{x}{x-1}$

(b) $g(x) = \frac{1}{2}e^{2x},\ x \in \mathbb{R}$

- $y = \frac{1}{2}e^{2x}$
- $2y = e^{2x}$

  $\ln(2y) = 2x$

  $x = \frac{1}{2}\ln(2y)$
- $g^{-1}(x) = \frac{1}{2}\ln(2x)$

(c) $h(x) = \ln(5-x),\ x \in \mathbb{R},\ x > 5$

- $y = \ln(5-x)$
- $e^{y} = 5 - x$

  $x = 5 - e^{y}$
- $h^{-1}(x) = 5 - e^{x}$

Question 8

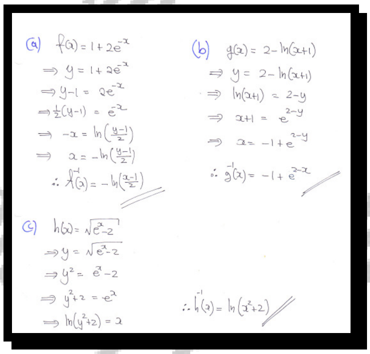
For each of the following functions find an expression for its inverse.

a) $f(x) = 1 + 2e^{-x}, \quad x \in \mathbb{R}$.

b) $g(x) = 2 - \ln(x+1), \quad x \in \mathbb{R},\ x > -1$.

c) $h(x) = \sqrt{e^x - 2}, \quad x \in \mathbb{R},\ x \geq \ln 2$.

$\boxed{f^{-1}(x) = -\ln\left(\frac{x-1}{2}\right)}$, $\boxed{g^{-1}(x) = e^{2-x} - 1}$, $\boxed{h^{-1}(x) = \ln\left(x^2 + 2\right)}$



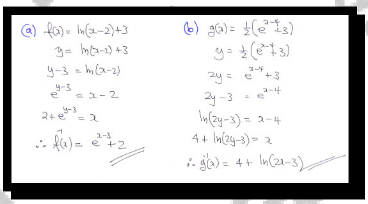
Question 9

For each of the following functions find an expression for its inverse.

a) $f(x) = \ln(x-2) + 3, \quad x \in \mathbb{R},\ x > 2$.

b) $g(x) = \frac{1}{2}\left(e^{x-4} + 3\right), \quad x \in \mathbb{R}$.

$\boxed{f^{-1}(x) = e^{x-3} + 2}$, $\boxed{g^{-1}(x) = 4 + \ln(2x - 3)}$



Question 10

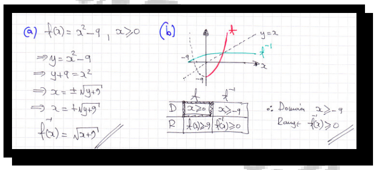
A function f is defined by

$$f(x) = x^2 - 9, \quad x \in \mathbb{R},\ x \geq 0.$$

a) Find an expression for $f^{-1}(x)$.

b) Find the domain and range of $f^{-1}(x)$.

$$\boxed{f^{-1}(x) = \sqrt{x+9}}, \ \boxed{x \geq -9}, \ \boxed{f^{-1}(x) \geq 0}$$



Question 11

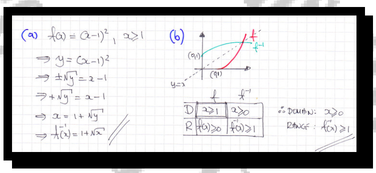
A function f is defined by

$$f(x) = (x-1)^2, \quad x \in \mathbb{R},\ x \geq 1.$$

a) Find an expression for $f^{-1}(x)$.

b) Find the domain and range of $f^{-1}(x)$.

$$\boxed{f^{-1}(x) = 1+\sqrt{x}}, \ \boxed{x \geq 0}, \ \boxed{f^{-1}(x) \geq 1}$$



Question 12

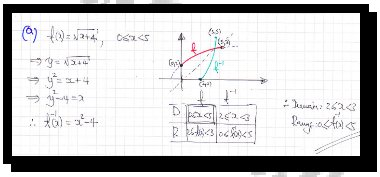
A function f is defined by

$$f(x) = \sqrt{x+4}, \quad x \in \mathbb{R},\ 0 \le x < 5.$$

a) Find an expression for $f^{-1}(x)$.

b) Find the domain and range of $f^{-1}(x)$.

$\boxed{f^{-1}(x) = x^2 - 4}$, $\boxed{2 \le x < 3}$, $\boxed{0 \le f^{-1}(x) < 5}$



Question 13

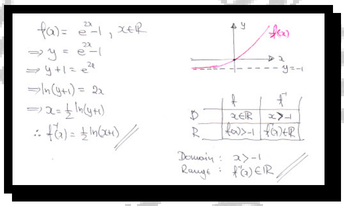
A function f is defined by

$$f(x) = \mathrm{e}^{2x} - 1, \quad x \in \mathbb{R}.$$

a) Find an expression for $f^{-1}(x)$.

b) Find the domain and range of $f^{-1}(x)$.

$\boxed{f^{-1}(x) = \frac{1}{2}\ln(x+1)}$, $\boxed{x > -1}$, $\boxed{f^{-1}(x) \in \mathbb{R}}$



Question 14

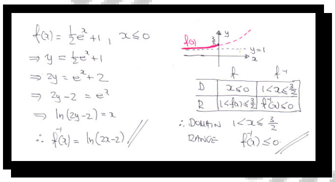
A function f is defined by

$$f(x) = \frac{1}{2}e^{x} + 1, \quad x \in \mathbb{R}, \ x \le 0.$$

a) Find an expression for $f^{-1}(x)$.

b) Find the domain and range of $f^{-1}(x)$.

$\boxed{f^{-1}(x) = \ln(2x-2)}$, $\boxed{1 < x \le \frac{3}{2}}$, $\boxed{f^{-1}(x) \le 0}$



Question 15

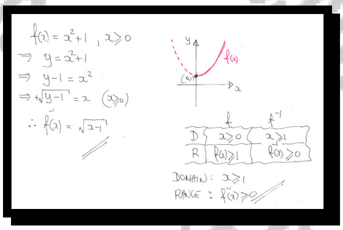
A function f is defined by

$$f(x) = x^{2} + 1, \quad x \in \mathbb{R}, \ x \ge 0.$$

a) Find an expression for $f^{-1}(x)$.

b) Find the domain and range of $f^{-1}(x)$.

$\boxed{f^{-1}(x) = \sqrt{x-1}}$, $\boxed{x \ge 1}$, $\boxed{f^{-1}(x) \ge 0}$



Question 16

A function f is defined by

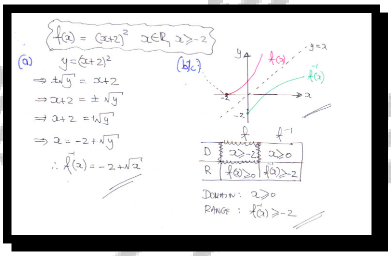
$$f(x) = (x+2)^2, \quad x \in \mathbb{R},\ x \geq -2.$$

a) Find an expression for $f^{-1}(x)$.

b) Sketch in the same diagram the graphs of $f(x)$ and $f^{-1}(x)$.

c) Find the domain and range of $f^{-1}(x)$.

$\boxed{f^{-1}(x) = -2+\sqrt{x}}$, $\boxed{x \geq 0}$, $\boxed{f^{-1}(x) \geq -2}$



Question 17

A function f is defined by

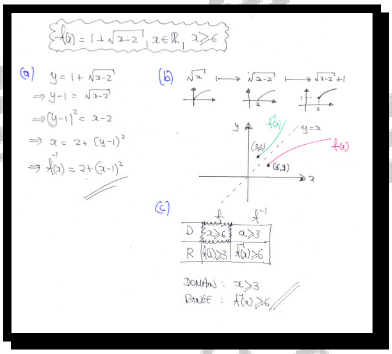
$$f(x) = 1 + \sqrt{x-2}, \quad x \in \mathbb{R}, \ x \geq 6.$$

a) Find an expression for $f^{-1}(x)$.

b) Sketch in the same diagram the graphs of $f(x)$ and $f^{-1}(x)$.

c) Find the domain and range of $f^{-1}(x)$.

$\boxed{f^{-1}(x) = 2 + (x-1)^2}$, $\boxed{x \geq 3}$, $\boxed{f^{-1}(x) \geq 0}$



Question 18

A function f is defined by

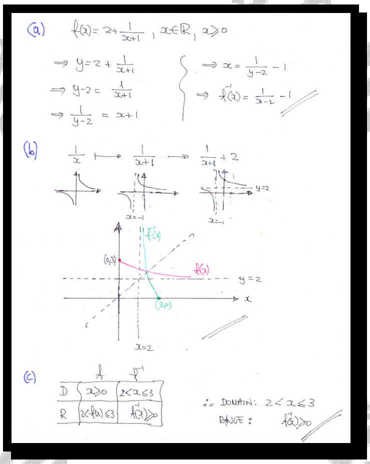
$$f(x) = 2 + \frac{1}{x+1}, \quad x \in \mathbb{R},\ x \geq 0.$$

a) Find an expression for $f^{-1}(x)$.

b) Sketch in the same diagram the graphs of $f(x)$ and $f^{-1}(x)$.

c) Find the domain and range of $f^{-1}(x)$.

$$\boxed{f^{-1}(x) = \frac{3-x}{x-2}},\ \boxed{2 < x \leq 3},\ \boxed{f^{-1}(x) \geq 0}$$



Question 19

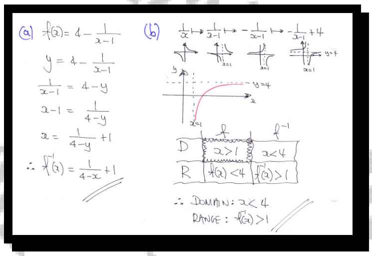
A function f is defined by

$$f(x) = 4 - \frac{1}{x-1}, \quad x \in \mathbb{R},\ x > 1.$$

a) Find an expression for $f^{-1}(x)$.

b) Find the domain and range of $f^{-1}(x)$.

$\boxed{f^{-1}(x) = \frac{x-5}{x-4}}$, $\boxed{x < 4}$, $\boxed{f^{-1}(x) > 1}$



Question 20

A function f is defined by

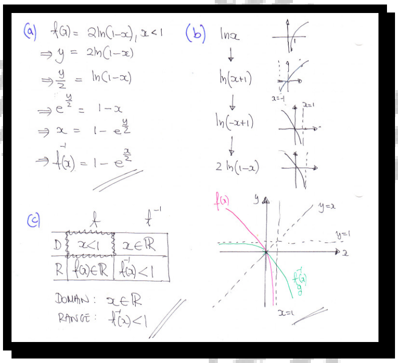
$$f(x) = 2\ln(1-x), \quad x < 1.$$

a) Find an expression for $f^{-1}(x)$.

b) Sketch in the same diagram the graphs of $f(x)$ and $f^{-1}(x)$.

c) Find the domain and range of $f^{-1}(x)$.

$\boxed{f^{-1}(x) = 1 - e^{\frac{1}{2}x}}$, $\boxed{x \in \mathbb{R}}$, $\boxed{f^{-1}(x) < 1}$



Question 21

A function f is defined by

$$f(x) = 2 - 3\sin x, \quad -\frac{\pi}{2} \leq x < \frac{\pi}{2}.$$

a) Find an expression for $f^{-1}(x)$.

b) Find the domain and range of $f^{-1}(x)$.

$\boxed{f^{-1}(x) = \arcsin\left(\frac{2-x}{3}\right)}$, $\boxed{-1 \leq x \leq 1}$, $\boxed{0 \leq f^{-1}(x) < 2\pi}$

(a) $f(x) = 2 - 3\sin x$

$\Rightarrow y = 2 - 3\sin x$

$\Rightarrow 3\sin x = 2 - y$

$\Rightarrow \sin x = \frac{2-y}{3}$

$\Rightarrow x = \arcsin\left(\frac{2-y}{3}\right)$

$\Rightarrow f^{-1}(x) = \arcsin\left(\frac{2-x}{3}\right)$

(b)

	f	f^{-1}
D	$-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$	$-1 \leq x \leq 5$
R	$-1 \leq f(x) \leq 5$	$-\frac{\pi}{2} \leq f^{-1}(x) \leq \frac{\pi}{2}$

$\therefore$ DOMAIN: $-1 \leq x \leq 5$

RANGE $-\frac{\pi}{2} \leq f^{-1}(x) \leq \frac{\pi}{2}$

Question 22

A function f is defined by

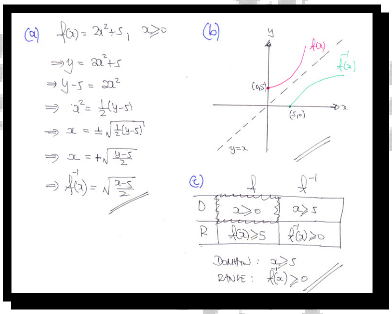
$$f(x) = 2x^2 + 5, \quad x \in \mathbb{R}, \; x \geq 0.$$

a) Find an expression for $f^{-1}(x)$.

b) Sketch in the same diagram the graphs of $f(x)$ and $f^{-1}(x)$.

c) Find the domain and range of $f^{-1}(x)$.

$$\boxed{f^{-1}(x) = \sqrt{\frac{x-5}{2}}}, \; \boxed{x \geq 5}, \; \boxed{f^{-1}(x) \geq 0}$$



Question 23

A function f is defined by

$$f(x) = x^2 - 4x - 1, \quad x \in \mathbb{R}, \ x < 0.$$

a) By completing the square, or otherwise, find an expression for $f^{-1}(x)$.

b) Sketch in the same diagram the graphs of $f(x)$ and $f^{-1}(x)$.

c) Find the domain and range of $f^{-1}(x)$.

$\boxed{f^{-1}(x) = 2 - \sqrt{x+5}}$, $\boxed{x > -1}$, $\boxed{f^{-1}(x) < 0}$